EX.4 :

(i) $y'' + 4y = 8x\,(2\cos(2x) - \sin(2x)) = e^{mx}\,(A(x)\cos(\omega x) + B(x)\sin(\omega x))$

Equ. caractéristique : $\lambda^2 + 4 = 0 \iff \lambda^2 = -4 \iff \lambda = \pm 2i$

$m = 0$ $A(x)$ et $B(x)$ polynômes de deg. 1

$\omega = 2$ et $i\omega + m = 2i$ racine simple de l'equ. caractéristique.

on cherche une solution particulière de la forme :

$$y(x) = x\,(\alpha(x)\cos(2x) + \beta(x)\sin(2x)) \quad \text{où} \quad \deg\alpha = \deg\beta = 1.$$

$\alpha(x) = ax + b$

$\beta(x) = cx + d.$

$$y(x) = x\,(ax+b)\cos(2x) + x\,(cx+d)\sin(2x)$$

$$y'(x) = (2ax+b)\cos(2x) + (2cx+d)\sin(2x)$$
$$- 2(ax^2+bx)\sin(2x) + 2(cx^2+dx)\cos(2x)$$

~~$y'(x) = (-2ax^2 + 2(a-b)x + b)$~~

$$y'(x) = (2cx^2 + 2dx + 2ax + b)\cos(2x) + (-2ax^2 - 2bx + 2cx + d)\sin(2x)$$

$$= (2cx^2 + 2(d+a)x + b)\cos(2x) + (-2ax^2 + 2(c-b)x + d)\sin(2x)$$

$$y''(x) = (4cx + 2(d+a))\cos(2x) + (-4ax^2 + 4(c-b)x + 2d)\cos(2x)$$

$$+ (-4ax + 2(c-b))\sin(2x) + (-4cx^2 - 4(d+a)x - 2b)\sin(2x)$$

$$y''(x) = (-4ax^2 + (8c-4b)x + 4d + 2a)\cos(2x) \qquad \times 1$$

$$+ (-4cx^2 + (-4d-8a)x + 2c - 4b)\sin(2x). \qquad +$$

$$y(x) = (ax^2+bx)\cos(2x) + (cx^2+d)\sin(2x). \qquad \times 4$$

$$y'' + 4y = (8cx + 4d + 2a)\cos(2x) + (-8ax + 2c - 4b)\sin(2x) = 16x\cos(2x) - 8x\sin(2x)$$

$$8c = 16 \;;\; 4d + 2a = 0 \;;\; -8a = -8 \;;\; 2c - 4b = 0$$

$\boxed{c=2}$; $\boxed{a=1}$; $4d = -2a = -2 \Rightarrow \boxed{d = -\frac{1}{2}}$

$2c - 4b = 0 \Rightarrow \boxed{b=1}$

$$\boxed{y_p(x) = x\,(x+1)\cos(2x) + x\left(2x - \tfrac{1}{2}\right)\sin(2x)}$$

(ii) $y'' + 4y = 9x\,(\sin(x) - 2\cos(x)) = e^{mx}\left(A(x)\sin(x) + B(x)\cos(x)\right)$

$m = 0$; $\deg A = \deg B = 1$; $\underline{\omega = 1}$

Equ. caractéristique : $\lambda = \pm 2i$

$i = m + i\omega \neq 2i$.

donc on cherche $y_p(x) = \left(\alpha(x)\sin(x) + \beta(x)\cos(x)\right)$; $\alpha(x) = ax + b$, $\beta(x) = cx + d$.

$$y_p'(x) = a\sin(x) + (ax+b)\cos(x)$$
$$c\cos(x) - (cx+d)\sin(x)$$

$$y_p'(x) = (-cx - d + a)\sin(x) + (ax + b + c)\cos(x)$$

$$y_p''(x) = -c\sin(x) + (-cx - d + a)\cos(x)$$
$$a\cos(x) - (ax + b + c)\sin(x)$$

$$y_p''(x) = (-ax - b - 2c)\sin(x) + (-cx - d + 2a)\cos(x) \quad \times 1$$

$$y_p(x) = (ax + b)\sin(x) + (cx + d)\cos(x) \quad \times 4$$

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$$y_p'' + 4y_p = (3ax + 3b - 2c)\sin(x) + (3cx + 3d + 2a)\cos(x) = 9x\sin(x) - 18x\cos(x)$$

$3a = 9$; $3b - 2c = 0$; $3c = -18$; $3d + 2a = 0$

$\boxed{a=3}$; $\boxed{c=-6}$; $3b = -12 \Rightarrow \boxed{b=-4}$; $3d = -2a = -6 \Rightarrow \boxed{d=-2}$

$$\boxed{y_p(x) = (3x - 4)\sin(x) + (-6x - 2)\cos(2x).}$$