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Scattering theory for a class of fermionic Pauli–Fierz models

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Abstract

The scattering theory for a class of fermionic Pauli–Fierz models is considered. We give a proof of the asymptotic completeness of the dynamics in the case of massive fermions. The result applied to the Hamiltonian of a quantized spin-$\frac{1}{2}$ Dirac particle interacting with an external field through a cutoff Yukawa interaction and to the Hamiltonian of a system of finitely many confined particles coupled to a fermionic field with a quadratic interaction.

Keywords: Quantum field theory; Fermion; Scattering theory; Asymptotic completeness

1. Introduction

In this paper we study the scattering theory for a class of fermionic Pauli–Fierz models. An example is an interacting spin–fermion model. The spin system describes a system of finitely many confined particles. Its Hamiltonian is a bounded from below self-adjoint operator K with compact resolvent acting on a Hilbert space $\mathscr{K}$. Let $\mathfrak{g}$ be a finite-dimensional Hilbert space describing internal degrees of freedom (e.g. spin) of a fermion field. We denote by $\mathfrak{h} := L^2(\mathbb{R}^d, dk) \otimes \mathfrak{g}$ the one-particle space of this fermion field. The state space of the fermionic system is the anti-symmetric Fock space $\Lambda(\mathfrak{h})$. Let ω be a positive, operator-valued function in $C(\mathbb{R}^d, \mathscr{B}(\mathfrak{g}))$, representing the dispersion relation of a single fermion. The free field Hamiltonian is

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given by the second quantization of ω,

$$\mathrm{d}\Gamma(\omega) = \int_{\mathbb{R}^d} b^*(k)\omega(k)b(k)\, dk.$$

We assume the interaction between the spin and the fermion system to be a $K^{\frac{1}{2}}$-bounded, even Wick polynomial P given by a family of kernels $w_{p,q}$ which are continuous linear maps from the Schwartz space $S(\mathbb{R}^{d(p+q)})$ into $\mathscr{B}(\mathscr{D}(K^{\frac{1}{2}}) \otimes \otimes^p \mathfrak{g}, \mathscr{K} \otimes \otimes^q \mathfrak{g})$,

$$P := \sum_{p,q \in \Xi} \int_{\mathbb{R}^{d(p+q)}} b^*(k_q)\dots b^*(k_1) w_{p,q}(k_1, \dots, k_q, k'_p, \dots, k'_1) \times b(k'_p)\dots b(k'_1) dk_1 \dots dk_q\, dk'_p \dots dk'_1. \tag{1.1}$$

Here b^*, b are the usual creation and annihilation operators representing the CAR on $\mathfrak{h}$, and Ξ is a finite subset of $\{(p,q) \in \mathbb{N}^2 \mid p+q \in 2\mathbb{N}\}$. We impose an ultraviolet cutoff on $w_{p,q}$ and require a smoothness condition on $w_{p,q}$ such that P is a symmetric $K^{\frac{1}{2}}$-bounded operator (see Section 2.2). We obtain such interaction by starting from a formal local and transition invariant interaction then introducing an ultraviolet and space cutoffs. The construction of the perturbed Hamiltonian is obvious in this case, and self-adjointness follows, for example, by the Kato–Rellich Theorem. The interacting Hamiltonian is given by

$$H := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + P, \quad \text{acting on } \mathscr{H} := \mathscr{K} \otimes \Lambda(\mathfrak{h}). \tag{1.2}$$

Let us mention some typical examples belonging to this class of models:

(i) The first is a quantized spin-$\frac{1}{2}$ Dirac particle interacting through a momentum cutoff Yukawa interaction with an external neutral scalar field [20]. Let $\mathscr{K} = \mathbb{C}$ and $\mathfrak{K} = L^2(\mathbb{R}^3, \mathbb{S})$ where $\mathbb{S}$ is a four-dimensional spinor space. Let $\alpha^1, \alpha^2, \alpha^3$ and β be the usual Dirac matrices [29]. We denote by $\mathfrak{D}$ the Dirac operator

$$\mathfrak{D} := -\mathrm{i}\alpha \cdot \nabla + \beta m.$$

Recall that $\mathfrak{K}$ decomposes into two parts $\mathfrak{K}_{\pm}$, which are the subspaces of positive and negative energy. We denote by $\mathbf{P}_{\pm}$ the corresponding projections on $\mathfrak{K}_{\pm}$. Let U_C be the unitary operator given by $\mathrm{i}\beta\alpha^2$ in the standard representation. The charge conjugation C is the operator acting on $L^2(\mathbb{R}^3, \mathbb{S})$ and defined by $C\psi := U_C\overline{\psi}$. C interchanges the subspaces $\mathfrak{K}_{\pm}$, and according to the Dirac theory the one-particle space decomposes as

$$\mathfrak{h} := \mathfrak{K}_+ \oplus \mathfrak{K}_+.$$

The Hilbert space of the quantized Dirac particle is the Fock space $\Lambda(\mathfrak{h})$. The gauge transformations $f \mapsto \mathrm{e}^{\mathrm{i}\theta} f$, which leave the Dirac equation invariant are implemented in the Fock space by $\mathrm{e}^{\mathrm{i}\theta Q}$, where $Q := \mathrm{d}\Gamma(\mathbb{1} \oplus -\mathbb{1})$ is the total charge operator. For

$v \in L^2(\mathbb{R}^3, \mathbb{S})$, we define the field operator to be the following bounded operator:

$$\Phi(v) := b(\mathbf{P}_+ v \oplus 0) + b^*(0 \oplus C\mathbf{P}_- v).$$

For $f \in L^2(\mathbb{R}^3)$ and $\mathrm{u}_j,\ j = 1, \dots, 4$, an orthonormal basis of $\mathbb{S}$, we set

$$\Phi(f(\cdot - x)) := \sum_{j=1}^{4} \mathrm{u}_j \otimes \Phi(f(\cdot - x)u_j), \quad \text{where } f(\cdot - x) := f(y - x) \in L^2(\mathbb{R}^3, dy).$$

We denote by $\mathfrak{D}_+ := \mathbf{P}_+ \mathfrak{D} \mathbf{P}_+$. For $g \in L^1(\mathbb{R}^3)$ a real-valued function, the Hamiltonian is given by

$$H_1 := \mathrm{d}\Gamma(\mathfrak{D}_+ \oplus \mathfrak{D}_+) + \int_{\mathbb{R}^3} : \Phi(f(\cdot - x))^* \beta \Phi(f(\cdot - x)) : g(x)\, dx,$$

where the interaction between dots :: denotes the Wick-ordered monomial with all b^* to the left and b to the right.

(ii) The second example is a system of a finitely many particles interacting with a quantized spin-$\frac{1}{2}$ Dirac particle. Consider the N-body Schrödinger operator on $\mathscr{K} = L^2(\mathbb{R}^{3P}, d\mathrm{x})$ given by

$$K := -\frac{1}{2m} \sum_{j=1}^{P} \Delta_{\mathrm{x}_j} + V(\mathrm{x}_1, \dots, \mathrm{x}_P),$$

with V such that $\lim_{|\mathrm{x}| \to \infty} V(\mathrm{x}) = \infty$. Let $f \in L^2(\mathbb{R}^3)$ and $\mathfrak{K}, \mathfrak{h}, \Phi$ be the same objects defined in the example (i). The interaction between the quantum system and the quantized Dirac field is given by

$$\mathbf{I}(\mathrm{x}_j) := \ : \Phi(f(\cdot - \mathrm{x}_j))^* \beta \Phi(f(\cdot - \mathrm{x}_j)) : .$$

The Hamiltonian of the interacting quantum system with the quantized Dirac particle is given by

$$H_2 := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\mathfrak{D}_+ \oplus \mathfrak{D}_+) + \sum_{j=1}^{P} \mathbf{I}(\mathrm{x}_j).$$

The main result of this work is the proof of existence of the wave operators and their completeness for the class of Hamiltonians defined by (1.1)–(1.2), including the examples (i)–(ii). This holds true under some hypotheses which are given in detail in Section 3. We briefly describe that hypotheses. Let $\mathscr{K}_1$, $\mathscr{K}_{1/2}$ denotes respectively, the Hilbert spaces $\mathscr{D}(K)$, $\mathscr{D}(K^{\frac{1}{2}})$ endowed with their corresponding graph norms and

$\mathscr{K}_1^*, \mathscr{K}_{1/2}^*$ their topological duals. For $\mathfrak{B}$ a Banach space we define the Schwartz norms on $S(\mathbb{R}^n, \mathfrak{B})$:

$$||f||_m := \sum_{|\alpha|+|\beta| \leqslant m} \sup_{x \in \mathbb{R}^n} ||x^\alpha D^\beta f||_{\mathfrak{B}}, \tag{1.3}$$

where α, β are multindices in $\mathbb{N}^n$ and $x^\alpha := (x_1^{\alpha_1}, \dots, x_n^{\alpha_n}), D^\beta := (\frac{1}{\mathrm{i}} \partial_{x_1}^{\beta_1}, \dots, \frac{1}{\mathrm{i}} \partial_{x_n}^{\beta_n})$. Let $S_m(\mathbb{R}^n, \mathfrak{B})$ be the Banach space defined by $||.||_m$. We set

$$\mathscr{B}_1 := \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K} \otimes \otimes^q \mathfrak{g}), \quad \overline{\mathscr{B}}_1 := \mathscr{B}(\mathscr{K} \otimes \otimes^p \mathfrak{g}, \mathscr{K}_1^* \otimes \otimes^q \mathfrak{g}),$$

and

$$\mathscr{B}_{1/2} := \mathscr{B}(\mathscr{K}_{1/2} \otimes \otimes^p \mathfrak{g}, \mathscr{K} \otimes \otimes^q \mathfrak{g}), \quad \overline{\mathscr{B}}_{1/2} := \mathscr{B}(\mathscr{K} \otimes \otimes^p \mathfrak{g}, \mathscr{K}_{1/2}^* \otimes \otimes^q \mathfrak{g}).$$

For $p, q \in \mathbb{N}$, $\varrho := [\frac{3}{2} d(p+q)] + 3$, we introduce the classes of symbols $\mathrm{S}_{p,q}^\varepsilon$, $\varepsilon = 1, 1/2$, to be the Banach spaces $S_\varrho(\mathbb{R}^{d(p+q)}, \mathscr{B}_\varepsilon \oplus \overline{\mathscr{B}}_\varepsilon)$.

Let G denote the conjugate operator introduced in Section 4.3 given by

$$G := \mathrm{d}\Gamma(-\frac{1}{2}(\nabla\omega(k).D_k + D_k.\nabla\omega(k))), \quad \text{acting on } \Lambda(\mathfrak{h}).$$

We assume the following:

- The particles system is confined, i.e:

$$(K + \mathrm{i})^{-1} \text{ is compact.} \quad (\mathscr{C})$$

- The fermion dispersion relation is smooth, massive, may have only one critical point $k = 0$ and moreover we assume that ω has a smooth diagonalization with eigenvalues $\lambda_1(k), \dots, \lambda_s(k)$ with constant uniform multiplicity for $k \in \mathbb{R}^d$, i.e:

$$\begin{cases} \partial_k^\beta \lambda(k) \in L^\infty(\mathbb{R}^d), \ |\beta| \geqslant 1, \\ \nabla\omega(k) \neq 0, \ \text{for } k \neq 0, \\ \lim\limits_{|k| \to \infty} ||\omega(k)||_{\mathscr{B}(\mathfrak{g})} = \infty, \qquad (\mathscr{M}) \\ \omega(k) \geqslant m \mathbb{1}_{\mathfrak{g}}, m > 0, \\ \inf\limits_{k \in \mathbb{R}^d, i \neq j} |\lambda_i(k) - \lambda_j(k)| > 0. \end{cases}$$

- The interaction is regular and of short-range type, i.e:

$$\begin{cases} w_{p,q} \in \mathrm{S}^{1/2}_{p,q}, \quad (\mathscr{R}_0) \\ \sum_{p,q\in\Xi} |||[w_{p,q}, G]|||_{\mathrm{S}^{1/2}_{p,q}} < \infty. \quad (\mathscr{R}_1) \\ \sum_{p,q\in\Xi} \sum_{i=1}^{p+q} ||\mathbb{1}_{[R,\infty[}(|\nabla_{k_i}|)\, w_{p,q}||_{\mathrm{S}^1_{p,q}} \leqslant CR^{-\mu}, \mu > 1. \quad (\mathscr{S}) \end{cases}$$

We now state our main result, Theorem 1.2. The following proposition is proved in Proposition 5.1.

Proposition 1.1. *Assume* $(\mathscr{M}), (\mathscr{R}_0)$, *and* $(\mathscr{S})$ *hold. Then the following strong limits*:

$$b^{\pm}(h) := s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} b(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}, \quad h\in\mathfrak{h}.$$

exist.

We define the space of asymptotic *vacua*

$$\mathscr{K}^{\pm} := \{\Psi\in\mathscr{H} \mid b^{\pm}(h)\Psi = 0, h\in\mathfrak{h}\}.$$

Let define the wave operators by

$$\Omega^{\pm} : \mathscr{H}\otimes\Lambda(\mathfrak{h})\to\mathscr{H},$$

$$\psi\otimes\Pi_{i=1}^{n} b^*(f_i)\Omega \mapsto \Pi_{i=1}^{n} b^{\pm}(f_i)^*\psi, \quad f_i\in\mathfrak{h},$$

where Ω is the vacuum vector in the Fock space $\Lambda(\mathfrak{h})$. We denote by $\mathscr{H}_{\mathrm{bd}}(H)$ the space of bound states of H.

Theorem 1.2. *Assume that* $(\mathscr{C}), (\mathscr{M}), (\mathscr{R}_{0,1})$, *and* $(\mathscr{S})$ *hold. Then* $\Omega^{\pm}$ *is unitary and the* asymptotic completeness *holds*, *i.e*:

$$\mathscr{H}^{\pm} := \mathscr{K}^{\pm}\otimes\Lambda(\mathfrak{h}) = \mathscr{H}, \quad \text{and} \quad \mathscr{K}^{\pm} = \mathscr{H}_{\mathrm{bd}}(H).$$

To prove the completeness of the wave operator, we mainly use the method developed in the work [11]. The main tool is the construction of the asymptotic velocity using propagation estimates and taking advantage of the structure of Fock space. This allows the identification of asymptotic vacua as states with zero asymptotic velocity and then as bound states using a Mourre theory. Let us mention that asymptotic completeness holds for certain QED models such as almost-solvable massless spin boson [28], massive spin boson [11], space cutoff $P(\varphi)_2$ [12], ultraviolet renormalized Nelson model [1], Rayleigh scattering [14], Compton scattering [13].

The particularity of our example is that the interaction is of arbitrary higher order on Wick monomials with kernels taking into account the degrees of freedom of the spin system. To deal with such interaction we use an abstract fermionic Wick formalism and study commutation relations, estimates on commutators and introduce Wick tensor products and their properties, see Section 2.2. This allows to study the scattering theory with the same strategy as in [11], and similar to the N-body Schrödinger Hamiltonians. Note that the class of models considered in this paper has features such that it is an interesting example in the study of dynamical stability of zero temperature KMS states and where asymptotic completeness could apply. Moreover, it serves as a non-trivial example for the non-existence of the Møller morphisms [2].

The paper is organized as follows. In Section 2 we introduce notation and recall some related material to the fermionic Fock space. Essentially, we study the fermionic exponential law and construct the scattering identification operator I, as well as its right inverse. We introduce Wick polynomials in Section 2.2 and establish some commutator relations and estimates. We state the hypotheses and the main theorem in Section 3. We prove a HVZ-type theorem and a Mourre estimate in Section 4. Section 5 is devoted to the proof of asymptotic completeness. We construct the wave operator in Section 5.1. We establish in Section 5.2 some propagation estimates among them a minimal velocity estimate. In Section 5.3 we construct the asymptotic velocity and finish the proof of the main theorem Theorem 3.2.

2. Basic theory

In this section we introduce the notation which will be used in the sequel. We recall the definitions of some operators acting on the fermionic Fock space, especially those related to the study of the scattering theory. Among this operators we cite the *scattering identification operator*, introduced in [11,23] and which play a fundamental role in the proof of asymptotic completeness. In Section 2.2, we follow the formalism in [12], defining Wick polynomials in the fermionic case. Furthermore, we derive some commutation relations and estimates satisfied by Wick polynomials.

2.1. Fermionic Fock spaces

Let $\mathfrak{h}$ be a Hilbert space. Let $\Lambda^n(\mathfrak{h})$ be the anti-symmetric n-fold tensor product of $\mathfrak{h}$. The *fermionic Fock space* over $\mathfrak{h}$ is

$$\Lambda(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Lambda^n(\mathfrak{h}),$$

where $\Lambda^0(\mathfrak{h}) := \mathbb{C}$. The *vacuum vector* $(1, 0, \dots)$ will be denoted by Ω. Let $\mathfrak{h}_0$ be a subspace of $\mathfrak{h}$, we denote by $\otimes^n_{\mathrm{Alg}} \mathfrak{h}_0$ the algebraic n-fold tensor product of $\mathfrak{h}_0$, and by $\Lambda^n_{\mathrm{Alg}}(\mathfrak{h}_0)$ the space of anti-symmetric vectors in $\otimes^n_{\mathrm{Alg}} \mathfrak{h}_0$. We set $\Lambda^{\mathrm{fin}}(\mathfrak{h}_0)$ to be the vector space generated by the union of $\Lambda^n_{\mathrm{Alg}}(\mathfrak{h}_0)$, $n \in \mathbb{N}$.

Let $S_n \ni \sigma \mapsto \pi_\sigma \in \mathscr{U}(\otimes^n \mathfrak{h})$ be the unitary representation of the permutation group S_n defined by

$$\pi_\sigma f_1 \otimes \cdots \otimes f_n = f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n}, \quad \text{where } f_i \in \mathfrak{h}, \text{ for } i = 1, \dots, n.$$

Let $\bigwedge_n$ be the orthogonal projection from $\otimes^n \mathfrak{h}$ into $\Lambda^n(\mathfrak{h})$. It acts as follows:

$$\bigwedge_n := \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \pi_\sigma,$$

where $\varepsilon(\sigma)$ is the signature of σ. For $\psi \in \Lambda^n(\mathfrak{h}), \phi \in \Lambda^m(\mathfrak{h})$, we set

$$\psi \wedge \phi := \bigwedge_{n+m} \psi \otimes \phi.$$

We have $\psi \wedge \phi = (-1)^{nm} \phi \wedge \psi$ and $\pi_\sigma \psi \wedge \phi = \varepsilon(\sigma) \psi \wedge \phi$, $\sigma \in S_{n+m}$. We denote by $\bigwedge$ the orthogonal projection given by $\sum_{n \geqslant 0} \bigwedge_n$ and we consider $\Lambda^n(\mathfrak{h})$ as a subspace of $\Lambda(\mathfrak{h})$.

Let I be a set of indices. By means of Cantor's well-ordering principal we equip I with a total order. If $J \subset I$ is finite, then J is an increasing sequence $j_s, s = 1, \dots, \sharp J$. We set for $(f_j)_{j \in J} \subset \mathfrak{h}$:

$$\wedge_{j \in J} f_j := \bigwedge \otimes_{j \in J} f_j,$$

where $\otimes_{j \in J} f_j = f_{j_1} \otimes \dots \otimes f_{j_{\sharp J}}$ written in the increasing order w.r.t. the order relation on J. If $A_i \in \mathscr{B}(\Lambda^{n_i}(\mathfrak{h}))$, $i = 1, 2$ we set:

$$A_1 \wedge A_2 := \bigwedge A_1 \otimes A_2 \in \mathscr{B}(\Lambda^{n_1}(\mathfrak{h}) \otimes \Lambda^{n_2}(\mathfrak{h}), \Lambda^{n_1+n_2}(\mathfrak{h})).$$

dΓ *operators*. We define the second quantization of a one-particle operator A denoted by $\mathrm{d}\Gamma(A)$ as

$$\mathrm{d}\Gamma(A)_{|\Lambda^n(\mathfrak{h})} := \sum_{j=1}^n A_j,$$

where A_j is the operator acting as A in the jth component and as the identity in the others. An example is the number operator $N = \mathrm{d}\Gamma(\mathbb{1})$. We mention that in general $\mathrm{d}\Gamma(A)$ is an unbounded operator even if A is bounded and it is bounded if and only if A is trace class.

2.1.1. Creation–annihilation operators

We recall respectively the creation and annihilation operators:

$$b^*(h)\psi = \sqrt{N}\, h \wedge \psi,$$

$$b(h)\psi = \sqrt{N+1}(h|\psi), \quad \psi \in \Lambda(\mathfrak{h}).$$

We will use the notation $b^\sharp(.)$ to simplify the writing of statements which hold for both $b(.)$ and $b^*(.)$. Note that $b^\sharp(h)$ is bounded operator with $||b^\sharp(h)|| = ||h||$, satisfying the canonical anti-commutation relations:

$$\{b^\sharp(h), b^\sharp(g)\} = 0, \tag{2.1}$$

$$\{b(h), b^*(g)\} = (h|g)\mathbb{1}. \tag{2.2}$$

Let J be a totally ordered finite set of indices we set $\prod_{j\in J} b^\sharp(h_j)$, $\prod_{J\ni j} b^\sharp(h_j)$ to be respectively the increasing and the decreasing product of $b^\sharp(h_j), j\in J$, w.r.t. to the ordering on J. For any orthonormal basis $\{e_i\}_{i\in I}$ of $\mathfrak{h}$ and a total ordering in I the family $\{e_J | J \subset I, J \text{ finite}\}$ given by

$$\begin{aligned} e_J &:= \sqrt{\sharp J!} \wedge_{j\in J} e_j \\ &= \prod_{j\in J} b^*(e_j)\Omega, \end{aligned}$$

defines an orthonormal basis of $\Lambda(\mathfrak{h})$. In the same way $\prod_{J\ni j} b^*(e_j)\Omega$ defines an orthonormal basis which we obtain by the above procedure by inversion of the total order.

Let $\mathfrak{g}$ be a finite-dimensional Hilbert space describing some internal degrees of freedom of fermions. If $\mathfrak{h} := L^2(\mathbb{R}^d, dk)\otimes \mathfrak{g}$, then any vector in $\Lambda^n(\mathfrak{h})$ can be considered as a function in $L^2(\mathbb{R}^{nd}, \otimes^n \mathfrak{g})$. Let $\alpha_\lambda, \lambda = 1, \ldots, \ell := \dim(\mathfrak{g})$ be an orthonormal basis of $\mathfrak{g}$. Let $\Psi, \Phi \in \Lambda(\mathfrak{h})$, we denote by

$$\Psi_\lambda^{(n)}(k,.) = \langle \alpha_\lambda | \otimes \mathbb{1}_{\otimes^{n-1}\mathfrak{h}} \Psi^{(n)},$$

the $L^2(\mathbb{R}^d)$ function with values on $\otimes^{n-1}\mathfrak{h}$. The map $\mathfrak{h}\times\Lambda(\mathfrak{h}) \ni (h,\Phi) \mapsto (b(h)\Psi|\Phi)$ is a continuous bilinear form and hence there exist a unique vector in $\mathfrak{h}\otimes\Lambda(\mathfrak{h})$ which we denote by $b(k)\Psi$ such that

$$(b(h)\Psi|\Phi) = \int_{\mathbb{R}^d} (b(k)\Psi|h(k)\otimes\Phi)_{\mathfrak{g}\otimes\Lambda(\mathfrak{h})}\, dk.$$

In fact, $b(k)$ is an operator-valued function in $L^2(\mathbb{R}^d, dk)$ acting from $\Lambda(\mathfrak{h})$ into $\mathfrak{g}\otimes\Lambda(\mathfrak{h})$. Moreover, one can define the annihilation–creation distributions in this case by

$$(b_\lambda(k)\Psi)^{(n)}(k_1,\dots,k_n) := (n+1)^{\frac{1}{2}}\,\Psi_\lambda^{(n+1)}(k,\dots,k_n)\in\otimes^n\mathfrak{g},$$

$$(b_\lambda^*(k)\Psi)^{(n)}(k_1,\dots,k_n) := n^{-\frac{1}{2}}\sum_{j=1}^{n}(-1)^{j-1}\delta(k-k_j)\Psi_\lambda^{(n-1)}(k_1,\dots,\widehat{k_j},\dots,k_n)\in\otimes^n\mathfrak{g},$$

where $\widehat{k_j}$ means that k_j is omitted and δ is the Dirac function. We have

$$b(k)\Psi=\sum_{\lambda=1}^{\ell}\alpha_\lambda\otimes b_\lambda(k)\Psi,\quad \Psi\in\Lambda(\mathfrak{h}).$$

We retrieve the annihilation and creation operators $b(h), b^*(h), h\in\mathfrak{h}$ using the integral representations:

$$b(h):=\sum_{\lambda=1}^{\ell}\int_{\mathbb{R}^d} b_\lambda(k)\,\overline{h_\lambda(k)}\,dk,\quad b^*(h):=\sum_{\lambda=1}^{\ell}\int_{\mathbb{R}^d} b_\lambda^*(k)\,h_\lambda(k)\,dk.$$

In the same way if $f\in L^2(\mathbb{R}^d)$, the map $\mathfrak{g}\times\mathfrak{h}\ni(g,\Phi)\mapsto(b(f\otimes g)\Psi|\Phi)$ is a continuous bilinear form and we define $b(f)\Psi$ to be the vector satisfying

$$(b(f)\Psi|g\otimes\Phi)=(b(f\otimes g)\Psi|\Phi).$$

The operator $b(f)$ is bounded from $\Lambda(\mathfrak{h})$ into $\mathfrak{g}\otimes\Lambda(\mathfrak{h})$ and furthermore we have

$$b(f)=\sum_{\lambda=1}^{\ell}\alpha_\lambda\otimes b(f\otimes\alpha_\lambda)=\int_{\mathbb{R}^d} b(k)\overline{f(k)}\,dk.$$

Let $\mathscr{K}_i,\ i=1,2$ be two Hilbert spaces describing a given system of particles. Let $\mathscr{H}_i:=\mathscr{K}_i\otimes\Lambda(\mathfrak{h}),\ i=1,2$ be the Hilbert space of the joint system. We can extend $b^\sharp(h)$ to an operator acting from $\mathscr{H}_1$ into $\mathscr{H}_2$. Let $v\in\mathscr{B}(\mathscr{K}_1,\mathscr{K}_2\otimes\mathfrak{h})$, we define

$$b^*(v)\phi\otimes\psi:=\sqrt{N}\,v\phi\wedge\psi,$$

$$b(v)\phi\otimes\psi:=\sqrt{N+1}\bigwedge v^*\phi\otimes\psi,\quad \psi\in\Lambda(\mathfrak{h}),\ \phi\in\mathscr{K}_1.$$

Note that in the definition of $b(v)$, v^* acts only in the first component $\mathscr{K}_1\otimes\mathfrak{h}$ and in the rest as the identity. In the case $\mathfrak{h}:=L^2(\mathbb{R}^d,dk)\otimes\mathfrak{g}$, and $\mathscr{K}_1,\mathscr{K}_2$ separable we can represent $v\in\mathscr{B}(\mathscr{K}_1,\mathscr{K}_2\otimes\mathfrak{h})$ as a function $\mathbb{R}^d\ni k\mapsto v(k)\in\mathscr{B}(\mathscr{K}_1,\mathscr{K}_2\otimes\mathfrak{g})$ defined almost every where such that

$$v(k)\phi = (v\phi)(k), \quad \text{for all } \phi \in \mathscr{K}_1, \text{ and } k \text{ a.e. in } \mathbb{R}^d,$$

$$\mathscr{K}_1 \times \mathscr{K}_1 \ni (\phi_1, \phi_2) \mapsto \int_{\mathbb{R}^d} (v(k)^* v(k)\phi_1 | \phi_2)_{\mathscr{K}_1} \, dk = (v\phi_1 | v\phi_2)_{\mathscr{K}_2 \otimes \mathfrak{h}},$$

is a continuous quadratic form. We have the following anti-commutation relations for all $v, w \in \mathscr{B}(\mathscr{K}_1, \mathscr{K}_1 \otimes \mathfrak{h})$ such that $[v^*(k), w(k')] = 0, v(k) \otimes \mathbb{1}_{\mathfrak{g}} w(k') = w(k') \otimes \mathbb{1}_{\mathfrak{g}} v(k)$, for a.e. k, k' in $\mathbb{R}^d$:

$$\{b^\sharp(v), b^\sharp(w)\} = 0,$$

$$\{b(v), b^*(w)\} = v^* w \otimes \mathbb{1}_{\Lambda(\mathfrak{h})}.$$

2.1.2. Other operators

Let B be an operator acting on $\mathfrak{h}$. The operator $\Gamma(B)$ is defined by

$$\Gamma(B) : \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$

$$\Gamma(B)_{|\Lambda^n(\mathfrak{h})} := B^{\otimes(n)},$$

with the notation $B^{\otimes(n)} := B \otimes \cdots \otimes B$, n times.

Let A, B be two operators on $\mathfrak{h}$. We define

$$\mathrm{d}\Gamma(A, B) : \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$

$$\mathrm{d}\Gamma(A, B)_{|\Lambda^n(\mathfrak{h})} := \sum_{j=1}^{n} A^{\otimes(j-1)} \otimes B \otimes A^{\otimes(n-j)}.$$

Note that $\Gamma(A)$ and $\mathrm{d}\Gamma(A, B)$ preserve the projection $\bigwedge$ and the number operator N. We recall a useful estimate, proved in [11, Lemma 2.8 (vi)] and which extend straightforward to the fermionic case.

Lemma 2.3. *Let A, B and C be three operators on $\mathfrak{h}$ such that $||A|| \leqslant 1$. Let $u, v \in \mathfrak{h}$, we have*

(i)

$$|(\mathrm{d}\Gamma(A, BC)u|v)| \leqslant || \, \mathrm{d}\Gamma(B^*B)^{1/2}v|| || \, \mathrm{d}\Gamma(C^*C)^{1/2}u||, \tag{2.3}$$

(ii)

$$||(N+1)^{-\frac{1}{2}} \, \mathrm{d}\Gamma(A, B)u|| \leqslant || \, \mathrm{d}\Gamma(B^*B)^{1/2}u||. \tag{2.4}$$

2.1.3. CAR representation

Let (L, S) be an orthogonal space (i.e: a real topological vector space endowed with a continuous positive definite symmetric bilinear form S on L). A CAR representation over (L, S) is a pair $(\mathscr{D}, \Phi_\pi)$ consisting of a Hilbert space $\mathscr{D}$ and a linear map $L \ni h \mapsto \Phi_\pi(h) \in \mathscr{B}(\mathscr{D})$ into self-adjoint bounded operators and satisfying

$$\{\Phi_\pi(h), \Phi_\pi(g)\} = S(h, g)\mathbb{1} \quad (\textit{Clifford relations}).$$

Assume that (L, S) is equipped with a complex structure consisting of an anti-involution $\mathscr{I} : L \to L, \mathscr{I}^2 = -\mathbb{1}$ compatible with the symmetric bilinear form S in the following sense:

(i)

$$S(h, \mathscr{I}g) + S(\mathscr{I}h, g) = 0.$$

This allows the construction of creation–annihilation operators

$$B^*_\pi(h) := \frac{1}{\sqrt{2}}(\Phi_\pi(h) - \mathrm{i}\Phi_\pi(\mathscr{I}h)),$$

$$B_\pi(h) := \frac{1}{\sqrt{2}}(\Phi_\pi(h) + \mathrm{i}\Phi_\pi(\mathscr{I}h)).$$

Moreover, $B_\pi(h), B^*_\pi(h)$ satisfy canonical anti-commutation relations as in (2.1)–(2.2), with the complex structure $\mathrm{i}h := \mathscr{I}h$ and the inner product $(h|g) := S(h, g) + \mathrm{i}S(h, \mathscr{I}g)$. For more details, see [7,8]. Note that a Hilbert space endowed with the bilinear form $\mathrm{Re}(.|.)$ and the anti-involution i is an orthogonal space with a compatible complex structure.

A vector Ω is called a vacua for a CAR representation $(\mathscr{D}, \Phi_\pi)$ over a Hilbert space $\mathfrak{H}$ iff $\Omega \in \mathscr{D}$ and satisfies $B_\pi(h)\Omega = 0, \forall h \in \mathfrak{H}$.

Lemma 2.4. *Let $(\mathscr{D}_i, \Phi_{\pi_i})$, $i = 1, 2$ be two CAR representations over a Hilbert space $\mathfrak{H}$ and $\Omega_i \in \mathscr{D}_i$ be cyclic vacua for Φ_{π_i}. Then the map*

$$U : \mathscr{D}_1 \to \mathscr{D}_2,$$

$$U \prod_{j=1}^{n} B^*_{\pi_1}(h_j)\Omega_1 = \prod_{j=1}^{n} B^*_{\pi_2}(h_j)\Omega_2$$

extends as a unitary map from $\mathscr{D}_1$ to $\mathscr{D}_2$.

Proof. The lemma asserts that $(\mathscr{D}_i, \Phi_{\pi_i})$ are unitary equivalent which follows by computing scalar products using the CAR relations preserved by U. $\square$

The Fermionic exponential law.[1] Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Hilbert spaces. We define the two following maps labelled by l, r referring to left/right:

$$U_{\mathrm{l/r}} : \Lambda(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Lambda(\mathfrak{h}_1) \otimes \Lambda(\mathfrak{h}_2),$$

such that

$$\begin{aligned} U_{\mathrm{l/r}}\, \Omega &:= \Omega \otimes \Omega, \\ U_{\mathrm{l}}\, b^{\sharp}(h_1 \oplus h_2) &:= (b^{\sharp}(h_1) \otimes \mathbb{1} + (-\mathbb{1})^{N} \otimes b^{\sharp}(h_2))\, U_{\mathrm{l}}, \\ U_{\mathrm{r}}\, b^{\sharp}(h_1 \oplus h_2) &:= (b^{\sharp}(h_1) \otimes (-\mathbb{1})^{N} + \mathbb{1} \otimes b^{\sharp}(h_2))\, U_{\mathrm{r}}, \end{aligned} \tag{2.5}$$

where $h_i \in \mathfrak{h}_i,\ i = 1,2$. Clearly, $U_{\mathrm{l/r}}$ extend to unitary maps using Lemma 2.4. Let $p_i,\ i = 1,2$ be the projection from $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ into $\mathfrak{h}_i$ and i_i be the canonical injection of $\mathfrak{h}_i$ into $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. We set

$$N_1 := N \otimes \mathbb{1}, \quad N_2 := \mathbb{1} \otimes N, \ \text{acting on } \Lambda(\mathfrak{h}_1) \otimes \Lambda(\mathfrak{h}_2).$$

Lemma 2.5. *Let $b_i \in B(\mathfrak{h}_i),\ i = 1,2$. We have*

$$U_{\mathrm{r}} = U_{\mathrm{l}}(-1)^{\mathrm{d}\Gamma(p_1)\mathrm{d}\Gamma(p_2)} = (-1)^{N_1 N_2} U_{\mathrm{l}}. \tag{2.6}$$

$$U_{\mathrm{l}|\Lambda^n \mathfrak{h}_1 \oplus \mathfrak{h}_2} = \sum_{m=0}^{n} \sqrt{\frac{n!}{(n-m)!m!}}\, p_1^{\otimes m} \otimes p_2^{\otimes n-m}. \tag{2.7}$$

$$\begin{aligned} U_{\mathrm{l}}^* u_1 \otimes u_2 &= \sqrt{\frac{(m+k)!}{m!k!}}\, \Gamma(i_1)u_1 \wedge \Gamma(i_2)u_2, \\ U_{\mathrm{r}}^* u_1 \otimes u_2 &= \sqrt{\frac{(m+k)!}{m!k!}}\, \Gamma(i_2)u_2 \wedge \Gamma(i_1)u_1, \quad \textit{for } u_1 \in \Lambda^m(\mathfrak{h}_1), u_2 \in \Lambda^k(\mathfrak{h}_2). \end{aligned} \tag{2.8}$$

$$U_{\mathrm{l/r}}\, \mathrm{d}\Gamma\left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right) = (\mathrm{d}\Gamma(b_1) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(b_2))\, U_{\mathrm{l/r}}. \tag{2.9}$$

Proof. To prove Eqs. (2.6)–(2.8) it is enough to check them in a basis of $\Lambda(h_1 \oplus \mathfrak{h}_2)$. Let $\{e_i\}_{i \in I_1}$ (resp. $\{f_i\}_{i \in I_2}$) be an orthonormal basis of $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$) with a total order on I_1, I_2. Then $\{e_i \oplus 0\}_{i \in I_1} \cup \{0 \oplus f_j\}_{j \in I_2}$ is an orthonormal basis of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ which we consider as indexed by the set $I = I_1 \cup I_2$ endowed with the total order relation induced by those of I_1, I_2 and $i \leqslant j, \forall i \in I_1, j \in I_2$. We set $e_j^0 := e_j \oplus 0,\ f_j^0 := 0 \oplus f_j$.

[1] I thank C. Gérard for pointing out to me the right map U_{r} and its properties [16].

Hence we get two bases of $\Lambda(h_1 \oplus \mathfrak{h}_2)$ given by

$$\mathfrak{L}_{J_1,J_2} := \wedge_{j\in J_1} e_j^0 \wedge \wedge_{j\in J_2} f_j^0,$$

and

$$\mathfrak{R}_{J_1,J_2} := \wedge_{j\in J_2} f_j^0 \wedge \wedge_{j\in J_1} e_j^0,$$

where $J_1 \subset I_1, J_2 \subset I_2$, finite. Furthermore, using the CAR we have

$$\mathfrak{R}_{J_1,J_2} = (-1)^{\sharp J_1 \sharp J_2} \mathfrak{L}_{J_1,J_2}.$$

Now using the above identity and (2.5) we obtain

$$U_\mathrm{l} \mathfrak{L}_{J_1,J_2} = \wedge_{j\in J_1} e_j \otimes \wedge_{j\in J_2} f_j = U_\mathrm{r} \mathfrak{R}_{J_1,J_2}. \tag{2.10}$$

This proves (2.6). Moreover if $\sharp J_1 = k, \sharp J_2 = m$, Eq. (2.10) leads to

$$\begin{aligned} U_\mathrm{l} \mathfrak{L}_{J_1,J_2} &= \bigwedge_k \left(\otimes_{j\in J_1} e_j\right) \otimes \bigwedge_m \left(\otimes_{j\in J_2} f_j\right) \\ &= \sqrt{\frac{(k+m)!}{k!m!}} p_1^{\otimes k} \otimes p_2^{\otimes m} \bigwedge_{k+m} \otimes_{j\in J_1} e_j^0 \otimes \otimes_{j\in J_2} f_j^0. \end{aligned} \tag{2.11}$$

One can write $\bigwedge_{k+m} = \frac{1}{(k+m)!} \sum_{\sigma\in S_{k+m}} \pi_\sigma$, and hence the r.h.s. of (2.11) follows by noticing that only the $\sigma \in S_k \otimes S_m$ have non-zero contribution. This proves the first identity of (2.8). The second holds using the first and (2.6).

Now, to prove Eq. (2.9) it suffices to show it for a rank one operator. Let $b_1 = |h_1\rangle\langle h_2|, b_2 = |g_1\rangle\langle g_2|$, then using (2.5) and the fact that $\mathrm{d}\Gamma(b_1) = b^*(h_1)b(h_2), \mathrm{d}\Gamma(b_2) = b^*(g_1)b(g_2)$, we see that (2.9) holds true. □

2.1.4. Scattering identification operator

Let i be the map defined by

$$i : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},$$

$$(h_0, h_\infty) \mapsto h_0 + h_\infty.$$

Set

$$N_\infty := \mathbb{1} \otimes N \text{ and } N_0 := N \otimes \mathbb{1} \text{ acting on } \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h}).$$

We define the scattering identification operator I:

$$I : \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$

$$I := \Gamma(i) U_\mathrm{l}^* \mathscr{J} = \Gamma(i) U_\mathrm{r}^*, \quad \mathscr{J} := (-\mathbb{1})^{N_0 N_\infty} \otimes \mathbb{1}, \text{ acting on } \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h}).$$

We have the following formula:

$$I\prod_{i=1}^{n} b^*(h_i)\Omega\otimes\prod_{j=1}^{p} b^*(g_j)\Omega=\prod_{j=1}^{p} b^*(g_j)\prod_{i=1}^{n} b^*(h_i)\Omega.$$

Notice that I is an unbounded operator since it contains an operator $\Gamma(i)$ and $||i||=\sqrt{2}$. In the case $\mathfrak{h}:=L^2(\mathbb{R}^d,dk)\otimes\mathfrak{g}$, we can express I using the following integral formula:

$$Iu\otimes\psi=\frac{1}{\sqrt{n!}}\int_{\mathbb{R}^{dn}}\psi(k_1,\dots,k_n)\,b^*(k_1)\dots b^*(k_n)u\,dk_1\dots dk_n,\quad u\in\Lambda(\mathfrak{h}),\ \ \psi\in\Lambda^n_{\mathrm{Alg}}(\mathfrak{h}).$$

The linear map $I:u\mapsto Iu\otimes\psi$ is bounded on $\Lambda(\mathfrak{h})$ for all $\psi\in\Lambda^n_{\mathrm{Alg}}(\mathfrak{h})$ fixed. Moreover $I(N+1)^{-\frac{n}{2}}\otimes\mathbb{1}$ is bounded on $\Lambda(\mathfrak{h})\otimes\Lambda^n(\mathfrak{h})$.

We would construct a right inverse for the scattering identification operator. This is the subject of the following paragraph.

$\check{\Gamma}(j)$ *Operators*: Let $j:=(j_0,j_\infty)$ a map such that

$$j:\mathfrak{h}\to\mathfrak{h}\oplus\mathfrak{h},$$

$$h\mapsto(j_0h,j_\infty h).$$

We define the operators $\check{\Gamma}_{1/\mathrm{r}}(j)$ by

$$\check{\Gamma}_{1/\mathrm{r}}(j):\Lambda(\mathfrak{h})\to\Lambda(\mathfrak{h})\otimes\Lambda(\mathfrak{h}),$$

$$\check{\Gamma}_{1/\mathrm{r}}(j):=U_{1/\mathrm{r}}\Gamma(j).$$

Lemma 2.6. *We have*:

(i)
$$\check{\Gamma}_1(j)\prod_{i=1}^{n} b^*(h_i)\Omega=\prod_{i=1}^{n}(b^*(j_0h_i)\otimes\mathbb{1}+(-1)^N\otimes b^*(j_\infty h_i))\Omega\otimes\Omega,$$

$$\check{\Gamma}_{\mathrm{r}}(j)\prod_{i=1}^{n} b^*(h_i)\Omega=\prod_{i=1}^{n}(b^*(j_0h_i)\otimes(-1)^N+\mathbb{1}\otimes b^*(j_\infty h_i))\Omega\otimes\Omega.$$

(ii)
$$\mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}_{\mathrm{r}}(j)_{|\Lambda^n(\mathfrak{h})}=(-1)^{k(n-k)}\sqrt{\frac{n!}{(n-k)!k!}}j_0^{\otimes(n-k)}\otimes j_\infty^{\otimes k},$$

$$\mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}_1(j)_{|\Lambda^n(\mathfrak{h})}=\sqrt{\frac{n!}{(n-k)!k!}}j_0^{\otimes(n-k)}\otimes j_\infty^{\otimes k}.$$

(iii) *If* $j_0 + j_\infty = \mathbb{1}$,

$$I\check{\Gamma}_{\rm r}(j) = \mathbb{1}, \quad I\mathscr{J}\check{\Gamma}_{\rm l}(j) = \mathbb{1}.$$

(iv) $$I\mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}_{\rm r}(j)_{|\Lambda^n(\mathfrak{h})} = I\mathscr{J}\mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}_{\rm l}(j)_{|\Lambda^n(\mathfrak{h})} = \sum_{\sharp\{\varepsilon_i=\infty\}=k} j_{\varepsilon_1}\otimes\cdots\otimes j_{\varepsilon_n},$$

where the sum runs over the set of $\varepsilon \in \{1,\dots,n\}^{\{0,\infty\}}$ *such that* $\sharp\{\varepsilon_i = \infty\} = k$.

Proof. Points (i)–(iii) follow by simple computation and Lemma 2.5. Let us prove (iv). It is enough to check (iv) on a pure wedge product on $\Lambda^n(\mathfrak{h})$. Note that using (2.9) we get

$$U_{\rm r}^*\mathbb{1}_{\{k\}}(N_\infty)U_{\rm r} = \mathbb{1}_{\{k\}}\left({\rm d}\Gamma\left(\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}\end{bmatrix}\right)\right).$$

Let $h_i \in \mathfrak{h}$, $i = 1,\dots,n$, we have

$$\Gamma(j)\wedge_{i=1}^n h_i = \wedge_{i=1}^n (j_0 h_i \oplus j_\infty h_i) = \sqrt{n!}\sum_\varepsilon \prod_{i=1}^n A_{i,\varepsilon_i}\Omega,$$

where $\varepsilon \in \{1,\dots,n\}^{\{0,\infty\}}$ and

$$A_{i,\varepsilon_i} \coloneqq \begin{cases} b^*(j_0 h_i \oplus 0) & \textit{if } \ \varepsilon_i = 0, \\ b^*(0 \oplus j_\infty h_i) & \textit{if } \ \varepsilon_i = \infty. \end{cases}$$

Then it follows that we have

$$\mathbb{1}_{\{k\}}\left({\rm d}\Gamma\left(\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}\end{bmatrix}\right)\right)\Gamma(j)\bigwedge_{i=1}^n h_i = \sqrt{n!}\sum_{\sharp\{\varepsilon_i=\infty\}=k}\prod_{i=1}^n A_{i,\varepsilon_i}\Omega,$$

and hence

$$I\mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}_{\rm r}(j)\bigwedge_{i=1}^n h_i = \sqrt{n!}\sum_{\sharp\{\varepsilon_i=\infty\}=k}\prod_1^n b^*(j_{\varepsilon_i}h_i)\Omega,$$

which proves (iv). $\square$

Let $j = (j_0, j_\infty)$, $l = (l_0, l_\infty)$ be two maps from $\mathfrak{h}$ into $\mathfrak{h}\oplus\mathfrak{h}$. We define the operator ${\rm d}\check{\Gamma}(j,l)$ by

$${\rm d}\check{\Gamma}(j,l) : \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h})\otimes\Lambda(\mathfrak{h}),$$

$${\rm d}\check{\Gamma}(j,l) \coloneqq U_{\rm r}\,{\rm d}\Gamma(j,l).$$

Lemma 2.7. *Let $j = (j_0, j_\infty)$, $l = (l_0, l_\infty)$ be two maps in $\mathscr{B}(\mathfrak{h}, \mathfrak{h}\oplus\mathfrak{h})$.*

(i) *Let $j(t) \in C^1(\mathbb{R}, \mathscr{B}(\mathfrak{h}, \mathfrak{h}\oplus\mathfrak{h}))$. We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\Gamma}(j(t)) = \mathrm{d}\check{\Gamma}(j(t), \frac{\mathrm{d}}{\mathrm{d}t}j(t)).$$

(ii) *Let b be an operator on $\mathfrak{h}$. We have*

$$(\mathrm{d}\Gamma(b)\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(b))\check{\Gamma}_{\mathrm{r}}(j) - \check{\Gamma}_{\mathrm{r}}(j)\,\mathrm{d}\Gamma(b) = \mathrm{d}\check{\Gamma}(j, \check{\mathrm{ad}}_b(j)),$$

where $\check{\mathrm{ad}}_b(j) := ([b, j_0], [b, j_\infty])$.

(iii) *Let $u, v \in \mathscr{D}(\mathrm{d}\Gamma(|l_\varepsilon|^{\frac{1}{2}})), \varepsilon = 0, \infty$. We have*

$$|(\mathrm{d}\check{\Gamma}(j,l)u|v)| \leqslant ||\,\mathrm{d}\Gamma(|l_0|)^{1/2}u||\,||\,\mathrm{d}\Gamma(|l_0|)^{1/2}v|| + ||\,\mathrm{d}\Gamma(|l_\infty|)^{1/2}u||\,||\,\mathrm{d}\Gamma(|l_\infty|)^{1/2}v||.$$

(iv) *If $j_0^* j_0 + j_\infty^* j_\infty \leqslant 1$. We have*

$$||(N_0 + N_\infty)^{-1/2}\,\mathrm{d}\check{\Gamma}(j,l)u|| \leqslant ||\,\mathrm{d}\Gamma(l_0^* l_0 + l_\infty^* l_\infty)u||.$$

Proof. Part (i) is elementary. Part (ii) follows using Eq. (2.9) in Lemma 2.5. To prove part (iii) we apply estimate (2.3) with $B = (|l_\varepsilon|^{1/2}, 0)$, $C = (\mathrm{sgn}(l_\varepsilon)|l_\varepsilon|^{1/2}, 0)$, $\varepsilon = 0, \infty$. Using estimate (2.4) we obtain (iv). $\square$

Let $j := (j_0, j_\infty)$ be a pair of maps such that $j_0, j_\infty : \mathfrak{h}\to\mathfrak{h}$. We set $I(j)$ to be the map

$$I(j) : \Lambda^{\mathrm{fin}}(\mathfrak{h})\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h})\to\Lambda^{\mathrm{fin}}(\mathfrak{h}),$$

$$I(j) := I\Gamma(j_0)\otimes\Gamma(j_\infty).$$

Clearly $I(j) = \check{\Gamma}_{\mathrm{r}}(j^*)^*$ with the identification $j : \mathfrak{h}\oplus\mathfrak{h}\to\mathfrak{h}$, $j(h_0\oplus h_\infty) = j_0 h_0 + j_\infty h_\infty$.

2.2. *Wick polynomials*

Let $w \in \mathscr{B}(\otimes^p\mathfrak{h}, \otimes^q\mathfrak{h})$. We define the *Wick monomial* of order (p,q) with *symbol* w to be the operator $\mathrm{Wick}(w)$ given by

$$\mathrm{Wick}(w) : \Lambda^{\mathrm{fin}}(\mathfrak{h})\to\Lambda^{\mathrm{fin}}(\mathfrak{h}),$$

$$\mathrm{Wick}(w)_{|\Lambda^n(\mathfrak{h})} := \begin{cases} \dfrac{\sqrt{n!(n+q-p!)}}{(n-p)!}\, w\wedge\mathbb{1}^{\otimes(n-p)} & \text{if } n\geqslant p,\\ 0 & \text{otherwise.}\end{cases}$$

We call *Wick polynomial* a finite sum of Wick monomials with symbols belonging to $\mathscr{B}(\otimes^{p_j}\mathfrak{h}, \otimes^{q_j}\mathfrak{h}), j\in J$ a finite family of indices. We mention that a Wick monomial can be defined even if the symbol is an unbounded operator, but we restrict ourselves to bounded symbols.

We introduce the following 'contracted' symbols to simplify the notation. Let $u\in\Lambda^m(\mathfrak{h})$, $v\in\Lambda^n(\mathfrak{h})$, $w\in\mathscr{B}(\otimes^p\mathfrak{h}, \otimes^q\mathfrak{h})$ with $m\leqslant p$, $n\leqslant q$, we set

$$(v|w := (\langle v|\otimes \mathbb{1}^{\otimes(q-n)})w\in\mathscr{B}(\otimes^p\mathfrak{h}, (\otimes^{q-n}\mathfrak{h}),$$

$$w|u) := w(|u\rangle\otimes\mathbb{1}^{\otimes(p-m)})\in\mathscr{B}(\otimes^{p-m}\mathfrak{h}, \otimes^q\mathfrak{h}),$$

$$(v|w|u) := (\langle v|\otimes\mathbb{1}^{\otimes(q-n)})w(|u\rangle\otimes\mathbb{1}^{\otimes(p-m)})\in\mathscr{B}(\otimes^{p-m}\mathfrak{h}, \otimes^{q-n}\mathfrak{h}).$$

The following lemma should enlighten the motivation behind the above definition.

Lemma 2.8. *We have*

(i) $\lambda\in\mathscr{B}(\otimes^0\mathfrak{h})\simeq\mathbb{C}$, $\mathrm{Wick}(\lambda) = \lambda\mathbb{1}$,
(ii) $w\in\mathscr{B}(\mathfrak{h})$, $\mathrm{Wick}(w) = \mathrm{d}\Gamma(w)$,
(iii) $w\in\mathscr{B}(\otimes^p\mathfrak{h}, \otimes^q\mathfrak{h})$, $\mathrm{Wick}(\bigwedge w) = \mathrm{Wick}(w) = \mathrm{Wick}(w\bigwedge)$,
(iv) $h_1, \dots, h_n\in\mathfrak{h}$, $\mathrm{Wick}(|h_1\wedge\cdots\wedge h_n)) = b^*(h_1)\dots b^*(h_n)$, $\mathrm{Wick}((h_1\wedge\cdots\wedge h_n|) = b(h_n)\dots b(h_1)$.

Let $\mathscr{K}_i$, $i = 1,2$ be two auxiliary Hilbert spaces. We extend the definition of Wick polynomials to symbols acting on $\mathscr{K}_i$. Let $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p\mathfrak{h}, \mathscr{K}_2\otimes\otimes^q\mathfrak{h})$, we define the extended Wick monomial $\mathrm{Wick}(w)$ with symbol w as follows:

$$\mathrm{Wick}(w) : \mathscr{K}_1\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h})\to\mathscr{K}_2\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}),$$

$$\mathrm{Wick}(w)_{|\mathscr{K}_1\otimes\Lambda^n(\mathfrak{h})} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\mathbb{1}_{\mathscr{K}_2}\otimes\bigwedge_{n+q-p} w\otimes\mathbb{1}^{\otimes(n-p)}.$$

If $v\in\mathscr{B}(\mathscr{K}_1, \mathscr{K}_2\otimes\mathfrak{h})$, we have

$$\mathrm{Wick}(v) = b^*(v), \quad \mathrm{Wick}(v^*) = b(v),$$

where $b^\sharp$ are the extended creation–annihilation operators in Section 2.1.

Let us summarize properties of these Wick polynomials in the following lemma.

Lemma 2.9. *Let* $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p\mathfrak{h}, \mathscr{K}_2\otimes\otimes^q\mathfrak{h}), u\in\Lambda^n(\mathfrak{h})$, $v\in\Lambda^m(\mathfrak{h})$. *We have*

(i) $\mathrm{Wick}(w)^* = \mathrm{Wick}(w^*)$.
(ii) $\mathrm{Wick}(|u)\wedge w\wedge(v|) = \mathrm{Wick}(|u))\,\mathrm{Wick}(w)\,\mathrm{Wick}((v|)$.

An alternative way to express Wick polynomials is to decompose the symbol on a basis. Let $\{e_i\}_{i\in I}$ be a basis of $\mathfrak{h}$ with I a totally ordered set of indices. If $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p\mathfrak{h},\mathscr{K}_2\otimes\otimes^q\mathfrak{h})$, then $\mathrm{Wick}(w)$ can be written as a weakly convergent sum on $\mathscr{K}_1\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h})\times\mathscr{K}_2\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h})$:

$$\mathrm{Wick}(w)=\sum_{J,J'\subset I|\sharp J=p,\sharp J'=q} p!q!(\wedge_{j\in J'}e_j|w|\wedge_{j\in J}e_j)\prod_{j\in J'}b^*(e_j)\prod_{J\ni j}b(e_j), \tag{2.12}$$

where $(\wedge_{j\in J'}e_j|w|\wedge_{j\in J}e_j)\in\mathscr{B}(\mathscr{K}_1,\mathscr{K}_2)$ and the sum runs over ordered subsets $J,J'\subset I$ w.r.t the order relation of I.

Let us consider the case $\mathfrak{h}:=L^2(\mathbb{R}^d,dk)\otimes\mathfrak{g}$ where $\mathfrak{g}$ is a finite-dimensional Hilbert space with a given basis $\{\alpha_\lambda\}$. Let $f_i\in L^2(\mathbb{R}^d)$, $i=1,\dots,n$, we denote by

$$(f_i|:\mathfrak{h}\to\mathfrak{g},(f_i|:=\sum_\lambda\alpha_\lambda\otimes\langle f_i\otimes\alpha_\lambda|$$

and

$$(\otimes_{i=1}^n f_i|:\otimes^n\mathfrak{h}\to\otimes^n\mathfrak{g},(\otimes_{i=1}^n f_i|:=\otimes_{i=1}^n(f_i|.$$

Notice that $\Pi_{i=1}^n\mathbb{1}_{\otimes^{n-i}\mathfrak{g}}\otimes b(f_i)$ is a bounded operator from $\Lambda(\mathfrak{h})$ into $\otimes^n\mathfrak{g}\otimes\Lambda(\mathfrak{h})$, which we shortly denote by $\Pi_{i=1}^n b(f_i)$, and where $b(f_i)$ is introduced in Section 2.1. Considering $(\otimes_{i=1}^n f_i|$ as a symbol in $\mathscr{B}(\otimes^p\mathfrak{h},\otimes^p\mathfrak{g}\otimes\mathbb{C})$ it follows that $\mathrm{Wick}((\otimes_{i=1}^n f_i|)=\Pi_{i=1}^n b(f_i)$.

Let $\{e_i\}_{i\in I}$ be a basis of $L^2(\mathbb{R}^d)$ and $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p\mathfrak{h},\mathscr{K}_2\otimes\otimes^q\mathfrak{h})$, then w can be decomposed as the following weak sum

$$w=\sum_{J,J'\subset I|\sharp J=p,\sharp J'=q}|\otimes_{j\in J}e_j)(\otimes_{j\in J}e_j|w|\otimes_{j\in J'}e_j)(\otimes_{j\in J'}e_j|.$$

Thus we have

$$\mathrm{Wick}(w)=\sum_{J,J'\subset I|\sharp J=p,\sharp J'=q}\prod_{j\in J}b^*(e_j)(\otimes_{i\in J}e_j|w|\otimes_{i\in J'}e_j)\prod_{j\in J'}b(e_j), \tag{2.13}$$

where the sum runs over all subsets $J,J'\subset I$.

We denote by $S(\mathbb{R}^d),S'(\mathbb{R}^d)$ the Schwartz space and its dual and we set for $\mathfrak{B}$ a Banach space $S'(\mathbb{R}^d,\mathfrak{B}):=\mathscr{B}(S(\mathbb{R}^d),\mathfrak{B})$. Let $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p\mathfrak{h},\mathscr{K}_2\otimes\otimes^q\mathfrak{h})$, $\psi\in\mathscr{K}_1\otimes\otimes^p\mathfrak{g}$, $\phi\in\mathscr{K}_2\otimes\otimes^q\mathfrak{g}$, and $u\in L^2(\mathbb{R}^{dp})$. The vector $u\otimes\psi$ is in $\mathscr{K}_1\otimes\otimes^p\mathfrak{h}$ and the map

$$L^2(\mathbb{R}^{dq})\times L^2(\mathbb{R}^{dp})\ni(v,u)\mapsto(v\otimes\phi|wu\otimes\psi)_{\mathscr{K}_2\otimes\otimes^q\mathfrak{h}} \tag{2.14}$$

is a bounded sesquilinear form. Hence it defines an operator in $\mathscr{B}(L^2(\mathbb{R}^{dp}), L^2(\mathbb{R}^{dq}))$ and using the kernel theorem we can associate to $(.\otimes\phi|w.\otimes\psi)$ an element of $S'(\mathbb{R}^{d(p+q)})$. Therefore, we associate to $w\in\mathscr{B}(\mathscr{K}_1\otimes\Lambda^p(\mathfrak{h}), \mathscr{K}_2\otimes\Lambda^q(\mathfrak{h})))$ a kernel in $S'(\mathbb{R}^{d(p+q)}, \mathscr{B}(\mathscr{K}_1\otimes \otimes^p \mathfrak{g}, \mathscr{K}_2\otimes \otimes^q \mathfrak{g}))$, which we still denote by w.

Recall that $b(k)$ introduced in Section 2.1 is an operator-valued function acting from $\Lambda(\mathfrak{h})$ into $\mathfrak{g}\otimes\Lambda(\mathfrak{h})$. In fact, one can see that $\Pi_{i=1}^p b(k_i)$ maps $\Lambda^{\text{fin}}(\mathrm{S}(\mathbb{R}^d))$ into $\otimes^p \mathfrak{g}\otimes\Lambda^{\text{fin}}(\mathrm{S}(\mathbb{R}^d))$ as an operator-valued function in $S(\mathbb{R}^{dp})$. This gives meaning to the quadratic form

$$\mathscr{K}_2\otimes\Lambda^{\text{fin}}(\mathrm{S}(\mathbb{R}^d)) \times \mathscr{K}_1\otimes\Lambda^{\text{fin}}(\mathrm{S}(\mathbb{R}^d))\to\mathbb{C}$$

$$(\Phi,\Psi)\mapsto \int_{\mathbb{R}^{d(p+q)}} (\Pi_{i=1}^q b(k_i)\Phi|w(k_1,\dots,k_q,k'_p,\dots,k'_1) \\ \Pi_{i=p}^1 b(k_i)\Psi)_{\mathscr{K}_2\otimes\otimes^q \mathfrak{h}\otimes\Lambda(\mathfrak{h})}\, dk\, dk'.$$

Thus $\mathrm{Wick}(w)$ has the following weak integral representation:

$$\mathrm{Wick}(w) = \int b^*(k_1)\dots b^*(k_q)\, w(k_1,\dots,k_q,k'_p,\dots,k'_1) \\ b(k'_p)\dots b(k'_1)\, dk_1\dots dk_q\, dk'_p\dots dk'_1. \tag{2.15}$$

In the sequel we will use decomposition (2.13) with a particular choice of the basis of $L^2(\mathbb{R}^d, dk)$. Let $e_j(s) := -\frac{1}{\sqrt{2^j j!}}P_j(s)e_0(s)$, $j\in\mathbb{N}$, where P_j are the Hermite polynomials satisfying $\frac{d^j}{ds^j}\,\mathrm{e}^{-\frac{s^2}{2}} = (-1)^j P_j(s)\,\mathrm{e}^{-\frac{s^2}{2}}$. The family of Hermite functions

$$\{\tilde{e}_J := \otimes_{j\in J}\, e_j, J\subset\{1,\dots,d\}^{\mathbb{N}}\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$. Let $w\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p \mathfrak{h}, \mathscr{K}_2\otimes\otimes^q \mathfrak{h})$, we have

$$\mathrm{Wick}(w) = \sum_{I\in\mathbb{N}^{dq}, J\in\mathbb{N}^{dp}} \prod_{i=1}^q b^*(\tilde{e}_{I_i})\lambda_{I,J}\prod_{j=1}^p b(\tilde{e}_{J_j}),$$

$$\lambda_{I,J} = \left(\otimes_{i=1}^q \tilde{e}_{I_i}|w|\otimes_{j=1}^p \tilde{e}_{J_j}\right)\in\mathscr{B}(\mathscr{K}_1\otimes\otimes^p \mathfrak{g}, \mathscr{K}_2\otimes\otimes^q \mathfrak{g}), \tag{2.16}$$

where the sum runs over all $I = (I_i)_{1\dots q}$, $J = (J_i)_{1\dots p}$, $I_i, J_i\in\mathbb{N}^d$. The r.h.s. in (2.16) is weakly convergent on $\mathscr{K}_1\otimes\Lambda^{\text{fin}}(\mathfrak{h}) \times \mathscr{K}_2\otimes\Lambda^{\text{fin}}(\mathfrak{h})$.

2.2.1. Boundedness of Wick polynomials

In the case $\mathfrak{h} = L^2(\mathbb{R}^d, dk)\otimes\mathfrak{g}$, we recall a well-known estimate [17] giving a sufficient condition to the boundedness of Wick monomials. For $p, q, \rho\in\mathbb{N}$, we denote by $\mathscr{S}^{\rho}_{p,q}$ the class of symbols given by the Banach space $S_\rho(\mathbb{R}^{d(p+q)}, \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{g}))$, defined in Section 1, by the norm in (1.3).

Theorem 2.10. *Let* $w \in \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{h}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{h})$ *and* $\varrho := [\frac{3}{2} d(p+q)] + 3$. *There exist* $C_{p,q}$ *such that*:

$$||\mathrm{Wick}(w)|| \leqslant C_{p,q}\, ||w(k_1, \dots, k_p, k'_1, \dots, k'_q)||_{\mathscr{S}^{\varrho}_{p,q}}.$$

Proof. Using the decomposition (2.16), we see that

$$||\mathrm{Wick}(w)|| \leqslant pq \sum_{I,J} ||\lambda_{I,J}||_{\mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{g})}.$$

Notice that $\widehat{e}_J := \otimes_{j=1}^{p} \widetilde{e}_{J_j}, J = (J_j)_{1..p} \in \mathbb{N}^{dp}$ are eigenvectors of the harmonic oscillator $\Theta_{dp} := \frac{1}{2}\sum_{i=1}^{dp} -\frac{d^2}{ds_i^2} + s_i^2 - 1$ with $\Theta_{dp}\widehat{e}_J = |J|\,\widehat{e}_J$. In the sequel we omit the subscript in the norm of $\mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K}_2 \otimes \otimes^p \mathfrak{g})$.

Using Hölder inequality, we have the following estimates for $\alpha := [\frac{d}{2}(p+q)] + 1$.

$$\sum_{I,J} ||\lambda_{I,J}|| \leqslant \sum_{I,J} ||\lambda_{I,J}||\, (|I| + |J|)^{\alpha} (|I| + |J|)^{-\alpha}, \tag{2.17}$$

$$\leqslant \left(\sum_{I,J} ||\lambda_{I,J}||^2 (|I| + |J|)^{2\alpha} \right)^{\frac{1}{2}} \left(\sum_{I,J} (|I| + |J|)^{-2\alpha} \right)^{\frac{1}{2}}, \tag{2.18}$$

$$\leqslant C \left(\sum_{I,J} \left|\left| \sum_{\ell=0}^{\alpha} C^{\alpha}_{\ell} (\widehat{e}_I | \Theta^{\ell}_{dq} w \Theta^{\alpha-\ell}_{dp} | \widehat{e}_J) \right|\right|^2 \right)^{\frac{1}{2}}, \tag{2.19}$$

$$\leqslant C||(\Theta_{dp} + \Theta_{dq})^{\alpha} w(k_1, \dots, k_p, k'_1, \dots, k'_q)||_{L^2(\mathbb{R}^{d(p+q)}, \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{h}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{h}))}, \tag{2.20}$$

$$\leqslant C||w(k_1, \dots, k_p, k'_1, \dots, k'_q)||_{\mathscr{S}^{\varrho}_{p,q}}. \tag{2.21}$$

Note that α is chosen such that the second term in (2.18) is bounded by a finite constant. In (2.19) we have a Hilbert–Schmidt norm which we replace by the L^2 norm of kernel (2.20). The last inequality follows by estimating the norm of the operator $(\Theta_{dp} + \Theta_{dq})^{\alpha}$ by a Schwartz norm through a L^{∞} estimate, (see e.g. [26, Example 7]). □

Hence, the map $\mathscr{S}^{\varrho}_{p,q} \ni w \mapsto \mathrm{Wick}(w) \in \mathscr{B}(\mathscr{K}_1 \otimes \Lambda(\mathfrak{h}), \mathscr{K}_2 \otimes \Lambda(\mathfrak{h}))$ is continuous. It is also equivalent to the continuity of the application which maps a sequence $\lambda_{I,J}$

with values in $\mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{g})$, satisfying $\sum_{I,J} ||\lambda_{I,J}||^2 \; (|I| + |J|)^{2\alpha} < \infty$, into the Wick monomial (2.16).

2.2.2. Wick commutation relations

We give now some commutation relations satisfied by the Wick polynomials.

Proposition 2.11. (i) *Let* $w \in B(\mathscr{K}_1 \otimes \otimes^p \mathfrak{h}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{h})$ *and* A *a linear operator on* $\mathfrak{h}$. *Then*

$$[\mathrm{d}\Gamma(A), \mathrm{Wick}(w)] = \mathrm{Wick}([\mathrm{d}\Gamma(A), w]).$$

(ii) *Let* $q \in B(\mathfrak{h})$. *Then*

$$\Gamma(q)\mathrm{Wick}(w\Gamma(q)) = \mathrm{Wick}(\Gamma(q)w)\Gamma(q),$$

$$\Gamma(q)\mathrm{Wick}(w\Gamma(q)) = \mathrm{Wick}(\Gamma(q)w\Gamma(q^*))\Gamma(q), \quad \textit{for } q \textit{ isometric},$$

$$\Gamma(q)\mathrm{Wick}(w)\Gamma(q^{-1}) = \mathrm{Wick}(\Gamma(q)w\Gamma(q^{-1})), \quad \textit{for } q \textit{ unitary},$$

$$[\Gamma(q), \mathrm{Wick}(w)] = \Gamma(q)\mathrm{Wick}(w(1 - \Gamma(q))) + \mathrm{Wick}((\Gamma(q) - 1)w)\Gamma(q).$$

(iii) *Let* $h \in \mathfrak{h}$, $w \in B(\mathscr{K}_1 \otimes \otimes^p \mathfrak{h}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{h})$ *and* $p + q$ *even. Then*

$$[\mathrm{Wick}(w), b^*(h)] = p\mathrm{Wick}(w|h)), \quad [\mathrm{Wick}(w), b(h)] = q\mathrm{Wick}((h|w).$$

Proof. To prove (i) it is enough to show it for $w = |\wedge_1^p h_i\rangle\langle\wedge_1^q g_i|$. Note that

$$\begin{aligned}[\mathrm{d}\Gamma(A), \Pi_{i=1}^p b^*(h_i)\Pi_{i=q}^1 b(g_i)] = {} & \sum_{j=1}^p \Pi_{i<j} b^*(h_i) b^*(Ah_i) \Pi_{i>j} b^*(h_i) \Pi_{i=q}^1 b(g_i) \\ & + \Pi_{i=1}^p b^*(h_i) \sum_{j=1}^p \Pi_{j<i} b(h_i) b(Ah_i) \Pi_{j>i} b(h_i). \quad (2.22)\end{aligned}$$

Hence (i) holds since the r.h.s. in (2.22) is a Wick monomial of order (p, q) with the symbol $[\mathrm{d}\Gamma(A), w]$. Statements (ii), (iii) are easy and left to the reader. □

2.2.3. Wick tensor product

The idea of adding a copy of Fock space describing asymptotic free particles had already proved its depth. In this way, we need to construct observables with an another copy of Fock space.

Let for $n=(n_1,n_2)$, $p=(p_1,p_2)$, $\mathscr{K}=\mathscr{K}_1$ or $\mathscr{K}_2$, T_n^p be the unitary map defined by

$$T_n^p:\ \mathscr{K}\otimes\otimes^{n_1}\mathfrak{h}_1\otimes\otimes^{n_2}\mathfrak{h}_2\to\otimes^{n_1-p_1}\mathfrak{h}_1\otimes\otimes^{n_2-p_2}\mathfrak{h}_2\otimes\mathscr{K}\otimes\ \otimes^{p_1}\mathfrak{h}_1\otimes\otimes^{p_2}\mathfrak{h}_2,$$

$$\psi\otimes\otimes_1^{n_1}h_j\otimes\otimes_1^{n_2}g_j\mapsto\otimes_{p_1+1}^{n_1}h_j\ \otimes\otimes_{p_2+1}^{n_2}g_j\otimes\psi\otimes\otimes_1^{p_1}h_j\otimes\otimes_1^{p_2}g_j.$$

We set $\alpha_{q_1,q_2}(N)\coloneqq Nq_2+q_2q_1$, $\alpha_{p_1,p_2}(N)\coloneqq Np_2+p_2p_1$.

Let $w\in\mathscr{B}(\mathscr{K}_1\otimes\Lambda^{p_1}(\mathfrak{h}_1)\otimes\Lambda^{p_2}(\mathfrak{h}_2),\mathscr{K}_2\otimes\Lambda^{q_1}(\mathfrak{h}_1)\otimes\Lambda^{q_2}(\mathfrak{h}_2))$, we define the left/right *tensor Wick monomial* with *symbol w* by

$$\mathrm{Wick}_{\mathrm{l}}^{\otimes}(w):\ \mathscr{K}_1\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_2)\to\mathscr{K}_2\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_2),$$

$$\mathrm{Wick}_{\mathrm{l}}^{\otimes}(w)_{|\mathscr{K}_1\otimes\Lambda^n(\mathfrak{h}_1)\otimes\Lambda^m(\mathfrak{h}_2)}\coloneqq\frac{\sqrt{n!(n-p_1+q_1)!}}{(n-p_1)!}\frac{\sqrt{m!(m-p_2+q_2)!}}{(m-p_2)!}$$

$$\left(\mathbb{1}_{\mathscr{K}_2}\otimes\bigwedge_{n-p_1+q_1}\otimes\bigwedge_{(m-p_2+q_2)}\right)\times(T_n^{p_1})^{-1}\times(\mathbb{1}_{\mathscr{K}_2}\otimes(-\mathbb{1})^{\alpha_{q_1,q_2}(N)}\otimes\mathbb{1}_{\Lambda(\mathfrak{h}_2)})$$

$$(\mathbb{1}_{\otimes^{n-p_1}\mathfrak{h}_1}\otimes w\otimes\mathbb{1}_{\otimes^{n-p_2}\mathfrak{h}_2})\times T_n^{p_1}\times((-\mathbb{1})^{\alpha_{p_1,p_2}(N)}\otimes\mathbb{1}_{\Lambda(\mathfrak{h}_2)}),$$

and

$$\mathrm{Wick}_{\mathrm{r}}^{\otimes}(w):\mathscr{K}_1\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_2)\to\mathscr{K}_2\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h}_2),$$

$$\mathrm{Wick}_{\mathrm{r}}^{\otimes}(w)_{|\mathscr{K}_1\otimes\Lambda^n(\mathfrak{h}_1)\otimes\Lambda^m(\mathfrak{h}_2)}\coloneqq\frac{\sqrt{n!(n-p_1+q_1)!}}{(n-p_1)!}\frac{\sqrt{m!(m-p_2+q_2)!}}{(m-p_2)!}$$

$$\left(\mathbb{1}_{\mathscr{K}_2}\otimes\bigwedge_{n-p_1+q_1}\otimes\bigwedge_{(m-p_2+q_2)}\right)\times(T_n^{p_1})^{-1}\times(\mathbb{1}_{\mathscr{K}_2}\otimes\mathbb{1}_{\Lambda^n(\mathfrak{h}_2)}\otimes(-1)^{Nq_1})$$

$$\times(\mathbb{1}_{\otimes^{n-p_1}\mathfrak{h}_1}\otimes w\otimes\mathbb{1}_{\otimes^{m-p_2}\mathfrak{h}_2})\times T_n^{p_1}\times(\mathbb{1}_{\mathscr{K}_1}\otimes\mathbb{1}_{\Lambda^n(\mathfrak{h}_1)}\otimes(-\mathbb{1})^{Np_1}).$$

In the case where $\mathfrak{h}_1=\mathfrak{h}_2=L^2(\mathbb{R}^d,dk)\otimes\mathfrak{g}$, we have the following integral representation for tensor Wick monomial with symbol $w\in S'(\mathbb{R}^{d(p_1+q_1+p_2+q_2)}$,

$\mathscr{B}(\mathscr{K}_1 \otimes \otimes^{p_1+p_2} \mathfrak{g}, \mathscr{K}_2 \otimes \otimes^{q_1+q_2} \mathfrak{g}))$:

$$\mathrm{Wick}_{\mathrm{l}}^{\otimes}(w) = \int w(k_{1,1},..,k'_{1,1},..,k_{2,1},..,k'_{2,1},..) \prod_{i=1}^{q_1} b^*(k_{1,i}) \otimes \mathbb{1} \prod_{i=1}^{q_2} (-\mathbb{1})^N \otimes b^*(k_{2,i}) \times \prod_{i=p_2}^{1} (-\mathbb{1})^N \otimes b(k'_{2,i}) \prod_{i=p_1}^{1} b(k'_{1,i}) \otimes \mathbb{1} \, dk\, dk', \tag{2.23}$$

where we integrate over all variables $k_{i,j}, k'_{k,l}$.

We collect some properties of tensor Wick monomials in the following lemma.

Lemma 2.12.

(i) *Let* $w_1 \in \mathscr{B}(\mathscr{K}_1 \otimes \Lambda^{p_1}(\mathfrak{h}_1), \mathscr{K}_2 \otimes \Lambda^{q_1}(\mathfrak{h}_1))$, *and* $w_2 \in \mathscr{B}(\Lambda^{p_2}(\mathfrak{h}_2), \Lambda^{q_2}(\mathfrak{h}_2))$. *We have*

$$\mathrm{Wick}_{\mathrm{l}}^{\otimes}(w_1 \otimes w_2) = (-\mathbb{1})^{\alpha_{q_1,q_2}(N)} \mathrm{Wick}(w_1)(-\mathbb{1})^{\alpha_{p_1,p_2}(N)} \otimes \mathrm{Wick}(w_2).$$

(ii) *Let* $w = |\wedge_{i=1}^{q_1} h_{1,i}\rangle\langle \wedge_{i=1}^{p_1} g_{1,i}| \otimes |\wedge_{i=1}^{q_2} h_{2,i}\rangle\langle \wedge_{i=1}^{p_2} g_{2,i}|$. *Then*

$$\mathrm{Wick}_{\mathrm{l}}^{\otimes}(w) = \prod_{1}^{q_1} b^*(h_{1,i}) \otimes \mathbb{1} \prod_{1}^{q_2} (-\mathbb{1})^N \otimes b^*(h_{2,i}) \times \prod_{p_2}^{1} (-\mathbb{1})^N \otimes b(g_{2,i}) \prod_{p_1}^{1} b(g_{1,i}) \otimes \mathbb{1}.$$

$$\mathrm{Wick}_{\mathrm{r}}^{\otimes}(w) = \prod_{1}^{q_1} b^*(h_{1,i}) \otimes (-\mathbb{1})^N \prod_{1}^{q_2} \mathbb{1} \otimes b^*(h_{2,i}) \prod_{p_2}^{1} \mathbb{1} \otimes b(g_{2,i}) \prod_{p_1}^{1} b(g_{1,i}) \otimes (-\mathbb{1})^N.$$

(iii) *Let* $j_i \in \mathscr{B}(\mathfrak{h}_i)$, $i = 1, 2$. *Then*

$$\Gamma(j_1) \otimes \Gamma(j_2)\, \mathrm{Wick}_{\mathrm{l/r}}^{\otimes}(w\, \Gamma(j_1) \otimes \Gamma(j_2)) = \mathrm{Wick}_{\mathrm{l/r}}^{\otimes}(\Gamma(j_1) \otimes \Gamma(j_2) w)\, \Gamma(j_1) \otimes \Gamma(j_2).$$

(iv)
$$\mathrm{Wick}_{\mathrm{l/r}}^{\otimes}(w^*) = \mathrm{Wick}_{\mathrm{l/r}}^{\otimes}(w)^*.$$

(v) *If* $\mathfrak{h} = L^2(\mathbb{R}^d, dk) \otimes \mathfrak{g}$, $\varrho := [\frac{3}{2} d(p+q)] + 3$, *then the map*

$$\mathscr{S}^{\varrho}_{p_1+p_2, q_1+q_2} \ni w \mapsto \mathrm{Wick}_{\mathrm{l/r}}^{\otimes}(w) \in \mathscr{B}(\mathscr{K}_1 \otimes \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h}), \mathscr{K}_2 \otimes \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h})),$$

is continuous.

Proof. (i)–(iv) follows by a simple computation on each sector $\mathscr{K}_1 \otimes \Lambda^n(\mathfrak{h}) \otimes \Lambda^m(\mathfrak{h})$. We can decompose a symbol of a tensor Wick monomial using Hermite functions in a similar way as in (2.13). Now applying the same argument as in the proof of Theorem 2.10 we obtain (v). □

Let us summarize some properties of tensor Wick monomials in the following proposition.

Proposition 2.13. *The following assertions hold*:

(i) *Let* $w \in \mathscr{B}(\mathscr{K}_1 \otimes \Lambda^p(\mathfrak{h}_1 \oplus \mathfrak{h}_2), \mathscr{K}_2 \otimes \Lambda^q(\mathfrak{h}_1 \oplus \mathfrak{h}_2))$.

$$U_{1/\mathrm{r}}\,\mathrm{Wick}(w)\,U_{1/\mathrm{r}}^* = \mathrm{Wick}_{1/\mathrm{r}}^{\otimes}(\widetilde{U} w \widetilde{U}^*),$$

where

$$\widetilde{U}^* u \otimes v := \frac{(p+q)!}{p!q!}\Gamma(i_1)u \wedge \Gamma(i_2)v, \quad u \in \Lambda^p(\mathfrak{h}_1), \ v \in \Lambda^q(\mathfrak{h}_2),$$

and

$$\widetilde{U}_{|\Lambda^n(\mathfrak{h}_1 \oplus \mathfrak{h}_2)} := \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} p_1^{\otimes k} \otimes p_2^{\otimes (n-k)}.$$

(ii) *Let* $w \in B(\mathscr{K}_1 \otimes \Lambda^p(\mathfrak{h}), \mathscr{K}_2 \otimes \Lambda^q(\mathfrak{h}))$.

$$\mathrm{Wick}(w)I = I\mathrm{Wick}_{\mathrm{r}}^{\otimes}(Pw\widetilde{I}),$$

$$\widetilde{I} u \otimes v = \frac{(p+q)!}{p!q!} u \wedge v, \quad u \in \Lambda^p(\mathfrak{h}), \ v \in \Lambda^q(\mathfrak{h}),$$

and

$$P : \Lambda^{\mathrm{fin}}(\mathfrak{h}) \to \Lambda^{\mathrm{fin}}(\mathfrak{h}) \otimes \Lambda^{\mathrm{fin}} h, \quad Pu = u \otimes \Omega.$$

(iii) *Let* $j = (j_0, j_\infty)$ *be a pair of maps acting on* $\mathfrak{h}$ *and* $w \in B(\mathscr{K}_1 \otimes \Lambda^p(\mathfrak{h}), \mathscr{K}_2 \otimes \Lambda^q(\mathfrak{h}))$.

$$\mathrm{Wick}(\Gamma(j_0^*)w)\check{\Gamma}_{\mathrm{r}}(j)^* = \check{\Gamma}_{\mathrm{r}}(j)^* \mathrm{Wick}_{\mathrm{r}}^{\otimes}(Pw\widetilde{I}\Gamma(j_0^*) \otimes \Gamma(j_\infty^*)).$$

Proof. We first assume without loss of generality that $\mathscr{K}_1 = \mathscr{K}_2 = \mathbb{C}$. To prove (i) it suffices using linearity and anti-symmetry to consider w a rank one operator such that:

$$w = |\wedge_1^q h_j\rangle\langle\wedge_1^{q'} g_j|, h_j, g_j \in \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

$$p_2 h_j = 0, 1 \leqslant j \leqslant q_1, p_1 h_j = 0, q_1 + 1 \leqslant j \leqslant q,$$

$$p_2 g_j = 0, 1 \leqslant j \leqslant q_1', p_1 g_j = 0, q_1' + 1 \leqslant j \leqslant q'.$$

We have

$$\begin{aligned}\widetilde{U} \wedge_1^q h_j &= \sum_{k=0}^{q} \frac{1}{k!(q-k)!} \sum_{\sigma \in S_q} \varepsilon(\sigma) p_1^{\otimes k} \otimes p_2^{\otimes q-k} \pi_\sigma h_1 \otimes \cdots \otimes h_q \\ &= \wedge_1^{q_1} p_1 h_j \otimes \wedge_{q_1+1}^{q} p_2 h_j.\end{aligned}$$

Hence we obtain

$$\widetilde{U} w \widetilde{U}^* = |\wedge_1^{q_1} p_1 h_j\rangle\langle\wedge_1^{q_1'} p_1 g_j \otimes |\wedge_{q_1+1}^{q} p_2 h_j\rangle\langle\wedge_{q_1'+1}^{q'} p_2 g_j|.$$

This yields

$$\begin{aligned}U_{\mathrm{l}} \mathrm{Wick}(w) U_{\mathrm{l}}^* = &\Pi_1^{q_1} b^*(p_1 h_j) \otimes \mathbb{1} \Pi_{q_1+1}^{q} (-\mathbb{1})^N \otimes b^*(p_2 h_j) \\ &\Pi_{q'}^{q_1'+1} (-\mathbb{1})^N \otimes b(p_2 g_j) \Pi_{q_1'+1}^{1} b(p_1 g_j) \otimes \mathbb{1};\end{aligned}$$

$$\begin{aligned}U_{\mathrm{r}} \mathrm{Wick}(w) U_{\mathrm{r}}^* = &\Pi_1^{q_1} b^*(p_1 h_j) \otimes (-\mathbb{1})^N \Pi_{q_1+1}^{q} \mathbb{1} \otimes b^*(p_2 h_j) \\ &\Pi_{q'}^{q_1'+1} \mathbb{1} \otimes b(p_2 g_j) \Pi_{q_1'}^{1} b(p_1 g_j) \otimes (-\mathbb{1})^N.\end{aligned}$$

Therefore using Lemma 2.12 (ii), we obtain (i).

One can reduce (ii) to the case where $w = |\wedge_1^q h_j\rangle\langle\wedge_1^p g_j|$, $h_j, g_j \in \mathfrak{h}$. Let us first compute $Pw\widetilde{I}$. For $u^\pm \in \Lambda^{p^\pm}(\mathfrak{h})$, $p^+ + p^- = p$, we have

$$\begin{aligned}(\wedge_1^p g_j, \widetilde{I} u^+ \otimes u^-) &= \frac{(p^+ + p^-)!}{p^+! p^-!} (\wedge_1^p g_j, u^+ \otimes u^-) \\ &= \frac{1}{p^+! p^-!} \sum_{\sigma \in S_p} \varepsilon(\sigma) (\pi_\sigma \bigotimes_1^p g_j, u^+ \otimes u^-).\end{aligned}$$

For each $I^+ \subset \{1, \dots, p\}$, $I^- := I \backslash I^+$, $p^+ := \sharp I^+$, there is a unique permutation σ_{I^+} sending $\{1, \dots, p^+\}$ onto I^+ and $\{p^+ + 1, \dots, p\}$ onto I^- and respecting the order relation. Set $\varepsilon(I^+) := \varepsilon(\sigma_{I^+})$. The set S_p splits into classes of permutations such that

$\sigma(\{1,\dots,p\}) = I^+$. In each class a permutation σ can be written as the product of σ_{I^+} by a permutation of I^+ and a permutation of I^-. Using this fact and the anti-symmetry of $U^\pm$, we obtain

$$(\wedge_1^p g_j, \widetilde{I}u^+ \otimes u^-) = \sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+)(\wedge_{j\in I^+} g_j \otimes \wedge_{j\in I^-} g_j, u^+ \otimes u^-),$$

i.e:

$$\widetilde{I}^* \wedge_1^p g_j = \sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+) \wedge_{j\in I^+} g_j \otimes \wedge_{j\in I^-} g_j.$$

This yields

$$\begin{aligned} Pw\widetilde{I} &= \sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+)|\wedge_1^q h_j \otimes \Omega\rangle\langle \wedge_{j\in I^+} g_j \otimes \wedge_{j\in I^-} g_j|, \\ &= \sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+)|\wedge_1^q h_j\rangle\langle \wedge_{j\in I^+} g_j| \otimes |\Omega\rangle\langle \wedge_{j\in I^-} g_j|. \end{aligned} \tag{2.24}$$

Let us now compute $\mathrm{Wick}(w)I$. We have

$$b^*(h)\Gamma(i) = \Gamma(i)b^*(i^\pm h), \quad b(h)\Gamma(i) = \Gamma(i)b(i^* h), \;\; h\in\mathfrak{h},$$

and hence

$$\begin{aligned} \mathrm{Wick}(w) &= \Pi_1^q b^*(h_j)\Pi_p^1 b(g_j) \\ \mathrm{Wick}(w)\Gamma(i) &= \Gamma(i)\Pi_1^q b^*(i^+ h_j)\Pi_p^1 b(i^* g_j) \\ \mathrm{Wick}(w)I &= I\Pi_1^q b^*(i^+ h_j) \otimes (-1)^N \Pi_p^1(\mathbb{1} \otimes b(g_j) + b(g_j) \otimes (-1)^N). \end{aligned}$$

Let now $A_j^\pm$ be operators. Then

$$\Pi_p^1(A_j^+ + A_j^-) = \sum_{\alpha\in\{1,\dots,p\}^{\{+,-\}}} \Pi_p^1 A_j^{\alpha_j}.$$

To $\alpha \in \{1,\dots,p\}^{\{+,-\}}$ we associate $I^\pm = \{j | \alpha_j = \pm\}$. This yields with $A_j^+ = b(g_j) \otimes (-1)^N$, $A_j^- = \mathbb{1} \otimes b(g_j)$ to

$$\Pi_p^1(A_j^+ + A_j^-) = \sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+)\Pi_{I^+\ni i}\mathbb{1} \otimes b(g_j)\Pi_{I^+\ni i} b(g_j) \otimes (-1)^N,$$

Hence

$$\begin{aligned}\mathrm{Wick}(w)I &= I\sum_{I^+\subset\{1,\dots,p\}} \varepsilon(I^+)\Pi_{I^+\ni i}\mathbb{1}\otimes b(g_j)\Pi_{I^+\ni i}b(g_j)\otimes(-1)^N\\ &= I\mathrm{Wick}_{\mathrm{r}}^{\otimes}(Pw\widetilde{I}),\end{aligned}$$

using (2.24).

Part (iii) follows by using the fact that $\check{\Gamma}_{\mathrm{r}}(j)^* = I\Gamma(j_0^*)\otimes\Gamma(j_\infty^*)$ and by combining part (ii) with part (iii) of Lemma 2.12. $\square$

Proposition 2.14. *Let $w\in\mathscr{B}(\mathscr{K}\otimes\Lambda^p(\mathfrak{h}),\mathscr{K}\otimes\Lambda^q(\mathfrak{h}))$, $p+q$ even and $j=(j_0,j_\infty)$: $\mathfrak{h}\to\mathfrak{h}\oplus\mathfrak{h}$. We have*

$$\begin{aligned}I^*(j)\mathrm{Wick}(w) &= \mathrm{Wick}(w)\otimes\mathbb{1}\, I^*(j)\\ &\quad + I^*(j)\mathrm{Wick}(w(1-\Gamma(j_0^*)))\\ &\quad + \mathrm{Wick}_{\mathrm{r}}^{\otimes}((\Gamma(j_0^*)-1)\otimes\Gamma(j_\infty)\widetilde{I}^*wP^*)I^*(j)\\ &\quad + \mathrm{Wick}_{\mathrm{r}}^{\otimes}(\mathbb{1}-(\Gamma(j_\infty^*)-|\Omega\rangle\langle\Omega|)\widetilde{I}^*wP^*)I^*(j).\end{aligned}$$

The proof follows from simple computation, we leave it for the reader.

We finish this subsection with some commutator estimates in the case $\mathfrak{h} = L^2(\mathbb{R}^d,dk)\otimes\mathfrak{g}$. Recall that we define the Schwartz semi-norms on $S(\mathbb{R}^n)$ as

$$||f||_m := \sum_{|\alpha|+|\beta|\leqslant m} ||k^\alpha D_k^\beta f||_{L^\infty(\mathbb{R}^n)}.$$

Let $q\in C_0^\infty(\mathbb{R}^d)$, and set $q^R := q(\frac{D_k}{R}), R\geqslant 1$. Let $j_0,j_\infty\in C^\infty(\mathbb{R}^d)$ and set $j_0^R := j_0(\frac{D_k}{R}), j_\infty^R := j_\infty(\frac{D_k}{R})$. We denote by $\mathscr{W}^s$ the Sobolev space of functions in $L^\infty(\mathbb{R}^d)$ such that all their derivatives of order equal or less than s are in $L^\infty(\mathbb{R}^d)$.

Theorem 2.15. *Let $w\in\mathscr{B}(\mathscr{K}_1\otimes\Lambda^{p_1}(\mathfrak{h}),\mathscr{K}_2\otimes\Lambda^{p_2}(\mathfrak{h}))$, $\varrho := [\frac{3}{2}d(p_1+p_2)]+3$, and $\varrho_0 := d(p_1+p_2)+2$. We have*

(i) *If $||q||_{L^\infty(\mathbb{R}^d)}\leqslant 1$. Then*

$$\begin{aligned}||[\Gamma(q^R),\mathrm{Wick}(w)]|| &\leqslant C_{p_1,p_2}\left(||q\left(\frac{k}{R}\right)||_{\mathscr{W}^{\varrho_0}}+1\right)^{\max(p_1,p_2)-1}\\ &\quad\times\sum_{i=1}^{p_1+p_2} ||\mathbb{1}_{\otimes_{\mathfrak{h}}^{i-1}}\otimes(1-q^R)\otimes\mathbb{1}_{\otimes_{\mathfrak{h}}^{p_1+p_2-i}}w||_{\mathscr{S}^{\varrho}_{p_1,p_2}}.\end{aligned}$$

(ii) *Let* j_0, j_∞ *as above such that* $j_0 j_0^* + j_\infty j_\infty^* \leqslant 1$. *Then*

$$\begin{aligned}||I^*(j^R)\mathrm{Wick}(w) - \mathrm{Wick}(w)\otimes \mathbb{1} I^*(j^R)|| \\ \leqslant C_{p_1,p_2}\left(\left\|j_0\left(\frac{k}{R}\right)\right\|_{\mathscr{W}^{\varrho_0}} + \left\|j_\infty\left(\frac{k}{R}\right)\right\|_{\mathscr{W}^{\varrho_0}} + 1\right)^{\max(p_1,p_2)-1} \\ \left(\sum_{i=1}^{p_1+p_2} ||\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}} \otimes (1 - j_0^R) \otimes \mathbb{1}_{\mathfrak{h}^{\otimes(p+q-i)}} w||_{\mathscr{S}^{\varrho}_{p_1,p_2}} \right. \\ \left. + \sum_{i=1}^{p_1+p_2} ||\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}} \otimes j_\infty^R \otimes \mathbb{1}_{\mathfrak{h}^{\otimes(p_1+p_2-i)}} w||_{\mathscr{S}^{\varrho}_{p_1,p_2}}\right).\end{aligned}$$

Proof. Using Lemma 2.11 (ii), we see that the commutator is equal to

$$[\Gamma(q^R), \mathrm{Wick}(w)] = \Gamma(q^R)\mathrm{Wick}(w(1-\Gamma(q^R))) + \mathrm{Wick}((\Gamma(q^R)-1)w)\Gamma(q^R).$$

Since $||q^R \otimes \mathbb{1}_{\mathfrak{g}}||_{\mathfrak{h}} \leqslant 1$, we have only to estimate $\mathrm{Wick}((\Gamma(q^R)-1)w)$ and $\mathrm{Wick}(w(1-\Gamma(q^R)))$ using Theorem 2.10. One can write

$$w(1-\Gamma(q^R)) = \sum_{i=1}^{p_1} w(q^R \otimes \cdots \otimes \underbrace{(1-q^R)}_{i} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}),$$

$$(\Gamma(q^R)-1)w = \sum_{i=1}^{p_2} (q^R \otimes \cdots \otimes \underbrace{(q^R-1)}_{i} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})w.$$

Let $Q_i \coloneqq \otimes_{j=1}^{i-1} q^R \otimes \mathbb{1} \in \mathscr{B}(\Lambda^{p_2}(\mathfrak{h}))$, $\overline{Q}_i \coloneqq \otimes_{j=1}^{i-1} q^{R,*} \otimes \mathbb{1} \in \mathscr{B}(\Lambda^{p_1}(\mathfrak{h}))$, $W_i = (q^R-1)_i w$ and $\overline{W}_i \coloneqq (1-q^{R,*})w^*$. We have

$$||[\Gamma(q), \mathrm{Wick}(w)]|| \leqslant \sum_{i=1}^{p_1} ||\mathrm{Wick}(\overline{Q}_i \overline{W}_i)|| + \sum_{i=1}^{p_2} ||\mathrm{Wick}(Q_i W_i)||.$$

Hence (i) reduces to estimate the terms $||\mathrm{Wick}(Q_i W_i)||$ and $||\mathrm{Wick}(\overline{Q}_i \overline{W}_i)||$.

Using inequality (2.19) in the proof of Theorem 2.10, we obtain

$$||\mathrm{Wick}(Q_i W_i)|| \leqslant C_{p_1,p_2} \sum_{|\alpha|+|\beta| \leqslant \varrho_0} ||k^\alpha \partial_k^\beta Q_i\, W_i||_{L^2(\mathbb{R}^{d(p+q)}, \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{h}, \mathscr{K}_2 \otimes \otimes^q \mathfrak{h}))}.$$

By simple computation one can write $k^\alpha Q_i = \sum_{|\delta| \leqslant \varrho_0} c_\delta \otimes_{j=1}^{i-1} \frac{1}{R^{|\delta|}} \partial^\delta q^R(\frac{D_k}{R}) k^{\alpha-\delta}$. Therefore, we get

$$||\mathrm{Wick}(Q_i W_i)|| \leqslant \widetilde{C}_{p_1,p_2} \left\|q\left(\frac{k}{R}\right)\right\|_{\mathscr{W}^{\varrho_0}}^{i-1} ||W_i||_{\mathscr{S}^{\varrho}_{p_1,p_2}}.$$

This leads to the bound given in (i). The second statement (ii) is similar to the above one. The fact that $j_0 j_0^* + j_\infty j_\infty^* \leqslant 1$ yields that $I(j^R)$ is bounded and by Proposition 2.14 one have only to estimate

$$\mathrm{Wick}((1-\Gamma(j_0))w), \quad \mathrm{Wick}_{\mathrm{r}}^{\otimes}(Pw\widetilde{I}(\Gamma(j_0)-1)\otimes\Gamma(j_\infty)),$$
$$\mathrm{Wick}_{\mathrm{r}}^{\otimes}(Pw\widetilde{I}\mathbb{1}\otimes(\Gamma(j_\infty)-|\Omega\rangle\langle\Omega|)).$$

This can be done using Theorem 2.10 with the kernels written as follows:

$$\begin{aligned} &Pw\widetilde{I}(\Gamma(j_0)-1)\otimes\Gamma(j_\infty) \\ &\quad = \sum_{s=0}^{p_1} \frac{p_1!}{s!(p_1-s)!} \sum_{i=1}^{s} Pw(j_0^{\otimes(i-1)}\otimes(j_0-1)\wedge j_\infty^{\otimes(s-i)})\mathbb{1}_{\Lambda^s(\mathfrak{h})}\otimes\mathbb{1}_{\Lambda^{p_1-s}(\mathfrak{h})}, \end{aligned}$$

$$Pw\widetilde{I}(\Gamma(j_\infty)-|\Omega\rangle\langle\Omega|) = \sum_{s=0}^{p_1-1} \frac{p_1!}{s!(p_1-s)!} w\mathbb{1}^{\otimes s}\wedge j_\infty^{\otimes p_1-s}\mathbb{1}_{\Lambda^s(\mathfrak{h})}\otimes\mathbb{1}_{\Lambda^{p_1-s}(\mathfrak{h})}. \qquad \square$$

3. Hamiltonians and main result

The class of fermionic Pauli–Fierz models considered in this paper describes in physics a confined finite many particles system interacting with a fermionic field. The free and the perturbed dynamics are described by a free and an interacting Hamiltonians. In this section we will introduce those Hamiltonians and state our hypotheses and our main theorem.

3.1. The model

3.1.1. The free dynamic

Let $\mathscr{K}$ be a Hilbert space and K a bounded below self-adjoint operator on $\mathscr{K}$ describing the atomic Hamiltonian. Let $\mathfrak{g}$ be a Hilbert space, we consider $\mathfrak{h} := L^2(\mathbb{R}^d, dk)\otimes\mathfrak{g}$ to be the one-particle space of the fermionic system. We denote by the italic letter k the one particle fermionic momentum and by the italic letter $x = \mathrm{i}\nabla_k$ the one particle fermionic position.

In all the sequel we assume

$$(\mathrm{H0}) \quad \begin{cases} K \text{ bounded below,} \\ (K+\mathrm{i})^{-1} \text{ compact,} \\ \mathfrak{g} \text{ finite dimensional.} \end{cases}$$

We can assume without loss of generality that K is a positive operator and we denote by $\mathscr{K}_1$, $\mathscr{K}_{1/2}$ respectively, the Hilbert spaces $\mathscr{D}(K)$, $\mathscr{D}(K^{\frac{1}{2}})$ endowed with their graph norms.

Let ω a matrix-valued function in $C(\mathbb{R}^d, \mathscr{B}(\mathfrak{g}))$ be the dispersion relation of fermions. Let $\lambda_1(k), \dots, \lambda_s(k)$ be the eigenvalues of $\omega(k)$ with uniform constant multiplicity for $k \in \mathbb{R}^d$. We assume that ω satisfies the following hypothesis:

$$(\mathrm{H1}) \quad \begin{cases} \nabla\omega(k) \neq 0, \ \text{for } k \neq 0, \\ \lim\limits_{|k| \to \infty} ||\omega(k)||_{\mathfrak{g}} = +\infty, \\ \inf\limits_{k \in \mathbb{R}^d, ||\psi||_{\mathfrak{g}} = 1} (\omega(k)\psi|\psi)_{\mathfrak{g}} = m > 0, \\ \dim(\mathrm{Ker}(\omega(k) - \lambda_i(k)\mathbb{1}_{\mathfrak{g}})) \text{ is } k \text{ independent}, \\ \inf\limits_{k \in \mathbb{R}^d, i \neq j} |\lambda_i(k) - \lambda_j(k)| > 0, \\ \partial_k^\beta \lambda_i(k) \in L^\infty(\mathbb{R}^d), |\beta| \geqslant 1. \end{cases}$$

We set

$$\mathscr{H} := \mathscr{K} \otimes \Lambda(\mathfrak{h}), \quad \mathscr{H}_{1/2} := \mathscr{K}_{1/2} \otimes \Lambda(\mathfrak{h}),$$

where $\mathscr{H}$ denotes the Hilbert space of the joint system spin-fermion. The free Hamiltonian is defined by

$$H_0 := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \quad \text{acting on } \mathscr{H}.$$

We mention that H_0 is essentially self-adjoint on $\mathscr{D}(K) \otimes \Lambda^{\mathrm{fin}}(C_0(\mathbb{R}^d) \otimes \mathfrak{g})$.

3.1.2. Perturbation

Let P be an element in $\mathscr{B}(\mathscr{H}_{1/2}, \mathscr{H}) \cap \mathscr{B}(\mathscr{H}, \mathscr{H}_{1/2}^*)$ such that $P^* = P$. Then we can construct the perturbed Hamiltonian H respectively to the interaction P by Kato–Rellich theorem [25]. Namely, we have

$$H := H_0 + P, \quad \text{acting on } \mathscr{H},$$

is a well-defined self-adjoint operator with domain $\mathscr{D}(H) = \mathscr{D}(H_0)$. Set

$$\mathscr{B}_1 := \mathscr{B}(\mathscr{K}_1 \otimes \otimes^p \mathfrak{g}, \mathscr{K} \otimes \otimes^q \mathfrak{g}), \quad \overline{\mathscr{B}}_1 := \mathscr{B}(\mathscr{K} \otimes \otimes^p \mathfrak{g}, \mathscr{K}_1^* \otimes \otimes^q \mathfrak{g}),$$

and

$$\mathscr{B}_{1/2} := \mathscr{B}(\mathscr{K}_{1/2} \otimes \otimes^p \mathfrak{g}, \mathscr{K} \otimes \otimes^q \mathfrak{g}), \quad \overline{\mathscr{B}}_{1/2} := \mathscr{B}(\mathscr{K} \otimes \otimes^p \mathfrak{g}, \mathscr{K}_{1/2}^* \otimes \otimes^q \mathfrak{g}).$$

Let us consider a perturbation given by an even Wick polynomial having the following form:

$$P := \sum_{p,q \in \Xi} \mathrm{Wick}(w_{p,q}) + \mathrm{Wick}(w_{p,q}^*),$$

where

$$\Xi \subset \{(p,q)\in\mathbb{N}^2 \mid p+q\in 2\mathbb{N}\}, \quad \text{finite and } w_{p,q}\in L^2(\mathbb{R}^{d(p+q)}, \mathscr{B}_{1/2}\cap\overline{\mathscr{B}}_{1/2}). \tag{3.1}$$

One can assume

$$\text{(I0)} \sum_{p,q\in\Xi} \sum_{I\in\mathbb{N}^{dq}, J\in\mathbb{N}^{dp}} ||(\widetilde{e}_I|w_{p,q}|\widetilde{e}_J)||_{\mathscr{B}_{1/2}\oplus\overline{\mathscr{B}}_{1/2}} < \infty,$$

where $\{\widetilde{e}_I\}_{I\in\mathbb{N}^{dq}}$ is the basis of $L^2(\mathbb{R}^{dq})$ defined in Section 2.2. Hypothesis (I0) ensures that P is $K^{\frac{1}{2}}$-bounded and therefore the perturbed dynamic is well defined by the Hamiltonian H which is a bounded below self-adjoint operator acting on $\mathscr{H}$.

In all the following we will assume without loss of generality that $\mathscr{B}(\mathfrak{g})\simeq\mathscr{M}_{\dim(\mathfrak{g})}(\mathbb{C})$ such that ω is a diagonal matrix-valued function satisfying hypothesis (H1). Indeed, if it is not the case one could diagonalize $\omega(k)$ by means of unitary transforms $\mathrm{u}(k)\in C(\mathbb{R}^d, \mathscr{B}(\mathfrak{g}))$, since it is symmetric. Therefore we could unitarily transform the Hamiltonian H by $\Gamma(\mathrm{u}(k))$ and obtain

$$\Gamma(\mathrm{u}(k))H\Gamma(\mathrm{u}(k))^{-1} = K\otimes\mathbb{1} + \mathrm{d}\Gamma(\mathrm{u}(k)\omega(k)\mathrm{u}(k)^{-1}) + \sum_{(p,q)\in\Xi} \mathrm{Wick}(\Gamma(\mathrm{u}(k))w_{p,q}\Gamma(\mathrm{u}(k))^{-1}).$$

Hypothesis (H1)–(I0) are stable under this change of representation. This is obvious for (I0) however some care is needed for (H1), see Lemma A.1.

We will assume a slightly stronger condition for P than (I0) which has the advantage to be more convenient to check since it involves the kernels of the symbols $w_{p,q}$ instead of their operator norms. We recall for $p,q\in\mathbb{N}$ and $\varrho := [\frac{3}{2}d(p+q)]+3$, the definition of the particular classes of symbols

$$\mathrm{S}^{1/2}_{p,q} := \mathscr{S}_\varrho(\mathbb{R}^{d(p+q)}, \mathscr{B}_{1/2}\oplus\overline{\mathscr{B}}_{1/2}),$$

$$\mathrm{S}^{1}_{p,q} := \mathscr{S}_\varrho(\mathbb{R}^{d(p+q)}, \mathscr{B}_{1}\oplus\overline{\mathscr{B}}_{1}).$$

Hence, we will assume

$$\text{(I0}'\text{)} \sum_{p,q\in\Xi} ||w_{p,q}||_{\mathrm{S}^{1/2}_{p,q}} < \infty.$$

Proposition 3.1. *Let P given by* (3.1) *and assume that* (I0′) *is satisfied. Then*

$$P(H_0+1)^{-\frac{1}{2}}\in\mathscr{B}(\mathscr{H}).$$

Moreover $H := H_0 + P$ is a bounded below self-adjoint operator with $\mathscr{D}(H) = \mathscr{D}(H_0)$.

Proof. Using Theorem 2.10 with $\mathscr{K}_1 = \mathscr{K}_2 = \mathscr{K}$, and $\varrho = [3/2d(p+q)] + 3$ we obtain

$$||P(H_0+1)^{-\frac{1}{2}}|| \leqslant C \sum_{p,q \in \Xi} ||w_{p,q}(K+1)^{-\frac{1}{2}}||_{\mathscr{S}^{\varrho}_{p,q}} + ||w^*_{p,q}(K+1)^{-\frac{1}{2}}||_{\mathscr{S}^{\varrho}_{p,q}}$$
$$\leqslant C \sum_{p,q \in \Xi} ||w_{p,q}||_{\mathrm{S}^{1/2}_{p,q}}. \qquad \square$$

We assume also an hypothesis concerning the Mourre estimate. Let

$$G := \mathrm{d}\Gamma\left(-\frac{1}{2}\nabla\omega(k).D_k + D_k.\nabla\omega(k)\right),$$

be the conjugate operator introduced in Section 4.3. The Mourre hypothesis is

$$(\mathrm{M0}) \quad \sum_{p,q \in \Xi} ||[G, w_{p,q}]||_{\mathrm{S}^{1/2}_{p,q}} < \infty.$$

To prove asymptotic completeness we need more restrictive hypothesis on the interaction. We assume a decay of the perturbation of *short range* type, similar to the one appearing in the bosonic case [11].

$$\sum_{p,q \in \Xi} \sum_{i=1}^{p+q} ||\mathbb{1}_{[R,\infty[}(|x_i|)\, w_{p,q}||_{\mathrm{S}^1_{p,q}} \in \begin{cases} O(R^{\mu}), \mu > 1 & (\mathrm{SR}), \\ o(R^0) & (\mathrm{I1}). \end{cases}$$

We introduce the following notations:

$$\mathscr{H}^{\mathrm{ext}} := \mathscr{H} \otimes \Lambda(\mathfrak{h}),$$

$$H_0^{\mathrm{ext}} := H_0 \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \quad H^{\mathrm{ext}} := H \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \text{ acting on } \mathscr{H}^{\mathrm{ext}},$$

$$N_0 := N \otimes \mathbb{1},\; N_\infty := \mathbb{1} \otimes N, \quad \text{acting on } \mathscr{H}^{\mathrm{ext}}.$$

3.2. Result

We formulate the main result in the next theorem. We define the asymptotic *vacua* space

$$\mathscr{K}^{\pm} := \{\Psi \in \mathscr{H}, b^{\pm}(h)\Psi = 0, h \in \mathfrak{h}\},$$

where $b^{\pm}$ are asymptotic annihilation operators defined by

$$b^{\pm}(h) := s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}b(h)\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}, \quad h \in \mathfrak{h}.$$

Their existence will be proved in Section 5.1. Let Δ be an interval of $\mathbb{R}$, we denote by $\mathbb{1}_\Delta(H)$, $\mathbb{1}_\Delta^{\mathrm{pp}}(H)$ the spectral projection of H, respectively, on Δ and on the pure point spectrum of H in Δ.

We set

$$\mathscr{H}^{\pm} := \mathscr{K}^{\pm} \otimes \Lambda(\mathfrak{h}), \quad \mathscr{H}_{\mathrm{bd}}(H) := \mathrm{Ran}\mathbb{1}^{\mathrm{pp}}(H)\mathscr{H}.$$

Theorem 3.2. *Assume hypotheses* (H0), (H1), (I0′), (M0) and (SR) are satisfied. Then *asymptotic completeness* holds for the fermionic Pauli–Fierz model, i.e:

$$\mathscr{K}^{\pm} = \mathscr{H}_{\mathrm{bd}}(H), \quad \textit{and} \quad \mathscr{H}^{\pm} = \mathscr{H}.$$

The proof is given in Section 5.4.

4. Spectral analysis

In this section we study the spectral properties of the fermionic Pauli–Fierz Hamiltonian. We prove in Theorem 4.3 the existence of a ground state below the essential spectrum with a gap equal to m. Such property was extensively studied especially in the massless case, see [4–6,15,19,27]. In this paper we follow the proof in [11].

We establish in Section 4.3 a Mourre estimate which has as consequence the local finiteness of point spectrum outside thresholds. This is an important step in the proof of asymptotic completeness as it is known for the N-body Schrödinger operators [10].

We start by collecting some commutator estimates which will be often useful in the sequel.

4.1. Commutator estimates

In this subsection, we will often use the following functional calculus formula for a self-adjoint operator A and a function $\chi \in C_0^\infty(\mathbb{R})$.

$$\chi(A) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \bar{z}}(z)(z-A)^{-1} dz \wedge d\bar{z}, \tag{4.1}$$

where $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of χ satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi, \ |\partial_{\bar{z}}\chi(z)| \leqslant c_n |\mathrm{Im}(z)|^n, \quad n \in \mathbb{N}.$$

Let $q \in C_0^\infty(\mathbb{R}^d)$, satisfying $0 \leqslant q \leqslant 1$, $q = 1$ near 0, in particular $q \in \mathscr{W}^s, \forall s \in \mathbb{N}$ and $||q(\frac{k}{R})||_{\mathscr{W}^s}$ are uniformly bounded for $R \geqslant 1$. We set $q^R := q(\frac{x}{R}) \otimes \mathbb{1}_{\mathfrak{g}}$.

Lemma 4.1. *Let* $\chi \in C_0^\infty(\mathbb{R})$. *We have for* R *large*

$$[\chi(H), \Gamma(q^R)] \in \begin{cases} O(R^{-1}) & \textit{under } (\mathrm{SR}), \\ o(R^0) & \textit{under } (\mathrm{I1}). \end{cases}$$

Proof. We first compute $[H, \Gamma(q^R)]$ as a quadratic form on $\mathscr{D}(H_0)$.

$$[H_0, \Gamma(q^R)] = \mathrm{d}\Gamma(q^R, [\omega, q^R]),$$

$$[P, \Gamma(q^R)] = \sum_{p,q \in \Xi} \Gamma(q^R)\mathrm{Wick}(w_{p,q}(1 - \Gamma(q^R))) + \mathrm{Wick}((\Gamma(q^R) - 1)w_{p,q})\Gamma(q^R) + hc.$$

We notice that $[H, \Gamma(q^R)]$ extends to a bounded operator on $\mathscr{D}(H_0)$.

Applying formula (4.1), we obtain

$$[\chi(H), \Gamma(q^R)] = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \bar{z}}(z)(z - H)^{-1}[H, \Gamma(q^R)](z - H)^{-1} dz \wedge d\bar{z}.$$

Then, it is enough to estimate $(\mathrm{i} + H_0)^{-1}[H, \Gamma(q^R)](\mathrm{i} + H_0)^{-1}$. We have the matrix $[\omega, q^R] = ([\omega_{ii}(k), q^R])_{ii} \in O(R^{-1})$ since $[\omega_{ii}(k), q^R] \in O(R^{-1})$. Using Theorem 2.15 (i) in the two cases (SR), (I1), we obtain the claimed estimate. □

Let $j_0 \in C_0^\infty(\mathbb{R}^d), j_\infty \in C^\infty(\mathbb{R}^d)$, such that $0 \leqslant j_0, j_\infty \leqslant 1$, $j_0 = 1$ near 0, j_∞ constant near ∞, $j_0^2 + j_\infty^2 \leqslant 1$. Hence $j_0, j_\infty \in \mathscr{W}^s, \forall s \in \mathbb{N}$ and $||j_\varepsilon(\frac{k}{R})||_{\mathscr{W}^s}, \varepsilon = 0, \infty$, are uniformly bounded for $R \geqslant 1$. We set $j_0^R := j_0(\frac{x}{R}), j_\infty^R := j_\infty(\frac{x}{R})$ and $j^R := (j_0^R, j_\infty^R)$.

Lemma 4.2. *Let* $\chi \in C_0^\infty(\mathbb{R})$. *We have for* R *large*

$$\chi(H^{\mathrm{ext}})I^*(j^R) - I^*(j^R)\chi(H) \in \begin{cases} O(R^{-1}) & \textit{under } (\mathrm{SR}), \\ o(R^0) & \textit{under } (\mathrm{I1}). \end{cases}$$

Proof. The proof is similar to the above one and it is based on commutation relation on Proposition 2.11 and estimate in Theorem 2.15. We have

$$\begin{aligned} &\chi(H^{\mathrm{ext}})I^*(j^R) - I^*(j^R)\chi(H) \\ &= \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \bar{z}}(z)(z - H^{\mathrm{ext}})^{-1}(H^{\mathrm{ext}}I^*(j^R) - I^*(j^R)H)(z - H)^{-1} dz \wedge d\bar{z}. \end{aligned} \tag{4.2}$$

Theorem 2.15 gives us

$$
\begin{aligned}
&||(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}(H^{\mathrm{ext}}I^*(j^R)-I^*(j^R)H)(\mathrm{i}+H_0)^{-1}|| \leqslant C \\
&\Bigg(||(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}\mathrm{d}\check{\Gamma}(j^R,[j^R,\omega])(\mathrm{i}+H_0)^{-1}|| \\
&+ \sum_{p,q\in\Xi}\sum_{i=1}^{p+q} ||\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}}\otimes(1-j_0^R)\otimes\mathbb{1}_{\mathfrak{h}^{\otimes(p+q-i)}}w_{p,q}||_{\mathrm{S}^{1/2}_{p,q}} \\
&+ \sum_{p,q\in\Xi}\sum_{i=1}^{p+q} ||\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}}\otimes j_\infty^R\otimes\mathbb{1}_{\mathfrak{h}^{\otimes(p+q-i)}}w_{p,q}||_{\mathrm{S}^{1/2}_{p,q}}\Bigg).
\end{aligned}
$$

The fact that $[j^R,\omega]\in O(R^{-1})$ and hypothesis (SR) and (I1) imply the following:

$$
||(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}(H^{\mathrm{ext}}I^*(j^R)-I^*(j^R)H)(\mathrm{i}+H_0)^{-1}|| \in \begin{cases} O(R^{-1}) & \text{under (SR)}, \\ o(R^0) & \text{under (I1)}. \end{cases} \tag{4.3}
$$

Now, combining (4.2) and (4.3) we prove the lemma. □

4.2. *The essential spectrum*

Next, we prove a HVZ-like theorem describing the essential spectrum of H.

Theorem 4.3. *Assume* (H0), (H1), (I0′) *and* (I1) *satisfied, then*

$$
\sigma_{\mathrm{ess}}(H) = [\inf\sigma(H)+m,+\infty[.
$$

Consequently, $\inf\sigma(H)$ *is a discrete eigenvalue*.

Proof. Let $\chi\in C_0^\infty(]-\infty,\inf\sigma(H)+m[)$. Since $H^{\mathrm{ext}}\mathbb{1}_{[1,+\infty[}(N_\infty)\geqslant\inf\sigma(H)+m$, we conclude that we have

$$
\chi(H^{\mathrm{ext}}) = \chi(H^{\mathrm{ext}})\mathbb{1}_{\{0\}}(N_\infty).
$$

We use Lemma 4.2 to write

$$
\begin{aligned}
\chi(H) &= \chi(H)I(j^R)I^*(j^R) \\
&= I(j^R)\mathbb{1}_{\{0\}}(N_\infty)\chi(H^{\mathrm{ext}})I^*(j^R)+o(R^0) \\
&= I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R)\chi(H)+o(R^0) \\
&= \Gamma((j_0^R)^2)\chi(H)+o(R^0).
\end{aligned}
$$

We claim that $\Gamma((j_0^R)^2)\chi(H)$ is a compact operator. In fact, we first see that $\prod_{i=1}^n j_0^2(\frac{x_i}{R})\ (K+\sum_{i=1}^n \omega(D_{x_i})+\mathrm{i})^{-1}$ is compact for every $n\in\mathbb{N}$. Next since $m>0$

$$\mathbb{1}_{[n,\infty[}(N)(H_0+\mathrm{i})^{-1}\to 0 \text{ when } n\to\infty \text{ in norm},$$

and using the fact that $\chi(H)(H_0+\mathrm{i})$ is bounded we obtain the compactness of $\Gamma((j_0^R)^2)\chi(H)$ and hence of $\chi(H)$. This shows

$$\sigma_{\mathrm{ess}}(H)\subset[\inf\sigma(H)+m,+\infty[.$$

To prove the converse inclusion, it is enough to find a Weyl sequence for every energy level $\lambda\in]\inf\sigma(H)+m,+\infty[$. Let Ψ be a ground state. The existence of Ψ is ensured by the first step of the proof. Let $h\in C_0^\infty(\mathbb{R}^d)$ with $||h||=1$ and k_0 such that $\omega(k_0)+\inf\sigma(H)=\lambda$. Let $R_j\in\mathbb{R}^d$ be a sequence satisfying $\lim_{j\to\infty} j^{-1}|R_j|=\infty$. We set

$$h_j(k)=j^{\frac{d}{2}}h(j(k-k_0))\mathrm{e}^{\mathrm{i}R_j.k}.$$

Then $w-\lim_{j\to\infty} h_j=0$, $||h_j||=1$ and $\lim_{j\to\infty}(\omega(k)-\omega(k_0))h_j=0$. Let $\Psi_j=b^*(h_j)\Psi$.

We have

$$\begin{aligned}(H-\lambda)\Psi_j &= b^*(h_j)(H-\lambda)\Psi+b^*(\omega h_j)\Psi+[P,b^*(h_j)]\Psi\\ &= b^*((\omega(k)-\omega(k_0))h_j)\Psi+[P,b^*(h_j)]\Psi.\end{aligned}$$

Since $\lim_{j\to\infty}[P,b^*(h_j)]\Psi=0$, we conclude that $\lim_{j\to\infty}(H-\lambda)\Psi_j=0$. Clearly Ψ_j is a Weyl sequence for the energy level λ. □

4.3. Mourre theory

Let G_0 be the operator acting on $\mathfrak{h}$ defined by

$$G_0 := -\frac{1}{2}(\nabla\omega(k).D_k+D_k.\nabla\omega(k)),\quad \text{on } C_0^\infty(\mathbb{R}^d).$$

The closure of G_0 is a self-adjoint operator with $C_0^\infty(\mathbb{R}^d)$ as a core [3, Proposition 4.2.3]. It is also the infinitesimal generator of the strong continuous group Φ_t associated to the vector field $\nabla\omega$, given by

$$\Phi_t F(k)=[\det\nabla\phi_{-t}(k)]^{\frac{1}{2}}F(\phi_{-t}(k)),\quad \text{for } F\in L^2(\mathbb{R}^d),$$

where ϕ_t is the flow of the vector field $\nabla\omega$. We denote the *threshold* set by

$$\tau := \sigma_{\mathrm{pp}}(H)+m\mathbb{N}^*,$$

where $\mathbb{N}^*$ is the set of positive integers. We define the conjugate operator

$$G := \mathrm{d}\Gamma(G_0).$$

It is essentially self-adjoint on $\Lambda^{\mathrm{fin}}(C_0^\infty(\mathbb{R}^d)\otimes\mathfrak{g})$.

Before considering the Mourre estimate we prove the following preliminary lemma.

Lemma 4.4. *Assume that* (H0)–(H1), (I0′) *and* (M0) *hold. Then*

(i) $\mathrm{e}^{\mathrm{i}tG}$ *preserves* $\mathscr{D}(H_0)$,
(ii) $|(H\Psi, G\Psi) - (G\Psi, H\Psi)| \leqslant c(||H_0\Psi||^2 + ||\Psi||^2), \Psi \in \mathscr{D}(H_0)\cap\mathscr{D}(G)$.

Proof. Let check Mourre hypothesis (i)–(ii). Let $H_{0,t} := \mathrm{e}^{\mathrm{i}tG}H_0\mathrm{e}^{-\mathrm{i}tG}$. It suffices to show that $\mathscr{D}(H_0)\subset\mathscr{D}(H_{0,t})$ to prove (i). We have

$$H_{0,t} = K\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(\mathrm{e}^{-\mathrm{i}tG_0}\omega\mathrm{e}^{\mathrm{i}tG_0})$$

$$= K\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(\omega(\phi_{-t}(k))).$$

Using the fact that $|\sum_{i=1}^N \omega(\phi_{-t}(k_i)) - \omega(k_i)| \leqslant N\sup_k ||\nabla\omega(k)||_{\mathfrak{g}}$, we obtain (i). We have as quadratic form on $\mathscr{D}(H_0)\cap\mathscr{D}(G)$,

$$[H, \mathrm{i}G] = \mathrm{d}\Gamma(|\nabla\omega|) + \sum_{p,q\in\Xi} \mathrm{Wick}([w_{p,q}, \mathrm{i}\,\mathrm{d}\Gamma(G_0)]) + hc.$$

Assuming hypothesis (M0) and using Theorem 2.10, we see that $[H, \mathrm{i}G]$ extends as a bounded operator on $\mathscr{D}(H_0)$. This proves (ii). □

Theorem 4.5. *Assume that* (H0), (H1), (I0′), (I1) *and* (M0) *hold. The following three assertions hold*:

(i) *Let* $\lambda\in\mathbb{R}\backslash\tau$. *Then there exist* $\varepsilon>0, C_0>0$ *and a compact operator* K_0 *such that*

$$\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H)[H, \mathrm{i}G]\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H) \geqslant C_0\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H) + K_0.$$

(ii) *For all* $[\lambda_1, \lambda_2]$ *such that* $[\lambda_1, \lambda_2]\cap\tau = \emptyset$, *one has*

$$\dim \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(H)\mathscr{H} < \infty.$$

Consequently $\sigma_{\mathrm{pp}}(H)$ *can accumulate only at* τ, *which is a closed countable set*.

(iii) *Let* $\lambda\in\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$. *Then there exists* $\varepsilon>0, C_0>0$ *such that*

$$\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H)[H, \mathrm{i}G]\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H) \geqslant C_0\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H).$$

Proof. We will follow the proof of the Mourre estimate in the case of a Pauli–Fierz Hamiltonian for bosons [11].

We set

$$d(\lambda) := \inf\left\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2; \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda, n = 1, 2\ldots, \tau \in \sigma_{\mathrm{pp}}(H)\right\},$$

$$\widetilde{d}(\lambda) := \inf\left\{\sum_{i=1}^{n} |\nabla\omega(k_i)|^2; \tau + \sum_{i=1}^{n} \omega(k_i) = \lambda, n = 0, 1, \ldots, \tau \in \sigma_{\mathrm{pp}}(H)\right\}.$$

$$\Delta_\lambda^\mu := [\lambda - \mu, \lambda + \mu], \quad \mu > 0,$$

$$d^\mu(\lambda) := \inf_{\nu \in \Delta_\lambda^\mu} d(\nu),$$

$$\widetilde{d}^\mu(\lambda) := \inf_{\nu \in \Delta_\lambda^\mu} \widetilde{d}(\nu),$$

$$E_0 := \inf \sigma(H).$$

We give statements, implying Theorem 4.5, that we will prove by induction in n:$H_1(n)$: Let $\varepsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists a compact operator K_0, an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(H)[H, \mathrm{i}G]\mathbb{1}_\Delta(H) \geqslant (d(\lambda) - \varepsilon)\mathbb{1}_\Delta(H) + K_0.$$

$H_2(n)$: Let $\varepsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists an interval $\Delta \ni \lambda$ such that

$$\mathbb{1}_\Delta(H)[H, \mathrm{i}G]\mathbb{1}_\Delta(H) \geqslant (\widetilde{d}(\lambda) - \varepsilon)\mathbb{1}_\Delta(H).$$

$H_3(n)$: Let $\mu > 0, \varepsilon_0 > 0$ and $\varepsilon > 0$. There exists $\delta > 0$ such that for all $\lambda \in [E_0, E_0 + nm - \varepsilon_0]$, one has

$$\mathbb{1}_{\Delta_\lambda^\delta}(H)[H, \mathrm{i}G]\mathbb{1}_{\Delta_\lambda^\delta}(H) \geqslant (\widetilde{d}^\mu(\lambda) - \varepsilon)\mathbb{1}_{\Delta_\lambda^\delta}(H).$$

$S_1(n)$: τ is closed countable set in $[E_0, E_0 + nm]$.$S_2(n)$: For all $\lambda_1 \leqslant \lambda_2 \leqslant E_0 + nm$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, one has

$$\dim \mathbb{1}^{\mathrm{pp}}_{[\lambda_1, \lambda_2]}(H)\mathscr{H} < \infty.$$

The sketch of the proof is given by

$$S_2(n-1) \Rightarrow S_1(n),$$

$$(S_1(n), H_3(n-1)) \Rightarrow H_1(n),$$

$$H_1(n) \Rightarrow H_2(n),$$

$$H_2(n) \Rightarrow H_3(n),$$

$$H_1(n) \Rightarrow S_2(n).$$

We notice that $H(1)$ and $S(1)$ are immediate because the spectrum of H is discrete in $[E_0, E_0 + m[$. $S_2(n-1) \Rightarrow S_1(n)$ is obvious. Recall that the hypotheses of Mourre (part (i)–(ii) of Lemma 4.4) implies the weaker condition $H \in C^1(A)$ introduced in [3] (i.e.: the map $\mathbb{R} \ni s \mapsto e^{isA}(z-H)^{-1}e^{-isA}, z \in \mathbb{C}\backslash\sigma(H)$ is strongly $C^1(\mathbb{R}, \mathscr{B}(\mathscr{H}))$). Under the later condition proved by Lemma 4.4, the fact that $H_1(n) \Rightarrow H_2(n)$, $H_2(n) \Rightarrow H_3(n)$ follow from standard arguments in [3,9,24]. The implication $H_1(n) \Rightarrow S_2(n)$ is based in the Virial theorem (see e.g. [10]), which holds also under the hypotheses of Mourre shown in Lemma 4.4. Then, we have only to prove the implication $(S_1(n), H_3(n-1) \Rightarrow H_1(n))$.

Let $\chi \in C_0^\infty(\mathbb{R})$.

$$\chi(H) = I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R)\chi(H) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)I^*(j^R)\chi(H) \tag{4.4}$$

$$= \Gamma((j_0^R)^2)\chi(H) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)\chi(H^{\text{ext}})I^*(j^R) + o(R^0). \tag{4.5}$$

Eq. (4.4) follows from the fact that $I(j^R)I^*(j^R) = \mathbb{1}$ and $I(j^R)$ is bounded. Eq. (4.5) follows from the fact that one has

$$\Gamma((j_0^R)^2) = I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R), \quad I(j^R) = \Gamma(j^R)U,$$

and application of Lemma 4.2. We recall that the term $\Gamma((j_0^R)^2)\chi(H)$ is compact since the operator $\Gamma((j_0^R)^2)(H_0+1)^{-\frac{1}{2}}$ is compact.

Let $\lambda \in [E_0, E_0 + nm[$. Since $S_2(n-1) \Rightarrow S_1(n)$, the set τ is closed in $[E_0, E_0 + nm]$, which gives $d(\lambda) = \sup_{\mu > 0} d^\mu(\lambda)$. So we can choose μ such that $d^\mu(\lambda) \geqslant d(\lambda) - \frac{\varepsilon}{3}$. $H_3(n-1)$ gives for $\lambda_1 < E_0 + (n-1)m$

$$\mathbb{1}_{\Delta_{\lambda_1}^\delta}(H)[H, iG]\mathbb{1}_{\Delta_{\lambda_1}^\delta}(H) \geqslant (\widetilde{d}^\mu(\lambda_1) - \frac{\varepsilon}{3})\mathbb{1}_{\Delta_{\lambda_1}^\delta}(H).$$

Replacing λ_1 with $\lambda - d\Gamma(\omega(k))$, we obtain

$$\mathbb{1}_{\Delta_\lambda^\delta}(H + \mathbb{1} \otimes d\Gamma(\omega(k)))([H, iG] + \mathbb{1} \otimes d\Gamma(|\nabla\omega|^2))$$
$$\times \mathbb{1}_{\Delta_\lambda^\delta}(H + \mathbb{1} \otimes d\Gamma(\omega(k)))\mathbb{1}_{[1,\infty[}(N_\infty)$$

$$\geqslant \mathbb{1}_{\Delta_\lambda^\delta}(H + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega(k)))$$
$$\times (\widetilde{d}^\mu(\lambda - \mathbb{1}\otimes \mathrm{d}\Gamma(\omega(k))) + \mathbb{1}\otimes \mathrm{d}\Gamma(|\nabla\omega|^2) - \frac{\varepsilon}{3})\mathbb{1}_{[1,\infty[}(N_\infty)$$
$$\geqslant (d^\mu(\lambda) - \frac{\varepsilon}{3})\mathbb{1}_{\Delta_\lambda^\delta}(H + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega(k)))(H + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega))\mathbb{1}_{[1,\infty[}(N_\infty)$$
$$\geqslant (d^\mu(\lambda) - \frac{2\varepsilon}{3})\mathbb{1}_{\Delta_\lambda^\delta}(H + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega(k)))(H + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega))\mathbb{1}_{[1,\infty[}(N_\infty)$$

Let $\chi \in C_0^\infty(\mathbb{R}), \chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. One has

$$\chi(H)[H, \mathrm{i}G]\chi(H)$$
$$= \Gamma((j_0^R)^2)\chi(H)[H, \mathrm{i}G]\chi(H)$$
$$+ I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(H^{\mathrm{ext}})I^*(j^R)\chi_1(H)[H, \mathrm{i}G]\chi(H) + o(R^0) \tag{4.6}$$

$$= \Gamma((j_0^R)^2)\chi(H)[H, \mathrm{i}G]\chi(H)$$
$$+ I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(H^{\mathrm{ext}})I^*(j^R)[H, \mathrm{i}G]\chi(H) + o(R^0) \tag{4.7}$$

$$= \Gamma((j_0^R)^2)\chi(H)[H, \mathrm{i}G]\chi(H)$$
$$+ I^*(j^R)\mathbb{1}_{[1,+\infty[}(N_\infty)\chi(H^{\mathrm{ext}})[H^{\mathrm{ext}}, \mathrm{i}G^{\mathrm{ext}}]\chi(H^{\mathrm{ext}})I^*(j^R) + o(R^0). \tag{4.8}$$

Eq. (4.6) follows by (4.5). Lemma 4.2 gives (4.7). Since $[H, \mathrm{i}G]$ is similar to the Hamiltonian H, one can easily derive an equivalent estimate for $[H, \mathrm{i}A]$ as in Lemma 4.2. This will imply (4.8) with $G^{\mathrm{ext}} := G\otimes\mathbb{1} + \mathbb{1}\otimes G$.

Since the operator $K_1^R := \Gamma((j_0^R)^2)\chi(H)[H, \mathrm{i}G]\chi(H)$ is compact, this gives for χ such that $\mathrm{supp}\,\chi \subset [\lambda - \delta, \lambda + \delta[$

$$\chi(H)[H, \mathrm{i}G]\chi(H) \geqslant \left(d(\lambda) - \frac{2\varepsilon}{3}\right)\chi^2(H) + K_1(R) + o(R^0).$$

Choosing R large enough, we obtain $H_1(n)$. Properties (ii), (iii) are standard consequences of (i). $\square$

5. Asymptotic completeness

In this section we prove asymptotic completeness for the fermionic Pauli–Fierz Hamiltonian H. We first study the asymptotic limits of Wick polynomials, then we construct the wave operator and we show that it is an unitary map. In Section 5.2, we prove a minimal velocity estimate using the Mourre estimate and some other propagation estimates. We construct the asymptotic velocity projection in Section 5.3 and then we finish the proof of asymptotic completeness.

5.1. The wave operator

We prove existence of asymptotic limits of Wick polynomials, using a Cook method [21]. We are interested in both limits in time $\mp\infty$, but since they are equivalent we will consider only the limit $+\infty$.

Theorem 5.1. (i) *The following strong limits*

$$b^{\sharp,\pm}(h) := s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}b^{\sharp}(h)\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}, \quad h\in\mathfrak{h}, \tag{5.1}$$

exist and are called asymptotic creation–annihilation operators.

(ii) *The map*

$$\mathfrak{h}\ni h\mapsto b^{\pm,\sharp}(h) \text{ are norm continuous.}$$

(iii) *The CAR relations hold*:

$$\{b^{\pm}(g), b^{\pm^*}(h)\} = (g|h),$$

$$\{b^{\pm^\sharp}(g), b^{\pm^\sharp}(h)\} = 0, \quad h,g\in\mathfrak{h}.$$

(iv) *The Hamiltonian preserves the asymptotic creation-annihilation operators*:

$$\mathrm{e}^{\mathrm{i}tH}b^{\pm,\sharp}(h)\mathrm{e}^{-\mathrm{i}tH} = b^{\pm,\sharp}(\mathrm{e}^{-\mathrm{i}t\omega}h), \quad \text{for } h\in\mathfrak{h}.$$

Proof. We use a Cook argument to prove the existence of limit (5.1). Let $\mathfrak{h}_0$ denotes the subspace $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ and $h_t := \mathrm{e}^{-\mathrm{i}t\omega}h, h\in\mathfrak{h}$. We have for $h\in\mathfrak{h}_0$,

$$\mathrm{e}^{\mathrm{i}tH}b^*(h_t)\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-1} = b^*(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}[P, b^*(h_s)](H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}sH}\,ds. \tag{5.2}$$

Now, using commutation relation in Proposition 2.11 and the short range condition (SR), we see that for $h\in\mathfrak{h}_0$ we have

$$\sum_{p,q\in\Xi}\sum_{i=1}^{p} p(w_{p,q}(k_i)|h(k_i))(K+\mathrm{i})^{-1}\in O(t^{-\mu}), \quad \mu>1.$$

This yields the strong limit (5.1) for $h\in\mathfrak{h}_0$. We extend it to $h\in\mathfrak{h}$ using density argument and the fact that $||b^{\sharp,\pm}(h)||\leqslant||h||, h\in\mathfrak{h}_0$. The case of $b(h)$ is similar and follows by taking the adjoint. The continuity of the map in (ii) is a consequence of $||b^{\sharp,\pm}(h)||\leqslant||h||$. The anti-commutation relations and the action of $\mathrm{e}^{\mathrm{i}tH}$ on the asymptotic creation annihilation operators are elementary. Note that (iii) yields that $||b^{\sharp,\pm}(h)|| = ||h||$. □

Proposition 5.2. *We have for* $h \in \mathfrak{h}$,

$$b^{\pm}(h)\mathbb{1}_{]-\infty,\lambda]}(H)\mathscr{H} \subset \mathbb{1}_{]-\infty,\lambda-m]}(H)\mathscr{H}.$$

Proof. The proof follows using a standard argument due to Høegh–Krohn [22]. □

We define the space of *asymptotic vacua* associated to the CAR representation given by the asymptotic creation-annihilation operators

$$\mathscr{K}^{\pm} := \{\Psi \in \mathscr{H} \mid b^{\pm}(h)\Psi = 0, h \in \mathfrak{h}\}.$$

We denote by $\mathscr{H}^{\pm}$ the space $\mathscr{K}^{\pm} \otimes \Lambda(\mathfrak{h})$.

Proposition 5.3. *The following assertions hold*:

(i) $\mathscr{K}^{\pm}$ *is closed* H*-invariant space.*
(ii) *One has*

$$\mathrm{Ran}\mathbb{1}^{\mathrm{pp}}(H) \subset \mathscr{K}^{\pm}.$$

Proof. The fact that $\mathscr{K}^{\pm}$ is H-invariant follows from Theorem 5.1. Let us prove now (ii). Let $u \in \mathscr{H}$ such that $Hu = Eu$, one has

$$\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} b(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}u = 0,$$

since $s - \lim_{t \to \pm\infty} b(\mathrm{e}^{-\mathrm{i}t\omega}h) = 0$ and $\mathrm{e}^{\mathrm{i}tH} b(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}u = \mathrm{e}^{\mathrm{i}t(H-E)} b(\mathrm{e}^{-\mathrm{i}t\omega}h)u$. This gives $b^{\pm}(h)u = 0$. □

We define

$$H^{\pm} := H_{|\mathscr{K}^{\pm}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega),$$

and the *wave operator*

$$\Omega^{\pm} : \mathscr{H}^{\pm} \to \mathscr{H},$$

$$\Omega^{\pm}\psi \otimes \prod_{i=1}^{n} b^{*}(h_i)\Omega := \prod_{i=1}^{n} b^{\pm *}(h_i)\psi, \quad \text{for } \psi \in \mathscr{K}^{\pm}, \ h_i \in \mathfrak{h}.$$

Theorem 5.4. $\Omega^{\pm}$ *is a unitary map satisfying*

$$\prod_{i=1}^{n} b^{\sharp,\pm}(h_i)\Omega^{\pm} = \Omega^{\pm}\mathbb{1}_{\mathscr{H}} \otimes \prod_{i=1}^{n} b^{\sharp}(h_i), \quad \text{for } h_i \in \mathfrak{h},$$
$$H\Omega^{\pm} = \Omega^{\pm}H^{\pm}.$$

Proof. The intertwining relations in Proposition 5.1 gives that $\Omega^{\pm}$ is isometric and satisfies properties stated in the theorem. Let prove that $\Omega^{\pm}$ is unitary. Let $\Psi \in (\mathrm{Ran}(\Omega^{\pm}))^{\perp}$. Since H preserves the later subspace one may assume that Ψ is localized in energy i.e: $\Psi = \mathbb{1}_{]-\infty,\lambda]}(H)\Psi$. Then, if $nm > \inf \sigma(H)$ and using Proposition 5.2, we obtain that

$$\prod_{i=1}^{n} b^{\pm}(h_i)\Psi = 0, \quad \forall h_1, \dots, h_n \in \mathfrak{h}. \tag{5.3}$$

Let n_0 be the smallest positive integer such that identity (5.3) is satisfied for $n = n_0$ but do not hold for $n = n_0 - 1$. Therefore, $\prod_{i=2}^{n_0} b^{\pm}(h_i)\Psi \in \mathscr{K}^{\pm}$ and $\prod_{i=n_0}^{2} b^{*\pm}(h_i)\prod_{i=2}^{n_0} b^{\pm}(h_i)\Psi \in \mathrm{Ran}(\Omega^{\pm})$. Using the fact that $\Psi \in \mathrm{Ran}(\Omega^{\pm})^{\perp}$, we obtain

$$0 = \left(\Psi, \prod_{i=n_0}^{2} b^{*\pm}(h_i) \prod_{i=2}^{n_0} b^{\pm}(h_i)\Psi\right) = \left\|\prod_{i=2}^{n_0} b^{\pm}(h_i)\Psi\right\|.$$

This is in contradiction with the choice of n_0 and hence leads to the fact that $\Psi = 0$. □

We define an *extended wave operator*

$$\Omega^{\mathrm{ext},\pm} : \mathscr{H}^{\mathrm{ext}} \to \mathscr{H},$$

$$\Omega^{\mathrm{ext},\pm}\psi \otimes \prod_{i=1}^{n} b^{*}(h_i)\Omega := \prod_{i=1}^{n} b^{\pm *}(h_i)\psi.$$

We notice that $\Omega^{\mathrm{ext},\pm}_{|\mathscr{H}^{\pm}} = \Omega^{\pm}$. This suggests to treat sometimes $\Omega^{\mathrm{ext},\pm}$ as a partial isometry. Another construction of the extended wave operator is given by the following theorem, see [11, Theorem 5.7].

Theorem 5.5. (i) *Let* $u \in \mathscr{D}(\Omega^{\mathrm{ext},\pm})$. *Then one has*

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} u = \Omega^{\mathrm{ext},\pm} u,$$

where I is the scattering identification operator defined in the Section 2.1.

(ii) *Let* $\chi \in C_0^\infty(\mathbb{R})$. *Then* $\mathrm{Ran}\chi(H^{\mathrm{ext}}) \subset \mathscr{D}(\Omega^{\mathrm{ext},\pm})$ *and the operators* $I\chi(H^{\mathrm{ext}})$, $\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})$ *are bounded. Moreover*

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} \chi(H^{\mathrm{ext}}) = \Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}}).$$

5.2. Minimal velocity estimate

In this subsection we will derive a minimal velocity estimate, following the approach of Graf [18]. It is based on the Mourre estimate and it does not need a double commutator estimate. We first prove a maximal velocity propagation estimate in Proposition 5.7, then we state phase space propagation estimates, which will be used to show the minimal velocity estimate.

We will use the following notation for Heisenberg derivatives:

$$\mathbb{d}_0 := \partial_t + \mathrm{i}[\omega(k), .],$$

$$\mathbb{D}_0 := \partial_t + \mathrm{i}[\mathrm{d}\Gamma(\omega), .],$$

$$\mathbb{D} := \partial_t + \mathrm{i}[H, .].$$

We recall a standard argument often used to prove propagation estimates, see e.g. [10, Lemma B.4.1].

Lemma 5.6. *Let* $\Phi(t)$ *be a bounded family of self-adjoint operators. Assume that there exist* $C_0 > 0$ *and operator valued function* $B(t)$ *and* $B_i(t)$, $i = 1, \dots, n$, *such that*

$$\mathbb{D}\Phi(t) \geqslant C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t),$$

$$\int_1^\infty ||B_i(t)\mathrm{e}^{-\mathrm{i}tH}u||^2 dt \leqslant C||u||^2, \quad i = 1, \dots, n.$$

Then there exists C_1 *such that*

$$\int_1^\infty ||B(t)\mathrm{e}^{-\mathrm{i}tH}u||^2 dt \leqslant C_1||u||^2.$$

Proposition 5.7. *Let* $\chi \in C_0^\infty(\mathbb{R})$. *For* $R' > R > v_{\max} := \sup||\nabla\omega(k)||_{\mathscr{B}(\mathfrak{g})}$, *there exists c such that we have*:

$$\int_1^\infty ||\mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2 \frac{dt}{t} \leqslant c||u||^2.$$

Proof. Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near ∞, to 0 near the origin, with $F'(s) \geqslant \mathbb{1}_{[R,R']}(s)$. We consider the observable

$$\Phi(t) := \chi(H)\,\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\chi(H).$$

We claim that

$$\mathbb{D}\Phi(t) \geqslant t^{-1} C_0 \chi(H)\,\mathrm{d}\Gamma\left(F'\left(\frac{|x|}{t}\right)\right)\chi(H) + O(t^{-\mu}), \quad \mu > 1. \tag{5.4}$$

One has

$$\mathbb{D}\Phi(t) = \chi(H)\,\mathrm{d}\Gamma\left(\mathbb{d}_0 F\left(\frac{|x|}{t}\right)\right)\chi(H) + \chi(H)\left[P, \mathrm{i}\,\mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right]\chi(H).$$

Using the fact that $\mathbb{d}_0 F\left(\frac{|x|}{t}\right) \geqslant \frac{c_0}{t} F'\left(\frac{|x|}{t}\right) + O(t^{-2})$, it is sufficient to show that the second term in the previous identity is $O(t^{-\mu}), \mu > 1$, to have (5.4).

We have

$$\begin{aligned}\left[P, \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] &= \sum_{p,q \in \Xi} \left[\mathrm{Wick}(w_{p,q}), \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right] \\ &= \sum_{p,q \in \Xi} \mathrm{Wick}\left(\left[w_{p,q}, \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right]\right).\end{aligned} \tag{5.5}$$

Using Proposition 2.11, it is enough to estimate the kernel of the operator $\left[w_{p,q}, \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right]$ which is equal to $\sum_{i=1}^{q} F\left(\frac{|x_i'|}{t}\right) w_{p,q}(k,k') - \sum_{i=1}^{q} F\left(\frac{|x_i|}{t}\right) w_{p,q}(k,k')$. Now hypothesis (SR) gives us

$$||(H+\mathrm{i})^{-1}\left[P, \mathrm{d}\Gamma\left(F\left(\frac{|x|}{t}\right)\right)\right](H+\mathrm{i})^{-1}|| \in O(t^{-\mu}), \quad \mu > 1.$$

This proves (5.4). Then using Lemma 5.6 and inequality (5.4), we obtain the claimed estimate announced in the proposition. □

Proposition 5.8. *Let* $\chi \in C_0^\infty(R), 0 < c_0 < c_1$, *and*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma\left(\left\langle \frac{x}{t} - \nabla\omega(k), \mathbb{1}_{[c_0,c_1]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t} - \nabla\omega(k)\right)\right\rangle\right).$$

One has

$$\int_1^\infty ||\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2 \frac{dt}{t} \leqslant c||u||^2.$$

Proof. Let $R_0(x) \in C^\infty$ be a function such that:

$$R_0(x) = 0, \quad \text{for } |x| \leqslant \frac{c_0}{2},$$

$$R_0(x) = \frac{1}{2}x^2 + c, \quad \text{for } |x| \geqslant 2c_1,$$

$$\nabla_x^2 R_0 \geqslant \mathbb{1}_{[c_0,c_1]}(|x|).$$

We choose $c_1 > 2$, $c_2 > c_1 + 1$ and we define the function

$$R(x) := F(|x|)R_0(x),$$

where $F(s) = 1$, if $s \leqslant c_1, F(s) = 0$, if $s \geqslant c_2$.

We set

$$b(t) := R\left(\frac{x}{t}\right) - \frac{1}{2}\left(\left\langle \nabla R\left(\frac{x}{t}\right), \frac{x}{t} - \nabla\omega(k)\right\rangle + hc\right).$$

We consider the observable

$$\Phi(t) := \chi(H)\, \mathrm{d}\Gamma(b(t))\chi(H).$$

Pseudo-differential calculus gives

$$\begin{aligned}&\chi(H)\mathbb{D}_0\, \mathrm{d}\Gamma(b(t))\chi(H)\\ &\quad \geqslant \chi(H)\left(\frac{1}{t}\Theta_{[c_0,c_2]}(t) - \frac{1}{t}\mathrm{d}\Gamma\left(\mathbb{1}_{[v_{max}+1,c_2]}\left(\frac{|x|}{t}\right)\right)\right)\chi(H) + O(t^{-2}).\end{aligned}$$

The first term will serves in the application of Lemma 5.6 and the second is integrable along the evolution using Proposition 5.7. To complete the proof of the proposition, it suffices to show that:

$$\chi(H)[P, \mathrm{i}\, \mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-\mu}), \quad \mu > 1. \tag{5.6}$$

As in the Proposition 5.7, it is enough to see that $[P, \mathrm{d}\Gamma(b(t))]$ has the kernel $\sum_{p,q\in\Xi} \sum_{i=1}^{p} b(t)_i w_{p,q}(k,k') - \sum_{i=1}^{q} b(t)_i w_{p,q}(k,k')$ which is using (SR) condition is estimated by $O(t^{-\mu})$, $\mu > 1$. $\square$

Proposition 5.9. *Let* $0 < c_0 < c_1, J \in C_0^\infty(\{c_0 < |x| < c_1\}), \chi \in C_0^\infty(\mathbb{R})$. *For* $1 \leqslant i \leqslant d$, *one has*

$$\int_1^\infty \left|\left|\mathrm{d}\Gamma\left(\left|J\left(\frac{x}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + hc\right|\right)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right|\right|^2 \frac{dt}{t} \leqslant c||u||^2.$$

Proof. We set

$$A := \left(\frac{x}{t} - \nabla\omega(k)\right)^2 + t^{-\delta},$$
$$b(t) := J\left(\frac{x}{t}\right)A^{\frac{1}{2}}J\left(\frac{x}{t}\right).$$

Let $J_1 \in C_0^\infty(\{c_0 < |x| < c_1\}), 0 \leqslant J \leqslant 1$, $J = 1$ near the support of J_1. We consider the observable

$$\Phi(t) := -\chi(H)\,\mathrm{d}\Gamma(b(t))\chi(H).$$

One has

$$\chi(H)\mathbb{D}_0\,\mathrm{d}\Gamma(b(t))\chi(H) = -\chi(H)\,\mathrm{d}\Gamma(\mathbb{d}_0 b(t))\chi(H),$$

and we have using [11, Lemma 6.4]

$$\begin{aligned}\chi(H)\mathbb{D}_0\,\mathrm{d}\Gamma(b(t))\chi(H)\\ \geqslant \frac{c_0}{t}\chi(H)\,\mathrm{d}\Gamma\left(|J_1\left(\frac{x}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + hc|\right)\chi(H)\\ -\frac{c}{t}\chi(H)\,\mathrm{d}\Gamma\left(\left\langle\frac{x}{t} - \nabla\omega, J_2\left(\frac{x}{t}\right)\left(\frac{x}{t} - \nabla\omega\right)\right\rangle\right)\chi(H) + O(t^{-1-\mu}).\end{aligned}$$

The second term is integrable along the evolution by Proposition 5.8. Then, it is enough to show that

$$\chi(H)[P, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-\mu}), \quad \mu > 1.$$

This follows by using (5.5), the fact that $J(\frac{x}{t})A^{\frac{1}{2}} \in O(1)$. The proof ends by application of Lemma 5.6. □

Proposition 5.10. *Let $\chi \in C_0^\infty(\mathbb{R})$, supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. There exist $\varepsilon > 0, C$ such that we have*

$$\int_1^\infty \left\|\Gamma\left(\mathbb{1}_{[0,\varepsilon]}\left(\frac{|x|}{t}\right)\right)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

Proof. Let $\varepsilon > 0$. Let $q \in C_0^\infty(|x| \leqslant 2\varepsilon)$ such that $0 \leqslant q \leqslant 1, q(x) = 1$, if $|x| \leqslant \varepsilon$.

We set

$$\Phi(t) := \chi(H)\Gamma(q^t)\frac{G}{t}\Gamma(q^t)\chi(H),$$

where G is the conjugate operator considered in Section 4.3. The Heisenberg derivative of $\Phi(t)$ is

$$\begin{aligned}\mathbb{D}\Phi(t) &= \chi(H)\, \mathrm{d}\Gamma(q^t, \mathbb{d}_0 q^t)\, \frac{G}{t}\, \Gamma(q^t)\chi(H) + hc \\ &\quad +\ \chi(H)[P, \mathrm{i}\Gamma(q^t)]\, \frac{G}{t}\, \Gamma(q^t)\chi(H) + hc \\ &\quad +\ t^{-1}\chi(H)\Gamma(q^t)[H, \mathrm{i}G]\Gamma(q^t)\chi(H) \\ &\quad -\ t^{-1}\chi(H)\Gamma(q^t)\, \frac{G}{t}\, \Gamma(q^t)\Gamma(q^t)\chi(H) \\ &=: R_1 + R_2 + R_3 + R_4.\end{aligned}$$

We claim that

$$R_2 \in O(t^{-\mu}), \quad \mu > 1. \tag{5.7}$$

To prove (5.7) it suffices to use Lemma 4.1 and the fact that $\frac{G}{t}\Gamma(q^t)(N+1)^{-2} \in O(1)$.

We consider now R_1. We have

$$\mathbb{d}_0 q^t = -\frac{1}{2t}\left\langle \frac{x}{t} - \nabla\omega(k), \nabla q\left(\frac{x}{t}\right)\right\rangle + hc + r^t =: \frac{1}{t} g^t + r^t,$$

where $r^t \in O(t^{-2})$. We have

$$||\chi(H)\, \mathrm{d}\Gamma(q^t, r^t)\frac{G}{t}\Gamma(q^t)\chi(H)|| \in O(t^{-2}).$$

We set

$$B_1 := \chi(H)\, \mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}}, \quad B_2 := (N+1)^{\frac{1}{2}}\frac{G}{t}\Gamma(q^t)\chi(H).$$

So we obtain the inequality

$$R_1 \geqslant -\varepsilon_0^{-1} t^{-1} B_1 B_1^* - \varepsilon_0 t^{-1} B_2 B_2^*.$$

Using arguments in [11, Proposition 6.5], we obtain

$$-B_2 B_2^* \geqslant -C_1 \chi(H)\Gamma(q^t)^2\chi(H) - Ct^{-1},$$

$$\int_1^\infty ||B_1 \mathrm{e}^{-\mathrm{i}tH} u||^2 \frac{dt}{t} \leqslant C||u||^2.$$

Using Lemma 4.1 and Theorem 4.5 (iii), we have

$$R_3 \geqslant C_0 t^{-1}\chi(H)\Gamma(q^t)^2\chi(H) - Ct^{-2}.$$

We have

$$-R_4 \leqslant C_2 \frac{\varepsilon}{t} \chi(H)\Gamma(q^t)^2\chi(H) + Ct^{-2}.$$

Collecting the four terms, we obtain

$$\begin{aligned}\mathbb{D}\phi(t) &\geqslant -\varepsilon_0 t^{-1} B_2 B_2^* - \varepsilon_0^{-1} t^{-1} B_1 B_1^* + R_2 + R_3 + R_4 \\ &\geqslant (C_0 - \varepsilon_0 C_1 - \varepsilon C_2) t^{-1} \chi(H)\Gamma(q^t)^2\chi(H) + R(t) \\ &\geqslant \widetilde{C}_0 \chi(H) t^{-1} \Gamma(q^t)^2 \chi(H) - R(t),\end{aligned}$$

where $R(t)$ is integrable. By Lemma 5.6, we obtain the inequality announced in the proposition for χ supported near a one energy level λ. Then we complete the proof for an arbitrary χ using a standard argument, see e.g. [10, Proposition 4.4.7]. □

5.3. Asymptotic velocity

In this subsection we construct asymptotic velocity projection and we give the proof of asymptotic completeness (Theorem 3.2).

Theorem 5.11. *Let $q, \widetilde{q} \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leqslant q, \widetilde{q} \leqslant 1, q, \widetilde{q} = 1$ on a neighborhood of 0 and $q^t := q(\frac{x}{t})$*

(i) *The following limits exist*:

$$\Gamma^\pm(q) := s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}tH}.$$

(ii) *We have*

$$\begin{aligned}&\Gamma^\pm(q\widetilde{q}) = \Gamma^\pm(q)\Gamma^\pm(\widetilde{q}), \\ &0 \leqslant \Gamma^\pm(q) \leqslant \Gamma^\pm(\widetilde{q}) \leqslant \mathbb{1}, \quad \text{if } 0 \leqslant q \leqslant \widetilde{q}, \\ &[H, \Gamma^\pm(q)] = 0.\end{aligned}$$

Proof. It is sufficient using a density argument and Lemma 4.1 to show for $\chi \in C_0^\infty(\mathbb{R})$ the existence of the limit

$$s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \chi(H) \Gamma(q^t) \chi(H) \mathrm{e}^{-\mathrm{i}tH}. \tag{5.8}$$

In general, showing the existence of asymptotic limits amounts to bound Heisenberg derivatives. We have on $\mathscr{H}$:

$$\partial_t(e^{itH}\chi(H)\,\Gamma(q^t)\chi(H)e^{-itH}) = e^{itH}\chi(H)\,d\Gamma(q^t, \mathbb{d}_0 q^t)\chi(H)e^{-itH} + e^{itH}\chi(H)[P, i\Gamma(q^t)]\chi(H)e^{-itH}. \quad (5.9)$$

We know from the proof of Lemma 4.1 that

$$\chi(H)[P, i\Gamma(q^t)]\chi(H) \in O(t^{-\mu}), \quad \mu > 1. \quad (5.10)$$

Let now analyze the r.h.s. in (5.9). One can write

$$\mathbb{d}_0 q^t = \frac{1}{t} g^t + r^t,$$

where

$$g^t = -\frac{1}{2}((\frac{x}{t} - \nabla\omega(k))\nabla q(\frac{x}{t}) + hc),$$

and

$$r^t \in O(t^{-2}).$$

Then we estimate the part of r^t, using inequality (ii) in Lemma 2.3, we obtain

$$||\chi(H)\,d\Gamma(q^t, r^t)\chi(H)|| \in O(t^{-2}). \quad (5.11)$$

The term coming from $\frac{1}{t}g^t$, will be estimated as follows:

$$|(e^{-itH}u, \chi(H)\,d\Gamma(q^t, g^t)\chi(H)e^{-itH}u)| \leqslant ||d\Gamma(|g^t|)^{\frac{1}{2}}\chi(H)e^{-itH}u||^2, \quad u \in \mathscr{H}. \quad (5.12)$$

We notice that the r.h.s. in (5.12) equals the observable in the propagation estimate Proposition 5.9. Now, combining (5.10)–(5.12) and Proposition 5.9, we prove the integrability of the Heisenberg derivative (5.9). Thus we prove the existence of limit (5.8) using a general argument (see e.g. [11, Lemma A.1]).

The first statement in (ii) follows from the fact that

$$\Gamma(q^t\tilde{q}^t) = \Gamma(q^t)\Gamma(\tilde{q}^t).$$

The second statement follows by

$$0 \leqslant \Gamma(q^t) \leqslant \Gamma(\tilde{q}^t) \leqslant 1 \quad \text{if } 0 \leqslant q \leqslant \tilde{q}.$$

The last statement is a consequence of (i) and Lemma 4.1. □

Let us state a corollary of the above theorem, giving the construction of the asymptotic velocity projection $P_0^{\pm}$.

Corollary 5.12. *Let* $\{q_n\} \in C_0^\infty(\mathbb{R}^d)$ *be a decreasing sequence of functions such that* $0 \leqslant q_n \leqslant 1$, $q_n = 1$ *on a neighborhood of* 0 *and* $\bigcap_{n=1}^\infty \operatorname{supp} q_n = \{0\}$. *Then the following limit exist and it does not depend in the choice of the sequence*

(i) $P_0^\pm := \lim_{n\to\infty} \Gamma^\pm(q_n)$,
(ii) $\operatorname{Ran} P_0^\pm \subset \mathscr{K}^\pm$.

Moreover $P_0^\pm$ *is an orthogonal projection.*

Proof. The existence of the limit (i) follows from Theorem 5.11 (ii) and the argument that the limit of a decreasing sequence of self-adjoint operators satisfying $\Gamma^\pm(q_{n+1})\Gamma^\pm(q_n) = \Gamma^\pm(q_{n+1})$ exist in the strong limit, see [10, Lemma A.3]. The independence from the choice of the sequence follows from the fact that there exists an index m_n such that $q_n \geqslant \widetilde{q}_{m_n}, \widetilde{q}_n \geqslant q_{m_n}; \lim_{n\to\infty} m_n = +\infty$ and Theorem 5.11 (ii).

Let us prove statement (ii). Let $u \in \mathscr{H}, h \in \mathfrak{h}$, we have

$$\begin{aligned} b^\pm(h)\Gamma^\pm(q)u &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} b(h_t)\mathrm{e}^{-\mathrm{i}tH}\Gamma^\pm(q)u \\ &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} b(h_t)\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH}u \\ &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t) b(q^t h_t)\mathrm{e}^{-\mathrm{i}tH}u. \end{aligned} \tag{5.13}$$

In order to show (ii) it suffices to prove that $b^\pm(h)u = 0$ for $u \in \operatorname{Ran}(P_0^\pm)$ and $h \in C_0^\infty(\mathbb{R}^d \backslash \{0\}) \otimes \mathfrak{g}$. This is due to the fact that for $h \in \mathfrak{h}$ there exists a sequence $h_n \in C_0^\infty(\mathbb{R}^d \backslash \{0\}) \otimes \mathfrak{g}$, such that $\lim_{n\to\infty} h_n = h$ and $h \mapsto b^\pm(h)$ is norm continuous. Now let $h \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$, one can choose $q \in C_\infty(\mathbb{R})$, $0 \leqslant q \leqslant 1$, $q(0) = 1$ supported in a small enough neighborhood of 0 such that by a stationary phase argument $q^t h_t \in o(1)$, $t \to \pm\infty$. Hence by (5.13) and the fact that the map $h \mapsto b(h_t)$ is continuous, we obtain that $b^\pm(h)\Gamma^\pm(q)u = 0$. □

5.4. Proof of Theorem 3.2

Let $j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_\infty \in C^\infty(\mathbb{R}^d)$, $0 \leqslant j_0, 0 \leqslant j_\infty, j_0^2 + j_\infty^2 \leqslant 1$, $j_0 = 1$ near 0 and j_∞ is constant near ∞. Set $j := (j_0, j_\infty)$ and $j^t := (j_0^t, j_\infty^t)$, where $j_0^t := j_0(\frac{x}{t}), j_\infty^t := j_\infty(\frac{x}{t})$.

Theorem 5.13. (i) *The following limits exist*:

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} I^*(j^t)\mathrm{e}^{-\mathrm{i}tH} =: W^\pm(j),$$

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} = W^\pm(j)^*.$$

(ii) *For a bounded Borel function* F, *we have*

$$W^{\pm}(j)F(H) = F(H^{\text{ext}})W^{\pm}(j).$$

(iii) *Let* $q_0 \in C_0^\infty(\mathbb{R}^d), q_\infty \in C^\infty(\mathbb{R}^d), \nabla q_\infty \in C_0^\infty(\mathbb{R}^d), 0 \leqslant q_0, q_\infty \leqslant 1, q_0 = 1$ *near* 0. *Set* $\widetilde{j} := (q_0 j_0, q_\infty j_\infty)$. *Then*

$$\Gamma^{\pm}(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))W^{\pm}(j) = W^{\pm}(\widetilde{j}).$$

(iv) *Let* $q \in C_0^\infty(\mathbb{R}^d), 0 \leqslant q \leqslant 1, q = 1$ *near* 0. *Then*

$$W^{\pm}(j)\Gamma^{\pm}(q) = W^{\pm}(qj), \quad \text{where } qj = (qj_0, qj_\infty).$$

(v) *Let* $\widetilde{j} = (\widetilde{j}_0, \widetilde{j}_\infty)$ *be another pair satisfying the conditions stated in the theorem. Then*

$$W^{\pm}(\widetilde{j})^* W^{\pm}(j) = \Gamma(\widetilde{j}_0 j_0 + \widetilde{j}_\infty j_\infty),$$

in particular if $j_0^2 + j_\infty^2 = 1$, *then* $W^{\pm}(j)$ *is isometric*.

(vi) *Let* $j_0 + j_\infty = 1$. *If* $\chi \in C_0^\infty(\mathbb{R})$, *then*

$$\Omega^{\text{ext},\pm}\chi(H^{\text{ext}})W^{\pm}(j) = \chi(H).$$

Proof. To prove (i) we use the same arguments as in Theorem 5.11. It is enough to prove the existence of the following limit for some $\chi \in C_0^\infty(\mathbb{R})$,

$$s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH^{\text{ext}}}\chi(H^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H), \tag{5.14}$$

We compute

$$\partial_t(\mathrm{e}^{\mathrm{i}tH^{\text{ext}}}\chi(H^{\text{ext}})I^*(j^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}) = \mathrm{e}^{\mathrm{i}tH^{\text{ext}}}(\chi(H^{\text{ext}})\mathfrak{D}_0 I^*(j^t)\chi(H)$$
$$+ \mathrm{i}\chi(H^{\text{ext}})(P \otimes \mathbb{1}I^*(j^t) - I^*(j^t)P)\chi(H))\mathrm{e}^{-\mathrm{i}tH},$$

where $\mathfrak{D}_0$ is the asymmetric Heisenberg derivative defined by $\partial_t + \mathrm{i}H_0^{\text{ext}}. - .\mathrm{i}H_0$. We have $\mathfrak{D}_0 I^*(j^t) = \check{\Gamma}(j^t, \mathfrak{d}_0 j^t)$.

Pseudo-differential calculus gives

$$\mathfrak{d}_0 j^t = \frac{1}{t}g^t + r^t,$$

$$g^t = (g_0^t, g_\infty^t), \quad g_\varepsilon^t = -\frac{1}{2}\left(\left(\frac{x}{t} - \nabla\omega(k)\right)\nabla j_\varepsilon\left(\frac{x}{t}\right) + hc\right), \quad \varepsilon = 0, \infty$$

with $r^t \in O(t^{-2})$. We obtain using estimate (iv) Lemma 2.7:

$$\|\chi(H^{\text{ext}})\, \mathrm{d}\check{\Gamma}(j^t, r^t)\chi(H)\| \in O(t^{-2}). \tag{5.15}$$

Now with $u_i^t := \mathrm{e}^{\mathrm{i}tH} u_i$, one obtain using estimate (iii) Lemma 2.7:

$$\begin{aligned} &|(u_1^t \mid \chi(H^{\text{ext}})\, \mathrm{d}\check{\Gamma}(j^t, g^t)\chi(H)u_2^t)| \\ &\quad \leqslant \|\, \mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}} \otimes \mathbb{1}\chi(H^{\text{ext}})u_2^t\| \, \|\, \mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}}\chi(H)u_1^t\| \\ &\qquad + \|(\mathbb{1} \otimes \mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}})\chi(H^{\text{ext}})u_2^t\| \, \|\, \mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}}\chi(H)u_1^t\|. \end{aligned}$$

Then the integrability of the term $\chi(H^{\text{ext}})\mathfrak{D}_0 I^*(j^t)\chi(H)$ follows using Proposition 5.7.

Using Lemma 4.2 we obtain

$$\chi(H^{\text{ext}})(P \otimes \mathbb{1}I^*(j^t) - I^*(j^t)P)\chi(H) \in O(t^{-\mu}), \quad \mu > 1.$$

Then, the existence of the limit in (i) follows.

(ii) Follows by spectral theorem and point (i). Point (iii) follows using the fact that

$$\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}t\, \mathrm{d}\Gamma(\omega)}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\, \mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)),$$

$$\Gamma(q_0^t) \otimes \Gamma(q_\infty^t)I^*(j^t) = I^*(\widetilde{j}^{\,t}).$$

(iv) Is true since

$$I^*(j^t)\Gamma(q^t) = I^*((jq)^t).$$

(v) Is a consequence of the fact

$$I(\widetilde{j}^{\,t})I^*(j^t) = \Gamma(\widetilde{j}_0^t j_0^t + \widetilde{j}_\infty^t j_\infty^t).$$

(vi) One has

$$H^{\text{ext}}\mathbb{1}_{[k,\infty[}(N_\infty) \geqslant mk + \inf\sigma(H).$$

Then for $\chi \in C_0^\infty(\mathbb{R})$, there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$\chi(H^{\text{ext}})\mathbb{1}_{]n,\infty[}(N_\infty) = 0. \tag{5.16}$$

We have

$$\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^{\pm}(j)$$

$$= \Omega^{\mathrm{ext},\pm}\mathbb{1}_{[0,n]}(N_\infty)\chi(H^{\mathrm{ext}})W^{\pm}(j) \tag{5.17}$$

$$= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I\mathbb{1}_{[0,n]}(N_\infty)\chi(H^{\mathrm{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}tH} \tag{5.18}$$

$$= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I\mathbb{1}_{[0,n]}(N_\infty)I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H). \tag{5.19}$$

Eq. (5.16) follows from (5.16). Eq. (5.18) follows from limit (i) and Theorem 5.5. Lemma 4.2 and the boundedness of the operator $I\mathbb{1}_{[0,n]}(N_\infty)(N_0+1)^{-\frac{n}{2}}$ gives (5.19). We use now an estimate proved in [12] and which extends also to our case.

$$||I\mathbb{1}_{]n,\infty[}(N_\infty)I^*(j^t)(N+1)^{-1}|| \leqslant (n+1)^{-1}. \tag{5.20}$$

Since $II^*(j^t) = \mathbb{1}$, letting $n \to \infty$ we obtain $\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^{\pm}(j) = \chi(H)$. This completes the proof. $\square$

Theorem 5.14. *Let $j_n = (j_{0,n}, j_{\infty,n})$ be a sequence satisfying the hypothesis stated in the beginning of Theorem 5.13 such that $j_0 + j_\infty = 1$ and for any $\varepsilon > 0$ there exists $m, \forall n > m, \operatorname{supp} j_{0,n} \subset [-\varepsilon, \varepsilon]$. Then*

$$\Omega^{\pm *} = w - \lim_{n\to+\infty} W^{\pm}(j_n),$$

$$\mathscr{K}^{\pm} = \operatorname{Ran} P_0^{\pm}.$$

Proof. Let $q \in C_0^\infty(\mathbb{R}), 0 \leqslant q \leqslant 1$ and $q = 1$ in a neighborhood of zero such that $q j_{0,n} = j_{0,n}$ for n large enough. Using Theorem 5.13 (iii) and Corollary 5.12 we obtain

$$\Gamma^{\pm}(q)\otimes\mathbb{1}W^{\pm}(j_n) = W^{\pm}(j_n),$$

$$w - \lim_{n\to+\infty} P_0^{\pm}\otimes\mathbb{1}W^{\pm}(j_n) - W^{\pm}(j_n) = 0. \tag{5.21}$$

Let $\chi \in C_0^\infty(\mathbb{R})$. We have

$$\Omega^{\pm *}\chi(H) = \Omega^{\pm *}\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^{\pm}(j_n) \tag{5.22}$$

$$= w - \lim_{n\to\infty} \Omega^{\pm *}\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})P_0^{\pm}\otimes\mathbb{1}W^{\pm}(j_n) \tag{5.23}$$

$$= w - \lim_{n\to\infty} P_0^{\pm}\otimes\mathbb{1}\chi(H^{\mathrm{ext}})W^{\pm}(j_n) \tag{5.24}$$

$$= w - \lim_{n\to\infty} \chi(H^{\text{ext}})W^{\pm}(j_n) \tag{5.25}$$

$$= w - \lim_{n\to\infty} W^{\pm}(j_n)\chi(H). \tag{5.26}$$

Eq. (5.22) follows from Theorem 5.13 (iv). Eq. (5.23) follows by (5.21). Eq. (5.24) is true since $P_0^{\pm}$ commutes with H^{ext} and that $\text{Ran}\, P_0^{\pm} \subset \mathscr{K}^{\pm}, \Omega^{\text{ext},\pm}\mathbb{1}_{\mathscr{K}^{\pm}} \otimes \mathbb{1} = \Omega^{\pm}$ and $\Omega^{\pm *}\Omega^{\pm} = \mathbb{1}_{\mathscr{K}^{\pm}} \otimes \mathbb{1}$. (5.25) follows from the fact that $P_0^{\pm} \otimes \mathbb{1}$ commutes with H^{ext} and (5.21). Eq. (5.26) is Theorem 5.13 (ii). So we conclude by a density argument that

$$\Omega^{\pm *} = w - \lim_{n\to+\infty} W^{\pm}(j_n),$$

$$P_0^{\pm} \otimes \mathbb{1}\Omega^{\pm *} = \Omega^{\pm *}.$$

So, we obtain

$$\text{Ran}\, \Omega^{\pm *} = \mathscr{K}^{\pm} \otimes \Gamma(\mathfrak{h}) \subset \text{Ran}\, P_0^{\pm} \otimes \Gamma(\mathfrak{h}) \subset \mathscr{K}^{\pm} \otimes \Gamma(\mathfrak{h}).$$

Thus, we conclude that

$$\mathscr{K}^{\pm} = \text{Ran}\, P_0^{\pm}. \qquad \square$$

Proof of Theorem 3.2. It suffices to show that $\mathscr{K}^{\pm} \subset \text{Ran}\, \mathbb{1}_{\text{pp}}(H)$, to prove Theorem 3.2.

Proposition 5.10 gives the existence of $\varepsilon > 0$ and $C > 0$ such that

$$\int_1^{+\infty} ||\Gamma(q^t)\chi(H)e^{-itH}u||^2 \frac{dt}{t} \leqslant C||u||^2,$$

where $\chi \in C_0^{\infty}(\mathbb{R}\backslash(\tau \cup \sigma_{\text{pp}}(H)))$ and $q \in C_0^{\infty}([-\varepsilon, \varepsilon]), q = 1$ for $|x| < \frac{\varepsilon}{2}$. Theorem 5.11 gives that

$$||\Gamma(q^t)\chi(H)e^{-itH}u|| \to ||\Gamma^{\pm}(q)\chi(H)u|| = 0, \quad t \to \pm\infty.$$

Hence $\Gamma^{\pm}(q)\chi(H) = 0$. Therefore, we have $\text{Ran}\, P_0^{\pm} \subset \text{Ran}\, \mathbb{1}_{\tau\cup\sigma_{\text{pp}}(H)}(H)$. Now, Theorem 4.3 gives that τ is a closed countable set and $\sigma_{\text{pp}}(H)$ can accumulate only at τ, so $\mathbb{1}_{\text{pp}}(H) = \mathbb{1}_{\tau\cup\sigma_{\text{pp}}(H)}(H)$. This proves

$$\mathscr{H}_{\text{bd}}(H) = \mathscr{K}^{\pm}.$$

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Appendix A

Let $\mathbb{R}^d \ni \lambda \mapsto A(\lambda) \in \mathcal{M}_n(\mathbb{C})$ be a symmetric matrices-valued function. Assume that $A(\lambda)$ has $E_1(\lambda), \dots, E_s(\lambda)$ eigenvalues with constant uniform multiplicity for $\lambda \in \mathbb{R}^d$ and satisfying:

$$\inf_{k \in \mathbb{R}^d, i \neq j} |E_i(\lambda) - E_j(\lambda)| > 0.$$

We diagonalize $A(\lambda)$ by means of an unitary matrix $\mathrm{u}(\lambda)$ such that $\mathrm{u}(\lambda)A(\lambda)\mathrm{u}(\lambda)^{-1} =: D(\lambda)$.

Lemma A.1. *The three following assertions hold*:

(i) *If* $A \in C^s(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))$ *then* $\mathrm{u}(\lambda), D(\lambda) \in C^s(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))$.
(ii) *If* $\nabla A(\lambda) \in L^\infty(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))^d$ *then* $\nabla D(\lambda) \in L^\infty(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))^d$.
(iii) *If* $\nabla A(\lambda) \neq 0$ *then* $\nabla D(\lambda) \neq 0$.

Proof. The map $T_\lambda : \mathbb{R} \ni \xi \mapsto (A(\lambda) - \xi)^{-1}$ is analytic on ξ except for some finite simple poles which depend from λ. Let $(\lambda_0, \xi_0), \xi_0 \notin \sigma(A(\lambda_0))$ be fixed. The map $\lambda \to T_\lambda$ is in $C^s(\mathcal{O}_{\lambda_0}, \mathcal{M}_n(\mathbb{C}))$ for $\mathcal{O}_{\lambda_0}$ a small neighborhood of λ_0. Let $E_i(\lambda), \Phi_i(\lambda)$, $i = 1, \dots, n$ be, respectively, the eigenvalues and the normalized eigenvectors of $A(\lambda)$. Hence

$$P_i(\lambda) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma (A(\lambda) - \xi)^{-1} d\xi$$

is in $C^s(\mathcal{O}_{\lambda_0}, \mathcal{M}_n(\mathbb{C}))$, where Γ is oriented closed contour containing only the eigenvalue $E_i(\lambda)$. Therefore, that eigenvectors and eigenvalues of $A(\lambda)$ are in $C^s(\mathbb{R}^d)$. This proves (i).

We have the following estimate:

$$\frac{1}{\sqrt{n}} ||\partial_j A(\lambda)|| \leqslant \sup_{i=1\dots n} |(\partial_j A(\lambda)\Phi_i(\lambda), \Phi_i(\lambda))| \leqslant ||\partial_j A(\lambda)||.$$

Moreover,

$$||\partial_j D(\lambda)|| = \sup_{i=1\dots n} |\partial_j E_i(\lambda)| = \sup_{i=1\dots n} |(\partial_j A(\lambda)\Phi_i(\lambda), \Phi_i(\lambda))|.$$

Thus we obtain (ii) and (iii). □

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