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Propagation of chaos for many-boson systems in one dimension with a point pair-interaction

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Abstract. We study the mean field limit of nonrelativistic quantum many-boson systems with delta potential in one-dimensional space. Such problem is related to the semiclassical limit of a second quantized Hamiltonian with an interaction given by a quartic Wick product. In this framework, we show that the evolution of coherent states is semiclassically given by squeezed coherent states under the action of a time-dependent affine Bogoliubov transformation. Results similar to those stated by Hepp [*Comm. Math. Phys.* **35** (1974), 265–277] and Ginibre-Velo [*Comm. Math. Phys.* **66** (1979), 37–76 and **68** (1979), 45–68] are proved. Furthermore, we show propagation of chaos for Schrödinger dynamics in the mean field limit using the argument of Rodnianski–Schlein [*Comm. Math. Phys.* **291** (2009), 31–61]. Thus, we provide a derivation of the cubic NLS equation in one dimension.

Keywords: classical limit, coherent state, Fock space, non-linear Schrödinger equation, mean field limit, non-autonomous Schrödinger equation

1. Introduction

We consider a non-relativistic quantum system of $\mathbf{N}$ bosons moving in d-dimensional space in the mean field scaling with a two-body point interaction. The heuristic Hamiltonian of the system is given by

$$\mathrm{H}_{\mathbf{N}} := \sum_{i=1}^{\mathbf{N}} -\Delta_{x_i} + \frac{1}{\mathbf{N}} \sum_{1\leqslant i<j\leqslant \mathbf{N}} \delta(x_i - x_j), \quad x_i, x_j \in \mathbb{R}^d. \tag{1}$$

Here δ stands for the Dirac distribution and the Hamiltonian $\mathrm{H}_{\mathbf{N}}$ formally acts on the space of symmetric square-integrable functions $L^2_s(\mathbb{R}^{d\mathbf{N}})$. This model is widely studied in the physical literature, particularly in the one dimension case $d = 1$ (see [17]). There are at least two subjects in which such a Hamiltonian is of interest, namely in nuclear physics and in quantum statistical mechanics. The Hamiltonian (1) indeed describes a simplified model of stripping reaction in nuclear physics where the main motivation is the calculation of cross-section and scattering matrix. In quantum statistical mechanics, (1) is related to Bose gases with hard-sphere interaction and Fermi pseudopotential (see [27]).

To the best of our knowledge there are, from a mathematical point of view, only few results available in dimension $d = 2, 3$. They are mainly restricted to the three-body problem. In contrast, the one-dimensional case $d = 1$ is quite simple. For instance, we can prove self-adjointness of $\mathrm{H}_{\mathbf{N}}$ by standard

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quadratic form technics. Consequently, the dynamic of the system is well defined in this case and it is given by the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t \Psi_{\mathbf{N}}^t = \mathrm{H}_{\mathbf{N}} \Psi_{\mathbf{N}}^t. \tag{2}$$

Beyond the simplicity of the one-dimensional case, the delta interaction in (1) is only form-bounded with respect to the kinetic energy. This means that the potential is quite singular. It is even not a Lebesgue measurable function. As we will explain it later, this feature of the model motivates our analysis.

Our goal in this paper is the justification of the chaos conservation hypothesis for the quantum many-body Hamiltonian (1) in one dimension $d = 1$. This well-know hypothesis finds its roots in statistical physics of classical many-particle systems (see [22] and references therein). Roughly speaking, we want to show that if the initial state of the system is uncorrelated

$$\Psi_{\mathbf{N}}^0 = \varphi_0^{\otimes \mathbf{N}} \in L_s^2(\mathbb{R}^{d\mathbf{N}}) \quad \text{with } \|\varphi_0\|_{L^2(\mathbb{R}^d)} = 1,$$

then the evolved state, at any fixed time t, is in a certain sense asymptotically uncorrelated, i.e.,

$$\Psi_{\mathbf{N}}^t \simeq \varphi_t^{\otimes \mathbf{N}} \quad \text{when } \mathbf{N} \to \infty$$

and φ_t solves the one particle non-linear cubic Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \varphi = -\Delta\varphi + |\varphi|^2 \varphi, \\ \varphi_{|t=0} = \varphi_0. \end{cases} \tag{3}$$

More precisely, consider the k-particle correlation functions of the state $\Psi_{\mathbf{N}}^t$

$$\gamma_{k,\mathbf{N}}^t(x_1, \ldots, x_k; y_1, \ldots, y_k) = \int_{\mathbb{R}^{d(\mathbf{N}-k)}} \Psi_{\mathbf{N}}^t(x_1, \ldots, x_k; z) \overline{\Psi_{\mathbf{N}}^t(y_1, \ldots, y_k; z)} \, \mathrm{d}z. \tag{4}$$

The kernel $\gamma_{k,\mathbf{N}}^t$ is symmetric with respect to any permutation over the variables $(x_1, \ldots, x_k)$ or $(y_1, \ldots, y_k)$. Furthermore, it defines a positive trace class operator over $L^2(\mathbb{R}^{dk})$, which we still denote by $\gamma_{k,\mathbf{N}}^t$. The chaos conservation hypothesis (also called propagation of chaos) is the property of convergence in the trace norm of $\gamma_{k,\mathbf{N}}^t$ to the projector $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$, i.e.,

$$\lim_{\mathbf{N}\to\infty} \operatorname{Tr} \left|\gamma_{k,\mathbf{N}}^t - |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\right| = 0.$$

By duality, this convergence of correlation functions $\gamma_{k,\mathbf{N}}^t$ is also equivalent to the statement:

$$\lim_{\mathbf{N}\to\infty} \langle \Psi_{\mathbf{N}}^t, A \otimes 1^{(\mathbf{N}-k)} \Psi_{\mathbf{N}}^t \rangle = \langle \varphi_t^{\otimes k}, A \varphi_t^{\otimes k} \rangle \tag{5}$$

for any bounded operator $A : L^2(\mathbb{R}^{dk}) \to L^2(\mathbb{R}^{dk})$. Here $1^{(\mathbf{N}-k)}$ denotes the identity operator acting on $L^2(\mathbb{R}^{d(\mathbf{N}-k)})$.

In the recent years, mainly motivated by the study of Bose–Einstein condensates, there is a renewed and growing interest in the analysis of many-body quantum dynamics in the mean field limit (see [1,5, 6,10–12,14–16,28,33], etc.).

A statement similar to (5) was first proved in [33] for bounded potentials (i.e., where δ in (1) is replaced by a real-valued function V in $L^\infty(\mathbb{R}^d)$ and the cubic Schrödinger equation (3) by a non-linear Hartree equation). Then it was extended in [6,12] to the Coulomb potential using the so-called BBGKY hierarchy method. This approach (named after Bogoliubov, Born, Green, Kirkwood and Yvon) is based on the analysis of the Heisenberg equation,

$$\begin{cases} \partial_t \rho_t = \mathrm{i}[\rho_t, \mathrm{H}_{\mathbf{N}}], \\ \rho_{|t=0} = |\varphi_0^{\otimes \mathbf{N}}\rangle\langle\varphi_0^{\otimes \mathbf{N}}|, \end{cases} \tag{6}$$

together with the finite chain of equations obtained from (6) by taking partial traces on k variables with $0 \leqslant k \leqslant \mathbf{N}$. For a general presentation on this method and its connection to the mean field problem for classical particles, we refer the reader to the reviews [21,33]. More recently, other approaches have emerged (e.g., [4,14,26,31]) and error estimates were also derived.

One of the alternative approaches to the chaos conservation hypothesis uses the second quantization framework (details on this notions are recalled in Section 2). We consider the Hamiltonian,

$$H_\varepsilon = \varepsilon \int_{\mathbb{R}^d} \nabla a^*(x) \nabla a(x)\,\mathrm{d}x + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^{2d}} a^*(x) a^*(y) \delta(x-y) a(x) a(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{7}$$

where a, a^* are the usual creation-annihilation operator-valued distributions in the Fock space over $L^2(\mathbb{R}^d)$. Recall that a and a^* satisfy the canonical commutation relations

$$\big[a(x), a^*(y)\big] = \delta(x-y), \qquad \big[a^*(x), a^*(y)\big] = 0 = \big[a(x), a(y)\big].$$

A simple computation leads to the following identity

$$\varepsilon^{-1} H_{\varepsilon_{|L^2_s(\mathbb{R}^{d\mathbf{N}})}} = \mathbf{H}_{\mathbf{N}} \quad \text{if } \varepsilon = \frac{1}{\mathbf{N}}.$$

Thus, the statement (5) on propagation of chaos can be written (up to an unessential factor) as

$$\lim_{\varepsilon \to 0} \langle \mathrm{e}^{-\mathrm{i}t\varepsilon^{-1} H_\varepsilon} \Psi^0_{\mathbf{N}}, b^{Wick} \mathrm{e}^{-\mathrm{i}t\varepsilon^{-1} H_\varepsilon} \Psi^0_{\mathbf{N}} \rangle = \langle \varphi_t^{\otimes k}, A \varphi_t^{\otimes k} \rangle, \quad \varepsilon = \frac{1}{\mathbf{N}},$$

where b^{Wick} denotes ε-dependent Wick observables defined by

$$b^{Wick} = \varepsilon^k \int_{\mathbb{R}^{2kd}} \prod_{i=1}^k a^*(x_i) A(x_1, \ldots, x_k; y_1, \ldots, y_k) \prod_{j=1}^k a(y_j)\,\mathrm{d}x_1 \cdots \mathrm{d}x_k\,\mathrm{d}y_1 \cdots \mathrm{d}y_k.$$

Here $A(x_1, \ldots, x_k; y_1, \ldots, y_k)$ denotes the distribution kernel of the bounded operator A on $L^2(\mathbb{R}^{kd})$. Therefore, the mean field limit ($\mathbf{N} \to \infty$) for the Hamiltonian $\mathbf{H}_{\mathbf{N}}$ can be related to the semiclassical limit ($\varepsilon \to 0$) for H_ε. The study of the semiclassical limit for many-boson systems started with the work of Hepp [23]. It was subsequently improved by Ginibre and Velo [19,20]. This analysis uses coherent states

$$\Psi^0_\varepsilon = \mathrm{e}^{-|\varphi|^2/(2\varepsilon)} \sum_{n=0}^\infty \varepsilon^{-n/2} \frac{\varphi^{\otimes n}}{\sqrt{n!}}, \quad \varphi \in L^2(\mathbb{R}^d),$$

instead of chaos states $\Psi_{\mathbf{N}}^0 = \varphi_0^{\otimes \mathbf{N}}$. However, a clever argument in the work of Rodnianski and Schlein [31] shows that propagation of chaos can be deduced form the semiclassical analysis with coherent states. They also provided error estimates on the k-particle correlation functions.

In this context, our purpose in considering this problem is to weaken the assumptions on the two-body potential. We expect that the statement (5) holds true for the general situation where δ is replaced by a quadratic form Q which is infinitesimally form bounded with respect to $-\Delta$. As one steep forward, we look at the specific problem of delta potential in one dimension which fits the above setting. The best available result in this direction, at our knowledge, is the work of Ginibre and Velo [20]. However, it is only valid for coherent states and dimension $d \geqslant 3$. The recent work of [26] apply to potentials V in $L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ while the work [31] assumes $(-\Delta + 1)^{1/2}$-bounded potential (i.e., $V(-\Delta + 1)^{-1/2}$ is bounded operator). Some other works are specific for the Coulomb potential (e.g., [9,16]). To briefly enlighten the comparison, we consider, in three dimension, the potential

$$V(x) = \frac{1}{|x|^\alpha}.$$

Then the work [31] apply for $\alpha \leqslant 1$ and [26] for $\alpha < 3/2$ while V is $-\Delta$-form bounded for $\alpha < 2$. So far, the Hamiltonian (1) has not been considered to our knowledge, except in [1] where a partial result is proved (i.e., convergence of BBGKY hierarchy is proved but not the uniqueness).

Another point of view consists of replacing the delta interaction in (1) by a smooth scaled potential $V_{\mathbf{N}}(x) = \mathbf{N}^{d\beta} V(\mathbf{N}^\beta x)$ with $\beta \in (0,1)$. In this case, propagation of chaos was proved (under some assumptions) in [2] for $d = 1$ and in [10] for $d = 3$. Although, $V_{\mathbf{N}}$ converges in a distribution sense to $\delta \int V(x)\,\mathrm{d}x$, it is not clear how this can be related to our problem. We remark that the case $\beta = 1$ is related to the Gross–Pitaevskii equation derived in [11].

For reader convenience we give a glimpse of our main results. Essentially, our work is divided into three parts:

(i) Propagation of coherent states.
(ii) Propagation of chaos.
(iii) Non-autonomous abstract Schrödinger equation.

In (i) we give a semiclassical approximation of the evolved coherent states $\mathrm{e}^{-\mathrm{i}t\varepsilon^{-1}H_\varepsilon}\Psi_\varepsilon^0$ using the unitary propagator $U_2(t,s)$ of a time-dependent quadratic Hamiltonian $A_2(t)$ related to H_ε. It is the most technical part of the paper where subtle points about form domains come into the play. The main result is Theorem 6.1 which is actually slightly stronger than the usual formulation of the semiclassical approximation in [19,20,23]. This will be helpful to achieve part (ii). We also identify $U_2(t,s)$ as a time-dependent Bogoliubov transformation in Proposition 6.7. Once Theorem 6.1 is proved, the part (ii) follows from an argument in [31] which can be made abstract. For completeness we provide a proof for this fact (see Theorem 2.3). In part (iii) we establish some abstract results on the existence of unitary propagator for non-autonomous Schrödinger equations which may have their own interest. For instance, we prove in Corollary C.4 a result which can be considered as time-dependent counterpart of the well-known Nelson commutator theorem [29] (see Appendix C). This is a key point which allows to construct the unitary propagator $U_2(t,s)$ with crucial estimates (see Proposition 5.5).

For the sake of clarity, we restricted ourselves in this paper to the specific case of point interaction potential in one dimension. We hope that this will be helpful for further improvement. We believe indeed that such simple example sums up the principal difficulties on the problem. We also remark that the results here can be easily extended to the case $V(x) = -\delta(x)$ by working locally in time.

Finally, we outline the content of this article. We recall the basic definitions for the Fock space framework in Section 2 and state two of our main results (Theorem 2.3 and Proposition 2.4). Then we accurately introduce the quantum dynamics of the considered many-boson system and its classical counterpart, namely the cubic NLS equation. The study of the semiclassical limit through Hepp's method is carried out in Section 6 where we use results on the time-dependent quadratic approximation derived in Section 5. In the last Section 7 we apply the argument of [31] to prove the chaos propagation result.

2. Preliminaries and main results

Let $\mathfrak{H}$ be a Hilbert space. We denote by $\mathcal{L}(\mathfrak{H})$ the space of all linear bounded operators on $\mathfrak{H}$. For a linear unbounded operator L acting on $\mathfrak{H}$, we denote by $\mathcal{D}(L)$ (respectively $\mathcal{Q}(L)$) the operator domain (respectively form domain) of L. Let D_{x_j} denotes the differential operator $-\mathrm{i}\partial_{x_j}$ on $L^2(\mathbb{R}^n)$ where $(x_1,\dots,x_n)\in\mathbb{R}^n$. Let $H^s(\mathbb{R}^m)$ denotes the Sobolev spaces.

In the following we recall the second quantization framework. We denote by $L^2_s(\mathbb{R}^{nd})$ the space of symmetric square integrable functions, i.e.,

$$\Psi_n\in L^2_s(\mathbb{R}^{nd})\quad\text{iff } \Psi_n\in L^2(\mathbb{R}^{nd})\quad\text{and}\quad \Psi_n(x_1,\dots,x_n)=\Psi_n(x_{\sigma_1},\dots,x_{\sigma_n})\quad\text{a.e.}$$

for any permutation σ on the symmetric group $\mathrm{Sym}(n)$. We can see $L^2_s(\mathbb{R}^{nd})$ as a closed subspace of $L^2(\mathbb{R}^{nd})$ characterized by the orthogonal projection $\mathfrak{S}_n$, given by

$$\mathfrak{S}_n\Psi_n(x_1,\dots,x_n)=\frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)}\Psi_n(x_{\sigma(1)},\dots,x_{\sigma(n)}),\quad \Psi_n\in L^2(\mathbb{R}^{nd}).$$

The symmetric Fock space over $L^2(\mathbb{R})$ is defined as the Hilbert space,

$$\mathcal{F}=\bigoplus_{n=0}^{\infty}L^2_s(\mathbb{R}^{nd}),$$

endowed with the inner product

$$\langle\Psi,\Phi\rangle=\sum_{n=0}^{\infty}\int_{\mathbb{R}^{nd}}\overline{\Psi_n(x_1,\dots,x_n)}\Phi_n(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where $\Psi=(\Psi_n)_{n\in\mathbb{N}}$ and $\Phi=(\Phi_n)_{n\in\mathbb{N}}$ are two arbitrary vectors in $\mathcal{F}$. The vacuum is the vector $\Omega_0=(1,0,\dots)$ in $\mathcal{F}$. We will use the notation

$$\mathcal{S}_s(\mathbb{R}^{nd}):=\mathfrak{S}_n\mathcal{S}(\mathbb{R}^{nd}),$$

where $\mathcal{S}(\mathbb{R}^{nd})$ is the Schwartz space on $\mathbb{R}^{nd}$. A convenient subspace of $\mathcal{F}$ is given by the algebraic direct sum

$$\mathcal{S}:=\bigoplus_{n=0}^{\mathrm{alg}}\mathcal{S}_s(\mathbb{R}^{nd}).$$

The most common operators on $\mathcal{F}$ are determined by their action on the family of vectors

$$\varphi^{\otimes n}(x_1, \dots, x_n) = \prod_{i=1}^{n} \varphi(x_i), \quad \varphi \in L^2(\mathbb{R}^d),$$

which spans the space $L^2_s(\mathbb{R}^{nd})$ thanks to the polarization identity,

$$\mathfrak{S}_n \prod_{i=1}^{n} \varphi_i(x_i) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \prod_{i=1}^{n} \left(\sum_{j=1}^{n} \varepsilon_j \varphi_j(x_i) \right).$$

For example, the creation and annihilation operators $a^*(f)$ and $a(f)$, parameterized by $\varepsilon > 0$, are defined by

$$a(f)\varphi^{\otimes n} = \sqrt{\varepsilon n}\langle f, \varphi\rangle \varphi^{\otimes(n-1)},$$
$$a^*(f)\varphi^{\otimes n} = \sqrt{\varepsilon(n+1)}\mathfrak{S}_{n+1}\big(f \otimes \varphi^{\otimes n}\big), \quad \forall \varphi, f \in L^2(\mathbb{R}^d).$$

They can also by written as

$$a(f) = \sqrt{\varepsilon} \int_{\mathbb{R}^d} \overline{f(x)} a(x)\, \mathrm{d}x, \qquad a^*(f) = \sqrt{\varepsilon} \int_{\mathbb{R}^d} f(x) a^*(x)\, \mathrm{d}x,$$

where $a^*(x), a(x)$ are the canonical creation-annihilation operator-valued distributions. Recall that for any $\Psi = (\Psi^{(n)})_{n\in\mathbb{N}} \in \mathcal{S}$, we have

$$\big[a(x)\Psi\big]^{(n)}(x_1, \dots, x_n) = \sqrt{(n+1)}\Psi^{(n+1)}(x, x_1, \dots, x_n),$$
$$\big[a^*(x)\Psi\big]^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j)\Psi^{(n-1)}(x_1, \dots, \hat{x}_j, \dots, x_n),$$

where δ is the Dirac distribution at the origin and $\hat{x}_j$ means that the variable x_j is omitted. The Weyl operators are given for $f \in L^2(\mathbb{R}^d)$ by

$$W(f) = \mathrm{e}^{(\mathrm{i}/\sqrt{2})[a^*(f)+a(f)]}$$

and they satisfy the Weyl commutation relations,

$$W(f_1)W(f_2) = \mathrm{e}^{-(\mathrm{i}\varepsilon/2)\,\mathrm{Im}\langle f_1, f_2\rangle} W(f_1 + f_2), \tag{8}$$

with $f_1, f_2 \in L^2(\mathbb{R}^d)$.

The interaction term in the Hamiltonian (7) is a quartic Wick product which we shall consider as a quadratic form on $\mathcal{S}$. From a general point view, this is better understood as a Wick quantization of a polynomial symbol. Let us briefly recall this Wick quantization procedure.

Definition 2.1. We say that a function $b : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is a continuous (p, q)-homogenous polynomial on $\mathcal{S}(\mathbb{R}^d)$ iff it satisfies:

(i) $b(\lambda z) = \bar{\lambda}^q \lambda^p b(z)$ for any $\lambda \in \mathbb{C}$ and $z \in \mathcal{S}(\mathbb{R}^d)$,
(ii) there exists a (unique) continuous hermitian form $\mathfrak{Q} : \mathcal{S}_s(\mathbb{R}^{dq}) \times \mathcal{S}_s(\mathbb{R}^{dp}) \to \mathbb{C}$ such that

$$b(z) = \mathfrak{Q}\big(z^{\otimes q}, z^{\otimes p}\big).$$

We denote by $\mathcal{E}$ the vector space spanned by all those polynomials.

The Schwartz kernel theorem ensures for any continuous (p,q)-homogenous polynomial b, the existence of a kernel $\tilde{b}(\cdot,\cdot) \in \mathcal{S}'(\mathbb{R}^{d(p+q)})$ such that

$$b(z) = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}\big(k_1',\dots,k_q';k_1,\dots,k_p\big)\overline{z\big(k_1'\big)\cdots z\big(k_q'\big)}z(k_1)\cdots z(k_p)\,\mathrm{d}k'\,\mathrm{d}k$$

in the distribution sense. The set of (p,q)-homogenous polynomials $b \in \mathcal{E}$ such that the kernel $\tilde{b}$ defines a bounded operator from $L^2_s(\mathbb{R}^{dp})$ into $L^2_s(\mathbb{R}^{dq})$ will be denoted by $\mathcal{P}_{p,q}(L^2(\mathbb{R}^d))$. Those classes of polynomial symbols are studied and used in [3,4].

Definition 2.2. The Wick quantization is the map which associate to each continuous (p,q)-homogenous polynomial $b \in \mathcal{E}$, a quadratic form b^{Wick} on $\mathcal{S}$ given by

$$\begin{aligned}\big\langle \Psi, b^{Wick}\Phi\big\rangle &= \varepsilon^{(p+q)/2}\int_{\mathbb{R}^{d(p+q)}} \tilde{b}\big(k',k\big)\big\langle a\big(k_1'\big)\cdots a\big(k_q'\big)\Psi, a(k_1)\cdots a(k_p)\Phi\big\rangle_{\mathcal{F}}\,\mathrm{d}k\,\mathrm{d}k' \\ &= \sum_{n=p}^{\infty} \varepsilon^{(p+q)/2}\frac{\sqrt{n!(n-p+q)!}}{(n-p)!} \\ &\quad\times \int_{\mathbb{R}^{d(n-p)}} \mathrm{d}x \int_{\mathbb{R}^{d(p+q)}} \mathrm{d}k\,\mathrm{d}k'\,\tilde{b}\big(k',k\big)\overline{\Psi^{(n)}(k,x)}\Phi^{(n-p+q)}\big(k',x\big)\end{aligned}$$

for any $\Phi, \Psi \in \mathcal{S}$.

We have, for example,

$$a^*(f) = \langle z, f\rangle^{Wick} \quad \text{and} \quad a(f) = \langle f, z\rangle^{Wick}.$$

Furthermore, for any self-adjoint operator A on $L^2(\mathbb{R}^d)$ such that $\mathcal{S}(\mathbb{R}^d)$ is a core for A, the Wick quantization

$$\mathrm{d}\Gamma(A) := \langle z, Az\rangle^{Wick}$$

defines a self-adjoint operator on $\mathcal{F}$. In particular, if A is the identity we get the ε-dependent number operator

$$N := \langle z, z\rangle^{Wick}.$$

We recall the standard number estimate (see, e.g., [3], Lemma 2.5),

$$\big|\big\langle \Psi, b^{Wick}\Phi\big\rangle\big| \leqslant \|\tilde{b}\|_{\mathcal{L}(L^2_s(\mathbb{R}^{dp}),L^2_s(\mathbb{R}^{dq}))}\big\|N^{q/2}\Psi\big\| \times \big\|N^{p/2}\Phi\big\|, \tag{9}$$

which holds uniformly in $\varepsilon \in (0,1]$ for $b \in \mathcal{P}_{p,q}(L^2(\mathbb{R}^d))$ and any $\Psi, \Phi \in \mathcal{D}(N^{\max(p,q)/2})$.

In this section, we state two of our main results. Other results as Corollary 6.5 and 6.6 are not less interesting, in our opinion, but they are postponed to Section 6 to keep the presentation fairly simple. The rigorous meaning of the Hamiltonian (7) is explained in Section 3, while the existence and uniqueness of solutions for the nonlinear Schrödinger equation (3) are recalled in Section 4. The first result is propagation of chaos.

Theorem 2.3. *For any $\varphi_0 \in H^2(\mathbb{R})$ such that $\|\varphi_0\|_{L^2(\mathbb{R})} = 1$ and any $b \in \mathcal{P}_{p,p}(L^2(\mathbb{R}))$, we have*

$$\lim_{n\to\infty} \langle \varphi_0^{\otimes n}, \mathrm{e}^{\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} b^{Wick} \mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} \varphi_0^{\otimes n} \rangle = b(\varphi_t),$$

where $n\varepsilon_n = 1$ and φ_t solves the NLS equation (3) *with initial data φ_0.*

The proof is given in Section 7 and follows the argument of [31]. In fact, we express $\varphi_0^{\otimes n}$ in terms of coherent states,

$$\varphi_0^{\otimes n} = \frac{\mathrm{e}^{n/2}\sqrt{n!}}{2\pi n^{n/2}} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta n} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon_n}\mathrm{e}^{\mathrm{i}\theta}\varphi_0\right) \Omega_0 \,\mathrm{d}\theta \quad \text{with } \varepsilon_n = 1/n$$

and then use the semiclassical propagation estimate for coherent states given in the following proposition.

Proposition 2.4. *For any $\varphi_0 \in H^2(\mathbb{R})$ there exists $c > 0$ depending only on φ_0 such that*

$$\left\| \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Omega_0 - \mathrm{e}^{\mathrm{i}\omega(t)/\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right) U_2(t,0)\Omega_0 \right\|_{\mathcal{F}} \leqslant \mathrm{e}^{c\mathrm{e}^{c|t|}} \varepsilon^{1/8},$$

holds for any $t \in \mathbb{R}$. Here φ_t solves the NLS equation (3) *with the initial condition φ_0, $\omega(t) = \frac{1}{2}\int_0^t \|\varphi_s^2\|_{L^2}^2 \,\mathrm{d}s$ and $U_2(t,s)$ is the unitary propagator given by Proposition* 5.5.

Actually, we will prove a stronger result in Section 6 (see Theorem 6.1). This part of the paper is the most technical. However, the idea behind Proposition 2.4 is rather simple. In fact, we write a Taylor expansion of the Hamiltonian H_ε around the classical solution φ_t and consider only the terms of order less or equal to ε. Such quantity defines a time-dependent quadratic Hamiltonian which provides an approximation for the evolution of coherent states. If we attempt to show Proposition 2.4, we formally differentiate the quantity

$$\mathcal{Y}(t) = \mathrm{e}^{\mathrm{i}(t/\varepsilon)H_\varepsilon} \mathrm{e}^{\mathrm{i}\omega(t)/\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right) U_2(t,0).$$

So, we obtain

$$-\mathrm{i}\varepsilon\partial_t\mathcal{Y}(t) = \mathrm{e}^{\mathrm{i}(t/\varepsilon)H_\varepsilon} \mathrm{e}^{\mathrm{i}\omega(t)/\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right) \left[\tilde{H}_\varepsilon - A_0(t) - \sqrt{\varepsilon}A_1(t) - \varepsilon A_2(t)\right] U_2(t,0),$$

where $\tilde{H}_\varepsilon = W(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t)^* H_\varepsilon W(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t)$ and $A_0(t)$, $A_1(t)$, $A_2(t)$ are ε-independent operators. It is possible to expand $\tilde{H}_\varepsilon$ in the form $\tilde{H}_\varepsilon = \sum_{k=0}^{4} \varepsilon^{k/2} A_k(t)$, where again $A_k(t)$ are ε-independent operators, so that

$$\mathcal{Y}(t)\Omega_0 - \mathcal{Y}(0)\Omega_0 = \mathrm{i}\varepsilon^{1/2}\int_0^t \mathrm{e}^{\mathrm{i}(s/\varepsilon)H_\varepsilon}\mathrm{e}^{\mathrm{i}\omega(s)/\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_s\right)\left[A_3(s) + \sqrt{\varepsilon}A_4(s)\right]U_2(s,0)\Omega_0 \,\mathrm{d}s. \quad (10)$$

Then, we estimate the left-hand side of (10), for $t > 0$, by

$$\left\|\mathcal{Y}(t)\Omega_0 - W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Omega_0\right\|_{\mathcal{F}} \leqslant \sqrt{\varepsilon}\int_0^t \left\|\left[A_3(s) + \sqrt{\varepsilon}A_4(s)\right]U_2(s,0)\Omega_0\right\|_{\mathcal{F}} \mathrm{d}s. \quad (11)$$

Hence, we get the coherent state estimate when the integrand in the right-hand side of (11) is bounded uniformly in ε. This holds true if we replace the δ interaction in (12) by a bounded potential V. But, in our case we end up with the problem that $U_2(s,0)\Omega_0$ is not in the domain of $A_3(s) + \sqrt{\varepsilon}A_4(s)$. So, we cannot proceed in this way. Instead, we use a careful decomposition by means of appropriate cutoffs (see Lemma 6.4) and exploit a crucial uniform estimates derived in Lemma 6.3 and Proposition 5.5. Other results concerning abstract non-autonomous Schrödinger equation are stated in Appendix C. They may be considered as an improvement of Kisynski's work [25]. In particular, Corollary C.4 is used to prove Lemma 6.3 and Proposition 5.5.

3. Many-boson system

In non-relativistic many-body theory, boson systems are described by the second quantized Hamiltonian in the symmetric Fock space $\mathcal{F}$ formally given by

$$H_\varepsilon = -\varepsilon\int_{\mathbb{R}^d} a^*(x)\Delta a(x)\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a^*(x)a^*(y)\delta(x-y)a(x)a(y)\,\mathrm{d}x\,\mathrm{d}y. \quad (12)$$

The rigorous meaning of formula (12) is as a quadratic form on $\mathcal{S}$, which we denote by h^{Wick}, obtained by Wick quantization of the classical energy functional

$$h(z) = \int_{\mathbb{R}^d} \left|\nabla z(x)\right|^2 \mathrm{d}x + P(z), \quad \text{where } P(z) = \frac{1}{2}\int_{\mathbb{R}^d} \left|z(x)\right|^4 \mathrm{d}x, z \in \mathcal{S}(\mathbb{R}^d). \quad (13)$$

More explicitly, we have for $\Psi \in \mathcal{S}$

$$\begin{aligned}\langle \Psi, h^{Wick}\Psi\rangle = {} & \varepsilon\sum_{n=1}^{\infty} n\int_{\mathbb{R}^{dn}} \left|\partial_{x_1}\Psi^{(n)}(x_1,\ldots,x_n)\right|^2 \mathrm{d}x_1\cdots\mathrm{d}x_n \\ & + \varepsilon^2\sum_{n=2}^{\infty}\frac{n(n-1)}{2}\int_{\mathbb{R}^{d(n-1)}} \left|\Psi^{(n)}(x_2,x_2,\ldots,x_n)\right|^2 \mathrm{d}x_2\cdots\mathrm{d}x_n.\end{aligned}$$

Moreover, in one-dimensional space (i.e., $d = 1$) one can show the existence of a unique self-adjoint operator bounded from below, which we denote by H_ε, such that

$$\langle \Psi, H_\varepsilon\Psi\rangle = \langle \Psi, h^{Wick}\Psi\rangle \quad \text{for any } \Psi \in \mathcal{S}.$$

This is proved in Proposition 3.3.

In all the sequel we restrict our analysis to space dimension $d = 1$ and consider the small parameter ε such that $\varepsilon \in (0, 1]$. The ε-independent self-adjoint operator,

$$S_\mu \Psi := \Psi + \sum_{n=1}^{\infty} \left[n^\mu \Psi^{(n)} + \sum_{j=1}^{n} -\Delta_{x_j} \Psi^{(n)} \right] = \left(\varepsilon^{-1} \, \mathrm{d}\Gamma(-\Delta) + \varepsilon^{-\mu} N^\mu + 1 \right) \Psi,$$

with $\mu > 0$, defines the Hilbert space $\mathcal{F}_+^\mu$ given as the linear space $\mathcal{D}(S_\mu^{1/2})$ equipped with the inner product

$$\langle \Psi, \Phi \rangle_{\mathcal{F}_+^\mu} := \langle S_\mu^{1/2} \Psi, S_\mu^{1/2} \Phi \rangle_{\mathcal{F}}.$$

We denote by $\mathcal{F}_-^\mu$ the completion of $\mathcal{D}(S_\mu^{-1/2})$ with respect to the norm associated to the following inner product

$$\langle \Psi, \Phi \rangle_{\mathcal{F}_-^\mu} := \langle S_\mu^{-1/2} \Psi, S_\mu^{-1/2} \Phi \rangle_{\mathcal{F}}.$$

Therefore, we have the Hilbert rigging

$$\mathcal{F}_+^\mu \subset \mathcal{F} \subset \mathcal{F}_-^\mu.$$

Note that the form domain of the ε-dependent self-adjoint operator $\mathrm{d}\Gamma(-\Delta) + N^\mu$ with $\mu > 0$ is

$$\mathcal{Q}\left(\mathrm{d}\Gamma(-\Delta) + N^\mu \right) = \mathcal{F}_+^\mu \quad \text{for any } \varepsilon \in (0, 1].$$

Lemma 3.1. *For any $\Psi, \Phi \in \mathcal{S}$,*

$$\left| \langle \Psi, P^{\mathit{Wick}} \Phi \rangle \right| \leqslant \frac{1}{4} \left\| \left[\mathrm{d}\Gamma(-\Delta) + N^3 \right]^{1/2} \Psi \right\| \times \left\| \left[\mathrm{d}\Gamma(-\Delta) + N^3 \right]^{1/2} \Phi \right\|.$$

Proof. A simple computation yields for any $\Psi, \Phi \in \mathcal{S}$

$$\langle \Psi, P^{\mathit{Wick}} \Phi \rangle = \sum_{n=2}^{\infty} \varepsilon^2 \frac{n(n-1)}{2} \int_{\mathbb{R}^{n-1}} \overline{\Psi^{(n)}(x_2, x_2, x_3, \ldots, x_n)} \Phi^{(n)}(x_2, x_2, x_3, \ldots, x_n) \, \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

Cauchy–Schwarz inequality yields

$$\left| \langle \Psi, P^{\mathit{Wick}} \Phi \rangle \right| \leqslant \left[\sum_{n=2}^{\infty} \varepsilon^2 \frac{n(n-1)}{2} \int_{\mathbb{R}^{n-1}} \left| \Psi^{(n)}(x_2, x_2, x_3, \ldots, x_n) \right|^2 \mathrm{d}x_2 \cdots \mathrm{d}x_n \right]^{1/2} \\ \times \left[\sum_{n=2}^{\infty} \varepsilon^2 \frac{n(n-1)}{2} \int_{\mathbb{R}^{n-1}} \left| \Phi^{(n)}(x_2, x_2, x_3, \ldots, x_n) \right|^2 \mathrm{d}x_2 \cdots \mathrm{d}x_n \right]^{1/2}.$$

Using Lemma A.1, we get for any $\alpha(n) > 0$

$$\big|\langle \Psi, P^{Wick}\Phi\rangle\big| \leqslant \left[\sum_{n=2}^{\infty} \varepsilon^2 \frac{n(n-1)}{2\sqrt{2}}\Big(\alpha(n)\big\langle D_{x_1}^2 \Psi^{(n)}, \Psi^{(n)}\big\rangle + \frac{\alpha(n)^{-1}}{2}\big\langle \Psi^{(n)}, \Psi^{(n)}\big\rangle\Big)\right]^{1/2} \times \left[\sum_{n=2}^{\infty} \varepsilon^2 \frac{n(n-1)}{2\sqrt{2}}\Big(\alpha(n)\big\langle D_{x_1}^2 \Phi^{(n)}, \Phi^{(n)}\big\rangle + \frac{\alpha(n)^{-1}}{2}\big\langle \Phi^{(n)}, \Phi^{(n)}\big\rangle\Big)\right]^{1/2}.$$

Hence, by choosing $\alpha(n) = \frac{1}{\sqrt{2}\varepsilon(n-1)}$, it follows that

$$\begin{aligned}\big|\langle \Psi, P^{Wick}\Phi\rangle\big| &\leqslant \frac{1}{4}\left[\sum_{n=2}^{\infty} \varepsilon n\big\langle D_{x_1}^2 \Psi^{(n)}, \Psi^{(n)}\big\rangle + \sum_{n=2}^{\infty} \varepsilon^3 n(n-1)^2\big\langle \Psi^{(n)}, \Psi^{(n)}\big\rangle\right]^{1/2} \\ &\quad \times \left[\sum_{n=2}^{\infty} \varepsilon n\big\langle D_{x_1}^2 \Phi^{(n)}, \Phi^{(n)}\big\rangle + \sum_{n=2}^{\infty} \varepsilon^3 n(n-1)^2\big\langle \Phi^{(n)}, \Phi^{(n)}\big\rangle\right]^{1/2} \\ &\leqslant \frac{1}{4}\sqrt{\big\langle \Psi, \big[\mathrm{d}\Gamma(-\Delta) + N^3\big]\Psi\big\rangle} \times \sqrt{\big\langle \Phi, \big[\mathrm{d}\Gamma(-\Delta) + N^3\big]\Phi\big\rangle}.\end{aligned}$$

This leads to the claimed estimate. □

Remark 3.2. Note that, as in Lemma 3.1, the estimate

$$\big|\langle \Psi, P^{Wick}\Phi\rangle\big| \leqslant \frac{\varepsilon^2}{4}\|\Psi\|_{\mathcal{F}_+^3}\|\Phi\|_{\mathcal{F}_+^3} \tag{14}$$

holds true for any $\Psi, \Phi \in \mathcal{S}$ and $\varepsilon \in (0, 1]$.

We can show that h^{Wick} is associated to a self-adjoint operator by considering its restriction to each sector $L_s^2(\mathbb{R}^n)$, however we will prefer the following point of view.

Proposition 3.3. *There exists a unique self-adjoint operator H_ε such that*

$$\big\langle \Psi, h^{Wick}\Phi\big\rangle = \langle \Psi, H_\varepsilon\Phi\rangle \quad \textit{for any } \Psi \in \mathcal{F}_+^3, \Phi \in \mathcal{D}(H_\varepsilon) \cap \mathcal{F}_+^3.$$

Moreover, $\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon}$ *preserves* $\mathcal{F}_+^3$.

Proof. We first use the KLMN theorem [30], Theorem X17, and Lemma 3.1 to show that the quadratic form $h^{Wick} + N^3 + 1$ is associated to a unique (positive) self-adjoint operator L with

$$\mathcal{Q}(L) = \mathcal{Q}\big(\mathrm{d}\Gamma(-\Delta) + N^3\big) = \mathcal{F}_+^3.$$

Observe that we also have

$$\big\|\big[\mathrm{d}\Gamma(-\Delta) + N^3\big]^{1/2}\Psi\big\| \leqslant \big\|L^{1/2}\Psi\big\| \quad \text{for any } \Psi \in \mathcal{F}_+^3. \tag{15}$$

Next, by the Nelson commutator theorem (Theorem B.2) we can prove that the quadratic form h^{Wick} is uniquely associated to a self-adjoint operator denoted by H_ε with $\mathcal{D}(L) \subset \mathcal{D}(H_\varepsilon) \cap \mathcal{F}^3_+$ and deduce the invariance of $\mathcal{F}^3_+$. Indeed, we easily check using Lemma 3.1 and (15) that

$$\left|\langle \Psi, h^{Wick}\Phi\rangle\right| \leqslant \frac{5}{4}\|L^{1/2}\Psi\|\|L^{1/2}\Phi\| \quad \text{for any } \Psi, \Phi \in \mathcal{F}^3_+. \tag{16}$$

Furthermore, we have for $\Psi, \Phi \in \mathcal{F}^3_+$ and $\lambda > 0$

$$\left\langle L(\lambda L+1)^{-1}\Psi, h^{Wick}(\lambda L+1)^{-1}\Phi\right\rangle - \left\langle(\lambda L+1)^{-1}\Psi, h^{Wick}L(\lambda L+1)^{-1}\Phi\right\rangle = 0. \tag{17}$$

The statements (16) and (17) with the help of Lemma B.3, allow to use Theorem B.2. □

Remark 3.4. The same argument as in Proposition 3.3 shows that the quadratic form on $\mathcal{F}^3_+$ given by

$$G := \varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta) + \varepsilon^{-2}P^{Wick} + \varepsilon^{-1}N + 1,$$

is associated to a unique (positive) self-adjoint operator which we denote by the same symbol G.

4. The cubic NLS equation

The energy functional h given by (13) has the associated vector field

$$X : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}),$$
$$z \mapsto X(z) = -\Delta z + \partial_{\bar z}P(z),$$

which leads to the non-linear classical field equation

$$\mathrm{i}\partial_t\varphi = X(\varphi) = -\Delta\varphi + |\varphi|^2\varphi \tag{18}$$

with initial data $\varphi_{|t=0} = \varphi_0 \in H^1(\mathbb{R})$. It is well known that the above cubic defocusing NLS equation is globally well-posed on $H^s(\mathbb{R})$ for $s \geqslant 0$. In particular, the equation (18) admits a unique global solution on $C^0(\mathbb{R}, H^m(\mathbb{R})) \cap C^1(\mathbb{R}, H^{m-2}(\mathbb{R}))$ for any initial data $\varphi \in H^m(\mathbb{R})$ when $m = 1$ and $m = 2$ (see [18] for $m = 1$ and [34] for $m = 2$). Moreover, we have energy and mass conservations i.e.,

$$h(\varphi_t) = h(\varphi_0) \quad \text{and} \quad \|\varphi_t\|_{L^2(\mathbb{R})} = \|\varphi_0\|_{L^2(\mathbb{R})}$$

for any initial data $\varphi_0 \in H^1(\mathbb{R})$ and φ_t solution of (18). It is not difficult to prove the following estimates

$$\begin{aligned}
&\|\varphi\|^2_{L^\infty(\mathbb{R})} \leqslant 2\|\varphi\|_{L^2(\mathbb{R})}\|\partial_x\varphi\|_{L^2(\mathbb{R})} \leqslant 2\|\varphi\|_{L^2(\mathbb{R})}h(\varphi)^{1/2},\\
&\|\varphi\|^p_{L^p(\mathbb{R})} \leqslant 2^{(p-2)/2}\|\varphi\|^{(p+2)/2}_{L^2(\mathbb{R})}\|\partial_x\varphi\|^{(p-2)/2}_{L^2(\mathbb{R})} \leqslant 2^{(p-2)/2}\|\varphi\|^{(p+2)/2}_{L^2(\mathbb{R})}h(\varphi)^{(p-2)/4}
\end{aligned} \tag{19}$$

for $p \geqslant 2$ and any $\varphi \in H^1(\mathbb{R})$. Furthermore, using Gronwall's inequality we show for any $\varphi_0 \in H^2(\mathbb{R})$ the existence of $c > 0$ depending only on φ_0 such that

$$\|\varphi_t\|_{H^2(\mathbb{R})} \leqslant \mathrm{e}^{c|t|}\|\varphi_0\|_{H^2(\mathbb{R})}, \tag{20}$$

where φ_t is a solution of the NLS equation (18) with initial condition φ_0.

5. Time-dependent quadratic dynamics

In this section we construct a time-dependent quadratic approximation for the Schrödinger dynamics. We prove existence of a unique unitary propagator for this approximation using the abstract results for non-autonomous linear Schrödinger equation stated in the Appendix C. This step will be useful for the study of propagation of coherent states in the semiclassical limit in Section 6.

The polynomial P has the following Taylor expansion for any $z_0 \in H^1(\mathbb{R})$

$$P(z+z_0) = \sum_{j=0}^{4} \frac{D^{(j)}P}{j!}(z_0)[z].$$

Let φ_t be a solution of the NLS equation (18) with an initial data $\varphi_0 \in H^1(\mathbb{R})$. Consider the time-dependent quadratic polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$P_2(t)[z] := \frac{D^{(2)}P}{2}(\varphi_t)[z] = \operatorname{Re}\int_{\mathbb{R}} \overline{z(x)}^2 \varphi_t(x)^2 \,\mathrm{d}x + 2\int_{\mathbb{R}} \left|z(x)\right|^2 |\varphi_t(x)|^2 \,\mathrm{d}x.$$

Let $\{A_2(t)\}_{t\in\mathbb{R}}$ be the ε-independent family of quadratic forms on $\mathcal{S}$ defined by

$$\varepsilon A_2(t) := \mathrm{d}\Gamma(-\Delta) + P_2(t)^{\mathit{Wick}}. \tag{21}$$

Lemma 5.1. *For $\varphi_0 \in H^1(\mathbb{R})$ let*

$$\vartheta_1 := 16^2\left(\|\varphi_0\|_{L^2(\mathbb{R})}+1\right)^3\left(h(\varphi_0)+1\right) \quad \text{and} \quad \vartheta_2 := 16^2\left(\|\varphi_0\|_{L^2(\mathbb{R})}+1\right)^{3/2}\sqrt{h(\varphi_0)+1}.$$

The quadratic forms on $\mathcal{S}$ defined by

$$S_2(t) := A_2(t) + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1, \quad t \in \mathbb{R},$$

are associated to unique self-adjoint operators, still denoted by $S_2(t)$, satisfying

- $S_2(t) \geqslant 1$,
- $\mathcal{D}(S_2(t)^{1/2}) = \mathcal{F}^1_+$ *for any $t \in \mathbb{R}$.*

Proof. The case $\varphi_0 = 0$ is trivial. By definition of Wick quantization we have for $\Psi, \Phi \in \mathcal{S}$,

$$
\begin{aligned}
\langle \Phi, P_2(t)^{Wick}\Psi\rangle = {} & 2\sum_{n=1}^{\infty} \varepsilon n \int_{\mathbb{R}^n} \left|\varphi_t(x_1)\right|^2 \overline{\Phi^{(n)}(x_1,\dots,x_n)}\Psi^{(n)}(x_1,\dots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
& + \sum_{n=0}^{\infty} \varepsilon\sqrt{(n+1)(n+2)} \int_{\mathbb{R}^n} \overline{\Phi^{(n)}(x_1,\dots,x_n)} \\
& \quad \times \left(\int_{\mathbb{R}} \overline{\varphi_t(x)}^2 \Psi^{(n+2)}(x,x,x_1,\dots,x_n)\,\mathrm{d}x\right)\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
& + \sum_{n=0}^{\infty} \varepsilon\sqrt{(n+1)(n+2)} \int_{\mathbb{R}^n} \Psi^{(n)}(x_1,\dots,x_n) \\
& \quad \times \left(\int_{\mathbb{R}} \varphi_t(x)^2 \overline{\Phi^{(n+2)}(x,x,x_1,\dots,x_n)}\,\mathrm{d}x\right)\mathrm{d}x_1 \cdots \mathrm{d}x_n. \qquad (22)
\end{aligned}
$$

Therefore, using Cauchy–Schwarz inequality, we show

$$
\begin{aligned}
\left|\langle \Phi, P_2(t)^{Wick}\Psi\rangle\right| \leqslant {} & 2\|\varphi_t\|^2_{L^\infty(\mathbb{R})}\|N^{1/2}\Phi\| \times \|N^{1/2}\Psi\| \\
& + \|\varphi_t\|^2_{L^4(\mathbb{R})}\|(N+\varepsilon)^{1/2}\Phi\| \\
& \times \left[\sum_{n=0}^{\infty} \varepsilon(n+2)\|\Psi^{(n+2)}(x,x,x_1,\dots,x_n)\|^2_{L^2(\mathbb{R}^{n+1})}\right]^{1/2} \\
& + \|\varphi_t\|^2_{L^4(\mathbb{R})}\|(N+\varepsilon)^{1/2}\Psi\| \\
& \times \left[\sum_{n=0}^{\infty} \varepsilon(n+2)\|\Phi^{(n+2)}(x,x,x_1,\dots,x_n)\|^2_{L^2(\mathbb{R}^{n+1})}\right]^{1/2}.
\end{aligned}
$$

Now we prove, by Lemma A.1, the crude estimate

$$
\begin{aligned}
\left|\langle \Phi, P_2(t)^{Wick}\Psi\rangle\right| \leqslant {} & \max\left(\|\varphi_t\|^2_{L^4(\mathbb{R})}, \|\varphi_t\|^2_{L^\infty(\mathbb{R})}\right) \\
& \times \big[2\|N^{1/2}\Phi\| \times \|N^{1/2}\Psi\| \\
& \quad + \|(N+\varepsilon)^{1/2}\Phi\| \times \left\|\left(\alpha\,\mathrm{d}\Gamma(-\Delta) + \alpha^{-1}N\right)^{1/2}\Psi\right\| \\
& \quad + \|(N+\varepsilon)^{1/2}\Psi\| \times \left\|\left(\alpha\,\mathrm{d}\Gamma(-\Delta) + \alpha^{-1}N\right)^{1/2}\Phi\right\|\big].
\end{aligned}
$$

This yields for any $\alpha > 0$

$$
\begin{aligned}
\left|\langle \Phi, P_2(t)^{Wick}\Psi\rangle\right| \leqslant {} & \alpha\max\left(\|\varphi_t\|^2_{L^4(\mathbb{R})}, \|\varphi_t\|^2_{L^\infty(\mathbb{R})}\right) \\
& \times \left\|\left[\mathrm{d}\Gamma(-\Delta) + \left(\alpha^{-1}+3\right)\alpha^{-1}N + \alpha^{-1}\varepsilon 1\right]^{1/2}\Phi\right\| \\
& \times \left\|\left[\mathrm{d}\Gamma(-\Delta) + \left(\alpha^{-1}+3\right)\alpha^{-1}N + \alpha^{-1}\varepsilon 1\right]^{1/2}\Psi\right\|. \qquad (23)
\end{aligned}
$$

Remark now that (19) yields

$$\max\left(\|\varphi_t\|^2_{L^4(\mathbb{R})}, \|\varphi_t\|^2_{L^\infty(\mathbb{R})}\right) \leqslant 2\left(\|\varphi_0\|_{L^2(\mathbb{R})} + 1\right)^{3/2}\sqrt{h(\varphi_0)+1}.$$

Hence, for $\alpha^{-1} = 3(\|\varphi_0\|_{L^2(\mathbb{R})} + 1)^{3/2}\sqrt{h(\varphi_0)+1} > 0$, we obtain

$$\varepsilon^{-1}\left|\langle \Phi, P_2(t)^{\mathrm{Wick}}\Psi\rangle\right| \leqslant \frac{2}{3}\left\|\left[\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta) + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1\right]^{1/2}\Phi\right\| \times \left\|\left[\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta) + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1\right]^{1/2}\Psi\right\|. \tag{24}$$

Applying now the KLMN theorem (see [30], Theorem X.17) with the help of inequality (24) we show that

$$S_2(t) = A_2(t) + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1 \quad \text{with } \vartheta_1 > \left(\alpha^{-1}+3\right)\alpha^{-1} \text{ and } \vartheta_2 > \alpha^{-1}+1,$$

are associated to unique self-adjoint operators $S_2(t)$ satisfying $S_2(t) \geqslant 1$. Furthermore, we have that the form domains of those operators are time-independent, i.e.,

$$\mathcal{Q}\left(S_2(t)\right) = \mathcal{F}^1_+$$

for any $t \in \mathbb{R}$. □

Remark 5.2. The choice of ϑ_1, ϑ_2 in the previous lemma takes into account the use of KLMN's theorem in the proof of Lemma 6.3.

We consider the non-autonomous Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u = A_2(t)u, & t \in \mathbb{R}, \\ u(t=s) = u_s. \end{cases} \tag{25}$$

Here $\mathbb{R} \ni t \mapsto A_2(t)$ is considered as a norm continuous $\mathcal{L}(\mathcal{F}^1_+, \mathcal{F}^1_-)$-valued map (see Lemma 5.3). We show in Proposition 5.5 the existence of a unique solution for any initial data $u_s \in \mathcal{F}^1_+$ using Corollary C.4. Moreover, the Cauchy problem's features allow to encode the solutions on a *unitary propagator* mapping $(t,s) \mapsto U_2(t,s)$ such that

$$U_2(t,s)u_s = u_t,$$

satisfying Definition C.1 with $\mathcal{H} = \mathcal{F}$, $\mathcal{H}_\pm = \mathcal{F}^1_\pm$ and $I = \mathbb{R}$.

In the following two lemmas we check the assumptions in Corollary C.4.

Lemma 5.3. *For any $\varphi_0 \in H^1(\mathbb{R})$ and $t \in \mathbb{R}$ the quadratic form $A_2(t)$ defines a symmetric operator on $\mathcal{L}(\mathcal{F}^1_+, \mathcal{F}^1_-)$ and the mapping $t \in \mathbb{R} \mapsto A_2(t) \in \mathcal{L}(\mathcal{F}^1_+, \mathcal{F}^1_-)$ is norm continuous.*

Proof. Using (24) we show for any $\Psi, \Phi \in \mathcal{S}$

$$\begin{aligned}\left|\langle \Phi, A_2(t)\Psi\rangle\right| &\leqslant \left|\langle \Phi, \varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta)\Psi\rangle\right| + \left|\langle \Phi, \varepsilon^{-1}P_2(t)^{\mathit{Wick}}\Psi\rangle\right| \\ &\leqslant \|S_1^{1/2}\Phi\|\|S_1^{1/2}\Psi\| + \frac{2}{3}\vartheta_1\|S_1^{1/2}\Phi\|\|S_1^{1/2}\Psi\| \\ &\leqslant \frac{5}{3}\vartheta_1\|\Psi\|_{\mathcal{F}_+^1}\|\Phi\|_{\mathcal{F}_+^1}, \end{aligned} \tag{26}$$

where ϑ_1, ϑ_2 are the parameters introduced in Lemma 5.1. Hence, this allows to consider $A_2(t)$ as a bounded operator in $\mathcal{L}(\mathcal{F}_+^1, \mathcal{F}_-^1)$. Since $A_2(t)$ is a symmetric quadratic form it follows that it is also symmetric as an operator in $\mathcal{L}(\mathcal{F}_+^1, \mathcal{F}_-^1)$.

Now, using a similar estimate as (23) we prove norm continuity. Indeed, we have

$$\begin{aligned}\left|\langle \Phi, \left[A_2(t) - A_2(s)\right]\Psi\rangle\right| &= \varepsilon^{-1}\left|\langle \Phi, \left[P_2(t) - P_2(s)\right]^{\mathit{Wick}}\Psi\rangle\right| \\ &\leqslant 4\max\left(\|\varphi_t^2 - \varphi_s^2\|_{L^2(\mathbb{R})}, \left\||\varphi_t|^2 - |\varphi_s|^2\right\|_{L^\infty(\mathbb{R})}\right)\|\Psi\|_{\mathcal{F}_+^1}\|\Phi\|_{\mathcal{F}_+^1}.\end{aligned}$$

Note that it is not difficult to prove that

$$\max\left(\|\varphi_t^2 - \varphi_s^2\|_{L^2(\mathbb{R})}, \left\||\varphi_t|^2 - |\varphi_s|^2\right\|_{L^\infty(\mathbb{R})}\right) \to 0 \quad \text{when } t \to s.$$

This follows by (19) and the fact that $\varphi_t \in C^0(\mathbb{R}, H^1(\mathbb{R}))$. □

Lemma 5.4. *For any $\varphi_0 \in H^2(\mathbb{R})$ there exists $c > 0$ (depending only on φ_0) such that the two statements below hold true.*

(i) *For any $\Psi \in \mathcal{F}_+^1$, we have*

$$\left|\partial_t\langle \Psi, S_2(t)\Psi\rangle\right| \leqslant \mathrm{e}^{c(|t|+1)}\left\|S_2(t)^{1/2}\Psi\right\|_{\mathcal{F}}.$$

(ii) *For any $\Psi, \Phi \in \mathcal{D}(S_2(t)^{3/2})$, we have*

$$\left|\langle \Psi, A_2(t)S_2(t)\Phi\rangle - \langle S_2(t)\Psi, A_2(t)\Phi\rangle\right| \leqslant c\left\|S_2(t)^{1/2}\Psi\right\|_{\mathcal{F}}\left\|S_2(t)^{1/2}\Phi\right\|_{\mathcal{F}}.$$

Proof. (i) Let $\Psi \in \mathcal{S}$, we have

$$\partial_t\langle \Psi, S_2(t)\Psi\rangle = \varepsilon^{-1}\partial_t\langle \Psi, P_2(t)^{\mathit{Wick}}\Psi\rangle = \varepsilon^{-1}\langle \Psi, \left[\partial_t P_2(t)\right]^{\mathit{Wick}}\Psi\rangle,$$

where $\partial_t P_2(t)$ is a continuous polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$\partial_t P_2(t)[z] = 2\,\mathrm{Re}\int_{\mathbb{R}} \overline{z(x)}^2\varphi_t(x)\partial_t\varphi_t(x)\,\mathrm{d}x + 4\,\mathrm{Re}\int_{\mathbb{R}} \left|z(x)\right|^2\overline{\varphi_t(x)}\partial_t\varphi_t(x)\,\mathrm{d}x.$$

A simple computation yields

$$\begin{aligned}\langle\Psi,[\partial_t P_2(t)]^{Wick}\Psi\rangle &= 4\,\mathrm{Re}\sum_{n=1}^{\infty} n\varepsilon \overbrace{\int_{\mathbb{R}^n}\overline{\varphi_t(x_1)}\partial_t\varphi_t(x_1)|\Psi^{(n)}(x_1,\dots,x_n)|^2\,\mathrm{d}x_1\cdots\mathrm{d}x_n}^{(1)}\\ &\quad+\sum_{n=0}^{\infty}\varepsilon\sqrt{(n+2)(n+1)}\int_{\mathbb{R}^n}\overline{\Psi^{(n)}(x_1,\dots,x_n)}\\ &\qquad\times\left(\int_{\mathbb{R}}\overline{\varphi_t(x)\partial_t\varphi_t(x)}\Psi^{(n+2)}(x,x,x_1,\dots,x_n)\,\mathrm{d}x\right)\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &\quad+hc.\end{aligned}$$

From (19) we get

$$\begin{aligned}|(1)| &\leqslant \|\varphi_t\partial_t\varphi_t\|_{L^1(\mathbb{R})}\int_{\mathbb{R}^{n-1}}\sup_{x_1\in\mathbb{R}}|\Psi^{(n)}(x_1,\dots,x_n)|^2\,\mathrm{d}x_2\cdots\mathrm{d}x_n\\ &\leqslant \|\varphi_t\|_{L^2(\mathbb{R})}\times\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\langle(1-\partial_{x_1}^2)\Psi^{(n)},\Psi^{(n)}\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Now we apply Cauchy–Schwarz inequality,

$$\begin{aligned}|\langle\Psi,[\partial_t P_2(t)]^{Wick}\Psi\rangle| &\leqslant 4\|\varphi_t\|_{L^2(\mathbb{R})}\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\left(\sum_{n=1}^{\infty}\varepsilon n\langle(1-\partial_{x_1}^2)\Psi^{(n)},\Psi^{(n)}\rangle_{L^2(\mathbb{R}^n)}\right)\\ &\quad+2\|\varphi_t\|_{L^\infty(\mathbb{R})}\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\left(\sum_{n=0}^{\infty}\varepsilon(n+2)\|\Psi^{(n+2)}(x,x,\cdot)\|^2_{L^2(\mathbb{R}^{n+1})}\right)^{1/2}\\ &\quad\times\left(\sum_{n=0}^{\infty}\varepsilon(n+1)\|\Psi^{(n)}\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.\end{aligned}$$

In the same spirit as in (23), we obtain a rough inequality

$$\begin{aligned}|\langle\Psi,[\partial_t P_2(t)]^{Wick}\Psi\rangle| &\leqslant \max(\|\varphi_t\|_{L^\infty(\mathbb{R})},\|\varphi_t\|_{L^2(\mathbb{R})})\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\\ &\quad\times[4\|(\mathrm{d}\Gamma(-\Delta)+N)^{1/2}\Psi\|^2+2\|(\mathrm{d}\Gamma(-\Delta)+N+1)^{1/2}\Psi\|^2].\end{aligned}$$

Observe that (24) implies $S_1\leqslant 3S_2(t)$ for all $t\in\mathbb{R}$. Hence, we have

$$\begin{aligned}\varepsilon^{-1}|\langle\Psi,[\partial_t P_2(t)]^{Wick}\Psi\rangle| &\leqslant 6\max(\|\varphi_t\|_{L^\infty(\mathbb{R})},\|\varphi_t\|_{L^2(\mathbb{R})})\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\|\Psi\|^2_{\mathcal{F}^1_+}\\ &\leqslant 18\max(\|\varphi_t\|_{L^\infty(\mathbb{R})},\|\varphi_t\|_{L^2(\mathbb{R})})\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\|S_2(t)^{1/2}\Psi\|^2_{\mathcal{F}}.\end{aligned}$$

This proves (i) since (19) and (20) ensure the existence of $c>0$ (depending only on φ_0) such that

$$\max(\|\varphi_t\|_{L^\infty(\mathbb{R})},\|\varphi_t\|_{L^2(\mathbb{R})})\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\leqslant \mathrm{e}^{c(|t|+1)}.$$

(ii) If $\Psi, \Phi \in \mathcal{D}(S_2(t)^{3/2})$ the quantity

$$\mathcal{C} := \langle \Psi, A_2(t) S_2(t) \Phi \rangle - \langle S_2(t)\Psi, A_2(t)\Phi \rangle$$

is well defined since $A_2(t) \in \mathcal{L}(\mathcal{F}^1_+, \mathcal{F}^1_-)$ and $S_2(t)\mathcal{D}(S_2(t)^{3/2}) \subset \mathcal{D}(S_2(t)^{1/2}) = \mathcal{F}^1_+$. Note that $N \in \mathcal{L}(\mathcal{F}^1_+, \mathcal{F}^1_-)$. Hence, we can write

$$\begin{aligned}\mathcal{C} &= \langle \Psi, [S_2(t) - \vartheta_1 \varepsilon^{-1} N - \vartheta_2 1] S_2(t)\Phi \rangle - \langle S_2(t)\Psi, [S_2(t) - \vartheta_1 \varepsilon^{-1} N - \vartheta_2 1]\Phi \rangle \\ &= \vartheta_1 \big(\langle S_2(t)\Psi, \varepsilon^{-1} N \Phi \rangle - \langle \varepsilon^{-1} N \Psi, S_2(t)\Phi \rangle \big).\end{aligned}$$

Observe that, for $\lambda > 0$, $\varepsilon^{-1}N(\lambda\varepsilon^{-1}N + 1)^{-1}\mathcal{F}^1_+ \subset \mathcal{F}^1_+$ and that

$$s - \lim_{\lambda \to 0^+} \varepsilon^{-1} N \big(\lambda \varepsilon^{-1} N + 1\big)^{-1} = \varepsilon^{-1} N \quad \text{in } \mathcal{L}\big(\mathcal{F}^1_+, \mathcal{F}^1_-\big).$$

Therefore, we have

$$\mathcal{C} = \vartheta_1 \lim_{\lambda \to 0^+} \underbrace{\langle S_2(t)\Psi, \varepsilon^{-1} N \big(\lambda\varepsilon^{-1}N + 1\big)^{-1}\Phi \rangle - \langle \varepsilon^{-1} N \big(\lambda\varepsilon^{-1}N + 1\big)^{-1}\Psi, S_2(t)\Phi \rangle}_{\mathcal{C}_\lambda}.$$

Let N_λ denote $\varepsilon^{-1}N(\lambda\varepsilon^{-1}N + 1)^{-1}$. A simple computation yields

$$\varepsilon \mathcal{C}_\lambda = \langle \Psi, P_2(t)^{\mathit{Wick}} N_\lambda \Phi \rangle - \langle N_\lambda \Psi, P_2(t)^{\mathit{Wick}} \Phi \rangle = \langle \Psi, g(t)^{\mathit{Wick}} N_\lambda \Phi \rangle - \langle N_\lambda \Psi, g(t)^{\mathit{Wick}} \Phi \rangle,$$

where $g(t)$ is the polynomial given by

$$g(t)[z] = \mathrm{Re} \int_{\mathbb{R}} \overline{z(x)}^2 \varphi_t(x)^2 \, \mathrm{d}x.$$

A similar computation as (22) yields

$$\begin{aligned}\mathcal{C}_\lambda = &\sum_{n=0}^{\infty} \kappa(n) \int_{\mathbb{R}^n} \overline{\Psi^{(n)}(x_1, \ldots, x_n)} \bigg(\int_{\mathbb{R}} \overline{\varphi_t(x)}^2 \Phi^{(n+2)}(x, x, x_1, \ldots, x_n) \, \mathrm{d}x \bigg) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &- \sum_{n=0}^{\infty} \kappa(n) \int_{\mathbb{R}^n} \Phi^{(n)}(x_1, \ldots, x_n) \bigg(\int_{\mathbb{R}} \varphi_t(x)^2 \overline{\Psi^{(n+2)}(x, x, x_1, \ldots, x_n)} \, \mathrm{d}x \bigg) \mathrm{d}x_1 \cdots \mathrm{d}x_n,\end{aligned}$$

where

$$\kappa(n) = \frac{(n+2)\sqrt{(n+1)(n+2)}}{(\lambda(n+2)+1)} - \frac{n\sqrt{(n+1)(n+2)}}{(\lambda n + 1)}.$$

Note that $\kappa(n) \leqslant 2(n+2)$. Hence, using Cauchy–Schwarz inequality, we show

$$|\mathcal{C}_\lambda| \leqslant 2\|\varphi_t\|^2_{L^4(\mathbb{R})}\Bigg[\sum_{n=0}^{\infty}(n+2)\big\|\Psi^{(n)}\big\|^2_{L^2(\mathbb{R}^n)}\Bigg]^{1/2}\Bigg[\sum_{n=0}^{\infty}(n+2)\big\|\Phi^{(n+2)}(x,x,\cdot)\big\|^2_{L^2(\mathbb{R}^{n+1})}\Bigg]^{1/2}$$
$$+ 2\|\varphi_t\|^2_{L^4(\mathbb{R})}\Bigg[\sum_{n=0}^{\infty}(n+2)\big\|\Phi^{(n)}\big\|^2_{L^2(\mathbb{R}^n)}\Bigg]^{1/2}\Bigg[\sum_{n=0}^{\infty}(n+2)\big\|\Psi^{(n+2)}(x,x,\cdot)\big\|^2_{L^2(\mathbb{R}^{n+1})}\Bigg]^{1/2}.$$

Using Lemma A.1, with $\alpha = \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)\big\|\Psi^{(n+2)}(x,x,\cdot)\big\|^2_{L^2(\mathbb{R}^{n+1})} &\leqslant \frac{1}{2}\sum_{n=0}^{\infty}(n+2)\big\langle D^2_{x_1}\Psi^{(n+2)},\Psi^{(n+2)}\big\rangle \\ &\qquad + (n+2)\big\|\Psi^{(n+2)}\big\|^2_{L^2(\mathbb{R}^{n+2})} \\ &\leqslant \frac{1}{2}\langle\Psi, S_1\Psi\rangle,\end{aligned}$$

together with an analogue estimate where Ψ is replaced by Φ. Now, we conclude that there exists $c > 0$ depending only on φ_0 such that

$$\vartheta_1|\mathcal{C}_\lambda| \leqslant c\|\Psi\|_{\mathcal{F}^1_+}\|\Phi\|_{\mathcal{F}^1_+}. \tag{27}$$

This proves part (ii). □

Proposition 5.5. *Let $\varphi_0 \in H^2(\mathbb{R})$ and $A_2(t)$ given by* (21). *Then the non-autonomous Cauchy problem*

$$\begin{cases} \mathrm{i}\partial_t u = A_2(t)u, & t \in \mathbb{R}, \\ u(t=s) = u_s, \end{cases}$$

admits a unique unitary propagator $U_2(t,s)$ in the sense of Definition C.1 *with $I = \mathbb{R}$ and $\mathcal{H}_\pm = \mathcal{F}^1_\pm$. Moreover, there exists $c > 0$ depending only on φ_0 such that*

$$\big\|U_2(t,0)\big\|_{\mathcal{L}(\mathcal{F}^1_+)} \leqslant \mathrm{e}^{c\mathrm{e}^{c|t|}}.$$

Proof. The proof immediately follows using Corollary C.4 with the help of Lemma 5.3 and 5.4 and the inequality

$$c_1 S_1 \leqslant S_2(t) \leqslant c_2 S_1,$$

which holds true using (26). □

6. Propagation of coherent states

In finite dimensional phase-space, coherent state analysis is a well-developed powerful tool, see for instance [7]. Here we study, using the ideas of Ginibre and Velo in [20], the asymptotics when $\varepsilon \to 0$ of the time-evolved coherent states

$$e^{-it/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{i\varepsilon}\varphi_0\right)\Psi$$

for Ψ in a dense subspace $\mathcal{G}_+ \subset \mathcal{F}$ defined below. We consider the following Hilbert rigging

$$\mathcal{G}_+ \subset \mathcal{F} \subset \mathcal{G}_-,$$

defined via the ε-independent self-adjoint operator (see Remark 3.4) given by

$$G := \varepsilon^{-1}\, d\Gamma(-\Delta) + \varepsilon^{-2} P^{Wick} + \varepsilon^{-1} N + 1,$$

as the completion of $\mathcal{D}(G^{\pm 1/2})$ with the respect to the inner product

$$\langle \Psi, \Phi\rangle_{\mathcal{G}_\pm} := \langle G^{\pm 1/2}\Psi, G^{\pm 1/2}\Phi\rangle_{\mathcal{F}}.$$

We have the continuous embedding

$$\mathcal{F}_+^3 \subset \mathcal{G}_+ \subset \mathcal{F}_+^1.$$

The main result of this section is Theorem 6.1 which describes the propagation of coherent states in the semiclassical limit.

Theorem 6.1. *For any $\varphi_0 \in H^2(\mathbb{R})$ there exists $c > 0$ depending only on φ_0 such that*

$$\left\| e^{-it/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{i\varepsilon}\varphi_0\right)\Psi - e^{i\omega(t)/\varepsilon} W\left(\frac{\sqrt{2}}{i\varepsilon}\varphi_t\right) U_2(t,0)\Psi \right\|_{\mathcal{F}} \leqslant e^{ce^{c|t|}} \varepsilon^{1/8} \|\Psi\|_{\mathcal{G}_+},$$

holds for any $t \in \mathbb{R}$ and $\Psi \in \mathcal{G}_+$ where φ_t solves the NLS equation (18) *with the initial condition φ_0 and $\omega(t) = \int_0^t P(\varphi_s)\, ds$. Here $U_2(t,s)$ is the unitary propagator given by Proposition* 5.5.

To prove this theorem we need several preliminary lemmas.

Lemma 6.2. *The following three assertions hold true*:

(i) *For any $\xi \in L^2(\mathbb{R})$ and $k \in \mathbb{N}$, the Weyl operator $W(\xi)$ preserves $\mathcal{D}(N^{k/2})$. If in addition $\xi \in H^1(\mathbb{R})$ then $W(\xi)$ preserves also $\mathcal{F}_+^\mu$ when $\mu \geqslant 1$.*
(ii) *For any $\xi \in H^1(\mathbb{R})$, we have in the sense of quadratic forms on $\mathcal{F}_+^3$,*

$$W\left(\frac{\sqrt{2}}{i\varepsilon}\xi\right)^* h^{Wick} W\left(\frac{\sqrt{2}}{i\varepsilon}\xi\right) = h(\cdot + \xi)^{Wick}.$$

(iii) *Let* $(\mathbb{R} \ni t \mapsto \varphi_t) \in C^1(\mathbb{R}, L^2(\mathbb{R}))$, *then for any* $\Psi \in \mathcal{D}(N^{1/2})$ *we have in* $\mathcal{F}$

$$\mathrm{i}\varepsilon\partial_t W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\Big)\Psi = W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\Big)\big[\mathrm{Re}\langle\varphi_t, \mathrm{i}\partial_t\varphi_t\rangle + 2\mathrm{Re}\langle z, \mathrm{i}\partial_t\varphi_t\rangle^{\mathit{Wick}}\big]\Psi$$
$$= \big[-\mathrm{Re}\langle\varphi_t, \mathrm{i}\partial_t\varphi_t\rangle + 2\mathrm{Re}\langle z, \mathrm{i}\partial_t\varphi_t\rangle^{\mathit{Wick}}\big]W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\Big)\Psi.$$

Proof. (i) Let $\mathcal{F}_0$ be the linear space spanned by vectors $\Psi \in \mathcal{F}$ such that $\Psi^{(n)} = 0$ for any n except for a finite number. It is known that for any $\xi \in L^2(\mathbb{R})$ and $\Psi \in \mathcal{F}_0$

$$\tilde{N}\Psi := W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\xi\Big)^* N W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\xi\Big)\Psi = \big(N + 2\,\mathrm{Re}\langle z, \xi\rangle^{\mathit{Wick}} + \|\xi\|^2 1\big)\Psi. \tag{28}$$

For a proof of the latter identity see [3], Lemma 2.10(iii). Hence, by Cauchy–Schwarz inequality it follows that

$$\begin{aligned}\Big\|N^{1/2}W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\xi\Big)\Psi\Big\|^2 &= \big\langle\Psi, \big[N + 2\,\mathrm{Re}\langle z, \xi\rangle^{\mathit{Wick}} + \|\xi\|^2 1\big]\Psi\big\rangle \\ &= \big\langle\Psi, \big(N + \|\xi\|^2_{L^2(\mathbb{R})}1\big)\Psi\big\rangle \\ &\quad + \sum_{n=0}^{\infty}\sqrt{\varepsilon(n+1)}\int_{\mathbb{R}^n}\overline{\Psi^{(n)}(y)}\Big(\int_{\mathbb{R}}\overline{\xi(x)}\Psi^{(n+1)}(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y + hc \\ &\leqslant \big(1 + \|\xi\|_{L^2(\mathbb{R})}\big)^2\big\|(N+1)^{1/2}\Psi\big\|^2.\end{aligned}$$

Now, for $k \geqslant 1$ we show the existence of an ε-independent constant $C_k > 0$ depending only on k and $\|\xi\|_{L^2(\mathbb{R})}$ such that

$$\Big\|N^{k/2}W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\xi\Big)\Psi\Big\|^2 = \langle\Psi, \tilde{N}^k\Psi\rangle \leqslant C_k\big\|(N+1)^{k/2}\Psi\big\|^2. \tag{29}$$

This is a consequence of the number operator estimate (9) and the fact that $\tilde{N}^k$ is a Wick polynomial in $\sum_{0\leqslant r,s\leqslant k}\mathcal{P}_{r,s}(L^2(\mathbb{R}))$ (see, e.g., [3], Proposition 2.7(i)). Thus, we have proved the invariance of $\mathcal{D}(N^{k/2})$ since $\mathcal{F}_0$ is a core of $N^{k/2}$.

Now the invariance of $\mathcal{F}^{\mu}_{+}$, $\mu \geqslant 1$, follows by Faris–Lavine Theorem B.1 where we take the operator

$$A = \sqrt{2}\,\mathrm{Re}\langle z, \xi\rangle^{\mathit{Wick}} \quad \text{and} \quad S = S_\mu = \varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta) + \varepsilon^{-\mu}N^\mu + 1,$$

and remember that

$$W(\xi) = \mathrm{e}^{\mathrm{i}\sqrt{2}\,\mathrm{Re}\langle z,\xi\rangle^{\mathit{Wick}}}.$$

In fact, assuming $\xi \in H^1(\mathbb{R})$ we have to check assumptions (i) and (ii) of Theorem B.1. For any $\Psi \in \mathcal{F}_+^\mu$, we have by Wick quantization

$$2\,\mathrm{Re}\langle z,\xi\rangle^{\mathit{Wick}}\Psi = \sum_{n=0}^{\infty} \sqrt{\varepsilon(n+1)} \int_{\mathbb{R}} \overline{\xi(x)}\Psi^{(n+1)}(x,x_1,\ldots,x_n)\,\mathrm{d}x + \sum_{n=1}^{\infty} \sqrt{\frac{\varepsilon}{n}} \sum_{j=1}^{n} \xi(x_j)\Psi^{(n-1)}(x_1,\ldots,\hat{x}_j,\ldots,x_n).$$

Therefore, it is easy to show

$$\|\mathrm{Re}\langle z,\xi\rangle^{\mathit{Wick}}\Psi\| \leqslant \sqrt{\varepsilon}\|\xi\|_{L^2(\mathbb{R})}\|(\varepsilon^{-1}N+1)^{1/2}\Psi\| \leqslant \sqrt{\varepsilon}\|\xi\|_{L^2(\mathbb{R})}\|S_1\Psi\|$$

and hence we obtain that $\mathcal{D}(S_\mu) \subset \mathcal{D}(A)$. Let $\Psi \in \mathcal{D}(S_\mu)$, a standard computation yields

$$\sqrt{2}\big(\langle A\Psi, S_\mu\Psi\rangle - \langle S_\mu\Psi, A\Psi\rangle\big) = \langle a(-\Delta\xi)\Psi,\Psi\rangle - \langle\Psi, a(-\Delta\xi)\Psi\rangle + \left\langle\left[\left(\frac{N}{\varepsilon}+1\right)^{\mu} - \left(\frac{N}{\varepsilon}\right)^{\mu}\right]\Psi, a^*(\xi)\Psi\right\rangle - hc. \tag{30}$$

Each two terms in the same line of (30) are similar and it is enough to estimate only one of them. We have by Cauchy–Schwarz inequality

$$\begin{aligned}\big|\langle a(-\Delta\xi)\Psi,\Psi\rangle\big| &\leqslant \left|\sum_{n=0}^{\infty}\sqrt{\varepsilon(n+1)}\int_{\mathbb{R}^n}\overline{\Psi^{(n)}(y)}\left(\int_{\mathbb{R}}\overline{-\Delta\xi(x)}\Psi^{(n+1)}(x,y)\,\mathrm{d}x\right)\mathrm{d}y\right| \\ &\leqslant \|\xi\|_{H^1(\mathbb{R})}\|S_1^{1/2}\Psi\|^2\end{aligned}$$

and for $1 \leqslant \theta \leqslant \mu - 1$

$$\begin{aligned}\big|\langle\varepsilon^{-\theta}N^\theta\Psi, a^*(\xi)\Psi\rangle\big| &\leqslant \left|\sum_{n=0}^{\infty}\sqrt{\varepsilon(n+1)}(n+1)^\theta\int_{\mathbb{R}^n}\Psi^{(n)}(y)\left(\int_{\mathbb{R}}\xi(x)\overline{\Psi^{(n+1)}(x,y)}\,\mathrm{d}x\right)\mathrm{d}y\right| \\ &\leqslant 2^\mu\|\xi\|_{L^2(\mathbb{R})}\|S_\mu^{1/2}\Psi\|^2.\end{aligned}$$

This shows for any $\Psi \in \mathcal{D}(S_\mu)$,

$$\pm\mathrm{i}\langle\Psi,[A,S_\mu]\Psi\rangle \leqslant C\|S_\mu^{1/2}\Psi\|^2.$$

Part (ii) follows by a similar argument as [3], Lemma 2.10(iii), and part (iii) is a well-known formula, see [19], Lemma 3.1(3). □

Set

$$\mathcal{W}(t) = W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)^{*}\mathrm{e}^{-\mathrm{i}\omega(t)/\varepsilon}\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon}W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right).$$

Lemma 6.3. *For any $\varphi_0 \in H^2(\mathbb{R})$ there exists $c > 0$ such that the inequality*

$$\|\mathcal{W}(t)\|_{\mathcal{L}(\mathcal{G}_+,\mathcal{F}_+^1)} \leqslant \mathrm{e}^{c\mathrm{e}^{c|t|}}$$

holds for $t \in \mathbb{R}$ uniformly in $\varepsilon \in (0,1]$.

Proof. Observe that the subspace $\mathcal{D}_+$ given as the image of $\mathcal{D}(H_\varepsilon) \cap \mathcal{F}_+^3$ by $W(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0)^*$ is dense in $\mathcal{F}$. Let $\Psi \in \mathcal{D}_+$ and $\Phi \in \mathcal{G}_+$, then differentiating the quantity $\langle \Phi, \mathcal{W}(t)\Psi\rangle$ with the help of Lemma 6.2 and Proposition 3.3, we obtain

$$\begin{aligned}\mathrm{i}\varepsilon\partial_t\langle \Phi, \mathcal{W}(t)\Psi\rangle &= \langle \Phi, \big[P(\varphi_t) - \mathrm{Re}\langle \varphi_t, \mathrm{i}\partial_t\varphi_t\rangle - 2\,\mathrm{Re}\langle z, \mathrm{i}\partial_t\varphi_t\rangle^{Wick}\big]\mathcal{W}(t)\Psi\rangle \\ &\quad + \underbrace{\left\langle \Phi, W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)^* \mathrm{e}^{-\mathrm{i}\omega(t)/\varepsilon}\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} H_\varepsilon W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi\right\rangle}_{(1)}. \end{aligned} \tag{31}$$

Let $R_\nu := 1_{[0,\nu]}(\varepsilon^{-1}N)$ and remark that $s - \lim_{\nu\to\infty} R_\nu = 1$. Furthermore, we have that $R_\nu\mathcal{G}_+ \subset \mathcal{F}_+^3$ since it easily holds that

$$\|R_\nu\Phi\|^2_{\mathcal{F}_+^3} \leqslant \nu^3\|\Phi\|^2_{\mathcal{G}_+}.$$

Therefore, since $W(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t)R_\nu\Phi$ and $W(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0)\Psi$ belong to $\mathcal{F}_+^3$, we have

$$\begin{aligned}(1) &= \lim_{\nu\to\infty}\left\langle R_\nu\Phi, W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)^* \mathrm{e}^{-\mathrm{i}\omega(t)/\varepsilon}\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} H_\varepsilon W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi\right\rangle \\ &= \lim_{\nu\to\infty}\langle R_\nu\Phi, h(\cdot + \varphi_t)^{Wick}\mathcal{W}(t)\Psi\rangle.\end{aligned}$$

So, we get

$$\begin{aligned}\mathrm{i}\varepsilon\partial_t\langle \Phi, \mathcal{W}(t)\Psi\rangle &= (1) + \lim_{\nu\to\infty}\langle R_\nu\Phi, \big[P(\varphi_t) - \mathrm{Re}\langle \varphi_t, \mathrm{i}\partial_t\varphi_t\rangle - 2\,\mathrm{Re}\langle z, \mathrm{i}\partial_t\varphi_t\rangle^{Wick}\big]\mathcal{W}(t)\Psi\rangle \\ &= \lim_{\nu\to\infty}\left\langle R_\nu\Phi, \big(\underbrace{\varepsilon A_2(t) + P_3(t)^{Wick} + P^{Wick}}_{=:\varepsilon\Theta(t)}\big)\mathcal{W}(t)\Psi\right\rangle,\end{aligned}$$

where we denote

$$P_3(t)[z] := \frac{D^{(3)}P}{3!}(\varphi_t)[z] = 2\,\mathrm{Re}\int_{\mathbb{R}}\varphi_t(x)\overline{z(x)}\big|z(x)\big|^2\,\mathrm{d}x$$

and

$$P(z) = \frac{D^{(4)}P}{4!}(\varphi_t)[z] = \frac{1}{2}\int_{\mathbb{R}}\big|z(x)\big|^4\,\mathrm{d}x.$$

A simple computation yields

$$\begin{aligned}\langle \Phi, P_3(t)^{Wick}\Psi\rangle &= \sum_{n=1}^{\infty}\sqrt{n^2(n+1)\varepsilon^3}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\overline{\varphi_t(x)\Phi^{(n)}(x,y)}\Psi^{(n+1)}(x,x,y)\,\mathrm{d}x\right)\mathrm{d}y \\ &\quad+\sum_{n=1}^{\infty}\sqrt{n^2(n+1)\varepsilon^3}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\varphi_t(x)\overline{\Phi^{(n+1)}(x,x,y)}\Psi^{(n)}(x,y)\,\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Using Cauchy–Schwarz inequality and Lemma A.1, we obtain

$$\begin{aligned}\left|\langle \Phi, P_3(t)^{Wick}\Psi\rangle\right| &\leqslant 2\sqrt{2}\frac{\|\varphi_t\|_{L^\infty(\mathbb{R})}}{\sqrt{\vartheta_2}}\sqrt{\langle \Phi, \left[\varepsilon^{-1}P^{Wick}+\vartheta_1\varepsilon^{-1}N+\vartheta_2 1\right]\Phi\rangle} \\ &\quad\times\sqrt{\langle \Psi, \left[\varepsilon^{-1}P^{Wick}+\vartheta_1\varepsilon^{-1}N+\vartheta_2 1\right]\Psi\rangle},\end{aligned} \tag{32}$$

where ϑ_1, ϑ_2 are the parameters in Lemma 5.1. Hence, $\Theta(t)$ extends to a bounded operator in $\mathcal{L}(\mathcal{G}_+, \mathcal{G}_-)$ since $A_2(t)$ and P^{Wick} belong to $\mathcal{L}(\mathcal{G}_+, \mathcal{G}_-)$. As an immediate consequence we obtain

$$\mathrm{i}\varepsilon\partial_t\langle \Phi, \mathcal{W}(t)\Psi\rangle = \langle \Phi, \varepsilon\Theta(t)\mathcal{W}(t)\Psi\rangle. \tag{33}$$

Now, we consider the quadratic form $\Lambda(t)$ on $\mathcal{G}_+$ given by

$$\Lambda(t) := \Theta(t) + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1.$$

It is easily follows, by (19) and (32), that

$$\begin{aligned}\left|\langle \Phi, P_3(t)^{Wick}\Psi\rangle\right| &\leqslant \frac{1}{4}\left\|\left(-\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta)+\varepsilon^{-1}P^{Wick}+\vartheta_1\varepsilon^{-1}N+\vartheta_2 1\right)^{1/2}\Phi\right\| \\ &\quad\times\left\|\left(-\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta)+\varepsilon^{-1}P^{Wick}+\vartheta_1\varepsilon^{-1}N+\vartheta_2 1\right)^{1/2}\Psi\right\|.\end{aligned} \tag{34}$$

Therefore, using (24) and (34) we show that

$$\varepsilon^{-1}\left[\frac{D^{(2)}P}{2}(\varphi_t)[z] + \frac{D^{(3)}P}{3!}(\varphi_t)[z]\right]^{Wick}$$

is form bounded by $\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta) + \varepsilon^{-1}P^{Wick} + \vartheta_1\varepsilon^{-1}N + \vartheta_2 1$ with a form-bound less than 1 uniformly in $\varepsilon \in (0, 1]$. Hence, by the KLMN Theorem [30], Theorem X17, the quadratic form $\Lambda(t)$ is associated to a unique self-adjoint operator which we still denote by $\Lambda(t)$, satisfying $\mathcal{Q}(\Lambda(t)) = \mathcal{G}_+$ and $\Lambda(t) \geqslant 1$. Moreover, it is not difficult to show the existence of $c_1, c_2 > 0$ such that

$$c_1 S_1 \leqslant \Lambda(t) \leqslant c_2 G \tag{35}$$

uniformly in $\varepsilon \in (0, 1]$ for any $t \in \mathbb{R}$. Now, we consider the non-autonomous Schrödinger equation

$$\mathrm{i}\partial_t u_t = \Theta(t)u_t, \tag{36}$$

with initial data $u_0 \in \mathcal{G}_+$. Next, we prove existence and uniqueness of a unitary propagator $\mathcal{V}(t,s)$ of the Cauchy problem (36). This will be done if we can check assumptions of Corollary C.4 with $\mathcal{G}_\pm = \mathcal{H}_\pm$, $A(t) = \Theta(t)$ and $S(t) = \Lambda(t)$. Thus, we will conclude that

$$\|\Lambda(t)^{1/2}\mathcal{V}(t,0)\Psi\|_{\mathcal{F}} \leqslant \mathrm{e}^{c\mathrm{e}^{c|t|}}\|\Lambda(0)^{1/2}\Psi\|_{\mathcal{F}}. \tag{37}$$

Observe that $\mathbb{R} \ni t \mapsto \Theta(t) \in \mathcal{L}(\mathcal{G}_+, \mathcal{G}_-)$ is norm continuous since

$$\begin{aligned}\big|\langle \Phi, (\Theta(t) - \Theta(s))\Psi\rangle\big| &\leqslant \|\Phi\|_{\mathcal{G}_+}\|A_2(t) - A_2(s)\|_{\mathcal{L}(\mathcal{F}^1_+,\mathcal{F}^1_-)}\|\Psi\|_{\mathcal{G}_+} \\ &\quad + \big|\langle \Phi, \varepsilon^{-1}\big(P_3(t) - P_3(s)\big)^{\mathit{Wick}}\Psi\rangle\big|\end{aligned}$$

and an estimate similar to (32) yields

$$\big|\langle \Phi, \varepsilon^{-1}\big(P_3(t) - P_3(s)\big)^{\mathit{Wick}}\Psi\rangle\big| \leqslant 2\sqrt{2}\|\varphi_t - \varphi_s\|_{L^\infty(\mathbb{R})}\|\Phi\|_{\mathcal{G}_+}\|\Psi\|_{\mathcal{G}_+}.$$

Let us check assumption (i) of Corollary C.4. We have for $\Psi \in \mathcal{G}_+ \subset \mathcal{F}^1_+$,

$$\partial_t\langle \Psi, \Lambda(t)\Psi\rangle = \partial_t\langle \Psi, S_2(t)\Psi\rangle + \partial_t\langle \Psi, \varepsilon^{-1}P_3(t)^{\mathit{Wick}}\Psi\rangle.$$

A simple computation yields

$$\begin{aligned}&\partial_t\langle \Psi, \varepsilon^{-1}P_3(t)^{\mathit{Wick}}\Psi\rangle \\ &\quad = 2\ \mathrm{Re}\Bigg[\sum_{n=1}^\infty \sqrt{n^2(n+1)\varepsilon}\int_{\mathbb{R}^{n-1}}\bigg(\int_{\mathbb{R}}\partial_t\varphi_t(x)\Psi^{(n)}(x,y)\overline{\Psi^{(n+1)}(x,x,y)}\,\mathrm{d}x\bigg)\,\mathrm{d}y\Bigg].\end{aligned}$$

So, by Cauchy–Schwarz inequality and Lemma A.1, we get

$$\begin{aligned}\big|\partial_t\langle \Psi, \varepsilon^{-1}P_3(t)^{\mathit{Wick}}\Psi\rangle\big| &\leqslant 2\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\Bigg[\sum_{n=1}^\infty (n+1)\Big\|\sup_{x\in\mathbb{R}}\big|\Psi^{(n)}(x,\cdot)\big|\Big\|^2_{L^2(\mathbb{R}^{n-1})}\Bigg]^{1/2} \\ &\quad \times \Bigg[\sum_{n=1}^\infty n^2\varepsilon\big\|\Psi^{(n+1)}(x,x,\cdot)\big\|^2_{L^2(\mathbb{R}^n)}\Bigg]^{1/2} \\ &\leqslant 2\sqrt{2}\|\partial_t\varphi_t\|_{L^2(\mathbb{R})}\big\|\Lambda(t)^{1/2}\Psi\big\|^2.\end{aligned}$$

The latter estimate with Lemma 5.4 (i) and (19) and (20) give us

$$\big|\partial_t\langle \Psi, \Lambda(t)\Psi\rangle\big| \leqslant \mathrm{e}^{c(|t|+1)}\big\|\Lambda(t)^{1/2}\Psi\big\|^2.$$

Now, we check assumption (ii) of Corollary C.4. We follow the same lines of the proof of Lemma 5.4(ii) by replacing $S_2(t)$ by $\Lambda(t)$ and $A_2(t)$ by $\Theta(t)$. So, we arrive at the step where we have to estimate for $\Psi, \Phi \in \mathcal{D}(\Lambda(t)^{3/2})$ and $\lambda > 0$, the quantity

$$\mathcal{C}_\lambda\big[g(t)\big] := \langle \Psi, \varepsilon^{-1}g(t)^{\mathit{Wick}}N_\lambda\Phi\rangle - \langle N_\lambda\Psi, \varepsilon^{-1}g(t)^{\mathit{Wick}}\Phi\rangle,$$

where $N_\lambda := \varepsilon^{-1}N(\lambda\varepsilon^{-1}N+1)^{-1}$ and $g(t)$ is the continuous polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$g(t)[z] = P_2(t)[z] + P_3(t)[z].$$

Note that the part $\mathcal{C}_\lambda[P_2(t)]$ involving only the symbol $P_2(t)$ is already bounded by (27). Thus, we need only to consider $\mathcal{C}_\lambda[P_3(t)]$. A simple computation yields

$$\begin{aligned}\mathcal{C}_\lambda\big[P_3(t)\big] &= \sum_{n=1}^{\infty}\kappa(n)\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\overline{\varphi_t(x)}\varPhi^{(n+1)}(x,x,y)\overline{\varPsi^{(n)}(x,y)}\,\mathrm{d}x\right)\mathrm{d}y\\ &\quad-\sum_{n=1}^{\infty}\kappa(n)\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\varphi_t(x)\varPhi^{(n)}(x,y)\overline{\varPsi^{(n+1)}(x,x,y)}\,\mathrm{d}x\right)\mathrm{d}y,\end{aligned}$$

where

$$\kappa(n) = \frac{(n+1)\sqrt{\varepsilon n^2(n+1)}}{(\lambda(n+1)+1)} - \frac{n\sqrt{\varepsilon n^2(n+1)}}{(\lambda n+1)}$$

satisfying $|\kappa(n)| \leqslant \sqrt{n^2(n+1)}$ uniformly in $\varepsilon \in (0,1]$ and $\lambda > 0$. So, using a similar estimate as (32), we obtain

$$\big|\mathcal{C}_\lambda\big[P_3(t)\big]\big| \leqslant \frac{1}{\sqrt{2}}\|\varphi_t\|_{L^\infty(\mathbb{R})}\big\|\varLambda(t)^{1/2}\varPsi\big\|\big\|\varLambda(t)^{1/2}\varPhi\big\|.$$

This proves assumption (ii) of Corollary C.4. Now, we check that

$$\mathcal{W}(t) = \mathcal{V}(t,0).$$

In fact, for $\varPhi \in \mathcal{G}_+$ and $\varPsi \in \mathcal{D}_+$ we have

$$\mathrm{i}\partial_r\big\langle\varPhi, \mathcal{V}(0,r)\mathcal{W}(r)\varPsi\big\rangle = -\big\langle\varTheta(r)\mathcal{V}(r,0)\varPhi, \mathcal{W}(r)\varPsi\big\rangle + \mathrm{i}\lim_{s\to 0}\left\langle\mathcal{V}(r+s,0)\varPhi, \frac{\mathcal{W}(r+s)-\mathcal{W}(r)}{s}\varPsi\right\rangle$$

and since by (31) we know that $\lim_{s\to 0}\frac{\mathcal{W}(r+s)-\mathcal{W}(r)}{s}\varPsi$ exists in $\mathcal{F}$, we conclude using (33) that

$$\partial_r\big\langle\varPhi, \mathcal{V}(0,r)\mathcal{W}(r)\varPsi\big\rangle = 0.$$

This identifies $\mathcal{W}(t)$ as the unitary propagator of the non-autonomous Schrödinger equation (36). Therefore, by (35)–(37) we get

$$\sqrt{c_1}\big\|\mathcal{W}(t)\varPsi\big\|_{\mathcal{F}^1_+} \leqslant \big\|\varLambda(t)^{1/2}\mathcal{W}(t)\varPsi\big\|_{\mathcal{F}} \leqslant \mathrm{e}^{c\mathrm{e}^{c|t|}}\big\|\varLambda(0)^{1/2}\varPsi\big\|_{\mathcal{F}} \leqslant \sqrt{c_2}\mathrm{e}^{c\mathrm{e}^{c|t|}}\|\varPsi\|_{\mathcal{G}_+}$$

for any $t \in \mathbb{R}$ uniformly in $\varepsilon \in (0,1]$. □

Lemma 6.4. *For any $\varphi_0 \in H^2(\mathbb{R})$ and $\Psi \in \mathcal{G}_+$ we have*

$$\left\|\mathcal{W}(t)\Psi - U_2(t,0)\Psi\right\|_{\mathcal{F}}^2 = 2\langle\Psi,(1-R_\nu)\Psi\rangle - 2\,\mathrm{Re}\,\langle\mathcal{W}(t)\Psi,(1-R_\nu)U_2(t,0)\Psi\rangle + 2\,\mathrm{Im}\int_0^t \langle\mathcal{W}(s)\Psi,\left[\Theta(s)R_\nu - R_\nu A_2(s)\right]U_2(s,0)\Psi\rangle\,\mathrm{d}s,$$

where $R_\nu := \sigma(\frac{\varepsilon^{-1}N}{\nu})$ with σ any bounded Borel function on $\mathbb{R}_+$ with compact support and here

$$\Theta(s) = A_2(s) + \varepsilon^{-1}Q_s(z)^{\mathit{Wick}},$$

with $Q_s(z)$ the continuous polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$Q_s(z) = \frac{D^{(3)}P}{3!}(\varphi_s)[z] + \frac{D^{(4)}P}{4!}(\varphi_s)[z].$$

Proof. We have

$$\begin{aligned}\left\|\mathcal{W}(t)\Psi - U_2(t,0)\Psi\right\|_{\mathcal{F}}^2 &= 2\|\Psi\|_{\mathcal{F}}^2 - 2\,\mathrm{Re}\,\langle\mathcal{W}(t)\Psi,U_2(t,0)\Psi\rangle \\ &= 2\langle\Psi,(1-R_\nu)\Psi\rangle - 2\,\mathrm{Re}\,\langle\mathcal{W}(t)\Psi,(1-R_\nu)U_2(t,0)\Psi\rangle \\ &\quad + 2\,\mathrm{Re}\langle\Psi,R_\nu\Psi\rangle - 2\,\mathrm{Re}\,\langle\mathcal{W}(t)\Psi,R_\nu U_2(t,0)\Psi\rangle. \end{aligned} \tag{38}$$

Hence, to prove the lemma it is enough to show that

$$\mathbb{R} \ni s \mapsto \mathrm{Re}\,\langle\mathcal{W}(s)\Psi,R_\nu U_2(s,0)\Psi\rangle \in C^1(\mathbb{R}) \tag{39}$$

and compute its derivative. Recall that the propagator $U_2(s,0) \in C^0(\mathbb{R},\mathcal{L}(\mathcal{F}_+^1))$, by Proposition 5.5 and that $\mathcal{W}(s) \in C^0(\mathbb{R},\mathcal{L}(\mathcal{G}_+))$ since it is the unitary propagator of the Cauchy problem (36). It is easily seen that

$$s \mapsto R_\nu U_2(s,0)\Psi$$

are in $\in C^0(\mathbb{R},\mathcal{G}_+)$ since R_ν maps continuously $\mathcal{F}_+^1$ into $\mathcal{G}_+$. We also have that

$$s \mapsto \mathcal{W}(s)\Psi \in C^1(\mathbb{R},\mathcal{G}_-) \quad \text{and} \quad s \mapsto U_2(s,0)\Psi \in C^1(\mathbb{R},\mathcal{F}_-^1).$$

This proves the statement (39). Therefore, we have

$$2\,\mathrm{Re}\langle\Psi,R_\nu\Psi\rangle - 2\,\mathrm{Re}\,\langle\mathcal{W}(t)\Psi,R_\nu U_2(t,0)\Psi\rangle = -\frac{2}{\varepsilon}\,\mathrm{Im}\int_0^t \mathrm{i}\varepsilon\partial_s\langle\mathcal{W}(s)\Psi,R_\nu U_2(s,0)\Psi\rangle\,\mathrm{d}s. \tag{40}$$

The fact that $\mathcal{W}(t)$ is the unitary propagator of (36) with Proposition 5.5 yields

$$\begin{aligned}&\mathrm{i}\varepsilon\partial_s\langle\mathcal{W}(s)\Psi,R_\nu U_2(s,0)\Psi\rangle \\ &\quad = -\langle\varepsilon\Theta(s)\mathcal{W}(s)\Psi,R_\nu U_2(s,0)\Psi\rangle + \langle\mathcal{W}(s)\Psi,R_\nu\varepsilon A_2(s)U_2(s,0)\Psi\rangle.\end{aligned} \tag{41}$$

Now, collecting (38), (40) and (41) we obtain the claimed identity. □

Proof of Theorem 6.1. We are now ready to prove Theorem 6.1.

First observe that we have

$$\left\| \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi - \mathrm{e}^{\mathrm{i}\omega(t)/\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right) U_2(t,0)\Psi \right\|_{\mathcal{F}}^2 = \left\|\mathcal{W}(t)\Psi - U_2(t,0)\Psi\right\|_{\mathcal{F}}^2.$$

Now, using Lemma 6.4 one obtains for $t > 0$ (the case $t < 0$ is similar) the estimate

$$\begin{aligned} \left\|\mathcal{W}(t)\Psi - U_2(t,0)\Psi\right\|_{\mathcal{F}}^2 &\leqslant 2\left|\langle \Psi, (1-R_\nu)\Psi\rangle\right| + 2\left|\langle \mathcal{W}(t)\Psi, (1-R_\nu)U_2(t,0)\Psi\rangle\right| \\ &\quad + 2\int_0^t \left|\langle \mathcal{W}(s)\Psi, \left[\Theta(s)R_\nu - R_\nu A_2(s)\right]U_2(s,0)\Psi\rangle\right| \mathrm{d}s. \end{aligned}$$

Here we consider σ to be in the class $C^1(\mathbb{R}_+)$, decreasing and satisfying $\sigma(s) = 1$ if $s \leqslant 1$ and $\sigma(s) = 0$ if $s \geqslant 2$. We have for ν positive integer,

$$\begin{aligned} \langle \Psi, (1-R_\nu)\Psi\rangle &\leqslant \frac{1}{\nu}\sum_{n=\nu+1}^{\infty} n\langle \Psi^{(n)}, (D_{x_1}^2+1)\Psi^{(n)}\rangle \\ &\leqslant \frac{1}{\nu}\langle \Psi, \varepsilon^{-1}\left[\mathrm{d}\Gamma(-\Delta)+N\right]\Psi\rangle \leqslant \frac{1}{\nu}\|\Psi\|_{\mathcal{F}_+^1}^2. \end{aligned}$$

Hence, we easily check with the help of Proposition 5.5 and Lemma 6.3 that

$$\begin{aligned} \left|\langle \mathcal{W}(t)\Psi, (1-R_\nu)U_2(t,0)\Psi\rangle\right| &\leqslant \frac{1}{\nu}\left\|U_2(t,0)\Psi\right\|_{\mathcal{F}_+^1}\left\|\mathcal{W}(t)\Psi\right\|_{\mathcal{F}_+^1} \\ &\leqslant \frac{1}{\nu}\mathrm{e}^{c_1\mathrm{e}^{c_1 t}}\|\Psi\|_{\mathcal{F}_+^1}\|\Psi\|_{\mathcal{G}_+} \leqslant \frac{1}{\nu}\mathrm{e}^{c_1\mathrm{e}^{c_1 t}}\|\Psi\|_{\mathcal{G}_+}^2. \end{aligned}$$

Next, we show that there exists $C > 0$ depending only on φ_0 such that

$$\left\|\varepsilon^{-1}Q_s(z)^{Wick}R_\nu\right\|_{\mathcal{L}(\mathcal{F}_+^1,\mathcal{F}_-^1)} \leqslant C\left(\nu\varepsilon^{1/2}+\nu^2\varepsilon\right).$$

The latter bound follows by Cauchy–Schwarz inequality, Lemma A.1 and (19),

$$\begin{aligned} &\left|\left\langle \Phi, \frac{P_3(s)}{\varepsilon}^{Wick} R_\nu\Psi\right\rangle\right| \\ &\quad\leqslant \sqrt{\varepsilon}\|\varphi_t\|_{L^\infty(\mathbb{R})}\left[\sum_{n=1}^{2\nu}(n+1)\left\|\Phi^{(n)}\right\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2}\left[\sum_{n=1}^{2\nu}n^2\left\|\Psi^{(n+1)}(x,x,\cdot)\right\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2} \\ &\qquad + \sqrt{\varepsilon}\|\varphi_t\|_{L^\infty(\mathbb{R})}\left[\sum_{n=1}^{2\nu}(n+1)\left\|\Psi^{(n)}\right\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2}\left[\sum_{n=1}^{2\nu}n^2\left\|\Phi^{(n+1)}(x,x,\cdot)\right\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2} \\ &\quad\leqslant 2\nu\sqrt{\varepsilon}\|\varphi_t\|_{L^\infty(\mathbb{R})}\left\|\left(\varepsilon^{-1}N+1\right)^{1/2}\Phi\right\|_{\mathcal{F}}\|\Psi\|_{\mathcal{F}_+^1} \\ &\qquad + 2\nu\sqrt{\varepsilon}\|\varphi_t\|_{L^\infty(\mathbb{R})}\left\|\left(\varepsilon^{-1}N+1\right)^{1/2}\Psi\right\|_{\mathcal{F}}\|\Phi\|_{\mathcal{F}_+^1} \end{aligned}$$

and a similar estimate for P^{Wick},

$$\left|\langle \Phi, P^{Wick} R_\nu \Psi\rangle\right| \leqslant \nu^2\varepsilon^2 \|\Phi\|_{\mathcal{F}^1_+}\|\Psi\|_{\mathcal{F}^1_+}.$$

Hence we can check that

$$\int_0^t \left|\langle \mathcal{W}(s)\Psi, \varepsilon^{-1} Q_s(z)^{Wick} R_\nu U_2(s,0)\Psi\rangle\right| \mathrm{d}s \leqslant C\left(\nu\varepsilon^{1/2}+\nu^2\varepsilon\right)\int_0^t \left\|\mathcal{W}(s)\Psi\right\|_{\mathcal{F}^1_+}\left\|U_2(s,0)\Psi\right\|_{\mathcal{F}^1_+} \mathrm{d}s.$$

Now, by Lemma 6.3 and Proposition 5.5 we obtain

$$\begin{aligned}\int_0^t \left\|\mathcal{W}(s)\Psi\right\|_{\mathcal{F}^1_+}\left\|U_2(s,0)\Psi\right\|_{\mathcal{F}^1_+} \mathrm{d}s &\leqslant \int_0^t \mathrm{e}^{c_1\mathrm{e}^{c_1 s}}\|\Psi\|_{\mathcal{G}_+}\left\|U_2(s,0)\Psi\right\|_{\mathcal{F}^1_+}\mathrm{d}s \\ &\leqslant \int_0^t \mathrm{e}^{c_2\mathrm{e}^{c_2 s}}\|\Psi\|_{\mathcal{G}_+}\|\Psi\|_{\mathcal{F}^1_+}\mathrm{d}s \\ &\leqslant \mathrm{e}^{c\mathrm{e}^{cs}}\|\Psi\|^2_{\mathcal{G}_+}.\end{aligned}$$

A simple computation yields

$$\begin{aligned}A_2(s)R_\nu - R_\nu A_2(s) = {}&\frac{1}{2}\left[\sigma\left(\frac{\varepsilon^{-1}N+2}{\nu}\right)-\sigma\left(\frac{\varepsilon^{-1}N}{\nu}\right)\right]\left(\int_{\mathbb{R}}\varphi_t(x)^2\overline{z(x)}^2\,\mathrm{d}x\right)^{Wick} \\ &+\frac{1}{2}\left[\sigma\left(\frac{\varepsilon^{-1}N-2}{\nu}\right)-\sigma\left(\frac{\varepsilon^{-1}N}{\nu}\right)\right]\left(\int_{\mathbb{R}}\overline{\varphi_t(x)}^2 z(x)^2\,\mathrm{d}x\right)^{Wick}.\end{aligned}$$

We easily check that

$$\left\|\sigma\left(\frac{\varepsilon^{-1}N\pm 2}{\nu}\right)-\sigma\left(\frac{\varepsilon^{-1}N}{\nu}\right)\right\|_{\mathcal{L}(\mathcal{F}^1_+)} \leqslant \frac{2}{\nu}\|\sigma'\|_{L^\infty(\mathbb{R}_+)},$$

since $\varepsilon^{-1}\,\mathrm{d}\Gamma(-\Delta)+\varepsilon^{-1}N$ commute with $\varepsilon^{-1}N$. Thus, using (24) there exists $c_0, c>0$ such that

$$\begin{aligned}\int_0^t \left|\langle \mathcal{W}(s)\Psi, [A_2(s), R_\nu]U_2(s,0)\Psi\rangle\right| \mathrm{d}s &\leqslant \frac{c_0}{\nu}\int_0^t \left\|\mathcal{W}(s)\Psi\right\|_{\mathcal{F}^1_+}\left\|U_2(s,0)\Psi\right\|_{\mathcal{F}^1_+}\mathrm{d}s \\ &\leqslant \frac{1}{\nu}\mathrm{e}^{c\mathrm{e}^{ct}}\|\Psi\|^2_{\mathcal{G}_+}.\end{aligned}$$

Finally, the claimed inequality in Theorem 6.1 follows by collecting the previous estimates and letting $\nu=\varepsilon^{-1/4}$. □

We have the following two corollaries.

Corollary 6.5. *For any $\varphi_0\in H^2(\mathbb{R})$ and any $\xi\in L^2(\mathbb{R})$ we have the strong limit*

$$s-\lim_{\varepsilon\to 0} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)^* \mathrm{e}^{\mathrm{i}t/\varepsilon H_\varepsilon} W(\xi)\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right) = \mathrm{e}^{\mathrm{i}\sqrt{2}\mathrm{Re}\langle\xi,\varphi_t\rangle}1,$$

where φ_t solves the NLS equation (18) *with initial data φ_0.*

Proof. It is enough to prove for any $\Psi, \Phi \in \mathcal{G}_+$ the limit:

$$\lim_{\varepsilon\to 0}\left\langle \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi, W(\xi)\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Phi\right\rangle = \mathrm{e}^{\mathrm{i}\sqrt{2}\,\mathrm{Re}\langle\xi,\varphi_t\rangle}\langle\Psi,\Phi\rangle. \tag{42}$$

Indeed, using Theorem 6.1, we show

$$\begin{aligned}&\left\langle \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi, W(\xi)\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Phi\right\rangle \\ &\quad= \left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Psi, W(\xi)W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Phi\right\rangle + \mathrm{O}\big(\varepsilon^{1/8}\big).\end{aligned}$$

Therefore, by Weyl commutation relations we have

$$\left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Psi, W(\xi)W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Phi\right\rangle = \langle U_2(t,0)\Psi, W(\xi)U_2(t,0)\Phi\rangle \mathrm{e}^{\mathrm{i}\sqrt{2}\mathrm{Re}\langle\xi,\varphi_t\rangle}.$$

Thus the limit is proved since $s-\lim_{\varepsilon\to 0} W(\xi) = 1$. □

Recall that $\mathcal{F}_0$ is the subspace of $\mathcal{F}$ spanned by vectors $\Psi \in \mathcal{F}$ such that $\Psi^{(n)} = 0$ for any index $n \in \mathbb{N}$ except for finite number. Note that $\mathcal{F}_0 \cap \mathcal{G}_+$ is dense in $\mathcal{F}$.

Corollary 6.6. *For any $\varphi_0 \in H^2(\mathbb{R})$ and any $\Psi, \Phi \in \mathcal{F}_0 \cap \mathcal{G}_+$ and $b \in \mathcal{P}_{p,q}(L^2(\mathbb{R}))$, we have*

$$\lim_{\varepsilon\to 0}\left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi, \mathrm{e}^{\mathrm{i}t/\varepsilon H_\varepsilon} b^{Wick}\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Phi\right\rangle = b(\varphi_t)\langle\Psi,\Phi\rangle,$$

where φ_t solves the NLS equation (18) *with initial data φ_0.*

Proof. Consider a (p,q)-homogenous polynomial $b \in \mathcal{P}_{p,q}(L^2(\mathbb{R}))$. We have

$$\begin{aligned}\mathcal{A} &:= \left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi, \mathrm{e}^{\mathrm{i}t/\varepsilon H_\varepsilon} b^{Wick}\mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Phi\right\rangle \\ &= \left\langle (N+1)^q W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Psi, \mathrm{e}^{\mathrm{i}t/\varepsilon H_\varepsilon} B_\varepsilon \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon}(N+1)^p W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)\Phi\right\rangle,\end{aligned}$$

where $B_\varepsilon := (N+1)^{-q} b^{Wick}(N+1)^{-p}$. The number estimate (9) yields

$$\|B_\varepsilon\| \leqslant \|\tilde{b}\|_{\mathcal{L}(L^2_s(\mathbb{R}^p),L^2_s(\mathbb{R}^q))},$$

uniformly in $\varepsilon \in (0,1]$. Let $\tilde{N}_t$ be the positive operator given by

$$\tilde{N}_t = N + 2\,\mathrm{Re}\langle z,\varphi_t\rangle^{Wick} + \|\varphi_t\|^2_{L^2(\mathbb{R})}.$$

By (28), we get

$$\mathcal{A} = \left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)(\tilde{N}_0+1)^q\Psi, \mathrm{e}^{\mathrm{i}t/\varepsilon H_\varepsilon} B_\varepsilon \mathrm{e}^{-\mathrm{i}t/\varepsilon H_\varepsilon} W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\right)(\tilde{N}_0+1)^p\Phi\right\rangle.$$

Now, observe that

$$\lim_{\varepsilon\to 0}(\tilde{N}_0+1)^p\Phi = \left(1+\|\varphi\|^2_{L^2(\mathbb{R})}\right)^p\Phi \quad \text{and} \quad \lim_{\varepsilon\to 0}(\tilde{N}_0+1)^q\Psi = \left(1+\|\varphi\|^2_{L^2(\mathbb{R})}\right)^q\Psi.$$

So, using Theorem 6.1 we obtain

$$\begin{aligned}\mathcal{A} &= \left(1+\|\varphi_0\|^2_{L^2(\mathbb{R})}\right)^{p+q}\left\langle W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Psi, B_\varepsilon W\left(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_t\right)U_2(t,0)\Phi\right\rangle + \mathrm{O}\left(\varepsilon^{1/8}\right)\\ &= \left\langle U_2(t,0)\Psi, (\tilde{N}_t+1)^{-q} b(\cdot+\varphi_t)^{Wick}(\tilde{N}_t+1)^{-p}U_2(t,0)\Phi\right\rangle + \mathrm{O}\left(\varepsilon^{1/8}\right).\end{aligned}$$

We set $\Psi_\varepsilon = (N+1)^q(\tilde{N}_t+1)^{-q}U_2(t,0)\Psi$ and $\Phi_\varepsilon = (N+1)^p(\tilde{N}_t+1)^{-p}U_2(t,0)\Phi$ and remark that we can show for $\varphi_0 \neq 0$ and μ a positive integer the following strong limit

$$s - \lim_{\varepsilon\to 0}(N+1)^\mu(\tilde{N}_t+1)^{-\mu} = \frac{1}{(1+\|\varphi_t\|^2_{L^2(\mathbb{R})})^\mu}. \tag{43}$$

This holds since we have by explicit computation

$$\left\|\left(a(\varphi_t)+a^*(\varphi_t)\right)\left(N+\|\varphi_t\|^2+1\right)^{-1}\right\| \leqslant \frac{\|\varphi_t\|}{2\sqrt{\|\varphi_t\|^2+1}} + \frac{\|\varphi_t\|}{2\sqrt{\|\varphi_t\|^2+1-\varepsilon}} < 1$$

for ε sufficiently small and hence we can write

$$(N+1)(\tilde{N}_t+1)^{-1} = (N+1)\left(N+\|\varphi_t\|^2+1\right)^{-1}\big[\overbrace{\left(a(\varphi_t)+a^*(\varphi_t)\right)\left(N+\|\varphi_t\|^2+1\right)^{-1}}^{\mathcal{R}_\varepsilon}+1\big]^{-1}.$$

This proves (43) for $\mu = 1$ since $s - \lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon = 0$. Now, we proceed by induction on μ using a commutator argument

$$\begin{aligned}(N+1)^{\mu+1}(\tilde{N}_t+1)^{-(\mu+1)} &= (N+1)^\mu(\tilde{N}_t+1)^{-\mu}(N+1)(\tilde{N}_t+1)^{-1}\\ &\quad + (N+1)^\mu(\tilde{N}_t+1)^{-\mu}\left[(\tilde{N}_t+1)^\mu, N\right](\tilde{N}_t+1)^{-(\mu+1)},\end{aligned}$$

with the observation that the second term of (r.h.s.) converges strongly to 0. Therefore, we obtain

$$\lim_{\varepsilon\to 0}\Psi_\varepsilon = \frac{1}{(1+\|\xi\|^2_{L^2(\mathbb{R})})^q}U_2(t,0)\Psi \quad \text{and} \quad \lim_{\varepsilon\to 0}\Phi_\varepsilon = \frac{1}{(1+\|\xi\|^2_{L^2(\mathbb{R})})^p}U_2(t,0)\Phi.$$

It is also easy to show by explicit computation that

$$w - \lim_{\varepsilon\to 0}(N+1)^{-q}b_{r,s}^{Wick}(N+1)^{-p} = 0$$

for any $b_{r,s} \in \mathcal{P}_{r,s}(L^2(\mathbb{R}))$ such that $0 < r \leqslant p$ and $0 < s \leqslant q$. Hence, letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} \mathcal{A} = \left(1 + \|\varphi_0\|^2_{L^2(\mathbb{R})}\right)^{p+q} \lim_{\varepsilon\to 0} \langle \Psi_\varepsilon, (N+1)^{-q} b(\varphi_t)(N+1)^{-p}\Phi_\varepsilon\rangle$$
$$= b(\varphi_t)\langle U_2(t,0)\Psi, U_2(t,0)\Phi\rangle = b(\varphi_t)\langle \Psi, \Phi\rangle,$$

since $\|\varphi_t\|_{L^2(\mathbb{R})} = \|\varphi_0\|_{L^2(\mathbb{R})}$ and $s - \lim_{\varepsilon\to 0}(N+1)^{-\mu} = 1$ for $\mu > 0$. □

We identify the propagator $U_2(t,s)$ as a time-dependent Bogoliubov's transform on the Fock representation of the Weyl commutation relations.

Proposition 6.7. *Let $\varphi_0 \in H^2(\mathbb{R})$ and consider the propagator $U_2(t,0)$ given in Proposition* 5.5. *For a given $s \in \mathbb{R}$ let $\xi_s \in H^2(\mathbb{R})$, we have*

$$U_2(t,s) W\left(\frac{\xi_s}{\mathrm{i}\sqrt{\varepsilon}}\right) U_2(s,t) = W\left(\frac{\beta(t,s)\xi_s}{\mathrm{i}\sqrt{\varepsilon}}\right),$$

where $\beta(t,s)$ is the symplectic propagator on $L^2(\mathbb{R})$, solving the equation

$$\begin{cases} \mathrm{i}\partial_t \xi_t(x) = \left[-\Delta + 2\left|\varphi_t(x)\right|^2\right]\xi_t(x) + \varphi_t(x)^2 \overline{\xi_t(x)}, \\ \xi_{|t=s} = \xi_s, \end{cases} \tag{44}$$

such that $\beta(t,s)\xi_s = \xi_t$.

Proof. Observe that if $\varphi_0 \in H^2(\mathbb{R})$ then the solution φ_t of the NLS equation (18) with initial condition φ_0 satisfies $\varphi_t \in C^0(\mathbb{R}, L^\infty(\mathbb{R}))$. Hence, by standard arguments the equation (44) admits a unique solution $\xi_t \in C^0(\mathbb{R}, H^2(\mathbb{R})) \cap C^1(\mathbb{R}, L^2(\mathbb{R}))$ for any $\xi_s \in H^2(\mathbb{R})$. Moreover, the propagator

$$\beta(t,s)\xi_s = \xi_t,$$

defines a symplectic transform on $L^2(\mathbb{R})$ for any $t, s \in \mathbb{R}$. This follows by differentiating

$$\operatorname{Im}\left\langle \beta(t,s)\xi, \beta(t,s)\eta\right\rangle,$$

with respect to t for $\xi, \eta \in H^2(\mathbb{R})$. Furthermore, β satisfies the laws

$$\beta(s,s) = 1, \qquad \beta(t,s)\beta(s,r) = \beta(t,r) \quad \text{for } t, r, s \in \mathbb{R}.$$

Now, we differentiate with respect to t the quantity

$$U_2(s,t) W\left(\frac{\xi_t}{\mathrm{i}\sqrt{\varepsilon}}\right) U_2(t,s)$$

in the sense of quadratic forms on $\mathcal{F}_+^1$, with ξ_t solution of (44). Hence, using Lemma 6.2(ii), we get

$$\begin{aligned}\partial_t&\Big[U_2(s,t)W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)U_2(t,s)\Big]\\&=U_2(s,t)W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)\Big[W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)^*\mathrm{i}A_2(t)W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)-\mathrm{i}A_2(t)\\&\qquad-\mathrm{i}\Big(\mathrm{Re}\langle\xi_t,\mathrm{i}\partial_t\xi_t\rangle+\frac{2}{\sqrt{\varepsilon}}\mathrm{Re}\langle z,\mathrm{i}\partial_t\xi_t\rangle^{Wick}\Big)\Big]U_2(t,s).\end{aligned}\tag{45}$$

Now, by [3], Lemma 2.10, we obtain

$$W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)^*A_2(t)W\Big(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\varepsilon}}\xi_t\Big)=\varepsilon^{-1}m(t)\big[z+\sqrt{\varepsilon}\xi_t\big]^{Wick},$$

where $m(t)[z]$ is the continuous polynomial on $\mathcal{S}(\mathbb{R})$ given by

$$m(t)[z]=\langle z,-\Delta z\rangle+P_2(t)[z].$$

Therefore, the r.h.s. of (45) is null if we show that

$$m(t)\big[z+\sqrt{\varepsilon}\xi_t\big]-m(t)[z]-\big(\varepsilon\,\mathrm{Re}\langle\xi_t,\mathrm{i}\partial_t\xi_t\rangle+2\sqrt{\varepsilon}\,\mathrm{Re}\langle z,\mathrm{i}\partial_t\xi_t\rangle\big)=0.$$

This follows by straightforward computation. □

7. Propagation of chaos

Propagation of chaos for a many-boson system with point pair-interaction in one dimension was studied in [1] (see also the related work [2]). Here we prove this conservation hypothesis for such quantum system using the method in [31]. Thus, we are led to study the asymptotics of time-evolved Hermite states

$$\mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}}\varphi_0^{\otimes n}\quad\text{with }\varphi_0\in H^2(\mathbb{R}),\ \|\varphi_0\|_{L^2(\mathbb{R})}=1,$$

when $n\to\infty$ with $n\varepsilon_n=1$. We denote the coherent states by

$$E(\varphi_0):=W\Big(\frac{\sqrt{2}}{\mathrm{i}\varepsilon}\varphi_0\Big)\Omega_0,$$

where $\Omega_0=(1,0,\dots)$ is the vacuum vector in the Fock space $\mathcal{F}$. To pass from coherent states to Hermite states we use the integral representation proved in [31],

$$\varphi_0^{\otimes n}=\frac{\gamma_n}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\theta n}E\big(\mathrm{e}^{\mathrm{i}\theta}\varphi_0\big)\,\mathrm{d}\theta,\quad\text{where }\gamma_n:=\frac{\mathrm{e}^{1/2\varepsilon_n}\sqrt{n!}}{\varepsilon_n^{-n/2}}.\tag{46}$$

Asymptotically, the factor γ_n grows as $(2\pi n)^{1/4}$ when $n\to\infty$. In the following we prove the chaos conservation hypothesis.

Proof of Theorem 2.3. It is known that if a sequence of positive trace-class operators ρ_n on $L^2(\mathbb{R})$ converges in the weak operator topology to ρ such that $\lim_{n\to\infty} \mathrm{Tr}[\rho_n] = \mathrm{Tr}[\rho] < \infty$ then ρ_n converges in the trace norm to ρ (see, for instance [8]). This argument reduces the proof to the case

$$b(z) = \prod_{i=1}^{p} \langle z, f_i\rangle\langle g_i, z\rangle,$$

where $f_i, g_i \in L^2(\mathbb{R})$. For shortness, we set

$$E_\theta = E\bigl(\mathrm{e}^{\mathrm{i}\theta}\varphi_0\bigr) \quad \text{and} \quad E_\theta^t = \mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} E_\theta.$$

Using formula (46), we get

$$\Gamma_n := \bigl\langle \varphi_0^{\otimes n}, \mathrm{e}^{\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} b^{\mathit{Wick}} \mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} \varphi_0^{\otimes n}\bigr\rangle = \frac{\gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}n(\theta-\theta')} \bigl\langle E_{\theta'}^t, b^{\mathit{Wick}} E_\theta^t\bigr\rangle \,\mathrm{d}\theta\,\mathrm{d}\theta'.$$

It is easily seen that

$$(N+1)^{-p} \mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} \varphi_0^{\otimes n} = 2^{-p} \mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} \varphi_0^{\otimes n}.$$

Therefore, we write

$$\Gamma_n = \frac{4^p\gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}n(\theta-\theta')} \left\langle E_{\theta'}^t, (N+1)^{-p} \prod_{i=1}^{p} a^*(f_i) \prod_{j=1}^{p} a(g_j) (N+1)^{-p} E_\theta^t \right\rangle \mathrm{d}\theta\,\mathrm{d}\theta'.$$

Now, we use the decomposition

$$\prod_{i=1}^{p} a^*(f_i) \prod_{j=1}^{p} a(g_j) = \sum_{I,J\subset\mathcal{N}_p} \prod_{i\in I^c} \bigl[a^*(f_i) - \langle \varphi_t^{\theta'}, f_i\rangle\bigr] \prod_{j\in J^c} \bigl[a(g_j) - \langle g_j, \varphi_t^{\theta}\rangle\bigr] \mathrm{e}^{-\mathrm{i}(\#I\theta' - \#J\theta)}$$
$$\times \prod_{i\in I} \overline{\langle f_i, \varphi_t\rangle} \prod_{j\in J} \langle g_j, \varphi_t\rangle,$$

where the sum runs over all subsets I, J of $\mathcal{N}_p := \{1, \dots, p\}$. Thus, we can write

$$\Gamma_n - b(\varphi_t) = \sum_{I,J\subset\mathcal{N}_p}^{\#I+\#J<2p} \frac{4^p\gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}[(n-\#J)\theta - (n-\#I)\theta']} \bigl\langle \tilde{E}_{\theta'}^t, B_{I,J}^{\mathit{Wick}} \tilde{E}_\theta^t\bigr\rangle \,\mathrm{d}\theta\,\mathrm{d}\theta', \tag{47}$$

where $\tilde{E}_\theta^t := (N+1)^{-p} E_\theta^t$ and $B_{I,J}(z)$ are sums of homogenous polynomials such that

$$\bigl\langle \tilde{E}_{\theta'}^t, B_{I,J}^{\mathit{Wick}} \tilde{E}_\theta^t\bigr\rangle = \prod_{i\in I} \langle \varphi_t, f_i\rangle \prod_{j\in J} \langle g_j, \varphi_t\rangle$$
$$\times \left\langle \prod_{i\in I^c} \bigl[a(f_i) - \langle f_i, \varphi_t^{\theta'}\rangle\bigr] \tilde{E}_{\theta'}^t, \prod_{j\in J^c} \bigl[a(g_j) - \langle g_j, \varphi_t^{\theta}\rangle\bigr] \tilde{E}_\theta^t \right\rangle.$$

We have, for $0 \leqslant \#I, \#J < p$, by Cauchy–Schwarz inequality

$$\begin{aligned}
\big|\langle \tilde{E}^t_{\theta'}, B^{Wick}_{I,J} \tilde{E}^t_{\theta}\rangle\big| \leqslant & \prod_{i\in I, j\in J} \|g_j\|_{L^2(\mathbb{R})} \|f_i\|_{L^2(\mathbb{R})} \\
& \times \Big\| \prod_{i\in I^c} \big[a(f_i) - \langle f_i, \varphi_t^{\theta'}\rangle\big] \tilde{E}^t_{\theta'} \Big\|_{\mathcal{F}} \times \Big\| \prod_{j\in J^c} \big[a(g_j) - \langle g_j, \varphi_t^{\theta}\rangle\big] \tilde{E}^t_{\theta} \Big\|_{\mathcal{F}}.
\end{aligned}$$

In the following we make use of the positive self-adjoint operator

$$\tilde{N} := N + 2\operatorname{Re}\langle z, \varphi_t\rangle^{Wick} + \|\varphi_t\|^2 1.$$

Observe that we have for any $\theta' \in [0, 2\pi]$ and $r \geqslant 1$,

$$\begin{aligned}
& \Big\| \prod_{i=1}^{r} \big[a(f_i) - \langle f_i, \varphi_t^{\theta'}\rangle\big] \tilde{E}^t_{\theta'} \Big\|_{\mathcal{F}} \\
& \quad = \Big\| \prod_{i=1}^{r} a(f_i)(\tilde{N}+1)^{-p} \mathcal{W}(t) \Omega_0 \Big\|_{\mathcal{F}} \\
& \quad \leqslant \Big\| \prod_{i=1}^{r-1} a(f_i)(\tilde{N}+1)^{-p} a(f_r) \mathcal{W}(t) \Omega_0 \Big\|_{\mathcal{F}} + \Big\| \prod_{i=1}^{r-1} a(f_i) \big[a(f_r), (\tilde{N}+1)^{-p}\big] \mathcal{W}(t)\Omega_0 \Big\|_{\mathcal{F}}.
\end{aligned}$$

We easily show that

$$\big\|a(f_r)\mathcal{W}(t)\Omega_0\big\|_{\mathcal{F}} \leqslant \|f_r\|_{L^2(\mathbb{R})} \sqrt{\varepsilon_n} \big\|\mathcal{W}(t)\big\|_{\mathcal{L}(\mathcal{G}_+, \mathcal{F}^1_+)}.$$

Furthermore, we have

$$\big\| \big[a(f_r), (\tilde{N}+1)^p\big](\tilde{N}+1)^{-p} \big\|_{\mathcal{L}(\mathcal{F})} \leqslant C\varepsilon_n$$

using (29) and the fact that $[a(f_r), (\tilde{N}+1)^p]$ is a Wick polynomial where we gained ε_n in its symbol, see [3], Proposition 2.7(ii). Recall also that we have by the number estimate (9) and (29),

$$\Big\| \prod_{i=1}^{r-1} a(f_i)(\tilde{N}+1)^{-p} \Big\|_{\mathcal{L}(\mathcal{F})} \leqslant C,$$

uniformly in n and $\theta' \in [0, 2\pi]$. Therefore, we have

$$\Big| \sum_{0\leqslant \#I,\#J<p} \frac{4^p \gamma_n^2}{(2\pi)^2} \int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}[(n-\#J)\theta - (n-\#I)\theta']} \langle \tilde{E}^t_{\theta'}, B^{Wick}_{I,J} \tilde{E}^t_{\theta}\rangle \,\mathrm{d}\theta \,\mathrm{d}\theta' \Big| \leqslant C\gamma_n^2 \varepsilon_n \overset{n\to\infty}{\longrightarrow} 0. \tag{48}$$

It still to control the terms $\#I = p, \#J = p-1$ and $\#I = p-1, \#J = p$ which are similar. In fact, remark that we have

$$\frac{4^p\gamma_n^2}{(2\pi)^2}\int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}[(n-p)\theta-(n-p+1)\theta']}\langle \tilde{E}^t_{\theta'}, B^{Wick}_{I,\mathcal{N}_p}\tilde{E}^t_\theta\rangle \,\mathrm{d}\theta\,\mathrm{d}\theta'$$
$$= \frac{4^p\gamma_n}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-p+1)\theta'}\langle \tilde{E}^t_{\theta'}, B^{Wick}_{I,\mathcal{N}_p}\mathrm{e}^{\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}}\varphi_0^{\otimes(n-p)}\rangle \,\mathrm{d}\theta'.$$

Now, a similar estimate as (48) yields that

$$\left|\frac{4^p\gamma_n^2}{(2\pi)^2}\int_{[0,2\pi]^2} \mathrm{e}^{-\mathrm{i}[(n-p)\theta-(n-p+1)\theta']}\langle \tilde{E}^t_{\theta'}, B^{Wick}_{I,\mathcal{N}_p}\tilde{E}^t_\theta\rangle \,\mathrm{d}\theta\,\mathrm{d}\theta'\right| \leqslant C\gamma_n\sqrt{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, we conclude that $\lim_{n\to\infty}\Gamma_n - b(\varphi_t) = 0$. □

Remark 7.1. (1) Let $\gamma^t_{k,n}$ be the k-particle correlation functions, defined by (4), associated to the states $\mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}}\varphi_0^{\otimes n}$. Then Theorem 2.3 implies the following convergence in the trace norm

$$\lim_{n\to\infty}\gamma^t_{k,n} = \varphi_t(x_1)\cdots\varphi_t(x_k)\overline{\varphi_t(y_1)\cdots\varphi_t(y_k)}.$$

(2) In terms of Wigner measures, introduced in [3,4], Theorem 2.3 says that the sequence $(\mathrm{e}^{-\mathrm{i}t/\varepsilon_n H_{\varepsilon_n}} \times \varphi_0^{\otimes n})_{n\in\mathbb{N}}$ admits a unique (Borel probability) Wigner measure μ_t given by

$$\mu_t = \frac{1}{2\pi}\int_0^{2\pi}\delta_{\mathrm{e}^{\mathrm{i}\theta}\varphi_t}\,\mathrm{d}\theta,$$

where $\delta_{\mathrm{e}^{\mathrm{i}\theta}\varphi_t}$ is the Dirac measure on $L^2(\mathbb{R})$ at the point $\mathrm{e}^{\mathrm{i}\theta}\varphi_t$.

Appendix A: Elementary estimate

Lemma A.1. *For any $\alpha > 0$ and any $\Psi^{(n)} \in \mathcal{S}_s(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^{n-1}}\left|\Psi^{(n)}(x_2,x_2,\ldots,x_n)\right|^2\mathrm{d}x_2\cdots\mathrm{d}x_n \leqslant \frac{\alpha}{\sqrt{2}}\langle D^2_{x_1}\Psi^{(n)},\Psi^{(n)}\rangle_{L^2(\mathbb{R}^n)} + \frac{\alpha^{-1}}{2\sqrt{2}}\left|\Psi^{(n)}\right|^2_{L^2(\mathbb{R}^n)}. \tag{49}$$

Proof. Let $x', \xi' \in \mathbb{R}^{n-1}$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Let us denote the Fourier transform of g by

$$\hat{g}(\xi) = \int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}x\xi}g(x)\,\mathrm{d}x.$$

We have

$$g(0,x') = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}}\mathrm{e}^{\mathrm{i}x'\xi'}\left(\frac{1}{2\pi}\int_{\mathbb{R}}\hat{g}(\xi_1,\xi')\,\mathrm{d}\xi_1\right)\mathrm{d}\xi'.$$

Cauchy–Schwarz inequality yields

$$\left|\int_{\mathbb{R}} \hat{g}(\xi_1,\xi')\,\mathrm{d}\xi_1\right|^2 \leqslant \int_{\mathbb{R}} \left|\hat{g}(\xi_1,\xi')\right|^2(\alpha^{-1}+\alpha\xi_1^2)\,\mathrm{d}\xi_1 \times \int_{\mathbb{R}} \frac{\mathrm{d}\xi_1}{\alpha^{-1}+\alpha\xi_1^2}.$$

Therefore, we get

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \left|g(0,x')\right|^2\mathrm{d}x' &= \frac{1}{4\pi^2(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}}\hat{g}(\xi_1,\xi')\,\mathrm{d}\xi_1\right|^2\mathrm{d}\xi' \\ &\leqslant \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{g}(\xi_1,\xi')\right|^2(\alpha^{-1}+\alpha\xi_1^2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi'.\end{aligned}$$

Set $g(x_1,\dots,x_n)=\Psi^{(n)}(\frac{x_1+x_2}{\sqrt{2}},\frac{x_2-x_1}{\sqrt{2}},x_3,\dots,x_n)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\left|\Psi^{(n)}(x_2,x_2,\dots,x_n)\right|^2\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{n-1}}\left|g^{(n)}(0,x_2,\dots,x_n)\right|^2\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n \\ &\leqslant \frac{(2\pi)^{-n}}{2\sqrt{2}}\int_{\mathbb{R}^n}\left|\hat{g}^{(n)}(\xi_1,\xi')\right|^2(\alpha^{-1}+\alpha\xi_1^2+\alpha\xi_2^2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi' \\ &\leqslant \frac{(2\pi)^{-n}}{2\sqrt{2}}\int_{\mathbb{R}^n}\left|\hat{\Psi}^{(n)}(\xi_1,\xi')\right|^2(\alpha^{-1}+\alpha\xi_1^2+\alpha\xi_2^2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi'.\end{aligned}$$

Thus, by Plancherel's identity we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\left|\Psi^{(n)}(x_2,x_2,\dots,x_n)\right|^2\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n \\ &\quad\leqslant \frac{\alpha}{2\sqrt{2}}\left\langle(D_{x_1}^2+D_{x_2}^2)\Psi^{(n)},\Psi^{(n)}\right\rangle_{L^2(\mathbb{R}^n)}+\frac{\alpha^{-1}}{2\sqrt{2}}\left|\Psi^{(n)}\right|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Thanks to the symmetry of $\Psi^{(n)}$, it is easy to see that

$$\left\langle(D_{x_1}^2+D_{x_2}^2)\Psi^{(n)},\Psi^{(n)}\right\rangle = 2\left\langle D_{x_1}^2\Psi^{(n)},\Psi^{(n)}\right\rangle.$$

Hence, we arrive at the claimed estimate (49). □

Appendix B: Commutator theorems

Here we first recall an abstract regularity argument from Faris–Lavine work [13], Theorem 2.

Theorem B.1. *Let A be a self-adjoint operator and let S be a positive self-adjoint operator satisfying*:

- $\mathcal{D}(S)\subset\mathcal{D}(A)$,
- $\pm\mathrm{i}[\langle A\Psi,S\Psi\rangle-\langle S\Psi,A\Psi\rangle]\leqslant c\|S^{1/2}\Psi\|^2$ *for all* $\Psi\in\mathcal{D}(S)$.

Then $\mathcal{Q}(S)$ is invariant by $\mathrm{e}^{-\mathrm{i}tA}$ for any $t \in \mathbb{R}$ and the inequality

$$\left\|S^{1/2}\mathrm{e}^{-\mathrm{i}tA}\Psi\right\| \leqslant \mathrm{e}^{c|t|}\left\|S^{1/2}\Psi\right\|$$

holds true.

Next we recall the Nelson commutator theorem (see, e.g., [29,30], Theorem X.36') with a useful regularity property added as a consequence of Faris–Lavine's Theorem B.1.

Theorem B.2. *Let S be a self-adjoint operator on a Hilbert space $\mathcal{H}$ such that $S \geqslant 1$. Consider a quadratic form $a(\cdot,\cdot)$ with $\mathcal{Q}(a) = \mathcal{D}(S^{1/2})$ and satisfying:*

(i) $|a(\Psi,\Phi)| \leqslant c_1\|S^{1/2}\Psi\|\|S^{1/2}\Phi\|$ *for any* $\Psi,\Phi \in \mathcal{D}(S^{1/2})$;
(ii) $|a(\Psi,S\Phi) - a(S\Psi,\Phi)| \leqslant c_2\|S^{1/2}\Psi\|\|S^{1/2}\Phi\|$ *for any* $\Psi,\Phi \in \mathcal{D}(S^{3/2})$.

Then the linear operator $A:\mathcal{D}(A) \to \mathcal{H}$, $\mathcal{D}(A) = \{\Phi \in \mathcal{D}(S^{1/2}):\mathcal{H} \ni \Psi \mapsto a(\Psi,\Phi)$ continuous$\}$ associated to the quadratic form $a(\cdot,\cdot)$ through the relation

$$\langle\Psi, A\Phi\rangle_{\mathcal{H}} = a(\Psi,\Phi) \quad \text{for all } \Psi \in \mathcal{D}\big(S^{1/2}\big), \Phi \in \mathcal{D}(A)$$

is densely defined and satisfies:

(1) $\mathcal{D}(S) \subset \mathcal{D}(A)$ *and* $\|A\Psi\| \leqslant c\|S\Psi\|$ *for any* $\Psi \in \mathcal{D}(S)$;
(2) A *is essentially self-adjoint on any core of* S;
(3) $\mathrm{e}^{-\mathrm{i}t\tilde{A}}$ *preserves* $\mathcal{D}(S^{1/2})$ *with the inequality*

$$\left\|S^{1/2}\mathrm{e}^{-\mathrm{i}t\tilde{A}}\Psi\right\| \leqslant \mathrm{e}^{c_2|t|}\left\|S^{1/2}\Psi\right\|,$$

where $\tilde{A}$ denotes the self-adjoint extension of A.

Proof. The point (3) follows from Theorem B.1 since its assumptions:

- $\mathcal{D}(S) \subset \mathcal{D}(A)$,
- $\pm\mathrm{i}[\langle A\Psi, S\Psi\rangle - \langle S\Psi, A\Psi\rangle] \leqslant c_2\|S^{1/2}\Psi\|^2$, for any $\Psi \in \mathcal{D}(S)$,

hold true using items (1), (2) and hypothesis (ii). □

We naturally associate to a self-adjoint operator $S \geqslant 1$ acting on a Hilbert space $\mathcal{H}$, a Hilbert rigging $\mathcal{H}_{\pm 1}$ where $\mathcal{H}_{+1}$ is defined as $\mathcal{D}(S^{1/2})$ endowed with the inner product

$$\langle\psi,\phi\rangle_{\mathcal{H}_{+1}} := \big\langle S^{1/2}\psi, S^{1/2}\phi\big\rangle_{\mathcal{H}}$$

and $\mathcal{H}_{-1}$ is the completion of $\mathcal{D}(S^{-1/2})$ with respect to the inner product

$$\langle\psi,\phi\rangle_{\mathcal{H}_{-1}} := \big\langle S^{-1/2}\psi, S^{-1/2}\phi\big\rangle_{\mathcal{H}}.$$

Assumption (ii) of Theorem B.2 can be reformulated in some other slightly different ways.

Lemma B.3. *Consider a self-adjoint operator S satisfying $S \geqslant 1$ with the associated Hilbert rigging $\mathcal{H}_{\pm 1}$ defined above. Let A be a symmetric bounded operator in $\mathcal{L}(\mathcal{H}_{+1}, \mathcal{H}_{-1})$, then the three following statements are equivalent,*

(1) *there exists $c > 0$ such that for any $\Psi, \Phi \in \mathcal{D}(S^{3/2})$,*

$$\big|\langle S\Psi, A\Phi\rangle - \langle A\Psi, S\Phi\rangle\big| \leqslant c\|\Psi\|_{\mathcal{H}_{+1}}\|\Phi\|_{\mathcal{H}_{+1}},$$

(2) *there exists $c > 0$ such that for any $\Psi, \Phi \in \mathcal{D}(S^{1/2})$ and $\lambda > 0$,*

$$\big|\big\langle(\lambda S+1)^{-1}S\Psi, A(\lambda S+1)^{-1}\Phi\big\rangle - \big\langle A(\lambda S+1)^{-1}\Psi, (\lambda S+1)^{-1}S\Phi\big\rangle\big| \leqslant c\|\Psi\|_{\mathcal{H}_{+1}}\|\Phi\|_{\mathcal{H}_{+1}},$$

(3) *there exists $c > 0$ such that for any $\Psi, \Phi \in \mathcal{D}(S^{1/2})$ and $\lambda > 0$,*

$$\big|\big\langle(\lambda S+1)^{-1}S\Psi, A\Phi\big\rangle - \big\langle A\Psi, (\lambda S+1)^{-1}S\Phi\big\rangle\big| \leqslant c\|\Psi\|_{\mathcal{H}_{+1}}\|\Phi\|_{\mathcal{H}_{+1}}.$$

Proof. (1) $\Leftrightarrow$ (2): Observe that if $\lambda > 0$ then $(\lambda S+1)^{-1}\mathcal{D}(S^{1/2}) \subset \mathcal{D}(S^{3/2})$. Assume (1) and let us prove (2) for $\Psi, \Phi \in \mathcal{D}(S^{1/2})$. Using (1) with $\tilde{\Psi} = (\lambda S+1)^{-1}\Psi \in \mathcal{D}(S^{3/2})$ and $\tilde{\Phi} = (\lambda S+1)^{-1}\Phi \in \mathcal{D}(S^{3/2})$, we obtain

$$\big|\langle S\tilde{\Psi}, A\tilde{\Phi}\rangle - \langle A\tilde{\Psi}, S\tilde{\Phi}\rangle\big| \leqslant c\big\|(\lambda S+1)^{-1}\Psi\big\|_{\mathcal{H}_{+1}} \times \big\|(\lambda S+1)^{-1}\Phi\big\|_{\mathcal{H}_{+1}}. \tag{50}$$

It is easy to see that the right-hand side of (50) is bounded by $c\|\Psi\|_{\mathcal{H}_{+1}}\|\Phi\|_{\mathcal{H}_{+1}}$. Thus, we obtain (2). Now, to prove (2) $\Rightarrow$ (1), we observe that $(\lambda S+1)\mathcal{D}(S^{3/2}) \subset \mathcal{D}(S^{1/2})$ and use (2) with $\Psi_\lambda = (\lambda S+1)\Psi \in \mathcal{D}(S^{1/2})$, $\Phi_\lambda = (\lambda S+1)\Phi \in \mathcal{D}(S^{1/2})$ such that $\Psi, \Phi \in \mathcal{D}(S^{3/2})$. Therefore, we get for $\lambda > 0$

$$\big|\langle S\Psi, A\Phi\rangle - \langle A\Psi, S\Phi\rangle\big| \leqslant c\|\Psi_\lambda\|_{\mathcal{H}_{+1}} \times \|\Phi_\lambda\|_{\mathcal{H}_{+1}}. \tag{51}$$

Letting $\lambda \to 0$ in the right-hand side of (51), we obtain (2).

(2) $\Leftrightarrow$ (3): Let $\Psi, \Phi \in \mathcal{D}(S^{1/2})$ and $\lambda > 0$, we have as identity in $\mathcal{L}(\mathcal{H}_{+1}, \mathcal{H}_{-1})$

$$A(\lambda S+1)(\lambda S+1)^{-1} = A\lambda S(\lambda S+1)^{-1} + A(\lambda S+1)^{-1},$$

since $\lambda S(\lambda S+1)^{-1} \in \mathcal{L}(\mathcal{H}_{+1})$ and $(\lambda S+1)^{-1} \in \mathcal{L}(\mathcal{H}_{+1})$. Therefore, since $(\lambda S+1)^{-1}S\Psi \in \mathcal{H}_{+1}$ and $(\lambda S+1)^{-1}S\Phi \in \mathcal{H}_{+1}$, the following computation is justified

$$\begin{aligned}
&\big\langle(\lambda S+1)^{-1}S\Psi, A\Phi\big\rangle - \big\langle A\Psi, (\lambda S+1)^{-1}S\Phi\big\rangle \\
&\quad = \big\langle(\lambda S+1)^{-1}S\Psi, A(\lambda S+1)(\lambda S+1)^{-1}\Phi\big\rangle - \big\langle A(\lambda S+1)(\lambda S+1)^{-1}\Psi, (\lambda S+1)^{-1}S\Phi\big\rangle \\
&\quad = \big\langle(\lambda S+1)^{-1}S\Psi, A(\lambda S+1)^{-1}\Phi\big\rangle - \big\langle A(\lambda S+1)^{-1}\Psi, (\lambda S+1)^{-1}S\Phi\big\rangle.
\end{aligned}$$

So, this shows the equivalence of the statements (2) and (3). □

Appendix C: Non-autonomous Schrödinger equation

Consider the Hilbert rigging

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-.$$

This means that $\mathcal{H}$ is a Hilbert space with an inner product $(\cdot,\cdot)_{\mathcal{H}}$ and $\mathcal{H}_+$ is a dense subspace of $\mathcal{H}$ which is itself a Hilbert space with respect to another inner product $(\cdot,\cdot)_{\mathcal{H}_+}$ such that

$$\|u\|_{\mathcal{H}} := \sqrt{(u,u)_{\mathcal{H}}} \leqslant \|u\|_{\mathcal{H}_+} := \sqrt{(u,u)_{\mathcal{H}_+}} \quad \forall u \in \mathcal{H}_+.$$

The Hilbert space $\mathcal{H}_-$ is defined as the completion of $\mathcal{H}$ with respect to the norm

$$\|u\|_{\mathcal{H}_-} := \sup_{f\in\mathcal{H}_+, \|f\|_{\mathcal{H}_+}=1} \big|(f,u)_{\mathcal{H}}\big|. \tag{52}$$

This extends by continuity the inner product $(\cdot,\cdot)_{\mathcal{H}}$ to a sesquilinear form on $\mathcal{H}_- \times \mathcal{H}_+$ satisfying

$$\big|(\xi,u)_{\mathcal{H}}\big| \leqslant \|u\|_{\mathcal{H}_+}\|\xi\|_{\mathcal{H}_-} \quad \forall u \in \mathcal{H}_+, \forall \xi \in \mathcal{H}_-.$$

Furthermore, we have

$$\|u\|_{\mathcal{H}_+} = \sup_{\xi\in\mathcal{H}_-, \|\xi\|_{\mathcal{H}_-}=1} \big|(\xi,u)_{\mathcal{H}}\big|. \tag{53}$$

Let I be a closed interval of $\mathbb{R}$ and let $(A(t))_{t\in I}$ denote a family of self-adjoint operators on $\mathcal{H}$ such that $\mathcal{D}(A(t)) \cap \mathcal{H}_+$ is dense in $\mathcal{H}_+$ and $A(t)$ are continuously extendable to bounded operators in $\mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$. We aim to solve the following abstract non-autonomous Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u = A(t)u, & t \in I, \\ u(t=0) = u_0, \end{cases} \tag{54}$$

where $u_0 \in \mathcal{H}_+$ is given and $t \mapsto u(t) \in \mathcal{H}_+$ is the unknown. This is a particular case of the more general topic of solving non-autonomous Cauchy problems where $-\mathrm{i}A(t)$ are infinitesimal generators of C_0-semigroups (see [25,32]). We provide here a useful result (Theorem C.2) which follows from the work of Kato [24].

Definition C.1. We say that the map

$$I \times I \ni (t,s) \mapsto U(t,s)$$

is a unitary propagator of the problem (54) iff:

(a) $U(t,s)$ is unitary on $\mathcal{H}$,
(b) $U(t,t) = 1$ and $U(t,s)U(s,r) = U(t,r)$ for all $t,s,r \in I$,

(c) The map $t \in I \mapsto U(t,s)$ belongs to $C^0(I, \mathcal{L}(\mathcal{H}_+)) \cap C^1(I, \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-))$ and satisfies

$$\mathrm{i}\partial_t U(t,s)\psi = A(t)U(t,s)\psi, \quad \forall \psi \in \mathcal{H}_+, \forall t,s \in I.$$

Here $C^k(I, \mathfrak{B})$ denotes the space of k-continuously differentiable $\mathfrak{B}$-valued functions where $\mathfrak{B}$ is endowed with the strong operator topology.

Theorem C.2. *Let I be a compact interval and let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be a Hilbert rigging with $(A(t))_{t\in I}$ a family of self-adjoint operators on $\mathcal{H}$ as above satisfying*:

(i) $I \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$ *is norm continuous.*
(ii) $\mathbb{R} \ni \tau \mapsto \mathrm{e}^{\mathrm{i}\tau A(t)} \in \mathcal{L}(\mathcal{H}_+)$ *is strongly continuous.*
(iii) *There exists a family of Hilbertian norms* $(\|\cdot\|_t)_{t\in I}$ *on* $\mathcal{H}_+$ *equivalent to* $\|\cdot\|_{\mathcal{H}_+}$ *such that*:

$$\exists c > 0, \forall \psi \in \mathcal{H}_+\colon\ \|\psi\|_t \leqslant \mathrm{e}^{c|t-s|}\|\psi\|_s \quad \textit{and} \quad \big\|\mathrm{e}^{\mathrm{i}\tau A(t)}\psi\big\|_t \leqslant \mathrm{e}^{c|\tau|}\|\psi\|_t.$$

Then the non-autonomous Cauchy problem (54) *admits a unique unitary propagator* $U(t,s)$.

Moreover, the following estimate holds

$$\forall \psi \in \mathcal{H}_+, \quad \big\|U(t,s)\psi\big\|_t \leqslant \mathrm{e}^{2c|t-s|}\|\psi\|_s.$$

Proof. We follow the same strategy as in [24] and split the proof into three steps. We assume, for reading convenience, that the interval I is of the form $[0,T], T > 0$, however the proof works exactly in the same way for any compact interval. Remark also that there is no restriction if we assume that $\|\cdot\|_{\mathcal{H}_+} = \|\cdot\|_0$.

Propagator approximation: Let $(t_0, \dots, t_n)$ be a regular partition of the interval I with

$$t_j = \frac{jT}{n}, \quad j = 0, \dots, n.$$

Consider the sequence of operator-valued step functions defined by

$$A_n(t) := A(T)1_{\{T\}}(t) + \sum_{j=0}^{n-1} A(t_j)1_{[t_j,t_{j+1}[}(t)$$

for any $n \in \mathbb{N}^*$ and $t \in I$. Assumption (i) ensures that

$$\lim_{n\to\infty} \big\|A_n(t) - A(t)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)} = 0$$

uniformly in $t \in I$. We now construct an approximating unitary propagator $U_n(t,s)$ as follows:

$$\begin{cases} \text{if } t_j \leqslant t,s \leqslant t_{j+1} \text{ then } U_n(t,s) = \mathrm{e}^{-\mathrm{i}(t-s)A(t_j)}, \\ \text{if } t_j < s \leqslant t_{j+1} < \cdots < t_l \leqslant t < t_{l+1} \text{ then } U_n(t,s) = \mathrm{e}^{-\mathrm{i}(t-t_l)A(t_l)} \cdots \mathrm{e}^{-\mathrm{i}(t_{j+1}-s)A(t_j)}, \\ \text{if } t_j < t \leqslant t_{j+1} < \cdots < t_l \leqslant s < t_{l+1} \text{ then } U_n(t,s) = \mathrm{e}^{-\mathrm{i}(t-t_{j+1})A(t_j)} \cdots \mathrm{e}^{-\mathrm{i}(t_l-s)A(t_l)}, \end{cases} \tag{55}$$

for any $j = 0, \dots, n-1$ and $l = 1, \dots, n$ with $j < l$.

By definition, the operators $U_n(t,s)$ are unitary on $\mathcal{H}$ for $t,s \in I$ and satisfy

$$U_n(t,t) = 1, \qquad U_n(t,s)^* = U_n(s,t). \tag{56}$$

Moreover, one can first check that

$$U_n(t,s)U_n(s,r) = U_n(t,r) \quad \text{for } r \leqslant s \leqslant t, \text{ with } t,s,r \in I$$

and then extend it for any $(t,s,r) \in I^3$ with the help of (56). Therefore, $U_n(t,s)$ satisfy the properties (a) and (b) of Definition C.1. Again by (55) and assumptions (i) and (ii) we have

$$\mathrm{i}\partial_t U_n(t,s)\psi = A_n(t)U_n(t,s)\psi \quad \text{and} \quad -\mathrm{i}\partial_s U_n(t,s)\psi = U_n(t,s)A_n(s)\psi \tag{57}$$

for any $\psi \in \mathcal{H}_+$ and any $t,s \neq t_j$, $j = 0,\dots,n$. In fact, we have for $\psi \in \mathcal{H}_+$ as identity in $\mathcal{H}_-$

$$\mathrm{e}^{-\mathrm{i}\tau A(s)}\psi = \psi - \mathrm{i}A(s)\int_0^\tau \mathrm{e}^{-\mathrm{i}rA(s)}\psi\,\mathrm{d}r, \tag{58}$$

since this holds first for $\psi \in \mathcal{D}(A(s)) \cap \mathcal{H}_+$ and then extends by density of $\mathcal{D}(A(s)) \cap \mathcal{H}_+$ in $\mathcal{H}_+$ using the uniform boundedness principal. By (58) we have

$$\left\| \frac{\mathrm{e}^{-\mathrm{i}\tau A(s)}\psi - \psi}{\tau} + \mathrm{i}A(s)\psi \right\|_{\mathcal{H}_-} \leqslant \frac{1}{\tau}\|A(s)\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)} \left| \int_0^\tau \|\mathrm{e}^{-\mathrm{i}rA(s)}\psi - \psi\|_{\mathcal{H}_+}\,\mathrm{d}r \right|$$

and hence using assumption (ii), we show the differentiability of $\tau \mapsto \mathrm{e}^{-\mathrm{i}\tau A(s)}\psi$ for $\psi \in \mathcal{H}_+$.

Convergence of the approximation: Assumption (iii) implies that

$$\|\mathrm{e}^{-\mathrm{i}s_nA(t_n)} \cdots \mathrm{e}^{-\mathrm{i}s_1A(t_1)}\psi\|_T \leqslant \mathrm{e}^{cT}\mathrm{e}^{c(s_1+\cdots+s_n)}\|\psi\|_0,$$

and

$$\|\mathrm{e}^{-\mathrm{i}s_1A(t_1)} \cdots \mathrm{e}^{-\mathrm{i}s_nA(t_n)}\psi\|_0 \leqslant \mathrm{e}^{cT}\mathrm{e}^{c(s_1+\cdots+s_n)}\|\psi\|_T$$

for any $s_j \geqslant 0$, $j = 1,\dots,n$. Hence, using the equivalence of the norms $\|\cdot\|_0 = \|\cdot\|_{\mathcal{H}_+}$ and $\|\cdot\|_T$ one shows the existence of $M > 0$ ($M = \mathrm{e}^{2cT}$) such that

$$\|U_n(t,s)\|_{\mathcal{L}(\mathcal{H}_+)} \leqslant M\mathrm{e}^{c|t-s|} \quad \text{and by duality } \|U_n(t,s)\|_{\mathcal{L}(\mathcal{H}_-)} \leqslant M\mathrm{e}^{c|t-s|}. \tag{59}$$

Furthermore, the same argument above yields

$$\|U_n(t,s)\psi\|_t \leqslant \mathrm{e}^{2c(|t-s|+T/n)}\|\psi\|_s. \tag{60}$$

Using (57) we obtain for any $\psi \in \mathcal{H}_+$

$$\partial_r\big[U_n(t,r)U_m(r,s)\psi\big] = \mathrm{i}U_n(t,r)\big[A_n(r) - A_m(r)\big]U_m(r,s)\psi \tag{61}$$

for $r \neq \frac{jT}{n}, r \neq \frac{jT}{m}$ with $j = 1, \dots, \max(n, m)$. Integrating (61) we get the identity

$$U_m(t,s)\psi - U_n(t,s)\psi = \mathrm{i}\int_s^t U_n(t,r)\big[A_n(r) - A_m(r)\big]U_m(r,s)\psi\,\mathrm{d}r.$$

Now (59) yields

$$\big\|U_m(t,s) - U_n(t,s)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)} \leqslant M^2|t-s|\mathrm{e}^{2c|t-s|}\sup_{r\in I}\big\|A_m(r) - A_n(r)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)}. \tag{62}$$

Therefore, for any $t, s \in I$, the sequence $U_n(t,s)$ converges in norm to a bounded linear operator $U(t,s) \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$. Since $U_n(t,s)$ are norm bounded operators on $\mathcal{H}_-$ uniformly in n, it follows by (59) that they converge strongly to an operator in $\mathcal{L}(\mathcal{H}_-)$ continuously extending $U(t,s)$. Moreover, this strong convergence yields

$$\lim_{n\to\infty}\big(\phi, U_n(t,s)\psi\big)_{\mathcal{H}} = \big(\phi, U(t,s)\psi\big)_{\mathcal{H}} \quad \forall\psi \in \mathcal{H}_+, \forall\phi \in \mathcal{H}_+.$$

Thus, using (59), we obtain

$$\big|(\phi, U(t,s)\psi)_{\mathcal{H}}\big| \leqslant M\mathrm{e}^{c|t-s|}\|\phi\|_{\mathcal{H}_-}\|\psi\|_{\mathcal{H}_+}.$$

Hence, it is easy to see by (53) that

$$\big\|U(t,s)\big\|_{\mathcal{L}(\mathcal{H}_+)} \leqslant M\mathrm{e}^{c|t-s|}.$$

A similar argument yields

$$\big\|U(t,s)\big\|_{\mathcal{L}(\mathcal{H})} \leqslant 1. \tag{63}$$

Now, since $U_n(t,s)$ satisfy part (b) of Definition C.1, we easily conclude that

$$U(t,t) = 1, \qquad U(t,r)U(r,s) = U(t,s), \quad t,s,r \in I, \tag{64}$$

by strong convergence in $\mathcal{L}(\mathcal{H}_-)$. Furthermore, combining (63) and (64) we show the unitarity of $U(t,s)$ on $\mathcal{H}$. Thus, we have proved that $U(t,s)$ satisfy (a) and(b) of Definition C.1.

For any $\psi \in \mathcal{H}_+$, the continuity of the map $I \ni t \mapsto U_n(t,s)\psi \in \mathcal{H}_-$ follows from the definition of $U_n(t,s)$. Now, we prove

$$\lim_{t\to s}\big(\phi, U(t,s)\psi\big)_{\mathcal{H}} = (\phi,\psi)_{\mathcal{H}} \quad \forall\psi \in \mathcal{H}_+, \forall\phi \in \mathcal{H}_-,$$

by applying an $\varepsilon/3$ argument when writing

$$\begin{aligned}\big|\big(\phi, U(t,s)\psi\big)_{\mathcal{H}} - (\phi,\psi)_{\mathcal{H}}\big| &\leqslant \|\phi - \phi_\kappa\|_{\mathcal{H}_-}\big\|U(t,s)\psi\big\|_{\mathcal{H}_+} + \big|\big(\phi_\kappa, \big[U(t,s) - U_n(t,s)\big]\psi\big)_{\mathcal{H}}\big| \\ &\quad + \big|\big(\phi_\kappa, \big[U_n(t,s) - 1\big]\psi\big)_{\mathcal{H}}\big| + \|\phi - \phi_\kappa\|_{\mathcal{H}_-}\|\psi\|_{\mathcal{H}_+},\end{aligned}$$

where $\phi_\kappa \to \phi$ in $\mathcal{H}_-$ and $\phi_\kappa \in \mathcal{H}_+$. Therefore, by the duality $(\mathcal{H}_+)' \simeq \mathcal{H}_-$, we get the weak limit

$$w - \lim_{t\to s} U(t,s) = 1$$

in $\mathcal{L}(\mathcal{H}_+)$. Now, observe that when $t \to s$ we can show by (59) that

$$\limsup_{t\to s} \|U(t,s)\psi\|_{\mathcal{H}_+} \leqslant \|\psi\|_{\mathcal{H}_+}.$$

So, we conclude that

$$\limsup_{t\to s} \|U(t,s)\psi - \psi\|^2_{\mathcal{H}_+} \leqslant \limsup_{t\to s} \big(\|\psi\|^2_{\mathcal{H}_+} + \|U(t,s)\psi\|^2_{\mathcal{H}_+} - 2\,\mathrm{Re}\big(\psi, U(t,s)\psi\big)_{\mathcal{H}_+}\big) = 0.$$

This gives the continuity of $I \ni t \mapsto U(t,s)\psi \in \mathcal{H}_+$ since we have in $\mathcal{H}_+$

$$s - \lim_{t\to r} U(t,s) = s - \lim_{t\to r} U(t,r)U(r,s) = U(r,s).$$

By differentiating $\mathrm{e}^{-\mathrm{i}(t-r)A(s)}U_m(r,s)\psi$ with $\psi \in \mathcal{H}_+$ and then integrating w.r.t. r, we get

$$U_m(t,s)\psi - \mathrm{e}^{-\mathrm{i}(t-s)A(s)}\psi = \mathrm{i}\int_s^t \mathrm{e}^{-\mathrm{i}(t-r)A(s)}\big[A(s) - A_m(r)\big]U_m(r,s)\psi\,\mathrm{d}r.$$

Letting $m \to \infty$ in the latter identity and estimating as in (62), one obtains

$$\big\|U(t,s)\psi - \mathrm{e}^{-\mathrm{i}(t-s)A(s)}\psi\big\|_{\mathcal{H}_-} \leqslant M^2\mathrm{e}^{2c|t-s|}\left|\int_s^t \big\|A(s) - A(r)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)}\,\mathrm{d}r\right|\|\psi\|_{\mathcal{H}_+}.$$

Using the fact that

$$\lim_{t\to s}\frac{1}{|t-s|}\int_s^t \big\|A(s) - A(r)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)}\,\mathrm{d}r = 0 \quad \text{and} \quad \lim_{t\to s}\frac{\mathrm{e}^{-\mathrm{i}(t-s)A(s)}\psi - \psi}{t-s} = -\mathrm{i}A(s)\psi$$

it holds that

$$\lim_{t\to s}\left\|\frac{U(t,s)\psi - \psi}{t-s} + \mathrm{i}A(s)\psi\right\|_{\mathcal{H}_-} = 0.$$

Thus, we obtain with the help of (64)

$$\mathrm{i}\partial_s U(s,r)\psi = \lim_{t\to s}\frac{U(t,s)U(s,r)\psi - U(s,r)\psi}{t-s} = A(s)U(s,r)\psi$$

for any $\psi \in \mathcal{H}_+$ and any $r, s \in I$. Hence we have proved the existence of a unitary propagator $U(t,s)$ for the non-autonomous Cauchy problem (54).

Uniqueness: Suppose that $V(t,s)$ is a unitary propagator for (54). By differentiating $U_n(t,r)V(r,s)\psi$, $\psi \in \mathcal{H}_+$ with respect to r we get

$$V(t,s)\psi - U_n(t,s)\psi = \mathrm{i}\int_s^t U_n(t,r)\big[A_n(r) - A(r)\big]V(r,s)\psi.$$

Using a similar estimate as (62) we obtain

$$\begin{aligned}&\big\|V(t,s)\psi - U_n(t,s)\psi\big\|_{\mathcal{H}_+}\\ &\quad\leqslant M\mathrm{e}^{c|t-s|}\sup_{r\in[s,t]}\big\|V(r,s)\big\|_{\mathcal{L}(\mathcal{H}_+)}\left|\int_s^t \big\|A(r) - A_n(r)\big\|_{\mathcal{L}(\mathcal{H}_+,\mathcal{H}_+)}\,\mathrm{d}r\right|\|\psi\|_{\mathcal{H}_+}\end{aligned}$$

and since the r.h.s. vanishes when $n \to \infty$ we conclude that $V(t,s) = U(t,s)$.

Finally, the uniform boundedness principle, equivalence of norms $\|\cdot\|_t$, $\|\cdot\|_{\mathcal{H}_+}$ and the inequality (60) give us the claimed estimate,

$$\forall\psi \in \mathcal{H}_+,\quad \big\|U(t,s)\psi\big\|_t \leqslant \liminf_{n\to\infty}\big\|U_n(t,s)\psi\big\|_t \leqslant \mathrm{e}^{2c|t-s|}\|\psi\|_s. \qquad \square$$

Remark C.3. It also follows that $(t,s) \mapsto U(t,s) \in \mathcal{L}(\mathcal{H}_+)$ is jointly strongly continuous.

In the following we provide a more effective formulation of the above result (Theorem C.2) which appears as a time-dependent version of the Nelson commutator theorem (see, e.g., [29,30] and Theorem B.2).

We associate to each family of self-adjoint operators $\{S(t)_{t\in I}, S\}$ on $\mathcal{H}$ such that $S \geqslant 1$, $S(t) \geqslant 1$ and $\mathcal{D}(S(t)^{1/2}) = \mathcal{D}(S^{1/2})$ for any $t \in I$, a Hilbert rigging $\mathcal{H}_{\pm 1}$ defined as the completion of $\mathcal{D}(S^{\pm 1/2})$ with respect to the inner product

$$\langle\psi,\phi\rangle_{\mathcal{H}_{\pm 1}} = \big\langle S^{\pm 1/2}\psi, S^{\pm 1/2}\phi\big\rangle_{\mathcal{H}}. \tag{65}$$

Corollary C.4. *Let $I \subset \mathbb{R}$ be a closed interval and let $\{S(t)_{t\in I}, S\}$ be a family of self-adjoint operators on a Hilbert space $\mathcal{H}$ such that*:

- $S \geqslant 1$ *and* $S(t) \geqslant 1$, $\forall t \in I$,
- $\mathcal{D}(S(t)^{1/2}) = \mathcal{D}(S^{1/2})$, $\forall t \in I$, *and consider the associated Hilbert rigging* $\mathcal{H}_{\pm 1}$ *given by* (65).

Let $\{A(t)\}_{t\in I}$ be a family of symmetric bounded operators in $\mathcal{L}(\mathcal{H}_{+1}, \mathcal{H}_{-1})$ satisfying:

- $t \in I \mapsto A(t) \in \mathcal{L}(\mathcal{H}_{+1}, \mathcal{H}_{-1})$ *is norm continuous.*

Assume that there exists a continuous function $f : I \to \mathbb{R}_+$ such that for any $t \in I$, we have:

(i) *for any* $\psi \in \mathcal{D}(S(t)^{1/2})$,

$$\big|\partial_t\langle\psi, S(t)\psi\rangle\big| \leqslant f(t)\big\|S(t)^{1/2}\psi\big\|^2;$$

(ii) *for any* $\Phi, \Psi \in \mathcal{D}(S(t)^{3/2})$,

$$\big|\big\langle S(t)\Psi, A(t)\Phi\big\rangle - \big\langle A(t)\Psi, S(t)\Phi\big\rangle\big| \leqslant f(t)\big\|S(t)^{1/2}\Psi\big\|\big\|S(t)^{1/2}\Phi\big\|.$$

Then the non-autonomous Cauchy problem (54) *admits a unique unitary propagator* $U(t,s)$*. Moreover, we have*

$$\|S(t)^{1/2}U(t,s)\psi\| \leqslant \mathrm{e}^{2|\int_s^t f(\tau)\,\mathrm{d}\tau|}\|S(s)^{1/2}\psi\|.$$

In addition, if we have $c_1, c_2 > 0$ *such that* $c_1 S \leqslant S(t) \leqslant c_2 S$ *for* $t \in I$*, then there exists* $c > 0$ *such that*

$$\|U(t,s)\|_{\mathcal{L}(\mathcal{H}_{+1})} \leqslant c\mathrm{e}^{2|\int_s^t f(\tau)\,\mathrm{d}\tau|}, \quad \forall t \in I. \tag{66}$$

Proof. First observe that the operator $A(t)$ satisfies the hypothesis of Nelson's commutator theorem (Theorem B.2) for any $t \in I$. Hence, we conclude that $A(t)$ is essentially self-adjoint on $\mathcal{D}(S(t)^{3/2})$ which is dense in $\mathcal{H}_{+1}$. We keep the same notation for its closure. Moreover, the unitary group $\mathrm{e}^{\mathrm{i}\tau A(t)}$ preserves $\mathcal{H}_{+1}$ and we have the estimate

$$\|S(t)^{1/2}\mathrm{e}^{\mathrm{i}\tau A(t)}\psi\|_{\mathcal{H}} \leqslant \mathrm{e}^{f(t)|\tau|}\|\psi\|_{\mathcal{H}}. \tag{67}$$

Now, observe that $t \mapsto \mathrm{e}^{-\mathrm{i}tA(s)}\psi \in \mathcal{H}_{+1}$ is weakly continuous for any $\psi \in \mathcal{H}_+$. This holds using a $\eta/3$-argument with the help of the estimate

$$|\langle f, (\mathrm{e}^{-\mathrm{i}tA(s)} - 1)\psi\rangle| \leqslant (1 + \mathrm{e}^{c(|t|+1)})\|f - f_\kappa\|_{\mathcal{H}_{-1}}\|\psi\|_{\mathcal{H}_{+1}} + |\langle(\mathrm{e}^{\mathrm{i}tA(s)} - 1)f_\kappa, \psi\rangle|,$$

where $f_\kappa \in \mathcal{H}$ is a sequence convergent to f in $\mathcal{H}_{-1}$ and t is near 0. Since strong and weak continuity of the group of bounded operators $\mathrm{e}^{-\mathrm{i}tA(s)}$ in $\mathcal{L}(\mathcal{H}_{+1})$ are equivalent, we conclude that assumption (ii) of Theorem C.2 holds true.

By assumption (ii), we also have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)^{1/2}\psi\|^2\right| \leqslant f(t)\|S(t)^{1/2}\psi\|^2.$$

Hence, by Gronwall's inequality we have

$$\|S(t)^{1/2}\psi\|^2 \leqslant \mathrm{e}^{|\int_s^t f(\tau)\,\mathrm{d}\tau|}\|S(s)^{1/2}\psi\|^2, \quad \forall t, s \in I. \tag{68}$$

Now, we use Theorem C.2 with the Hilbert rigging

$$\mathcal{H}_+ = \mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_- = \mathcal{H}_{-1}$$

and the family of equivalent norms on $\mathcal{H}_+$ given by

$$\|\psi\|_t := \|S(t)^{1/2}\psi\|_{\mathcal{H}}.$$

Indeed, assumptions (i)–(iii) of Theorem C.2 are satisfied in any compact subinterval of I with the help of (67) and (68). Therefore, we obtain existence and uniqueness of a unitary propagator $U(t,s)$ of the Cauchy problem (54) in the whole interval I with the following estimate

$$\|U(t,s)\psi\|_t \leqslant \mathrm{e}^{2|t-s|\max_{\tau\in\Delta(t,s)} f(\tau)}\|\psi\|_s$$

for any $t, s \in I$ and where $\Delta(t, s)$ stands for the interval of extremities t, s.

Using the multiplication law of the propagator, we obtain for any partition $(t_0, \ldots, t_n)$ of the interval $\Delta(t, s)$ the inequality

$$\|U(t,s)\psi\|_t \leqslant \prod_{j=0}^{n-1} \mathrm{e}^{2(|t-s|/n)\max_{\tau\in\Delta_j} f(\tau)} \|\psi\|_s,$$

where Δ_j are the subintervals $[t_j, t_{j+1}]$. Since f is continuous, by letting $n \to \infty$, we get

$$\|U(t,s)\psi\|_t \leqslant \mathrm{e}^{2|\int_s^t f(\tau)\,\mathrm{d}\tau|} \|\psi\|_s.$$

Finally, the assumption $c_1 S \leqslant S(t) \leqslant c_2 S$ for $t \in I$, allows to involve the norm $\|\cdot\|_{\mathcal{H}_{+1}}$. Thus we have

$$\|U(t,s)\psi\|_{\mathcal{H}_{+1}} \leqslant \frac{1}{\sqrt{c_1}}\|U(t,s)\psi\|_t \leqslant \frac{1}{\sqrt{c_1}}\mathrm{e}^{2|\int_s^t f(\tau)\,\mathrm{d}\tau|}\|\psi\|_s \leqslant \sqrt{\frac{c_2}{c_1}}\mathrm{e}^{2|\int_s^t f(\tau)\,\mathrm{d}\tau|}\|\psi\|_{\mathcal{H}_{+1}}. \qquad \square$$

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