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Asymptotic Completeness for a Renormalized Nonrelativistic Hamiltonian in Quantum Field Theory: The Nelson Model

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Abstract. Scattering theory for the Nelson model is studied. We show Rosen estimates and we prove the existence of a ground state for the Nelson Hamiltonian. Also we prove that it has a locally finite pure point spectrum outside its thresholds. We study the asymptotic fields and the existence of the wave operators. Finally we show asymptotic completeness for the Nelson Hamiltonian.

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Table of Contents

1. Introduction

Recently there has been a renewed interest in quantum field theory models that describe a system of nonrelativistic particles interacting with a bosonic field. The main physical example is a nonrelativistic atom interacting with photons. For this model the existence of a ground state was established in [BFS], [AH]. The absence of excited states was also shown in [BFS] using a renormalization group analysis and in [BFSS], [DJ] using a positive commutator method.

Another result related to the present work is [DG2] where for a model of a confined atom interacting with massive bosons the asymptotic completeness of wave operators was proved.

In all these works, the model contains an ultraviolet cutoff which switches off the interaction between the nucleons and the bosonic field above a certain momentum scale. This can be justified physically by the fact that nucleons interacting with bosons of very high energy will become relativistic and in such a situation the model will anyhow loose its validity.

With a cutoff these models are free of ultraviolet divergences and, hence, can be easily constructed rigorously by rather elementary methods.

However the presence of a cutoff implies that the interaction term is now non local and that quantitative results depend on the choice of the cutoff scale. Therefore it would be more satisfactory to remove the ultraviolet cutoff from the model under consideration.

When the interaction term is linear in the field variables, the removal of the ultraviolet cutoff was done long time ago by Nelson [Ne]. This was probably the first model which was rigorously constructed using a renormalization procedure. It consists in considering cutoff Hamiltonians H_κ, where κ is some ultraviolet cutoff parameter and applying a cutoff-dependent unitary transformation U_κ. After substracting a divergent self-energy term E_κ, the sequence of Hamiltonians $U_\kappa(H_\kappa - E_\kappa)U_\kappa^*$ converges in norm resolvent sense to a Hamiltonian $\hat{H}_\infty$ when $\kappa \to \infty$ while U_κ converges strongly to a unitary transformation U_∞ (in other words, no change of representation is necessary). The Hamiltonian $H := U_\infty^* \hat{H}_\infty U_\infty$ is called the *Nelson Hamiltonian*. After Nelson's paper, the Nelson model was studied by Cannon [Ca] and Fröhlich [Fr].

In this paper we consider the Nelson model for a confined atom and massive bosons and study its spectral and scattering theory. Our main result is the asymptotic completeness of the wave operators, which implies the unitarity of the S matrix. The strategy and the proofs of our paper follow closely those of [DG2], which is devoted to a similar model with an ultraviolet cutoff. Nevertheless there are new difficulties coming from the fact that the Nelson Hamiltonian is only defined as the resolvent limit of the cutoff Hamiltonians.

Let us now describe the content of the paper. In Section 2 we recall classical notations related to Fock spaces, introduce some definitions and prove some extensions of Glimm–Jaffe's N_τ estimates.

In Section 3 we recall the construction of the Nelson Hamiltonian following [Ne]. In Section 4 we prove the so-called *higher order estimates*, following an idea of Rosen [Ro]. In Section 5 we study the spectral theory for the confined Nelson Hamiltonian. We prove an HVZ theorem and a positive commutator estimate. Section 6 is devoted to the scattering theory for the Nelson model. We show the existence of asymptotic fields and that the CCR representation they define is of Fock type, using an argument from [DG3]. In Section 7 we prove various propagation estimates for the Nelson Hamiltonian. Finally the asymptotic completeness of the wave operators is shown in Section 8.

2. Presentation of the Model

In this section we define the Nelson model. We start with a review of the basic construction and the main notations related to bosonic Fock spaces. For a more detailed exposition we refer the reader to [Be], [BR], [BSZ]. In Subsection 2.2 we give some technical estimates obtained by adaptation of N_τ-estimates [GJ]. Finally in Subsection 2.3 we introduce the formal Hamiltonian of the interacting system of P confined nonrelativistic particles (nucleons) with a relativistic scalar field (mesons). In order to give sense to the formal Hamiltonian, we put a high-momentum cutoff in the interaction and we show that the cutoff Hamiltonian is a well defined selfadjoint operator.

2.1. BASIC DEFINITIONS AND NOTATIONS

Let $\mathfrak{h}$ be a complex Hilbert space. Let $\otimes_s^n \mathfrak{h}$ denote the symmetric n-fold tensor power of $\mathfrak{h}$. We introduce the *bosonic Fock space* by $\Gamma(\mathfrak{h}) := \bigoplus_{n\geqslant 0} \otimes_s^n \mathfrak{h}$. $\otimes_s^0 \mathfrak{h} := \mathbb{C}$, identified as subspace of $\Gamma(\mathfrak{h})$, represents the space of zero-particle states. We denote by Ω the vector $(1, 0, \ldots)$ usually called *vacuum vector* and by $\Gamma_{\text{fin}}(\mathfrak{h})$ the subspace of finite particle states, which is the subspace of finite sum of vectors in $\otimes_s^n \mathfrak{h}$. Among the main operators acting on $\Gamma(\mathfrak{h})$, we will first recall the definitions of the most familiar as *number operator* N given in its spectral decomposition $N|_{\otimes_s^n \mathfrak{h}} := n\mathbb{1}$. Creation operators, which are unbounded operators densely defined on $\mathcal{D}(N^{\frac{1}{2}})$, are given by

$$a^*(h)|_{\otimes_s^n \mathfrak{h}} := \sqrt{(n+1)} S_{n+1} h \otimes \mathbb{1}_{\otimes_s^n \mathfrak{h}},$$

where S_n denotes the orthogonal projection from $\otimes^n \mathfrak{h}$ into $\otimes_s^n \mathfrak{h}$. The annihilation operator $a(h)$ is the adjoint of $a^*(h)$. We will use the notation $a^\sharp$ for a or a^*.

We define the *field operator* by

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)), \quad h \in \mathfrak{h}.$$

$\phi(h)$ is essentially selfadjoint on $\Gamma_{\text{fin}}(\mathfrak{h})$. We still denote by $\phi(h)$ its closure. By functional calculus we get unitary operators called *Weyl operators*, defined as $W(h) := e^{i\phi(h)}$. We recall a useful differentiation estimate for $W(h)$:

Let $0 \leqslant \epsilon \leqslant 1$,

$$\begin{aligned}&\|(W(h_1) - W(h_2))u\| \\ &\quad \leqslant C_\epsilon \|h_1 - h_2\|^\epsilon ((\|h_1\|^2 + \|h_2\|^2)^{\frac{\epsilon}{2}} \|u\| + \|(N+1)^{\frac{\epsilon}{2}} u\|),\end{aligned} \tag{2.1}$$

$$\lim_{s \to 0} \sup_{\|h\| \leqslant c} s^{-1} \|(W(sh) - \mathbb{1} - \mathrm{i}s\phi(h))(N+1)^{-\frac{1}{2}-\epsilon}\| = 0, \quad \epsilon > 0. \tag{2.2}$$

Let $\mathfrak{h}$ be a Hilbert space. Let $\mathfrak{f} : \mathfrak{h} \to \mathfrak{h}$ be an (unbounded) operator. We denote by $\mathrm{d}\Gamma(\mathfrak{f})$ the amplification of $\mathfrak{f}$ to the whole space $\Gamma(\mathfrak{h})$

$$\mathrm{d}\Gamma(\mathfrak{f})|_{\otimes_s^n \mathfrak{h}} := \sum_{j=1}^{n} \mathbb{1}^{\otimes(j-1)} \otimes \mathfrak{f} \otimes \mathbb{1}^{\otimes(n-j)}.$$

Let $\mathfrak{h}_i$, $i = 1, 2$ be two Hilbert spaces. Let $\mathfrak{g} : \mathfrak{h}_1 \to \mathfrak{h}_2$ be a bounded operator. We define the operator $\Gamma(\mathfrak{g})$ by

$$\begin{aligned}&\Gamma(\mathfrak{g}) : \mathfrak{h}_1 \to \mathfrak{h}_2, \\ &\Gamma(\mathfrak{g})|_{\otimes_s^n \mathfrak{h}_1} := \mathfrak{g}^{\otimes(n)}.\end{aligned}$$

A less familiar operator is $\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})$, where $\mathfrak{f}$, $\mathfrak{g}$ are two operators on $\mathfrak{h}_1$ into $\mathfrak{h}_2$. It is defined as in [DG2]:

$$\begin{aligned}&\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g}) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ &\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})|_{\otimes_s^n \mathfrak{h}_1} := \sum_{j=1}^{n} \mathfrak{f}^{\otimes(j-1)} \otimes \mathfrak{g} \otimes \mathfrak{f}^{\otimes(n-j)}.\end{aligned}$$

We notice that $\mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{f}) = N\Gamma(\mathfrak{f})$ and if $\mathfrak{h}_1 = \mathfrak{h}_2$, we have $\mathrm{d}\Gamma(\mathbb{1}, \mathfrak{g}) = \mathrm{d}\Gamma(\mathfrak{g})$. If $\|\mathfrak{f}\| \leqslant 1$, the following inequality holds

$$\|N^{-\frac{1}{2}} \mathrm{d}\Gamma(\mathfrak{f}, \mathfrak{g})u\| \leqslant \|\mathrm{d}\Gamma(\mathfrak{g}^*\mathfrak{g})^{\frac{1}{2}} u\|. \tag{2.3}$$

Let i_1 (resp. i_2) be the injection of $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$) into $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. There exists a unitary transformation U identifying $\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$ with $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$, defined as follows

$$Uu \otimes v := \sqrt{\frac{(p+q)!}{p!q!}} S_{p+q} \Gamma(i_1)u \otimes \Gamma(i_2)v, \quad u \in \otimes_s^p \mathfrak{h}_1,\ v \in \otimes_s^q \mathfrak{h}_2.$$

This transformation has the following properties:

(i) $U\Omega \otimes \Omega = \Omega$.

(ii) Let $h_1 \in \mathfrak{h}_1, h_2 \in \mathfrak{h}_2$

$$\begin{aligned}&a^\sharp(h_1 \oplus h_2)U = U(a^\sharp(h_1) \otimes \mathbb{1} + \mathbb{1} \otimes a^\sharp(h_2)), \\ &\phi(h_1 \oplus h_2)U = U(\phi(h_1) \otimes \mathbb{1} + \mathbb{1} \otimes \phi(h_2)).\end{aligned}$$

(iii) Let $\mathfrak{f}_i : \mathfrak{h}_i \to \mathfrak{h}_i$, $i = 1, 2$ be two operators

$$\mathrm{d}\Gamma(\mathfrak{f}_1 \oplus \mathfrak{f}_2)U = U(\mathrm{d}\Gamma(\mathfrak{f}_1) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\mathfrak{f}_2)),$$
$$U\,\Gamma(\mathfrak{f}_1) \otimes \Gamma(\mathfrak{f}_2) = \Gamma(\mathfrak{f}_1 \oplus \mathfrak{f}_2).$$

We define the *scattering identification operator* I

$$I : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$
$$Iu \otimes v := \sqrt{\frac{(p+q)!}{p!q!}}\, S_{p+q}\, u \otimes v, \quad u \in \otimes_s^p \mathfrak{h},\ v \in \otimes_s^q \mathfrak{h}.$$

We can also define I by the following formula:

$$I \prod_{i=1}^{p} a^*(h_i)\Omega \otimes \prod_{i=1}^{q} a^*(g_i)\Omega := \prod_{i=1}^{q} a^*(g_i) \prod_{i=1}^{p} a^*(h_i)\Omega, \quad h_i, g_i \in \mathfrak{h}.$$

Let π be the following map

$$\pi : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},$$
$$(h_0, h_\infty) \to h_0 + h_\infty.$$

Then we can express I as following $I = \Gamma(\pi)U$. We notice that I is unbounded since $\|\pi\| = \sqrt{2}$.

Let $i = (i_0, i_\infty)$ be a pair of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$I(i) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$
$$I(i) := I\,\Gamma(i_0) \otimes \Gamma(i_\infty).$$

Let $i = (i_0, i_\infty)$, $j = (j_0, j_\infty)$ be two pairs of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$\mathrm{d}I(i, j) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}),$$
$$\mathrm{d}I(i, j) := I\,(\mathrm{d}\Gamma(i_0, j_0) \otimes \Gamma(i_\infty) + \Gamma(i_0) \otimes \mathrm{d}\Gamma(i_0, j_\infty)).$$

If $i_0 i_0^* + i_\infty i_\infty^* \leqslant 1$ we have the estimates

$$\|(N_0 + N_\infty)^{-\frac{1}{2}} \mathrm{d}I^*(i, j)u\| \leqslant \|\mathrm{d}\Gamma(j_0 j_0^* + j_\infty j_\infty^*)^{\frac{1}{2}} u\|, \tag{2.4}$$

$$\begin{aligned} &|(u_2 | \mathrm{d}I^*(i, j)u_1)| \\ &\quad \leqslant \|\mathrm{d}\Gamma(|j_0|)^{\frac{1}{2}} \otimes \mathbb{1}\, u_2\|\ \|\mathrm{d}\Gamma(|j_0|)^{\frac{1}{2}} u_1\| + \\ &\qquad + \|\mathbb{1} \otimes \mathrm{d}\Gamma(|j_\infty|)^{\frac{1}{2}} u_2\|\ \|\mathrm{d}\Gamma(|j_\infty|)^{\frac{1}{2}} u_1\|. \end{aligned} \tag{2.5}$$

For other properties and equivalent definitions of these operators, we refer the reader to [DG3].

Let $\mathcal{K}$ be an auxiliary Hilbert space. Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. We define an extended creation operator:

$$a^*(v) : \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h}) \to \mathcal{K} \otimes \Gamma_{\text{fin}}(\mathfrak{h}),$$
$$a^*(v)_{|\mathcal{K}\otimes\otimes_s^n \mathfrak{h}} := \sqrt{(n+1)}\ (\mathbb{1}_{\mathcal{K}} \otimes S_{n+1})v \otimes \mathbb{1}_{\otimes_s^n \mathfrak{h}}.$$

$a^*(v)$ is closable densely defined operator since its adjoint $a(v)$ is densely defined. We define the field operator $\phi(v)$ as in the scalar case. When $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ can be represented as a function $k \to v(k) \in \mathcal{B}(\mathcal{K})$, such that for $x \in \mathcal{K}$, $v(k)x := vx(k)$, k-a.e. and

$$\mathcal{K} \times \mathcal{K} \ni (x, y) \to \int (v(k)^* v(k)x|\, y)_{\mathcal{K}}\ \mathrm{d}k = (vx|vy)_{\mathcal{K}\otimes\mathfrak{h}}$$

is a continuous quadratic form. A stronger condition is $v \in L^2(\mathbb{R}^d, \mathcal{B}(\mathcal{K}))$, i.e.:

$$\int \|v(k)\|^2_{\mathcal{B}(\mathcal{K})}\, \mathrm{d}k \ < \ \infty.$$

Assume that $[v_1^*(k),\ v_2(k')] = 0,\ \forall k, k'$. Then:

$$[a(v_1), a^*(v_2)] = v_1^* v_2 \otimes \mathbb{1}_{\Gamma(\mathfrak{h})},$$
$$[\phi(v_1), \phi(v_2)] = \mathrm{i}\mathrm{Im}(v_1^* v_2) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})},$$
$$[\phi(v_1), W(v_2)] = \mathrm{Im}(v_1^* v_2) \otimes\ W(v_2).$$

2.2. TECHNICAL ESTIMATES

In this subsection we will collect some technical estimates which are adaptation of Glimm–Jaffe's N_τ-estimates.

We recall the symbolic annihilation and creation operators in the case of a Fock space constructed over the space of square integrable functions $\mathfrak{h} := L^2(\mathbb{R}^d, \mathrm{d}k)$. Let $\Psi \in \Gamma_{\text{fin}}(\mathfrak{h})$:

$$(a(k)\Psi)^{(n)}(k_1, \ldots, k_n) := (n+1)^{\frac{1}{2}}\ \Psi^{(n+1)}(k, k_1, \ldots, k_n),$$
$$(a^*(k)\ \Psi)^{(n)}(k_1, \ldots, k_n)$$
$$:= n^{-\frac{1}{2}} \sum_{j=1}^{n} \delta(k - k_j)\ \Psi^{(n-1)}(k_1, \ldots, \hat{k}_j, \ldots, k_n),$$

where $\hat{k}_j$ means that k_j is omitted. Let $S(\mathbb{R}^d)$ be the Schwartz space. We can define the monomial $a(k_1) \ldots a(k_s)$ as an operator from $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ into $S(\mathbb{R}^{ds}) \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^d))$.

Let $\mathcal{K}$ be an auxiliary Hilbert space. Let w be unbounded operator from $\mathcal{K}\otimes\mathfrak{h}^{\otimes s}$ into $\mathcal{K}\otimes\mathfrak{h}^{\otimes r}$, with a domain containing $\mathcal{K}\otimes S(\mathbb{R}^{ds})$. A *Wick monomial* with *symbol* w is the following sesquilinear form on $\mathcal{K} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^d))$

$$W_{r,s} :=\ a^*(k_1') \ldots a^*(k_r')\ w\, a(k_1) \ldots a(k_s).$$

Let ω be a positive regular function satisfying:

$$\omega \in C^\infty(\mathbb{R}^d),$$
$$|\partial_k^\alpha \omega(k)| \leqslant c_\alpha (1+|k|)^{N_\alpha} \quad \text{for } \alpha \in \mathbb{N}^d,$$
$$\omega(k) \geqslant m > 0.$$

We set $N_\tau := \mathrm{d}\Gamma(\omega^\tau)$.

LEMMA 2.1. *Let r be a positive integer and $\tau := (\tau_i)_{1\ldots r}$ be a sequence of real numbers. For $\psi \in \mathcal{D}(\prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$:*

$$\left\| \prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}} \psi \right\|^2 = \sum_{j=1}^r \int P_{r,j}^\tau (k_1, \ldots, k_j) \left\| \prod_{i=1}^j a(k_i)\psi \right\|^2 \mathrm{d}k_1 \ldots \mathrm{d}k_j, \tag{2.6}$$

where $P_{r,j}^\tau$ is a sum of homogeneous functions in the variables $\omega(k_i)$ of degree $\sum_{i=1}^r \tau_i$ and satisfying

$$P_{r,j}^\tau(k_1, \ldots, k_j) = \sum_{(i_s)_{1\ldots r} \in \mathfrak{S}_{r,j}} \omega(k_{i_1})^{\tau_1} \ldots \omega(k_{i_r})^{\tau_r}, \tag{2.7}$$

where $\mathfrak{S}_{r,j}$ is the set, constructed by induction, of surjective maps i from $\{1, \ldots, r\}$ into $\{1, \ldots, j\}$, such that $i_s \leqslant s$ and $(i_s)_{1\ldots r-1}$ is in $\mathfrak{S}_{r-1,j}$ or in $\mathfrak{S}_{r-1,j-1}$ and $\mathfrak{S}_{j,j+1} = \mathfrak{S}_{1,0} = \emptyset$.

Proof. $\prod_{i=1}^j a(k_i)$ can be defined as operator on $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ into $S(\mathbb{R}^{dj}) \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^d))$. For $\psi \in \Gamma_{\text{fin}}(S(\mathbb{R}^d))$,

$$(k_1, \ldots, k_j) \to \left\| \prod_{i=1}^j a(k_i)\psi \right\|^2$$

is a function in $S(\mathbb{R}^{dj})$. This implies that the right-hand side of (2.6) is well defined for $\psi \in \Gamma_{\text{fin}}(S(\mathbb{R}^d))$. The hypothesis on ω imply that $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ is a core for $N_\tau^{\frac{1}{2}}$ and for $\prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}}$. If the lemma holds for $\psi \in \Gamma_{\text{fin}}(S(\mathbb{R}^d))$ then it can be extended to $\psi \in \mathcal{D}(\prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$. In fact since $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ is a core for $\prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}}$, then we can extend $\prod_{i=1}^j a(k_i)$ to bounded operator from $\mathcal{D}(\prod_{i=1}^r N_{\tau_i}^{\frac{1}{2}})$ into $L^2(\mathbb{R}^{dj}, (P_{r,j}^\tau)^{\frac{1}{2}}\,\mathrm{d}k) \otimes \Gamma(\mathfrak{h})$. Let us prove the lemma for $\psi \in \Gamma_{\text{fin}}(S(\mathbb{R}^3))$ by induction in r.

For $r = 1$,

$$\begin{aligned} \|N_{\tau_1}^{\frac{1}{2}}\psi\|^2 &= (\psi | N_{\tau_1}\psi) \\ &= \int \omega(k)^{\tau_1} \|a(k)\psi\|^2\,\mathrm{d}k. \end{aligned}$$

We see that $P_{1,1}^{\tau_1}(k) = \omega(k)^{\tau_1}$. (2.6), (2.7) are satisfied for $r = 1$. Assume that (2.6), (2.7) hold for r. Using the fact that N_τ preserves $\Gamma_{\text{fin}}(S(\mathbb{R}^d))$ and the induction hypothesis, we have

$$\begin{aligned}
&\left\|\prod_{i=1}^{r+1} N_{\tau_i}^{\frac{1}{2}}\psi\right\|^2\\
&\quad=\sum_{j=1}^{r}\int P_{r,j}^{\tau}(k_1,\dots,k_j)\left\|\prod_{i=1}^{j} a(k_i)N_{\tau_{r+1}}^{\frac{1}{2}}\psi\right\|^2 \mathrm{d}k\\
&\quad=\sum_{j=1}^{r}\int P_{r,j}^{\tau}\left\|\left(N_{\tau_{r+1}}+\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}\right)^{\frac{1}{2}}\prod_{i=1}^{j} a(k_i)\psi\right\|^2 \mathrm{d}k\\
&\quad=\sum_{j=1}^{r}\int P_{r,j}^{\tau}\left\{\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}\left\|\prod_{i=1}^{j} a(k_i)\psi\right\|^2+\left\|N_{\tau_{r+1}}^{\frac{1}{2}}\prod_{i=1}^{j} a(k_i)\psi\right\|^2\right\}\mathrm{d}k\\
&\quad=\sum_{j=1}^{r+1}\int\left[P_{r,j}^{\tau}\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}+P_{r,j-1}^{\tau}\omega(k_j)^{\tau_{r+1}}\right]\left\|\prod_{i=1}^{j} a(k_i)\psi\right\|^2 \mathrm{d}k,
\end{aligned}$$

where $P_{r,r+1}=P_{r,0}=0$. Then we obtain the following iterated relation

$$\begin{aligned}
&P_{r+1,j}^{\tau}(k_1,\dots,k_j)\\
&\quad=P_{r,j}^{\tau}(k_1,\dots,k_j)\sum_{i=1}^{j}\omega(k_i)^{\tau_{r+1}}+P_{r,j-1}^{\tau}(k_1,\dots,k_{j-1})\omega(k_j)^{\tau_{r+1}}. \qquad (2.8)
\end{aligned}$$

We note that $P_{r,r}^{\tau}(k_1,\dots,k_r)=\prod_{i=1}^{r}\omega(k_i)^{\tau_i}$. It is easy to see, using induction hypothesis (2.7) for r and (2.8), that

$$P_{r+1,j}^{\tau}(k_1,\dots,k_{r+1})=\sum_{(i_s)_{1\dots r+1}\in\mathfrak{S}_{r+1,j}}\omega(k_{i_1})^{\tau_1}\dots\omega(k_{i_{r+1}})^{\tau_{r+1}}. \qquad \square$$

COROLLARY 2.2. *Let* $\alpha,\nu\ ,\tau\ :=\ (\tau_i)_{1\dots r}$ *be a sequence of real numbers. For* $\psi\in\mathcal{D}(N_\nu^{\alpha}\prod_{i=1}^{r}N_{\tau_i}^{\frac{1}{2}})$

$$\begin{aligned}
&\left\|N_\nu^{\alpha}\prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}}\psi\right\|^2\\
&\quad=\sum_{j=1}^{r}\int P_{r,j}^{\tau}(k_1,\dots,k_j)\left\|\left(N_\nu+\sum_{i=1}^{j}\omega(k_i)^{\nu}\right)^{\alpha}\prod_{i=1}^{j} a(k_i)\psi\right\|^2\times\\
&\qquad\times\,\mathrm{d}k_1\dots\mathrm{d}k_j, \qquad (2.9)
\end{aligned}$$

where $P_{r,j}^{\tau}$ *is the function defined by* (2.7).

LEMMA 2.3. *Let* $p, q \in \mathbb{N}$ *and* $\tau' := (\tau_i')_{1\dots p}, \tau := (\tau_i)_{1\dots q}$ *be two sequences of real numbers. Let* $W_{r,s}$ *be a Wick monomial such that* $r \leqslant p, s \leqslant q$. *Then*

$$\left\| \left(\prod_{i=1}^{p} (N_{\tau_i'} + 1)^{-\frac{1}{2}} \right) W_{r,s} \prod_{i=1}^{q} (N_{\tau_i} + 1)^{-\frac{1}{2}} \right\| \leqslant \|(P_{p,r}^{\tau'})^{-\frac{1}{2}}\, w\, (P_{q,s}^{\tau})^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s}, \mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

Proof. Let $\psi, \phi \in \mathcal{K} \otimes \Gamma_{\mathrm{fin}}(S(\mathbb{R}^d))$.

$$\begin{aligned}
&\left|(W_{r,s}\, \psi|\phi)\right| \\
&\quad = \left| \left(w \prod_{i=1}^{s} a(k_i)\psi \,\middle|\, \prod_{i=1}^{r} a(k_i')\phi \right)_{\mathcal{K}\otimes\mathfrak{h}^{\otimes r}\otimes\Gamma(\mathfrak{h})} \right| \\
&\quad \leqslant \left| \left((P_{p,r}^{\tau'})^{-\frac{1}{2}}\, w\, (P_{q,s}^{\tau})^{-\frac{1}{2}}\, (P_{s,q}^{\tau})^{\frac{1}{2}} \prod_{i=1}^{s} a(k_i)\psi \,\middle|\, (P_{p,r}^{\tau'})^{\frac{1}{2}} \prod_{i=1}^{r} a(k_i')\phi \right) \right| \\
&\quad \leqslant \|(P_{p,r}^{\tau'})^{-\frac{1}{2}}\, w\, (P_{q,s}^{\tau})^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s}, \mathcal{K}\otimes\mathfrak{h}^{\otimes r})} \left(\int P_{q,s}^{\tau} \left\| \prod_{i=1}^{s} a(k_i)\psi \right\|^2 \mathrm{d}k \right)^{\frac{1}{2}} \times \\
&\qquad \times \left(\int P_{p,r}^{\tau'} \left\| \prod_{i=1}^{r} a(k_i')\phi \right\|^2 \mathrm{d}k' \right)^{\frac{1}{2}} \\
&\quad \leqslant \|(P_{p,r}^{\tau'})^{-\frac{1}{2}}\, w\, (P_{q,s}^{\tau})^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s}, \mathcal{K}\otimes\mathfrak{h}^{\otimes r})} \left\| \prod_{i=1}^{q} N_{\tau_i}^{\frac{1}{2}} \psi \right\| \times \left\| \prod_{i=1}^{p} N_{\tau_i'}^{\frac{1}{2}} \phi \right\|.
\end{aligned}$$

This inequality shows that the quadratic form

$$\prod_{i=1}^{p} (N_{\tau_i'} + 1)^{-\frac{1}{2}} W_{r,s} \prod_{i=1}^{q} (N_{\tau_i} + 1)^{-\frac{1}{2}}$$

can be extended to a bounded operator with norm less than

$$\left\| \left(P_{p,r}^{\tau'}\right)^{-\frac{1}{2}} w\, (P_{q,s}^{\tau})^{-\frac{1}{2}} \right\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s}, \mathcal{K}\otimes\mathfrak{h}^{\otimes r})}. \qquad \square$$

COROLLARY 2.4. *Let* $v_i \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$; $i = 1, \dots, n$. *Then there exists* $c > 0$ *such that*

(i) $$\left\| (N+1)^p \prod_{i=1}^{n} a^{\sharp}(v_i)(N+1)^{-p-\frac{n}{2}} \right\| \leqslant c \prod_{i=1}^{n} \|v_i\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.$$

(ii) $$\left\| (N+1)^p \prod_{i=1}^{n} \phi(v_i)(N+1)^{-p-\frac{n}{2}} \right\| \leqslant c \prod_{i=1}^{n} \|v_i\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.$$

(iii) *Let* $r, s \in \mathbb{N}$, $(\tau_i')_{1\dots r}$, $(\tau_i)_{1\dots s}$ *be sequences of real numbers. Then*

$$\left\| \left(\prod_{i=1}^{r} (N_{\tau'_i}+1)^{-\frac{1}{2}} \right) W_{r,s} \prod_{i=1}^{s} (N_{\tau_i}+1)^{-\frac{1}{2}} \right\| \leqslant \left\| \prod_{i=1}^{r} \omega(k'_i)^{-\frac{\tau'_i}{2}} \; w \; \prod_{i=1}^{s} \omega(k_i)^{-\frac{\tau_i}{2}} \right\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

Proof. Clearly (i) gives (ii). For $p = 0$ and $n = 1$, (i) follows from the last lemma by taking $\omega = 1$ and $r = p = 1, s = q = 0$ or $r = p = 0, s = q = 1$. For $p, n \in \mathbb{N}$, commutation properties reduces the inequality to the case $p = 0, n = 1$. (iii) is a direct application of Lemma 2.3. □

We recall now well known estimates, see [Ro].

LEMMA 2.5. *Let b be a positive operator. Then*

$$\mathrm{d}\Gamma(b^{\alpha_1}) \leqslant N^{1-\alpha_1}\mathrm{d}\Gamma(b)^{\alpha_1}, \quad where\ \alpha_1 \leqslant 1.$$

$$\mathrm{d}\Gamma(b^{\tau_1})^{\alpha_1} \leqslant \mathrm{d}\Gamma(b^{\tau_2})^{\alpha_2}\mathrm{d}\Gamma(b^{\tau_3})^{\alpha_3},$$

where $\alpha_1 = \alpha_2 + \alpha_3$, and $\alpha_1\tau_1 = \alpha_2\tau_2 + \alpha_3\tau_3$.

Combining Lemma 2.5 and Lemma 2.3 we obtain a slightly more general estimate.

LEMMA 2.6. *Let $r, s, p, q \in \mathbb{N}$. Let $\tau'_j := (\tau'^i_j)_{1\ldots p}$, $\tau_j := (\tau^i_j)_{1\ldots q}$, $j = 1\ldots 3$ be sequences of real numbers such that $\tau'^i_1 = \tau'^i_2 + \tau'^i_3$, $\tau^i_1 = \tau^i_2 + \tau^i_3$. Then*

$$\left\| \prod_{i=1}^{p} (N_{\tau'^i_2}+1)^{-\frac{1}{2}} (N_{\tau'^i_3}+1)^{-\frac{1}{2}} W_{r,s} \prod_{i=1}^{q} (N_{\tau^i_2}+1)^{-\frac{1}{2}} (N_{\tau^i_3}+1)^{-\frac{1}{2}} \right\| \leqslant \|(P^{\tau'_1}_{p,r})^{-\frac{1}{2}} \; w \; (P^{\tau_1}_{q,s})^{-\frac{1}{2}}\|.$$

The following estimate is an immediate application of Lemma 2.6.

COROLLARY 2.7. *Let r, s, α, β be positive integers such that $\alpha \leqslant r$, $\beta \leqslant s$. Then the following assertion holds*

$$\|(N+1)^{-\frac{r-\alpha}{2}} (\mathrm{d}\Gamma(\omega)+1)^{-\frac{\alpha}{2}} \, W_{r,s} \, (N+1)^{-\frac{s-\beta}{2}} (\mathrm{d}\Gamma(\omega)+1)^{-\frac{\beta}{2}}\| \leqslant \inf_{\substack{\{\tau'_i,\tau_i\in[0,1]|\\ \sum\tau'_i=\alpha,\ \sum\tau_i=\beta\}}} \left\| \prod_{i=1}^{r} \omega(k'_i)^{-\frac{\tau'_i}{2}} \; w \; \prod_{i=1}^{s} \omega(k_i)^{-\frac{\tau_i}{2}} \right\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h}^{\otimes s},\mathcal{K}\otimes\mathfrak{h}^{\otimes r})}.$$

2.3. THE NELSON MODEL

The Nelson model [Ne] describes a system of P nonrelativistic particles coupled to a scalar relativistic field of bosons by a local, translation invariant interaction. It exhibits a relatively mild ultraviolet divergence, and was the first QFT model, on which a renormalization procedure was rigorously carried on.

We consider the atomic Hamiltonian of the system of P nonrelativistic confined particles as follows

$$K := \frac{1}{2M}\sum_{j=1}^{P} D_j^2 + V(\mathrm{x}_1, \ldots, \mathrm{x}_P).$$

It acts on the Hilbert space $L^2(\mathbb{R}^{3P}, \mathrm{dx})$ which we denote in the sequel by $\mathcal{K}$. We assume that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^{3P})$ and $V \geqslant 0$. Kato's inequality gives that K is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{3P})$, see [RS, I–IV, Thm. X.28]. We set $\langle \mathrm{x}\rangle := (|\mathrm{x}|^2+1)^{\frac{1}{2}}$. We will also assume

$$V \geqslant c\sum_i \langle \mathrm{x}_i\rangle^\alpha, \quad \alpha > 2.$$

We notice that for $0 \leqslant \beta \leqslant 1$, $\langle D\rangle^\beta(K+1)^{-\frac{\beta}{2}}$, $\langle \mathrm{x}\rangle^\beta(K+1)^{-\frac{\beta}{2}}$ are bounded operators.

The boson one particle space is the Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathrm{d}k)$, where k denotes the boson momentum observable. The boson position observable $-(\nabla_k/\mathrm{i})$ will be denotes by the italic letter x. This should not be confused with the nucleon position observable denoted by the roman letter x. The free bosonic Hamiltonian is defined by the second quantization of a single boson energy. It acts on the bosonic Fock space $\Gamma(\mathfrak{h})$.

$$H_b := \mathrm{d}\Gamma(\omega),$$

$$\omega(k) := (|k|^2+m^2)^{\frac{1}{2}}, \quad m > 0.$$

It is essentially self-adjoint on $\Gamma_{\mathrm{fin}}(\mathcal{D}(\omega))$.

The Hilbert space of the joint system is $\mathcal{H} := \mathcal{K}\otimes\Gamma(\mathfrak{h})$. The Hamiltonian without interaction is given by

$$H_0 := K\otimes\mathbb{1} + \mathbb{1}\otimes H_b.$$

This is a self-adjoint operator since $K\otimes\mathbb{1}$ and $\mathbb{1}\otimes H_b$ commute on $\mathcal{H}$. The local translation invariant interaction between nucleons and bosons is given by the formal expression

$$\sum_{j=1}^{P}\varphi(\mathrm{x}_j), \quad \text{where } \varphi(\mathrm{x}) := [2(2\pi)^3]^{-\frac{1}{2}}\int \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}(a^*(k)+a(-k))\frac{\mathrm{d}k}{\omega(k)^{\frac{1}{2}}}.$$

The interaction term cannot be defined as an operator on $\mathscr{H}$ with a dense domain. This comes from the fact that $\omega^{-\frac{1}{2}} \notin L^2(\mathbb{R}^3, \mathrm{d}k)$, because the integral diverges for large k. This phenomenon is known as an *ultraviolet problem*. In order to have a well defined operator, one introduces cutoff interactions:

$$\varphi_\kappa(\mathrm{x}) := [2(2\pi)^3]^{-\frac{1}{2}} \int \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}(a^*(k) + a(-k))\frac{\chi_\kappa(k)}{\omega(k)^{\frac{1}{2}}}\,\mathrm{d}k,$$

$$I_\kappa := \sum_{j=1}^{P} \varphi_\kappa(\mathrm{x}_j), \qquad v_\kappa := [(2\pi)^3]^{-\frac{1}{2}}\frac{\chi_\kappa(k)}{\omega(k)^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}.$$

Here χ is a positive function in $C^\infty(\mathbb{R}^3)$ such that $0 \leqslant \chi(k) \leqslant 1$, $\chi(k) = 1$ for $|k| \leqslant 1$, $\chi(k) = 0$ for $|k| \geqslant 2$ and $\chi(-k) = \chi(k)$. We set $\chi_\kappa(k) := \chi(k/\kappa)$.

LEMMA 2.8. *One has for* $\alpha \geqslant 0$

(i) $\mathrm{d}\Gamma(\omega)^\alpha \leqslant H_0^\alpha$,

(ii) $K^\alpha \leqslant H_0^\alpha$.

Proof. For $\psi \in C_0^\infty(\mathbb{R}^3) \otimes \Gamma_{\mathrm{fin}}(S(\mathbb{R}^3))$, which is a core for H_0, one has

$$(\mathrm{d}\Gamma(\omega)\psi, \psi) \leqslant (H_0\psi, \psi),$$
$$(K\psi, \psi) \leqslant (H_0\psi, \psi).$$

This means that $\mathrm{d}\Gamma(\omega) \leqslant H_0$ and $K \leqslant H_0$. Since H_0, K and $\mathrm{d}\Gamma(\omega)$ commute, the spectral theorem gives the inequalities announced in the lemma. □

I_κ are well-defined operators on $\mathcal{D}((N+1)^{\frac{1}{2}})$ as long as $\kappa < \infty$ and they are H_0-bounded with infinitesimal bound. We set $H_\kappa := H_0 + I_\kappa$.

THEOREM 2.9. *For* $\kappa < \infty$, H_κ *is a self-adjoint operator on* $\mathcal{D}(H_0)$.

Proof. Using Corollary 2.4 and Lemma 2.8, we prove for c independent from λ

$$\|I_\kappa(H_0 + \lambda)^{-1}\| \leqslant c\,\lambda^{-\frac{1}{2}}\,\|v_\kappa\|.$$

Then by the Kato–Rellich theorem, one sees that H_κ is a selfadjoint operator on $\mathcal{D}(H_0)$. □

3. Construction of the Nelson Hamiltonian

In this section we recall the construction in [Ne] of the Nelson Hamiltonian. It consists in applying to the cutoff Hamiltonians H_κ a cutoff dependent unitary transformation U_κ, letting then κ to ∞.

3.1. DRESSING TRANSFORMATION

For a fixed κ_0 and $\kappa < \infty$, we define

$$g_\kappa(k) := -\mathrm{i}\frac{(2\pi)^{-\frac{3}{2}}}{\omega^{\frac{1}{2}}(k)}\,\frac{\chi_\kappa(k)-\chi_{\kappa_0}(k)}{\omega(k)+\frac{k^2}{2M}} \in C_0^\infty(\mathbb{R}^3), \tag{3.1}$$

$$G_\kappa := \sum_{j=1}^{P} \mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}\, g_\kappa(k) \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}), \tag{3.2}$$

$$C_\kappa := \frac{1}{2}\sum_{1\leqslant j,\ell\leqslant P}\int |g_\kappa(k)|^2\, \sin\langle k, (\mathrm{x}_j-\mathrm{x}_\ell)\rangle\,\mathrm{d}k \in S(\mathbb{R}^{3P}), \tag{3.3}$$

$$\begin{aligned} E_\kappa := {}& \frac{P}{2(2\pi)^3}\int \frac{1}{\omega(k)}\,\frac{(\chi_\kappa(k)-\chi_{\kappa_0}(k))^2}{(\omega(k)+\frac{k^2}{2M})}\,\mathrm{d}k - \\ & -\frac{P}{(2\pi)^3}\int\frac{\chi_\kappa(k)}{\omega(k)}\frac{\chi_\kappa(k)-\chi_{\kappa_0}(k)}{(\omega(k)+\frac{k^2}{2M})}\,\mathrm{d}k, \end{aligned} \tag{3.4}$$

$$r_\kappa(\mathrm{x}) := -\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle} g_\kappa(k), \tag{3.5}$$

$$U_\kappa := \mathrm{e}^{\mathrm{i}\phi(\mathrm{G}_\kappa)+\mathrm{i}\mathrm{C}_\kappa}, \tag{3.6}$$

$$\hat{H}_\kappa := U_\kappa(H_\kappa - E_\kappa)U_\kappa^*. \tag{3.7}$$

In order to simplify the writing of some formulas, we will replace often $r_\kappa(\mathrm{x}_j)$, $v_\kappa(\mathrm{x}_j)$ by r_κ^j, v_κ^j.

LEMMA 3.1. *For a fixed κ_0 and for $\kappa_0 < \kappa < \infty$, $\hat{H}_\kappa$ is a selfadjoint operator on the domain $\mathcal{D}(H_0)$ and equal to*

$$\hat{H}_\kappa = H_0 + \sum_{1\leqslant i<j\leqslant P} V_\kappa(\mathrm{x}_i-\mathrm{x}_j)\otimes\mathbb{1} + \hat{I}_\kappa, \tag{3.8}$$

where

$$\hat{I}_\kappa := I_{\kappa_0} + \frac{1}{2M}\sum_{j=1}^{P} R_j(r_\kappa(\mathrm{x}_j)),$$

$$R_j(v) := \frac{1}{2}a^2(v) + \frac{1}{2}a^{*^2}(v) + a^*(v)\,a(v) - \sqrt{2}D_j\,a(v) - \sqrt{2}a^*(v)D_j,$$

$$V_\kappa(\mathrm{x}) := \mathrm{Re}\int \omega(k)\,|g_\kappa(k)|^2\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k - 2\mathrm{Im}\int \bar{g}_\kappa(k)\,v_\kappa(k)\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k.$$

Proof. We compute $U_\kappa H_\kappa U_\kappa^*$ using commutation relations. We notice that terms coming from H_b and K have been mixed. This creates a translation invariant potential and second-order interacting terms:

$$U_\kappa H_b U_\kappa^* = H_b - \phi(\mathrm{i}\omega G_\kappa) + \\ + \sum_{1\leqslant i<j\leqslant P} V_\kappa^{(1)}(\mathrm{x}_i - \mathrm{x}_j) + \frac{P}{2}\int \omega(k)|g_\kappa(k)|^2\,\mathrm{d}k, \tag{3.9}$$

where

$$V_\kappa^{(1)}(\mathrm{x}) := \mathrm{Re}\int \omega(k)|g_\kappa(k)|^2\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k.$$

$$U_\kappa K U_\kappa^* = \frac{1}{2M}\sum_{j=1}^{P}(D_j + \phi(\mathrm{i}k\, g_\kappa(k)\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}))^2 + \\ + V(\mathrm{x}_1,\ldots,\mathrm{x}_P). \tag{3.10}$$

$$U_\kappa I_\kappa U_\kappa^* = I_\kappa + \sum_{1\leqslant i<j\leqslant P} V_\kappa^{(2)}(\mathrm{x}_i - \mathrm{x}_j) - \\ - \frac{P}{(2\pi)^3}\int \frac{\chi_\kappa(k)}{\omega(k)}\,\frac{\chi_\kappa(k) - \chi_{\kappa_0}(k)}{\omega(k) + \frac{k^2}{2M}}\,\mathrm{d}k,$$

where

$$V_\kappa^{(2)}(\mathrm{x}) := -2\mathrm{Im}\int \bar{g}_\kappa(k)\,v_\kappa(k)\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k.$$

Using the following computation

$$(D_j - \phi(r_\kappa^j))^2 \\ = D_j^2 + \frac{1}{2}a^2(r_\kappa^j) + \frac{1}{2}a^{*^2}(r_\kappa^j) + a^*(r_\kappa^j)\,a(r_\kappa^j) - \sqrt{2}D_j a(r_\kappa^j) - \\ - \sqrt{2}a^*(r_\kappa^j)D_j - \phi(\mathrm{i}k^2\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}g_\kappa) + \frac{1}{2}\int k^2|g_\kappa(k)|^2\,\mathrm{d}k,$$

in the second term (3.10) and collecting similar terms together, we obtain (3.8).

It is easy to see, by (3.9)–(3.10), that U_κ preserves $\mathcal{D}(H_0)$ for $\kappa < \infty$, hence $\mathcal{D}(\hat{H}_\kappa) = U_\kappa\mathcal{D}(H_0) = \mathcal{D}(H_0)$. □

3.2. REMOVAL OF THE ULTRAVIOLET CUTOFF

We set

$$
\begin{aligned}
g_\infty(k) &:= -\mathrm{i}\frac{(2\pi)^{-\frac{3}{2}}}{\omega^{\frac{1}{2}}(k)}\frac{1-\chi_{\kappa_0}(k)}{(\omega(k)+\frac{k^2}{2M})}\,,\\
G_\infty &:= \sum_{j=1}^{P}\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}_j\rangle}g_\infty.\\
V_\infty &:= \mathrm{Re}\int \omega(k)\,|g_\infty(k)|^2\,\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k\; -\\
&\quad -2\mathrm{Im}\int \bar{g}_\infty(k)\,\frac{1}{\omega(k)^{\frac{1}{2}}}\mathrm{e}^{-\mathrm{i}\langle k,\mathrm{x}\rangle}\,\mathrm{d}k,\\
C_\infty &:= \frac{1}{2}\sum_{1\leqslant j,\ell\leqslant P}\int |g_\infty(k)|^2\,\sin\langle k,(\mathrm{x}_j-\mathrm{x}_\ell)\rangle\,\mathrm{d}k,\\
U_\infty &:= \mathrm{e}^{\mathrm{i}\phi(G_\infty)+\mathrm{i}C_\infty}.
\end{aligned}
$$

LEMMA 3.2. *Letting κ to ∞, we obtain the following limits:*

(i) $E_\kappa \to -\infty$,
(ii) $g_\kappa \to g_\infty$ in $\mathfrak{h}$,
(iii) $C_\kappa \to C_\infty$ in $L^\infty(\mathbb{R}^{3P})$,
(iv) $V_\kappa \to V_\infty$ in $L^\infty(\mathbb{R}^3) + L^s(\mathbb{R}^3)$ for $s \in\,]2,+\infty[$,
(v) $U_\kappa \to U_\infty$ strongly in $\mathcal{B}(\mathcal{H})$.

Proof. (i) is obvious. (ii) follows from the monotone convergence theorem. (iii) follows from (ii), since $|g_\kappa|^2$ converges in $L^1(\mathbb{R}^3)$. $V_\kappa^{(1)}$ converges in $L^\infty(\mathbb{R}^3)$, which follows from the fact that its integrand converges in $L^1(\mathbb{R}^3)$. $V_\kappa^{(2)}$ converges in $L^s(\mathbb{R}^3)$ for $s \in\,]2,+\infty[$, by using Hausdorff–Young inequality. This proves (iv). Using the fact that the map $\mathfrak{h} \ni v \to \mathrm{e}^{\mathrm{i}\phi(v)}$ is strongly continuous and the limits (ii), (iii), we see that $s\text{-}\lim_{k\to\infty} U_\kappa$ exists and is equal to $\mathrm{e}^{\mathrm{i}\phi(G_\infty)+\mathrm{i}C_\infty}$. □

We give now a lemma which will be useful in this section and in Section 4.

LEMMA 3.3. *For $s \in [0,1]$, and $v_i \in \mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, $i = 1,2$ we have:*

(i) $\|(N+1)^{-\frac{s}{2}}\,a(v_1)\,(H_0+1)^{-\frac{1-s}{2}}\| \leqslant \|\omega^{\frac{s-1}{2}}\,v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

(ii) $\|(H_0+1)^{-\frac{s}{2}}\,a^*(v_1)\,(N+1)^{-\frac{1-s}{2}}\| \leqslant \|\omega^{-\frac{s}{2}}\,v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

(iii) $\|(N+1)^{-s}\,a(v_1)\,a(v_2)\,(H_0+1)^{-1+s}\|$
$\leqslant \|\omega^{-\frac{1-s}{2}}\,v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}\,\|\omega^{-\frac{1-s}{2}}\,v_2\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

(iv) $\|(H_0+1)^{-s}\,a^*(v_1)a^*(v_2)(N+1)^{-1+s}\|$
$\leqslant \|\omega^{-\frac{s}{2}}\,v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}\,\|\omega^{-\frac{s}{2}}\,v_2\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

Proof. We see clearly that suitable choice of r, s, α, and β in Corollary 2.7 gives similar inequalities with $d\Gamma(\omega)$ in the place of H_0. Now using Lemma 2.8 we obtain (i)–(iv). □

LEMMA 3.4. *One has for $v_i \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, such that $[V, v_i] = 0$, where V is the potential of the atomic Hamiltonian and $s, \beta \in [0, 1]$:*

(i) $\|(N+1)^{-\frac{1}{2}} a(v_1)\,(H_0+1)^{-\frac{1}{2}}\| \leqslant c\,\|(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}}\, v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

(ii) $\|(H_0+1)^{-\frac{1}{2}} a^*(v_1)\,(N+1)^{-\frac{1}{2}}\| \leqslant c\,\|(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}}\, v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}$.

(iii) $$\|(N+1)^{-1} a(v_1) a(v_2) (H_0+1)^{-\frac{1}{2}}\| \leqslant c\,\|(V+1)^{-\frac{\beta s}{2}}\,\omega^{\frac{s-1}{4}}\, v_1\| \times \|(V+1)^{-\frac{(1-\beta)s}{2}}\,\omega^{\frac{s-1}{4}}\, v_2\|.$$

(iv) $$\|(H_0+1)^{-\frac{1}{2}} a^*(v_1) a^*(v_2) (N+1)^{-1}\| \leqslant c\,\|(V+1)^{-\frac{\beta s}{2}}\,\omega^{\frac{s-1}{4}}\, v_1\| \times \|(V+1)^{-\frac{(1-\beta)s}{2}}\,\omega^{\frac{s-1}{4}}\, v_2\|.$$

(v) $$\|(N+1)^{-\frac{1}{2}} (H_0+1)^{-\frac{1}{2}} a^*(v_1) a(v_2) (H_0+1)^{-\frac{1}{2}}\| \leqslant c\|(V+1)^{-\frac{1-s}{2}} \omega^{-\frac{s}{2}} v_1\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \times \|(V+1)^{-\frac{s}{2}} \omega^{-\frac{1-s}{2}} v_2\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}.$$

Proof. Using Lemma 3.3, we see that (i)–(v) are true with $(H_0+1)^{-\frac{1}{2}}$ replaced by $(H_0+1)^{-\frac{1-s}{2}}(V+1)^{-\frac{s}{2}}$. We use then the fact that $(V+1)^{\frac{s}{2}}(H_0+1)^{-\frac{s}{2}}$ is bounded for $s \in [0, 1]$. □

We set for $\kappa < \infty$:

$$B_\kappa(\phi) := \left(\sum_{1 \leqslant i < j \leqslant P} V_\kappa(\mathrm{x}_i - \mathrm{x}_j) + \hat{I}_\kappa \phi | \phi \right), \quad \phi \in \mathcal{D}(H_0^{\frac{1}{2}}).$$

LEMMA 3.5. *There exists κ_0, $0 \leqslant a < 1$, $0 \leqslant b$ independent from κ, such that for $\kappa \geqslant 2\kappa_0$*

$$|B_\kappa(\phi)| \leqslant a\|H_0^{\frac{1}{2}}\phi\|^2 + b\|\phi\|^2, \quad \phi \in \mathcal{D}(H_0^{\frac{1}{2}}).$$

Proof. One has using Lemma 3.3

$$\|(H_0+\lambda)^{-\frac{1}{2}} \hat{I}_\kappa (H_0+\lambda)^{-\frac{1}{2}}\| \leqslant c(\lambda^{-\frac{1}{2}}\|\omega^{-\frac{1}{2}} v_{\kappa_0}\| + \|\omega^{-\frac{1}{4}} r_\kappa\|^2 + \|\omega^{-\frac{1}{2}} r_\kappa\|^2 + \|\omega^{-\frac{1}{2}} r_\kappa\|). \tag{3.11}$$

We notice that $\omega^{-\frac{1}{4}} r_\kappa$ contains the term $\chi_\kappa - \chi_{\kappa_0}$, for $\kappa \geqslant 2\kappa_0$, which is arbitrarily small for κ_0 large enough. Using this fact and (3.11) we see that there exists κ_0 such that $\hat{I}_\kappa$ has a small H_0-form bound for $\kappa > 2\kappa_0$. The integrand of $V_\kappa^{(1)}$, $V_\kappa^{(2)}$ contain the term $\chi_\kappa - \chi_{\kappa_0}$, then there exist κ_0 such that $V_\kappa^{(1)}$ (resp. $V_\kappa^{(2)}$) has a small $L^\infty(\mathbb{R}^3)$ (resp. $L^s(\mathbb{R}^3)$, $s > 2$) norm for $\kappa > 2\kappa_0$. After a change of variables to separate the motion of the center of mass, the term $V_\kappa^{(2)}$ becomes $V_\kappa^{(3)} + V_\kappa^{(4)} \in$

$L^\infty + L^q$ where $2 \leqslant q < \infty$. Using the Sobolev injection $H^1(\mathbb{R}^3) \to L^{q'}(\mathbb{R}^3)$, $2 \leqslant q' \leqslant 6$, we obtain, by a convenient choice of q and q', that $V_\kappa^{(2)}$ is H_0-form bounded [Ne]. □

In all the sequel a limit of a sequence of operators written without a prefix shall be understand as a norm limit.

THEOREM 3.6. *There is a unique selfadjoint operator $\hat{H}_\infty$ acting on $\mathcal{H}$, satisfying*

(i) $\lim_{\kappa\to\infty}(\hat{H}_\kappa - z)^{-1} = (\hat{H}_\infty - z)^{-1}$,

(ii) $s\text{-}\lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} = \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}$ *for* $t \in \mathbb{R}$.

The domain of $\hat{H}_\infty$ satisfies $\mathcal{D}(\hat{H}_\infty) \subset \mathcal{D}(\hat{H}_\infty^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$.

Proof. The proof is based on the Theorem A.1 in the appendix. Let us apply now this theorem with

$$B_\kappa(\phi) = \left(\sum_{1\leqslant i<j\leqslant P} V_\kappa(\mathrm{x}_i - \mathrm{x}_j) + \hat{I}_\kappa \phi | \, \phi \right).$$

Using Lemma 3.3, one has for $\phi \in \mathcal{D}(H_0^{\frac{1}{2}})$

$$\begin{aligned} &|B_\kappa(\phi) - B_{\kappa'}(\phi)| \\ &\quad \leqslant c(\|V_\kappa^{(1)} - V_{\kappa'}^{(1)}\|_{L^\infty} + \|V_\kappa^{(2)} - V_{\kappa'}^{(2)}\|_{L^s} + \|\omega^{-\frac{1}{2}}(r_\kappa - r_{\kappa'})\| + \\ &\qquad + \|\omega^{-\frac{1}{4}}(r_\kappa - r_{\kappa'})\| + \|\omega^{-\frac{1}{4}}(r_\kappa - r_{\kappa'})\|^2) \times \|(H_0+1)^{\frac{1}{2}}\phi\|^2. \end{aligned} \tag{3.12}$$

We see clearly, using Lemma 3.2(ii) and (iv) in the right-hand side of (3.12), that B_κ satisfies hypothesis of Theorem A.1. The KLMN theorem applied for B_∞, gives that $\mathcal{D}(\hat{H}_\infty^{\frac{1}{2}}) = \mathcal{D}(H_0^{\frac{1}{2}})$. So this proves the theorem. □

DEFINITION 3.7. The Hamiltonian $H := U_\infty^* \hat{H}_\infty U_\infty$ is called the Nelson Hamiltonian and $\hat{H}_\infty$ the modified Hamiltonian.

THEOREM 3.8. *One has*

(i) $\lim_{\kappa\to\infty}(H_\kappa - E_\kappa - z)^{-1} = (H - z)^{-1}$,

(ii) $s\text{-}\lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}t(H_\kappa - E_\kappa)} = \mathrm{e}^{-\mathrm{i}tH}$ *for* $t \in \mathbb{R}$.

(iii) $\mathcal{D}(H) \subset \mathcal{D}(H^{\frac{1}{2}}) = U_\infty^* \, \mathcal{D}(H_0^{\frac{1}{2}})$.

Proof. One has

$$\begin{aligned}
&\|U_\kappa^*(\hat{H}_\kappa - z)^{-1}U_\kappa - U_\infty^*(\hat{H}_\infty - z)^{-1}U_\infty\| \\
&\quad \leqslant \|(U_\kappa^* - U_\infty^*)(\hat{H}_\kappa - z)^{-1}\| + \\
&\qquad + \|(U_\kappa^* - U_\infty^*)(\hat{H}_\infty - \bar{z})^{-1}\| + \\
&\qquad + \|(\hat{H}_\kappa - z)^{-1} - (\hat{H}_\infty - z)^{-1}\|.
\end{aligned} \tag{3.13}$$

Using (3.13), Theorem 3.6 and the fact that the map $h \to W(h)(N+1)^{-\epsilon}$, $\epsilon > 0$, is norm continuous, we obtain

$$\begin{aligned}
&\|U_\kappa^*(\hat{H}_\kappa - z)^{-1}U_\kappa - U_\infty^*(\hat{H}_\infty - z)^{-1}U_\infty\| \\
&\quad \leqslant c(\|(W(G_\kappa) - W(G_\infty))(N+1)^{-\epsilon}\| + \\
&\qquad + \|(\hat{H}_\kappa - z)^{-1} - (\hat{H}_\infty - z)^{-1}\| + \\
&\qquad + \|\mathrm{e}^{\mathrm{i}C_\kappa} - \mathrm{e}^{\mathrm{i}C_\infty}\|).
\end{aligned}$$

The application of Lemma 3.2 completes the proof of (i). (ii) follows from the equivalence of the convergence in the strong resolvent sense and the strong convergence of unitary groups (Trotter theorem). (iii) is obvious. □

Let χ'_κ be another cutoff function and define H' to be the Nelson Hamiltonian constructed using the later cutoff.

PROPOSITION 3.9. *There exists a finite constant E, such that*

$$H' = H + E.$$

Proof. We define H'' (resp. $\hat{H}''_\kappa$) to be the Nelson Hamiltonian (resp. the cutoff modified Hamiltonian) obtained using the dressing transformation given by

$$g''_\kappa := -\mathrm{i}\frac{(2\pi)^{-\frac{3}{2}}}{\omega^{\frac{1}{2}}(k)} \frac{\chi'_\kappa(k) - \chi_{\kappa_0}(k)}{\omega(k) + \frac{k^2}{2M}}.$$

It is easy to see that $H' = H'' + E$, where E is a finite constant. Using similar calculus with (3.12), we obtain that $\hat{H}_\infty = \hat{H}''_\infty$. Since $U_\infty = U''_\infty$ we have $H = H''$. Then $H' = H + E$. □

4. Higher Order Estimates

In this section we prove some estimates which allow to bound powers of N and H_0 by powers of H. They play an important role. Fröhlich has proved a higher estimates in the massless case [Fr], but they are different from what we intend to prove. Their proofs are based in the following principle of cutoff independence [Ro].

LEMMA 4.1. *Let* $\{N_j\}$ *and* $\{H_j\}$ *be sequences of operators such that, for c independent of j,*

$$\|N_j\psi\| \leqslant c\,\|H_j\psi\|, \quad \textit{for } \psi \in \mathcal{D}(H_j).$$

Suppose that N_j *is self-adjoint and that* $N_j \to N$ *in the strong resolvent sense, where N is self-adjoint, and* $H_j \to H$ *in the strong graph limit. Then*

$$\|N\psi\| \leqslant c\,\|H\psi\|, \quad \textit{for } \psi \in \mathcal{D}(H).$$

Let Γ (resp. Γ_j) denotes the graph of H (resp. H_j). We recall that $H_j \to H$ in the strong graph sense if for all $(\psi, \varphi) \in \Gamma$, there exist a sequence $(\psi_j, \varphi_j) \in \Gamma_j$ which converges to (ψ, φ) in $\mathcal{H} \times \mathcal{H}$.

The following easy first order estimate follows from the proof of Theorem 3.6.

LEMMA 4.2. *There exists* $c > 0$ *independent for* κ *such that for* $\kappa_0 \leqslant \kappa \leqslant \infty$

$$H_0 \leqslant (\hat{H}_\kappa + c).$$

4.1. ROSEN ESTIMATES

In this subsection we prove higher order estimates using a technique due to Glimm and Jaffe to prove similar estimates for the Y_2 and $(\varphi^4)_2$ models. It has been taken up by Rosen in [Ro] for the general $(\varphi^{2n})_2$ model and Fröhlich [Fr]. This technique is based in the so called *pull-through formula* which is the identity that we obtain, in a formal way, when we move the resolvent through a product of annihilation operators $a(k_i)$. But some care must be taken when we want to rigorously prove the pull through formula, since we need a dense subspace $\mathcal{H}_0 \subset \mathcal{D}(H_0)$ on which $\prod_i a(k_i)$ acts as an operator and satisfies $\prod_i a(k_i)\mathcal{H}_0 \subset \mathcal{D}(H_0)$ and $\prod_i a(k_i)$ can be defined as operator on $H_0\mathcal{H}_0$. We need also a resolvent control of the commutator of the modified cutoff interaction with $a(k_i)$, which allows to define it as locally integrable function with values in bounded operators on $\mathcal{D}(H_0)$. This requires $H_0\mathcal{H}_0$ to be dense.

Let $\{J_i\}_{1\ldots j+1}$ be a set of disjoint subsets of $\{1, \ldots, n\}$ so that within each subset J_i the elements are taken in their natural order. We introduce the notation:

$$H_\kappa^J := [a(k_{i_1}), \ldots, [a(k_{i_j}), \hat{I}_\kappa]\ldots], \quad \text{where } J = \{i_1, \ldots, i_j\},$$

$$R(z) := (z - \hat{H}_\kappa)^{-1},$$

$$R_\ell(z) := \left(z - \sum_i \omega(k_i) - \hat{H}_\kappa\right)^{-1},$$

where the sum runs over $i \in J_\ell \cup J_{\ell+1} \cup \cdots \cup J_{j+1}$.

LEMMA 4.3. *Let* α ,$\tau := (\tau_i)_{1\ldots r}$ *be a sequence of real numbers. For*

$$\psi \in \mathcal{D}\left((H_0+1)^{\alpha} \prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}}\right)$$

$$\begin{aligned}
&\left\|(H_0+1)^{\alpha} \prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}} \psi\right\|^2 \\
&\quad = \sum_{j=1}^{r} \int P_{r,j}^{\tau}(k_1,\ldots,k_j) \left\|\left(H_0+\sum_{i=1}^{j}\omega(k_i)+1\right)^{\alpha} \times \right. \\
&\qquad \left. \times \prod_{i=1}^{j} a(k_i)\psi\right\|^2 \mathrm{d}k_1\ldots\mathrm{d}k_j. \qquad (4.1)
\end{aligned}$$

Proof. This lemma is similar to Corollary 2.2. We prove (4.1) for $\psi \in \mathcal{D}(K)\otimes \Gamma_{\mathrm{fin}}(S(\mathbb{R}^3))$ and then we extend it to $\psi \in \mathcal{D}((H_0+1)^{\alpha}\prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}})$. This extends $\prod_{i=1}^{j} a(k_i)$ to bounded operator from $\mathcal{D}((H_0+1)^{\alpha}\prod_{i=1}^{r} N_{\tau_i}^{\frac{1}{2}})$ into $(H_0+\sum_{i=1}^{j}\omega(k_i)+1)^{-\alpha} L^2(\mathbb{R}^{3j}, (P_{r,j}^{\tau})^{\frac{1}{2}}\,\mathrm{d}k)\otimes\mathcal{H}$. □

Using the fact that $(H_0+\sum_{i=1}^{j}\omega(k_i)+1)^{-1}H_0$ is bounded and (4.1), we have for $\psi \in (H_0+1)^{-1}\mathcal{D}(N^{\frac{r}{2}})$

$$\begin{aligned}
&\int \left\|N^{\frac{p-r}{2}} H_0 \prod_{i=1}^{r} a(k_i)\psi\right\|^2 \mathrm{d}k \\
&\quad \leqslant c \int \left\|\left(H_0+\sum_{i=1}^{r}\omega(k_i)+1\right)\prod_{i=1}^{r} a(k_i) N^{\frac{p-r}{2}}\psi\right\|^2 \mathrm{d}k \\
&\quad \leqslant c \int \left\|\prod_{i=1}^{r} a(k_i)(H_0+1)N^{\frac{p-r}{2}}\psi\right\|^2 \mathrm{d}k \\
&\quad \leqslant c\, \|N^{\frac{p}{2}}(H_0+1)\psi\|^2.
\end{aligned}$$

Hence $\prod_{i=1}^{r} a(k_i)$ can be defined as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $L^2(\mathbb{R}^{3r}, \mathrm{d}k)\otimes(H_0+1)^{-1}\mathcal{D}(N^{\frac{p-r}{2}})$. It is easy to see, using commutation relations and Lemma 3.1, that $(N+1)^{-\frac{p}{2}}\hat{I}_{\kappa} N^{\frac{p}{2}}(H_0+1)^{-1}$ is bounded for $\kappa < \infty$, hence $\hat{I}_{\kappa}$ can be defined as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $\mathcal{D}(N^{\frac{p}{2}})$. So the commutator $H_{\kappa}^{\{1\}}$ acts as bounded operator from $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into

$L^2(\mathbb{R}^3, \mathrm{d}k_1) \otimes \mathcal{D}(N^{\frac{p-1}{2}})$. Now for $J := \{1, \dots, r\}$ we can prove by induction on r that

$$H_\kappa^{\{J\}} : (H_0 + 1)^{-1}\mathcal{D}(N^{\frac{p}{2}}) \to L^2\left(\mathbb{R}^{3r}, \prod_{i=1}^r dk_i\right) \otimes \mathcal{D}(N^{\frac{p-r}{2}}). \tag{4.2}$$

For $\#J = 1$ (4.2) is already done. Assume that (4.2) holds for $\#J = r$. Using the fact that $a(k_{r+1})$ maps $L^2(\mathbb{R}^{3r}, \prod_{i=1}^r \mathrm{d}k_i) \otimes \mathcal{D}(N^{\frac{p-r}{2}})$ into $L^2(\mathbb{R}^{3(r+1)}, \prod_{i=1}^{r+1} \mathrm{d}k_i) \otimes \mathcal{D}(N^{\frac{p-r-1}{2}})$ and maps $(H_0+1)^{-1}\mathcal{D}(N^{\frac{p}{2}})$ into $L^2(\mathbb{R}^3, \mathrm{d}k_{r+1}) \otimes (H_0+1)^{-1}\mathcal{D}(N^{\frac{p-1}{2}})$, we prove (4.2).

A simple computation gives:

$$\begin{aligned}
H_\kappa^{\{1\}} &= \frac{1}{\sqrt{2}} \sum_{j=1}^P v_{\kappa_0}^j(k_1) + \frac{1}{2M} \sum_{j=1}^P r_\kappa^j(k_1) a^*(r_\kappa^j) + \\
&\quad + r_\kappa^j(k_1) a(r_\kappa^j) - \sqrt{2} D_j r_\kappa^j(k_1), \quad \text{on } (H_0 + 1)^{-1}\mathcal{D}(N^{\frac{1}{2}}), \\
H_\kappa^{\{1,2\}} &= \frac{1}{2M} \sum_{j=1}^P r_\kappa^j(k_1) r_\kappa^j(k_2), \quad \text{on } (H_0 + 1)^{-1}\mathcal{D}(N), \\
H_\kappa^J &= 0, \quad \text{for all } J \text{ such that } \#J \geqslant 3.
\end{aligned}$$

Before starting with the pull-through formula we will prove two lemma which will be useful in the sequel.

LEMMA 4.4. *Let r be an integer and $z \notin \sigma(\hat{H}_\kappa)$, then*

$$(H_0 + 1)(\hat{H}_\kappa - z)^{-1} : \mathcal{D}(N^{\frac{r}{2}}) \to \mathcal{D}(N^{\frac{r}{2}}),$$

is a bijective map.

Proof. Since $(N + 1)^{-\frac{r}{2}} \hat{I}_\kappa N^{\frac{r}{2}} (H_0 + 1)^{-1}$ is bounded for $\kappa < \infty$, we see that $(\hat{H}_\kappa - z)(H_0 + 1)^{-1}$ maps $\mathcal{D}(N^{\frac{r}{2}})$ into $\mathcal{D}(N^{\frac{r}{2}})$. So it is enough to show that $(H_0 + 1) N^{\frac{r}{2}} (\hat{H}_\kappa - z)^{-1} (1 + N)^{\frac{-r}{2}}$ is bounded. We define $[N, \mathrm{i}\hat{H}_\kappa]$ as quadratic form on $\mathcal{D}(H_0)$

$$\begin{aligned}
[N, \mathrm{i}\hat{H}_\kappa] \;=\; &\sum_{j=1}^P \phi(\mathrm{i}v_{\kappa_0}^j) + \frac{1}{2M} \sum_{j=1}^P \mathrm{i}a^{*^2}(r_\kappa^j) - \mathrm{i}a^2(r_\kappa^j) \;+ \\
&+ \mathrm{i}\sqrt{2} D_j a(r_\kappa^j) - \mathrm{i}\sqrt{2} a^*(r_\kappa^j) D_j.
\end{aligned}$$

For $\kappa < \infty$, using Corollary 2.4 and the fact that $\langle D \rangle (K + 1)^{-\frac{1}{2}}$ is bounded, we see that $[N, \mathrm{i}\hat{H}_\kappa]$ can be defined as a bounded operator on $\mathcal{D}(H_0)$.

We set $\mathrm{ad}_N^\ell . := [N, \mathrm{i}\,\mathrm{ad}_N^{\ell-1} .]$ and $\mathrm{ad}_N^1 . := [N, \mathrm{i}.]$. So $\mathrm{ad}_N^\ell \hat{H}_\kappa$, which is similar to $\mathrm{ad}_N^1 \hat{H}_\kappa$ is defined by induction in ℓ as a bounded operator on $\mathcal{D}(H_0)$ and equal to:

$$\begin{aligned}
\mathrm{ad}_N^\ell \hat{H}_\kappa \;=\; &\sum_{j=1}^P \phi(\mathrm{i}^\ell\, v_{\kappa_0}^j) + \frac{1}{2M} \sum_{j=1}^P \mathrm{i}^\ell 2^{\ell-1} a^{*^2}(r_\kappa^j) - \mathrm{i}^\ell (-2)^{\ell-1} a^2(r_\kappa^j) - \\
&- (-\mathrm{i})^\ell \sqrt{2} D_j\, a(r_\kappa^j) - \mathrm{i}^\ell \sqrt{2} a^*(r_\kappa^j) D_j.
\end{aligned}$$

Since $\mathcal{D}(N) \supset \mathcal{D}(\hat{H}_\kappa) = \mathcal{D}(H_0)$, the resolvent $(z - \hat{H}_\kappa)^{-1}$ preserves the domain of N. This means that the following identity

$$N(z - \hat{H}_\kappa)^{-1} = (z - \hat{H}_\kappa)^{-1}N + (z - \hat{H}_\kappa)^{-1}[N, \hat{H}_\kappa](z - \hat{H}_\kappa)^{-1}, \tag{4.3}$$

holds in the sense of bounded operators on $\mathcal{H}$. Using repeatedly (4.3) we notice that $(z - \hat{H}_\kappa)^{-1}$ preserves $\mathcal{D}(N^p)$ and we obtain on $\mathcal{D}(N^p)$

$$\begin{aligned} &N^p(z - \hat{H}_\kappa)^{-1} \\ &\quad = N^{p-1}(z - \hat{H}_\kappa)^{-1}N - \mathrm{i}N^{p-1}(z - \hat{H}_\kappa)^{-1}\mathrm{ad}_N^1 \hat{H}_\kappa \,(z - \hat{H}_\kappa)^{-1}. \end{aligned} \tag{4.4}$$

We move now all factors of N in each term to the right, we obtain the following identity between bounded operators

$$N^p\,(z - \hat{H}_\kappa)^{-1}\,N^{-p} = (z - \hat{H}_\kappa)^{-1} + \sum_{\ell=1}^{k}(z - \hat{H}_\kappa)^{-1}B_\ell(z)N^{-\ell},$$

where $B_\ell(z)$ is a polynomial in $\mathrm{ad}_N^j \hat{H}_\kappa\,(z - \hat{H}_\kappa)^{-1},\ j \leqslant \ell$. Using Lemma 3.3 with $s = 0$, we see that $B_\ell(z)$ is bounded for $\kappa < \infty$. Hence $(z - \hat{H}_\kappa)^{-1}(H_0 + 1)$ is a bijective map from $\mathcal{D}(N^p)$ into $\mathcal{D}(N^p)$. Using Hadamard's three lines lemma [RS, I–IV] for

$$f(\zeta) := ((z - \hat{H}_\kappa)^{-1}H_0N^{-\zeta}\psi, N^{\bar{\zeta}}\phi),$$

$$\text{in } \mathfrak{S} := \{\zeta \in \mathbb{C},\, p \leqslant \mathrm{Re}(\zeta) \leqslant p + 1\},$$

where $\psi, \phi \in \mathcal{K} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$. $f(\zeta)$ is a bounded continuous analytic function on $\mathfrak{S}$, satisfying

$$|f(p + \mathrm{i}\lambda)| \leqslant c\,\|\phi\|\,\|\psi\|, \quad \lambda \in \mathbb{R},$$

$$|f(p + 1 + \mathrm{i}\lambda)| \leqslant c\,\|\phi\|\,\|\psi\|, \quad \lambda \in \mathbb{R}.$$

We obtain

$$|f(\zeta)| \leqslant c\,\|\phi\|\,\|\psi\|, \quad \text{for } \zeta \in \mathfrak{S}.$$

Let $2p \leqslant r \leqslant 2p + 2$. Since $\mathcal{K} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ is a core for $N^{\frac{r}{2}}$ then $(z - \hat{H}_\kappa)^{-1}H_0(1 + N)^{-\frac{r}{2}}\psi \in \mathcal{D}(N^{\frac{r}{2}})$ and we have also

$$\|N^{\frac{r}{2}}(z - \hat{H}_\kappa)^{-1}H_0(1 + N)^{-\frac{r}{2}}\psi\| \leqslant c\,\|\psi\|, \quad \psi \in \mathcal{K} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}).$$

Then we have $(H_0 + 1)N^{\frac{r}{2}}(\hat{H}_\kappa - z)^{-1}(1 + N)^{\frac{r}{2}}$ bounded. □

LEMMA 4.5. *There exist* $\epsilon > 0$, $b < 0$ *and* c *independent for* κ

(i) $$\|R_\ell^{\frac{1}{2}}(b)H_\kappa^{\{1\}}\,R_{\ell+1}^{\frac{1}{2}}(b)\| \leqslant c\left(\sum_{j=1}^{P}|v_{\kappa_0}^j(k_1)| + |r_\kappa^j(k_1)\omega(k_1)^{-\epsilon}|\right).$$

(ii) $$\|R_\ell^{\frac{1}{2}}(b)H_\kappa^{\{1,2\}}\,R_{\ell+1}^{\frac{1}{2}}(b)\| \leqslant c\left(\sum_{j=1}^{P}|r_\kappa^j(k_1)\omega(k_1)^{-\epsilon}r_\kappa^j(k_2)\omega(k_2)^{-\epsilon}|\right).$$

Proof. Let $\mathcal{D}$ be a dense set of analytic vectors for K. $H_\kappa^{J_i}$ can be defined on $\mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$, then $H_\kappa^{J_i} R_{\ell+1}^{\frac{1}{2}}(b)$ is well defined on $\mathcal{D}_\kappa(b) := (\hat{H}_\kappa - b)^{\frac{1}{2}} \mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$. Furthermore since $\mathcal{D} \otimes \Gamma_{\text{fin}}(S(\mathbb{R}^3))$ is a core for $(\hat{H}_\kappa - b)$, $\mathcal{D}_\kappa(b)$ is dense in $\mathcal{D}((\hat{H}_\kappa - b)^{\frac{1}{2}})$, which is dense in $\mathcal{H}$. Hence it is enough to show (i)–(ii) in $\mathcal{D}_\kappa(b)$. Lemma 3.3 and Lemma 2.8 give the bounds uniformly in κ. Then $R_\ell^{\frac{1}{2}}(b)\, H_\kappa^J\, R_{\ell+1}^{\frac{1}{2}}(b)$ extends from $\mathcal{D}_\kappa(b)$ to a bounded operator on $\mathcal{H}$. □

The following lemma is the *generalized pull through formula*.

LEMMA 4.6. *The following identity holds for all $\phi \in \mathcal{D}(N^{\frac{r}{2}})$:*

$$
\begin{aligned}
&\prod_{i=1}^{r} a(k_i)(z - \hat{H}_\kappa)^{-1}\phi \\
&\quad = R_1(z) \prod_{i=1}^{r} a(k_i)\phi + \\
&\qquad + \sum_{\text{part.}} R_1(z) H_\kappa^{J_1} R_2(z) \ldots R_{\ell-1}(z) H_\kappa^{J_{\ell-1}} R_\ell(z) \prod_{j \in J_\ell} a(k_j)\phi + \\
&\qquad + \sum_{\text{part.}} R_1(z) H_\kappa^{J_1} R_2(z) \ldots R_{\ell-1}(z) H_\kappa^{J_\ell} R(z)\, \phi.
\end{aligned}
$$

The sum in right-hand side is taken over all the partitions of the set $\{1, \ldots, r\}$ *into ordered subsets.*

Proof. We prove this lemma by induction on r. By Lemma 4.4, we know that there exist $\psi \in (H_0 + 1)^{-1}\mathcal{D}(N^{\frac{1}{2}})$ such that $\phi = (z - \hat{H}_\kappa)\psi$. We consider $a(k_1)$ as bounded operator

$$a(k_1) : (H_0 + 1)^{-1}\mathcal{D}(N^{\frac{1}{2}}) \to (H_0 + \omega(k_1) + 1)^{-1} L^2(\mathbb{R}^3, \mathrm{d}k_1) \otimes \mathcal{H}.$$

Then we can write for $\kappa < \infty$

$$a(k_1)\psi = (z - \omega(k_1) - \hat{H}_\kappa)^{-1}(z - \omega(k_1) - \hat{H}_\kappa)a(k_1)\psi.$$

By the justification in the beginning of this subsection, we see that $H_\kappa^{\{1\}}\psi \in L^2(\mathbb{R}^3, \mathrm{d}k_1) \otimes \mathcal{H}$. We have the following identity on $L^2(\mathbb{R}^3, \mathrm{d}k_1) \otimes \mathcal{H}$

$$(z - \omega(k_1) - \hat{H}_\kappa)a(k_1)\psi = a(k_1)(z - \hat{H}_\kappa)\psi + H_\kappa^{\{1\}}\psi. \tag{4.5}$$

This proves the pull through formula for $r = 1$.

The formula (4.5) can be generalized by induction in r.

We claim that for $\psi \in (H_0+1)^{-1}\,\mathcal{D}(N^{\frac{r}{2}})$, $I_r := \{1\ldots r\}$, we have in

$$L^2\left(\mathbb{R}^{3r}, \prod_{i=1}^{r} \mathrm{d}k_i\right) \otimes \mathcal{H} :$$

$$\begin{aligned}
&\left(z - \sum_{i\in I_r} \omega(k_i) - \hat{H}_\kappa\right) \prod_{i\in I_r} a(k_i)\psi \\
&\quad = \prod_{i\in I_r} a(k_i)(z-\hat{H}_\kappa)\psi + H_\kappa^{I_r}\psi + \sum_{I_r=(J_1,J_2)} H_\kappa^{J_1} \prod_{i\in J_2} a(k_i)\psi,
\end{aligned} \tag{4.6}$$

where the sum is over all partitions (J_1, J_2) of I_r. Let us prove (4.6), by induction on r. We have for $\psi \in (H_0+1)^{-1}\mathcal{D}(N^{\frac{r+1}{2}})$:

$$\begin{aligned}
&\left(z - \sum_{i\in I_{r+1}} \omega(k_i) - \hat{H}_\kappa\right) \prod_{i\in I_{r+1}} a(k_i)\psi \\
&\quad = a(k_{r+1})\left(z - \sum_{i\in I_r} \omega(k_i) - \hat{H}_\kappa\right) \prod_{i\in I_r} a(k_i)\psi + H_\kappa^{\{r+1\}} \prod_{i\in I_r} a(k_i)\psi \\
&\quad = \prod_{i\in I_{r+1}} a(k_i)(z-\hat{H}_\kappa)\psi + H_\kappa^{\{r+1\}} \prod_{i\in I_r} a(k_i)\psi + \\
&\qquad + a(k_{r+1}) \sum_{I_r=(J_1,J_2)} H_\kappa^{J_1} \prod_{J_2} a(k_i)\psi + a(k_{r+1}) H_\kappa^{I_r}\psi.
\end{aligned}$$

Moving $a(k_{r+1})$ through $H_\kappa^{J_1}$ and $H_\kappa^{I_r}$ and using the identity

$$a(k_{r+1})H_\kappa^{J_1} = H_\kappa^{J_1} a(k_{r+1}) + H_\kappa^{J_1\cup\{r+1\}}, \quad \text{on } (H_0+1)^{-1}\mathcal{D}(N^{\frac{\#J_1+1}{2}}),$$

we obtain (4.6).

Now assume that the pull through formula holds for r, and let us prove it for $r+1$. We have

$$\begin{aligned}
&\prod_{i\in I_{r+1}} a(k_i)(z-\hat{H}_\kappa)^{-1}\phi \\
&\quad = \left(z - \sum_{i\in I_{r+1}} \omega(k_i) - \hat{H}_\kappa\right)^{-1} \left(z - \sum_{i\in I_{r+1}} \omega(k_i) - \hat{H}_\kappa\right) \prod_{i\in I_{r+1}} a(k_i)\psi.
\end{aligned}$$

Using now the iterated formula (4.6), we obtain

$$\begin{aligned}
&\prod_{i\in I_{r+1}} a(k_i)(z-\hat{H}_\kappa)^{-1}\phi \\
&\quad = \left(z - \sum_{i\in I_{r+1}} \omega(k_i) - \hat{H}_\kappa\right)^{-1} \prod_{i\in I_{r+1}} a(k_i)\phi +
\end{aligned}$$

$$+\left(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa\right)^{-1}H_\kappa^{I_{r+1}}(z-\hat{H}_\kappa)^{-1}\phi+$$

$$+\left(z-\sum_{i\in I_{r+1}}\omega(k_i)-\hat{H}_\kappa\right)^{-1}\sum_{I_{r+1}=(J_1,J_2)}H_\kappa^{J_1}\prod_{i\in J_2}a(k_i)(z-\hat{H}_\kappa)^{-1}\phi.$$

Using the induction hypothesis we obtain:

$$\sum_{I_{r+1}=(J_1,J_2)}H_\kappa^{J_1}\prod_{i\in J_2}a(k_i)(z-\hat{H}_\kappa)^{-1}$$

$$=\sum_{\text{part.}}R_1(z)H_\kappa^{J_1}R_2(z)\dots R_{\ell-1}(z)H_\kappa^{J_{\ell-1}}R_\ell(z)\prod_{i\in J_\ell}a(k_i).$$

This completes the proof. □

THEOREM 4.7. *Let* $\nu\leqslant 1$, $0\leqslant\tau$ *and* $r\in\mathbb{N}$. *Then*

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r-1}{2}}\psi\|\leqslant c\,\|(\hat{H}_\kappa+b)^{\frac{r}{2}}\psi\|,\quad \textit{for } \psi\in\mathcal{D}(\hat{H}_\kappa^{\frac{r}{2}}),\tag{4.7}$$

where $c,\ b$ *are constants independent of* κ.

Proof. We prove the theorem by induction on r. The case $r=1$ follows from Lemma 4.2. Assume that (4.7) holds for all $j\leqslant r$. Let $\phi\in(H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})\subset\mathcal{D}(N^{\frac{r}{2}})$ and $\psi:=R(-b)\phi$. Since $\phi\in\mathcal{D}(N^{\frac{r}{2}})$, by Lemma 4.4, we see that $\psi\in\mathcal{D}(N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}})$. Clearly, we have also $\phi\in\mathcal{D}(N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{j-2}{2}}),\ j\leqslant r$.

We have

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}}\psi\|^2$$

$$=\sum_{j=1}^{r}\int P_{r,j}^{-\tau}\left\|\left(N_\nu+\sum_{i=1}^{j}\omega(k_i)^\nu\right)^{\frac{1}{2}}\prod_{i=1}^{j}a(k_i)R(-b)\phi\right\|^2\mathrm{d}k.\tag{4.8}$$

(4.8) follows by Corollary 2.2. We recall that H_κ^J is an operator-valued function in variables $k_i, i\in J$. If we denote by $\mathrm{d}J:=\mathrm{d}k_{i_1}\dots\mathrm{d}k_{i_p}$ where $J=\{i_1,\dots,i_p\}$, then by Lemma 4.5 we have $\|R_\ell^{\frac{1}{2}}(-b)H_\kappa^J R_{\ell+1}^{\frac{1}{2}}(-b)\|\in L^2(\mathbb{R}^{3p},\mathrm{d}J)$. Using the pull through formula in the right-hand side of (4.8) and the fact that $(N_\nu+\sum_{i=1}^{j}\omega(k_i)^\nu)^{\frac{1}{2}}R_1^{\frac{1}{2}}(-b)$ is bounded, we obtain:

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}}\psi\|^2$$

$$\leqslant c\sum_{j=1}^{r}\int P_{r,j}^{-\tau}\left\|R_1^{\frac{1}{2}}(-b)\prod_{i=1}^{j}a(k_i)\phi\right\|^2\mathrm{d}k+$$

$$+\,c\sum_{j=1}^{r}\sum_{\text{part.}}\int P_{r,j}^{-\tau}\prod_{\ell=1}^{d}\|R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j}\,R_{\ell+1}^{\frac{1}{2}}(-b)\|^2\times$$

$$\times\left\|R_{d+1}^{\frac{1}{2}}(-b)\prod_{i\in J_{d+1}^j}a(k_i)\phi\right\|^2\prod_{\ell=1}^{d+1}\mathrm{d}J_\ell^j+$$

$$+\,c\sum_{j=1}^{r}\sum_{\text{part.}}\int P_{r,j}^{-\tau}\prod_{\ell=1}^{d'}\|R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j}\,R_{\ell+1}^{\frac{1}{2}}(-b)\|^2\times$$

$$\times\|R^{\frac{1}{2}}(-b)\phi\|^2\prod_{\ell=1}^{d'}\mathrm{d}J_\ell^j$$

$$=:\ \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

We recall that $P_{j,j}^{-\tau}=\prod_{i=1}^{j}\omega(k_i)^{-\tau}$ and by (2.7) we notice that

$$P_{r,j}^{-\tau}\leqslant c\,P_{j,j}^{-\tau}. \tag{4.9}$$

In II using (4.9) we can separate the integral in variables $k_i\notin J_{d+1}^j$ and $k_i\in J_{d+1}^j$. Since $\|R_\ell^{\frac{1}{2}}(-b)H_\kappa^{J_\ell^j}\,R_{\ell+1}^{\frac{1}{2}}(-b)\|\in L^2(\mathbb{R}^{3p},\mathrm{d}J)$, then we obtain

$$\mathrm{II}\leqslant c\sum_{j=1}^{r}\sum_{\text{part.}}\int\prod_{i\in J_{d+1}^j}\omega(k_i)^{-\tau}\left\|R_{d+1}^{\frac{1}{2}}(-b)\prod_{i\in J_{d+1}^j}a(k_i)\phi\right\|^2\mathrm{d}J_{d+1}^j. \tag{4.10}$$

Now reordering the terms in the right-hand side of (4.10) and taking into account the fact that $J_{d+1}^j\neq\emptyset$, we have

$$\mathrm{II}\leqslant c\sum_{j=1}^{r-1}\int P_{j,j}^{-\tau}\left\|\left(\hat{H}_\kappa+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi\right\|^2\mathrm{d}k. \tag{4.11}$$

Doing the same thing for III, we obtain

$$\mathrm{III}\leqslant c\,\|R^{\frac{1}{2}}(-b)\phi\|^2. \tag{4.12}$$

Collecting (4.11)–(4.12) we obtain

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}}\psi\|^2$$

$$\leqslant\ c\left(\sum_{j=1}^{r}\int P_{r,j}^{-\tau}\left\|\left(\hat{H}_\kappa+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi\right\|^2\mathrm{d}k+\right.$$

$$+\sum_{j=1}^{r-1}\int P_{j,j}^{-\tau}\left\|\left(\hat{H}_\kappa+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi\right\|^2\mathrm{d}k+$$

$$+\|R^{\frac{1}{2}}(-b)\phi\|^2\Bigg)$$

$$\leqslant c\left(\sum_{j=1}^{r}\int P_{j,j}^{-\tau}\left\|\left(\hat{H}_\kappa+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{-\frac{1}{2}}\prod_{i=1}^{j}a(k_i)\phi\right\|^2 \mathrm{d}k+ \right.$$

$$\left. +\|R^{\frac{1}{2}}(-b)\phi\|^2\right). \tag{4.13}$$

(4.13) follows by (4.9). Now using the fact that

$$\left(\hat{H}_\kappa+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{-\frac{1}{2}}\left(H_0+\sum_{i=1}^{j}\omega(k_i)^\nu+b\right)^{\frac{1}{2}}$$

is bounded uniformly in k_i and Corollary 2.2, we obtain

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}}R(b)\phi\|^2$$

$$\leqslant c\left(\sum_{j=1}^{r}\|(H_0+b)^{-\frac{1}{2}}N_{-\tau}^{\frac{j}{2}}\phi\|^2+\|R^{\frac{1}{2}}(-b)\phi\|^2\right) \tag{4.14}$$

$$\leqslant c\left(\sum_{j=1}^{r}\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{j-2}{2}}\phi\|^2+\|R^{\frac{1}{2}}(-b)\phi\|^2\right). \tag{4.15}$$

In fact, using the fact that $N_{-\tau}(H_0+b)^{-\frac{1}{2}}(1+N_\nu)^{-\frac{1}{2}}$ is bounded, we see that the right-hand side in (4.14) is less than (4.15), which holds for all $\phi\in(H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$. Since

$$\mathcal{D}\otimes\Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3))\subset(H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$$

is a core for $N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r-2}{2}}$, we see that $(H_0+1)^{-\frac{1}{2}}\mathcal{D}(N^{\frac{r-1}{2}})$ is dense in $\mathcal{D}(N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r-2}{2}})$ and hence (4.15) holds for all $\phi\in\mathcal{D}(N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r-2}{2}})$. Now let

$$\phi:=(\hat{H}_\kappa+b)\psi,\ \psi\in\mathcal{D}((\hat{H}_\kappa+b)^{\frac{r+1}{2}})$$

and $b>0$. Then

$$\phi\in\mathcal{D}((\hat{H}_\kappa+b)^{\frac{r-1}{2}})\subset\mathcal{D}(N_\nu^{\frac{1}{2}}N^{\frac{r-2}{2}}).$$

The induction hypothesis and (4.15) give

$$\|N_\nu^{\frac{1}{2}}N_{-\tau}^{\frac{r}{2}}\psi\|\leqslant c\,\|(\hat{H}_\kappa+b)^{\frac{r+1}{2}}\psi\|,\quad \psi\in\mathcal{D}((\hat{H}_\kappa+b)^{\frac{r+1}{2}}).\qquad\square$$

COROLLARY 4.8. *Let $\gamma \geqslant 0$ and $\epsilon < 1/r$, where $r \in \mathbb{N}$. Then*

$$N_\epsilon^r \leqslant c\,(\hat{H}_\kappa + b)^r, \tag{4.16}$$

$$H_0^{1-\gamma}\, N^{r-1+\gamma} \leqslant c\,(\hat{H}_\kappa + b)^r, \tag{4.17}$$

where c and b are constants independent of κ.

Proof. The inequalities follow from Lemma 2.5 and Theorem 4.7. (4.16) follows with $\nu = 1, \tau = (1-\epsilon r)/(1-r)$ in Theorem 4.7 and $b = \omega, \tau_1 = \epsilon, \tau_2 = 1, \tau_3 = -(1-\epsilon r)/(1-r), \alpha_1 = r, \alpha_2 = 1, \alpha_3 = r-1$ in Lemma 2.5. (4.17) follows with $\nu = 1, \tau = \gamma/(r-1)$ in Theorem 4.7 and $b = \omega, \tau_1 = 0, \tau_2 = 1, \tau_3 = -\gamma/(r-1), \alpha_1 = r-1+\gamma, \alpha_2 = \gamma, \alpha_3 = r-1$ in Lemma 2.5 and the fact that $\mathrm{d}\Gamma(\omega)^\gamma \leqslant H_0^{\gamma-1}\,\mathrm{d}\Gamma(\omega)$. □

Using the principle of the cutoff independence, formulated in Lemma 4.1, we deduce similar estimates for $\hat{H}_\infty$.

THEOREM 4.9. *Let $\gamma \geqslant 0$ and $\epsilon < 1/r$, where $r \in \mathbb{N}$. c, b are positive constants. Then*

$$N_\epsilon^r \leqslant c\,(\hat{H}_\infty + b)^r,$$

$$H_0^{1-\gamma}\, N^{r-1+\gamma} \leqslant c\,(\hat{H}_\infty + b)^r.$$

Theorem 4.9 for $\epsilon = 0$ and Hadamard's three lines lemma in [Ro] give the following corollary.

COROLLARY 4.10. *For $r \in \mathbb{R}^+$ there are c,b positive constants such that*

$$(N+1)^r \leqslant c\,(\hat{H}_\infty + b)^r.$$

COROLLARY 4.11. *For $r \in \mathbb{R}^+$ there are c,b positive constants such that*

$$(N+1)^r \leqslant c\,(H + b)^r.$$

Proof. If we prove that $N^r U_\infty (N+1)^{-r}$ is bounded for r positive integer, the corollary follows from Corollary 4.10.

We have on $\mathcal{D}(N)$ the identity

$$U_\infty^* N U_\infty = N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \tfrac{1}{2}\|G_\infty\|^2.$$

So U_∞ preserves $\mathcal{D}(N)$. By iteration we have on $\mathcal{D}(N^r)$

$$U_\infty^* N^r U_\infty = \big(N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \tfrac{1}{2}\|G_\infty\|^2\big)^r.$$

Then we obtain the boundness of the operator $N^r U_\infty (N+1)^{-r}$, since

$$\big(N - \mathrm{i}\phi(\mathrm{i}G_\infty) - \tfrac{1}{2}\|G_\infty\|^2\big)^r (N+1)^{-r},$$

is bounded. □

4.2. NUMBER-ENERGY ESTIMATES

We say that a sequence of operators $A_\kappa(R)$, $R \in \mathbb{R}$ or $\mathbb{C}$ is of class $\mathrm{O}(R^\gamma)$ (resp. $\mathrm{o}(R^\gamma)$) uniformly in κ, if there exists c constant independent from κ and R such that $\|A_\kappa(R)\| \leqslant cR^\gamma$ (resp. $\|A_\kappa(R)\|R^{-\gamma} \to 0$, when $R \to \infty$).

LEMMA 4.12. *We have uniformly in κ for z in a bounded set of $\mathbb{C} \setminus \mathbb{R}$ and $m \in \mathbb{N}$, $\gamma \geqslant 0$:*

(i) $(H_0+1)^{\frac{1-\gamma}{2}}(N+1)^{\frac{\gamma}{2}+m}(z-\hat{H}_\kappa)^{-k}(N+1)^{-m+k-1} \in \mathrm{O}(|\mathrm{Im}(z)|^{c_{m,k}})$.
(ii) $(N+1)^m(z-\hat{H}_\kappa)^{-k}(N+1)^{-m+k+\frac{\gamma-2}{2}}(H_0+1)^{\frac{1-\gamma}{2}} \in \mathrm{O}(|\mathrm{Im}(z)|^{c_{m,k}})$.
(iii) *Let $\chi \in C_0^\infty(\mathbb{R})$ and $n, q \in \mathbb{N}$:*

$$\|N^n\chi(\hat{H}_\kappa)N^q\| < \infty.$$

Proof. We recall the identity (4.4) on $\mathcal{D}(N^k)$, which had been proved in the proof of Lemma 4.4

$$N^k(z-\hat{H}_\kappa)^{-1} = N^{k-1}(z-\hat{H}_\kappa)^{-1}N - \mathrm{i}N^{k-1}(z-\hat{H}_\kappa)^{-1}\mathrm{ad}_N^1\hat{H}_\kappa(z-\hat{H}_\kappa)^{-1}.$$

We move now all factors of N in each term to the right, we obtain the following identity between bounded operators

$$N^k(z-\hat{H}_\kappa)^{-1}N^{-k} = (z-\hat{H}_\kappa)^{-1} + \sum_{\ell=1}^{k}(z-\hat{H}_\kappa)^{-\frac{1}{2}}B_\ell(z)(z-\hat{H}_\kappa)^{-\frac{1}{2}}N^{-\ell},$$

where $B_\ell(z)$ is a polynomial in $(z-\hat{H}_\kappa)^{-\frac{1}{2}}\,\mathrm{ad}_N^j\hat{H}_\kappa(z-\hat{H}_\kappa)^{-\frac{1}{2}}$, $j \leqslant \ell$. Using Lemmas 3.3 and 4.2, we see that $\|B_\ell(z)\| \leqslant c|\mathrm{Im}(z)|^{-c_\ell}$, uniformly in κ. Using Corollary 4.8, (4.17), we see that $\|H_0^{\frac{1-\gamma}{2}}N^{\frac{\gamma}{2}}(z-\hat{H}_\kappa)^{-\frac{1}{2}}\| \leqslant |\mathrm{Im}(z)|^{-\frac{1}{2}}$, which proves (i) for $k=1$. For $k \neq 1$, we write

$$\begin{aligned}&H_0^{\frac{1-\gamma}{2}}N^{m+\frac{\gamma}{2}}(z-\hat{H}_\kappa)^{-k}N^{-m+k-1}\\ &\quad = H_0^{\frac{1-\gamma}{2}}N^{m+\frac{\gamma}{2}}(z-\hat{H}_\kappa)^{-1}N^{-m}\prod_{\ell=0}^{k}N^{m-\ell}(z-\hat{H}_\kappa)^{-1}N^{-m+\ell-1}.\end{aligned}$$

This proves (i) for $k \neq 1$. The proof of (ii) is similar to (i). (iii) follows from the higher-order estimates in Theorem 4.7 with $\tau = \nu = 0$. □

We set:

$$\begin{aligned}\mathcal{H}^{\mathrm{ext}} &:= \mathcal{H}\otimes\Gamma(\mathfrak{h}), \qquad \hat{H}_\kappa^{\mathrm{ext}} = \hat{H}_\kappa\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(\omega),\\ N_0 &:= N\otimes\mathbb{1}, \qquad N_\infty := \mathbb{1}\otimes N, \quad \text{acting in } \mathcal{H}^{\mathrm{ext}}.\end{aligned}$$

LEMMA 4.13. *We have uniformly in κ and z in a bounded set of $\mathbb{C}\setminus\mathbb{R}$ and $m \in \mathbb{N}$*

(i) $(N_0+N_\infty)^m\,(z-\hat{H}_\kappa^{\rm ext})^{-1}\,(N_0+N_\infty)^{-m+1}\in {\rm O}(|{\rm Im}(z)|^{-c_m})$.
(ii) $(H_0^{\rm ext}+1)^{\frac12}\,(N_0+N_\infty)^m\,(z-\hat{H}_\kappa^{\rm ext})^{-1}\,(N_0+N_\infty)^{-m}\in {\rm O}(|{\rm Im}(z)|^{-c_m})$.

Proof. The proof is analogous to the proof of Lemma 4.12. In fact we have

$$\begin{aligned}&{\rm ad}^\ell_{N_0+N_\infty}\hat{H}_\kappa^{\rm ext}={\rm ad}^\ell_N\hat{H}_\kappa\otimes\mathbb{1},\\ &(N_0+N_\infty)^m\,(z-\hat{H}_\kappa^{\rm ext})^{-1}\,(N_0+N_\infty)^{-m}\\ &\quad=(z-\hat{H}_\kappa^{\rm ext})^{-1}+\sum_{\ell=1}^m(z-\hat{H}_\kappa^{\rm ext})^{-\frac12}B_\ell(z)\,(z-\hat{H}_\kappa^{\rm ext})^{-\frac12}\,(N_0+N_\infty)^{-\ell},\end{aligned}$$

where $B_\ell(z)\in {\rm O}(|{\rm Im}z|^{-c_\ell})$. □

4.3. COMMUTATOR ESTIMATES

Let $q\in C_0^\infty(\mathbb{R}^3)$, $0\leqslant q\leqslant 1$, $q=1$ near 0. We set $q^R:=q(x/R)$. We recall that we consider $\mathfrak{h}$ in its momentum representation $L^2(\mathbb{R}^3,{\rm d}k)$ and $x=\nabla_k/{\rm i}$.

We use the following functional calculus formula, see [DG1], for $\chi\in C_0^\infty(\mathbb{R})$ and A a self-adjoint operator:

$$\chi(A)=\frac{\rm i}{2\pi}\int_{\mathbb{C}}\partial_{\bar z}\breve{\chi}(z)(z-A)^{-1}\,{\rm d}z\wedge{\rm d}\bar z, \tag{4.18}$$

where $\breve{\chi}$ is an almost analytic extension of χ, such that

$$\breve{\chi}_{|\mathbb{R}}=\chi,\qquad |\partial_{\bar z}\breve{\chi}(z)|\leqslant c_n|{\rm Im}z|^n,\quad n\in\mathbb{N}.$$

LEMMA 4.14. *Let $\chi\in C_0^\infty(\mathbb{R})$, then one has uniformly for $\kappa\leqslant\infty$*

$$N^n[\chi(\hat{H}_\kappa),\Gamma(q^R)]N^m\in {\rm O}(R^{-1}).$$

Proof. Commutation relations allow to compute $[\hat{H}_\kappa,\Gamma(q^R)]$ as a sesquilinear form on $D(H_0)$, which by N_τ-estimates is a bounded operator on $D(H_0)$, when $\kappa<\infty$.

We have

$$\begin{aligned}&[H_0,\Gamma(q^R)]={\rm d}\Gamma(q^R,[\omega,q^R]),\\ &[\phi(v_{\kappa_0}),\Gamma(q^R)]=\frac{1}{\sqrt2}a^*((1-q^R)v_{\kappa_0})\Gamma(q^R)-\frac{1}{\sqrt2}\Gamma(q^R)a((1-q^R)v_{\kappa_0}),\\ &[a^2(r_\kappa^j),\Gamma(q^R)]=-\Gamma(q^R)a((1-q^R)r_\kappa^j)a((1+q^R)r_\kappa^j),\\ &[a^{*^2}(r_\kappa^j),\Gamma(q^R)]=a^*((1-q^R)r_\kappa^j)a^*((1+q^R)r_\kappa^j)\Gamma(q^R),\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&[a^*(r_\kappa^j)a(r_\kappa^j), \Gamma(q^R)] \\
&\quad = -a^*(r_\kappa^j)\Gamma(q^R)a((1-q^R)r_\kappa^j) + a^*((1-q^R)r_\kappa^j)\Gamma(q^R)a(r_\kappa^j), \\
&[a^*(r_\kappa^j)D_j, \Gamma(q^R)] = a^*((1-q^R)r_\kappa^j)\Gamma(q^R)D_j, \\
&[D_j a(r_\kappa^j), \Gamma(q^R)] = -D_j\Gamma(q^R)a((1-q^R)r_\kappa^j).
\end{aligned}$$

Let $\chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. Using (4.18), we have

$$\begin{aligned}
&N^n[\chi(\hat{H}_\kappa), \Gamma(q^R)]N^m \\
&\quad = N^n\chi_1(\hat{H}_\kappa)[\chi(\hat{H}_\kappa), \Gamma(q^R)]N^m + N^n[\chi_1(\hat{H}_\kappa), \Gamma(q^R)]\chi(\hat{H}_\kappa)N^m \\
&\quad = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\check{\chi}(z)N^n\chi_1(\hat{H}_\kappa)(z-\hat{H}_\kappa)^{-1} \times \\
&\qquad \times [\hat{H}_\kappa, \Gamma(q^R)](z-\hat{H}_\kappa)^{-1}N^m\, \mathrm{d}z\wedge \mathrm{d}\bar{z} + \\
&\qquad + \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\check{\chi}_1(z)N^n(z-\hat{H}_\kappa)^{-1}[\hat{H}_\kappa, \Gamma(q^R)] \times \\
&\qquad \times (z-\hat{H}_\kappa)^{-1}\chi(\hat{H}_\kappa)N^m\, \mathrm{d}z\wedge \mathrm{d}\bar{z}.
\end{aligned} \tag{4.20}$$

Moving the power of N toward $\chi(\hat{H}_\kappa)$, $\chi_1(\hat{H}_\kappa)$ and then using Lemma 4.12 and Corollary 4.8, we see that it is enough to show that for $b > 0$, $(N+1)^{-n}(H_0 + b)^{-\frac{1}{2}}[\hat{H}_\kappa, \Gamma(q^R)](H_0+b)^{-\frac{1}{2}} \in \mathrm{O}(R^{-1})$, uniformly in κ, to have the lemma. Using now Lemma 3.4, we obtain

$$\begin{aligned}
&\|(N+1)^{-n}(H_0+b)^{-\frac{1}{2}}[\hat{H}_\kappa, \Gamma(q^R)](H_0+b)^{-\frac{1}{2}}\| \\
&\quad \leqslant c(\|(1-q^R)\,(V+1)^{-\frac{1}{2}}v_{\kappa_0}\| + \\
&\qquad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}\,(1-q^R)r_\kappa\|\|\omega^{\frac{s-1}{4}}\,(1+q^R)r_\kappa\| + \\
&\qquad + \|N^{-1}\mathrm{d}\Gamma(q^R, [\omega, q^R])\| + \\
&\qquad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}\,(1-q^R)r_\kappa\|\|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}r_\kappa\| + \\
&\qquad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1-q^R)r_\kappa\|\|(K+\mathrm{i})^{-\frac{1}{2}}D\|).
\end{aligned} \tag{4.21}$$

Using the inequality (2.3) recalled in Subsection 2.1,

$$\|N^{-1}\,\mathrm{d}\Gamma(q^R, [\omega, q^R])\| \leqslant \|[\omega, q^R]\|,$$

and the fact that $[\omega, q^R] \in \mathrm{O}(R^{-1})$, we see that $\|N^{-1}\mathrm{d}\Gamma(q^R, [\omega, q^R])\| \in \mathrm{O}(R^{-1})$. Now for the other kind of terms we will proceed as follows. Since $V \geqslant \langle \mathrm{x}\rangle^\alpha$, $\alpha >$

2, we can pick $\mu > 0$ and $s < 1$, such that $(V+1)^{-\frac{s}{2}}\langle \mathrm{x}\rangle^{1+\mu}$ is bounded. Then using Lemma A.2, we obtain

$$\|\langle \mathrm{x}\rangle^{1+\mu}\omega^{\frac{s-1}{2}}(1-q^R)r_\kappa\| \in \mathrm{O}(R^{-1-\mu}),$$

$$\|\langle \mathrm{x}\rangle^{1+\mu}\omega^{\frac{s-1}{4}}(1-q^R)r_\kappa\| \in \mathrm{O}(R^{-1-\mu}).$$

Hence we have:

$$\|(N+1)^{-n}(H_0+b)^{-\frac{1}{2}}[\hat{I}_\kappa, \Gamma(q^R)](H_0+b)^{-\frac{1}{2}}\| \in \mathrm{O}(R^{-1-\mu}). \tag{4.22}$$

Then the integrand in (4.20) is $|\mathrm{Im}(z)|^{-2}\mathrm{O}(R^{-1})$. This ends the proof. □

Let $j_0 \in C_0^\infty(\mathbb{R}^3)$, $j_\infty \in C^\infty(\mathbb{R}^3)$, $0 \leqslant j_0$, $0 \leqslant j_\infty$, $j_0^2 + j_\infty^2 \leqslant 1$, $j_0 = 1$ near 0. We set for $R \geqslant 1$, $j^R := (j_0^R, j_\infty^R)$, where

$$j_0^R := j_0\left(\frac{x}{R}\right),\ j_\infty^R := j_\infty\left(\frac{x}{R}\right).$$

We set $j := j^1$.

LEMMA 4.15. *One has uniformly for* $\kappa \leqslant \infty$

(i) $\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\,\chi(\hat{H}_\kappa) \in \mathrm{O}(R^{-1})$.

(ii) *Let* $\chi \in C_0^\infty(\mathbb{R})$, *then*

$$(N_0+N_\infty)^n\,(\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\,\chi(\hat{H}_\kappa))\, N^m \in \mathrm{O}(R^{-1}).$$

Proof. The proof is similar to the previous one. Instead of (4.20), we use the identities:

$$\begin{aligned}
&H_0^{\mathrm{ext}} I^*(j^R) - I^*(j^R) H_0\\
&\quad = \mathrm{d}I^*(j^R, [\omega, j^R]), \quad \text{where } [\omega, j^R] = ([\omega, j_0^R], [\omega, j_\infty^R]),
\end{aligned}$$

$$\begin{aligned}
&\phi(v_{\kappa_0}) \otimes \mathbb{1} I^*(j^R) - I^*(j^R)\phi(v_{\kappa_0})\\
&\quad = \phi((1-j_0^R)v_{\kappa_0}) \otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes}\phi(j_\infty^R v_{\kappa_0}) I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
&a^*(r_\kappa^j)D_j \otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^*(r_\kappa^j)D_j\\
&\quad = a^*((1-j_0^R)r_\kappa^j)D_j \otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes} a^*(j_\infty^R r_\kappa^j)D_j I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
&D_j\, a(r_\kappa^j) \otimes \mathbb{1} I^*(j^R) - I^*(j^R) D_j\, a(r_\kappa^j)\\
&\quad = D_j\, a((1-j_0^R)r_\kappa^j) \otimes \mathbb{1} I^*(j^R) - \mathbb{1}\tilde{\otimes} D_j a(j_\infty^R r_\kappa^j) I^*(j^R),
\end{aligned}$$

$$\begin{aligned}
&a^2(r_\kappa^j) \otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^2(r_\kappa^j)\\
&\quad = -I^*(j^R)a((1-j_0^R)r_\kappa^j)a((1+j_0^R)r_\kappa^j),
\end{aligned}$$

$$a^{*^2}(r_\kappa^j) \otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^{*^2}(r_\kappa^j)$$

$$= (-2a^*(j_0^R r_\kappa^j) \otimes a^*(j_\infty^R r_\kappa^j) - \mathbb{1}\tilde{\otimes} a^{*^2}(j_\infty^R r_\kappa^j)) \times I^*(j^R),$$

$$a^*(r_\kappa^j)a(r_\kappa^j) \otimes \mathbb{1} I^*(j^R) - I^*(j^R)a^*(r_\kappa^j)a(r_\kappa^j)$$
$$= a^*(r_\kappa^j) \otimes \mathbb{1} I^*(j^R) a((-1 + j_0^R)r_\kappa^j) +$$
$$+ (a^*((1 - j_0^R)r_\kappa^j) \otimes \mathbb{1} + \mathbb{1}\tilde{\otimes} a^*(j_\infty^R r_\kappa^j)) \times I^*(j^R)a(r_\kappa^j).$$

We notice that for $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$, we define

$$\mathbb{1}\tilde{\otimes} a^\sharp(v) : \mathcal{K} \otimes \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \to \mathcal{K} \otimes \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2),$$

$$\mathbb{1}\tilde{\otimes} a^\sharp(v) := T^{-1}\, \mathbb{1}_{\Gamma(\mathfrak{h}_1)} \otimes a^\sharp(v)\, T,$$

where T is the natural identification:

$$T : \mathcal{K} \otimes \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1) \otimes \mathcal{K} \otimes \Gamma(\mathfrak{h}_2).$$

Using (2.4) with $j_0^2 + j_\infty^2 \leqslant 1$, we see that $(H_0^{\mathrm{ext}} I^*(j^R) - I^*(j^R)H_0)(N+1)^{-1}$ is bounded, which shows that $I^*(j^R) : \mathcal{D}(H_0) \to \mathcal{D}(H_0^{\mathrm{ext}})$. We obtain the following identity on $\mathcal{H}$

$$C(z) := (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1} I^*(j^R) - I^*(j^R)(z - \hat{H}_\kappa)^{-1}$$
$$= (z - \hat{H}_\kappa^{\mathrm{ext}})^{-1}(\hat{H}_\kappa^{\mathrm{ext}} I^*(j^R) - I^*(j^R)\hat{H}_\kappa)(z - \hat{H}_\kappa)^{-1}.$$

Using (4.18), we obtain:

$$\chi(\hat{H}_\kappa^{\mathrm{ext}})\, I^*(j^R) - I^*(j^R)\, \chi(\hat{H}_\kappa) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\check{\chi}(z)\; C(z)\, \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Using Corollary 4.8 and Lemma 3.4 we obtain for $b < 0$

$$\begin{aligned}
&\|C(b)\| \\
&\leqslant c(\|(N_\infty + N_0)^{-1}\mathrm{d}I^*(j^R, [j^R, \omega])\| + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R)v_{\kappa_0}\| + \\
&\quad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R v_{\kappa_0}\| + \\
&\quad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}(1 - j_0^R)r_\kappa\|\, \|\omega^{\frac{s-1}{4}}(1 + j_0^R)r_\kappa\| + \\
&\quad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}} j_\infty^R r_\kappa\|\, \|\omega^{\frac{s-1}{4}} j_0^R r_\kappa\| + \|(V+1)^{-\frac{s}{4}}\omega^{\frac{s-1}{4}} j_\infty^R r_\kappa\|^2 + \\
&\quad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} r_\kappa\| \times \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R)r_\kappa\| + \\
&\quad + (\|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}(1 - j_0^R)r_\kappa\| + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R r_\kappa\|) \times \\
&\quad \times \|(K + \mathrm{i})^{-\frac{1}{2}} D\| + \\
&\quad + \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} r_\kappa\| \times \|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}} j_\infty^R r_\kappa\|).
\end{aligned} \tag{4.23}$$

Applying (2.4) with $[j_0^R,\omega]^2+[j_\infty^R,\omega]^2\in \mathrm{O}(R^{-2})$, we obtain

$$\|(b-\hat H_\kappa^{\mathrm{ext}})^{-1}(H_0^{\mathrm{ext}}I^*(j^R)-I^*(j^R)H_0)(b-\hat H_\kappa)^{-1}\|\in \mathrm{O}(R^{-1}).$$

For the other terms of $C(b)$, we use (4.23), the fact that we can pick $\mu>0$ and $s<1$ such that $(V+1)^{-\frac{s}{2}}\langle \mathrm{x}\rangle^{1+\mu}$ is bounded and Lemma A.2. We obtain:

$$\begin{aligned}
&\|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{2}}f_\epsilon^R r_\kappa\|\in \mathrm{O}(R^{-1-\mu}),\\
&\|(V+1)^{-\frac{s}{2}}\omega^{\frac{s-1}{4}}f_\epsilon^R r_\kappa\|\in \mathrm{O}(R^{-1-\mu}),\qquad (4.24)\\
&\|(V+1)^{-\frac{s}{4}}\omega^{\frac{s-1}{4}}j_\infty^R r_\kappa\|^2\in \mathrm{O}(R^{-1-\mu}),
\end{aligned}$$

where f_ϵ^R denotes j_∞^R or $1-j_0^R$. Then we have

$$\begin{aligned}
&\|(b-\hat H_\kappa^{\mathrm{ext}})^{-1}\times(\hat I_\kappa\otimes \mathbb{1}\, I^*(j^R)-I^*(j^R)\hat I_\kappa)(b-\hat H_\kappa)^{-1}\|\\
&\quad\in\ \mathrm{O}(R^{-1-\mu}).\qquad (4.25)
\end{aligned}$$

Hence we have $\|C(z)\|\in|\mathrm{Im}(z)|^{-2}\mathrm{O}(R^{-1})$. This proves (i).

Let $\chi_1\in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi=\chi$. As in the previous lemma, we have using (4.18):

$$\begin{aligned}
&(N_0+N_\infty)^n\chi(\hat H_\kappa^{\mathrm{ext}})I^*(j^R)-I^*(j^R)\chi(\hat H_\kappa)N^m\\
&\quad=\ (N_0+N_\infty)^n\chi_1(\hat H_\kappa^{\mathrm{ext}})(\chi(\hat H_\kappa^{\mathrm{ext}})I^*(j^R)-I^*(j^R)\chi(\hat H_\kappa))N^m+\\
&\qquad+(N_0+N_\infty)^n(\chi_1(\hat H_\kappa^{\mathrm{ext}})I^*(j^R)-I^*(j^R)\chi_1(\hat H_\kappa))\chi(\hat H_\kappa)N^m\\
&\quad=\ \frac{\mathrm{i}}{2}\int_{\mathbb{C}}\partial_{\bar z}\check\chi(z)(N_0+N_\infty)^n\chi_1(\hat H_\kappa^{\mathrm{ext}})C(z)N^m\,\mathrm{d}z\wedge\mathrm{d}\bar z+\\
&\qquad+\frac{\mathrm{i}}{2}\int_{\mathbb{C}}\partial_{\bar z}\check\chi_1(z)(N_0+N_\infty)^nC(z)\chi(\hat H_\kappa)N^m\,\mathrm{d}z\wedge\mathrm{d}\bar z.
\end{aligned}$$

Moving $(N_0+N_\infty)^n$ (resp. N^m) toward $\chi(\hat H_\kappa)$ (resp. $\chi_1(\hat H_\kappa^{\mathrm{ext}})$) in the last expression and then using (i) and Lemma 4.13 we prove (ii). □

5. Spectral Theory for the Nelson Hamiltonian

We study in this section the spectral properties of both Nelson and modified Hamiltonians. In Subsection 5.1 we prove the existence of ground state for the Nelson Hamiltonian. We use essentially the fact that $H_\kappa-E_\kappa$ are Pauli–Fierz Hamiltonian which converge in the norm resolvent sense to H. This is the subject of Theorem 5.1. In Subsection 5.2 we prove a Mourre estimate for the modified Hamiltonian, which gives that pure point spectrum is locally finite outside its thresholds.

5.1. HVZ THEOREM

THEOREM 5.1. *One has*

$$\sigma_{\text{ess}}(H) = [\inf \sigma(H) + m, +\infty[,$$

and $\inf \sigma(H)$ *is a discrete eigenvalue of* H.

Proof. H_κ is an example of a Pauli–Fierz Hamiltonian, see [DG2, Section 3]. The HVZ theorem proved in [DG2] for the Pauli–Fierz Hamiltonians gives for $\kappa < \infty$

$$\sigma_{\text{ess}}(\hat{H}_\kappa) = \sigma_{\text{ess}}(H_\kappa - E_\kappa) = [\inf \sigma(\hat{H}_\kappa) + m, +\infty[.$$

Using the fact that $(z - \hat{H}_\kappa)^{-1} \to (z - \hat{H}_\infty)^{-1}$ and [RS, I–IV, Thm. VIII.23], we see that

$$\lim_{\kappa \to \infty} (\inf \sigma(\hat{H}_\kappa)) = \inf \sigma(\hat{H}_\infty). \tag{5.1}$$

Let $\chi \in C_0^\infty(] - \infty, \inf \sigma(\hat{H}_\infty) + m[)$, then $\chi \in C_0^\infty(] - \infty, \inf \sigma(\hat{H}_{\kappa_n}) + m[)$ for a sequence $\kappa_n \to +\infty$. By the HVZ theorem for $\hat{H}_{\kappa_n}$, $\chi(\hat{H}_{\kappa_n})$ is compact and by Theorem 3.6 $\chi(\hat{H}_\infty)$ is compact. So we obtain

$$\sigma_{\text{ess}}(\hat{H}_\infty) \subset [\inf \sigma(\hat{H}_\infty) + m, +\infty[.$$

Let us now show that $[\inf \sigma(\hat{H}_\infty) + m, +\infty[\subset \sigma_{\text{ess}}(\hat{H}_\infty)$. Let λ such that $\lambda > \inf \sigma(\hat{H}_\infty) + m$. By (5.1) and the HVZ theorem for $\hat{H}_\kappa$, there exists a sequence $\kappa_n \to +\infty$ such that $\lambda \in \sigma_{\text{ess}}(\hat{H}_{\kappa_n})$, or equivalently

$$\mu := (\lambda + c)^{-1} \in \sigma_{\text{ess}}((\hat{H}_{\kappa_n} + c)^{-1}), \quad c \gg 1.$$

Let us show that $\mu \in \sigma_{\text{ess}}((\hat{H}_\infty + c)^{-1})$, i.e. :$\lambda \in \sigma_{\text{ess}}(\hat{H}_\infty)$:

Assume the contrary and let $\chi \in C_0^\infty(\mathbb{R})$, such that $\chi(\mu) = 1$, $\chi((\hat{H}_\infty + c)^{-1})$ compact. Let $\varphi_{\kappa_n, j}$ be Weyl sequences for $(\hat{H}_{\kappa_n} + c)^{-1}$ at μ such that

$$\|\varphi_{\kappa_n, j}\| = 1, \qquad \lim_{j \to \infty} ((\hat{H}_{\kappa_n} + c)^{-1} - \mu)\varphi_{\kappa_n, j} = 0,$$

and

$$w\text{-}\lim_{j \to \infty} \varphi_{\kappa_n, j} = 0.$$

One has

$$\begin{aligned} &\|\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_n, j} - \varphi_{\kappa_n, j}\| \\ &\quad \leqslant \|\chi((\hat{H}_{\kappa_n} + c)^{-1}) - \chi((\hat{H}_\infty + c)^{-1})\| + \|\chi((\hat{H}_{\kappa_n} + c)^{-1})\varphi_{\kappa_n, j} - \varphi_{\kappa_n, j}\|. \end{aligned}$$

Since $\chi((\hat{H}_\infty + c)^{-1})$ compact, there exists for $\epsilon > 0$, a κ_1 and j_1 such that

$$\|\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_1, j_1} - \varphi_{\kappa_1, j_1}\| \leqslant 2\epsilon,$$

$$\|\chi((\hat{H}_\infty + c)^{-1})\varphi_{\kappa_1, j_1}\| \leqslant \epsilon.$$

We obtain $\|\varphi_{\kappa_1, j_1}\| \leqslant 3\epsilon$, this gives a contradiction, if we choose $\epsilon < 1/3$. □

5.2. MOURRE ESTIMATE

We denote by b the operator acting on $\mathfrak{h}$ defined by

$$b := \frac{1}{2}(\nabla\omega\,.\,D_k + D_k\,.\,\nabla\omega), \quad \text{on } C_0^\infty(\mathbb{R}^3).$$

[ABG, Prop. 4.2.3] yields that the closure of b is the infinitesimal generator of the strongly continuous unitary group U_t associated to the vector field $\nabla\omega$ in the following sense

$$U_t F := [\det\nabla\phi_{-t}(k)]^{\frac{1}{2}}\; F(\phi_{-t}(k)), \quad F \in S'(\mathbb{R}^3), \tag{5.2}$$

where ϕ_t is the flow of the vector field $\nabla\omega$. Moreover $C_0^\infty(\mathbb{R}^3)$ is a core for b. We denote in the sequel by b its closure. We set $B := \mathrm{d}\Gamma(b)$. Clearly B is essentially selfadjoint on $\Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3))$.

We denote by τ the set of *thresholds*,

$$\tau := \sigma_{\mathrm{pp}}(\hat{H}_\infty) + m\mathbb{N}^*.$$

Let S be a selfadjoint operator on $\mathcal{H}$, we say that S is of class $C^1(B)$, see [ABG], if the map

$$t \mapsto \mathrm{e}^{\mathrm{i}tB}(S-z)^{-1}\,\mathrm{e}^{-\mathrm{i}tB},$$

is strongly C^1 for some $z \in \mathbb{C} \setminus \sigma(S)$. By [ABG, Lemma 6.2.9] $S \in C^1(B)$ if and only if the sesquilinear form $[B, (z-S)^{-1}]$ on $\mathcal{D}(B)$ is continuous for the topology of $\mathcal{H}$, i.e:

$$|((S-z)^{-1}\varphi, B\varphi) - (B\varphi, (S-z)^{-1}\varphi)| \leqslant c\,\|\varphi\|^2 \quad \text{for } \varphi \in \mathcal{D}(B). \tag{5.3}$$

We recall here a well known theorem, see [ABG, Thm. 6.2.10].

THEOREM 5.2. *Let S, B two selfadjoint operators acting on Hilbert space. If $S \in C^1(B)$ then*

(i) *$\mathcal{D}(S) \cap \mathcal{D}(B)$ is dense in $\mathcal{D}(S)$,*
(ii) *$([B, S]u, u) \leqslant c\,\|Su\|^2, u \in \mathcal{D}(S) \cap \mathcal{D}(B)$,*
(iii) *$[B, (z-S)^{-1}] = (z-S)^{-1}[B, S](z-S)^{-1}$,*

where (iii) *is understood as identity between bounded operators in the following sense:*

$$\mathcal{H} \overset{(z-S)^{-1}}{\longrightarrow} \mathcal{D}(S) \overset{[B,S]}{\longrightarrow} \mathcal{D}(S)^* \overset{(z-S)^{-1}}{\longrightarrow} \mathcal{H}.$$

LEMMA 5.3. *$\hat{H}_\kappa$ is of class $C^1(B)$ if $\kappa < \infty$.*

Proof. We will prove

(i) e^{itB} preserves $\mathcal{D}(H_0)$.
(ii) $|(\hat{H}_\kappa u, Bu) - (Bu, \hat{H}_\kappa u)| \leqslant c\,\|H_0 u\|^2 + \|u\|^2,\ u \in \mathcal{D}(H_0) \cap \mathcal{D}(B)$.

[ABG, Thm. 6.3.4] and [ABG, Prop. 6.3.5] give that (i), (ii) implies $\hat{H}_\kappa \in C^1(B)$.

Let us prove (i). It is enough to show that $H_0 e^{itB}(H_0+1)^{-1}e^{-itB}$ is bounded to have (i). We have using (5.2)

$$\begin{aligned} e^{itB} d\Gamma(\omega) e^{-itB} &= d\Gamma(e^{itb}\omega e^{-itb}) \\ &= d\Gamma(\omega(\phi_{-\frac{t}{2}}(k))). \end{aligned}$$

Since $\nabla\omega$ is a bounded complete C^∞ vector field, we have $|\phi_t(k) - k| \leqslant c\,|t|$ uniformly in k. So this implies $|\sum_1^N \omega(\phi_t(k_i)) - \omega(k_i)| \leqslant c\,|t|\,N$ and, hence, $H_0 e^{itB}(H_0+1)^{-1}e^{-itB}$ is bounded. This prove (i).

Let us prove (ii). We compute $[\hat{H}_\kappa, iB]$:

$$\begin{aligned} &(iBu, \hat{H}_\kappa u) - (i\hat{H}_\kappa u, Bu) \\ &\quad = (d\Gamma(|\nabla\omega|^2)u, u) - \sum_{j=1}^{P} (\phi(ibv^j_{\kappa_0})u, u) + \frac{1}{2M}\sum_{j=1}^{P} (ia(r^j_\kappa)a(br^j_\kappa)u, u) - \\ &\qquad - (ia^*(r^j_\kappa)a^*(br^j_\kappa)u, u) + (ia^*(r^j_\kappa)a(br^j_\kappa) - ia^*(br^j_\kappa)a(r^j_\kappa)u, u) - \\ &\qquad - \sqrt{2}(iD_j a(br^j_\kappa)u, u) + \sqrt{2}(ia^*(br^j_\kappa)D_j u, u). \end{aligned}$$

A simple computation yields

$$\begin{aligned} &\text{(i)} \quad \langle x\rangle^{-1} bv_{\kappa_0} \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}), \\ &\text{(ii)} \quad \langle x\rangle^{-1} br_\kappa \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}) \quad \text{for } \kappa < \infty, \\ &\text{(iii)} \quad \langle k\rangle^{-\epsilon}\langle x\rangle^{-1} br_\kappa \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}) \quad \text{for } \epsilon > 0, \text{ uniformly in } \kappa. \end{aligned} \tag{5.4}$$

Now using Lemma 3.4 with $\beta = 1$ and (5.4)(i)–(ii) we obtain for $u \in \mathcal{D}(H_0) \cap \mathcal{D}(B)$

$$\begin{aligned} &|(iBu, \hat{H}_\kappa u) - (i\hat{H}_\kappa u, Bu)| \\ &\quad \leqslant c(\|r_\kappa\|\,\|(V+1)^{-\frac{1}{2}} br_\kappa\| + \|(V+1)^{-\frac{1}{2}} bv_{\kappa_0}\| + \\ &\qquad + \|(K+i)^{-\frac{1}{2}} D\|\,\|(V+1)^{-\frac{1}{2}} br_\kappa\|) \times (\|H_0 u\|^2 + \|u\|^2). \end{aligned}$$

Since $V \geqslant \sum_i \langle x_i\rangle^\alpha$, $\alpha > 2$, we prove (ii). This completes the proof. □

LEMMA 5.4. *$\hat{H}_\infty$ is of class $C^1(B)$.*

Proof. Since $\hat{H}_\kappa$ is of class $C^1(B)$ we know by Theorem 5.2 that $(z-\hat{H}_\kappa)^{-1}: \mathcal{D}(B) \to \mathcal{D}(B)$ and

$$(z-\hat{H}_\kappa)^{-1}[\hat{H}_\kappa, \mathrm{i}B](z-\hat{H}_\kappa)^{-1} = [(z-\hat{H}_\kappa)^{-1}, \mathrm{i}B], \quad \text{on } \mathcal{H}.$$

For $\phi \in D(B)$, one has

$$\begin{aligned}&(\mathrm{i}(\hat{H}_\kappa+\mathrm{i})^{-1}\phi, B\phi) - (\mathrm{i}B\phi, (\hat{H}_\kappa-\mathrm{i})^{-1}\phi)\\ &\quad = ((\hat{H}_\kappa+\mathrm{i})^{-1}\phi, [\hat{H}_\kappa, \mathrm{i}B](\hat{H}_\kappa-\mathrm{i})^{-1}\phi).\end{aligned}$$

Using Lemma 3.3 and (4.17) we obtain

$$\begin{aligned}&|((\hat{H}_\kappa+\mathrm{i})^{-1}\phi, B\phi) - (B\phi, (\hat{H}_\kappa-\mathrm{i})^{-1}\phi)|\\ &\quad \leqslant c\,(\|\mathrm{d}\Gamma(|\nabla\omega|^2)(N+1)^{-1}\| + \|\langle k\rangle^{-\epsilon} r_\kappa\|\,\|\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} b r_\kappa\| +\\ &\qquad + \|\langle \mathrm{x}\rangle^{-1} b v_{\kappa_0}\| + \|(K+\mathrm{i})^{-\frac{1}{2}} D\|\,\|\langle k\rangle^{-\epsilon}\langle \mathrm{x}\rangle^{-1} b r_\kappa\|)\|\phi\|^2, \quad \epsilon > 0,\end{aligned}$$

where c is independent from κ. Then letting $\kappa \to \infty$ and using Theorem 3.6 we obtain

$$|((\hat{H}_\infty+\mathrm{i})^{-1}\phi, \mathrm{i}B\phi) - (\mathrm{i}B\phi, (\hat{H}_\infty-\mathrm{i})^{-1}\phi)| \leqslant c\,\|\phi\|^2 \quad \text{for } \phi \in D(B).$$

This implies $\hat{H}_\infty \in C^1(B)$. □

LEMMA 5.5. *We have for* $\chi \in C_0^\infty(\mathbb{R})$

(i) $w\text{-}\lim_{\kappa\to\infty}[(\hat{H}_\kappa-\mathrm{i})^{-1}, \mathrm{i}B] = [(\hat{H}_\infty-\mathrm{i})^{-1}, \mathrm{i}B]$,
(ii) $\lim_{\kappa\to\infty}\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) = \chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty)$.

Proof. (i) follows from the proof of Lemma 5.4. Let us prove (ii):

$$\begin{aligned}&\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'})[\hat{H}_{\kappa'}, \mathrm{i}B]\chi(\hat{H}_{\kappa'})\\ &\quad = (\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'}))[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) +\\ &\qquad + \chi(\hat{H}_{\kappa'})([\hat{H}_\kappa, \mathrm{i}B] - [\hat{H}_{\kappa'}, \mathrm{i}B])\chi(\hat{H}_\kappa) +\\ &\qquad + \chi(\hat{H}_\kappa)[\hat{H}_{\kappa'}, \mathrm{i}B](\chi(\hat{H}_\kappa) - \chi(\hat{H}_{\kappa'})).\end{aligned} \tag{5.5}$$

We first claim that

$$\lim_{\kappa\to\infty}\chi(\hat{H}_\kappa)H_0^{\frac{1}{2}} = \chi(\hat{H}_\infty)H_0^{\frac{1}{2}}. \tag{5.6}$$

To have (5.6), we see using the functional calculus formula (4.18), that it is enough to show that

$$\lim_{\kappa\to\infty}(\hat{H}_\kappa - z)^{-1}H_0^{\frac{1}{2}} = (\hat{H}_\infty - z)^{-1}H_0^{\frac{1}{2}}.$$

For $\kappa, \kappa' < \infty$, we have the following operator identity on $\mathcal{H}$

$$(\hat{H}_\kappa - z)^{-1} - (\hat{H}_{\kappa'} - z)^{-1} = (\hat{H}_\kappa - z)^{-1}(\hat{H}_{\kappa'} - \hat{H}_\kappa)(\hat{H}_{\kappa'} - z)^{-1}.$$

It follows from the proof of Theorem 3.6 that

$$\lim_{\kappa,\kappa' \to \infty} (H_0 + 1)^{-\frac{1}{2}}(\hat{H}_{\kappa'} - \hat{H}_\kappa)(H_0 + 1)^{-\frac{1}{2}} = 0.$$

Then we obtain (5.6). We also claim that

$$\begin{aligned} &\lim_{\kappa \to \infty} (H_0 + 1)^{-\frac{1}{2}}[\hat{H}_\kappa, \mathrm{i}B](H_0 + 1)^{-\frac{1}{2}}(N + 1)^{-1} \\ &\quad = (H_0 + 1)^{-\frac{1}{2}}[\hat{H}_\infty, \mathrm{i}B](H_0 + 1)^{-\frac{1}{2}}(N + 1)^{-1}. \end{aligned} \tag{5.7}$$

In fact, using Lemma 3.4 and the fact that $V \geqslant c\ \sum_i \langle \mathrm{x}_i \rangle^2$, we get

$$\begin{aligned} &\|(H_0 + \mathrm{i})^{-\frac{1}{2}} a^*(br_{\kappa'} - br_\kappa) a^*(r_{\kappa'})(H_0 + \mathrm{i})^{-\frac{1}{2}}\, N^{-1}\| \\ &\quad \leqslant \|\langle k \rangle^{-\epsilon} \langle \mathrm{x} \rangle^{-1}(br_{\kappa'} - br_\kappa)\|\ \|\langle k \rangle^{-\epsilon} r_{\kappa'}\|, \\ &\|(H_0 + \mathrm{i})^{-\frac{1}{2}} a(r_{\kappa'} - r_\kappa) a(br_{\kappa'})(H_0 + \mathrm{i})^{-\frac{1}{2}}\, N^{-1}\| \\ &\quad \leqslant \|\langle k \rangle^{-\epsilon}(r_{\kappa'} - r_\kappa)\|\ \|\langle k \rangle^{-\epsilon} \langle \mathrm{x} \rangle^{-1} br_{\kappa'}\|, \\ &\|(H_0 + \mathrm{i})^{-\frac{1}{2}} a^*(br_{\kappa'} - br_\kappa) a(r_{\kappa'})(H_0 + \mathrm{i})^{-\frac{1}{2}}\, N^{-1}\| \\ &\quad \leqslant \|\langle k \rangle^{-\epsilon} \langle \mathrm{x} \rangle^{-1}(br_{\kappa'} - br_\kappa)\|\ \|\langle k \rangle^{-\epsilon} r_{\kappa'}\|, \\ &\|(H_0 + \mathrm{i})^{-\frac{1}{2}} a^*(br_{\kappa'} - br_\kappa) D_j (H_0 + \mathrm{i})^{-\frac{1}{2}}\, N^{-1}\| \\ &\quad \leqslant \|\langle k \rangle^{-\epsilon} \langle \mathrm{x} \rangle^{-1}(br_{\kappa'} - br_\kappa)\|\ \|(K + 1)^{-\frac{1}{2}} D\|, \\ &\|(H_0 + \mathrm{i})^{-\frac{1}{2}} D_j a(br_{\kappa'} - br_\kappa)(H_0 + \mathrm{i})^{-\frac{1}{2}}\, N^{-1}\| \\ &\quad \leqslant \|\langle k \rangle^{-\epsilon} \langle \mathrm{x} \rangle^{-1}(br_{\kappa'} - br_\kappa)\|\ \|(K + 1)^{-\frac{1}{2}} D\|. \end{aligned}$$

Using these estimates and (5.4), we obtain (5.7). Now using (5.5), (5.6) and (5.7) we obtain (ii). □

We have to prove a localization estimate for $[\hat{H}_\kappa, \mathrm{i}B]$ similar to the one in the Lemma 4.15.

LEMMA 5.6. *We have uniformly in* κ*:*

$$\chi(\hat{H}_\kappa^{\mathrm{ext}})\big([\hat{H}_\kappa^{\mathrm{ext}}, \mathrm{i}B^{\mathrm{ext}}] I^*(j^R) - I^*(j^R)[\hat{H}_\kappa, \mathrm{i}B]\big)\chi(\hat{H}_\kappa) \in \mathrm{o}(R^0).$$

Proof. We set

$$C(z) := (z - \hat{H}_\kappa^{\rm ext})^{-1}([\hat{H}_\kappa^{\rm ext}, {\rm i}B^{\rm ext}]I^*(j^R) - I^*(j^R)[\hat{H}_\kappa, {\rm i}B])(z - \hat{H}_\kappa)^{-1},$$

where $B^{\rm ext} := B \otimes \mathbb{1} + \mathbb{1} \otimes B$. A simple computation gives

$$\begin{aligned}
&[H_0^{\rm ext}, {\rm i}B^{\rm ext}]I^*(j^R) - I^*(j^R)[H_0, {\rm i}B] = {\rm d}I^*(j^R, [|\nabla\omega|^2, j^R]),\\
&a(r_\kappa^j)a(br_\kappa^j) \otimes \mathbb{1}I^*(j^R) - I^*(j^R)a(r_\kappa^j)a(br_\kappa^j)\\
&\quad = I^*(j^R)(a((j_0^R - 1)r_\kappa^j)a(j_0^R br_\kappa^j) + a(r_\kappa^j)a((j_0 - 1)br_\kappa^j)),\\
&a^*(r_\kappa^j)a^*(br_\kappa^j) \otimes \mathbb{1}I^*(j^R) - I^*(j^R)a^*(r_\kappa^j)a^*(br_\kappa^j)\\
&\quad = -(a^*(j_0^R r_\kappa^j) \otimes a^*(j_\infty^R br_\kappa^j) + a^*(j_0^R br_\kappa^j) \otimes a^*(j_\infty^R r_\kappa^j) +\\
&\qquad + \mathbb{1} \otimes a^*(j_\infty^R r_\kappa^j)a^*(j_\infty br_\kappa^j)).
\end{aligned}$$

We have also similar identities for $a^*a, aD_j, D_j a^*$, replacing r_κ^j by br_κ^j, in the proof of Lemma 4.15. As in the proof of Lemma 4.15 we use Corollary 4.8 and Lemma 3.3.

We have for $\beta < 0$:

$$\begin{aligned}
&\|C(\beta)\|\\
&\quad \leqslant c(\|(N_0 + N_\infty)^{-1}{\rm d}I^*(j^R, [|\nabla\omega|^2, j^R])\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}(1 - j_0^R)r_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_0^R br_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}(1 - j_0^R)br_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_0^R r_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_0^R r_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R br_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_0^R br_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R r_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R br_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R r_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}br_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}(1 - j_0^R)r_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}r_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}(1 - j_0^R)br_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}r_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R br_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}br_\kappa\| \, \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R r_\kappa\| +\\
&\qquad + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}(1 - j_0^R)br_\kappa\| + \|\langle k\rangle^{-\epsilon}\langle {\rm x}\rangle^{-1}j_\infty^R br_\kappa\|).
\end{aligned}$$

Using Lemma A.2, Lemma 3.3 and 5.4(iii) we obtain

$$C(z) \in |{\rm Im}(z)|^{-2}{\rm o}(R^0), \quad \text{uniformly in } \kappa. \qquad \square$$

THEOREM 5.7. *The following three assertions hold:*

(i) *Let* $\lambda \in \mathbb{R} \setminus \tau$. *Then there exist* $\epsilon > 0$, $C_0 > 0$ *and compact operator* K_0 *such that*

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) \geqslant C_0\, \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) + K_0.$$

(ii) *For all* $[\lambda_1, \lambda_2]$ *such that* $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, *one has*

$$\dim \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(\hat{H}_\infty)\mathcal{H} < \infty.$$

Consequently $\sigma_{\mathrm{pp}}(\hat{H}_\infty)$ *can accumulate only at* τ, *which is a closed countable set.*

(iii) *Let* $\lambda \in \mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(\hat{H}_\infty))$. *Then there exists* $\epsilon > 0$, $C_0 > 0$ *such that*

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty) \geqslant C_0\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(\hat{H}_\infty).$$

Proof. We set

$$d(\lambda) := \inf\left\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2;\, \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda,\ n = 1, 2 \ldots,\ \tau \in \sigma_{\mathrm{pp}}(\hat{H}_\infty)\right\},$$

$$\tilde{d}(\lambda) := \inf\left\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2;\, \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda,\ n = 0, 1, \ldots,\ \tau \in \sigma_{\mathrm{pp}}(\hat{H}_\infty)\right\}.$$

$$\Delta_\lambda^\mu := [\lambda - \mu, \lambda + \mu], \quad \mu > 0,$$

$$d^\mu(\lambda) := \inf_{\nu\in\Delta_\lambda^\mu} d(\nu),$$

$$\tilde{d}^\mu(\lambda) := \inf_{\nu\in\Delta_\lambda^\mu} \tilde{d}(\nu),$$

$$E_0 := \inf \sigma(\hat{H}_\infty).$$

We will follow the logic of the proof of Mourre estimate in the case of a Pauli–Fierz Hamiltonian [DG2]. Let us recall the statements that we will prove by induction in n:

$H_1(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists a compact operator K_0, an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_\Delta(\hat{H}_\infty) \geqslant (d(\lambda) - \epsilon)\mathbb{1}_\Delta(\hat{H}_\infty) + K_0.$$

$H_2(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_\Delta(\hat{H}_\infty) \geqslant (\tilde{d}(\lambda) - \epsilon)\mathbb{1}_\Delta(\hat{H}_\infty).$$

$H_3(n)$: Let $\mu > 0, \epsilon_0 > 0$. There exists $\delta > 0$ such that for all $\lambda \in [E_0, E_0 + nm - \epsilon_0]$, one has

$$\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty) \geqslant (\tilde{d}^\mu(\lambda) - \epsilon)\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty).$$

$S_1(n)$: τ is closed countable set in $[E_0, E_0 + nm]$.

$S_2(n)$: For all $\lambda_1 \leqslant \lambda_2 \leqslant E_0 + nm$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, one has

$$\dim \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(\hat{H}_\infty)\mathcal{H} < \infty.$$

The sketch of the proof is given by

$$S_2(n-1) \Rightarrow S_1(n),$$
$$(S_1(n), H_3(n-1)) \Rightarrow H_1(n),$$
$$H_1(n) \Rightarrow H_2(n),$$
$$H_2(n) \Rightarrow H_3(n),$$
$$H_1(n) \Rightarrow S_2(n).$$

$H(1)$ and $S(1)$ are immediate because the spectrum of $\hat{H}_\infty$ is discrete in $[E_0, E_0 + m[$. $S_2(n-1) \Rightarrow S_1(n)$ is obvious. $H_1(n) \Rightarrow H_2(n)$, $H_2(n) \Rightarrow H_3(n)$ follow using arguments in [CFKS], [Mr]. The implication $H_1(n) \Rightarrow S_2(n)$ is based in the Virial theorem which holds since $\hat{H}_\infty \in C^1(B)$, see [ABG, Prop. 7.2.10]. So we have only to prove the implication $(S_1(n), H_3(n-1)) \Rightarrow H_1(n)$.

Let $\chi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\chi(\hat{H}_\kappa) &= I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R)\chi(\hat{H}_\kappa) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)I^*(j^R)\chi(\hat{H}_\kappa) \quad (5.8)\\ &= \Gamma(q^R)\chi(\hat{H}_\kappa) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\mathrm{ext}})I^*(j^R) + o(R^0). \quad (5.9)\end{aligned}$$

(5.8) follows from the fact that $I(j^R)I^*(j^R) = \mathbb{1}$ and $I(j^R)$ is bounded. (5.9) follows from the fact that

$$\Gamma(q^R) = I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R), \quad q^R = (j^R)^2; \; I(j^R) = \Gamma(j^R)U,$$

and Lemma 4.15. We notice that the term $\Gamma(q^R)\chi(\hat{H}_\kappa)$ is compact since $\Gamma(q^R)(H_0 + 1)^{-\frac{1}{2}}$ is compact which is proved in [DG2, Lemma 4.2].

Let $\lambda \in [E_0, E_0 + nm[$. Since $S_2(n-1) \Rightarrow S_1(n)$, the set τ is closed in $[E_0, E_0 + nm]$, which gives $d(\lambda) = \sup_{\mu>0} d^\mu(\lambda)$. So we can choose μ such that $d^\mu(\lambda) \geqslant d(\lambda) - \frac{\epsilon}{3}$. $H_3(n-1)$ gives for $\lambda_1 < E_0 + (n-1)m$

$$\mathbb{1}_{\Delta_{\lambda_1}^\delta}(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\mathbb{1}_{\Delta_{\lambda_1}^\delta}(\hat{H}_\infty) \geqslant \left(\tilde{d}^\mu(\lambda_1) - \frac{\epsilon}{3}\right)\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty).$$

Replacing λ_1 with $\lambda - \mathrm{d}\Gamma(\omega(k))$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)))([\hat{H}_\infty, \mathrm{i}B] + \mathbb{1} \otimes \mathrm{d}\Gamma(|\nabla\omega|^2)) \times \\
&\quad \times \mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)))\mathbb{1}_{[1,\infty[}(N_\infty) \\
&\quad \geqslant \mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k))) \times \\
&\quad \times \left(\tilde{d}^\mu(\lambda - \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k))) + \mathbb{1} \otimes \mathrm{d}\Gamma(|\nabla\omega|^2) - \frac{\epsilon}{3}\right)\mathbb{1}_{[1,\infty[}(N_\infty) \\
&\quad \geqslant \left(d^\mu(\lambda) - \frac{\epsilon}{3}\right)\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)))(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega))\mathbb{1}_{[1,\infty[}(N_\infty) \\
&\quad \geqslant \left(d^\mu(\lambda) - \frac{2\epsilon}{3}\right)\mathbb{1}_{\Delta_\lambda^\delta}(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)))(\hat{H}_\infty + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega))\mathbb{1}_{[1,\infty[}(N_\infty).
\end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. One has uniformly in κ:

$$\begin{aligned}
&\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \\
&\quad = \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \ldots + \\
&\qquad + I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\text{ext}})I^*(j^R)\chi_1(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + \qquad (5.10) \\
&\qquad + \mathrm{o}(R^0) \\
&\quad = \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + \\
&\qquad + I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\text{ext}})I^*(j^R)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + \mathrm{o}(R^0) \qquad (5.11) \\
&\quad = \Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) + \\
&\qquad + I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\kappa^{\text{ext}})[\hat{H}_\kappa^{\text{ext}}, \mathrm{i}B^{\text{ext}}]\chi(\hat{H}_\kappa^{\text{ext}})I^*(j^R) + \qquad (5.12) \\
&\qquad + \mathrm{o}(R^0).
\end{aligned}$$

(5.10) follows by (5.9). Lemma 4.15(i) gives (5.11) and (5.12) follows by Lemma 5.6.

Lemma 5.5(ii) proves that

$$\Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \to \Gamma(q^R)\chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty)$$

norm limit. Now, letting $\kappa \to \infty$ in the expression (5.12) which holds uniformly in κ and using the fact that $\Gamma(q^R)\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa)$ is compact, we obtain:

$$\begin{aligned}
&\chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty) \\
&\quad = K_1(R) + I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(\hat{H}_\infty^{\text{ext}})[\hat{H}_\infty^{\text{ext}}, \mathrm{i}B^{\text{ext}}]\chi(\hat{H}_\infty^{\text{ext}})I^*(j^R) + \\
&\qquad + \mathrm{o}(R^0),
\end{aligned}$$

where $K_1(R)$ is a compact operator. This gives for χ such that supp $\chi \subset [\lambda - \delta, \lambda + \delta[$

$$\chi(\hat{H}_\infty)[\hat{H}_\infty, \mathrm{i}B]\chi(\hat{H}_\infty) \geqslant \left(d(\lambda) - \frac{2\epsilon}{3}\right)\chi^2(\hat{H}_\infty) + K_1(R) + \mathrm{o}(R^0).$$

Choosing R large enough, we obtain $H_1(n)$. Properties (ii), (iii) are standard consequences of (i). □

6. Construction of the Wave Operators

6.1. ASYMPTOTIC FIELDS

In this subsection we prove the existence of asymptotic fields using the Cook method, see, e.g., [H-K]. We set $h_t := \mathrm{e}^{-\mathrm{i}t\omega(k)}h$, for $h \in \mathfrak{h}$ and we denote by $\mathfrak{h}_0$ the space $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. We introduce Heisenberg derivatives:

$$\mathbb{D}_\kappa := \partial_t + \mathrm{i}[H_\kappa, \cdot],$$

$$\mathbb{D} := \partial_t + \mathrm{i}[H, \cdot].$$

Since the existence of asymptotic fields in time $\pm\infty$ is similar, we will restraint proofs of this subsection to the case $+\infty$.

THEOREM 6.1. (i) *For $h \in \mathfrak{h}$ the strong limits*

$$W^\pm(h) := s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} \tag{6.1}$$

exist and are called asymptotic Weyl operators.

(ii) *Furthermore*

$$W^\pm(h)(H+\mathrm{i})^{-1} = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)(H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tH}. \tag{6.2}$$

(iii) *The map*

$$\mathfrak{h} \ni h \mapsto W^\pm(h) \quad \textit{is strongly continuous},$$

$$\mathfrak{h} \ni h \mapsto W^\pm(h)(H+1)^{-1} \quad \textit{is norm continuous}.$$

(iv) *The Weyl commutation relations hold:*

$$W^\pm(h)\, W^\pm(g) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(h|g)}W^\pm(h+g),$$

$$W^\pm(h)^* = W^\pm(-h).$$

(v) *The Hamiltonian preserves the asymptotic Weyl operators:*

$$\mathrm{e}^{\mathrm{i}tH}W^\pm(h)\mathrm{e}^{-\mathrm{i}tH} = W^\pm(h_{-t}).$$

Proof. We have the relation on $\mathscr{H}$

$$W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0} W(h) \mathrm{e}^{\mathrm{i}tH_0}.$$

Hence we can define $\partial_t W(h_t)$ as quadratic form on $\mathscr{D}(H_0)$

$$\partial_t W(h_t) = -\mathrm{i}[H_0, W(h_t)]. \tag{6.3}$$

Using (6.3) and Theorem 2.9 we have, since $\mathrm{Im}(h_t|v_\kappa) \in \mathscr{B}(\mathscr{K})$, the following identity on $\mathscr{H}$

$$\partial_t(\mathrm{e}^{\mathrm{i}tH_\kappa} W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa}) = \mathrm{i}\mathrm{e}^{\mathrm{i}tH_\kappa} \mathrm{Im}(h_t|v_\kappa) W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa}. \tag{6.4}$$

We will first prove (6.1) and (6.2) for $h \in \mathfrak{h}_0$ then we extend to $h \in \mathfrak{h}$. Let $h \in \mathfrak{h}_0$, we notice in this case that $(h_s|v_\kappa) = (h_s|v_{\kappa_1})$, for $\kappa \geqslant \kappa_1$. Since $\mathrm{e}^{\mathrm{i}tH_\kappa}$ is a strongly continuous unitary group and using the inequality (2.1) we see that $t \mapsto \mathrm{e}^{\mathrm{i}tH_\kappa} \mathrm{Im}(h_t|v_{\kappa_1}) W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa}$ is strongly measurable. Hence, by integrating (6.4) we obtain on $\mathscr{H}$ the identity:

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}tH_\kappa} W(h_t) \mathrm{e}^{-\mathrm{i}tH_\kappa} \\ &\quad = W(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH_\kappa} \mathrm{Im}(h_s|v_{\kappa_1}) W(h_s) \mathrm{e}^{-\mathrm{i}sH_\kappa}\,\mathrm{d}s, \quad h \in \mathfrak{h}_0.\end{aligned} \tag{6.5}$$

Using Theorem 3.8 and the convergence dominated theorem, letting $\kappa \to \infty$ in (6.5), we obtain

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH} \\ &\quad = W(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH} \mathrm{Im}(h_s|v_{\kappa_1}) W(h_s) \mathrm{e}^{-\mathrm{i}sH}\,\mathrm{d}s, \quad h \in \mathfrak{h}_0.\end{aligned} \tag{6.6}$$

Moreover

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}tH} W(h_t)(H+\mathrm{i})^{-1} \mathrm{e}^{-\mathrm{i}tH} \\ &\quad = W(h)(H+\mathrm{i})^{-1} + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH} \mathrm{Im}(h_s|v_{\kappa_1}) W(h_s)(H+\mathrm{i})^{-1} \mathrm{e}^{-\mathrm{i}sH}\,\mathrm{d}s.\end{aligned} \tag{6.7}$$

Clearly there exists $\epsilon > 0$ such that $\sum_i \langle \mathrm{x}_i \rangle^{1+\epsilon}(H+\mathrm{i})^{-1}$ is bounded since $\sum_i \langle \mathrm{x}_i \rangle^{1+\epsilon}(\hat{H}_\infty+\mathrm{i})^{-1}$ is bounded. Lemma A.2 gives that $\sum_i \langle \mathrm{x}_i \rangle^{-1-\epsilon}\mathrm{Im}(h_s|v_{\kappa_1}) \in \mathrm{O}(s^{-1-\epsilon})$. Then the existence of the limits (6.2) and consequently (6.1) for $h \in \mathfrak{h}_0$ follows.

Let $h \in \mathfrak{h}$ and $h_n \in \mathfrak{h}_0$ a sequence such that $\lim_{n\to\infty} h_n = h$ in $\mathfrak{h}$. Corollary 4.11 and inequality (2.1) gives

$$\begin{aligned}&\|(\mathrm{e}^{\mathrm{i}sH} W(h_s) \mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH})\,(H+\mathrm{i})^{-1}\| \\ &\quad \leqslant c\,(\|(\mathrm{e}^{\mathrm{i}sH} W(h_{n,s}) \mathrm{e}^{-\mathrm{i}sH} - \mathrm{e}^{\mathrm{i}tH} W(h_{n,t}) \mathrm{e}^{-\mathrm{i}tH})(H+\mathrm{i})^{-1}\| + \|h_n - h\|^\epsilon).\end{aligned}$$

This inequality gives the existence of (6.2) and (6.1) for $h \in \mathfrak{h}$. This shows (i), (ii). Using (2.1) and Corollary 4.11 we have

$$\|e^{isH}(W(h_s) - W(g_s))e^{-isH}(H + i)^{-1}\| \leqslant c\,\|h - g\|^{-\epsilon}. \tag{6.8}$$

Taking the limit $s \to \infty$ in (6.8) we obtain

$$\|(W^+(h) - W^+(g))(H + 1)^{-1}\| \leqslant c\,\|h - g\|^{-\epsilon}.$$

This proves (iii). The rest follows from simple computations. □

THEOREM 6.2. *The five following assertions hold:*

(i) *There exist self-adjoint operators $\phi^{\pm}(h)$, called asymptotic fields, such that*

$$W^{\pm}(h) = e^{i\phi^{\pm}(h)} \text{ for } h \in \mathfrak{h}.$$

(ii) *For $h_i \in \mathfrak{h}$, $i = 1 \ldots n$. We have $\mathcal{D}((H + i)^{\frac{n}{2}}) \subset \mathcal{D}(\prod_1^n \phi^{\pm}(h_i))$ and*

$$\prod_{i=1}^{n} \phi^{\pm}(h_i)(H + i)^{-\frac{n}{2}} = \lim_{t\to\pm\infty} e^{itH} \prod_{i=1}^{n} \phi(h_{i,t})e^{-itH}(H + i)^{-\frac{n}{2}}.$$

(iii) *The map*

$$(h_1, \ldots, h_n) \mapsto \prod_{i=1}^{n} \phi^{\pm}(h_i)(H + i)^{-\frac{n}{2}} \quad \textit{is norm continuous}.$$

(iv) *The commutation relations hold as quadratic forms on $\mathcal{D}(\phi^{\pm}(h)) \cap \mathcal{D}(\phi^{\pm}(g))$*

$$[\phi^{\pm}(h), \phi^{\pm}(g)] = i\mathrm{Im}(h|g).$$

(v) *We have*

$$e^{itH}\phi^{\pm}(h)e^{-itH} = \phi^{\pm}(h_{-t}).$$

Proof. Since $s \to W^+(sh)$ is strongly continuous using Theorem 6.1(iii), (i) follows from Stone's theorem.

We intend to show the existence of the following limit for $h_i \in \mathfrak{h}$, $i = 1 \ldots n$

$$\lim_{t\to+\infty} e^{itH} \prod_{i=1}^{n} \phi(h_{i,t})e^{-itH}(H + 1)^{-\frac{n}{2}}. \tag{6.9}$$

Let $h_i \in \mathfrak{h}_0$, $i = 1 \ldots n$. As in the previous proof we have the following identity as quadratic form on $\mathcal{D}(H_0)$ which extends as an operator identity on $\mathcal{H}$:

$$\partial_t\left(\prod_{i=1}^{n} \phi(h_{i,t})(N + 1)^{-\frac{n}{2}}\right) = -i\left[H_0, \prod_{i=1}^{n} \phi(h_{i,t})\right](N + 1)^{-\frac{n}{2}}. \tag{6.10}$$

Now we compute the derivative

$$\begin{aligned}
&\partial_t\left(\mathrm{e}^{\mathrm{i}tH_\kappa}\prod_{i=1}^n\phi(h_{i,t})(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u,v\right)\\
&=\left(\partial_t\prod_{i=1}^n\phi(h_{i,t})(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u,\mathrm{e}^{-\mathrm{i}tH_\kappa}v\right)+\\
&\quad+\left(\prod_{i=1}^n\phi(h_{i,t})(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u,H_\kappa\mathrm{e}^{-\mathrm{i}tH_\kappa}v\right)-\\
&\quad-\left(H_\kappa(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}u,\prod_{i=1}^n\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH_\kappa}v\right).
\end{aligned}$$

We have $(h_{i,t}|v_\kappa)=(h_{i,t}|v_{\kappa_1})$ for $\kappa\geqslant\kappa_1$. Hence, we have on $\mathscr{H}$

$$\begin{aligned}
&\partial_t\left(\mathrm{e}^{\mathrm{i}tH_\kappa}\prod_{i=1}^n\phi(h_{i,t})(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}\right)\\
&=\mathrm{i}\mathrm{e}^{\mathrm{i}tH_\kappa}\sum_{j=1}^n\prod_{i\neq j}^n\mathrm{Im}(h_{j,t}|v_{\kappa_1})\phi(h_{i,t})(H_\kappa-E_\kappa+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH_\kappa}.
\end{aligned}\tag{6.11}$$

Letting $\kappa\to\infty$ in (6.11) we obtain

$$\begin{aligned}
&\partial_t\left(\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n\phi(h_{i,t})(H+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}\right)\\
&=\mathrm{i}\mathrm{e}^{\mathrm{i}tH}\sum_{j=1}^n\prod_{i\neq j}^n\mathrm{Im}(h_{j,t}|v_{\kappa_1})\phi(h_{i,t})(H+\mathrm{i})^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}.
\end{aligned}\tag{6.12}$$

Since $\sum_i\langle \mathrm{x}_i\rangle^{-1-\epsilon}\mathrm{Im}(h_{i,t}|v_{\kappa_1})\in\mathrm{O}(t^{-1-\epsilon})$, the dominated convergence theorem gives the existence of (6.9) for $h\in\mathfrak{h}_0$.

Let $h_i\in\mathfrak{h}$, $i=1\ldots n$ and $h_{i,\ell}\in\mathfrak{h}_0$ sequences such that $\lim_\ell h_{i,\ell}=h_i$, in $\mathfrak{h}$. Using Corollary 2.4 and Corollary 4.8, we obtain the inequality

$$\begin{aligned}
&\left\|\left(\mathrm{e}^{\mathrm{i}sH}\prod_{i=1}^n\phi(h_{i,s})\mathrm{e}^{-\mathrm{i}sH}-\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}\right)(H+1)^{-\frac{n}{2}}\right\|\\
&\leqslant c\left(\left\|\left(\mathrm{e}^{\mathrm{i}sH}\prod_{i=1}^n\phi(h_{i,\ell,s})\mathrm{e}^{-\mathrm{i}sH}-\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^n\phi(h_{i,\ell,t})\mathrm{e}^{-\mathrm{i}tH}\right)(H+1)^{-\frac{n}{2}}\right\|+\right.\\
&\quad\left.+\sum_{i=1}^n\|h_i-h_{i,\ell}\|\right).
\end{aligned}$$

Cauchy criterion for the convergence, proves the existence of the limit (6.9).

To complete (ii) it suffices to show by induction in n and for $u \in \mathcal{H}$, the existence of following limit

$$\lim_{s\to 0}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\bigg(\frac{1}{s}\Big(W(sh_{1,t})-1\Big)\prod_{i=2}^{n}\phi(h_{i,t})- \\ -\mathrm{i}\prod_{i=1}^{n}\phi(h_{i,t})\bigg)(H+1)^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH}u = 0. \tag{6.13}$$

We first prove (6.13) for $u \in \mathcal{D}((H+1)^{-\epsilon})$, $\epsilon > 0$, then by an argument of density we obtain (6.13) for $u \in \mathcal{H}$. We recall (2.2):

$$\lim_{s\to 0}\sup_{\|h\|\leqslant c} s^{-1}\|(W(sh)-\mathbb{1}-\mathrm{i}s\phi(h))(N+1)^{-\frac{1}{2}-\epsilon}\| = 0.$$

We see that

$$\lim_{s\to 0}\sup_{t\in\mathbb{R}}\left\|\bigg(\frac{1}{s}(W(sh_{1,t})-1)\prod_{i=2}^{n}\phi(h_{i,t})-\mathrm{i}\prod_{i=1}^{n}\phi(h_{i,t})\bigg)(H+1)^{-\frac{n}{2}-\epsilon}\right\| = 0.$$

This completes the proof of (ii). (iii) follows from (ii) and Corollary 2.4 (ii). (iv) follows from the properties of CCR representations. □

DEFINITION 6.3. We define the asymptotic creation and annihilation operators on $\mathcal{D}(\phi^{\pm}(h))\cap\mathcal{D}(\phi^{\pm}(\mathrm{i}h))$

$$a^{\pm}(h) := \frac{1}{\sqrt{2}}\,(\phi^{\pm}(h)-\mathrm{i}\phi^{\pm}(\mathrm{i}h)),$$
$$a^{\pm *}(h) := \frac{1}{\sqrt{2}}\,(\phi^{\pm}(h)+\mathrm{i}\phi^{\pm}(\mathrm{i}h)).$$

We denote by $a^{\pm\sharp}(h)$ the operator $a^{\pm *}(h)$ or $a^{\pm}(h)$.

We formulate now a theorem which follows from the previous one.

THEOREM 6.4. (i) *$a^{\pm *}(h)$ and $a^{\pm}(h)$ are closed operators.*

(ii) *For $h_i \in \mathfrak{h}$, $i = 1\dots n$. We have $\mathcal{D}((H+\mathrm{i})^{\frac{n}{2}}) \subset \mathcal{D}(\prod_{i=1}^{n} a^{\pm\sharp}(h_i))$ and*

$$\prod_{i=1}^{n} a^{\pm\sharp}(h_i)(H+\mathrm{i})^{-\frac{n}{2}} = \lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^{n} a^{\sharp}(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-\frac{n}{2}}.$$

(iii) *The map*

$$(h_1,\dots,h_n)\mapsto\prod_{i=1}^{n} a^{\pm\sharp}(h_i)(H+\mathrm{i})^{-\frac{n}{2}}\quad \textit{is norm continuous}.$$

(iv) *The commutation relations hold as quadratic form on* $\mathcal{D}(a^{\pm}(h)) \cap \mathcal{D}(a^{\pm}(g))$

$$[a^{\pm}(h), a^{\pm *}(g)] = (h|g)\mathbb{1},$$
$$[a^{\pm}(h), a^{\pm}(g)] = [a^{\pm *}(h), a^{\pm *}(g)] = 0.$$

(v) *We have*

$$\mathrm{e}^{\mathrm{i}tH} a^{\pm\sharp}(h) \mathrm{e}^{-\mathrm{i}tH} = a^{\pm\sharp}(h_{-t}).$$

Similar results hold for the modified Hamiltonian $\hat{H}_\infty$. We formulate this in the following theorem.

THEOREM 6.5. (i) *For* $h \in \mathfrak{h}$ *the following limit exists*

$$\hat{W}^{\pm}(h) := U_\infty^* W^{\pm}(h) U_\infty = s\text{-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} W(h_t) \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}.$$

(ii) $h \to \hat{W}^{\pm}(h)$ *is a CCR representation. We denote by* $\hat{a}^{\pm *}(h), \hat{a}^{\pm}(h)$ *the creation and annihilation operators associated to this representation.*

(iii) *For* $h_i \in \mathfrak{h}$, $i = 1 \ldots n$. *We have* $\mathcal{D}((\hat{H}_\infty + \mathrm{i})^{\frac{n}{2}}) \subset \mathcal{D}(\prod_{i=1}^n \hat{a}^{\pm\sharp}(h_i))$ *and*

$$\prod_{i=1}^{n} \hat{a}^{\pm\sharp}(h_i)(\hat{H}_\infty + \mathrm{i})^{-\frac{n}{2}} = \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \prod_{i=1}^{n} a^{\sharp}(h_{i,t}) \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} (\hat{H}_\infty + \mathrm{i})^{-\frac{n}{2}},$$

where $\hat{a}^{\pm\sharp}(h)$ *denote either* $\hat{a}^{\pm *}(h)$ *or* $\hat{a}^{\pm}(h)$.

(iv) *The map*

$$(h_1, \ldots, h_n) \mapsto \prod_{i=1}^{n} \hat{a}^{\pm\sharp}(h_i)(\hat{H}_\infty + \mathrm{i})^{-\frac{n}{2}} \quad \textit{is norm continuous}.$$

(v) *We have*

$$\mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \hat{a}^{\pm\sharp}(h) \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} = \hat{a}^{\pm\sharp}(h_{-t}).$$

Proof. The existence of the strong limit follows from (6.1) and the fact that

$$U_\infty^* W(h_t) U_\infty = \mathrm{e}^{-\mathrm{i}\mathrm{Im}(G_\infty|h_t)} W(h_t),$$

$$w\text{-}\lim_{t \to +\infty} h_t = 0.$$

This prove (i). Theorem 6.1(iv) and (i) give

$$\hat{W}^{\pm}(h)\, \hat{W}^{\pm}(g) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(h|g)} \hat{W}^{\pm}(h+g),$$

$$\hat{W}^{\pm}(h)^* = \hat{W}^{\pm}(-h).$$

This proves CCR representation. (iii) is a consequence of Theorem 6.2(ii) and the fact that

$$U_\infty^* \phi(h_t) U_\infty = \phi(h_t) + \mathrm{Im}(G_\infty | h_t).$$

The rest follows from Theorem 6.4. □

6.2. WAVE OPERATORS

We recall the construction of the Fock subrepresentation of a CCR representation. Details can be found in [BR], [DG3]. Let $\mathfrak{g}$ be pre-Hilbert space, and denote by $\bar{\mathfrak{g}}$ its completion. We define the space of *vacua* associated to a CCR representation π over $\bar{\mathfrak{g}}$:

$$\mathcal{K}_\pi := \{u \in \mathcal{H} \mid a_\pi(h)u = 0,\ h \in \mathfrak{g}\}.$$

PROPOSITION 6.6. (i) *$\mathcal{K}_\pi$ is a closed space.*
(ii) *$\mathcal{K}_\pi$ is contained in the set of analytic vectors of $\phi_\pi(h)$, $h \in \mathfrak{g}$.*

Let $\mathcal{H}_\pi := \mathcal{K}_\pi \otimes \Gamma(\bar{\mathfrak{g}})$. We define

$$\Omega_\pi : \mathcal{K}_\pi \otimes \Gamma_{\mathrm{fin}}(\bar{\mathfrak{g}}) \to \mathcal{H},$$
$$\Omega_\pi \psi \otimes \phi(h)^p \Omega := \phi_\pi(h)^p \psi, \quad h \in \mathfrak{g},\ \psi \in \mathcal{K}_\pi.$$

PROPOSITION 6.7. *The map Ω_π extends to an isometric map*

$$\Omega_\pi : \mathcal{H}_\pi \to \mathcal{H},$$

satisfies $\Omega_\pi \mathbb{1} \otimes a^\sharp(h) = a_\pi^\sharp(h), \quad h \in \mathfrak{g}$.

Theorem 6.1 shows that asymptotic Weyl operators define a CCR representation. Then we define the space of *vacua* in our case

$$\mathcal{K}^\pm := \{u \in \mathcal{H} \mid a^\pm(h)u = 0,\ h \in \mathfrak{h}\}.$$

We denote by $\mathcal{H}^\pm$ the space $\mathcal{K}^\pm \otimes \Gamma(\mathfrak{h})$.

PROPOSITION 6.8. *The following three assertions hold:*

(i) *$\mathcal{K}^\pm$ is closed H-invariant space.*
(ii) *For $h_i \in \mathfrak{h}$, $i = 1 \ldots n$. One has $\mathcal{K}^\pm \subset \mathcal{D}(\prod_{i=1}^n a^{\pm *}(h_i))$.*
(iii) $\mathrm{Ran}\, \mathbb{1}_{\mathrm{pp}}(H) \subset \mathcal{K}^\pm$.

Proof. The fact that $\mathcal{K}^{\pm}$ is H-invariant follows from Theorem 6.4(v). (i) and (ii) follow by Proposition 6.7. Let us prove now (iii). Let $u \in \mathcal{H}$ such that $Hu = Eu$, one has

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} a(h_t)\mathrm{e}^{-\mathrm{i}tH} u = 0,$$

since

$$s\text{-}\lim_{t\to\pm\infty} a(h_t) = 0$$

and

$$\mathrm{e}^{\mathrm{i}tH} a(h_t)\mathrm{e}^{-\mathrm{i}tH} u = (E+\mathrm{i})\,\mathrm{e}^{\mathrm{i}t(H-E)} a(h_t)(H+\mathrm{i})^{-1} u.$$

This means $a^{\pm}(h)u = 0$. □

We define

$$H^{\pm} := H|_{\mathcal{K}^{\pm}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega),$$

and the *wave operator*

$$\Omega^{\pm} : \mathcal{H}^{\pm} \to \mathcal{H},$$

$$\Omega^{\pm}\psi \otimes \prod_{i=1}^{n} a^{*}(h_i)\Omega := \prod_{i=1}^{n} a^{\pm *}(h_i)\psi, \quad \text{for } \psi \in \mathcal{K}^{\pm},\ h_i \in \mathfrak{h},\ i = 1\ldots n.$$

THEOREM 6.9. *$\Omega^{\pm}$ is a unitary map satisfying*

$$a^{\pm\sharp}(h)\Omega^{\pm} = \Omega^{\pm}\,\mathbb{1}\otimes a^{\sharp}(h), \quad \mathit{for}\ h \in \mathfrak{h},$$
$$H\Omega^{\pm} = \Omega^{\pm}H^{\pm}.$$

Proof. Proposition 6.8 gives that $\Omega^{\pm}$ is isometric and satisfies properties announced in the theorem. Let prove that $\Omega^{\pm}$ is unitary. Using [DG3, Thm. 3.3], it suffices to show that the CCR representation $h \to W^{\pm}(h)$ admits a densely defined number operator. For each finite-dimensional space $\mathfrak{f} \subset \mathfrak{h}$, we define as quadratic form the following expression

$$n^{\pm}_{\mathfrak{f}}(u) := \sum_{i=1}^{\dim\mathfrak{f}} \|a^{\pm}(h_i)u\|^2, \quad \{h_i\}\text{ is an orthonormal basis of } \mathfrak{f},\ u \in \mathcal{H}.$$

Now we show that $n^{\pm}(u) := \sup_{\mathfrak{f}} n^{\pm}_{\mathfrak{f}}(u)$ is densely defined:

$$\begin{aligned}\|n^{\pm}_{\mathfrak{f}}(u)\|^2 &\leqslant \lim_{t\to\pm\infty} \sum_{i=1}^{\dim\mathfrak{f}} \|a(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}u\|^2\\ &\leqslant \lim_{t\to\pm\infty} (\mathrm{e}^{-\mathrm{i}tH}u, N\mathrm{e}^{-\mathrm{i}tH}u).\end{aligned}$$

We conclude using Corollary 4.11 that $n^{\pm}(u) \leqslant c\,\|(H+b)^{\frac{1}{2}}u\|$. Thus $\mathcal{D}(H^{\frac{1}{2}}) \subset \mathcal{D}(n^{\pm})$ and $\operatorname{Ran}\Omega^{\pm} = \mathcal{H}$. □

We define an *extended wave operator*

$$\hat{\Omega}^{\mathrm{ext},\pm} : \bigoplus_{n=0}^{\infty} \mathcal{D}((\hat{H}_{\infty}+1)^{\frac{n}{2}}) \otimes \otimes_s^n \mathfrak{h} \to \mathcal{H},$$

$$\hat{\Omega}^{\mathrm{ext},\pm}\psi \otimes \prod_{i=1}^{n} a^*(h_i)\Omega := \prod_{i=1}^{n} \hat{a}^{\pm *}(h_i)\psi, \quad \psi \in \mathcal{D}((\hat{H}_{\infty}+1)^{\frac{n}{2}}).$$

We set

$$\hat{\mathcal{K}}^{\pm} := U_{\infty}^{*}\mathcal{K}^{\pm}, \qquad \hat{\mathcal{H}}^{\pm} := \hat{\mathcal{K}}^{\pm} \otimes \Gamma(\mathfrak{h}).$$

Then we have a wave operator of the modified Hamiltonian:

$$\hat{\Omega}^{\pm} : \hat{\mathcal{H}}^{\pm} \to \mathcal{H},$$

$$\hat{\Omega}^{\pm}\psi \otimes \prod_{i=1}^{n} a^*(h_i)\Omega := \prod_{i=1}^{n} \hat{a}^{\pm *}(h_i)\psi, \quad \text{for } \psi \in \hat{\mathcal{K}}^{\pm},\ h_i \in \mathfrak{h},\ i = 1 \ldots n.$$

We notice that $\hat{\Omega}^{\mathrm{ext},\pm}_{|\hat{\mathcal{H}}^{\pm}} = \hat{\Omega}^{\pm}$. This suggests to treat sometimes $\hat{\Omega}^{\mathrm{ext},\pm}$ as a partial isometry. Another construction of the extended wave operator is given by the following theorem, see [DG2, Thm. 5.7]:

THEOREM 6.10. (i) *Let* $u \in \mathcal{D}(\hat{\Omega}^{\mathrm{ext},\pm})$. *Then one has*

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_{\infty}} I \mathrm{e}^{-\mathrm{i}t\hat{H}_{\infty}^{\mathrm{ext}}} u = \hat{\Omega}^{\mathrm{ext},\pm} u,$$

where I is the scattering identification operator defined in the Subsection 2.1.

(ii) *Let* $\chi \in C_0^{\infty}(\mathbb{R})$. *Then* $\operatorname{Ran}\chi(\hat{H}_{\infty}^{\mathrm{ext}}) \subset \mathcal{D}(\hat{\Omega}^{\mathrm{ext},\pm})$ *and the operators* $I\chi(\hat{H}_{\infty}^{\mathrm{ext}})$, $\hat{\Omega}^{\mathrm{ext},\pm}\chi(\hat{H}_{\infty}^{\mathrm{ext}})$ *are bounded. Moreover*

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_{\infty}} I \mathrm{e}^{-\mathrm{i}t\hat{H}_{\infty}^{\mathrm{ext}}} \chi(\hat{H}_{\infty}^{\mathrm{ext}}) = \hat{\Omega}^{\mathrm{ext},\pm}\chi(\hat{H}_{\infty}^{\mathrm{ext}}).$$

7. Propagation Estimates

We make the following notations for the Heisenberg derivatives

$$\mathbf{d}_0 := \partial_t + \mathrm{i}[\omega(k), \cdot],$$

$$\mathbf{D}_0 := \partial_t + \mathrm{i}[\mathrm{d}\Gamma(\omega), \cdot],$$

$$\hat{\mathbf{D}}_{\kappa} := \partial_t + \mathrm{i}[\hat{H}_{\kappa}, \cdot].$$

PROPOSITION 7.1. *Let* $\chi \in C_0^\infty(\mathbb{R})$. *For* $R' > R > 1$, *there exists* c *independent from* κ *such that we have for* $\kappa \leqslant \infty$

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right)^{\frac{1}{2}} \chi(\hat{H}_\kappa)\, \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} u \right\|^2 \frac{\mathrm{d}t}{t} \leqslant c\|u\|^2.$$

Proof. We use a standard method in scattering theory of the N-body problem [Gr], [SS]. It is based on a technical lemma, see, e.g., [DG1, Lemma B.4.1].

Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near ∞, to 0 near the origin, with $F'(s) \geqslant \mathbb{1}_{[R,R']}(s)$. We consider the observable

$$\Phi(t) := \chi(\hat{H}_\kappa)\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\chi(\hat{H}_\kappa).$$

By Lemma A.3, it is enough to show that

$$\hat{\mathbb{D}}_\kappa \Phi(t) \geqslant t^{-1} C_0 \;\; \chi(\hat{H}_\kappa)\mathrm{d}\Gamma\left(F'\left(\frac{|x|}{t}\right)\right)\chi(\hat{H}_\kappa) + \mathrm{O}(t^{-1-\mu}) \tag{7.1}$$

uniformly in κ to have the inequality.

One has

$$\begin{aligned}\hat{\mathbb{D}}_\kappa \Phi(t) \;=\; & \chi(\hat{H}_\kappa)\, \mathrm{d}\Gamma\left(\mathbb{d}_0 F\left(\frac{|x|}{t}\right)\right)\chi(\hat{H}_\kappa) \;+ \\ & + \chi(\hat{H}_\kappa)\left[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right]\chi(\hat{H}_\kappa).\end{aligned}$$

Using the fact that

$$\mathbb{d}_0 F\left(\frac{|x|}{t}\right) \geqslant \frac{c_0}{t}\; F'\left(\frac{|x|}{t}\right) + \mathrm{O}(t^{-2}),$$

it is sufficient to show that the second term in the previous identity is $\mathrm{O}(t^{-1-\mu})$, $\mu > 0$ uniformly in κ, to have (7.1).

By simple commutation relations we obtain:

$$\begin{aligned}&\left[\phi(v_{\kappa_0}), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] = \mathrm{i}\phi\left(\mathrm{i}F\left(\frac{|x|}{t}\right)v_{\kappa_0}\right),\\ &\left[a^{*^2}(r_\kappa^j), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] = 2\, a^*(r_\kappa^j)a^*\left(F\left(\frac{|x|}{t}\right)r_\kappa^j\right),\\ &\left[a^2(r_\kappa^j), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] = -2\, a(r_\kappa^j)a\left(F\left(\frac{|x|}{t}\right)r_\kappa^j\right),\\ &\left[a^*(r_\kappa^j)a(r_\kappa^j), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right]\end{aligned} \tag{7.2}$$

$$= a^*(r_\kappa^j) a\left(F\left(\frac{|x|}{t}\right) r_\kappa^j\right) + a^*\left(F\left(\frac{|x|}{t}\right) r_\kappa^j\right) a(r_\kappa^j),$$

$$\left[D_j\, a(r_\kappa^j), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] = D_j\, a\left(F\left(\frac{|x|}{t}\right) r_\kappa^j\right),$$

$$\left[a^*(r_\kappa^j)\, D_j, \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] = a^*\left(F\left(\frac{|x|}{t}\right) r_\kappa^j\right) D_j.$$

Using the functional calculus formula (4.18) and Corollary 4.8, it is enough to estimate

$$(N+1)^{-n}(H_0+c)^{-\frac{1}{2}}\left[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right](H_0+c)^{-\frac{1}{2}}, \quad c > 0.$$

Using (7.2) and Lemma 3.4, we obtain:

$$\begin{aligned}
&\|(N+1)^{-n}(H_0+c)^{-\frac{1}{2}}\left[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right](H_0+c)^{-\frac{1}{2}}\| \\
&\leqslant c\left(\left\|(V+1)^{-\frac{1}{2}} F\left(\frac{|x|}{t}\right) v_{\kappa_0}\right\| + \right. \\
&\quad + \|\omega^{\frac{s-1}{4}} r_\kappa^j\| \left\|(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{4}} F\left(\frac{|x|}{t}\right) r_\kappa^j\right\| + \\
&\quad + \|(V+1)^{\frac{s-1}{2}} \omega^{-\frac{s}{2}} r_\kappa^j\| \left\|(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}} F\left(\frac{|x|}{t}\right) r_\kappa^j\right\| + \\
&\quad \left. + \|D\,(K+c)^{-\frac{1}{2}}\| \left\|(V+1)^{-\frac{s}{2}} \omega^{\frac{s-1}{2}} F\left(\frac{|x|}{t}\right) r_\kappa^j\right\|\right).
\end{aligned} \tag{7.3}$$

It remains to see that the terms

$$\left\|(V+1)^{-\frac{1}{2}} F\left(\frac{|x|}{t}\right) v_{\kappa_0}\right\|, \qquad \left\|(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{s-1}{2}} F\left(\frac{|x|}{t}\right) r_\kappa\right\| \quad \text{and}$$

$$\left\|(V+1)^{-\frac{s}{2}}\ \omega^{-\frac{s-1}{4}}\ F\left(\frac{|x|}{t}\right) r_\kappa\right\|$$

are integrable for $(1-s)$ small enough. They are $\mathrm{O}(t^{-1-\mu})$ by (4.24). Then using Lemma A.3 we finish the proof of the estimate announced in the proposition. □

PROPOSITION 7.2. *Let* $\chi \in C_0^\infty(\mathbb{R})$, $0 < c_0 < c_1$, *and*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma\left(\left\langle \frac{x}{t} - \nabla\omega(k), \mathbb{1}_{[c_0,c_1]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t} - \nabla\omega(k)\right)\right\rangle\right).$$

One has uniformly in $\kappa \leqslant \infty$

$$\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}} \chi(\hat{H}_\kappa) \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} u\|^2 \frac{\mathrm{d}t}{t} \leqslant c\|u\|^2.$$

Proof. Let $R_0(x) \in C^\infty$ be a function such that:

$$R_0(x) = 0, \quad \text{for } |x| \leqslant \frac{c_0}{2},$$

$$R_0(x) = \frac{1}{2}x^2 + c, \quad \text{for } |x| \geqslant 2c_1,$$

$$\nabla_x^2 R_0 \geqslant \mathbb{1}_{[c_0,c_1]}(|x|).$$

We choose $c_1 > 2,\ c_2 > c_1 + 1$ and we define the function

$$R(x) := F(|x|)R_0(x),$$

where $F(s) = 1$, if $s \leqslant c_1$, $F(s) = 0$, if $s \geqslant c_2$.

We set

$$b(t) := R\left(\frac{x}{t}\right) - \frac{1}{2}\left(\left\langle \nabla R\left(\frac{x}{t}\right), \frac{x}{t} - \nabla\omega(k)\right\rangle + hc\right).$$

We consider the observable

$$\Phi(t) := \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa).$$

Pseudodifferential calculus gives

$$\chi(\hat{H}_\kappa)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa)$$
$$\geqslant \chi(\hat{H}_\kappa)\left(\frac{1}{t}\Theta_{[c_0,c_2]}(t) - \frac{1}{t}\mathrm{d}\Gamma\left(\mathbb{1}_{[2,c_2]}\left(\frac{|x|}{t}\right)\right)\right)\chi(\hat{H}_\kappa) + \mathrm{O}(t^{-2}).$$

The first term will serves in the application of Lemma A.4 and the second is integrable along the evolution using Proposition 7.1. To complete the proof of the proposition, it suffices to show uniformly in κ that:

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(\hat{H}_\kappa) \in \mathrm{O}(t^{-1-\mu}), \quad \mu > 0. \tag{7.4}$$

As in the Proposition 7.1, using (7.2) and (7.3), we see that (7.4) is bounded by a sum of terms

$$\|(V+1)^{-\frac{1}{2}}b(t)v_{\kappa_0}\|, \qquad \|(V+1)^{-\frac{s}{2}}\,\omega^{-\frac{1-s}{2}}\,b(t)r_\kappa\|$$

and

$$\|(V+1)^{-\frac{s}{2}}\,\omega^{-\frac{1-s}{4}}\,b(t)r_\kappa\|.$$

By (4.24) these terms are $\mathrm{O}(t^{-1-\mu})$, $\mu > 0$, for $(1-s)$ small enough. We end the proof by using Lemma A.3. □

PROPOSITION 7.3. *Let* $0 < c_0 < c_1,\ J \in C_0^\infty(\{c_0 < |x| < c_1\}),\ \chi \in C_0^\infty(\mathbb{R})$. *For* $1 \leqslant i \leqslant 3$, *one has uniformly in* $\kappa \leqslant \infty$

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\left| J\left(\frac{x}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + hc\right|\right)^{\frac{1}{2}} \chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u\right\|^2 \frac{\mathrm{d}t}{t} \leqslant c\|u\|^2.$$

Proof. We set

$$A := \left(\frac{x}{t} - \nabla\omega(k)\right)^2 + t^{-\delta},$$
$$b(t) := J\left(\frac{x}{t}\right)A^{\frac{1}{2}}J\left(\frac{x}{t}\right).$$

Let $J_1 \in C_0^\infty(\{c_0 < |x| < c_1\})$, $0 \leqslant J \leqslant 1$, $J = 1$ near the support of J_1. We consider the observable

$$\Phi(t) := -\chi(\hat{H}_\kappa)\,\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa).$$

One has

$$\chi(\hat{H}_\kappa)\mathbb{D}_0\,\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa) = -\chi(\hat{H}_\kappa)\,\mathrm{d}\Gamma(\mathrm{d}_0 b(t))\chi(\hat{H}_\kappa),$$

and we have using [DG2, Lemma 6.4]

$$\begin{aligned}
&\chi(\hat{H}_\kappa)\mathbb{D}_0\,\mathrm{d}\Gamma(b(t))\chi(\hat{H}_\kappa)\\
&\quad\geqslant \frac{c_0}{t}\chi(\hat{H}_\kappa)\,\mathrm{d}\Gamma\left(\left|J_1\left(\frac{x}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + hc\right|\right)\chi(\hat{H}_\kappa) -\\
&\qquad - \frac{c}{t}\chi(\hat{H}_\kappa)\,\mathrm{d}\Gamma\left(\left\langle\frac{x}{t} - \nabla\omega, J_2\left(\frac{x}{t}\right)\left(\frac{x}{t} - \nabla\omega\right)\right\rangle\right)\chi(\hat{H}_\kappa) + \mathrm{O}(t^{-1-\mu}).
\end{aligned}$$

The second term is integrable along the evolution by Proposition 7.2. It's enough to show that

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(\hat{H}_\kappa) \in \mathrm{O}(t^{-1-\mu}), \quad \mu > 0, \text{ uniformly in } \kappa.$$

This follows by using (7.3), the fact that $J(\frac{x}{t})A^{\frac{1}{2}} \in \mathrm{O}(1)$ and Lemma A.2. Using Lemma A.3 we end the proof. □

PROPOSITION 7.4. *Let $\chi \in C_0^\infty(\mathbb{R})$, supported in $\mathbb{R}\setminus(\tau\cup\sigma_{\mathrm{pp}}(\hat{H}_\infty))$. There exist $\epsilon > 0$, C independent in κ and a sequence of $\hat{H}_\kappa$ such that for $\kappa \leqslant \infty$, we have*

$$\int_1^\infty \left\|\Gamma\left(\mathbb{1}_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right)\chi(\hat{H}_\kappa)\,\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u\right\|^2\frac{\mathrm{d}t}{t} \leqslant C\|u\|^2.$$

Proof. We notice that Proposition 7.4 is a minimal velocity estimate for a sequence of $\hat{H}_\kappa$ which is uniform in κ. Let χ supported near λ such that $\lambda \in \mathbb{R}\setminus(\tau\cup\sigma_{\mathrm{pp}}(\hat{H}_\infty))$. Then there exists a sequence $\hat{H}_\kappa$ such that $\lambda \in \mathbb{R}\setminus(\tau\cup\sigma_{\mathrm{pp}}(\hat{H}_\kappa))$. Lemma 5.5 in Mourre estimate section gives

$$\chi(\hat{H}_\kappa)[\hat{H}_\kappa, \mathrm{i}B]\chi(\hat{H}_\kappa) \geqslant c_\kappa\chi^2(\hat{H}_\kappa).$$

Let $\epsilon > 0$. Let $q \in C_0^\infty(|x| \leqslant 2\epsilon)$ such that $0 \leqslant q \leqslant 1$, $q(x) = 1$, if $|x| \leqslant \epsilon$.
We set

$$\Phi_\kappa(t) := \chi(\hat{H}_\kappa)\Gamma(q^t)\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa).$$

The Heisenberg derivative of $\Phi_\kappa(t)$ is

$$\begin{aligned}\hat{\mathbb{D}}_\kappa\Phi_\kappa(t) &= \chi(\hat{H}_\kappa)\,\mathrm{d}\Gamma(q^t, \mathbb{d}_0 q^t)\,\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa) + hc + \\ &\quad + \chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)]\,\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa) + hc + \\ &\quad + t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)[\hat{H}_\kappa, \mathrm{i}B]\Gamma(q^t)\chi(\hat{H}_\kappa) - \\ &\quad - t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)\frac{B}{t}\Gamma(q^t)\Gamma(q^t)\chi(\hat{H}_\kappa) \\ &=: R_\kappa^1 + R_\kappa^2 + R_\kappa^3 + R_\kappa^4.\end{aligned}$$

We claim that

$$R_\kappa^2 \in \mathrm{O}(t^{-1-\mu}), \quad \mu > 0. \tag{7.5}$$

To prove this, we use the estimates (4.22) in the proof of Lemma 4.14 to obtain

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)](\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}} \in \mathrm{O}(t^{-1-\mu}), \quad \mu > 0. \tag{7.6}$$

To prove (7.5) it suffices by Corollary 4.8 to show that

$$(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t)(\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}}(N+1)^{-2} \in \mathrm{O}(1). \tag{7.7}$$

A simple explicit calculus gives

$$\begin{aligned}&(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t) \\ &\quad = \frac{B}{t}\Gamma(q^t)(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}} + \left[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \frac{B}{t}\right]\Gamma(q^t) + \\ &\quad\quad + \frac{B}{t}[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \Gamma(q^t)].\end{aligned} \tag{7.8}$$

Clearly $\frac{B}{t}\Gamma(q^t)(N+1)^{-2} \in \mathrm{O}(1)$ and

$$\left[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \frac{B}{t}\right]\Gamma(q^t) = (\mathrm{d}\Gamma(\omega)+1)^{-\frac{1}{2}}\mathrm{d}\Gamma(|\nabla\omega|^2)\Gamma(q^t) \in \mathrm{O}(1).$$

To estimate the last term in (7.8), we write on the n-particle sector:

$$\begin{aligned}&[(\mathrm{d}\Gamma(\omega)+1)^{\frac{1}{2}}, \Gamma(q^t)]_{|\mathcal{K}\otimes_n^s \mathfrak{h}}\\&= \sum_{j=1}^{n}\prod_{i=1}^{j-1} q\left(\frac{x_i}{t}\right)\left[\left(\sum_{i=1}^{n}\omega(k_i)+1\right)^{\frac{1}{2}}, q\left(\frac{x_j}{t}\right)\right]\prod_{i=j+1}^{n} q\left(\frac{x_i}{t}\right)\\&=: \sum_{j=1}^{n} R_j(t).\end{aligned}$$

Pseudodifferential calculus gives that

$$x_k R_j(t) \in \mathrm{O}(1), \quad \text{uniformly in } k, j.$$

This proves (7.7). Now (7.6) and (7.7) imply that $R_\kappa^2 \in \mathrm{O}(t^{-1-\nu})$, for $\kappa < \infty$.
We consider now R_κ^1. We have:

$$\mathbf{d}_0 q^t = -\frac{1}{2t}\left\langle \frac{x}{t} - \nabla\omega(k), \nabla q\left(\frac{x}{t}\right)\right\rangle + hc + r^t =: \frac{1}{t} g^t + r^t,$$

where $r^t \in \mathrm{O}(t^{-2})$. We have using (2.3) and Corollary 4.8:

$$\left\| \chi(\hat{H}_\kappa)\, \mathrm{d}\Gamma(q^t, r^t)\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa)\right\| \in \mathrm{O}(t^{-2}).$$

We set

$$B_1 := \chi(\hat{H}_\kappa)\, \mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}}, \qquad B_2 := (N+1)^{\frac{1}{2}}\frac{B}{t}\Gamma(q^t)\chi(\hat{H}_\kappa).$$

So we obtain the inequality

$$R_\kappa^1 \geqslant -\epsilon_0^{-1} t^{-1} B_1 B_1^* - \epsilon_0 t^{-1} B_2 B_2^*.$$

Using arguments in [DG2, Prop. 6.5], we obtain

$$-B_2 B_2^* \geqslant -C_1 \chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) - Ct^{-1},$$

$$\int_1^\infty \|B_1 \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} u\| \leqslant C\|u\|^2.$$

Using Lemma 4.14 and Theorem 5.7, we have

$$R_\kappa^3 \geqslant C_0 t^{-1}\chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) - Ct^{-2}.$$

We have

$$-R_\kappa^4 \leqslant C_2 \frac{\epsilon}{t}\chi(\hat{H}_\kappa)\Gamma(q^t)^2\chi(\hat{H}_\kappa) + Ct^{-2}.$$

Collecting the four terms, we obtain

$$\begin{aligned}\hat{\mathbb{D}}_\kappa \phi_\kappa(t) &\geqslant -\epsilon_0 t^{-1} B_2 B_2^* + R_\kappa^2 + R_\kappa^3 + R_\kappa^4 \\ &\geqslant (C_0 - \epsilon_0 C_1 - \epsilon C_2) t^{-1} \chi(\hat{H}_\kappa) \Gamma(q^t)^2 \chi(\hat{H}_\kappa) + C t^{-2} \\ &\geqslant \tilde{C}_0 \chi(\hat{H}_\kappa) t^{-1} \Gamma(q^t)^2 \chi(\hat{H}_\kappa) - R(t),\end{aligned}$$

where $R(t)$ is integrable. By Lemma A.3 we obtain the inequality announced in the proposition for χ supported near a one energy level λ. Then we complete the proof for an arbitrary χ using a standard argument, see, e.g., [DG1, Proposition 4.4.7]. □

8. Asymptotic Completeness

In this section we prove the main result of this paper, which is the asymptotic completeness of the Nelson Hamiltonian. This is the subject of Theorem 8.5, where we prove $\operatorname{Ran} \mathbb{1}_{\mathrm{pp}}(H) = \mathcal{K}^\pm$.

THEOREM 8.1. *Let $q, \tilde{q} \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leqslant q, \tilde{q} \leqslant 1$, $q, \tilde{q} = 1$ on a neighborhood of zero and $q^t := q(\frac{x}{t})$.*

(i) *The following limits exist*

$$\begin{aligned}\Gamma^\pm(q) &:= s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} \\ &= \lim_{\kappa\to+\infty} s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}.\end{aligned}$$

(ii) *We have*

$$\Gamma^\pm(q\tilde{q}) = \Gamma^\pm(q)\Gamma^\pm(\tilde{q}),$$

$$0 \leqslant \Gamma^\pm(q) \leqslant \Gamma^\pm(\tilde{q}) \leqslant \mathbb{1}, \quad \text{if } 0 \leqslant q \leqslant \tilde{q},$$

$$[\hat{H}_\infty, \Gamma^\pm(q)] = 0.$$

(iii) *We have* $\operatorname{Ran} \Gamma^\pm(q) \subset \hat{\mathcal{K}}^\pm$.

Proof. It is sufficient using a density argument and Lemma 4.14 to show for $\chi \in C_0^\infty(\mathbb{R})$ the existence of the limit

$$s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} \chi(\hat{H}_\infty) \Gamma(q^t) \chi(\hat{H}_\infty) \mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}. \tag{8.1}$$

Using Lemma A.4, we see that as for all asymptotic limits these amounts to bound Heisenberg derivatives uniformly in κ. We have on $\mathcal{H}$:

$$\begin{aligned}&\partial_t (\mathrm{e}^{\mathrm{i}t\hat{H}_\kappa} \chi(\hat{H}_\kappa) \Gamma(q^t) \chi(\hat{H}_\kappa) \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}) \\ &\quad = \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa} \chi(\hat{H}_\kappa) \mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t) \chi(\hat{H}_\kappa) \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa} + \\ &\qquad + \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa} \chi(\hat{H}_\kappa) [\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)] \chi(\hat{H}_\kappa) \mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}.\end{aligned}$$

By (4.22) we have uniformly in κ:

$$\chi(\hat{H}_\kappa)[\hat{I}_\kappa, \mathrm{i}\Gamma(q^t)]\chi(\hat{H}_\kappa) \in \mathrm{O}(t^{-1-\epsilon}). \tag{8.2}$$

We use now an argument introduced in [DG2]:

$$\mathbb{d}_0 q^t = \frac{1}{t} g^t + r^t,$$

where

$$g^t = -\frac{1}{2}\left(\left(\frac{x}{t} - \partial\omega(k)\right)\partial q\left(\frac{x}{t}\right) + hc\right) \quad \text{and} \quad r^t \in \mathrm{O}(t^{-2}).$$

The estimate (2.3) and Corollary 4.8 give uniformly in κ

$$\|\chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, \mathbb{d}_0 r^t)\chi(\hat{H}_\kappa)\| \in \mathrm{O}(t^{-2}). \tag{8.3}$$

The other term will be estimated:

$$\begin{aligned} &|(\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u, \chi(\hat{H}_\kappa)\mathrm{d}\Gamma(q^t, g^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u)| \\ &\quad \leqslant \|\mathrm{d}\Gamma(|g^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u\|^2, \quad u \in \mathcal{H}. \end{aligned} \tag{8.4}$$

Using (8.2)–(8.4), Proposition 7.3 and Lemma A.4, we obtain the existence of the limit (8.1).

The first statement in (ii) follows from the fact that

$$\Gamma(q^t\tilde{q}^t) = \Gamma(q^t)\Gamma(\tilde{q}^t).$$

The second statement follows by

$$0 \leqslant \Gamma(q^t) \leqslant \Gamma(\tilde{q}^t) \leqslant 1 \quad \text{if } 0 \leqslant q \leqslant \tilde{q}.$$

The last statement is a consequence of (i) and Lemma 4.14.

Let us prove (iii). Since $\hat{H}_\infty, \Gamma^\pm(q)$ commute, $\Gamma^\pm(q)$ preserves $\mathcal{D}(\hat{H}_\infty)$. Consequently $\mathcal{D}(\hat{H}_\infty) \cap \operatorname{Ran}\Gamma^\pm(q)$ is dense in $\operatorname{Ran}\Gamma^\pm(q)$. Since $\hat{\mathcal{K}}^\pm$ is closed, it is enough to show that $\mathcal{D}(\hat{H}_\infty) \cap \operatorname{Ran}\Gamma^\pm(q) \subset \hat{\mathcal{K}}^\pm$ to prove (iii). Let $v \in \mathcal{D}(\hat{H}_\infty) \cap \operatorname{Ran}\Gamma^\pm(q)$, $v = \Gamma^\pm(q)u$.

$$\begin{aligned} &(\hat{H}_\infty + b)^{-1}\hat{a}^\pm(h)\Gamma^\pm(q)u \\ &\quad = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}a(h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\Gamma^\pm(q)u \\ &\quad = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}a(h_t)\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}u \\ &\quad = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty}(\hat{H}_\infty + b)^{-1}\Gamma(q^t)a(q^t h_t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}u. \end{aligned}$$

If $h \in \mathfrak{h}_0$, by a stationary phase argument we have $q^t h_t \in \mathrm{o}(1), t \to \pm\infty$. Using the fact that $h \to (\hat{H}_\infty + \mathrm{i})^{-1}a(h_t)$ is continuous, we obtain $\hat{a}^\pm(h)v = 0$ for all $h \in \mathfrak{h}$. This ends the proof. □

COROLLARY 8.2. *Let* $\{q_n\} \in C_0^\infty(\mathbb{R}^3)$ *be a decreasing sequence of functions such that* $0 \leqslant q_n \leqslant 1$, $q_n = 1$ *on a neighborhood of* 0 *and* $\bigcap_{n=1}^\infty \operatorname{supp} q_n = \{0\}$. *Then the following limit exist and it does not depend in the choose of the sequence*

(i) $P_0^\pm := \lim_{n\to\infty} \Gamma^\pm(q_n)$,

(ii) $\operatorname{Ran} P_0^\pm \subset \hat{\mathcal{K}}^\pm$.

Moreover, $P_0^\pm$ *is an orthogonal projection.*

Proof. The existence of the limit (i) follows from Theorem 8.1(ii) and Lemma A.5. The independence from the choose of the sequence follows from the fact that there exists an index m_n such that $q_n \geqslant \tilde{q}_{m_n}, \tilde{q}_n \geqslant q_{m_n}$; $\lim_{n\to\infty} m_n = +\infty$ and Theorem 8.1(ii). (ii) is a consequence of Theorem 8.1(iii) and (i). □

Let $j_0 \in C_0^\infty(\mathbb{R}^3)$, $0 \leqslant j_0$, $0 \leqslant j_\infty$, $j_0^2 + j_\infty^2 \leqslant 1$, $j_0 = 1$ near 0. Set $j := (j_0, j_\infty)$ and $j^t := (j_0^t, j_\infty^t)$, where $j_0^t := j_0(\frac{x}{t})$, $j_\infty^t := j_\infty(\frac{x}{t})$. We recall that $I(j^t)$ is the operator introduced in Subsection 2.1.

THEOREM 8.3. (i) *The following limits exist*

$$s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}} I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} =: \tilde{W}^\pm(j),$$

$$s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}} = \tilde{W}^\pm(j)^*.$$

(ii) *For a bounded Borel function F, we have*

$$\tilde{W}^\pm(j)\ F(\hat{H}_\infty) = F(\hat{H}_\infty^{\mathrm{ext}})\tilde{W}^\pm(j).$$

(iii) *Let* $q_0, q_\infty \in C^\infty(\mathbb{R}^3)$, $\nabla q_0, \nabla q_\infty \in C_0^\infty(\mathbb{R}^3)$, $0 \leqslant q_0, q_\infty \leqslant 1$, $q_0 = 1$ *near* 0. *Set* $\tilde{j} := (q_0 j_0, q_\infty j_\infty)$. *Then*

$$\Gamma^\pm(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))\tilde{W}^\pm(j) = \tilde{W}^\pm(\tilde{j}).$$

(iv) *Let* $q \in C^\infty(\mathbb{R}^3)$, $\nabla q \in C_0^\infty(\mathbb{R}^3)$, $0 \leqslant q \leqslant 1$, $q = 1$ *near* 0. *Then*

$$\tilde{W}^\pm(j)\Gamma^\pm(q) = \tilde{W}^\pm(qj), \quad \text{where } qj = (qj_0, qj_\infty).$$

(v) *Let* $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ *be another pair satisfying the conditions stated before the theorem. Then*

$$\tilde{W}^\pm(\tilde{j})^*\tilde{W}^\pm(j) = \Gamma(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty),$$

in particular if $j_0^2 + j_\infty^2 = 1$, *then* $\tilde{W}^\pm(j)$ *is isometric.*

(vi) *Let* $j_0 + j_\infty = 1$. *If* $\chi \in C_0^\infty(\mathbb{R})$, *then*

$$\hat{\Omega}^{\mathrm{ext},\pm}\chi(\hat{H}_\infty^{\mathrm{ext}})\tilde{W}^\pm(j) = \chi(\hat{H}_\infty).$$

Proof. To prove (i) we use the same arguments as in Theorem 8.1. Using Lemma A.4, it is enough to prove the existence of the limit

$$s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty^{\mathrm{ext}}}\chi(\hat{H}_\infty^{\mathrm{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\chi(\hat{H}_\infty), \tag{8.5}$$

for some $\chi \in C_0^\infty(\mathbb{R})$. We compute

$$\begin{aligned}
&\partial_t(\mathrm{e}^{\mathrm{i}t\hat{H}_\kappa^{\mathrm{ext}}}\chi(\hat{H}_\kappa^{\mathrm{ext}})I^*(j^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa})\\
&\quad= \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa^{\mathrm{ext}}}(\chi(\hat{H}_\kappa^{\mathrm{ext}})\mathfrak{D}_0 I^*(j^t)\chi(\hat{H}_\kappa)+\\
&\qquad+\mathrm{i}\chi(\hat{H}_\kappa^{\mathrm{ext}})(\hat{I}_\kappa\otimes\mathbb{1}I^*(j^t)-I^*(j^t)\hat{I}_\kappa)\chi(\hat{H}_\kappa))\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa},
\end{aligned}$$

where $\mathfrak{D}_0$ is the asymmetric Heisenberg derivative $\partial_t + \mathrm{i}H_0^{\mathrm{ext}}. - .\mathrm{i}H_0$. We have $\mathfrak{D}_0 I^*(j^t) = \mathrm{d}I^*(j^t, \mathfrak{d}_0 j^t)$.

Pseudodifferential calculus gives

$$\mathfrak{d}_0 j^t = \frac{1}{t}g^t + r^t,$$

$$g^t = (g_0^t, g_\infty^t), \qquad g_\epsilon^t = -\frac{1}{2}\left(\left(\frac{x}{t} - \partial\omega(k)\right)\partial j_\epsilon\left(\frac{x}{t}\right) + hc\right), \quad \epsilon = 0, \infty$$

with $r^t \in \mathrm{O}(t^{-2})$. Using Corollary 4.8 and (2.4) we obtain

$$\|\chi(\hat{H}_\kappa^{\mathrm{ext}})\mathrm{d}I^*(j^t, r^t)\chi(\hat{H}_\kappa)\| \in \mathrm{O}(t^{-2}). \tag{8.6}$$

Using now (2.5) with $u_i^t := \mathrm{e}^{\mathrm{i}t\hat{H}_\kappa}u_i$, one obtain

$$\begin{aligned}
&|(u_1^t|\chi(\hat{H}_\kappa^{\mathrm{ext}})\mathrm{d}I^*(j^t, g^t)\chi(\hat{H}_\kappa)u_2^t)|\\
&\quad\leqslant \|\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}}\otimes\mathbb{1}\chi(\hat{H}_\kappa^{\mathrm{ext}})u_2^t\|\ \|\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)u_1^t\| +\\
&\qquad+\|(\mathbb{1}\otimes\mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}})\chi(\hat{H}_\kappa^{\mathrm{ext}})u_2^t\|\ \|\mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}}\chi(\hat{H}_\kappa)u_1^t\|.
\end{aligned}$$

Then the κ-uniform integrability of the term $\chi(\hat{H}_\kappa^{\mathrm{ext}})\mathfrak{D}_0 I^*(j^t)\chi(\hat{H}_\kappa)$ follows using Proposition 7.1.

Using (4.25) we obtain uniformly in κ

$$\chi(\hat{H}_\kappa^{\mathrm{ext}})(\hat{I}_\kappa\otimes\mathbb{1}I^*(j^t) - I^*(j^t)\hat{I}_\kappa)\chi(\hat{H}_\kappa) \in \mathrm{O}(t^{-1-\mu}).$$

Then the existence of the limit in (i) follows.

(ii) follows by Lemma 4.15. (iii) follows using the fact that

$$\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)),$$

$$\Gamma(q_0^t)\otimes\Gamma(q_\infty^t)I^*(j^t) = I^*(\tilde{j}^t).$$

(iv) is true since

$$I^*(j^t)\Gamma(q^t) = I^*((jq)^t).$$

(v) is a consequence of the fact

$$I(\tilde{j}^t)I^*(j^t)\Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t).$$

(vi) One has

$$\hat{H}_\infty^{\text{ext}} \mathbb{1}_{[k,\infty[}(N_\infty) \geqslant mk + E_0.$$

Let $\chi \in C_0^\infty(\mathbb{R})$. There exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$\chi(\hat{H}_\infty^{\text{ext}})\mathbb{1}_{]n,\infty[}(N_\infty) = 0. \tag{8.7}$$

We have

$$\begin{aligned}
&\hat{\Omega}^{\text{ext},\pm}\chi(\hat{H}_\infty^{\text{ext}})\tilde{W}^\pm(j) \\
&\quad = \hat{\Omega}^{\text{ext},\pm}\mathbb{1}_{[0,n]}(N_\infty)\chi(\hat{H}_\infty^{\text{ext}})\tilde{W}^\pm(j) &(8.8)\\
&\quad = s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I \mathbb{1}_{[0,n]}(N_\infty)\chi(\hat{H}_\infty^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty} &(8.9)\\
&\quad = s\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}_\infty} I \mathbb{1}_{[0,n]}(N_\infty)I^*(j^t)\mathrm{e}^{-\mathrm{i}t\hat{H}_\infty}\chi(\hat{H}_\infty). &(8.10)
\end{aligned}$$

(8.8) follows from (8.7). (8.9) follows from the limit (i) and Theorem 6.10. Lemma 4.15 and the boundeness of the operator $I\mathbb{1}_{[0,n]}(N_\infty)(N_0+1)^{-\frac{n}{2}}$ gives (8.10). We use now an estimate proved in [DG3]:

$$\|I\mathbb{1}_{]n,\infty[}(N_\infty)I^*(j^t)(N+1)^{-1}\| \leqslant (n+1)^{-1}. \tag{8.11}$$

Since $II^*(j^t) = \mathbb{1}$, letting $n \to \infty$ we obtain $\hat{\Omega}^{\text{ext},\pm}\chi(\hat{H}_\infty^{\text{ext}})\tilde{W}^\pm(j) = \chi(\hat{H}_\infty)$. This completes the proof. □

THEOREM 8.4. *Let $j_n = (j_{0,n}, j_{\infty,n})$ be a sequence satisfying the hypothesis stated in the beginning of Theorem* 8.3 *such that $j_0 + j_\infty = 1$ and for any $\epsilon > 0$ there exists $m, \forall n > m,$ $\operatorname{supp} j_{0,n} \subset [-\epsilon, \epsilon]$. Then*

$$\hat{\Omega}^{\pm *} = w\text{-}\lim_{\kappa\to+\infty} \tilde{W}^\pm(j_n),$$

$$\hat{\mathcal{K}}^\pm = \operatorname{Ran} P_0^\pm.$$

Proof. Let $q \in C_0^\infty(\mathbb{R})$, $0 \leqslant q \leqslant 1$ and $q = 1$ in a neighborhood of zero such that $qj_{0,n} = j_{0,n}$ for n large enough. Using Theorem 8.3(iii) and Corollary 8.2 we obtain

$$\Gamma^\pm(q) \otimes \mathbb{1}\ \tilde{W}^\pm(j_n) = \tilde{W}^\pm(j_n),$$

$$w\text{-}\lim_{n\to+\infty} P_0^{\pm}\otimes \mathbb{1}\ \tilde{W}^{\pm}(j_n) - \tilde{W}^{\pm}(j_n) = 0. \tag{8.12}$$

Let $\chi \in C_0^{\infty}(\mathbb{R})$. We have

$$\begin{aligned}
\hat{\Omega}^{\pm *}\chi(\hat{H}_\infty) &= \hat{\Omega}^{\pm *}\hat{\Omega}^{\mathrm{ext},\pm}\chi(\hat{H}_\infty^{\mathrm{ext}})\tilde{W}^{\pm}(j_n) && (8.13)\\
&= w\text{-}\lim_{n\to\infty} \hat{\Omega}^{\pm *}\hat{\Omega}^{\mathrm{ext},\pm}\chi(\hat{H}_\infty^{\mathrm{ext}})P_0^{\pm}\otimes \mathbb{1}\ \tilde{W}^{\pm}(j_n) && (8.14)\\
&= w\text{-}\lim_{n\to\infty} P_0^{\pm}\otimes \mathbb{1}\chi(\hat{H}_\infty^{\mathrm{ext}})\tilde{W}^{\pm}(j_n) && (8.15)\\
&= w\text{-}\lim_{n\to\infty} \chi(\hat{H}_\infty^{\mathrm{ext}})\tilde{W}^{\pm}(j_n) && (8.16)\\
&= w\text{-}\lim_{n\to\infty} \tilde{W}^{\pm}(j_n)\chi(\hat{H}_\infty). && (8.17)
\end{aligned}$$

Formula (8.13) follows from Theorem 8.3(iv). (8.14) follows by (8.12). (8.15) is true since $P_0^{\pm}$ commutes with $\hat{H}_\infty^{\mathrm{ext}}$ and that $\mathrm{Ran} P_0^{\pm} \subset \hat{\mathcal{K}}^{\pm}$, $\hat{\Omega}^{\mathrm{ext},\pm}\mathbb{1}_{\hat{\mathcal{K}}^{\pm}}\otimes \mathbb{1} = \hat{\Omega}^{\pm}$ and $\hat{\Omega}^{\pm *}\hat{\Omega}^{\pm} = \mathbb{1}_{\hat{\mathcal{K}}^{\pm}}\otimes \mathbb{1}$. (8.16) follows from the fact that $P_0^{\pm}$ commutes with $\hat{H}_\infty^{\mathrm{ext}}$ and (8.12). (8.17) is Theorem 8.3(ii). So we conclude by a density argument that

$$\hat{\Omega}^{\pm *} = w\text{-}\lim_{n\to+\infty} \tilde{W}^{\pm}(j_n),$$

$$P_0^{\pm}\otimes \mathbb{1}\ \hat{\Omega}^{\pm *} = \hat{\Omega}^{\pm *}.$$

So we obtain

$$\mathrm{Ran}\,\hat{\Omega}^{\pm *} = \hat{\mathcal{K}}^{\pm}\otimes \Gamma(\mathfrak{h}) \subset \mathrm{Ran}\,P_0^{\pm}\otimes\Gamma(\mathfrak{h}) \subset \hat{\mathcal{K}}^{\pm}\otimes\Gamma(\mathfrak{h}).$$

Hence we prove that $\hat{\mathcal{K}}^{\pm} = \mathrm{Ran}\,P_0^{\pm}$. □

THEOREM 8.5. *We have*

$$\mathrm{Ran}\,\mathbb{1}_{\mathrm{pp}}(H) = \mathcal{K}^{\pm}.$$

Proof. By Proposition 6.8(iii) we have

$$\mathrm{Ran}\,\mathbb{1}_{\mathrm{pp}}(\hat{H}_\infty) \subset \hat{\mathcal{K}}^{\pm}.$$

Then it suffices to show that $\hat{\mathcal{K}}^{\pm} \subset \mathrm{Ran}\,\mathbb{1}_{\mathrm{pp}}(\hat{H}_\infty)$. Proposition 7.4 gives the existence of $\epsilon > 0$ and a sequence $\hat{H}_\kappa$ such that

$$\int_1^{+\infty} \|\Gamma(q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u\|^2\frac{\mathrm{d}t}{t} \leqslant C\|u\|^2,$$

where $\chi \in C_0^{\infty}(\mathbb{R}\setminus(\tau\cup\sigma_{\mathrm{pp}}(\hat{H}_\infty)))$ and $q \in C_0^{\infty}([-\epsilon,\epsilon])$, $q = 1$ for $|x| < \epsilon/2$. Theorem 8.1 gives that

$$\|\Gamma(q^t)\chi(\hat{H}_\kappa)\mathrm{e}^{-\mathrm{i}t\hat{H}_\kappa}u\| \to \|\Gamma^{\pm}(q)\chi(\hat{H}_\infty)u\| = 0, \quad t\to\pm\infty$$

then $\Gamma^{\pm}(q)\chi(\hat{H}_\infty) = 0$. So we have $\operatorname{Ran} P_0^{\pm} \subset \operatorname{Ran} \mathbb{1}_{\tau\cup\sigma_{pp}(\hat{H}_\infty)}(\hat{H}_\infty)$. Theorem 5.7 gives that τ is a closed countable set and $\sigma_{pp}(\hat{H}_\infty)$ can accumulate only at τ, so $\mathbb{1}_{pp}(\hat{H}_\infty) = \mathbb{1}_{\tau\cup\sigma_{pp}(\hat{H}_\infty)}(\hat{H}_\infty)$. This proves $\operatorname{Ran} \mathbb{1}_{pp}(\hat{H}_\infty) = \hat{\mathcal{K}}^{\pm}$. Then we prove the theorem. □

Appendix

The following theorem follows from the KLMN theorem and [RS, I–IV, Thm. VIII.25].

THEOREM A.1. *Let H_0 be a positive self-adjoint operator on $\mathcal{H}$. Let for $\kappa \leqslant \infty$, B_κ be quadratic forms on $D(H_0^{\frac{1}{2}})$ such that*

$$|B_\kappa(\psi, \psi)| \leqslant a \,\|H_0^{\frac{1}{2}}\psi\|^2 + b\,\|\psi\|^2,$$

where $a < 1$ uniformly in κ and $B_\kappa \to B_\infty$ on $D(H_0^{\frac{1}{2}})$. Then

(i) *There exist for $\kappa \leqslant \infty$ self-adjoint operators H_κ with $D(H_\kappa) \subset D(H_0^{\frac{1}{2}})$ and*

$$(H_\kappa\psi, \psi) = B_\kappa(\psi, \psi) + (H_0^{\frac{1}{2}}\psi, H_0^{\frac{1}{2}}\psi), \quad \psi \in D(H_\kappa),$$

(ii) $\lim_{\kappa\to\infty} (z - H_\kappa)^{-1} = (z - H_\infty)^{-1}$,

(iii) $s\text{-}\lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}tH_\kappa} = \mathrm{e}^{-\mathrm{i}tH_\infty}$.

LEMMA A.2. *Let $F \in C^\infty(\mathbb{R})$, equal to 0 near the origin and bounded near ∞. we denote by F^R the derivative operator $F(|x|/R)$. We recall that* x *denote the nucleon position observable. One has uniformly in κ*

(i) $\|\langle \mathrm{x}\rangle^{-s} F^R v_{\kappa_0}\| \in \mathrm{O}(R^{-s})$,

(ii) $\|\langle \mathrm{x}\rangle^{-s}\omega(k)^{-\epsilon} F^R r_\kappa\| \in \mathrm{O}(R^{-s})$, $\quad \epsilon > 0$.

Proof. We have $\langle \mathrm{x}\rangle^{-s} v_{\kappa_0} \in H^s(\mathbb{R}^3, \mathcal{B}(\mathcal{K}))$. Since we have

$$\left\| R^s F\left(\frac{|D_k|}{R}\right)\langle \mathrm{x}\rangle^{-s} v_{\kappa_0}\right\| \leqslant c\||D_k|^s\langle \mathrm{x}\rangle^{-s} v_{\kappa_0}\|$$

we obtain (i). Let us prove (ii). Pseudodifferential calculus gives

$$\left\|\omega(k)^{-\epsilon} F\left(\frac{|D_k|}{R} \geqslant 1\right) r_\kappa\right\| \leqslant c\left\| F\left(\frac{|D_k|}{R} \geqslant \frac{1}{2}\right)\omega(k)^{-\epsilon} r_\kappa\right\| + \frac{c}{R^2}\|\omega(k)^{-\epsilon} r_\kappa\|.$$

Then we obtain (ii) using (i) for $\omega^{-\epsilon} r_\kappa$, which is $L^2(\mathbb{R}^3, \mathrm{d}k)$ uniformly in κ. □

Let $\mathcal{H}$ a Hilbert space. Let $\{H_\kappa\}$ be a sequence of self-adjoint operators on a common domain $\mathcal{D} \subset \mathcal{H}$. We suppose that $s\text{-}\lim_{\kappa\to\infty} \mathrm{e}^{-\mathrm{i}tH_\kappa} = \mathrm{e}^{-\mathrm{i}tH_\infty}$, where H_∞ is a self-adjoint operator. We have for $\chi \in C_0^\infty(\mathbb{R})$, using [RS, I–IV, Thm. VIII.20; I–IV, Thm. VIII.21]

$$s\text{-}\lim_{\kappa\to\infty} \chi(H_\kappa) = \chi(H_\infty).$$

LEMMA A.3. *Let $t \to B_t \in \mathcal{B}(\mathcal{H})$ and $\chi \in C_0^\infty(\mathbb{R})$ such that for $\kappa \leqslant \infty$*

$$\|B_t^* B_t\, \chi(H_\kappa)\| \leqslant c_t,$$

where c_t is κ-independent locally integrable function in t. If there exist a constant c independent of κ such that for $\kappa < \infty$

$$\int_1^\infty \|B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u\|^2\, \mathrm{d}t \leqslant c\, \|u\|^2, \quad u \in \mathcal{H},$$

then

$$\int_1^\infty \|B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u\|^2\, \mathrm{d}t \leqslant c\, \|u\|^2.$$

Proof. We have to prove, uniformly in T, that:

$$\int_1^T \|B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u\|^2\, \mathrm{d}t \leqslant c\, \|u\|^2. \tag{A.1}$$

We apply the dominated convergence theorem. It is enough to show that

$$\lim_{\kappa\to\infty} \|B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u\|^2 = \|B_t\, \chi(H_\infty)\mathrm{e}^{-\mathrm{i}tH_\infty} u\|^2,$$

since we have $\|B_t\, \chi(H_\kappa)\mathrm{e}^{-\mathrm{i}tH_\kappa} u\|^2 \leqslant c'\, \|B_t^* B_t\, \chi(H_\kappa)\|\, \|u\|^2$. It is easy to see that to prove (A.1) it suffices to show

$$w\text{-}\lim_{\kappa\to\infty} \chi(H_\kappa) B_t^* B_t \chi(H_\kappa) = \chi(H_\infty) B_t^* B_t \chi(H_\infty).$$

This follows from the hypothesis $\|B_t^* B_t \chi(H_\kappa)\| \leqslant c_t$ for $\kappa \leqslant \infty$. □

Let $\mathcal{H}_i$, $i = 1, 2$ be two Hilbert spaces. Let $H_{i,\kappa}$, $i = 1, 2$ be two sequences of self-adjoint operators on $\mathcal{H}_i$, such that $H_{i,\kappa}$ converge in the strong resolvent sense to $H_{i,\infty}$.

The Lemma A.4 follows from the proof of Lemma A.3 and [DG1, Lemma B.4.2].

LEMMA A.4. *Let $t \to C(t) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\chi \in C_0^\infty(\mathbb{R})$. We suppose that the asymmetric Heisenberg derivatives:*

$$D_\kappa C(t) := \partial_t C(t) + \mathrm{i}(H_{1,\kappa} C(t) - C(t) H_{2,\kappa}),$$

satisfies for $\kappa < \infty$, the following hypothesis:

(i) $D_\kappa C(t) = B(t) + R_\kappa(t)$.

(ii) $\|\chi(H_{2,\kappa})\, R_\kappa(t)\, \chi(H_{1,\kappa})\| \leqslant c\, t^{-1-\epsilon},\ \epsilon > 0,$ *uniformly for $\kappa < \infty$.*

(iii) $\|\chi(H_{2,\kappa}) B(t)\| \leqslant c_t,\ \|B(t)\, \chi(H_{1,\kappa})\| \leqslant c_t,$ *uniformly for $\kappa \leqslant \infty$, where c_t is κ-independent locally integrable function in t.*

(iv) $|(u_2|B(t)u_1)| \leqslant c \sum_{j=1}^{n} \|B_{2,j}(t)u_2\| \|B_{1,j}(t)u_1\|,$ *where*

$$\int_1^\infty \|B_{i,j}(t)\, \chi(H_{i,\infty})\mathrm{e}^{-\mathrm{i}tH_{i,\infty}} u\|^2\, \mathrm{d}t \leqslant c\, \|u\|^2, \quad i = 1, 2 \text{ and } j = 1 \dots n.$$

Then the following limit exists

$$s\text{-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_{2,\infty}} \chi(H_{2,\infty})\, C(t)\, \chi(H_{1,\infty}) \mathrm{e}^{-\mathrm{i}tH_{1,\infty}}.$$

Proof. Let $u \in \mathcal{H}_1$, $v \in \mathcal{H}_2$. We have:

$$\begin{aligned}
&(u|\mathrm{e}^{\mathrm{i}t_2 H_{2,\infty}} \chi(H_{2,\infty}) C(t_2) \chi(H_{1,\infty}) \mathrm{e}^{\mathrm{i}t_2 H_{1,\infty}}) - \\
&\quad - (u|\mathrm{e}^{\mathrm{i}t_1 H_{2,\infty}} \chi(H_{2,\infty}) C(t_1) \chi(H_{1,\infty}) \mathrm{e}^{-\mathrm{i}t_1 H_{1,\infty}} v) \\
&= \lim_{\kappa\to\infty} (u|\mathrm{e}^{\mathrm{i}t_2 H_{2,\kappa}} \chi(H_{2,\kappa}) C(t_2) \chi(H_{1,\kappa}) \mathrm{e}^{-\mathrm{i}t_2 H_{1,\kappa}} v) - \\
&\quad - (u|\mathrm{e}^{\mathrm{i}t_1 H_{2,\kappa}} \chi(H_{2,\kappa}) C(t_1) \chi(H_{1,\kappa}) \mathrm{e}^{-\mathrm{i}t_1 H_{1,\kappa}} v) \\
&= \lim_{\kappa\to\infty} \int_{t_1}^{t_2} (\mathrm{e}^{-\mathrm{i}tH_{2,\kappa}} u|\chi(H_{2,\kappa}) \mathbb{D}_\kappa C(t) \chi(H_{1,\kappa}) \mathrm{e}^{-\mathrm{i}tH_{1,\kappa}} v)\, \mathrm{d}t \\
&= \lim_{\kappa\to\infty} \int_{t_1}^{t_2} (\mathrm{e}^{-\mathrm{i}tH_{2,\kappa}} u|\chi(H_{2,\kappa}) (B(t) + R(t)) \chi(H_{1,\kappa}) \mathrm{e}^{-\mathrm{i}tH_{1,\kappa}} v)\, \mathrm{d}t.
\end{aligned}$$

Using Lebesgue dominated convergence theorem and, as in the proof of Lemma A.3, the fact that

$$\text{(iii)} \Rightarrow w\text{-}\lim_{\kappa\to\infty} \chi(H_{2,\kappa}) B(t) \chi(H_{1,\kappa}) = \chi(H_{2,\infty}) B(t) \chi(H_{1,\infty}),$$

we obtain

$$\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}tH_{2,\kappa}}u|\chi(H_{2,\kappa})B(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}tH_{1,\kappa}}v)\,\mathrm{d}t \tag{A.2}$$

$$= \int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}tH_{2,\infty}}u|\chi(H_{2,\infty})B(t)\chi(H_{1,\infty})\mathrm{e}^{-\mathrm{i}tH_{1,\infty}}v)\,\mathrm{d}t.$$

We have by (ii):

$$\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}tH_{2,\kappa}}u|\chi(H_{2,\kappa})R(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}tH_{1,\kappa}}v)\,\mathrm{d}t$$

$$\leqslant c\,t_1^{-\epsilon}\,\|v\|\|u\|, \quad \text{if } t_1 < t_2. \tag{A.3}$$

Using (iv) we obtain:

$$\lim_{\kappa\to\infty}\int_{t_1}^{t_2}(\mathrm{e}^{-\mathrm{i}tH_{2,\infty}}u|\chi(H_{2,\kappa})B(t)\chi(H_{1,\kappa})\mathrm{e}^{-\mathrm{i}tH_{1,\infty}}v)\,\mathrm{d}t \tag{A.4}$$

$$\leqslant c\sum_{j=1}^{n}\int_{t_1}^{t_2}\|B_{2,j}(t)\mathrm{e}^{-\mathrm{i}tH_{2,\infty}}u\|^2\,\mathrm{d}t \times \int_{t_1}^{t_2}\|B_{1,j}(t)\mathrm{e}^{-\mathrm{i}tH_{1,\infty}}u\|^2\,\mathrm{d}t$$

$$\leqslant c\,\|u\|\,\|v\|.$$

(A.2), (A.3) and (A.4) give the existence of the claimed limit. □

We recall here a convergence lemma of positive operators, see, e.g., [DG2, Lemma A.3]

LEMMA A.5. *Let Q_n be a sequence of commuting self-adjoint operators. If*

$$0 \leqslant Q_n \leqslant 1, \qquad Q_{n+1} \leqslant Q_n, \qquad Q_{n+1}Q_n = Q_{n+1}.$$

Then there exist Q a projection

$$Q = s\text{-}\lim_n Q_n.$$

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