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On Odd Perturbations of Free Fermion Fields

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Abstract. We study the scattering theory of fermion systems subject to a smooth local perturbation with a non-vanishing odd part. We introduce a modified free fermion fields which have an appropriate commutation relations with the free Fock fermion fields. We construct the wave operators using the modified field and prove asymptotic completeness. Our work extends former results on Hilbert space asymptotic completeness.

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1. Introduction

The free and perturbed fermion systems are favorite models in QSM. This is related to the fact that fermion fields are bounded operators and, hence, perturbation theory in the spirit of Dyson–Schwinger series is well adapted to the investigation of those models. For example, the above approach allows the construction of perturbed $\mathcal{C}^*$-dynamical systems. Recall that if $\mathcal{U}$ is a $\mathcal{C}^*$-algebra and α_t is a one parameter group of *-automorphisms on $\mathcal{U}$, then we call the pair $(\mathcal{U}, \alpha_t)$ a $\mathcal{C}^*$-dynamical system. Let δ be the infinitesimal generator of α_t and $P \in \mathcal{U}$ such that $P = P^*$. We denote by $(\mathcal{U}, \alpha_t^P)$ the perturbed $\mathcal{C}^*$-dynamical system satisfying $\partial_t \alpha_t^P = \delta^P \circ \alpha_t^P$, $\delta^P := \delta + [\mathrm{i}P, \cdot]$ and can be expressed by means of Dyson–Schwinger expansion.

One of the questions arising in this subject is the unitary equivalence of free and perturbed dynamics, known as scattering theory in $\mathcal{C}^*$-algebras [12, 16]. Results on the equivalence of dynamics for fermion systems based on a perturbative approach are obtained in [1, 2, 4, 5], (see also the survey [6]). Under the condition of existence of a unique physical ground state for the perturbed system, equivalence of dynamics implies asymptotic completeness.

In this Letter, we present a new result on asymptotic completeness for a fermionic system describing a zero temperature Fermi ideal gas perturbed by a local interaction which may have a non vanishing odd part. To simplify the presentation we consider a simple model. However the result could be extended without much work to more general situations as spin-fermion models, quantized spin $\frac{1}{2}$ Dirac particle, see [3]. Our approach is nonperturbative, time depending and uses a similar strategy as in

[10]. The result we obtain applies for smooth local perturbations and extends those in [1, 3, 5] and [6, Theorems 2.8, 3.3].

Before making the above condition precise, let us first fix the notation. Let $\mathbb{H}$ be a Hilbert space, we denote the Banach space of bounded operators on $\mathbb{H}$ by $\mathcal{B}(\mathbb{H})$ and the group of unitary operators on $\mathbb{H}$ by $\mathcal{U}(\mathbb{H})$. We consider the one-particle fermion space in the momentum representation to be $\mathfrak{h} := L_2(\mathbb{R}^\nu, \mathrm{d}k)$ and denote by $x := \partial_k/\mathrm{i}$. Let $\mathcal{H}$ be the anti-symmetric Fock space over $\mathfrak{h}$ given by $\mathcal{H} := \bigoplus_{n=0}^{\infty} \bigwedge^{(n)} \mathfrak{h}$, where $\bigwedge^{(0)} \mathfrak{h} := \mathbb{C}$ and $\Omega = (1, 0, \ldots)$ is the vacuum vector. Let $\omega \in C^0(\mathbb{R}^\nu, \mathbb{R}_+)$ be the dispersion relation of fermion. The free Hamiltonian is defined by

$$H_0 := \mathrm{d}\Gamma(\omega),$$

where $\mathrm{d}\Gamma(\omega)$ denotes the second quantization of ω. Let b, b^* be the usual annihilation-creation operators representing the CAR on $\mathfrak{h}$. Let $\mathfrak{U}$ be the C^*-algebra generated by polynomials on $b(h)$, $b^*(h)$, $h \in \mathfrak{h}$. If $V \in \mathfrak{U}$ such that $V^* = V$, then we can construct the perturbed Hamiltonian H with respect to the interaction V using the Kato–Rellich theorem [15]. Namely, we have that $H := H_0 + V$, is a well defined self-adjoint operator with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$.

Let us collect the needed hypotheses. We assume that ω satisfies the following conditions:

$$(\mathcal{D}) \quad \begin{cases} \nabla\omega \in L_\infty(\mathbb{R}^\nu), \\ \nabla\omega(k) \neq 0, \ \text{for}\, k \neq 0, \\ \lim\limits_{|k|\to\infty} \omega(k) = +\infty, \\ \inf\limits_{k\in\mathbb{R}^\nu} \omega(k) =: m > 0. \end{cases}$$

Note that the last condition on ω is crucial on the proof of asymptotic completeness. We consider perturbations $V \in \mathfrak{U}$ of the following form:

$$V := \sum_{i=1}^{d} \int_{\mathbb{R}^{\nu(m_i+n_i)}} \mathcal{V}_i(k_1, \ldots, k_{m_i}, \ldots, k_{m_i+n_i}) \times \\ \times b^*(k_1) \cdots b^*(k_{m_i})\, b(k_{m_i+1}) \cdots b(k_{m_i+n_i}) \mathrm{d}k_1 \cdots \mathrm{d}k_{m_i+n_i}. \tag{1}$$

We introduce the Banach space $S_n(\mathbb{R}^\mu)$ as the completion of $C_0^n(\mathbb{R}^\mu)$ with respect to the Schwartz norm

$$\| f \|_n := \sum_{|\alpha|+|\beta|\leqslant n} \sup_{k\in\mathbb{R}^\mu} |k^\alpha \partial_k^\beta f(k)|,$$

where $\alpha, \beta \in \mathbb{N}^\mu$ and $k^\alpha \partial_k^\beta = \sum_{i=1}^{\mu} k_i^{\alpha_i} \partial_{k_i}^{\beta_i}$.

DEFINITION 1. Let $V \in \mathfrak{U}$ such that $V^* = V$. Then V is said to be in the class of smooth local perturbations $\mathcal{A}$, if V is given by (1) and

$$\mathcal{V}_i \in S_{\beta_i}(\mathbb{R}^{\nu(m_i+n_i)}), \qquad \beta_i \geqslant 2\nu(m_i + n_i + 2) + 2, \quad i = 1, \ldots, d.$$

Let A be an operator on $\mathfrak{h}$, we denote by $\Gamma(A)$ the operator on $\mathcal{H}$ given by

$$\Gamma(A)_{|\wedge^{(n)}\mathfrak{h}} := \underbrace{A \otimes \cdots \otimes A}_{n}.$$

Let N denote the number operator. The Fock CAR representation $\mathfrak{h} \ni h \mapsto \phi(h) := 1/\sqrt{2}(b^*(h) + b(h))$ is a free fermion field $(\mathcal{H}, \phi, \Gamma, \Omega)$ (see Definition 4). We set

$$a^*(h) := b^*(h)(-\mathbb{1})^N, \qquad a(h) := (-\mathbb{1})^N b(h), \quad h \in \mathfrak{h}.$$

By Proposition 5, we see that (a, a^*) define a free fermion field. Let $\tilde{\mathfrak{U}}$ be the C^*-algebra generated by polynomials on $a(h)$, $a^*(h), h \in \mathfrak{h}$. Let τ_t, τ_t^V denote the one-parameter groups of $*$-automorphisms given by the Heisenberg evolutions associated respectively to H_0, and H.

PROPOSITION 2. *Assume* $(\mathcal{D})$ *holds and* $V \in \mathcal{A}$. *The strong limits*

$$\gamma_{\pm} := s - \lim_{t \to \pm\infty} \tau_{-t}^V \circ \tau_t,$$

exist on $\tilde{\mathfrak{U}}$.

Proof. The proof follows by Proposition 9. □

Let $\mathcal{H}_{\mathrm{bd}}(H)$ be the space of bound states of H and set $a_{\pm}(h) := \gamma_{\pm}(a(h)), h \in \mathfrak{h}$. We define the asymptotic vacua space and the free space, respectively, by

$$\mathcal{K}_{\pm} := \{\Psi \in \mathcal{H} \mid a_{\pm}(h)\Psi = 0, h \in \mathfrak{h}\}, \qquad \mathcal{H}_0 := \mathcal{H}_{\mathrm{bd}}(H) \otimes \mathcal{H}.$$

The wave operators are given by

$$W_{\pm} \colon \mathcal{H}_0 \to \mathcal{H}, \qquad \psi \otimes \prod_{i=1}^{n} a^*(f_i)\Omega \mapsto \prod_{i=1}^{n} a_{\pm}^*(f_i)\psi, \quad f_i \in \mathfrak{h}.$$

THEOREM 3. *Assume that* $(\mathcal{D})$ *holds and* $V \in \mathcal{A}$. *Then* $W_{\pm}$ *is unitary and the* asymptotic completeness *holds, i.e.*

$$\mathcal{K}_{\pm} = \mathcal{H}_{\mathrm{bd}}(H) \quad \textit{and} \quad \mathcal{H}_{\pm} := \mathrm{Ran}\,(W_{\pm}) = \mathcal{H}.$$

2. CAR Representation

Let (L, S) be an orthogonal space (i.e: a real topological vector space L endowed with a continuous symmetric bilinear form S). A CAR representation over (L, S) is a pair $(\mathcal{D}, \Phi)$ consisting on a Hilbert space $\mathcal{D}$ and a linear map $L \ni h \mapsto \Phi(h) \in \mathcal{B}(\mathcal{D})$ into self-adjoint bounded operators and satisfying

$$\{\Phi(h), \Phi(g)\} = S(h, g)\mathbb{1} \quad \text{(Clifford relation)}.$$

Assume that (L, S) is equipped with a complex structure consisting on a conjugation $\mathcal{I} \colon L \to L, \mathcal{I}^2 = -\mathbb{1}$ compatible with the symmetric bilinear form S in the following sense:

(1) $S(h, \mathcal{I}g) + S(\mathcal{I}h, g) = 0$,
(2) $S(h, h) > 0$, for all $h = 0$.

This allows to have a complex structure $\mathrm{i}h := \mathcal{I}h, h \in L$ and an inner product $(h|g) := S(h, g) - \mathrm{i}S(h, \mathrm{i}g)$. Thus, we can construct the creation-annihilation operators:

$$B^*(h) := \frac{1}{\sqrt{2}}(\Phi(h) - \mathrm{i}\Phi(\mathrm{i}h)), \qquad B(h) := \frac{1}{\sqrt{2}}(\Phi(h) + \mathrm{i}\Phi(\mathrm{i}h)).$$

Furthermore, $B(h), B^*(h)$ satisfy the canonical anti-commutation relations:

$$\{B^\sharp(h), B^\sharp(g)\} = 0, \qquad \{B(h), B^*(g)\} = (h|g)\mathbb{1}, \tag{2}$$

where $B^\sharp$ stands for B or B^*. Note that a Hilbert space endowed with the bilinear form $\mathrm{Re}(.|.)$ and the conjugation $\mathrm{i} = \sqrt{-1}$ is an orthogonal space with a compatible complex structure. For more details, see [7, 9].

DEFINITION 4. Let $\mathfrak{H}$ be a Hilbert space and $(\mathcal{D}, \Phi)$ a CAR representation over $\mathfrak{H}$. $(\mathcal{D}, \Phi)$ is said to be a free fermion field if it is endowed with a continuous representation Γ of the unitary group $\mathcal{U}(\mathfrak{H})$ and a cyclic vector Ω such that

(1) $\mathcal{U}(\mathfrak{H}) \ni U \mapsto \Gamma(U) \in \mathcal{U}(\mathcal{D})$,
(2) $\Gamma(U)B(h)\Gamma(U)^{-1} = B(Uh), U \in \mathcal{U}(\mathfrak{H})$,
(3) $\Gamma(U)\Omega = \Omega$,
(4) $\frac{1}{\mathrm{i}}\partial_t \Gamma(\mathrm{e}^{\mathrm{i}tA})_{|t=0} \geqslant 0, \forall A \in \mathcal{B}(\mathfrak{H}), A \geqslant 0$.

We denote a free fermion field by the quadruple $(\mathcal{D}, \Phi, \Gamma, \Omega)$.

Let $(\mathcal{D}, \Phi, \Gamma, \Omega)$ be a free fermion field over a Hilbert space $\mathfrak{H}$. Consider $\mathfrak{U}$ to be the $\mathcal{C}^*$-algebra generated by $\Phi(h), h \in \mathfrak{h}$ and let $\mathfrak{U}_\mathrm{e}$ denote the even CAR algebra (i.e: the $\mathcal{C}^*$-algebra generated by even polynomials on $B(h), B^*(h), h \in \mathfrak{H}$). We define the involution $\mathbf{J}$ by

$$\mathbf{J}: \mathcal{D} \to \mathcal{D}, \qquad \mathbf{J} := \Gamma(-\mathbb{1}_{\mathfrak{H}}).$$

Clearly, we have $\mathbf{J}^* = \mathbf{J}, \mathbf{J}^2 = \mathbb{1}_{\mathcal{D}}$. We define an even/odd projection

$$\mathbf{P}_{\mathrm{e/o}}: \mathfrak{U} \to \mathfrak{U},$$
$$\mathbf{P}_\mathrm{e}A := \tfrac{1}{2}(A + \mathbf{J}A\mathbf{J}), \qquad \mathbf{P}_\mathrm{o} := \tfrac{1}{2}(A - \mathbf{J}A\mathbf{J}), \quad A \in \mathfrak{U}.$$

Hence the algebra $\mathfrak{U}$ decomposes to a direct sum of vector spaces $\mathfrak{U} = \mathfrak{U}_\mathrm{e} \otimes \mathfrak{U}_\mathrm{o}$, where $\mathfrak{U}_\mathrm{e} = \mathbf{P}_\mathrm{e}\mathfrak{U}$ and $\mathfrak{U}_\mathrm{o} := \mathfrak{U} \ominus \mathfrak{U}_\mathrm{e} = \mathbf{P}_\mathrm{o}\mathfrak{U}$.

PROPOSITION 5. *Let* $(\mathcal{D}, \Phi, \Gamma, \Omega)$ *be a free fermion field and* $\mathcal{J}$ *an involution on* $\mathcal{D}$ *such that*

$$\begin{aligned} &[\mathcal{J}, A] = 0, \quad \forall A \in \mathfrak{U}_\mathrm{e}, \qquad \{\mathcal{J}, A\} = 0, \quad \forall A \in \mathfrak{U}_\mathrm{o}, \\ &[\mathcal{J}, \Gamma(U)] = 0, \quad \forall U \in \mathcal{U}(\mathfrak{H}). \end{aligned} \tag{3}$$

Set

$$\Phi_{\mathcal{J}}(h) := \frac{1}{\sqrt{2}}(B^*(h)\mathcal{J}) + \mathcal{J}B(h)).$$

Then $(\mathcal{D}, \Phi_{\mathcal{J}}, \Gamma, \Omega)$ *defines a free fermion field satisfying*

$$[\Phi_{\mathcal{J}}(h), \Phi(g)] = \mathcal{J}\mathrm{Re}(h, g).$$

Proof. An elementary computation using the fact that $\mathcal{J}$ commutes with $A \in \mathfrak{U}_e$ and anti-commutes with $A \in \mathfrak{U}_o$ shows that $(\mathcal{D}, \Phi_{\mathcal{J}})$ is a CAR representation. Moreover, the representation Γ preserves the relation

$$\Gamma(U)\Phi_{\mathcal{J}}(h)\Gamma(U)^{-1} = \Phi_{\mathcal{J}}(Uh), \quad U \in \mathcal{U}(\mathfrak{h}).$$

Hence, $(\mathcal{D}, \Phi_{\mathcal{J}}, \Gamma, \Omega)$ defines a free fermion field. □

We call the above free fermion field $(\mathcal{D}, \Phi_{\mathcal{J}}, \Gamma, \Omega)$ a $\mathcal{J}$-free fermion field associated to $(\mathcal{D}, \Phi, \Gamma, \Omega)$. Note that **J** satisfies the hypotheses (3) of Proposition 5.

LEMMA 6. *Let* $\mathcal{J}$ *be a self-adjoint involution satisfying* (3) *for the Fock fermion field* $(\mathcal{H}, \phi, \Gamma, \Omega)$. *Then* $\mathcal{J} = \pm(-\mathbb{1}_{\mathfrak{h}})^N$.

Proof. Using the fact that $\{\mathcal{J}, b(f)\} = 0$, we obtain that $\mathcal{J}\Omega \in \mathbb{C}\Omega$. Since $\mathcal{J}^2 = \mathbb{1}$, it follows that $\mathcal{J}\Omega = \pm\Omega$. Furthermore, we have

$$\mathcal{J}\prod_{i=1}^{n} b^*(f_i)\Omega = (-\mathbb{1})^n \prod_{i=1}^{n} b^*(f_i)\mathcal{J}\Omega.$$

Hence $\mathcal{J} = \pm(-\mathbb{1})^N$. □

The above lemma and Proposition 5 provide us exactly with $\pm$**J**-Fock fermion field associated to $(\mathcal{H}, \phi, \Gamma, \Omega)$.
We recall a known theorem, see [7, Thm. 2.1].

THEOREM 7. *Let* $(\mathcal{D}_i, \Phi_i, \Gamma_i, \Omega_i)$, $i = 1, 2$ *be two free fermion fields over a Hilbert space* $\mathfrak{h}$. *The map:*

$$\mathbb{U}: \mathcal{D}_1 \to \mathcal{D}_2, \qquad \mathbb{U}\prod_{j=1}^{n} \Phi_1(h_j)\Omega_1 = \prod_{j=1}^{n} \Phi_2(h_j)\Omega_2$$

extends to an unitary map from D_1 *to* D_2.

Note that Theorem 7 yields that if $(\mathcal{D}, \Phi, \Gamma, \Omega)$ is a free fermion field over $\mathfrak{h}$ then it is unitarily equivalent to $(\mathcal{D}, \Phi_{\pm \mathrm{J}}, \Gamma, \Omega)$.

3. Scattering Theory

3.1. ASYMPTOTIC FIELDS

Let τ_t, τ_t^V denote the two one parameter group of $*$-automorphisms on $\mathfrak{U}$ defined by

$$\tau_t(A) := e^{itH_0} A e^{-itH_0}, \qquad \tau_t^V(A) := e^{itH} A e^{-itH}, \quad A \in \mathcal{B}(\mathcal{H}).$$

Clearly, the free dynamic τ_t preserves $\mathfrak{U}$. To see that the perturbed dynamic τ_t^V preserves $\mathfrak{U}$ it suffices to express τ_t^V by means of Dyson expansion, see [9, Prop. 5.4.1] and [8, Thm. 3.1.33], i.e:

$$\tau_t^V(A) := \tau_t(A) + \sum_{n=1}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n [\tau_{t_n}(V), [\cdots [\tau_{t_1}(V), \tau_t(A)]]]. \tag{4}$$

The Møller morphisms are defined by

$$\gamma \pm (A) := \lim_{t \to \pm\infty} \tau_{-t}^V \circ \tau_t(A), \quad A \in \mathfrak{U}.$$

The following definition has been introduced in [16].

DEFINITION 8. A C^*-dynamical system $(\mathcal{U}, \alpha_t)$ is said to be $L_1(\mathcal{U}_0)$-asymptotically Abelian relatively to a perturbation $P \in \mathcal{U}$ iff

$$\int_{-\infty}^{+\infty} ||[P, \alpha_t(A)]|| dt < \infty,$$

for all A in a norm dense $*$-subalgebra $\mathcal{U}_0$.

If one assumes that $(\mathcal{U}, \alpha_t)$ satisfies the above condition, then using Cook's argument we can prove the existence of the Møller morphisms, see [9, Prop. 5.4.10].

In the sequel, we use the above argument with the dynamics $(\tilde{\mathfrak{U}}, \tau_t), (\mathcal{B}(\mathcal{H}), \tau_t^V)$. Note that $\tilde{\mathfrak{U}}$ is only preserved by τ_t^V if $V \in \mathfrak{U}_e$.

PROPOSITION 9. (i) *The norm limits*

$$a_\pm^\sharp(h) := \lim_{t \to \pm\infty} \tau_{-t}^V \circ \tau_t a^\sharp(h), \quad h \in \mathfrak{h}, \tag{5}$$

exist and are called asymptotic annihilation-creation operators.

(ii) *Let $\mathfrak{U}_\pm$ be the C^*-algebra generated by $a_\pm^\sharp(h)$. The maps*

$$\gamma_\pm : \tilde{\mathfrak{U}} \to \mathfrak{U}_\pm, \qquad a^\sharp(h) \mapsto \gamma_\pm(a^\sharp(h)) := a_\pm^\sharp(h).$$

extend to $*$-morphisms on $\tilde{\mathfrak{U}}$.

(iii) *The dynamic τ_t^V preserves the asymptotic algebra $\mathfrak{U}_\pm$:*

$$\tau_t^V a_\pm^\sharp(h) = a_\pm^\sharp(e^{it\omega} h), \quad h \in \mathfrak{h}.$$

Proof. We use a Cook argument to prove (5). Let $\mathfrak{h}_0 := C_0^\infty(\mathbb{R}^\nu \backslash \{0\})$ and $h_t := e^{it\omega} h$, $h \in \mathfrak{h}$. For $h \in \mathfrak{h}_0$, we have that

$$\tau_t^V a^\sharp(h_t) = a^\sharp(h) + i \int_0^t \tau_s^V [V, a^\sharp(h_s)] ds. \tag{6}$$

Using commutation relations in Proposition 5 and that fact that $V \in \mathcal{A}$, one can prove by stationary phase argument that we have

$$[V, a^\sharp(h_s)] \in \mathrm{O}(t^{-2}), \quad \text{for } h \in \mathfrak{h}_0.$$

We extend the existence of the limit to $h \in \mathfrak{h}$, using density argument and the fact that the norm of $a^\sharp_\pm(h)$ is preserved, i.e: $||a^\sharp_\pm(h)|| = ||h||$. (ii)–(iii) are obvious. □

PROPOSITION 10. *We have for* $h \in \mathfrak{h}$,

$$a_\pm(h)\mathbb{1}_{]-\infty,\lambda]}(H)\mathcal{H} \subset \mathbb{1}_{]-\infty,\lambda-m]}(H)\mathcal{H}.$$

Proof. The proof follows using a standard argument due to Høegh-Krohn [14]. □

3.2. WAVE OPERATORS

PROPOSITION 11. *The following assertions hold:*

(i) $\mathcal{K}_\pm$ *is closed* H*-invariant space.*
(ii) *One has* $\mathcal{H}_{\mathrm{bd}}(H) \subset \mathcal{K}_\pm$.

Proof. The fact that $\mathcal{K}_\pm$ is H-invariant follows from Proposition 9. Let us prove (ii). Let $u \in \mathcal{H}$ such that $Hu = Eu$. One has $s - \lim_{t\to\pm\infty} a(h_t) = 0$, $h \in \mathfrak{h}_0$. Hence

$$\lim_{t\to\pm\infty} \tau_t^V a(h_t)u = 0,$$

since $\tau_t^V a(h_t)u = \mathrm{e}^{\mathrm{i}t(H-E)}a(h_t)u$. □

Let $\mathcal{H}^{\mathrm{ext}} := \mathcal{H} \otimes \mathcal{H}$. Set

$$H_{\mathrm{As}} := H_{|\mathcal{H}_{\mathrm{bd}}(H)} \otimes \mathbb{1} + \mathbb{1} \otimes H_0, \quad \text{on } \mathcal{H}_0,$$
$$H_0^{\mathrm{ext}} := H_0 \otimes \mathbb{1} + \mathbb{1} \otimes H_0, \qquad H^{\mathrm{ext}} := H_0^{\mathrm{ext}} + V \otimes \mathbb{1}, \quad \text{on } \mathcal{H}^{\mathrm{ext}}.$$

THEOREM 12. (i) $W_\pm$ *are isometric.*
(ii) $W_\pm$ *are unitary iff* $\mathcal{K}_\pm = \mathcal{H}_{\mathrm{bd}}(H)$.
(iii) *We have*

$$a^\sharp_\pm(h)W_\pm = W_\pm \mathbb{1} \otimes a^\sharp(h), \quad \textit{for } h \in \mathfrak{h},$$
$$HW_\pm = W_\pm H_{\mathrm{As}}.$$

Proof. The intertwining relations in Proposition 9 give that $W_\pm$ is isometric and satisfies (iii). Let us prove (ii). Let $\Psi \in (\mathrm{Ran}\,(W_\pm))^\perp$. Since H preserves the later subspace one may assume that Ψ is localized in energy i.e.: $\Psi = \mathbb{1}_{]-\infty,\lambda]}(H)\Psi$. If $nm > \inf\sigma(H)$ then using Proposition 10 we obtain that $\prod_{i=1}^n a_\pm(h_i)\Psi = 0$. Therefore, using the orthogonality

$$0 = \left(\Psi, \prod_{i=2}^{n} a_{\pm}^{*}(h_i) \prod_{i=2}^{n} a_{\pm}(h_i)\Psi \right) = \left\| \prod_{i=2}^{n} a_{\pm}(h_i)\Psi \right\|.$$

This yields $\Psi = 0$. □

We define an *extended wave operator*

$$W_{\pm}^{\text{ext}}: \mathcal{H}^{\text{ext}} \to \mathcal{H},$$

$$W_{\pm}^{\text{ext}} \psi \otimes \prod_{i=1}^{n} a^{*}(h_i)\Omega := \prod_{i=1}^{n} a_{\pm}^{*}(h_i)\psi.$$

We notice that $W_{\pm|\mathcal{H}_0}^{\text{ext}} = W_{\pm}$. We introduce the scattering identification operator $I: \mathcal{H}^{\text{ext}} \to \mathcal{H}$, defined by

$$I \prod_{i=1} a^{*}(f_i)\Omega \otimes \prod_{i=1} a^{*}(g_j)\Omega = \prod_{i=1} a^{*}(g_j) \prod_{i=1} a^{*}(f_i)\Omega.$$

The following theorem follows as in [10, Thm. 5.7].

THEOREM 13. (i) *Let $u \in \mathcal{D}(W_{\pm}^{\text{ext}})$, then one has*

$$\lim_{t \to \pm\infty} e^{itH} I e^{-itH^{\text{ext}}} u = W_{\pm}^{\text{ext}} u.$$

(ii) *Let $\chi \in C_0^{\infty}(\mathbb{R})$. Then* $\operatorname{Ran} \chi(H^{\text{ext}}) \subset \mathcal{D}(W_{\pm}^{\text{ext}})$ *and the operators* $I\chi(H^{\text{ext}})$, $W_{\pm}^{\text{ext}}\chi(H^{\text{ext}})$ *are bounded. Moreover,*

$$\lim_{t \to \pm\infty} e^{itH} I e^{-itH^{\text{ext}}} \chi(H^{\text{ext}}) = W_{\pm}^{\text{ext}} \chi(H^{\text{ext}}).$$

3.3. SPECTRAL ANALYSIS

In this subsection we collect some spectral results.

THEOREM 14. *Assume $(\mathcal{D})$ holds and $V \in \mathcal{A}$, then*

$$\sigma_{\text{ess}}(H) = [\inf \sigma(H) + m, +\infty[.$$

Consequently, $\inf \sigma(H)$ *is a discrete eigenvalue.*

Let G denote the conjugate operator given by

$$G := d\Gamma\left(-\tfrac{1}{2}(\nabla\omega(k).x + x.\nabla\omega(k))\right), \text{ acting on } \mathcal{H}.$$

We set $\tau := \sigma_{\text{pp}}(H) + m\mathbb{N}^{*}$. We have for $V \in \mathcal{A}, [V, G] \in \mathcal{B}(\mathcal{H})$. Let Δ be an interval of $\mathbb{R}$. Let $\mathbb{1}_{\Delta}(H)$ be the spectral projection of H on Δ and $\mathbb{1}_{\Delta}^{\text{pp}}(H) := \mathbb{1}_{\Delta \cap \sigma_{\text{pp}}(H)}(H)$.

THEOREM 15. *The following two assertions hold:*

(i) *For all $[\lambda_1, \lambda_2]$ such that $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, one has*

$$\dim \mathbb{1}_{[\lambda_1, \lambda_2]}^{\text{pp}}(H)\mathcal{H} < \infty.$$

Consequently, $\sigma_{pp}(H)$ can accumulate only at τ, which is a closed countable set.

(ii) *Let $\lambda \in \mathbb{R}\backslash(\tau \cup \sigma_{pp}(H))$. Then there exists $\epsilon > 0, C_0 > 0$ such that*

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H, \mathrm{i}G]\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geqslant C_0 \mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

3.4. GEOMETRIC ASYMPTOTIC COMPLETENESS

In this subsection we sketch the proof of asymptotic completeness. The reader is referred to [3] for details.

THEOREM 16. *Let $q \in C_0^\infty(\mathbb{R}^\nu)$ such that $0 \leqslant q \leqslant 1, q = 1$ on a neighbourhood of 0 and $q^t := q(x/t)$.*

(i) *The following limits exist:*

$$\Gamma^\pm(q) := s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH}.$$

(ii) *Let $q_n \in C_0^\infty(\mathbb{R}^\nu)$ be a decreasing sequence such that $0 \leqslant q_n \leqslant 1, q_n = 1$ near 0, and $\bigcap_n \operatorname{supp} q_n = \{0\}$, then*

$$P_0^\pm := s - \lim_{n\to\infty} \Gamma^\pm(q_n),$$

exists and defines an orthogonal projection.

(iii) *We have $\operatorname{Ran} P_0^\pm \subset \mathcal{K}_\pm$.*

Let $j_0 \in C_0^\infty(\mathbb{R}^\nu), 0 \leqslant j_0, j_\infty, j_0^2 + j_\infty^2 \leqslant 1$, and $j_0 = 1$ near 0. Set $j := (j_0, j_\infty)$ and $j^t := (j_0^t, j_\infty^t)$, where $j_0^t := j_0(x/t)$, $j_\infty^t := j_\infty(x/t)$. We denote $I(j^t) := I\Gamma(j_0^t) \otimes \Gamma(j_\infty^t)$.

THEOREM 17. (i) *The following limits exist*

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} I(j^t)^* \mathrm{e}^{-\mathrm{i}tH} =: \mathbb{W}^\pm(j).$$

(ii) *Let $q_0, q_\infty \in C_0^\infty(\mathbb{R}^\nu), 0 \leqslant q_0, q_\infty \leqslant 1, q_0 = 1$ near 0. Set $\tilde{j} := (q_0 j_0, q_\infty j_\infty)$, and $q_0 j = (q_0 j_0, q_0 j_\infty)$, then*

$$\mathbb{W}^\pm(j)\Gamma^\pm(q_0) = \mathbb{W}^\pm(q_0 j),$$
$$\Gamma^\pm(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))W^\pm(j) = W^\pm(\tilde{j}).$$

(iii) *Let $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ be another pair satisfying the conditions stated before the theorem. Then*

$$\mathbb{W}^\pm(\tilde{j})^* \mathbb{W}^\pm(j) = \Gamma(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty).$$

In particular if $j_0^2 + j_\infty^2 = 1$, then $\mathbb{W}^\pm(j)$ is isometric.

(iv) *Let $j_0 + j_\infty = 1$. If $\chi \in C_0^\infty(\mathbb{R})$, then*

$$W_\pm^{\mathrm{ext}} \chi(H^{\mathrm{ext}}) \mathbb{W}^\pm(j) = \chi(H).$$

(v) *Let* $j_n = (j_{0,n}, j_{\infty,n})$ *be a sequence as above and such that* $j_{0,n} + j_{\infty,n} = 1$ *and* $\forall\epsilon > 0, \exists n_0, \forall n > n_0$, supp $j_{0,n} \subset [-\epsilon, \epsilon]$. *We have*

$$W^{\text{ext}*}_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}} = w - \lim_{n\to+\infty} \mathbb{W}^{\pm}(j_n) \quad \textit{and} \quad \mathcal{K}_\pm = \text{Ran}\, P_0^{\pm}.$$

Proof. To prove (i) it is enough to show the existence of the following limit for some $\chi \in C_0^\infty(\mathbb{R})$.

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH^{\text{ext}}} \chi(H^{\text{ext}}) I^*(j^t) \mathrm{e}^{-\mathrm{i}tH} \chi(H). \tag{7}$$

We compute

$$\begin{aligned}
&\partial_t(\mathrm{e}^{\mathrm{i}tH^{\text{ext}}} \chi(H^{\text{ext}}) I^*(j^t) \chi(H) \mathrm{e}^{-\mathrm{i}tH}) \\
&\quad = \mathrm{e}^{\mathrm{i}tH^{\text{ext}}} (\chi(H^{\text{ext}}) \mathfrak{D}_0 I^*(j^t) \chi(H) + \\
&\qquad + \mathrm{i}\chi(H^{\text{ext}})(V \otimes \mathbb{1} I^*(j^t) - I^*(j^t)V)\chi(H)) \mathrm{e}^{-\mathrm{i}tH},
\end{aligned}$$

where $\mathfrak{D}_0$ is the asymmetric Heisenberg derivative defined by $\partial_t + \mathrm{i}H_0^{\text{ext}}. - .\mathrm{i}H_0$. Let $\mathrm{d}\check{\Gamma}(j^t, \mathfrak{d}_0 j^t) := \mathfrak{D}_0 I^*(j^t)$, with $\mathfrak{d}_0 j_\varepsilon^t := \partial_t j_\varepsilon + \mathrm{i}[\omega, j_\varepsilon^t], \varepsilon = 0, \infty$. Pseudo-differential calculus gives

$$\mathfrak{d}_0 j^t = \frac{1}{t} g^t + r^t, \quad \text{where } r^t \in \mathrm{O}(t^{-2}),\ \ g^t = (g_0^t, g_\infty^t),$$

and

$$g_\epsilon^t = -\frac{1}{2}\left(\left(\frac{x}{t} - \partial\omega(k)\right)\partial j_\epsilon\left(\frac{x}{t}\right) + hc\right), \quad \epsilon = 0, \infty$$

We obtain using the estimate in [3, Lemma 2.7]:

$$||\chi(H^{\text{ext}}) \mathrm{d}\check{\Gamma}(j^t, r^t) \chi(H)|| \in \mathrm{O}(t^{-2}). \tag{8}$$

Let $u_i^t := \mathrm{e}^{\mathrm{i}tH} u_i$, one obtains using the estimate (iii) in [3, Lemma 2.7] that

$$\begin{aligned}
&|(u_1^t | \chi(H^{\text{ext}}) \mathrm{d}\check{\Gamma}(j^t, g^t) \chi(H) u_2^t)| \\
&\quad \leqslant ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}} \otimes \mathbb{1} \chi(H^{\text{ext}}) u_2^t||\, ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}} \chi(H) u_1^t|| + \\
&\qquad + ||(\mathbb{1} \otimes \mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}}) \chi(H^{\text{ext}}) u_2^t||\, ||\mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}} \chi(H) u_1^t||.
\end{aligned}$$

Then the integrability of the term $\chi(H^{\text{ext}}) \mathfrak{D}_0 I^*(j^t) \chi(H)$ follows using Proposition A.1. Furthermore, one can prove as in [3, Lemma 4.2] that

$$\chi(H^{\text{ext}})(V \otimes \mathbb{1} I^*(j^t) - I^*(j^t)V)\chi(H) \in \mathrm{O}(t^{-2}).$$

This yields the existence of the limit (i).

(ii) follows using the fact that

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)),$$

and

$$\Gamma(q_0^t) \otimes \Gamma(q_\infty^t) I^*(j^t) = I^*(\tilde{j}^t).$$

(iii) is a consequence of the fact

$$I(\tilde{j}^t)I^*(j^t) = \Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t).$$

(iv) One has $H^{\text{ext}}\mathbb{1}_{[k,\infty[}(\mathbb{1}\otimes N) \geqslant mk + \inf\sigma(H)$. Then for $\chi \in C_0^\infty(\mathbb{R})$, there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$\chi(H^{\text{ext}})\mathbb{1}_{]n,\infty[}(\mathbb{1}\otimes N) = 0. \tag{9}$$

We have

$$\begin{aligned} &W_\pm^{\text{ext}}\chi(H^{\text{ext}})\mathbb{W}^\pm(j) \\ &\quad= W_\pm^{\text{ext}}\mathbb{1}_{[0,n]}(\mathbb{1}\otimes N)\chi(H^{\text{ext}})\mathbb{W}^\pm(j) && (10)\\ &\quad= s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I\mathbb{1}_{[0,n]}(\mathbb{1}\otimes N)\chi(H^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}tH} && (11)\\ &\quad= s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I\mathbb{1}_{[0,n]}(\mathbb{1}\otimes N)I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H). && (12)\end{aligned}$$

The right-hand side of (10) follows by (9). The part (11) follows by (i) and Thm. 13. Furthermore, by [3, Lemma 4.2] and the boundedness of the operator $I\mathbb{1}_{[0,n]}(1\otimes N)(N_0+1)^{-\frac{n}{2}}$ we obtain (12). We recall an estimate proved in [11] and which extends to our case.

$$||I\mathbb{1}_{]n,\infty[}(1\otimes N)I^*(j^t)(N+1)^{-1}|| \leqslant (n+1)^{-1}. \tag{13}$$

Since $II^*(j^t) = \mathbb{1}$, letting $n\to\infty$ we obtain (iv).

(v) Let $q \in C_0^\infty(\mathbb{R}), 0 \leqslant q \leqslant 1$ and $q = 1$ in a neighbourhood of zero such that $qj_{0,n} = j_{0,n}$ for n large enough. Using (ii) and Theorem 16 we obtain

$$\begin{aligned}&\Gamma^\pm(q)\otimes \mathbb{1}\mathbb{W}^\pm(j_n) = \mathbb{W}^\pm(j_n),\\ &w-\lim_{n\to+\infty} P_0^\pm\otimes\mathbb{1}\mathbb{W}^\pm(j_n) - \mathbb{W}^\pm(j_n) = 0.\end{aligned} \tag{14}$$

Let $\chi \in C_0^\infty(\mathbb{R})$. We have

$$\begin{aligned} &W_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}^{\text{ext}*}\chi(H) \\ &\quad= W_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}^{\text{ext}*}W_\pm^{\text{ext}}\chi(H^{\text{ext}})\mathbb{W}^\pm(j_n) && (15)\\ &\quad= w-\lim_{n\to\infty} W_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}^{\text{ext}*}W_\pm^{\text{ext}}\chi(H^{\text{ext}})P_0^\pm\otimes\mathbb{1}\mathbb{W}^\pm(j_n) && (16)\\ &\quad= w-\lim_{n\to\infty} P_0^\pm\otimes\mathbb{1}\chi(H^{\text{ext}})\mathbb{W}^\pm(j_n) && (17)\\ &\quad= w-\lim_{n\to\infty} \chi(H^{\text{ext}})\mathbb{W}^\pm(j_n) && (18)\\ &\quad= w-\lim_{n\to\infty} \mathbb{W}^\pm(j_n)\chi(H). && (19)\end{aligned}$$

Formula (15) follows from (iv), (16) follows by (14), and (17) is true since $P_0^\pm$ commutes with H^{ext} and that $\operatorname{Ran} P_0^\pm \subset \mathcal{K}_\pm$ and

$$W_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}^{\text{ext}*}W_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}^{\text{ext}} = \mathbb{1}_{\mathcal{K}_\pm}\otimes\mathbb{1}$$

Formula (18) follows from the fact that $P_0^\pm\otimes\mathbb{1}$ commutes with H^{ext} and (14) and (19) is in Theorem 17 (ii). We conclude that

$$W^{\text{ext}*}_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}} = w - \lim_{n\to+\infty} \mathbb{W}^{\pm}(j_n), \quad P_0^{\pm}\otimes 1\, W^{\text{ext}*}_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}} = W^{\text{ext}*}_{\pm|\mathcal{K}_\pm\otimes\mathcal{H}}.$$

Thus, we obtain (v). □

Proof of Theorem 3. In order to prove Theorem 3 it suffices to show that $\mathcal{K}_\pm \subset \mathcal{H}_{\text{bd}}(H)$. By Prop. A.2 there exist $\epsilon > 0$ and $C > 0$ such that

$$\int_1^{+\infty} ||\Gamma(q^t)\chi(H)\mathrm{e}^{\mathrm{i}tH}u||^2 \frac{\mathrm{d}t}{t} \leqslant C||u||^2,$$

where

$$\chi \in C_0^\infty(\mathbb{R}\backslash(\tau\cup\sigma_{\text{pp}}(H))) \quad \text{and} \quad q\in C_0^\infty([-\epsilon,\epsilon]), \quad q=1, \text{ for } |x|<\epsilon/2.$$

Theorem 16 gives that

$$||\Gamma(q^t)\chi(H)\mathrm{e}^{\mathrm{i}tH}u|| \to ||\Gamma^{\pm}(q)\chi(H)u|| = 0, \quad t\to\pm\infty.$$

Therefore, $\Gamma^{\pm}(q)\chi(H)=0$ and, hence, $\text{Ran}\, P_0^{\pm} \subset \text{Ran}\, 1_{\tau\cup\sigma_{\text{pp}}(H)}(H)$. By Theorem 15 we know that τ is a closed countable set and $\sigma_{\text{pp}}(H)$ can accumulate only at τ, so $1_{\text{pp}}(H) = 1_{\tau\cup\sigma_{\text{pp}}(H)}(H)$. This proves $\mathcal{H}_{\text{bd}}(H) = \mathcal{K}_\pm$.

Appendix

We state two propagation estimates which are crucial in the proof of Theorem 3.

PROPOSITION A.1. *Let* $\chi\in C_0^\infty(R)$, $0<c_0<c_1$, $c_1>\sup|\nabla\omega|$, *and*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma\left(\left\langle \frac{x}{t}-\nabla\omega(k),\, 1_{[c_0,c_1]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t}-\nabla\omega(k)\right)\right\rangle\right).$$

One has

$$\int_1^{\infty} ||\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{\mathrm{i}tH}u||^2 \frac{\mathrm{d}t}{t} \leqslant c||u||^2.$$

PROPOSITION A.2. *Let* $\chi\in C_0^\infty(\mathbb{R})$ *be supported in* $\mathbb{R}\backslash(\tau\cup\sigma_{\text{pp}}(H))$. *There exist* $\epsilon>0$, C *such that we have*

$$\int_1^{\infty} \left|\left|\Gamma\left(1_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right)\chi(H)\mathrm{e}^{\mathrm{i}tH}u\right|\right|^2 \frac{\mathrm{d}t}{t} \leqslant C||u||^2.$$

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