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Mean field limit for bosons and propagation of Wigner measures

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We consider the N-body Schrödinger dynamics of bosons in the mean field limit with a bounded pair-interaction potential. According to the previous work [Ammari, Z. and Nier, F., "Mean field limit for bosons and infinite dimensional phase-space analysis," Ann. Henri Poincare **9**, 1503 (2008)], the mean field limit is translated into a semiclassical problem with a small parameter $\varepsilon \to 0$, after introducing an ε-dependent bosonic quantization. The limits of quantum correlation functions are expressed as a push forward by a nonlinear flow (e.g., Hartree) of the associated Wigner measures. These object and their basic properties were introduced by Ammari and Nier in the infinite dimensional setting. The additional result presented here states that the transport by the nonlinear flow holds for a rather general class of quantum states in their mean field limit. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3115046]

I. INTRODUCTION

Consider the Schrödinger operator of a nonrelativistic boson system of N particles,

$$\mathbf{H}_N = \sum_{i=1}^{N} -\Delta_{x_i} + \frac{1}{N}\sum_{0\le i<j\le N} V(x_i - x_j), \quad \text{on } \mathbb{R}^{dN},$$

with a potential satisfying $V(x)=V(-x)$. The operator $\mathbf{H}_N$ acts on the space $L_s^2(\mathbb{R}^{dN})$ of symmetric square integrable functions since we deal with bosonic particles. This means that

$$\Psi_N \in L_s^2(\mathbb{R}^{dN}) \quad \text{iff} \quad \Psi_N \in L^2(\mathbb{R}^{dN}) \quad \text{and} \quad \Psi_N(x_1, \ldots, x_N) = \Psi_N(x_{\sigma_1}, \ldots, x_{\sigma_N}) \text{a.e.}$$

for any permutation σ on the symmetric group Sym(N). Under suitable conditions on the potential V the operator $\mathbf{H}_N$ is self-adjoint on $L_s^2(\mathbb{R}^{dN})$ for any positive integer N. By Stone's theorem the time-dependent Schrödinger equation,

$$i\partial_t \Psi_N(t) = \mathbf{H}_N \Psi_N(t),$$

admits for any initial data $\Psi_N(0)=\Psi_N \in L_s^2(\mathbb{R}^{dN})$ a unique solution given by

$$\Psi_N(t) = e^{-it\mathbf{H}_N}\Psi_N.$$

This paper studies the asymptotic behavior of some correlation functions of the N-body Schrödinger dynamics when the number of particles N is large. We aim precisely at proving convergence when $N \to \infty$ of the following quantities:

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$$\langle e^{-it\mathbf{H}_N}\Psi_N, B\otimes 1^{(N-k)}e^{-it\mathbf{H}_N}\Psi_N\rangle_{L^2(\mathbb{R}^{dN})}, \tag{1}$$

such that $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})}=1$ and B is any bounded operator on $L^2(\mathbb{R}^{kd})$ with k fixed. Here $B\otimes 1^{(N-k)}$ denotes the operator on $L^2(\mathbb{R}^{dN})$ defined by

$$B\otimes 1^{(N-k)}\varphi^{\otimes N} = (B\varphi^{\otimes k})\otimes\varphi^{\otimes(N-k)}, \quad \forall\, \varphi\in L^2(\mathbb{R}^d).$$

It is well known, under some assumptions, that correlation functions (1) converge for the specific choice of initial condition $\Psi_N=\varphi^{\otimes N}$ with $\|\varphi\|_{L^2(\mathbb{R}^d)}=1$. Moreover, the limit is

$$\lim_{N\to\infty}\langle\Psi_N(t), B\otimes 1^{(N-k)}\Psi_N(t)\rangle_{L^2(\mathbb{R}^{dN})} = \langle\varphi_t^{\otimes k}, B\varphi_t^{\otimes k}\rangle_{L^2(\mathbb{R}^{kd})}, \tag{2}$$

where φ_t solves the (nonlinear) Hartree equation

$$\begin{cases} i\partial_t\varphi_t = -\Delta\varphi_t + (V*|\varphi_t|^2)\varphi_t \ \text{ on } \ \mathbb{R}_t\times\mathbb{R}^d, \\ \varphi_{t=0}=\varphi. \end{cases}$$

This is commonly called the mean field limit of Schrödinger dynamics for boson systems. There exists an alternative way to formulate this problem using a Fock space framework. Indeed, consider the following Hamiltonian:

$$\varepsilon^{-1}H_\varepsilon = \left[\int_{\mathbb{R}^d}\nabla a^*(x)\,\nabla a(x)dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^{2d}}V(x-y)a^*(x)a^*(y)a(x)a(y)dxdy\right],$$

where a, a^* are the usual creation-annihilation operator-valued distributions in the Fock space over $L^2(\mathbb{R}^d)$, i.e.,

$$[a(x),a^*(y)] = \delta(x-y), \ \ [a^*(x),a^*(y)] = 0 = [a(x),a(y)].$$

The following relation

$$\varepsilon^{-1}H_{\varepsilon|L_s^2(\mathbb{R}^{dN})} = \mathbf{H}_N, \quad \text{if } \ \varepsilon = \frac{1}{N},$$

holds true. Thus, the correlation functions in (1) may be written (up to an unessential factor) as

$$\langle e^{-it\varepsilon^{-1}H_\varepsilon}\Psi_\varepsilon, b^{\mathrm{Wick}}e^{-it\varepsilon^{-1}H_\varepsilon}\Psi_\varepsilon\rangle, \tag{3}$$

where b^{Wick} denotes ε-dependent Wick observables defined by

$$b^{\mathrm{Wick}} = \varepsilon^k\int_{\mathbb{R}^{2kd}}\prod_{i=1}^k a^*(x_i)B(x_1,\ldots,x_k;y_1,\ldots,y_k)\prod_{j=1}^k a(y_j)dx_1\ldots dx_k dy_1\ldots dy_k, \tag{4}$$

with $B(x_1,\ldots,x_k;y_1,\ldots,y_k)$ denotes the distribution kernel of the operator B on $L^2(\mathbb{R}^{kd})$. This provides a bridge between the so-called semiclassical limit $\varepsilon\to 0$ for H_ε and the mean field limit $N\to\infty$ for $\mathbf{H}_N$. Except that the limit $\varepsilon\to 0$ of (3) is usually proven for coherent states initial condition,

$$\Psi_\varepsilon = e^{-|\varphi|^2/2\varepsilon}\sum_{n=0}^\infty \varepsilon^{-n/2}\frac{\varphi^{\otimes n}}{\sqrt{n!}}, \quad \varphi\in L^2(\mathbb{R}^d),$$

and uses the so-called Hepp method, different from the convergence analysis of (2) with Hermite states $\Psi_N=\varphi^{\otimes N}$. However, arguments in Refs. 19 and 1 show that the approaches with Hermite and coherent states are equivalent. In the sense that one approach implies the other one under suitable assumptions.

The mathematical analysis of the mean field (semiclassical) limit of the N-body quantum dynamics of bosons started with the work of Refs. 15 and 13 using coherent states. Since, the problem has experienced intensive investigations based mainly on the so-called BBGKY hierarchy method (explained in Ref. 20). This BBGKY approach essentially considers the evolution of Hermite states. Interest has more recently focused on the analysis of singular interaction potential related with the Gross–Pitaevskii equations for Bose–Einstein condensates (see, for example, Refs. 2, 6, 3, and 7).

Recently, a new method was given in Ref. 8 (see also Ref. 9) for a scalar bounded potential which, combined with our previous article,[1] inspires this work. This method avoids the use of specific choice of initial states. Instead, it gives the asymptotics of time-evolved Wick observables in the following way:

$$e^{it\varepsilon^{-1}H_\varepsilon}b^{\mathrm{Wick}}e^{-it\varepsilon^{-1}H_\varepsilon} = b(t)^{\mathrm{Wick}} + R(\varepsilon),$$

where $\lim_{\varepsilon\to 0}R(\varepsilon)=0$ in a particular sense (see Theorem III.1) and $b(t)^{\mathrm{Wick}}$ is an infinite sum of Wick operators as in (4) with time-dependent kernels. The latter result is stronger than convergence of correlation functions (1) and (3). In fact, we will show that it implies convergence of correlation functions (3) for a wide class of quantum states using a new tool, namely, Wigner measures introduced in Ref. 1.

In the work,[1] Wigner measures were extended to the infinite dimensional setting, as Borel probability measures under general assumptions. It was also explained how the previous formulations of the mean field limit are contained in the definition of these asymptotic Wigner measures, after a reformulation of the N-body problem as a semiclassical problem with the small parameter $\varepsilon = 1/N \to 0$.

In the present paper, the problem of the mean field dynamics is considered under some restrictive assumptions on the initial data. The convergence of correlation functions (3) will be proven for a class of families (or sequences) of density operator parametrized by $\varepsilon>0$, which contains all the common examples (coherent, Hermite states, etc.). Furthermore, the limit will be expressed as push forward by a nonlinear flow (e.g., Hartree) of Wigner measures associated with those density operator families. Remember that contrary to the finite dimensional case, no natural pseudodifferential calculus can be deformed by arbitrary nonlinear flows. Thus, the propagation of Wigner measures as dual objects cannot be straightforward in the infinite dimensional case. To enlighten the discussion let us give an example. Let $\Psi_N=\varphi^{\otimes N}$ with $\varphi\in L^2(\mathbb{R}^d)$, $\|\varphi\|_{L^2(\mathbb{R}^d)}=1$ and observe that the limit of (1) at time $t=0$ is

$$\lim_{N\to\infty}\langle \Psi_N, B\otimes 1^{(N-k)}\Psi_N\rangle_{L^2(\mathbb{R}^{dN})} = \int_{L^2(\mathbb{R}^d)} b(z)d\mu(z),$$

with $b(z)=\langle z^{\otimes k}, Bz^{\otimes k}\rangle_{L^2(\mathbb{R}^{kd})}$ and μ is the measure defined by

$$\mu = \frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}\varphi}d\theta,$$

where $\delta_{e^{i\theta}\varphi}$ is the delta function on $L^2(\mathbb{R}^d)$ at the point $e^{i\theta}\varphi$. Hence, the sequence of Hermite states $(\Psi_N)_{N\in\mathbb{N}^*}$ admits the Wigner measure on $L^2(\mathbb{R}^d)$, μ. The convergence of correlation functions (1) in the mean field limit states simply

$$\lim_{N\to\infty}\langle \Psi_N(t), B\otimes 1^{(N-k)}\Psi_N(t)\rangle_{L^2(\mathbb{R}^{dN})} = \int_{L^2(\mathbb{R}^d)} b(z_t)d\mu(z),$$

where z_t solves the Hartree equation with initial condition z. Thus, the sequence of quantum states $(\Psi_N(t)=e^{-it\mathbf{H}_N}\varphi^{\otimes N})_{N\in\mathbb{N}^*}$ admits the Wigner measure μ_t defined by

$$\int_{L^2(\mathbb{R}^d)} b(z)\mu_t(z) = \int_{L^2(\mathbb{R}^d)} b(z_t)d\mu(z).$$

Therefore the mean field limit can be interpreted as a transport of Wigner measures by a nonlinear flow.

The result here holds when the pair-interaction potential $V(x-y)$ is a bounded multiplication operator on $L^2(\mathbb{R}^{2d}_{x,y})$. This can be considered as a regular case and subsequent work will be devoted to more singular cases like in Ref. 10 with a Coulomb interaction $V(x-y)=1/|x-y|$ or in the derivation of cubic nonlinear Schrödinger equations with $V(x-y)=\delta(x-y)$ like in Ref. 7.

Since in the literature the nonrelativistic and the semirelativistic dynamics of bosons were both studied (see Ref. 5), an abstract setting for the free part of the quantum Hamiltonian seems relevant. Examples are reviewed in Sec. VI.

II. PRELIMINARIES

On a complex Hilbert space $\mathfrak{h}$, the identity operator is denoted by I and the set of linear bounded (trace class) operators by $\mathcal{L}(\mathfrak{h})$ [respectively $\mathcal{L}_1(\mathfrak{h})$]. The notation $\mathcal{S}(E)$, when E is a finite dimensional real vector space, stands for the Schwartz space of functions on E.

A. Fock space

The phase space, a complex separable Hilbert space, is denoted by $\mathcal{Z}$ with the scalar product $\langle\cdot,\cdot\rangle$. The symmetric Fock space over $\mathcal{Z}$ is defined as the following direct Hilbert sum:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \overset{n}{\vee}\mathcal{Z},$$

where $\vee^n\mathcal{Z}$ is the n-fold symmetric tensor product. The orthogonal projection from $\mathcal{Z}^{\otimes n}$ onto the closed subspace $\vee^n\mathcal{Z}$ is given by

$$\mathcal{S}_n(\xi_1\otimes\xi_2\cdots\otimes\xi_n) = \frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)} \xi_{\sigma(1)}\otimes\xi_{\sigma(2)}\cdots\otimes\xi_{\sigma(n)}.$$

Algebraic direct sums or tensor products are denoted with an "alg" superscript. Hence

$$\mathcal{H}_0 = \bigoplus_{n\in\mathbb{N}}^{\mathrm{alg}} \overset{n}{\vee}\mathcal{Z}$$

denotes the subspace of vectors with a finite number of particles. The creation and annihilation operators $a^*(z)$ and $a(z)$, parameterized by $\varepsilon>0$, are then defined by

$$a(z)\varphi^{\otimes n} = \sqrt{\varepsilon n}\langle z,\varphi\rangle\varphi^{\otimes(n-1)},$$

$$a^*(z)\varphi^{\otimes n} = \sqrt{\varepsilon(n+1)}\mathcal{S}_{n+1}(z\otimes\varphi^{\otimes n}), \quad \forall\, \varphi,z\in\mathcal{Z}.$$

They extend to closed operators and they are adjoint of one another. They also satisfy the ε-canonical commutation relations,

$$[a(z_1),a^*(z_2)] = \varepsilon\langle z_1,z_2\rangle I, \quad [a^*(z_1),a^*(z_2)] = 0 = [a(z_1),a(z_2)]. \tag{5}$$

The Weyl operators are given for $z\in\mathcal{Z}$ by

$$W(z) = e^{i/\sqrt{2}[a^*(z)+a(z)]},$$

and they satisfy Weyl commutation relations in the Fock space,

$$W(z_1)W(z_2) = e^{-(i\varepsilon/2)\mathrm{Im}\langle z_1,z_2\rangle}W(z_1+z_2), \quad z_1,z_2 \in \mathcal{Z}. \tag{6}$$

The coherent states,

$$E(z) = W\left(\frac{\sqrt{2}z}{i\varepsilon}\right)\Omega = e^{(1/\varepsilon)[a^*(z)-a(z)]}\Omega \quad \in \mathcal{H},$$

where Ω is the vacuum vector $(1,0,\ldots)\in\mathcal{H}$ and $z\in\mathcal{Z}$, form a total family in $\mathcal{H}$. Furthermore, $E(z)$ can be written explicitly,

$$E(z) = e^{-(|z|^2/2\varepsilon)}\sum_{n=0}^{\infty}\frac{1}{\varepsilon^n}\frac{a^*(z)^n}{n!}\Omega = e^{-(|z|^2/2\varepsilon)}\sum_{n=0}^{\infty}\varepsilon^{-n/2}\frac{z^{\otimes n}}{\sqrt{n!}}. \tag{7}$$

The number operator is also parametrized by $\varepsilon>0$,

$$\mathbf{N}_{|\vee^n\mathcal{Z}} = \varepsilon n I_{|\vee^n\mathcal{Z}},$$

where I denotes the identity operator. For any self-adjoint operator $A:\mathcal{Z}\supset\mathcal{D}(A)\to\mathcal{Z}$, the operator $d\Gamma(A)$ is the self-adjoint operator given by

$$d\Gamma(A)_{|\vee^{n,\mathrm{alg}}\mathcal{D}(A)} = \varepsilon\left[\sum_{k=1}^{n} I\otimes\cdots\otimes \underbrace{A}_{k}\otimes\cdots\otimes I\right].$$

B. Wick and Weyl quantized operators

For any p, $q\in\mathbb{N}$, the space $\mathcal{P}_{p,q}(\mathcal{Z})$ of complex-valued polynomials on $\mathcal{Z}$ is defined with the following continuity condition:

$$b \in \mathcal{P}_{p,q}(\mathcal{Z}) \quad \text{iff there exists a unique } \tilde{b} \in \mathcal{L}(\overset{p}{\vee}\mathcal{Z},\overset{q}{\vee}\mathcal{Z}),$$

such that

$$b(z) = \langle z^{\otimes q},\tilde{b}z^{\otimes p}\rangle.$$

The subspace of $\mathcal{P}_{p,q}(\mathcal{Z})$ made of polynomials b such that $\tilde{b}$ is a compact operator is denoted by $\mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$. The *Wick monomial* of a "symbol" $b\in\mathcal{P}_{p,q}(\mathcal{Z})$ is the linear operator

$$b^{\mathrm{Wick}}:\mathcal{H}_0\to\mathcal{H}_0$$

defined as

$$b^{\mathrm{Wick}}_{|\vee^n\mathcal{Z}} = 1_{[p,+\infty)}(n)\frac{\sqrt{n!\,(n+q-p)!}}{(n-p)!}\varepsilon^{(p+q)/2}\mathcal{S}_{n-p+q}(\tilde{b}\otimes 1^{(n-p)}), \tag{8}$$

where $\tilde{b}\otimes 1^{(n-p)}$ is the operator with the action $\tilde{b}\otimes 1^{(n-p)}\varphi^{\otimes n}=(\tilde{b}\varphi^{\otimes p})\otimes\varphi^{\otimes(n-p)}$. Notice that b^{Wick} depends on the scaling parameter ε.

Let $\mathbb{P}$ denote the set of all finite rank orthogonal projections on $\mathcal{Z}$ and for a given $\wp\in\mathbb{P}$ let $L_{\wp}(dz)$ denote the Lebesgue measure on the finite dimensional subspace $\wp\mathcal{Z}$. A function $f:\mathcal{Z}\to\mathbb{C}$ is said *cylindrical* if there exists $\wp\in\mathbb{P}$ and a function g on $\wp\mathcal{Z}$ such that $f(z)=g(\wp z)$ for all $z\in\mathcal{Z}$. In this case we say that f is based on the subspace $\wp\mathcal{Z}$. We set $\mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$ to be the cylindrical Schwartz space,

$$(f\in\mathcal{S}_{\mathrm{cyl}}(\mathcal{Z}))\Leftrightarrow(\exists\wp\in\mathbb{P},\ \exists\, g\in\mathcal{S}(\wp\mathcal{Z}),\ \ f(z)=g(\wp z)).$$

Introduce the Fourier transform of a function $f\in\mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$ based on the subspace $\wp\mathcal{Z}$ as

$$\mathcal{F}[f](z) = \int_{\wp\mathcal{Z}} f(\xi) e^{-2\pi i \,\mathrm{Re}\langle z,\xi\rangle} L_{\wp}(d\xi)$$

and its inverse Fourier transform is

$$f(z) = \int_{\wp\mathcal{Z}} \mathcal{F}[f](z) e^{2\pi i \,\mathrm{Re}\langle z,\xi\rangle} L_{\wp}(dz).$$

With any symbol $b \in \mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$ based on $\wp\mathcal{Z}$, a *Weyl observable* can be associated according to

$$b^{\mathrm{Weyl}} = \int_{\wp\mathcal{Z}} \mathcal{F}[b](z) W(\sqrt{2}\pi z) L_{\wp}(dz). \tag{9}$$

Notice that b^{Weyl} is a well defined bounded operator on $\mathcal{H}$ for all $b \in \mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$ and that this quantization of cylindrical symbols depends on the parameter ε.

C. Quantum and classical dynamics

Consider a polynomial $Q \in \mathcal{P}_{2,2}(\mathcal{Z})$ such that $\tilde{Q} \in \mathcal{L}(\vee^2\mathcal{Z})$ is bounded symmetric. In the sequel, the many-body quantum Hamiltonian of bosons is written as the ε-dependent operator,

$$H_\varepsilon = d\Gamma(A) + Q^{\mathrm{Wick}}, \tag{10}$$

where A is a given self-adjoint operator on $\mathcal{Z}$ with domain $\mathcal{D}(A)$. The commutation $[Q^{\mathrm{Wick}},\mathbf{N}]=0$ with the number operator $\mathbf{N}$ ensures the essential self-adjointness of H_ε on $\mathcal{D}(d\Gamma(A)) \cap \mathcal{H}_0$. The time evolution of the quantum system described by H_ε is given by $U_\varepsilon(t) = e^{-i(t/\varepsilon)H_\varepsilon}$ and $U_\varepsilon^0(t) = e^{-i(t/\varepsilon)d\Gamma(A)}$ is the free motion.

Let us now introduce the multiple Poisson brackets. Polynomials in $\mathcal{P}_{p,q}(\mathcal{Z})$ admit Fréchet differentials. For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, set

$$\partial_{\bar{z}} b(z)[u] = \bar{\partial}_r b(z+ru)_{|r=0}, \quad \partial_z b(z)[u] = \partial_r b(z+ru)_{|r=0},$$

where $\bar{\partial}_r$, ∂_r are the usual derivatives over $\mathbb{C}$. Moreover, $\partial_z^k b(z)$ naturally belongs to $(\vee^k\mathcal{Z})^*$ (i.e., k-linear symmetric functionals), while $\partial_{\bar{z}}^j b(z)$ is identified via the scalar product with an element of $\vee^j\mathcal{Z}$ for any fixed $z \in \mathcal{Z}$. For $b_i \in \mathcal{P}_{p_i,q_i}(\mathcal{Z})$, $i=1,2$, and $k \in \mathbb{N}$, set

$$\partial_z^k b_1 \cdot \partial_{\bar{z}}^k b_2(z) = \langle \partial_z^k b_1(z), \partial_{\bar{z}}^k b_2(z)\rangle_{(\vee^k\mathcal{Z})^*,\vee^k\mathcal{Z}} \in \mathcal{P}_{p_1+p_2-k,q_1+q_2-k}(\mathcal{Z}).$$

The multiple *Poisson brackets* are defined by

$$\{b_1,b_2\}^{(k)} = \partial_z^k b_1 \cdot \partial_{\bar{z}}^k b_2 - \partial_z^k b_2 \cdot \partial_{\bar{z}}^k b_1, \quad \{b_1,b_2\} = \{b_1,b_2\}^{(1)}.$$

We will also use the following notations:

$$b_t = b \circ e^{-itA} : \mathcal{Z} \ni z \mapsto b_t(z) = b(e^{-itA}z) \tag{11}$$

for any $b \in \oplus^{\mathrm{alg}}_{p,q\in\mathbb{N}} \mathcal{P}_{p,q}(\mathcal{Z})$ and any $t \in \mathbb{R}$. Notice that b_t belongs to $\oplus^{\mathrm{alg}}_{p,q\in\mathbb{N}} \mathcal{P}_{p,q}(\mathcal{Z})$.

Definition II.1: *For $m \in \mathbb{N}$ and $t_1,\cdots,t_m,t \in \mathbb{R}$, associate with any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ the polynomial,*

$$C_0^{(m)}(t_m,\ldots,t_1,t) = \{Q_{t_m},\ldots,\{Q_{t_1},b_t\}\cdots\} \in \mathcal{P}_{p-m,q-m}(\mathcal{Z}). \tag{12}$$

For conciseness, the dependence of $C_0^{(m)}(t_m,\cdots,t_1,t)$ on b is forgotten through the notation, with the even shorter abbreviation $C_0^{(m)}$. Conventionally, we set $C_0^{(0)}(t) = b_t$.

Consider now the description of the nonlinear classical dynamics analogs of (10). The energy functional,

$$h(z) = \langle z, Az\rangle + Q(z), \quad z \in \mathcal{D}(A),$$

has the associated vector field $X:\mathcal{D}(A) \to \mathcal{Z}$, $X(z) = Az + \partial_{\bar{z}} Q(z)$. Hence, the nonlinear field equation reads

$$i\partial_t z_t = X(z_t)$$

with initial condition $z_0 = z \in \mathcal{D}(A)$. For our purpose we only need the integral form of the latter equation,

$$z_t = e^{-itA}z - i\int_0^t e^{-i(t-s)A}\partial_{\bar{z}}Q(z_s)ds \quad \text{for } z \in \mathcal{Z}. \tag{13}$$

The standard fixed point argument implies that (13) admits a unique global C^0-flow on $\mathcal{Z}$ which is denoted by $\mathbf{F}:\mathbb{R}\times\mathcal{Z}\to\mathcal{Z}$ [i.e., $\mathbf{F}$ is a C^0-map satisfying $\mathbf{F}_{t+s}(z) = \mathbf{F}_t \circ \mathbf{F}_s(z)$ and $\mathbf{F}_t(z)$ solves (13) for any $z \in \mathcal{Z}$]. Moreover, if z_t solves (13), and $Q_t(z) = Q(e^{-itA}z)$, then $w_t = e^{itA}z_t$ solves the differential equation

$$\frac{d}{dt}w_t = -i\partial_{\bar{z}}Q_t(w_t).$$

Therefore for any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the following identity holds:

$$\frac{d}{dt}b(w_t) = \partial_{\bar{z}}b(w_t)[-i\partial_{\bar{z}}Q_t(w_t)] + \partial_z b(w_t)[-i\partial_{\bar{z}}Q_t(w_t)] = i\{Q_t, b\}(w_t).$$

Hence, we obtain the Duhamel formula,

$$b \circ \mathbf{F}_t(z) = b_t(z) + i\int_0^t \{Q, b_{t-t_1}\} \circ \mathbf{F}_{t_1}(z)dt_1, \tag{14}$$

by observing that $\{Q_{t_1}, b\}(w_{t_1}) = \{Q, b_{-t_1}\}(z_{t_1})$.

III. RESULTS

The main result of this paper (Theorem III.3) relies on two ingredients. Namely, the asymptotics of time-evolved Wick observables (Theorem III.1) and the construction of Wigner measures (Theorem III.2).

Theorem III.1: *Fix $p, q \in \mathbb{N}$, and assume $b \in \mathcal{P}_{p,q}(\mathcal{Z})$. Then the asymptotic expansion*

$$U_\varepsilon(t)^* b^{\mathrm{Wick}} U_\varepsilon(t) = \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [C_0^{(n)}(t_n, \ldots, t_1, t)]^{\mathrm{Wick}} + \varepsilon R(\varepsilon, t)$$

holds for any $\delta > 0$ in $\mathcal{L}(\vee^k \mathcal{Z}, \vee^{k-p+q}\mathcal{Z})$ with the uniform estimate

$$|R(\varepsilon,t)|_{\mathcal{L}(\vee^k\mathcal{Z},\vee^{k-p+q}\mathcal{Z})} \le C_\delta \quad \text{when } k\varepsilon \le 1 + \delta/2 \quad \text{and} \quad 4(1+2\delta)|t||V|_{L^\infty} \le 1.$$

Theorem III.2: *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, $\bar{\varepsilon} > 0$, be a family of normal states on $\mathcal{L}(\mathcal{H})$, parametrized by ε. Assume $\mathrm{Tr}[\mathbf{N}^\delta \rho^\varepsilon] \le C_\delta$ uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$ for some fixed $\delta > 0$ and $C_\delta \in (0, +\infty)$. Then for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \varepsilon_n = 0$, there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel probability measure μ, called Wigner measure, on $\mathcal{Z}$ such that*

$$\lim_{k\to\infty} \mathrm{Tr}[\rho_{\varepsilon_{n_k}} b^{\mathrm{Weyl}}] = \int_{\mathcal{Z}} b(z) d\mu(z)$$

for all $b \in \mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$.

Moreover this probability measure μ satisfies $\int_{\mathcal{Z}} |z|^{2\delta} d\mu(z) < \infty$.

Without loss of generality, the following hold.

(i) One can consider a countable family $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ of density matrices, $\varrho_{\varepsilon_n}\geq 0$, $\mathrm{Tr}[\varrho_{\varepsilon_n}]=1$, instead of considering a noncountable range $(0,\bar{\varepsilon})$, $\bar{\varepsilon}>0$, of values for ε.

(ii) One can consider a sequence ϱ_{ε_n} associated with a unique (or single) Wigner measure instead of several possible measures provided by Theorem III.2.

Notice that the Wick or Weyl observables are ε_n-quantized before taking the limit $\varepsilon_n\to 0$. For the sake of conciseness, the ε or ε_n parameter does not appear in the notations of quantized observables.

Let $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ be a sequence of non-negative trace class operators on $\mathcal{H}$ such that $\mathrm{Tr}[\rho_\varepsilon]=1$. The first condition which characterizes our class of ε_n-dependent density matrices says that there exists a fixed $\lambda>0$, such that

$$\forall k\in\mathbb{N},\quad \mathrm{Tr}[\mathbf{N}^k\rho_{\varepsilon_n}]\leq\lambda^k \quad \text{uniformly in } n\in\mathbb{N},\ (\mathbf{N}=\mathbf{N}_{\varepsilon_n}). \qquad (H_\lambda)$$

Wigner measures of $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ are constructed via Weyl quantization, however, with the help of condition (H_λ) it extends to a certain class of Wick observables. In fact, one can prove (possibly after extracting a subsequence) for $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ satisfying (H_λ) that the associated Wigner measure μ satisfies

$$\lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}b^{\mathrm{Wick}}]=\int_{\mathcal{Z}}b(z)d\mu(z)\quad \text{for any } b\in\mathcal{P}^{\infty}_{p,q}(\mathcal{Z}),p,q\in\mathbb{N}, \tag{15}$$

where the compactness assumption of the operators $\tilde{b}\in\mathcal{L}(\vee^p\mathcal{Z},\vee^q\mathcal{Z})$ is crucial. Only the information of the asymptotic measure μ essentially carried by finite dimensional spaces is accessible with Wick-quantized observables. This has been identified in Ref. 1 as a dimensional defect of compactness phenomenon and examples have been given. One of them is reproduced in Example 4 in Sec. VI. The separation of condensated and noncondensated phases for a gaz of bosons in the thermodynamic limit provides another, more physically relevant example, which is discussed also in Ref. 1.

The extension of (15) to the larger class of symbols $\oplus^{\mathrm{alg}}_{\alpha,\beta\in\mathbb{N}}\mathcal{P}_{\alpha,\beta}(\mathcal{Z})$ is actually a specific property of the sequence $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$. It turns out to be a decisive assumption in the present analysis of the mean field limit. In the following, a sequence $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ with a single Wigner measure μ will have the property (P) when

$$\lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}b^{\mathrm{Wick}}]=\int_{\mathcal{Z}}b(z)d\mu(z)\quad \text{for any } b\in\oplus^{\mathrm{alg}}_{\alpha,\beta\in\mathbb{N}}\mathcal{P}_{\alpha,\beta}(\mathcal{Z}). \qquad (P)$$

Here is the main theorem.

Theorem III.3: *Let the sequence* $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ of density matrices, $\varrho_{\varepsilon_n}\geq 0$, $\mathrm{Tr}[\varrho_{\varepsilon_n}]=1$, $\lim_{n\to\infty}\varepsilon_n=0$, *satisfy* (H_λ) and (P). *Then the limit,*

$$\lim_{n\to\infty}\mathrm{Tr}[\rho_{\varepsilon_n}e^{i(t/\varepsilon_n)H_{\varepsilon_n}}b^{\mathrm{Wick}}e^{-i(t/\varepsilon_n)H_{\varepsilon_n}}]=\int_{\mathcal{Z}}(b\circ\mathbf{F}_t)(z)d\mu, \tag{16}$$

holds for any $t\in\mathbb{R}$ *and any* $b\in\oplus^{\mathrm{alg}}_{\alpha,\beta\in\mathbb{N}}\mathcal{P}_{\alpha,\beta}(\mathcal{Z})$ *with* $b^{\mathrm{Wick}}=b^{\mathrm{Wick}}_{\varepsilon_n}$.

Remark III.4: *Since* $\mathbf{F}$ is a C^0*-map the right hand side of (16) can be written as*

$$\int_{\mathcal{Z}}(b\circ\mathbf{F}_t)(z)d\mu=\int_{\mathcal{Z}}b(z)d\mu_t,$$

where μ_t *is a push-forward measure defined by* $\mu_t(B)=\mu(\mathbf{F}_{-t}(B))$ *for any Borel set B.*

We formally sketch the strategy of the proof of Theorem III.3. The asymptotic expansion in Theorem III.1 allows the step

$$\mathrm{Tr}[\rho_{\varepsilon_n} e^{i(t/\varepsilon_n)H_{\varepsilon_n}} b^{\mathrm{Wick}} e^{-i(t/\varepsilon_n)H_{\varepsilon_n}}] = \mathrm{Tr}\left[\rho_{\varepsilon_n} \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [C_0^{(n)}(t_n, \ldots, t_1, t)]^{\mathrm{Wick}}\right] + O(\varepsilon).$$

We interchange formally the infinite sum and the trace

$$\mathrm{Tr}\left[\rho_{\varepsilon_n} \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [C_0^{(n)}(t_n, \cdots, t_1, t)]^{\mathrm{Wick}}\right]$$
$$= \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \, \mathrm{Tr}[\rho_{\varepsilon_n} [C_0^{(n)}(t_n, \ldots, t_1, t)]^{\mathrm{Wick}}].$$

Taking now the limit $\varepsilon_n \to 0$ and using Wigner measures, we obtain

$$\lim_{n\to\infty} \mathrm{Tr}[\rho_{\varepsilon_n} U_{\varepsilon_n}(t) b^{\mathrm{Wick}} U_{\varepsilon_n}(t)^*] = \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathcal{Z}} C_0^{(n)}(t_n, \ldots, t_1, t) d\mu(z).$$

The last step is the identification for small time of $b \circ \mathbf{F}_t(z)$ as an infinite sum of multiple Poisson brackets,

$$\int_{\mathcal{Z}} \left[\sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n C_0^{(n)}(t_n, \ldots, t_1, t)\right] d\mu(z) = \int_{\mathcal{Z}} (b \circ \mathbf{F}_t)(z) d\mu(z).$$

A similar result holds for Weyl observables with symbols on the cylindrical Schwartz space $\mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$.

Corollary III.5: *Let the sequence* $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ *of density matrices,* $\varrho_{\varepsilon_n} \geq 0$, $\mathrm{Tr}[\varrho_{\varepsilon_n}]=1$, $\lim_{n\to\infty}\varepsilon_n=0$, *satisfy* (H_λ) *and* (P). *Then the limit,*

$$\lim_{\varepsilon_n\to 0} \mathrm{Tr}[\rho_{\varepsilon_n} e^{i(t/\varepsilon_n)H_{\varepsilon_n}} b^{\mathrm{Weyl}} e^{-i(t/\varepsilon_n)H_{\varepsilon_n}}] = \int_{\mathcal{Z}} b \circ \mathbf{F}_t(z) d\mu, \tag{17}$$

holds for any $b \in \mathcal{S}_{\mathrm{cyl}}(\mathcal{Z})$ *and any* $t \in \mathbb{R}$.

Proof: A consequence of Theorem III.3 and Ref. 1, Proposition 6.15 is that the sequence

$$\rho_{\varepsilon_n}(t) = U_{\varepsilon_n}(t) \rho_{\varepsilon_n} U_{\varepsilon_n}(t)^*$$

admits a single Wigner measure given by μ_t. Hence, by definition

$$\lim_{\varepsilon_n\to 0} \mathrm{Tr}[\rho_{\varepsilon_n}(t) b^{\mathrm{Weyl}}] = \lim_{\varepsilon_n\to 0} \int_{\wp\mathcal{Z}} \mathcal{F}[b](\xi) \mathrm{Tr}[\rho_{\varepsilon_n}(t) W(\sqrt{2}\pi\xi)] L_{\wp}(d\xi)$$
$$= \int_{\wp\mathcal{Z}} \mathcal{F}[b](\xi) \int_{\mathcal{Z}} e^{2\pi i \,\mathrm{Re}(z,\xi)} d\mu_t(z) L_{\wp}(d\xi)$$
$$= \int_{\mathcal{Z}} b(z) d\mu_t(z).$$

□

Another formulation states that the Wigner measure μ_t satisfies a transport equation in an integral form.

Corollary III.6: *Let* $(\rho_{\varepsilon_n}(t))_{n\in\mathbb{N}}$ *satisfy the assumptions of Theorem III.3 and let* μ_t *denote its Wigner measure. Then* $t \in \mathbb{R} \mapsto \mu_t$ *is a solution to the transport equation*

$$\mu_t(b) = \mu_t^0(b) + i \int_0^t \mu_s(\{Q, b_{t-s}\}) ds \tag{18}$$

for any $b\in\oplus^{\text{alg}}_{p,q\in\mathbb{N}}\mathcal{P}(\mathcal{Z})$ *and where* $\mu_t^0(B)=\mu(e^{-itA}B)$ *for any Borel set B.*

Proof: The relation (18) is given by integrating (14) with respect to the measure $\mu=\mu_0$. □

IV. CRITERIA FOR THE PROPERTY (*P*)

In the following, two conditions which ensure the property (P) are formulated. Recall that for any $\wp\in\mathbb{P}$ the operator $\Gamma(\wp)$ acting on $\mathcal{H}$ is defined by

$$\Gamma(\wp)_{|\vee^n\mathcal{Z}}=\wp\otimes\wp\cdots\otimes\wp$$

and $\Gamma(\wp)$ is an orthogonal projector. The first criterion is a "tightness" assumption with respect to the trace norm of the state,

$$\forall\,\eta>0,\,\exists\,\wp_\eta\in\mathbb{P},\quad\forall\,n\in\mathbb{N},\ \ \mathrm{Tr}[(1-\Gamma(\wp_\eta))\rho_{\varepsilon_n}]<\eta. \qquad (T)$$

The dual version is formulated as an equicontinuity assumption with respect to the Wick symbols,

$$\forall\,p,q\in\mathbb{N},\,\forall\,\eta>0,\,\exists\,\mathcal{W}_0^\eta\subset\mathcal{L}(\overset{p}{\vee}\mathcal{Z},\overset{q}{\vee}\mathcal{Z}),\quad\forall\,\tilde{b}\in\mathcal{W}_0^\eta,\,\forall\,n\in\mathbb{N},\quad|\mathrm{Tr}[\rho_{\varepsilon_n}b^{\mathrm{Wick}}]|<\eta, \qquad (E)$$

where $\mathcal{W}_0^\eta$ is a neighborhood of zero in $\mathcal{L}(\vee^p\mathcal{Z},\vee^q\mathcal{Z})$ endowed with the σ-weak topology.

Lemma IV.1: *Assume that* $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ *satisfies* (H_λ). *Then*

(i) $(T)\Rightarrow(P)$,
(ii) $(E)\Rightarrow(P)$.

Proof: We want to prove (P) for $b\in\mathcal{P}_{p,q}(\mathcal{Z})$.

(i) Start with

$$\mathrm{Tr}[\rho_{\varepsilon_n}b^{\mathrm{Wick}}]=\mathrm{Tr}[\rho_{\varepsilon_n}\Gamma(\wp)b^{\mathrm{Wick}}\Gamma(\wp)]+\mathrm{Tr}[\rho_{\varepsilon_n}(1-\Gamma(\wp))b^{\mathrm{Wick}}\Gamma(\wp)]$$
$$+\mathrm{Tr}[\rho_{\varepsilon_n}\Gamma(\wp)b^{\mathrm{Wick}}(1-\Gamma(\wp))]+\mathrm{Tr}[\rho_{\varepsilon_n}(1-\Gamma(\wp))b^{\mathrm{Wick}}(1-\Gamma(\wp))].$$

Estimate all the terms containing $(1-\Gamma(\wp))$ like in the next example,

$$|\mathrm{Tr}[\rho_{\varepsilon_n}(1-\Gamma(\wp))b^{\mathrm{Wick}}\Gamma(\wp)]|=|\mathrm{Tr}[\langle\mathbf{N}\rangle^{(p+q)/2}\rho_{\varepsilon_n}(1-\Gamma(\wp))b^{\mathrm{Wick}}\langle\mathbf{N}\rangle^{-(p+q/2)}\Gamma(\wp)]| \qquad (19)$$

$$\le C_{p,q}(b)\|\langle\mathbf{N}\rangle^{(p+q)2}\rho_{\varepsilon_n}^{1/2}\rho_{\varepsilon_n}^{1/2}(1-\Gamma(\wp))\|_1 \qquad (20)$$

$$\le C_{p,q}(b)\|\langle\mathbf{N}\rangle^{(p+q)/2}\rho_{\varepsilon_n}\langle\mathbf{N}\rangle^{(p+q)/2}\|_1^{1/2}\|(1-\Gamma(\wp))\rho_{\varepsilon_n}$$
$$\times(1-\Gamma(\wp))\|_1^{1/2} \qquad (21)$$

$$\le\tilde{C}_{p,q}(b)\mathrm{Tr}[\rho_{\varepsilon_n}(1-\Gamma(\wp))]^{1/2}. \qquad (22)$$

First (20) comes from the number estimate

$$\|b^{\mathrm{Wick}}\langle\mathbf{N}\rangle^{-(p+q)/2}\|\le C_{p,q}(b), \qquad (23)$$

then Cauchy–Schwarz inequality yields (21). The last estimate (22) is possible with (H_λ). Notice that $\Gamma(\wp)b^{\mathrm{Wick}}\Gamma(\wp)=\Gamma(\wp)b(\wp z)^{\mathrm{Wick}}\Gamma(\wp)$ and that the polynomial $b(\wp z)\in\mathcal{P}^\infty_{p,q}(\mathcal{Z})$ when $\wp$ is finite rank orthogonal projector. The hypothesis (T) and the above argument allow to approximate $\mathrm{Tr}[\rho_{\varepsilon_n}b^{\mathrm{Wick}}]$ by the quantity $\mathrm{Tr}[\rho_{\varepsilon_n}b(\wp z)^{\mathrm{Wick}}]$ using $\eta/3$ argument. Now, write

$$|\mathrm{Tr}[\rho_{\varepsilon_n} b^{\mathrm{Wick}}] - \int_{\mathcal{Z}} b(z) d\mu| \leq |\mathrm{Tr}[\rho_{\varepsilon_n}(b^{\mathrm{Wick}} - b(\wp z)^{\mathrm{Wick}})] + \mathrm{Tr}[\rho_{\varepsilon_n} b(\wp z)^{\mathrm{Wick}}] - \int_{\mathcal{Z}} b(\wp z) d\mu$$
$$+ \int_{\mathcal{Z}} [b(\wp z) - b(z)] d\mu|.$$

So, the property (T) and (H_λ) imply (P).

(ii) *There exists a sequence* $b_\kappa \in \mathcal{P}^\infty_{p,q}(\mathcal{Z})$*, such that* $\tilde{b}_\kappa$ *converges in the* σ*-weak topology to* $\tilde{b}$*. We have*

$$|\mathrm{Tr}[\rho_{\varepsilon_n} b^{\mathrm{Wick}}] - \int_{\mathcal{Z}} b(z) d\mu| \leq |\mathrm{Tr}[\rho_{\varepsilon_n}(b^{\mathrm{Wick}} - b_\kappa^{\mathrm{Wick}})] + \left(\mathrm{Tr}[\rho_{\varepsilon_n} b_\kappa(z)^{\mathrm{Wick}}] - \int_{\mathcal{Z}} b_\kappa(z) d\mu \right)$$
$$+ \int_{\mathcal{Z}} [b_\kappa(z) - b(z)] d\mu|. \tag{24}$$

So, (P) holds by an $\eta/3$ argument and using, respectively, (E), (15) and dominated convergence for each term in (right hand side) (24). □

Remark IV.2:

(1) *The space of bounded operators* $\mathcal{L}(\vee^p \mathcal{Z}, \vee^q \mathcal{Z})$ endowed with the σ*-weak topology is not a Baire space when* $\mathcal{Z}$ is infinite dimensional. Otherwise, (E) and hence (P) would be fulfilled by any sequence $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ satisfying (H_λ), *according to Banach–Steinhaus theorem (uniform boundedness principle).*

(2) *The hypothesis* (H_λ) in the above lemma can be replaced by the weaker statement (see Ref. 1, *Proposition 6.15)*

$$\exists C > 0 : \forall\, k \in \mathbb{N}, \mathrm{Tr}[\mathbf{N}^k \rho_{\varepsilon_n} \mathbf{N}^k] \leq C(Ck)^k$$

uniformly in ε_n*. This can be roughly interpreted as an analyticity property of* $t \to \mathrm{Tr}[e^{it\mathbf{N}^2} \varrho_{\varepsilon_n}]$ *in* $\{|t| < 1/(eC)\}$*, uniformly with respect to* ε_n*.*

V. PROOF OF THEOREM III.3

We collect some statements from Ref. 1. The symbol $\tilde{b}$ denotes the operator

$$\tilde{b} = \frac{\partial_{\bar{z}}^q \partial_z^p}{q!\,p!} b(z) \in \mathcal{L}(\overset{p}{\vee}\mathcal{Z}, \overset{p}{\vee}\mathcal{Z})$$

associated with $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

Lemma V.1: *Let* $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

(i) *The following inequality holds true:*

$$|\widetilde{\{Q_s, b_t\}^{(2)}}|_{\mathcal{L}(\vee^p \mathcal{Z}, \vee^q \mathcal{Z})} \leq 2[p(p-1) + q(q-1)] |\tilde{Q}| |\tilde{b}|_{\mathcal{L}(\vee^p \mathcal{Z}, \vee^q \mathcal{Z})}.$$

(ii) *For any* $m \in \mathbb{N}$*, we have*

$$|\widetilde{C_0^{(m)}}|_{\mathcal{L}(\vee^{p+m}\mathcal{Z}, \vee^{q+m}\mathcal{Z})} \leq 2^{2m} \frac{(p+m-1)!}{(p-1)!} |\tilde{Q}|^m |\tilde{b}|_{\mathcal{L}(\vee^p \mathcal{Z}, \vee^q \mathcal{Z})},$$

when $p \geq q$ *with a similar expression when* $q \geq p$ *[replace* $(p+m, p-1)$ *with* $(q+m, q-1)$*].*

Proof: See Ref. 1, Lemma 5.8 and Lemma 5.9. □

Lemma V.2: *For any* $\delta > 0$ *there exists* $T > 0$ *such that for all* $0 < t < T$*,*

$$\sum_{m=0}^{\infty} \delta^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m |\widetilde{C_0^{(m)}}(t_m, \ldots, t_1, t)|_{\mathcal{L}(\vee^{p+m}\mathcal{Z}, \vee^{q+m}\mathcal{Z})} < \infty. \tag{25}$$

Proof: It is enough to bound (25) in the case $p \geq q$. Using Lemma V.1, part (ii), we obtain

$$\sum_{m=0}^{\infty} \delta^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m |\widetilde{C_0^{(m)}}(t_m, \ldots, t_1, t)| \leq 2^{p-1} |\tilde{b}| \sum_{m=0}^{\infty} (2^3 \delta t |\tilde{Q}|)^m.$$

The right hand side is finite whenever $0 < t < T = (2^3 \delta |\tilde{Q}|)^{-1}$. □

Proof of Theorem III.3: First consider the following expansion proven in Ref. (1), (50)–(52), for any positive integer M:

$$\begin{aligned} U_\varepsilon(t)^* b^{\mathrm{Wick}} U_\varepsilon(t) &= \sum_{m=0}^{M-1} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m [C_0^{(m)}(t_m, \ldots, t_1, t)]^{\mathrm{Wick}} \\ &+ \frac{\varepsilon}{2} \sum_{m=1}^{M} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m U_\varepsilon(t_m)^* U_\varepsilon^0(t_m) \\ &\times [\{Q_{t_m}, C_0^{(m-1)}(t_{m-1}, \ldots, t_1, t)\}^{(2)}]^{\mathrm{Wick}} U_\varepsilon^0(t_m)^* U_\varepsilon(t_m) \\ &+ i^M \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M U_\varepsilon(t_M)^* U_\varepsilon^0(t_M) [C_0^{(M)}(t_M, \ldots, t_1, t)]^{\mathrm{Wick}} U_\varepsilon^0(t_M)^* U_\varepsilon(t_M), \end{aligned}$$

where the equality holds in $\mathcal{L}(\vee^s \mathcal{Z}, \vee^{s+q-p} \mathcal{Z})$ for any $s \in \mathbb{N}$, $s \geq q-p$. Multiplying on the left the above identity by ρ_{ε_n} and then using the number estimates (23), with the help of (H_λ), yields an identity on $\mathcal{L}_1(\mathcal{H})$ on which we take the trace.

Set

$$\rho_{\varepsilon_n}(t) = U_{\varepsilon_n}(t) \rho_{\varepsilon_n} U_{\varepsilon_n}(t)^*.$$

This leads to

$$\mathrm{Tr}[\rho_{\varepsilon_n}(t) b^{\mathrm{Wick}}] = \sum_{m=0}^{M-1} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \mathrm{Tr}[\rho_{\varepsilon_n}(C_0^{(m)}(t_m, \ldots, t_1, t))^{\mathrm{Wick}}] \tag{26}$$

$$+ \frac{\varepsilon_n}{2} \sum_{m=1}^{M} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m$$

$$\mathrm{Tr}[\rho_{\varepsilon_n}(t_m) U_{\varepsilon_n}^0(t_m) (\{Q_{t_m}, C_0^{(m-1)}(t_{m-1}, \ldots, t_1, t)\}^{(2)})^{\mathrm{Wick}} U_{\varepsilon_n}^0(t_m)^*] \tag{27}$$

$$+ i^M \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M \, \mathrm{Tr}[\rho_{\varepsilon_n}(t_M) U_{\varepsilon_n}^0(t_M) (C_0^{(M)}(t_M, \ldots, t_1, t))^{\mathrm{Wick}} U_{\varepsilon_n}^0(t_M)^*]. \tag{28}$$

The interchange of trace and integrals on the right hand side is justified by the bounds on Lemma V.1. Lemma V.2 implies that the terms of (26) and (27) are bounded by

$$A_m = \lambda^{m+(p+q)/2} \, \mathrm{sign}(t)^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m |\widetilde{C_0^{(m)}}|,$$

$$B_m = \varepsilon_n|\tilde{Q}|(p+q+m-1)^2\lambda^{m-1+(p+q)/2}\,\mathrm{sign}(t)^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m|\widetilde{C_0^{(m-1)}}|,$$

while the remainder (28) is estimated by

$$|(28)| \le \mathrm{sign}(t)^M\int_0^t dt_1\cdots\int_0^{t_{M-1}} dt_M|\widetilde{C_0^{(M)}}| = C_M.$$

By Lemma V.1, the series $\Sigma_{m=0}^{\infty}A_m$ and $\Sigma_{m=0}^{\infty}B_m$ converge as soon as $|t|<T_0=(2^3\lambda|\tilde{Q}|)^{-1}$, while $\lim_{M\to\infty}C_M=0$. Hence the relation (26)–(28) holds with $M=\infty$ with a vanishing third term and a second term bounded by $\Sigma_{m=0}^{\infty}B_m=\mathcal{O}(\varepsilon_n)$. Therefore, we obtain

$$\lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}U_{\varepsilon_n}(t)^* b^{\mathrm{Wick}}U_{\varepsilon_n}(t)] - \sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m\,\mathrm{Tr}[\rho_{\varepsilon_n}(C_0^{(m)}(t_m,\ldots,t_1,t))^{\mathrm{Wick}}] = 0.$$

Owing to the condition (P) which provides the pointwise convergence and the uniform bound of $\Sigma_{m=0}^{\infty}A_m$, the Lebesgue's convergence theorem implies

$$\lim_{\varepsilon_n\to 0}\sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m\,\mathrm{Tr}[\rho_{\varepsilon_n}(C_0^{(m)}(t_m,\ldots,t_1,t))^{\mathrm{Wick}}]$$
$$= \sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m\int_{\mathcal{Z}} C_0^{(m)}(t_m,\ldots,t_1,t;z)d\mu. \qquad (29)$$

Now, we interchange the sum over m and the integrals on $(t_1,\ldots,t_m,t)$ with the integral over $\mathcal{Z}$ on (29) simply with a Fubini argument based on the absolute convergence (written here for $t>0$),

$$\sum_{m=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m\int_{\mathcal{Z}}|C_0^{(m)}(t_m,\ldots,t_1,t;z)|d\mu$$
$$\le \sum_{m=0}^{\infty}\left(\int_{\mathcal{Z}}|z|^{p+q+2m}d\mu\right)\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m|\widetilde{C_0^{(m)}}(t_m,\ldots,t_1,t)|.$$

Again (H_λ) and (P) imply that for all $k\in\mathbb{N}$ there exists $\lambda>0$, such that

$$\int_{\mathcal{Z}}|z|^{2k}d\mu = \lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}(|z|^{2k})^{\mathrm{Wick}}] = \lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}\mathbf{N}^k] \le \lambda^k.$$

Hence, Lemma V.2 yields for $|t|<T_0$

$$\lim_{\varepsilon_n\to 0}\mathrm{Tr}[\rho_{\varepsilon_n}U_{\varepsilon_n}(t)^* b^{\mathrm{Wick}}U_{\varepsilon_n}(t)] = \sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m\int_{\mathcal{Z}} C_0^{(m)}(t_m,\ldots,t_1,t;z)d\mu$$
$$= \int_{\mathcal{Z}}\sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m C_0^{(m)}(t_m,\ldots,t_1,t;z)d\mu,$$

where the integrand $\sum_{m=0}^{\infty} i^m\int_0^t dt_1\cdots\int_0^{t_{m-1}}dt_m C_0^{(m)}(z)$ is a convergent series in $L^1(\mu)$.

The last step is the identification of the limit with the right hand side of (16). Indeed, an iteration of (14) reads

$$b(z_t) = b_t(z) + i\int_0^t \{Q_{t_1}, b_t\}(z)dt_1 + i^2\int_0^t dt_1 \int_0^{t_1} dt_2 \{Q_{t_2},\{Q_{t_1},b_t\}\}(e^{it_2A}z_{t_2}),$$

after setting $z_t = \mathbf{F}_t(z)$ and defining the Wick symbols b_t and Q_t according to (11). By induction we obtain for any $M > 1$,

$$b \circ \mathbf{F}_t(z) = b_t(z) + \sum_{m=1}^{M-1} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m C_0^{(m)}(t_m, \ldots, t_1, t; z)$$
$$+ i^M \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M C_0^{(M)}(t_M, \ldots, t_1, t; e^{it_M A} z_{t_M}).$$

An integration with respect to the measure μ leads to

$$\int_{\mathcal{Z}} b \circ \mathbf{F}_t(z) d\mu = \sum_{n=0}^{M-1} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathcal{Z}} C_0^{(n)}(t_n, \ldots, t_1, t; z) d\mu$$
$$+ i^M \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M \int_{\mathcal{Z}} C_0^{(M)}(t_M, \ldots, t_1, t; e^{it_M A} z_{t_M}) d\mu.$$

Again the uniform estimate $\Sigma_{m=0}^{\infty} A_m$ when $|t| < T_0$ and $\lim_{M\to\infty} C_M = 0$ allows to take the limit as $M \to \infty$. This implies for $|t| < T_0$

$$\int_{\mathcal{Z}} b \circ \mathbf{F}_t(z) d\mu = \sum_{m=0}^{\infty} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \int_{\mathcal{Z}} C_0^{(m)}(t_m, \ldots, t_1, t; z) d\mu.$$

This proves the result for $|t| < T_0$, and it is extended to any time by the next iteration argument. Indeed, it is clear that $\rho_{\varepsilon_n}(t) = U_{\varepsilon_n}(t)\rho_{\varepsilon_n} U_{\varepsilon_n}(t)^*$ satisfies (H_λ) since $U_{\varepsilon_n}(t)$ commute with $\mathbf{N}$. The property (P) holds for $\rho_{\varepsilon_n}(t)$ when $|t| < T_0$ by Remark III.4 and Corollary III.5. For t, s such that $|t|, |s| < T_0$, the sequence $(\rho_{\varepsilon_n}(t))_{n\in\mathbb{N}}$ satisfies (H_λ) and (P). Therefore, the result for short times yields

$$\lim_{\varepsilon_n \to 0} \mathrm{Tr}[\rho_{\varepsilon_n}(t) U_{\varepsilon_n}(s)^* b^{\mathrm{Wick}} U_{\varepsilon_n}(s)] = \int_{\mathcal{Z}} b \circ \mathbf{F}_s(z) d\mu_t = \int_{\mathcal{Z}} b \circ \mathbf{F}_{t+s}(z) d\mu.$$

□

Remark V.3: *As by product we have for any* $b \in \oplus_{\alpha,\beta\in\mathbb{N}}^{\mathrm{alg}} \mathcal{P}_{\alpha,\beta}(\mathcal{Z})$

$$b \circ \mathbf{F}_t(z) = L^1(\mu) - \sum_{m=0}^{\infty} i^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m C_0^{(m)}(t_m, \ldots, t_1, t; z). \tag{30}$$

Moreover, the arguments used in the proof of Theorem III.3 cannot ensure the pointwise absolute convergence of the right hand side of (30) for all $z \in \mathcal{Z}$.

VI. EXAMPLES

A. Models

(M1) *Nonrelativistic boson system:* Let $\mathcal{Z} = L^2(\mathbb{R}^d, dx)$, $A = -\Delta_x + U(x)$ where $U(x)$ is a real potential such that A is self-adjoint and Q is a multiplication operator by $\frac{1}{2}V(x-y)$ with $V \in L^\infty(\mathbb{R}^d)$.

(M2) *Relativistic boson system:* Let $\mathcal{Z} = L^2(\mathbb{R}^d, dx)$, $A = \sqrt{-\Delta_x + m^2} + U(x)$ self-adjoint and Q as in (M1).

(M3) When $\mathcal{Z}=\mathbb{C}^d\sim\mathbb{R}^{2d}_{x,\xi}$, one recovers the standard semiclassical limit problem and the condition (P) is always satisfied if (H_λ) is satisfied. We refer, for example, the reader to Refs. 4, 11, 12, 14, and 16–18 for various results about this topic.

B. Density operator sequences

(1) Every sequence $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ valued in a compact set of the Banach space of trace class operators has the Wigner measure δ_0. Due to the compactness one can extract a subsequence $(\rho_{\varepsilon_{n_k}})_{n_k\in\mathbb{N}}$ which converges in the trace norm to a normal state ρ_∞. This leads to

$$\lim_{n_k\to\infty}\mathrm{Tr}[\rho_{\varepsilon_{n_k}}W(\sqrt{2}\pi\xi)]=1,\qquad\forall\,\xi\in\mathcal{Z}.$$

Using the definition of Weyl quantization, one can see that δ_0 is the only possible Wigner measure associated with the sequence $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$.

If in addition $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ satisfies (H_λ) then (P) holds true.

(2) Let $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ as in (1) and satisfying (H_λ) and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{Z}$ such that $\lim_{n\to\infty}|z_n-z|=0$. Then $\tilde{\rho}_{\varepsilon_n}=W((\sqrt{2}/i\varepsilon)z_n)\rho_{\varepsilon_n}W(-(\sqrt{2}/i\varepsilon)z_n)$ admits the unique Wigner measure $\mu=\delta_z$ and (P) holds true. The push-forward measure is $\mu_t=\delta_{z_t}$.

(3) Let $(z_n)_{n\in\mathbb{N}}$ be a sequence valued in a compact set of $\mathcal{Z}$. So

$$\rho_{\varepsilon_n}=|z_n^{\otimes[\varepsilon_n^{-1}]}\rangle\langle z_n^{\otimes[\varepsilon_n^{-1}]}|$$

satisfies (H_λ) and the property (P) and admits the Wigner measures

$$\frac{1}{2\pi}\int_0^{\pi}\delta_{e^{i\theta}z}d\theta,$$

where z is any cluster point of $(z_n)_{n\in\mathbb{N}}$. Several other examples can be obtained by superposition, see Ref. 1.

(4) Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that $|z_n|=1$ in $\mathcal{Z}$ converging weakly to 0. Then (P) fails for $\rho_{\varepsilon_n}=|E(z_n)\rangle\langle E(z_n)|$ with $E(z_n)=W((\sqrt{2}/i\varepsilon)z_n)|\Omega\rangle$, although (H_λ) holds. The equality

$$\lim_{n\to\infty}\langle E(z_n),b^{\mathrm{Wick}}E(z_n)\rangle=\lim_{n\to\infty}b(z_n)=0$$

for any $b\in\mathcal{P}^\infty_{p,q}(\mathcal{Z})$ says that $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ is associated with the Wigner measure δ_0. Now remark that (P) is false since

$$\lim_{n\to\infty}\langle E(z_n),\mathbf{N}E(z_n)\rangle=\lim_{n\to\infty}\langle E(z_n),b^{\mathrm{Wick}}E(z_n)\rangle=1\neq 0,$$

with $b(z)=|z|^2$.

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