Contents

Geometric Small Cancellation

Vincent Guirardel

IAS/Park City Mathematics Series
Volume 21, 2012

Geometric Small Cancellation

Vincent Guirardel

Introduction

The aim of these lectures is to present *geometric small cancellation*, also known as *very small cancellation*, introduced by Gromov in [**Gro01b, Gro01a**], and further developed by Gromov-Delzant, Arzhantseva-Delzant, Coulon, and Dahmani-Guirardel-Osin [**DG08, AD, Cou11, DGO**].

Starting from a group G acting on a hyperbolic space, together with a family of subgroups satisfying a small cancellation condition, this theory studies the quotient of G by the normal subgroup generated by the given subgroups. Applications of the theory include the construction of *monsters* (i.e. groups with pathological properties), by taking iterated small cancellation quotients. The Dehn filling theory of relatively hyperbolic groups can also be understood from this framework. Beyond hyperbolic and relatively hyperbolic groups, this small cancellation theory has applications concerning groups having nice actions on hyperbolic spaces such as the mapping class group of a surface, the outer automorphism group of a free group, or the group of birational transformations of the projective plane.

In the first lecture, we start by discussing a classical small cancellation condition, applications of small cancellation, and then state the geometric small cancellation theorem.

In the second lecture, we discuss weak proper discontinuity as a way to produce small cancellation subgroups, and in particular, we present an application to SQ-universality.

The next two lectures are devoted to the proof of the geometric small cancellation theorem. In this proof, one produces a suitable hyperbolic space by a *cone-off* construction, and one describes the normal subgroup generated by a small cancellation family via its action on this cone-off space. Lecture three is about this description of the normal subgroup via the theory of *very rotating families* on the

---

Institut de Recherche Mathématique de Rennes et Institut Universitaire de France
Université de Rennes 1 et CNRS (UMR 6625)
263 avenue du Général Leclerc, CS 74205
F-35042 RENNES Cédex
France.
http://perso.univ-rennes1.fr/vincent.guirardel

E-mail address: vincent.guirardel@univ-rennes1.fr
I would like to warmly thank Rémi Coulon. First for his help preparing the lectures (the material about the cone-off follows his own work), and then for his very useful comments on the text that helped a lot improve the exposition. I acknowledge support from the Institut Universitaire de France and ANR grant ANR-11-BS01-013

hyperbolic cone-off. Lecture four describes the construction and the properties of the hyperbolic cone-off.

LECTURE 1

What is small cancellation about?

1. The basic setting

The basic problem tackled by small cancellation theory is the following one.

Problem. *Let G be a group, and $R_1, \dots, R_n$ some subgroups of G. Give conditions under which you understand the normal subgroup $\langle\langle R_1, \dots, R_n\rangle\rangle \lhd G$ and the quotient $G/\langle\langle R_1, \dots, R_n\rangle\rangle$.*

In combinatorial group theory, there are various notions of small cancellation conditions for a finite presentation $\langle S|r_1, \dots, r_k\rangle$. In this case, G is the free group $\langle S\rangle$, and R_i is the cyclic group $\langle r_i\rangle$. Essentially, these conditions ask that any common subword between two relators has to be short compared to the length of the relators.

More precisely, a *piece* is a word u such that there exist cyclic conjugates $\tilde{r}_1, \tilde{r}_2$ of relators r_{i_1}, r_{i_2} ($i_1 = i_2$ is allowed) such that $\tilde{r}_i = ub_i$ (as concatenation of words) with $b_1 \neq b_2$. Then the $C'(1/6)$ small cancellation condition asks that in this situation, $|u| < \frac{1}{6}|r_1|$ and $|u| < \frac{1}{6}|r_2|$. One can replace $\frac{1}{6}$ by any $\lambda < 1$ to define the $C'(\lambda)$ condition.

Then small cancellation theory says, among other things, that the group $\langle S|r_1, \dots, r_k\rangle$ is a hyperbolic group, that it is torsion-free if no relator is a proper power. Moreover, when G is torsion-free, the 2-complex defined by the presentation is aspherical (meaning in some sense that there are no relations among relations), and in particular G is 2-dimensional.

There are many variants and generalizations of this condition. Building on Max Dehn's work on surface groups, this started in the 50's with the work of Tartakovskii, Greendlinger, and continued with Lyndon, Schupp, Rips, Olshanskii, and many others [**Tar49, Gre60, LS01, Ol'91a, Rip82**]. Small cancellation theory was generalized to hyperbolic and relatively hyperbolic groups by Olshanskii, Delzant, Champetier, and Osin [**Ol'91b, Del96, Cha94, Osi10**]. An important variant is Gromov's *graphical* small cancellation condition, where the presentation is given by killing the loops of a labelled graph, and one asks for pieces in this graph to be small [**Gro03**]. This lecture will be about *geometric* small cancellation (or *very small* cancellation) introduced by Gromov in [**Gro01a**], and further developed by Gromov-Delzant, Arzhantseva-Delzant, Coulon, and Dahmani-Guirardel-Osin [**DG08, AD, Cou11, DGO**].

There are other very interesting small cancellation theories, in particular, Wise's small cancellation theory for special cube complex [**Wis11**].

2. Applications of small cancellation

Small cancellation is a large source of examples of groups (the following list is very far from being exhaustive !).

Interesting hyperbolic groups

The Rips construction allows us to produce hyperbolic groups (in fact small cancellation groups) that map onto any given finitely presented group with finitely generated kernel. This allows us to encode many pathologies of finitely presented groups into hyperbolic groups. For instance, there are hyperbolic groups having a finitely generated subgroup whose membership problem is not solvable [**Rip82**]. There are many useful variants of this elegant construction, see for instance [**BO08, BW05, OW07, Wis03**].

Dehn fillings

Given a relatively hyperbolic group with respect to P, and $N \lhd P$ a normal subgroup, then if N is *deep enough* (i.e. avoids a finite subset $F \subset P \setminus \{1\}$ given in advance), then P/N embeds in $G/\langle\langle N\rangle\rangle$, and $G/\langle\langle N\rangle\rangle$ is relatively hyperbolic with respect to P/N [**GM08, Osi07**].

Normal subgroups

Small cancellation allows us to understand the structure of the corresponding normal subgroup. For instance, Delzant shows that for any hyperbolic group G there exists n such that for any hyperbolic element $h \in G$, the normal subgroup generated by $\langle h^n\rangle$ is free [**Del96**]. This is because $\langle h^n\rangle$ is a subgroup satisfying a small cancellation condition (see below). The same idea shows that if $h \in MCG$ is a pseudo-Anosov element of the mapping class group (or a fully irreducible automorphism of a free group), then for some $n \geq 1$, the normal subgroup generated by $\langle h^n\rangle$ is free and purely pseudo-Anosov [**DGO**]. This uses the fact that MCG acts on the curve complex, which is a hyperbolic space [**MM99**], and that $\langle h^n\rangle$ is a small cancellation subgroup when acting on the curve complex. Similar arguments work in the outer automorphism group of a free group $\mathrm{Out}(F_r)$ and in the Cremona group $Bir(\mathbb{P}^2\mathbb{C})$ because they have a nice action on hyperbolic space [**BF10, Can11, CL**].

Many quotients

Small cancellation theory allows us to produce *many* quotients of any non-elementary hyperbolic group G: it is SQ-universal [**Del96, Ol′95**]. This means that for any countable group A there exists a quotient $G \twoheadrightarrow Q$ in which A embeds (in particular, G has uncountably many non-isomorphic quotients). Small cancellation theory also allows us to prove SQ universality of Mapping Class Groups, $\mathrm{Out}(F_n)$, and the Cremona group $Bir(\mathbb{P}^2)$ [**DGO**]. More generally, this applies to groups with *hyperbolically embedded subgroups* [**DGO**] (we will not discuss this notion in this lecture, only the existence of hyperbolic elements with the WPD property, see Section 2). Abundance of quotients makes it difficult for a group with few quotients to embed in such a group. This idea can be used to prove that lattices in higher rank Lie groups don't embed in mapping class groups, or $\mathrm{Out}(F_n)$ [**DGO, BW11**], the original proof for mapping class group is due to Kaimanovich-Masur [**KM96**].

Monsters

The following monsters are (or can be) produced as limits of infinite chains of small cancellation quotients:

(1) Infinite Burnside groups. For n large enough, $r \geq 2$, the free Burnside group $B(r, n) = \langle s_1, \dots, s_r | \forall w, w^n = 1\rangle$ is infinite [**NA68, Iva94, Lys96, Ol′82, DG08**], see also the notes by Rémi Coulon [**Cou**].

(2) Tarski monster. For each prime p large enough, there is an infinite, finitely generated group all whose proper subgroups are cyclic of order p [**Ol′80**].
(3) Osin's monster. There is a finitely generated group not isomorphic to $\mathbb{Z}/2\mathbb{Z}$, such that all its non-trivial elements are conjugate [**Osi10**].
(4) Gromov's monster. This is a finitely generated group that contains a uniformly embedded expander, and which therefore does not uniformly embed in a Hilbert space [**Gro03, AD**]. This gives a counterexample to the strong form of the Baum-Connes conjecture [**HLS02**].

3. Geometric small cancellation

The goal of this lecture is to describe *geometric small cancellation*, introduced by Gromov in [**Gro01b, Gro01a**]. We give some preliminary definitions before stating the results.

3.1. Preliminaries and notations

A metric space X is *δ-hyperbolic* if it is geodesic, and if it satisfies the δ-hyperbolic 4-point inequality: for all $x, y, z, t \in X$,

$$d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\} + 2\delta.$$

This implies that for any geodesic triangle, any side is contained in the 4δ-neighbourhood of the two other sides. We denote by $[x, y]$ a geodesic between x and y; although there is no uniqueness of geodesics, this usually does not lead to confusion. An $\mathbb{R}$-tree is a 0-hyperbolic space.

We denote by $\delta_{\mathbb{H}^2}$ the hyperbolicity constant of the hyperbolic plane $\mathbb{H}^2$. A geodesic metric space is $CAT(-1)$ if its triangles are thinner than comparison triangles in $\mathbb{H}^2$ (see [**BH99**] for details). Such a space is $\delta_{\mathbb{H}^2}$-hyperbolic.

Given a subset Q of a hyperbolic space X and $r \geq 0$, we denote by Q^{+r} its r-neighbourhood. We say that Q is *almost convex* if for all $x, y \in Q$, there exist $x', y' \in Q$ and geodesics $[x, x'], [x', y'], [y', y]$ such that $d(x, x') \leq 8\delta$, $d(y, y') \leq 8\delta$, and $[x, x'] \cup [x', y'] \cup [y', y] \subset Q$. It follows that the path metric d_Q on Q induced by the metric d_X of X is close to d_X: for all $x, y \in Q$, $d_X(x, y) \leq d_Q(x, y) \leq d_X(x, y) + 32\delta$.

Recall that $Q \subset X$ is *K-quasiconvex* if for all $x, y \in Q$, any geodesic $[x, y]$ is contained in Q^{+K}. This notion is weaker as it does not say anything about d_Q (Q might even be disconnected). However, if Q is K-quasiconvex, then for all $r \geq K$, Q^{+r} is almost convex. Also note that δ-hyperbolicity implies that an almost convex subset is 8δ-quasiconvex.

3.2. Moving families and the geometric small cancellation

Let X be a δ-hyperbolic space, and G be a group acting on X by isometries. Consider $\mathcal{Q} = (Q_i)_{i \in I}$ a family of *almost convex* subspaces of X, and $\mathcal{R} = (R_i)_{i \in I}$ a corresponding family of subgroups such that R_i is a normal subgroup of the stabilizer of Q_i. This data should be G-invariant: G acts on I so that $Q_{gi} = gQ_i$, and $R_{gi} = gR_ig^{-1}$. Let us call such data a *moving family* $\mathcal{F}$.

We now define the *injectivity radius* and the *fellow traveling constant* of a moving family. The small cancellation hypothesis defined below will ask for a large injectivity radius and a small fellow traveling constant.

The injectivity radius measures the minimal displacement of all non-trivial elements of all R_i's:

$$\mathrm{inj}(\mathcal{F}) = \inf\{d(x,gx) | i \in I, x \in Q_i, g \in R_i \setminus \{1\}\}.$$

Here the infimum is taken only over $x \in Q_i$, but since Q_i is almost convex and R_i-invariant, the definition would not change much if we took the infimum over all $x \in X$.

The fellow traveling constant between two subspaces Q_i, Q_j measures how long they remain at a bounded distance from each other. Technically,

$$\Delta(Q_i, Q_j) = \mathrm{diam}\left(Q_i^{+20\delta} \cap Q_j^{+20\delta}\right).$$

Because Q_i, Q_j are almost convex in a hyperbolic space, any point of Q_i that is far from $Q_i^{+20\delta} \cap Q_j^{+20\delta}$ is far from Q_j, so this really measures what we want. The fellow traveling constant of $\mathcal{F}$ is defined by

$$\Delta(\mathcal{F}) = \sup_{i \neq j} \Delta(Q_i, Q_j).$$

Definition 1.1. *Assume that X is δ-hyperbolic, with $\delta > 0$. The moving family $\mathcal{F}$ satisfies the (A,λ)-small cancellation condition if it satisfies*

(1) *large injectivity radius:* $\mathrm{inj}(\mathcal{F}) \geq A\delta$, *and*
(2) *small fellow traveling compared to injectivity radius:* $\Delta(\mathcal{F}) \leq \lambda\, \mathrm{inj}(\mathcal{F})$.

Remark 1.2.

- It is convenient to say that some subgroup $R < G$ satisfies the (A,λ)-small cancellation condition if the family $\mathcal{R}$ of all conjugates of R together with a suitable family of subspaces of X, makes a small cancellation moving family.
- The (A,λ)-small cancellation hypothesis (for A large enough) implies that each R_i is torsion-free, because every element of $R_i \setminus \{1\}$ is hyperbolic.
- It is often convenient to take $I = \mathcal{Q}$, and to view $\mathcal{R}$ as a group attached to each subspace in $\mathcal{Q}$: $\mathcal{R} = (R_Q)_{Q\in\mathcal{Q}}$, or conversely, to take $I = \mathcal{R}$ and to view $\mathcal{Q}$ as a space attached to each group in $\mathcal{R}$: $\mathcal{Q} = (Q_H)_{H\in\mathcal{R}}$.
- We don't assume any properness on X, and no finiteness on I/G.

Relation with classical small cancellation

The small cancellation hypothesis (almost) covers the classical small cancellation condition $C'(\lambda)$ in the following way. The group G is the free group, acting on its Cayley graph X, $(R_i)_{i\in I}$ is the family of cyclic groups generated by the conjugates of the relators, and $(Q_i)_{i\in I}$ is the family of their axes. In this context, the injectivity radius is the length of the smallest relation, and the fellow traveling constant is the length of the largest piece between relators. The large injectivity radius assumption is empty because the Cayley graph of the free group is δ-hyperbolic for any $\delta > 0$. The small fellow traveling constant assumption is (a strengthening of) the $C'(\lambda)$ small cancellation assumption. However, contrary to classical small cancellation, the constants A_0, λ_0 in the small cancellation theorem below are not explicit and far from optimal.

Graphical small cancellation also fits in this context. In this case, the groups R_i's need not be cyclic any more, they are conjugates of the subgroups of G defined by some labelled subgraphs.

The small cancellation theorem

Theorem 1.3 (Small cancellation theorem)**.** *There exist A_0, λ_0 such that if $\mathcal{F}$ satisfies the (A_0,λ_0)-small cancellation hypothesis then*

(1) $\langle\langle R_i | i \in I \rangle\rangle$ *is a free product of a subfamily of the R_i's,*
(2) $\mathrm{Stab}(Q_i)/R_i$ *embeds in* $G/\langle\langle R_i | i \in I\rangle\rangle$
(3) *small elements survive: for every $C > 0$, there exists A_C, λ_c such that if $\mathcal{F}$ satisfies the (A_C, λ_c)-small cancellation condition, then any non-trivial element whose translation length is at most $C\delta$ is not killed in $G/\langle\langle R_i\rangle\rangle$.*
(4) $G/\langle\langle R_i | i \in I\rangle\rangle$ *acts on a suitable hyperbolic space.*

Remarks. In the setting of $C'(1/6)$ small cancellation, the groups R_i are conjugates of the cyclic groups generated by relators. Thus, if Q_i is the axis of some conjugate r of a relator, then $\mathrm{Stab}(Q_i)$ is the maximal cyclic subgroup containing r. In particular, $\mathrm{Stab}(Q_i)/R_i$ is trivial if r is not a proper power, and $\mathrm{Stab}(Q_i)/R_i \simeq \mathbb{Z}/k\mathbb{Z}$ if $r = u^k$ for some u that is not a proper power.

In (3), one can even prove that elements of translation length at most $\mathrm{inj}(\mathcal{F})(1-\max\{C_1\lambda, \frac{C_2}{A}\})$ are not killed.

It is difficult to state right now the properties of the *suitable hyperbolic space* $\overline{X}$ in (4). One of the main goals of these lectures is to describe this space $\overline{X}$. One can still say that one of its main properties is that $\overline{X}$ has a controlled geometry, including a controlled hyperbolicity constant. However, one can say more assuming that our initial space X is proper, and that the action of G is proper and cocompact (so that G is a hyperbolic group). If each $\mathrm{Stab}(Q_i)/R_i$ is finite, and I/G is finite, then $\overline{X}$ is also proper with a proper cocompact action of $G/\langle\langle R_i | i \in I\rangle\rangle$ so $G/\langle\langle R_i | i \in I\rangle\rangle$ is also a hyperbolic group.

LECTURE 2

Applying the small cancellation theorem

Assume that we have a group G acting on a space X. We are going to see how to produce small cancellation moving families, and how to use them.

1. When the theorem does not apply

Given a group G acting on a hyperbolic space, small cancellation families may very well not exist, except for trivial ones.

A first type of silly example is the solvable Baumslag-Solitar group $BS(1, n) = \langle a, t | tat^{-1} = a^n \rangle$, $n > 1$. This group acts on the Bass-Serre tree of the underlying HNN extension, but there is no small cancellation family.

Exercise 2.1. *Prove this assertion. Note that any two hyperbolic elements of $BS(1, n)$ share a half axis.*

Since we think of small cancellation families as a way to produce quotients, one major obstruction to the existence of such families occurs if G has very few quotients, for instance if it is simple. This is the case for the simple group $G = \mathrm{Isom}^+(\mathbb{H}^n)$ for example. If we restrict ourselves to finitely generated groups, an irreducible lattice in $\mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{Isom}^+(\mathbb{H}^2)$ acts on $\mathbb{H}^2$ (in two ways), but any non-trivial quotient is finite by the Margulis normal subgroup theorem [**Mar91**]. Similar, but more sophisticated examples include Burger-Mozes simple group [**BM00**], a lattice in the product of two trees viewed as a group acting on one of these two trees, or some Kac-Moody groups when the twin buildings are hyperbolic [**CR09**].

Exercise 2.2. *What are* trivial *small cancellation families? Here are examples:*

(1) *The empty family.*
(2) *Take $\mathcal{Q} = \{X\}$ consisting of the single subspace X, and $\mathcal{R} = \{N\}$ consists of a single normal subgroup of G, (including the case $N = \{1\}$ and $N = G$).*
(3) *Another way is to take $\mathcal{Q}$ a G-invariant family of subspaces that satisfy the fellow traveling condition (for instance bounded subspaces), and take $(R_Q)_{Q\in\mathcal{Q}}$ a copy of the trivial group for each subspace.*

More generally, a trivial small cancellation family is a family such that $R_i = \{1\}$ except for at most one index i.

Prove that if G is simple, then there exists A, λ such that any (A, λ)-small cancellation moving family is trivial in the above sense.

Hint: Consider a small cancellation moving family $(Q_i)_{i\in I}$, $(R_i)_{i\in I}$. Since G is simple, $R_i = \mathrm{Stab}(Q_i)$ by the small cancellation Theorem. If $h_1 \in R_{i_1} \setminus \{1\}$, $h_2 \in R_{i_2}$ for $i_1 \neq i_2$, prove that $h_1^N h_2^N$ satisfies the WPD property below, contradicting that G is simple.

2. Weak proper discontinuity

In hyperbolic groups, the easiest small cancellation family consists of the conjugates of a suitable power of a hyperbolic element. The proof is based on the properness of the action. In fact, a weaker notion, due to Bestvina-Fujiwara is sufficient.

Preliminaries about quasi-axes

Here we discuss the notion quasi-axes for hyperbolic elements $g \in G$. This is a g-invariant almost convex subset of X, that is quasi-isometric to $\mathbb{R}$, with constants depending only on δ. One could define such a quasi-axis in terms of the boundary at infinity of X, but because we don't assume properness of X, we prefer avoiding this.

To make many statements simpler, we will always assume that X is a metric graph, all of whose edges have the same length.

Define $[g] = \inf\{d(x, gx) | x \in X\}$ the translation length of g. Recall that g is *hyperbolic* if the orbit map $\mathbb{Z} \to X$ defined by $i \mapsto g^i x$ is a quasi-isometric embedding (for some x, equivalently for any x). This occurs if and only the stable norm of g, defined as $\|g\| = \lim_{i\to\infty} \frac{1}{i} d(x, g^i x)$ is not zero (the limit exists by subadditivity, and does not depend on x).

These are closely related as $[g] - 16\delta \leq \|g\| \leq [g]$ [**CDP90**, 10.6.4]. In particular, if $[g] > 16\delta$ then g is hyperbolic.

Consider a hyperbolic element g. Define the characteristic set of g as $C_g = \{x | d(x, gx) = [g]\}$ (a non-empty set since X is a graph). We want to say that if $[g]$ is large enough, C_g is close to being a bi-infinite line (with constants independent of g). Given $x \in C_g$, consider the bi-infinite path $l = l_{x,g} = \cup_{i\in\mathbb{Z}} [g^i x, g^{i+1} x]$. One easily checks that if $y \in l$, then $d(y, gy) = [g]$, so l is contained in C_g. Moreover, l a local geodesic: any subsegment of length $[g]$ is geodesic. By stability of 100δ-local geodesics [**BH99**, Th 1.13 p.405], there exists a constant C depending only on δ such that if $[g] \geq 100\delta$, $l_{x,g}$ and $l_{y,g}$ are at Hausdorff distance at most C. Similar arguments show that if $[g] \geq 100\delta$, for any k, C_g and C_{g^k} are at Hausdorff distance at most C for some constant C depending only on δ.

In this sense, if $[g] \geq 100\delta$, C_g is a good quasi-axis for g. If g is hyperbolic with $[g] \leq 100\delta$, then there is k such $[g^k] \geq 100\delta$, and a better quasi-axis for g would be C_{g^k} (note that it is g-invariant). Finally, we want the quasi-axis to be almost convex. One easily checks that C_{g^k} is $2C + 4\delta$-quasiconvex. Thus, we define the quasi-axis of g as $A_g = C_{g^k}^{+2C+4\delta}$ where k is the smallest power of g such that $[g^k] \geq 100\delta$.

Lemma 2.3. *There exists a constant C such that for all hyperbolic isometry g, for all $x \in A_g$ and all $i \in \mathbb{Z}$, $i\|g\| \leq d(x, g^i x) \leq i\|g\| + C$.*

This follows from the fact that the quasi-axes A_g and A_{g^i} are at bounded Hausdorff distance, and from the inequality $[g^i] - 16\delta \leq \|g^i\| = i\|g\| \leq [g^i]$.

Weak proper discontinuity

Definition 2.4. *We say that $g \in G$, acting hyperbolically on X, satisfies the WPD property (for* weak proper discontinuity*) if there exists r_0 such that for every pair of points $x, y \in A_g$ at distance at least r_0, the set of all elements $a \in G$ that move both x and y by at most 100δ is finite:*

$$\#\{a \in G | d(x, ax) \leq 100\delta, d(y, ay) \leq 100\delta\} < \infty.$$

Obviously, if the action of G on X is proper, then any hyperbolic element g satisfies the WPD property. In particular, any element of infinite order in a hyperbolic group satisfies the WPD property.

Here is an equivalent definition:

Definition 2.5. *g satisfies the WPD property if for all l, there exists r_l such that for every pair of points $x, y \in A_g$ at distance at least r_l, the set of all elements $a \in G$ that move both x and y by at most l is finite:*

$$\#\{a \in G | d(x, ax) \leq l, d(y, ay) \leq l\} < \infty.$$

Exercise 2.6. *Prove that the definitions are equivalent.*

A lot of interesting groups have such elements.

Example 2.7.

(1) If G is hyperbolic or relatively hyperbolic, then any hyperbolic element satisfies the WPD property for the action of G on its Cayley graph if G is hyperbolic, or Bowditch's space with horoballs.
(2) If G is a non-cyclic right-angled Artin group that is not a direct product, then G acts on a tree in which there is an element satisfying the WPD property (see [**DGO**, cor. 6.50]).
(3) If G is the mapping class group of a surface (of large enough complexity) acting on its curve complex, any pseudo-anosov element is a hyperbolic element satisfying the WPD property [**BF07**].
(4) If $G = \mathrm{Out}(F_n)$, or G is the Cremona group $Bir(\mathbb{P}^2)$, then G acts on a hyperbolic space with an element satisfying the WPD property [**BF10, CL**].
(5) If G acts properly on a proper $CAT(0)$ space Y, and if g is a rank one hyperbolic element (its axis does not bound a half plane), there is an element satisfying the WPD property for some action of G on some hyperbolic space (see [**Sis11**], based on [**BBF10**]).

Proposition 2.8. *Assume that g satisfies the WPD property. Then for all A, λ, there exists N such that the moving family consisting of the conjugates of $\langle g^N \rangle$, together with their quasi-axes, satisfies the (A, λ)-small cancellation condition.*

Corollary 2.9. *If G contains a hyperbolic element with the WPD property, then G is not simple.*

Exercise 2.10. *Prove the proposition.*

Hints: First prove that there exists a constant Δ such that if A_g fellow travels with $A_{hgh^{-1}} = hA_g$ on a distance at least Δ, then hA_g is at finite Hausdorff distance from A_g. For this, show that if the fellow-traveling distance $\Delta(A_g, A_{hgh^{-1}})$ is large, there is a large portion of A_g that is moved by bounded amount by $g^i.hg^{\pm i}h^{-1}$ for many i's. Then apply the WPD property to deduce that h commutes with some power of g, hence maps A_g at finite Hausdorff distance. To conclude that this gives a small cancellation family, prove that the subgroup $E(g) = \{h \in G | d_H(h.A_g, A_g) < \infty\}$ is virtually cyclic (where d_H denotes the Hausdorff distance). Now take N such that $[g^N]$ is large compared to $\Delta(A_g, A_{hgh^{-1}})$, and such that $\langle g^N \rangle \lhd E(g)$ (recall that the definition of a moving family requires the group R_i to be normal in the stabilizer of the corresponding space Q_i).

Remark 2.11 (Remark about torsion)**.** If G is not torsion-free, choosing N such that $[g^N]$ is large compared to $\Delta(A_g, A_{hgh^{-1}})$ is not sufficient, as is shown by the exercise below.

Exercise 2.12. *Let F be a finite group, $\varphi : F \to F$ a non-trivial automorphism, of order d. Let $G = (\mathbb{Z} \ltimes_\varphi F) * \mathbb{Z} =< a, b, F | \forall f \in F,\ afa^{-1} = \varphi(f) >$. Let X be a Cayley graph of this group.*

Show that the family of conjugates of $\langle a^k \rangle$ does not satisfy any small cancellation condition if k is not a multiple of d.

3. SQ-universality

We will greatly strengthen Corollary 2.9 saying that G is not simple if it contains a hyperbolic element with the WPD property.

Definition 2.13. *A group G is* SQ-universal[1] *if for any countable group A, there exists a quotient of G in which A embeds.*

Since there are uncountably many 2-generated groups, and since a given finitely generated group has only countably many 2-generated subgroups, a SQ-universal group has uncountably many non-isomorphic quotients.

Theorem 2.14. *If G is not virtually cyclic, acts on a hyperbolic space X, and contains a hyperbolic element satisfying the WPD property, then G is SQ-universal.*

The first step in the proof consists in producing a free subgroup satisfying the small cancellation condition.

Proposition 2.15. *Assume that G is not virtually cyclic, acts on a hyperbolic space X, and contains a hyperbolic element h satisfying the WPD property. Then for all (A, λ), there exists $H < G$ a free group of rank 2 and $Q_H \subset X$ an H-invariant almost convex subset, so that*

1. *the conjugates of H and the corresponding translates of Q_H form a moving family satisfying the (A, λ)-small cancellation condition, and*
2. *the stabilizer of Q_H is $H \times F$ for some finite subgroup F.*

Exercise 2.16. *Prove the proposition if G is torsion-free.*

Hint: prove that there is some conjugate k of h such that $\Delta(A_h, A_k)$ is finite. Replace h, k by large powers so that their translation length is large compared to $\Delta(A_h, A_k)$. The consider something like $a = h^{1000}k^{1000}h^{1001}k^{1001} \dots h^{1999}k^{1999}$, and $b = h^{2000}k^{2000}h^{2001}k^{2001} \dots h^{2999}k^{2999}$, and $H = \langle a, b \rangle$.

Note that such H might fail to satisfy the small cancellation condition in presence of torsion. Indeed, there may be some element of finite order that almost fixes only half of the axis of a, so that $\Delta(A_a, A_{cac^{-1}})$ might be large.

The proof sketched in the exercise works if $E(h) = \mathbb{Z} \times F$ for some finite subgroup F, and if $F < E(k)$ for all conjugate k of h. To prove the proposition in full generality, one constructs h such that this holds, see [**DGO**, Section 6.2].

Proof of the Theorem. It is a classical result that every countable group embeds in a two generated group [**LS01**]. Thus it is enough to prove that any two-generated group A embeds in some quotient of G. Let $F_2 \to A$ be an epimorphism,

[1]SQ stands for subquotient

and N be its kernel. Let $H < G$ be a free group of rank 2 satisfying the small cancellation hypothesis as in the proposition, and let $Q_H \subset X$ be the corresponding subspace in the moving family. View N as a normal subgroup of H.

We claim that N also satisfies the (A, λ)-small cancellation. Indeed, we assign the group gNg^{-1} to the subspace $g.Q_H$. For this to be consistent, we need N to be normal in $\mathrm{Stab}(Q_H)$. This is true because $\mathrm{Stab}(Q_H) = H \times F$.

Applying the small cancellation theorem, we see that $\mathrm{Stab}(Q_H)/N$ embeds in $G/\langle\langle N \rangle\rangle$. It follows that $A \simeq H/N$ embeds in $G/\langle\langle N \rangle\rangle$. □

4. Dehn fillings

Let G be a relatively hyperbolic group with respect to a subgroup P (we assume that there is one parabolic group only for notational simplicity). By definition, this means that G acts properly on a proper hyperbolic space X with the following properties: there is a G-invariant family $\mathcal{Q}$ of disjoint, almost-convex horoballs in X; all the horoballs in $\mathcal{Q}$ are in the same orbit, and their stabilizers are the conjugates of P; and G acts cocompactly on the complement of these horoballs.

In fact, we can additionally assume that the distance between any two distinct horoballs is as large as we want, in particular, greater than 40δ. This means that the fellow traveling constant for $\mathcal{Q}$ is zero! Given $R_0 \lhd P$, the family $\mathcal{R}$ of conjugates of R_0 defines a moving family $\mathcal{F} = (\mathcal{R}, \mathcal{Q})$.

Now for the small cancellation theorem to apply, we need the injectivity radius to be large. This clearly fails since elements of R_0 are parabolic, so their translation length is small. However, the following variant of the small cancellation theorem holds.

In the small cancellation hypothesis, replace the *large injectivity radius* (asking that all points of Q_i are moved a lot by each $g \in R_i \setminus \{1\}$), by the following one asking this only on the boundary of Q_i:

Theorem 2.17. *Consider a moving family on a hyperbolic space with the notations above. There exists A_0 a universal constant such that the following holds. Assume that $\Delta(\mathcal{Q}) = 0$ (the Q_i's don't come close to each other), and that*

$$\forall i \in I, \forall g \in R_i \setminus \{1\}, \forall x \in \partial Q_i, d(x, gx) > A_0\delta.$$

Then the conclusion of the small cancellation theorem still holds, where Assertion (3) is modified as follows: any non-trivial element whose translation length is at most $C\delta$, and which is not contained in a conjugate of some R_i is not killed in $G/\langle\langle R_i \rangle\rangle$.

Let $*$ be a base point on the horosphere ∂Q preserved by P. Since P acts cocompactly on ∂Q, consider $r > 0$ such that the P-orbit of $B(*, r)$ contains ∂Q. Now if R_0 avoids the finite set $S \subset P$ of all elements $g \in P \setminus \{1\}$ such that $d(*, g*) \leq 2r + A$ then R_0 satisfies this new assumption.

We thus get the Dehn filling theorem:

Theorem 2.18 (**[Osi07, GM08]**). *Let G be hyperbolic relative to P. Then there exists a finite set $S \subset P \setminus \{1\}$ such that for all $R_0 \lhd P$ avoiding S,*

- *P/R_0 embeds in $G/\langle\langle R_0 \rangle\rangle$*
- *$G/\langle R_0 \rangle$ is hyperbolic relative to P/R_0. In particular, if R_0 has finite index in P, then $G/\langle\langle R_0 \rangle\rangle$ is hyperbolic.*

In fact, the proof allows us to control the hyperbolicity constant of the hyperbolic space on which the quotient group acts. This can be a very useful property.

LECTURE 3

Rotating families

1. Road-map of the proof of the small cancellation theorem

The goal of the remaining two lectures is to prove the geometric small cancellation theorem.

There are essentially two main steps in the proof, each step involving only one of the two main hypotheses.

(1) Construct from the space X and the subspaces Q_i a cone-off $\dot{X}$ by coning all the subspaces Q_i, and prove its hyperbolicity. This step does not involve the groups R_i, so this is independent of the large injectivity radius hypothesis.

(2) Because the spaces Q_i have been coned, each subgroup R_i fixes a point in $\dot{X}$, and thus looks like a rotation. Our moving family becomes a *rotating family*. One studies the normal group $N = \langle\langle (R_i)_{i\in I} \rangle\rangle$ via its action on the cone-off. This is where the *large injectivity radius* assumption is used: it translates into a so-called *very rotating* assumption saying somehow that every non-trivial element of R_i rotates by a large angle. The group G/N naturally acts on the quotient space $\dot{X}/N$, and the hyperbolicity of the quotient space $\dot{X}/N$ is then easy to deduce.

In this lecture, we discuss the second step which involves the study of rotating families.

2. Definitions

Consider a group G acting on a δ-hyperbolic space X.

Definition 3.1. *A* rotating family *is a collection* $\{R_c, c \in C\}$ *of subgroups of* G *indexed by a subset* $C \subset X$ *such that*

- R_c *fixes* c *for all* $c \in C$
- C *is* G*-invariant*
- *and* $\forall g \in G$, $\forall c \in C$, $R_{gc} = gR_cg^{-1}$.

One says that the rotating family is ρ-separated *if any two distinct points in* C *are at distance at least* ρ.

The set C is called the set of apices of the family, and the groups R_c are called the rotation subgroups of the family. Note that this definition implies that R_c is a normal subgroup of the stabilizer $\mathrm{Stab}(c)$ of $c \in C$.

Let us reformulate this definition. Start with a group G, and consider $R_1,\dots,R_k$ some subgroups of G. The goal is to understand the quotient $G/\langle\langle R_1, ..., R_k\rangle\rangle$. We assume that every R_i fixes a point c_i such that R_i is a normal subgroup of the stabilizer of c_i, and that c_i is not in the G-orbit of c_j for $i \neq j$. Then one gets a rotating family by putting $C = G.\{c_1, \dots, c_k\}$, and by defining $\{R_c, c \in C\}$ by $R_{gc_i} = gR_ig^{-1}$. The fact that there no ambiguity in this definition is a consequence of the fact that R_i is normal in the stabilizer of c_i.

The following definition formalizes the fact that every non-trivial element of R_i rotates by *a large angle*.

Definition 3.2 (Very rotating condition: local version)**.** *We say that the rotating family is* very rotating *if the following holds. Consider $c \in C, g \in R_c \setminus \{1\}$, and $x, y \in B(c, 40\delta) \setminus B(c, 20\delta)$. If $d(x, y) \leq d(x, c) + d(c, y) - 10\delta$, then any geodesic between x and gy contains c.*

Intuitively, the very rotating condition says that if x, c, y make a *small angle* at c, and $g \in R_c \setminus \{1\}$, then x, c, gy makes a *large angle* at c. This somehow means that g rotates by a large angle. This is for instance the case if X is $CAT(-1)$, and if for all $x \in X \setminus \{c\}$, the geodesics $[c, x], [c, gx]$ make an angle of at least 2π (see Lemma 4.8). For exposition reasons, the definition above is slightly different from the one in **[DGO]**, but this does not change the nature of the results.

The very rotating condition is local around an apex. It implies the following global condition. This shows in particular that R_c acts freely and discretely on $X \setminus B(c, 20\delta)$.

Lemma 3.3 (Very rotating condition: global version)**.** *Consider $c \in C$, and $x, y \in X$ at distance at least 20δ from c such that $d(x, y) \leq d(x, c) + d(c, y) - 22\delta$. Then for any $g \in R_c \setminus \{1\}$, any geodesic between x and gy contains c. In particular, for any choice of geodesics $[x, c]$, $[c, gy]$, their concatenation $[x, c] \cup [c, gy]$ is geodesic.*

PROOF. To unify notations, write $x_1 = x, x_2 = y$. For $i \in \{1, 2\}$, let $p_i, q_i \in [c, x_i]$ be such that $d(p_i, c) = 20\delta$, and $d(q_i, c) = 11\delta$. By thinness of a triangle with vertices c, x_1, x_2, $d(q_1, q_2) \leq 4\delta$. In particular, $d(p_1, p_2) \leq d(p_1, q_1) + 4\delta + d(q_2, p_2) = d(p_1, c) + d(c, p_2) - 18\delta$. The local very rotating hypothesis says that $d(p_1, c) + d(c, gp_2) = d(p_1, gp_2)$. Consider any geodesic $[x_1, gx_2]$, and let $p'_1, q'_1, gq'_2, gp'_2 \in [x_1, gx_2]$ be such that $d(p'_i, x_i) = d(p_i, x_i)$ and $d(q'_i, x_i) = d(q_i, x_i)$. By the triangle inequality, $d(p'_i, c) \geq d(p_i, c) = 20\delta$. By thinness of the triangle x_1, c, gx_2, $d(q_i, q'_i) \leq 4\delta$, so $d(q'_1, q'_2) \leq 12\delta$, and $d(p'_1, p'_2) \leq d(p'_1, q'_1) + 12\delta + d(p'_2, q'_2) = 30\delta$. Thus, the local very rotating condition applies to the points p'_1, p'_2, and we get that $[p'_1, gp'_2]$ contains c, and so does $[x_1, gx_2]$. ☐

3. Statements

Now we state some results describing the structure of the normal subgroup generated by the rotating family.

Theorem 3.4. *Let $(R_c)_{c \in C}$ be a ρ-separated very rotating family, with ρ large enough compared to the hyperbolicity constant δ. Let $N = \langle\langle R_c | c \in C \rangle\rangle$. Then*

1. $\mathrm{Stab}(c)/R_c$ *embeds in G/N. More generally, if $[g] < \rho$ and $g \in N$ then $g \in R_c$ for some $c \in C$,*
2. *there exists a subset $S \subset C$ such that N is the free product of the collection of $(R_c)_{c \in S}$, and*
3. *X/N is hyperbolic.*

Remark 3.5. $\rho \geq 120\delta$ is enough for the first two assertions. For the last one, we need it to be large enough to apply the Cartan-Hadamard theorem (Theorem 3.11 below), see Proposition 3.12 below.

The first assertion follows from the following form of the Greendlinger lemma which we are going to prove together with the theorem. The classical Greendlinger Lemma says that if a cyclically reduced word w represents the trivial element in a small cancellation group, then it has a subword u that is a subword of a relator r with $|u| > |r|/2$, thus w can be shortened by replacing u by the inverse of the rest of the relator. This is the basis of Dehn's algorithm for solving the word problem in a small cancellation group.

Theorem 3.6. *(Greendlinger lemma) Every element g in N that does not lie in any R_c is loxodromic in X, it has a g-invariant geodesic line l, this line contains a point $c \in C$ such that there is a shortening element at c in l (as defined below).*

Definition 3.7. *Let l be a geodesic, and $c \in l$. A* shortening element *at c in l is an element $r \in R_c \setminus \{1\}$ such that if q_1, q_2 are the two points in l at distance 20δ from c, then $d(q_1, rq_2) \leq 10\delta$.*

Assume that l is a g-invariant geodesic line, and that there is a shortening element r at $c \in l$. Then up to exchanging the roles of q_1 and q_2, we can assume that q_1, q_2, gq_1 are aligned in this order in l. Since $d(q_1, gq_1) = [g]$, we get

$$\begin{aligned}[gr] &\leq d(r^{-1}q_1, gq_1) \leq d(r^{-1}q_1, q_2) + d(q_2, gq_1) \\ &\leq 10\delta + d(q_1, gq_1) - d(q_1, q_2) = [g] - 30\delta,\end{aligned}$$

so $[gr] \leq [g] - 30\delta$. Thus, Greendlingers's lemma gives a form of (relative) linear isoperimetric inequality: every element g of N is the product of at most $[g]/30\delta$ elements of the rotation subgroups.

4. Proof of Theorem 3.4

The proof is by an iterative process, described by Gromov in **[Gro01b]**, see **[DGO]**. To perform it, we construct inductively a sequence of subsets called *windmills* with a set of properties that remain true inductively (see Definition 3.8 and Figure 1). To each windmill $W \subset X$, we associate the group G_W generated by $\{R_c | c \in W\}$. As the windmills we construct are going to exhaust X (see Proposition 3.9), the groups G_W will exhaust the normal subgroup $N = \langle R_c | c \in C \rangle$.

Definition 3.8 (Windmill)**.** *A* windmill *is a subset $W \subset X$ satisfying the following axioms.*

(1) *W is almost convex,*
(2) *$W^{+40\delta} \cap C = W \cap C \neq \emptyset$,*
(3) *The group G_W generated by $\bigcup_{c \in W \cap C} R_c$ preserves W,*
(4) *There exists a subset $S_W \subset W \cap C$ such that G_W is the free product $*_{c \in S_W} R_c$.*
(5) *(Greendlinger) Every elliptic element of G_W lies in some $R_c, c \in W \cap C$, other elements of R_c have an invariant geodesic line l such that $l \cap C$ contains a point at which there is a shortening element (as in Definition 3.7).*

To initiate this inductive process, we choose $c \in C$, and we take as initial windmill $W_0 = \{c\}$ (we could also choose for instance $W_0 = B(c, r)$ with $r \leq \rho - 50\delta$)).

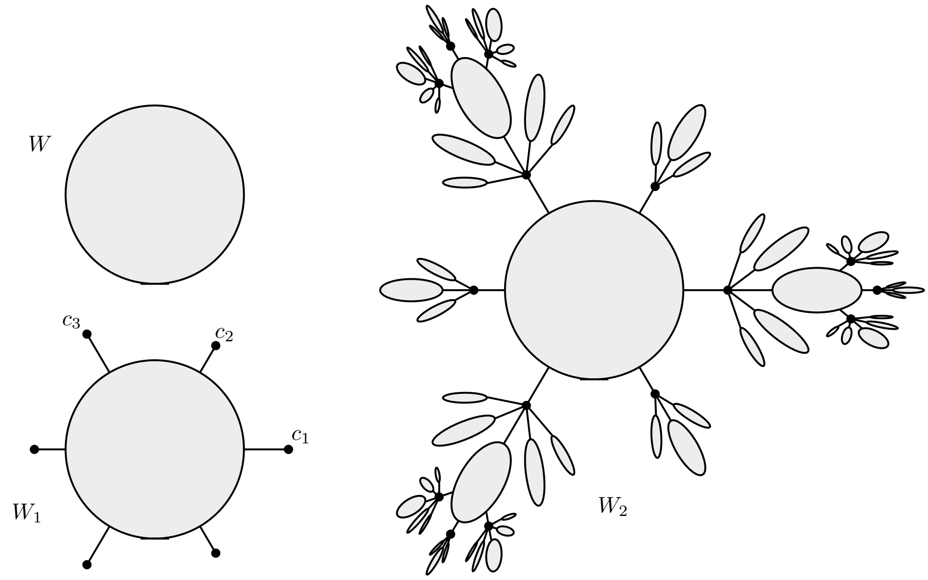


FIGURE 1. A windmill. The iterative step starts with a windmill W, constructs W'_1, W'_2, and the end of the iterative step, the new windmill W' is a thickening of W'_2.

Proposition 3.9 (Inductive procedure)**.** *For any windmill W, there exists a windmill W' containing $W^{+10\delta}$ and $W^{+60\delta} \cap C$, and such that $G_{W'} = G_W * (*_{x \in S} R_c)$ for some $S \subset C \cap (W' \setminus W)$.*

PROOF OF THEOREM 3.4 FROM PROPOSITION 3.9. Starting from $W_0 = \{c\}$, define W_{i+1} from W_i by applying the proposition. Since $\cup_i W_i = X$, $\cup_i G_{W_i} = N$. Greendlinger lemma follows, and so does the fact that N is a free product. □

PROOF OF PROPOSITION 3.9. If $W^{+60\delta}$ does not intersect C, we just inflate W by taking $W' = W^{+10\delta}$. Otherwise, we construct W' in several steps.

Step 1. Let $C_1 = C \cap (W^{+60\delta} \setminus W)$. For each $c \in C_1$ choose a projection p_c of c on W, and a geodesic $[c, p_c]$. This choice can be done G_W-equivariantly because G_W acts freely on C_1 (by Greendlinger hypothesis, and the very rotating assumption). Define $W'_1 = W \cup \bigcup_{c \in C_1} [c, p_c]$. Almost convexity of W easily implies that W'_1 is 12δ-quasiconvex. Note for future use that for any $c \in C_1$, $W'_1 \setminus [c, p_c]$ is also 12δ-quasiconvex for the same reason. Since C is ρ-separated with $\rho > 112\delta$, any point in $C \setminus \{c\}$ is at distance at least 52δ from $[c, p_c]$. This implies that $W_1'^{+52\delta} \cap C = W'_1 \cap C$.

Step 2. The group $G' = \langle R_c | c \in W'_1 \rangle$ is the group generated by G_W and by $\{R_c | c \in C_1\}$. We define $W'_2 = G'.W'_1$. Abstract nonsense shows that $G' = \langle R_c | c \in W'_2 \rangle$.

Step 3. We take $W' = W_2'^{+12\delta} = G'.(W_1'^{+12\delta})$.

Let us check that W' satisfies Axiom 2 of a windmill. We have $W'^{+40\delta} = W_2'^{+52\delta}$. Since $W_1'^{+52\delta} \cap C = W'_1 \cap C$, we get $W'^{+40\delta} \cap C = W'_2 \cap C \subset W' \cap C$. Axiom 2 follows. It also follows that $G' = G_{W'}$, and that W' is $G_{W'}$-invariant so Axiom 3 follows.

To prove that W' satisfies the other axioms, we first look at how W'_1 is rotated around some $c \in C_1$. So take $h \in R_c \setminus \{1\}$, and look at $W'_1 \cup hW'_1$. Consider $x \in W'_1 \setminus [c, p_c]$ and $y \in h(W'_1 \setminus [c, p_c])$. Consider $q_x \in [c, x]$ and $q'_x \in [c, p_c]$, both at distance 20δ from c. Define $q_y \in [c, y]$ and $q'_y \in [c, hp_c]$ similarly. By thinness of the triangle c, x, p_c, $d(q_x, q'_x) \leq 4\delta$: otherwise, there would be some $q''_x \in [p_c, x]$ such that $d(q_x, q''_x) \leq 4\delta$, and by 12δ-quasiconvexity of $W'_1 \setminus [c, p_c]$, $d(q_x, W'_1 \setminus [c, p_c]) \leq 16\delta$, so $d(c, W'_1 \setminus [c, p_c]) \leq 36\delta$, a contradiction. Similarly, $d(q_y, q'_y) \leq 4\delta$, and since $h^{-1}q'_y = q'_x$, $d(q_x, h^{-1}q_y) \leq 8\delta$. The global very rotating condition implies that any geodesic from x to y contains c. We note that h is a shortening element of $[x, y]$ at c. We have proved:

Lemma 3.10 (Key lemma)**.** *Fix $c \in C_1$, $h \in R_c \setminus 1$, $x \in W'_1 \setminus [c, p_c]$ and $y \in h(W'_1 \setminus [c, p_c])$. Then any geodesic from x to y contains c and h is a shortening element of $[x, y]$ at c.*

By 12δ-quasiconvexity of W'_1 and hW'_1, we get that for any x, y as in the lemma, $[x, y]$ is in the 12δ-neighbourhood of $W'_1 \cup hW'_1$. If x lies in $[c, p_c]$ or y lies in $h[c, p_c]$, we get similarly that $[x, y]$ lies in the 12δ-neighbourhood of $W'_1 \cup hW'_1$ so $W'_1 \cup hW'_1$ is 12δ-quasiconvex.

Now we prove that W'_2 has a tree-like structure (see Figure 1). Recall that $W'_2 = G'.W'_1$. Let Γ be the graph with vertex set $V = V_C(\Gamma) \sqcup V_W(\Gamma)$, where $V_C(\Gamma)$ is the set of apices in $G'.C_1$, and $V_W(\Gamma)$ is the set of translates of W under G'. We put an edge between gW and gc for any $g \in G'$ and $c \in C_1$.

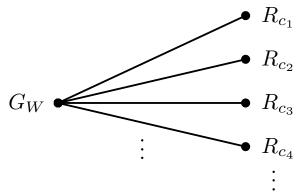


FIGURE 2. The graph of groups defining the tree T, where $c_1, c_2, \ldots$ is an enumeration of S_1

Since C_1 is G_W-invariant, we consider $S_1 \subset C_1$ a set of representatives of C_1/G_W (note that S_1 may be infinite). We define the free product $\hat{G} = G_W * (*_{c \in S_1} R_c)$, viewed as a tree of groups with trivial edge groups as in the Figure 2. Let T be the corresponding Bass-Serre tree. We denote by $u_W \in T$ the vertex stabilized by W, and for each $c \in S_1$, we denote by $u_c \in T$ the vertex stabilized by R_c. Denote by $V_W(T)$ the set of vertices of T in the orbit of u_W, and $V_C(T)$ the set of other vertices of T. Any edge of T has an endpoint in $V_W(T)$ and the other endpoint in $V_C(T)$. The inclusions $G_W \subset G'$ and $R_c \subset G'$ induce an epimorphism $\varphi : \hat{G} \to G'$, and there is a natural φ-equivariant map $f : T \to \Gamma$ sending u_W to W, and u_c to c for all $c \in S_1$.

We prove that φ and f are isomorphisms.

We first note that the key lemma implies that f does not identify any pair of vertices in $V_W(T)$ at distance 2 from each other.

Next, we claim that f is injective on the set of vertices adjacent to u_W. Otherwise, there are $c, c' \in S_1$ and $g, g' \in G_W$ such that $f(gu_c) = f(g'u_{c'})$, i.e. $gc = g'c'$.

Since S_1 is a set of representatives for the orbits of G_W, this implies that $c = c'$, and the element $g^{-1}g'$ fixes c. As noted above, the Greendlinger Axiom 5 for W implies that G_W acts freely on C_1 so $g^{-1}g' = 1$. This proves the claim, and shows more generally that f does not identify any pair of vertices in $V_C(T)$ at distance 2 from each other.

We now claim that f does not identify any pair of points $u \neq u' \in V_C(T)$. Consider the segment $[u, u']_T \subset T$, and let $u = u_1, u_2, \dots, u_n = u'$ be the points in $[u, u']_T \cap V_C(T)$. Let $c_i \in C$ be the image of u_i under f. Consider the path γ in X defined as a concatenation of geodesics $[c_1, c_2]_X, [c_2, c_3]_X, ..., [c_{n-1}, c_n]_X$. The key lemma (applied around c_2) shows that any geodesic from c_1 to c_3 contains c_2 so in particular, $\gamma_3 = [c_1, c_2] \cup [c_2, c_3]$ is a geodesic and there is a shortening element at c_2 for $[c_1, c_2] \cup [c_2, c_3]$. Similarly, $[c_2, c_3] \cup [c_3, c_4]$ is geodesic, and there is a shortening element at c_3 for $[c_2, c_3] \cup [c_3, c_4]$. Then the global very rotating condition at c_3 applies to $\gamma_3 \cup [c_3, c_4]$ and shows that $\gamma_3 \cup [c_3, c_4]$ is geodesic. By induction, we get that γ is geodesic so $c_1 \neq c_n$ hence $f(u) \neq f(u')$, which proves our claim.

Finally, a similar argument shows that f is injective in restriction to $V_W(T)$. Indeed, if $gW \neq g'W \in V_W(T)$, consider a path of the form

$$[x, c_1]_X.[c_1, c_2]_X \dots [c_n, y]_X$$

where $x \in gW$, $y \in g'W$ and $\{c_1, \dots, c_n\} = [gW, g'W]_T \cap V_C(T)$. The argument above shows that this path is geodesic. It follows that f is injective.

Injectivity of φ follows since an element of $\ker \varphi$ has to fix T pointwise, and is therefore trivial. Since f and φ are obviously onto, they are isomorphisms. This proves that G' can be written as a free product as in the Proposition, and that W' satisfies Axiom 4.

The paths $[x, c_1]_X.[c_1, c_2]_X \dots [c_n, y]_X$ considered above also have shortening pairs at c_i. The very rotating condition implies that any geodesic segment between x and y has to contain c_i and is therefore of this form. Since W'_1 is 12δ-quasiconvex, it follows that so is W'_2. It follows that W' is almost convex, and Axiom 1 holds.

The Greendlinger Axiom is similar: if $g \in G_{W'}$ is elliptic in the tree T, there is nothing to prove because W is a windmill. If g is hyperbolic in T, its axis contains a vertex in $u \in V_C(T)$. Let $u = u_1, u_2, \dots, u_n = gu$ be the points in $[u, gu]_T \cap V_C(T)$, and let $c_i \in C$ be the image of u_i under f. Then the g-translates of $[c_1, c_2]_X.[c_2, c_3]_X \dots [c_{n-1}, c_n]_X$ form a g-invariant bi-infinite geodesic, and there is a shortening element in at each c_i. □

5. Hyperbolicity of the quotient

The goal of this section is to prove the hyperbolicity of the quotient space X/N. We will prove local hyperbolicity, and use the Cartan-Hadamard Theorem.

5.1. The Cartan Hadamard Theorem

The Cartan-Hadamard theorem allows us to deduce global hyperbolicity from local hyperbolicity, see [**DG08**], and more detailed account in Coulon's notes [**Cou**], (see also [**OOS09**, Th 8.3]).

Theorem 3.11 (Cartan-Hadamard Theorem)**.** *There exist universal constants C_1, C_2 such that the following holds. Consider a geodesic space Y and some $\delta > 0$ such that*

- *Y is $C_1\delta$-locally δ-hyperbolic, and*

- *Y is 32δ-simply connected.*

Then Y is (globally) $C_2\delta$-hyperbolic.

We will denote by $R_{CH}(\delta) = C_1\delta$ and $\delta_{CH}(\delta) = C_2\delta$. The assumption that Y is locally δ-hyperbolic asks that for any subset $\{a, b, c, d\} \subset Y$ whose diameter is at most $R_{CH}(\delta)$, the 4-point inequality holds: $d(a, b) + d(c, d) \leq \max\{d(a, c) + d(b, d), d(a, d) + d(b, c)\} + 2\delta$.

The assumption that Y is 32δ-simply connected means that the fundamental group of Y is normally generated by free homotopy classes of loops of diameter at most 32δ. Equivalently, one may ask that the Rips complex $P_{32\delta}(Y)$ is simply connected.

We will apply this Theorem to $Y = X/N$. Note that since X is δ-hyperbolic, X is 4δ-simply connected [**CDP90**, Section 5, prop. 1.1]. Since N is generated by isometries fixing a point, X/N is also 4δ-simply connected. Indeed, let $\overline{\gamma}$ be a loop in X/N. Lift it to γ in X, joining x to gx with $g \in N$. Write $g = g_n ... g_1$ with g_i fixing a point. One can homotope γ rel endpoints to ensure that γ contains a fixed point c of g_n (just insert a path and its inverse). Then $\gamma = \gamma_1.\gamma_2$ where the endpoint of γ_1 and the initial point of γ_2 are c. Downstairs, this is gives a homotopy. Now change γ_2 to $g_n^{-1}\gamma_2$. Downstairs, this does not change the path. The new path $\gamma_1.g_n^{-1}\gamma_2$ joins x to $g_{n-1} \dots g_1 x$. Repeating, we can assume $g = 1$ so that γ is a loop in X. By hypothesis, $\gamma = \prod_i p_i l_i p_i^{-1}$ where p_i is a path with origin at x, and l_i is a loop of diameter at most 4δ. Projecting downstairs, we get the same property for the projection.

Thus, in view of the Cartan-Hadamard Theorem, it is enough to prove local hyperbolicity of the quotient.

5.2. Proof of local hyperbolicity

We will only prove the proposition in the particular case where X is a cone-off of radius ρ (see Corollary 4.3 in the next lecture). The main simplification is that in this case, the neighbourhood of an apex is a hyperbolic cone over a graph, and so is its quotient. Thus we can apply Proposition 4.6 saying that such a hyperbolic cone is locally $2\delta_{\mathbb{H}^2}$-hyperbolic, where $\delta_{\mathbb{H}^2}$ is the hyperbolicity constant of $\mathbb{H}^2$.

Proposition 3.12. *Under the assumptions of 3.4, assume that X and the rotating family are obtained by coning-off a small cancellation moving family, as described in the next section, where ρ is the radius of the cone-off. We denote by δ the hyperbolicity constant of X, N be the normal group generated by the rotating family, $\overline{X} = X/N$, and $\overline{C}$ the image of C in X/N. Let $\delta' = \max\{\delta, \delta_{\mathbb{H}_2}\}$, and assume that $\rho \geq 10 \max\{R_{CH}(\delta')\}$. Then*

(1) *for each apex $\overline{c} \in \overline{C}$, the $9\rho/10$-neighbourhood $\overline{c}$ in $\overline{X}$ is $\rho/10$-locally $2\delta_{\mathbb{H}^2}$-hyperbolic,*
(2) *the complement of the $8\rho/10$-neighbourhood $\overline{C}$ in $\overline{X}$ is $\rho/10$-locally δ-hyperbolic. In fact, any subset of diameter at most $\rho/10$ in $X \setminus C^{+8\rho/10}$ isometrically embeds in X/N, and*
(3) *X/N is $\delta_{CH}(\delta')$-hyperbolic.*

In the case where our rotating family is not obtained by coning-off, one can also prove that X/N is locally hyperbolic with worse constants, see [**DGO**].

Proof. The first assertion is a direct consequence of the fact that the hyperbolic cone over a graph is $2\delta_{\mathbb{H}^2}$-hyperbolic (Proposition 4.6).

For the second assertion, let $E \subset X \setminus C^{+8\rho/10}$ be a subset of diameter at most $\rho/10$, and let E' be its $\rho/10$-neighbourhood. We claim that E' injects into $\overline{X}$, so that E isometrically embeds in $\overline{X}$. Now assume on the contrary that there are $x, y \in E'$ and $g \in N \setminus \{1\}$ such that $y = gx$. In particular $[g] < \rho$, so by Assertion 1 of Theorem 3.4, $g \in R_c$ for some $c \in C$. Then the very rotating condition implies that any geodesic $[x, y]$ contains c, so $d(x, c) \leq 3\rho/10$, a contradiction.

To conclude, we have shown that X/N is $\rho/10$-locally δ'-hyperbolic with $\delta' = \max\{\delta, \delta_{\mathbb{H}_2}\}$. Since X/N is 4δ-simply connected, and since $\rho > 10R_{CH}(\delta')$, the Cartan-Hadamard Theorem says that X/N is globally $\delta_{CH}(\delta')$-hyperbolic. □

6. Exercises

Exercise 3.13. *Assume that $\rho \gg \delta$. Let $E \subset X$ be an almost convex subset, and assume that E does not intersect the $\rho/10$-neighbourhood of C. Prove that E isometrically embeds in X/N.*

Hint: prove that any subset of E of diameter $\rho/100$ isometrically injects in X/N. Then say that a $\rho/100$-local geodesic in X/N is close to a global geodesic.

Exercise 3.14. *Assume that G is torsion-free, and that for all c,* $\mathrm{Stab}(c)/R_c$ *is torsion-free. Prove that G/N is torsion-free.*

Hint: use the fact that an elliptic isometry of a δ-hyperbolic space has an orbit of diameter at most 16δ. Then given $g \in G/N$ of finite order, look for a lift in G with smallest translation length.

LECTURE 4

The cone-off

1. Presentation

The goal of this section is, given a hyperbolic space X and a family $\mathcal{Q}$ of almost convex subspaces, to perform a coning construction of these subspaces, thus obtaining a new hyperbolic space $\dot{X}$ called the *cone-off* space. The effect of this operation is to transform a small cancellation moving family on X into a very rotating family on this new space $\dot{X}$.

This construction has been introduced by Gromov in [**Gro01a, Gro03**], and further developed by Gromov, Delzant and Coulon [**DG08, Cou11, Cou**]. A construction of this type was introduced before by Bowditch in the context of relatively hyperbolic groups (with cone points at infinity). See also Farb's and Groves-Manning's constructions [**Far98, GM08**]. We follow [**Cou11**], with minor modifications and simplifications.

For simplicity we assume that X is a metric graph, all of whose edges have the same length. This is no loss of generality: if X is a length space, the graph Y with vertex set X where one connects x to y by an edge of length l if $d(x,y) \leq l$ satisfies $\forall x, y \in X,\ d_X(x,y) \leq d_Y(x,y) \leq d_X(x,y) + l$. We don't assume that X is locally compact.

Topologically, we are going to cone a family $\mathcal{Q}$ of subgraphs (see Figure 1), and to put a geometry by identifying the added triangles with sectors of $\mathbb{H}^2$ of fixed radius ρ (see Section 2 for details). Thus ρ is a parameter of this construction, to be chosen.

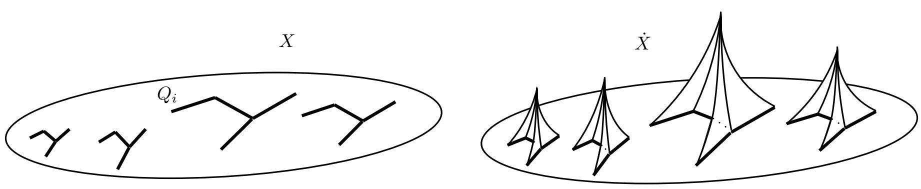


FIGURE 1. The cone-off.

The assumptions will be that X is δ-hyperbolic with δ very small, and that we have a family $\mathcal{Q}$ of almost convex subspaces having a small fellow traveling length.

The features of the resulting space will be as follows:

(1) $\dot{X}$ is a hyperbolic space, whose hyperbolicity constant is *good* (meaning: a universal constant; in particular, ρ can be chosen to be very large compared to this hyperbolicity constant).

(2) If a group G acts on X, preserving $\mathcal{Q}$, and if to each $Q \in \mathcal{Q}$ corresponds a group R_Q (in an equivariant way) preserving Q and with sufficiently large injectivity radius, then $(R_Q)_{Q\in\mathcal{Q}}$ is a very rotating family on $\dot{X}$.

With quantifiers, the main result of this section will be:

Theorem 4.1. *There exist constants $\delta_c, \Delta_c, \rho_0, \delta_U$ as follows. If X is δ_c-hyperbolic, and $\mathcal{Q}$ is a family of almost convex subspaces such that $\Delta(\mathcal{Q}) \leq \Delta_c$, then for all $\rho \geq \rho_0$, the corresponding cone-off $\dot{X}$ satisfies:*

(1) *$\dot{X}$ is R-locally $(2\delta_{\mathbb{H}^2})$-hyperbolic (where $\delta_{\mathbb{H}^2}$ is the hyperbolicity constant of $\mathbb{H}^2$, and $R = R_{CH}(2\delta_{\mathbb{H}^2})$ is the constant required by the Cartan-Hadamard theorem).*
(2) *It is globally δ_U-hyperbolic (with $\delta_U = \delta_{CH}(2\delta_{\mathbb{H}^2})$).*
(3) *If $\mathcal{F} = (R_Q)_{Q\in\mathcal{Q}}$ is a moving family whose injectivity radius is at least $2\pi\sinh(\rho)$, then $(R_Q)_{Q\in\mathcal{Q}}$ is a 2ρ-separated very rotating family on $\dot{X}$.*

Note that the hyperbolicity constant δ_U of $\dot{X}$ does not depend on X or ρ.

In fact, the geometry of the cone-off is even nicer than this δ_U-hyperbolicity. Indeed, this space is $CAT(-1,\varepsilon)$, meaning in a precise sense "almost $CAT(-1)$". This property introduced in **[Gro01a]** implies hyperbolicity with a hyperbolicity constant close to $\delta_{\mathbb{H}^2}$, but gives in particular a much better control of bigons than in a standard $\delta_{\mathbb{H}^2}$-hyperbolic space (at least when ε is small enough). We will not discuss this property here.

It is important that ρ is large compared to the hyperbolicity constant δ_U of $\dot{X}$, in particular to apply the theorem about rotating families. We have the freedom to do so in Theorem 4.1 since δ_U is independent of ρ.

The hypotheses on X in the theorem can be achieved by rescaling the metric if the fellow traveling constant $\Delta(\mathcal{Q})$ of $\mathcal{Q}$ is finite. However, if $\mathcal{F} = (R_Q)_{Q\in\mathcal{Q}}$ is a moving family, this rescaling scales down the injectivity radius accordingly. In order to get the very rotating condition on the cone-off, Assertion 3 of Theorem 4.1 requires R_Q to have a large injectivity radius after rescaling. To achieve this, the initial injectivity radius has to be large compared to the initial hyperbolicity constant and the initial fellow traveling constant. This is exactly what the small cancellation hypothesis asks for.

Corollary 4.2. *For any $\rho \geq \rho_0$, there exists $A_\rho, \lambda_\rho > 0$ such that if $(R_Q)_{Q\in\mathcal{Q}}$ is an (A_ρ, λ_ρ)-small cancellation moving family on X, then $(R_Q)_{Q\in\mathcal{Q}}$ is a 2ρ-separated very rotating family on $\dot{X}_\alpha$, the cone-off of radius ρ of X_α, where X_α is the rescaling of X by a factor $\alpha > 0$.*

Proof. One can take $A_\rho = \frac{2\pi\sinh(\rho)}{\delta_c}$ and $\lambda_\rho = \frac{\Delta_c}{2\pi\sinh(\rho)}$. Indeed, if X_α is the space X where the metric is multiplied by the factor $\alpha = \min\{\frac{\delta_c}{\delta}, \frac{\Delta_c}{\Delta}\}$, then Theorem 4.1 applies to X_α. Moreover, if $(R_Q)_{Q\in\mathcal{Q}}$ satisfies the (A_ρ, λ_ρ)-small cancellation condition, it acts on X_α with injectivity radius at least $2\pi\sinh\rho$. Assertion 3 of Theorem 4.1 implies that $(R_Q)_{Q\in\mathcal{Q}}$ acts on the cone-off $\dot{X}_\alpha$ of X_α as a 2ρ-separated very rotating family. □

Let $\rho_U \geq \rho_0$ be such that Theorem 3.4 about rotating families applies to any $2\rho_U$-separated very-rotating family on a δ_U-hyperbolic space (where δ_U is the hyperbolicity constant of the cone-off given by Theorem 4.1). Applying Corollary 4.2 to this value of ρ, we get:

Corollary 4.3. *There exists $A_0, \lambda_0 > 0$ such that if $\mathcal{F} = (R_Q)_{Q\in\mathcal{Q}}$ is an (A_0, λ_0)-small cancellation moving family on X, then $(R_Q)_{Q\in\mathcal{Q}}$ is a very rotating family on $\dot{X}_\alpha$, the cone-off of radius ρ_U of a rescaled version of X.*

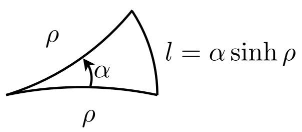
It is now easy to deduce the small cancellation Theorem 1.3.

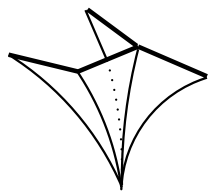
Proof of the small cancellation Theorem 1.3. All the assertions follow immediately from Corollary 4.3 and Theorem 3.4, except maybe Assertion (3) saying that given $C > 0$, elements with a translation length at most $C\delta$ survive in the quotients if the small cancellation constants are good enough.

Greendlinger's lemma says that any element $g \in G\setminus\{1\}$ with translation length less than 2ρ in $\dot{X}_\alpha$ survives, except if contained in some group of our small cancellation moving family $\mathcal{F}$. But if we take $A > C$, the small cancellation assumption says that elements of $R_i \setminus \{1\}$ act on X with translation length greater that $A\delta \geq C\delta$. Now given $C > 0$, consider ρ such that $C\delta_c < 2\rho$. Assume that our moving family $\mathcal{F}$ satisfies the (A_ρ, λ_ρ)-small cancellation condition with A_ρ, λ_ρ as in Corollary 4.2. Let $\dot{X}_\alpha$ be the space given by this corollary. Then any element acting on X with translation length at most $C\delta$, acts on the spaces X_α and $\dot{X}_\alpha$ with translation length at most $C\delta_c < 2\rho$, and is therefore not killed in the quotient. □

2. The hyperbolic cone of a graph

Given $\rho > 0$, and $\alpha \in (0, \pi)$, consider a hyperbolic sector of radius ρ and angle α in $\mathbb{H}^2$. The arclength of its boundary arc of circle is $l = \alpha \sinh \rho$. If $\alpha \geq \pi$, one can still define a hyperbolic sector of angle α by gluing several sectors of angle less than π.





If Q is a metric graph, all whose edges have length l, the hyperbolic cone over Q is the triangular 2-complex $C(Q) = ([0, \rho] \times Q)/\sim$ where $\sim$ is the equivalence relation that collapses $Q \times \{0\}$ to a point. The cone point $c = Q \times \{0\}$ is also called the *apex* of $C(Q)$. We define a metric on each 2-cell of $C(Q)$ by identifying it with the hyperbolic sector of radius ρ and arclength l. We identify Q with $Q \times \{\rho\}$, but we distinguish the original metric d_Q from the new metric $d_{C(Q)}$. If we want to emphasize the dependance in ρ, we will denote the cone by $C_\rho(Q)$.

For $t \in [0, \rho], x \in Q$ we denote by tx the image of (t, x) in $C(Q)$. There is an explicit formula for the distance in $C(Q)$ **[BH99**, Def 5.6 p.59]:

$$\cosh d(tx, t'x') = \cosh t \cosh t' - \sinh t \sinh t' \cos\left(\min\left\{\pi, \frac{d_Q(x, x')}{\sinh(\rho)}\right\}\right).$$

This formula allows one to define the hyperbolic cone over any metric space. We shall not use it directly (except in the proof of Fact 4.14). Instead, we will use the following basic facts.

Proposition 4.4 ([**BH99**, Chap I.5, Prop. 5.10])**.**

(1) *For each $x \in Q$, the radial segment $\{tx|\, t \in [0,\rho]\}$ is the only geodesic joining c to x;*
(2) *For each $x, y \in Q$ such that $d_Q(x,y) \geq \pi \sinh \rho$, then for any $t, s \in [0,\rho]$ the only geodesic joining tx to sy is the concatenation of the two radial segments $[tx, c] \cup [c, sy]$.*
(3) *For each $x, y \in Q$ such that $d_Q(x,y) < \pi \sinh \rho$, and all $s, t \in (0,\rho]$, there is a bijection between the set of geodesics between x and y in Q and the set of geodesics between tx and sy in $C(Q)$. None of these geodesics go through c.*

The map $C(Q) \setminus \{c\} \to Q$ defined by $tx \mapsto x$ is called the *radial projection.*

Exercise 4.5. *Prove that the radial projection is locally Lipschitz. Note that it is not globally Lipschitz in general.*

Prove that the local Lipschitz constant tends to 1 as one gets closer to Q: for each $\varepsilon > 0$, there exists $t_0 < \rho$ such that for each $tx \in C(Q)$, and with $t \geq t_0$, there exists a neighbourhood U of tx in $C(Q)$ such that the restriction of the radial projection on U is $(1+\varepsilon)$-Lipschitz.

The hyperbolic cone on a tripod is $CAT(-1)$ because it is obtained by gluing $CAT(-1)$ spaces over a convex subset. It follows that the cone over a tree is $CAT(-1)$ since by Proposition 4.4, any geodesic triangle is contained in the cone over a tripod. This extends to the hyperbolic cone over an $\mathbb{R}$-tree which can be defined by writing the $\mathbb{R}$-tree as an increasing union of metric trees (with edges of varying lengths), or by the distance formula above. One can also view this fact as a particular case of Beretosvkii's theorem saying that, writing $\kappa = \pi \sinh \rho$, the hyperbolic cone of radius ρ over any $CAT(\kappa)$-space, is $CAT(-1)$ [**BH99**, Chap I.5, Th 3.14]. In particular, the hyperbolic cone over an $\mathbb{R}$-tree is $\delta_{\mathbb{H}^2}$-hyperbolic.

Proposition 4.6. *The hyperbolic cone of any radius, over any graph, is $2\delta_{\mathbb{H}^2}$-hyperbolic.*

This is analogous to the hyperbolicity of a Groves-Manning combinatorial horoballs [**GM08**].

Remark 4.7. We don't want to assume local compactness of Q. The fact that Q is a graph whose edges have the same length is used to ensure that Q and $C(Q)$, (and later the cone-off) are geodesic spaces. Indeed, a theorem by Bridson shows that any connected simplicial complex whose cells are isometric to finitely many convex simplices in $\mathbb{H}^n$, and glued along their faces using isometries, is a geodesic space [**BH99**, Th 7.19]. This can be easily adapted to our situation where 2-cells are all isometric to the same 2-dimensional sector.

The following very simple proof is due to Coulon.

Proof. Let C be such a cone, and c its apex. One checks the hyperbolic 4-point inequality: given $x, y, z, t \in C$, we want to prove that one of the following inequalities holds

$$L : xy + zt \leq xz + yt + 4\delta_{\mathbb{H}^2} \quad R : xy + zt \leq xt + yz + 4\delta_{\mathbb{H}^2}$$

(where we use the notation $xy = d(x, y)$).

Since any 3-point set is isometric to a subset of a tree, and since the cone over a tree is $CAT(-1)$, for any 3 points $u, v, w \in C$, we know that u, v, w, c satisfy the $\delta_{\mathbb{H}^2}$-hyperbolic 4-point inequality. Consider the inequalities

$$\begin{aligned}
L_x &: cy + zt \leq cz + yt + 2\delta_{\mathbb{H}^2}, & R_x &: cy + zt \leq ct + yz + 2\delta_{\mathbb{H}^2}\\
L_y &: cx + zt \leq ct + xz + 2\delta_{\mathbb{H}^2}, & R_y &: cx + zt \leq cz + xt + 2\delta_{\mathbb{H}^2}\\
L_z &: ct + xy \leq cx + yt + 2\delta_{\mathbb{H}^2}, & R_z &: ct + xy \leq cy + xt + 2\delta_{\mathbb{H}^2}\\
L_t &: cz + xy \leq cy + xz + 2\delta_{\mathbb{H}^2}, & R_t &: cz + xy \leq cx + yz + 2\delta_{\mathbb{H}^2}.
\end{aligned}$$

We know that for each $u \in \{x, y, z, t\}$, either L_u or R_u holds.

If L_x and L_t hold, then summing, we see that L holds. Similarly, assuming that neither L nor R holds, we get

$$L_x \Rightarrow \neg L_t \Rightarrow R_t \Rightarrow \neg R_y \Rightarrow L_y \Rightarrow \neg L_z \Rightarrow R_z.$$

Up to exchanging the role of z and t, we may assume that L_x holds, and therefore that so do R_t, L_y and R_z. Summing up $L_x + R_t + L_y + R_z$, we get that $L + R$ holds, so either L or R holds. ☐

The definition of the hyperbolic cone generalizes naturally to $\rho = \infty$, where one glues on each edge a sector of horoball with arclength l (explicitly, each triangle is isometric to $[0, l] \times [1, \infty)$ in the upper half-plane model of $\mathbb{H}^2$). The same argument shows that the a horospheric cone over any graph is also $2\delta_{\mathbb{H}^2}$-hyperbolic.

Lemma 4.8 (Very rotating condition)**.** *Recall that c is the apex of $C(Q)$. Assume that some group R acts on Q, and that $d_Q(y, gy) \geq 2\pi \sinh \rho$ for all $y \in Q, g \in R \setminus \{1\}$. Then for all $x_1, x_2 \in C(Q)$ such that $d(x_1, gx_2) < d(x_1, c) + d(x_2, c)$, then any geodesic from x_1 to x_2 in $C(Q)$ contains c. In particular, R satisfies the very rotating condition on $C(Q)$.*

Proof. For $i = 1, 2$, denote $x_i = t_i y_i$ with $y_i \in Q$. To prove that any geodesic $[x_1, x_2]$ contains the apex c, we have to check that $d_Q(y_1, y_2) \geq \pi \sinh \rho$. By the triangle inequality, no geodesic $[x_1, gx_2]$ contains c so $d_Q(y_1, gy_2) \leq \pi \sinh \rho$. By hypothesis on g, $d_Q(y_1, y_2) \geq d_Q(y_2, gy_2) - d_Q(gy_2, y_1) \geq 2\pi \sinh \rho - \pi \sinh \rho \geq \pi \sinh \rho$. ☐

3. Cone-off of a space over a family of subspaces

Let X be a δ-hyperbolic metric graph, whose edges all have the same length. Let $\mathcal{Q}$ be a family of almost convex subgraphs. We fix some radius $\rho > 0$. For every $Q \in \mathcal{Q}$, $C(Q)$ is the hyperbolic cone of radius ρ over Q. We denote its apex by c_Q. Later, we will also consider a moving family $\mathcal{F} = (R_Q)_{Q \in \mathcal{Q}}$.

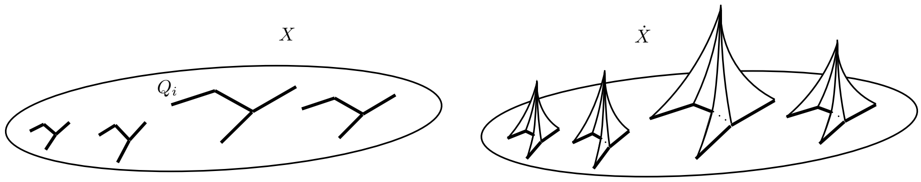
Definition 4.9. *The hyperbolic cone-off of X over $\mathcal{Q}$, of radius ρ, is the 2-complex*

$$\dot{X} = \left(X \sqcup \coprod_{Q \in \mathcal{Q}} (C(Q)) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation that identifies for each $Q \in \mathcal{Q}$, the subset of X defined by Q, and its image in $C(Q)$.

The metric on $\dot{X}$ is the corresponding path metric.

Although $C(Q)$ may fail to be isometrically embedded in $\dot{X}$, for any $R > 0$, any subset of $B(c_Q, \rho - R)$ of diameter at most R is isometrically embedded in $\dot{X}$.



Recall that we assume that every $Q \in \mathcal{Q}$ is *almost convex* in the following sense: for all $x, y \in Q$, there exist $x', y' \in Q$ such that $d(x, x') \leq 8\delta$, $d(y, y') \leq 8\delta$ and all geodesics $[x, x'], [x', y'], [y', y]$ are contained in Q. In particular, for all $x, y \in Q$, $d_X(x, y) \leq d_Q(x, y) \leq d_X(x, y) + 32\delta$.

Once hyperbolicity of $\dot{X}$ is established, Assertion 3 of Theorem 4.1 is immediate from Lemma 4.8. Indeed if $x \in Q$, and $\mathrm{inj}_X(\mathcal{F}) \geq 2\pi \sinh \rho$, then $d_Q(x, gx) \geq d_X(x, gx) \geq 2\pi \sinh \rho$ and Lemma 4.8 concludes that the very rotating property holds in $C(Q)$. Since the ball of radius $\rho/2$ in $C(Q)$ isometrically embeds in $\dot{X}$, and since the very rotating condition happens in the ball of radius 40δ around an apex, Assertion 3 of Theorem 4.1 holds as long as we take $\rho_0 \geq 80\delta_U$.

We note that if X is δ-hyperbolic, then it is 4δ-simply connected, hence so is $\dot{X}$ by the Van Kampen theorem (each $Q \in \mathcal{Q}$ is connected because it is almost convex). Thus, by the Cartan-Hadamard Theorem, to prove the hyperbolicity of $\dot{X}$, it is enough to prove that $\dot{X}$ is R-locally $2\delta_{\mathbb{H}^2}$-hyperbolic, with $R = R_{CH}(2\delta_{\mathbb{H}^2})$. In other words, Assertion 1 of Theorem 4.1 implies Assertion 2. Thus Theorem 4.1 follows from the following result.

Theorem 4.10. *Fix $R = R_{CH}(2\delta_{\mathbb{H}^2})$ as above. There exists $\delta_c, \Delta_c > 0$ such that for all δ_c-hyperbolic metric graph X whose edges have the same length, and for all Δ_c fellow-traveling family $\mathcal{Q}$ of almost convex subgraphs of X, and all $\rho > 7R$, the hyperbolic cone-off of radius ρ of X over $\mathcal{Q}$ is R-locally $(2\delta_{\mathbb{H}^2})$-hyperbolic.*

The limit case of the theorem is as follows.

Lemma 4.11. *Let T be an $\mathbb{R}$-tree, $\mathcal{Q}$ be a family of closed subtrees of T, any two of which intersect in at most one point. Then the cone-off $\dot{T}$ of T over $\mathcal{Q}$ is $\delta_{\mathbb{H}^2}$-hyperbolic (in fact $CAT(-1)$).*

Remark 4.12. We only defined the cone-off of a graph over a family of subgraphs, but the definition extends immediately to the setting of the lemma.

Proof of the lemma. If $\mathcal{Q}$ is finite and T is a finite metric tree, then $\dot{T}$ is $\delta_{\mathbb{H}^2}$-hyperbolic. For instance, this follows by induction on $\#\mathcal{Q}$ using the fact that the space obtained by gluing two $\delta_{\mathbb{H}^2}$-hyperbolic spaces over a point is $\delta_{\mathbb{H}^2}$-hyperbolic (see also [**BH99**, Th II.11.1]). For the general case, consider $x_1, x_2, x_3, x_4 \in \dot{T}$, and write $\dot{T}$ as an increasing union of cone-offs $\dot{S}_n$ of finite trees, with $\{x_1, x_2, x_3, x_4\} \subset \dot{S}_n$ for all n, such that $d_{\dot{S}_n}(x_i, x_j) \to d_{\dot{T}}(x_i, x_j)$. The 4-point inequality of $\dot{S}_n$ thus implies the 4-point inequality for $\dot{T}$. □

3.1. Ultralimits to prove local hyperbolicity

Let $\omega : 2^{\mathbb{N}} \to \{0,1\}$ be a non-principal ultrafilter. By definition, this is a finitely additive "measure" defined on all subsets of $\mathbb{N}$, such that $\omega(\mathbb{N}) = 1$, and $\omega(F) = 0$ for every finite subset $F \subset \mathbb{N}$. Zorn's Lemma shows that for any infinite subset E, there is a non-principal ultrafilter such that $\omega(E) = 1$. Given a sequence of properties P_i depending on $i \in \mathbb{N}$, we say that P_i holds for *ω-almost every i* if $\omega(\{i | P_i \textit{ true}\}) = 1$. Since ω takes values in $\{0,1\}$, if P_i does not hold for ω-almost every i, its negation holds for ω-almost every i. If $(t_i)_{i\in\mathbb{N}}$ is any sequence of real numbers, one can always define $\lim_\omega t_i \in [-\infty, \infty]$: this is the only $l \in [-\infty, \infty]$ such that for any neighborhood U of l, $t_i \in U$ for ω-almost every i. Thus, the ultrafilter ω *selects* an accumulation point of the sequence.

Let $(X_i, *_i)_{i\in\mathbb{N}}$ be a sequence of pointed metric spaces. Let $B \subset \prod_i X_i$ be the set of all sequences of points $(x_i)_{i\in\mathbb{N}}$ such that $d(x_i, *_i)$ is *bounded ω-almost everywhere*, i.e. on a subset of ω-measure 1 (equivalently $\lim_\omega d(x_i, *_i) < \infty$). By definition, the ultralimit of $(X_i, *_i)$ for ω is the metric space $X_\infty = B/\sim$ where $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ if $\lim_\omega d(x_i, y_i) = 0$, and where the distance between $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ is defined as $\lim_\omega d(x_i, y_i)$.

If $x_i \in X_i$ is a sequence of points such that $d(x_i, *_i)$ is bounded ω-almost everywhere, we define the ultralimit of x_i as the image of $(x_i)_{i\in\mathbb{N}}$ in X_∞.

We will use ultralimits in the following fashion. Note that we do not rescale our metric spaces, contrary to what one does in the construction of asymptotic cones. Assume that $(X_i)_{i\in\mathbb{N}}$ is a sequence of metric spaces such that any ultralimit of X_i is δ-hyperbolic (for any ultrafilter, and any base point $*_i$). Then for all $R, \varepsilon > 0$, X_i is R-locally $(\delta + \varepsilon)$-hyperbolic for i large enough. Indeed, if this does not hold, then there is a subsequence X_{i_k} and a subset $\{x_{i_k}, y_{i_k}, z_{i_k}, t_{i_k}\} \subset X_{i_k}$ of diameter at most R that contradicts the 4-point $(\delta+\varepsilon)$-hyperbolicity condition. Taking $*_i = x_i$ as a base point, and taking ω a non-principal ultrafilter such that $\omega(\{i_k\}_{k\in\mathbb{N}}) = 1$, we get an ultralimit X_∞ in which the ultralimit of the points $\{x_{i_k}, y_{i_k}, z_{i_k}, t_{i_k}\}$ contradicts δ-hyperbolicity.

3.2. Proof of the local hyperbolicity of the cone-off

PROOF OF THEOREM 4.10. Let R be given. We need to prove that any 4-point set $\{x, y, z, t\}$ of diameter at most R satisfies the $2\delta_{\mathbb{H}^2}$-hyperbolic inequality. We use that any subset of $B(c_Q, \rho - R)$ of diameter at most R is isometrically embedded in $\dot{X}$. Since $C(Q)$ is $2\delta_{\mathbb{H}^2}$-hyperbolic, we are done if $\{x, y, z, t\}$ is contained in $B(c_Q, \rho - R)$.

There remains to check that there exist Δ_c, δ_c such that the $2R$-neighborhood of X in $\dot{X}$ is R-locally $2\delta_{\mathbb{H}^2}$-hyperbolic. If not, there are two sequences δ_i, Δ_i converging to 0, such that for each $i \in \mathbb{N}$, one can find a counterexample as follows: there are

- a δ_i-hyperbolic space X_i,
- a family $\mathcal{Q}_i$ of almost convex subsets of X_i with $\Delta(\mathcal{Q}_i) \leq \Delta_i$, and
- a radius $\rho_i > 7R$,

so that the cone-off $\dot{X}_i$ of radius ρ_i of X_i over $\mathcal{Q}_i$ contains a subset $\{x_i, y_i, z_i, t_i\} \subset \dot{X}_i$ of diameter at most R for which the 4-point $2\delta_{\mathbb{H}^2}$-hyperbolicity inequality fails. Let $*_i \in X_i$ be a point at distance at most $2R$ from x_i. We note that $\{x_i, y_i, z_i, t_i\} \subset B(*_i, 3R)$.

Let ω be a non-principal ultrafilter, and $\dot{X}_\infty$ the ultralimit of $\dot{X}_i$ pointed at $*_i$. We denote by $* \in \dot{X}_\infty$ the ultralimit of the sequence $(*_i)_{i\in\mathbb{N}}$. Let $x, y, z, t \in \dot{X}_\infty$ be the ultralimit of the points x_i, y_i, z_i and t_i. Since $2\delta_{\mathbb{H}^2} > \delta_{\mathbb{H}^2}$, to get a contradiction, it is enough to prove that x, y, z, t satisfy the 4-point $\delta_{\mathbb{H}^2}$-hyperbolicity inequality.

We want to compare $\dot{X}_\infty$ with the cone-off on an $\mathbb{R}$-tree. Let T be the ultralimit of X_i pointed at $*_i$ (this is an $\mathbb{R}$-tree). We denote by $*_T \in T$ the ultralimit of $(*_i)_{i\in\mathbb{N}}$. To define a cone-off of T, we need to define a family of subtrees $\mathcal{Q}$ of T. Given a sequence of subspaces $\mathbf{Q} = (Q_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{Q}_i$ (each $Q_i \in \mathcal{Q}_i$ is a subset of X_i,), we say that this sequence is *non-escaping* if there exists $q_i \in Q_i$ such that $d(q_i, *_i)$ is bounded ω-almost everywhere, Let $\mathcal{Q}_\infty \subset (\prod_{i\in\mathbb{N}} \mathcal{Q}_i)/\sim_\omega$ be the set of non-escaping sequences up to equality ω-almost everywhere. Given $\mathbf{Q} = (Q_i)_{i\in\mathbb{N}}$ a non-escaping sequence, let $\mathbf{Q}_\omega$ be the ultralimit of $(Q_i)_{i\in\mathbb{N}}$ based at q_i. Note that this ultralimit does not depend on the choice of q_i as long as $d(q_i, *_i)$ is bounded ω-almost everywhere. There is a natural map $\mathbf{Q}_\omega \to T$ induced by the inclusions $Q_i \to X_i$. This map is an isometry because the inclusion $Q_{j_i} \to X_i$ is an isometry up to an additive constant bounded by $32\delta_i$, and δ_i converges to 0. Thus we identify $\mathbf{Q}_\omega$ with its image in T. Then we define the collection of all possible such subsets $\mathbf{Q}_\omega$ by $\mathcal{Q} = (\mathbf{Q}_\omega)_{\mathbf{Q}\in\mathcal{Q}_\infty}$, and we consider $\dot{T}$ the corresponding cone-off with radius $\rho = \lim_\omega \rho_i$ (note that ρ might be infinite, in which case we construct the corresponding horospheric cone-off).

Lemma 4.13.

1. *For $\mathbf{Q} \neq \mathbf{Q}' \in \mathcal{Q}_\infty$, $\mathbf{Q}_\omega \cap \mathbf{Q}'_\omega$ contains at most one point. In particular $\dot{T}$ is $\delta_{\mathbb{H}^2}$-hyperbolic.*
2. *There is a natural 1-Lipschitz map $\dot{\psi} : \dot{T} \to \dot{X}_\infty$ that maps isometrically $B_{\dot{T}}(*_T, 3R)$ to $B_{\dot{X}_\infty}(*, 3R)$.*

The lemma allows us to conclude the proof: $\{x, y, z, t\} \subset B_{\dot{X}_\infty}(*, 3R)$, which is isometric to a subset of the $\delta_{\mathbb{H}^2}$-hyperbolic space $\dot{T}$, so x, y, z, t satisfy the 4-point $\delta_{\mathbb{H}^2}$-hyperbolicity inequality. □

Proof of Lemma 4.13. For Assertion 1, consider $\mathbf{Q} = (Q_i)_{i\in\mathbb{N}}$, $\mathbf{Q}' = (Q'_i)_{i\in\mathbb{N}}$ with $Q_i \neq Q'_i$ for ω-almost every i. Given $x \in \mathbf{Q}_\omega \cap \mathbf{Q}'_\omega$, there are sequences $(x_i)_{i\in\mathbb{N}}$, $(x'_i)_{i\in\mathbb{N}}$ representing x such that $x_i \in Q_i, x'_i \in Q'_i$. In particular $\lim_\omega d(x_i, x'_i) = 0$. If $y \in \mathbf{Q}_\omega \cap \mathbf{Q}'_\omega$ is another point, there exist similarly, $y_i \in Q_i, y'_i \in Q'_i$ representing y, so that in particular, $\lim_\omega d(y_i, y'_i) = 0$. If $x \neq y$, then $d(x, y) > 0$, so $d(x_i, y_i)$ and $d(x'_i, y'_i)$ are bounded below by $d(x, y)/2$ for ω-almost every i. By almost convexity, we see that Q_i fellow travels Q'_i by at least $d(x, y)/4$ for ω-almost every i. Since Δ_i tends to 0, we get $Q_i = Q'_i$ for almost every i, so $\mathbf{Q} = \mathbf{Q}'$, a contradiction. This proves Assertion 1.

Now we define the map $\dot{\psi} : \dot{T} \to \dot{X}_\infty$. Inclusions $\varphi_{X_i} : X_i \to \dot{X}_i$ are 1-Lipschitz and define naturally a 1-Lipschitz map $\psi : T \to \dot{X}_\infty$. Similarly, given $\mathbf{Q} = (Q_i)_{i\in\mathbb{N}} \in \mathcal{Q}_\infty$, the inclusions $\varphi_{C(Q_i)} : C_{\rho_i}(Q_i) \to \dot{X}_i$ induce a 1-Lipschitz map $\psi_{C(\mathbf{Q}_\omega)} : C_\rho(\mathbf{Q}_\omega) \to \dot{X}_\infty$. Since for each $\mathbf{Q} \in \mathcal{Q}_\infty$, ψ coincides with $\psi_{C_\rho(\mathbf{Q}_\omega)}$ in restriction to $\mathbf{Q}_\omega$, these maps induce a 1-Lipschitz map $\dot{\psi} : \dot{T} \to X_\infty$. Note that in general, $\dot{\psi}$ may be not onto.

To prove Assertion 2, we define a partial inverse ψ' of $\dot{\psi}$. Given $x \in B_{\dot{X}_\infty}(*, 3R)$, represent x by a sequence $x_i \in \dot{X}_i$ with $d_{\dot{X}_i}(x_i, *_i) \leq 3R$.

If x_i lies in X_i (i.e. not in the interior of a cone) for ω-almost every i, we want to define $\psi'(x)$ as the ultralimit in T of x_i. For this ultralimit to exist, we have to prove that $d_{X_i}(x_i, *_i)$ is bounded ω-almost everywhere. But since $\rho_i > 3R$, any geodesic $[*_i, x_i]$ avoids the $\rho_i - 3R$ neighbourhood of any apex. Now there exists M such that the radial projection is locally M-Lipschitz (independently of ρ_i, see Exercise 4.15). It follows that the radial projection of this geodesic has length bounded by $3RM$, so the ultralimit of x_i in T exists.

Similarly, if x_i lies in a cone for ω-almost every i, write $x_i = s_i y_i$ for some $s_i < \rho_i$, and $y_i \in X_i$. The argument above shows that $d_{X_i}(*_i, y_i)$ is bounded, so the ultralimit of y_i in T exists, we denote it by y. Moreover, the sequence $\mathbf{Q}$ of cones Q_i containing x_i is non-escaping, so we can define $\psi'(x)$ as sy in the cone $\mathbf{Q}_\omega$, $s = \lim_\omega s_i$.

It is clear from the definition that $\psi'(x)$ is a preimage of x under $\dot{\psi}$. There remains to show that ψ' is 1-Lipschitz. It is based on the following technical fact, proved below.

Fact 4.14. *For any $\rho_0, \varepsilon, D_0 > 0$, there exists $n \in \mathbb{N}$ such that the following holds. Consider a graph X, and a cone-off $\dot{X}$ of radius $\rho \geq \rho_0$. Then for any pair of points $x, y \in \dot{X}$ with $d(x,y) \leq D_0$, there is a path p joining x to y in $\dot{X}$ such that*

- *the length of p is at most $d_{\dot{X}}(x,y) + \varepsilon$*
- *p is a concatenation of at most n paths, each of which is either contained in X or in a cone $C(Q)$.*

To conclude, take $\rho_0 = 7R$, $\rho > 7R$, $\varepsilon > 0$, and let $D_0 = 6R + 3\varepsilon$. We assume that ε is small enough so that $D_0 < 7R$. Consider n given by the fact. Consider $x, y \in B_{\dot{X}_\infty}(*, 3R)$, write x and y as an ultralimit of sequences $x_i, y_i \in B_{\dot{X}_i}(*_i, 3R+\varepsilon)$. Consider p_i a path joining x_i, y_i of length at most $d_{\dot{X}_i}(x_i, y_i) + \varepsilon \leq D_0$ and which is a concatenation of at most n sub-paths as in the fact. We can assume that p is a concatenation of exactly n subpaths: $p_i = p_i^1 \cdot p_i^2 \cdots p_i^n$. Because $D_0 < 7R < \rho$, p_i stays at distance at least $7R - D_0$ from the cone point. Fix $k \in \{1, \dots, n\}$. If $p_i^k \subset X_i$ for ω-almost every i, then the ultralimit of p_i^k defines a path $p_\infty^k \subset T$. Otherwise, p_i^k is contained in a cone for ω-almost every i. Since the distance from p_i^k to the cone point is bounded below by $7R - D_0 > 0$, its radial projection has bounded length by Exercise 4.15. It follows that the radial projections of p_i^k converge to a path of finite length in T. Thus, writing p_i^k as a map $p_i^k : [0,1] \to Q_{i,k} \times [0, \rho_i]$, and taking an ultralimit defines a path p_∞^k in $\dot{T}$. We thus get a path $p_\infty = p_\infty^1 \cdots p_\infty^n$ joining $\psi'(x)$ to $\psi'(y)$, and whose length is at most $\lim_\omega d_{\dot{X}_i}(x_i, y_i) + \varepsilon = d_{\dot{X}}(x,y) + \varepsilon$. □

Proof of Fact 4.14. Consider $p \subset C(Q)$ a geodesic path avoiding the apex and whose endpoints are in Q. Denote by l its length and L the length of its radial projection. We claim that $\frac{L}{l}$ goes to 1 as l tends to 0 independently of ρ. More precisely, we claim that for any $\lambda > 1$, there exists $\eta > 0$ such that if $l \leq \eta$, then $L \leq \lambda l$ where η does not depend on ρ as long as $\rho \geq R_0$.

To prove the claim, we use that l and L are related by the relation

$$\cosh l = \cosh^2 \rho - \sinh^2 \rho \cos\left(\frac{L}{\sinh \rho}\right)$$

which can be rewritten as

$$\sinh l/2 = \sinh\rho \sin\left(\frac{L}{2\sinh\rho}\right).$$

Since $\sin(x) \geq x - \frac{1}{6}x^3$ and $\operatorname{argsh}(x) \geq x - \frac{1}{6}x^3$ for all $x \geq 0$, one can deduce the estimate $L \geq l \geq L - \frac{1}{24}(1 + \frac{1}{\sinh^2\rho})L^3$. Since $\rho \geq \rho_0 > 0$, we get $\frac{l}{L} \geq 1 - \frac{1}{24}(1 + \frac{1}{\sinh^2\rho_0})L^2$ and the claim follows.

To prove the fact, consider a path p in $\dot{X}$ joining x to y, of length $d(x,y)+\varepsilon/2$. We can assume that p is a concatenation of paths $p_1,\dots,p_k$ where each p_i is either contained in a cone, or contained in X. If two consecutive paths are contained in the same cone or are both contained in X, we can replace them by their concatenation to decrease k. Thus for each $i \in \{2,\dots,k-1\}$, if p_i is contained in a cone $C(Q)$, then the endpoints of p_i are in Q. Let $\lambda = \frac{R_0+\varepsilon}{R_0+\varepsilon/2} \leq \frac{d(x,y)+\varepsilon}{d(x,y)+\varepsilon/2}$, and consider η as in the claim above. For each $i \in \{2,\dots,k-1\}$ such that p_i is contained in a cone and has length at most η, we replace it by its radial projection p'_i, and we define $p'_i = p_i$ for all other indices i. The length of the obtained new path p' is at most $\lambda(d(x,y)+\varepsilon/2) \leq d(x,y)+\varepsilon$. Since each p'_i that is not contained in X has length at least η, there are at most $n_0 = (R_0+\varepsilon)/\eta$ such sub-paths. By concatenation of consecutive paths contained in X, we get that p' is a concatenation of at most $2n_0+3$ paths, each of which is either contained in a cone, or contained in X. □

Exercise 4.15. *Given $\rho > 0$ denote by $p_\rho : B_{\mathbb{H}^2}(0,\rho) \setminus 0 \to S(0,\rho)$ the radial projection on $S(0,\rho)$, the circle of radius ρ. Prove that given $r, \rho_0 > 0$, there is a constant M such that for any $\rho \in [\rho_0 + r, \infty)$, the restriction of the radial projection to of $B(0,\rho) \setminus \mathring{B}(0,\rho - r)$ is locally M-Lipschitz.*

Hint: Since the closest point projection $\mathbb{H}^2 \to B(0,\rho - r)$ is distance decreasing, it is enough to bound the Lipschitz constant of the restriction of p_ρ to the circle of radius $\rho - r$. Using polar coordinates, prove that this follows from the fact that $\frac{\sinh\rho}{\sinh(\rho-r)}$ decreases with ρ.

Bibliography

[AD] Goulnara N. Arzhantseva and Thomas Delzant. Examples of random groups. Journal of Topology, to appear.

[BO08] Igor Belegradek and Denis Osin. Rips construction and Kazhdan property (T). *Groups Geom. Dyn.*, 2(1):1–12, 2008.

[BBF10] Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara. The asymptotic dimension of mapping class groups is finite, 2010.

[BF10] Mladen Bestvina and Mark Feighn. A hyperbolic Out(F_n)-complex. *Groups Geom. Dyn.*, 4(1):31–58, 2010.

[BF07] Mladen Bestvina and Koji Fujiwara. Quasi-homomorphisms on mapping class groups. *Glas. Mat. Ser. III*, 42(62)(1):213–236, 2007.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BW11] Martin R. Bridson and Richard D. Wade. Actions of higher-rank lattices on free groups. *Compos. Math.*, 147(5):1573–1580, 2011.

[BW05] Inna Bumagin and Daniel T. Wise. Every group is an outer automorphism group of a finitely generated group. *J. Pure Appl. Algebra*, 200(1-2):137–147, 2005.

[BM00] Marc Burger and Shahar Mozes. Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.*, (92):151–194 (2001), 2000.

[Can11] Serge Cantat. Sur les groupes de transformations birationnelles des surfaces. *Ann. of Math. (2)*, 174(1):299–340, 2011.

[CL] Serge Cantat and Stéphane Lamy. Normal subgroups in the cremona group. arXiv:1007.0895 [math.AG].

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy. Simplicity and superrigidity of twin building lattices. *Invent. Math.*, 176(1):169–221, 2009.

[Cha94] Christophe Champetier. Petite simplification dans les groupes hyperboliques. *Ann. Fac. Sci. Toulouse Math. (6)*, 3(2):161–221, 1994.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov.

[Cou] Rémi Coulon. Small cancellation theory and burnside problem. `www.math.vanderbilt.edu/~coulonrb/papiers/notes.pdf`.

[Cou11] Rémi Coulon. Asphericity and small cancellation theory for rotation families of groups. *Groups Geom. Dyn.*, 5(4):729–765, 2011.

[DGO] François Dahmani, Vincent Guirardel, and Denis Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. Memoirs of the A.M.S, to appear. arXiv:1111.7048 [math.GR].

[Del96] Thomas Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.*, 83(3):661–682, 1996.

[DG08] Thomas Delzant and Misha Gromov. Courbure mésoscopique et théorie de la toute petite simplification. *J. Topol.*, 1(4):804–836, 2008.

[Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.

[Gre60] Martin Greendlinger. Dehn's algorithm for the word problem. *Comm. Pure Appl. Math.*, 13:67–83, 1960.

[Gro01a] Misha Gromov. Mesoscopic curvature and hyperbolicity. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, volume 288 of *Contemp. Math.*, pages 58–69. Amer. Math. Soc., Providence, RI, 2001.

[Gro01b] Misha Gromov. CAT(κ)-spaces: construction and concentration. *Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI)*, 280(Geom. i Topol. 7):100–140, 299–300, 2001.

[Gro03] Misha Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[GM08] Daniel Groves and Jason Fox Manning. Dehn filling in relatively hyperbolic groups. *Israel J. Math.*, 168:317–429, 2008.

[HLS02] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[Iva94] Sergei V. Ivanov. The free Burnside groups of sufficiently large exponents. *Internat. J. Algebra Comput.*, 4(1-2):ii+308, 1994.

[KM96] Vadim A. Kaimanovich and Howard Masur. The Poisson boundary of the mapping class group. *Invent. Math.*, 125(2):221–264, 1996.

[LS01] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[Lys96] I. G. Lysënok. Infinite Burnside groups of even period. *Izv. Ross. Akad. Nauk Ser. Mat.*, 60(3):3–224, 1996.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[NA68] P. S. Novikov and S. I. Adjan. Infinite periodic groups. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 32:212–244, 1968.

[OW07] Yann Ollivier and Daniel T. Wise. Kazhdan groups with infinite outer automorphism group. *Trans. Amer. Math. Soc.*, 359(5):1959–1976, 2007.

[Ol′80] A. Ju. Ol′šanskiĭ. An infinite group with subgroups of prime orders. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(2):309–321, 479, 1980.

[Ol′82] A. Yu. Ol′shanskiĭ. The Novikov-Adyan theorem. *Mat. Sb. (N.S.)*, 118(160)(2):203–235, 287, 1982.

[Ol'91a] A. Yu. Ol'shanskiĭ. *Geometry of defining relations in groups*, volume 70 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 1989 Russian original by Yu. A. Bakhturin.

[Ol'91b] A. Yu. Ol'shanskiĭ. Periodic quotient groups of hyperbolic groups. *Mat. Sb.*, 182(4):543–567, 1991.

[Ol′95] A. Yu. Ol′shanskiĭ. SQ-universality of hyperbolic groups. *Mat. Sb.*, 186(8):119–132, 1995.

[OOS09] Alexander Yu. Ol′shanskii, Denis V. Osin, and Mark V. Sapir. Lacunary hyperbolic groups. *Geom. Topol.*, 13(4):2051–2140, 2009. With an appendix by Michael Kapovich and Bruce Kleiner.

[Osi10] Denis Osin. Small cancellations over relatively hyperbolic groups and embedding theorems. *Ann. of Math. (2)*, 172(1):1–39, 2010.

[Osi07] Denis V. Osin. Peripheral fillings of relatively hyperbolic groups. *Invent. Math.*, 167(2):295–326, 2007.

[Rip82] E. Rips. Subgroups of small cancellation groups. *Bull. London Math. Soc.*, 14(1):45–47, 1982.

[Sis11] Alessandro Sisto. Contracting elements and random walks, 2011.

[Tar49] V. A. Tartakovskiĭ. Solution of the word problem for groups with a k-reduced basis for $k > 6$. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 13:483–494, 1949.

[Wis03] Daniel T. Wise. A residually finite version of Rips's construction. *Bull. London Math. Soc.*, 35(1):23–29, 2003.

[Wis11] Daniel T. Wise. The structure of groups with quasiconvex hierarchy, 2011. available at `http://www.math.mcgill.ca/wise/papers.html`.