The art of modeling water waves
Habilitation à Diriger des Recherches

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**Why “modeling”?**

Replace “complicated’ set of equations with “simple” set of equations.

1. To enlighten the basic mechanisms of a phenomenon
   - Wavebreaking: \( \partial_t u + u \partial_x u = 0 \) (Hopf)
   - Solitary waves: \( \partial_t u + u \partial_x u + \partial_x^3 u = 0 \) (KdV)
   - Non-smooth solitary waves (or wave breaking and solitary waves):
     \[
     \partial_t u + \sqrt{\frac{\tanh(|D|)}{|D|}} \partial_x \zeta + \zeta \partial_x \zeta = 0
     \]
     \[
     \text{(Whitham)}
     \]

2. To produce approximate solutions (e.g. numerical)
   - \( O(\mu) \): \( \partial_t \zeta + \sqrt{gd} \left( \partial_x \zeta + \frac{3}{2d} \zeta \partial_x \zeta \right) = 0 \) (Hopf)
   - \( O(\mu^2 + \mu \epsilon) \): \( \partial_t \zeta + \sqrt{gd} \left( \partial_x \zeta + \frac{3}{2d} \zeta \partial_x \zeta + d^2 \partial_x^3 \zeta \right) = 0 \) (KdV)
   - \( O(\mu \epsilon) \): \( \partial_t \zeta + \sqrt{gd} \left( \sqrt{\frac{\tanh(d|D|)}{d|D|}} \partial_x \zeta + \frac{3}{2d} \zeta \partial_x \zeta \right) = 0 \) (Whitham)

3. To publish papers. To have fun.
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Why “water waves”?

Figure: Water waves, by Anouk and Lucie Duchêne

[Feynman] “[water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...] ; they have all the complications that waves can have.”

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Why “the art”?  

There will be traps. Avoiding them will have a cost. We will make choices, with benefits and downsides.

A useful tool: theorems.
Why “the art”? 

There will be traps. Avoiding them will have a cost. We will make choices, with benefits and downsides.

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3 Shallow water models
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All you need to know about the water waves system (today)

Warning: the following applies only to inviscid, incompressible, homogeneous, irrotational flows. Serving suggestion.

Zakharov/Craig-Sulem formulation [Zakharov ’68, Craig&Sulem ’93]

\[
\begin{align*}
\partial_t \zeta - \frac{\delta \mathcal{H}}{\delta \psi} &= 0, \\
\partial_t \psi + \frac{\delta \mathcal{H}}{\delta \zeta} &= 0,
\end{align*}
\]

(WW)

with

\[
\mathcal{H} (\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi G^{\mu} [\epsilon \zeta] \psi \, dx
\]

where the Dirichlet-to-Neumann operator \( G^{\mu} [\epsilon \zeta] \psi \) is defined by

\[
G^{\mu} [\epsilon \zeta] \psi = (\frac{1}{\mu} \partial_z \Phi - \epsilon \nabla \zeta \cdot \nabla_x \Phi) |_{z=\epsilon \zeta}
\]

with \( \Phi \) solution to

\[
\begin{align*}
\mu \Delta_x \Phi + \partial_z^2 \Phi &= 0 & \text{in } \{(x, z), -1 < z < \epsilon \zeta (t, x)\}, \\
\Phi &= \psi & \text{on } \{(x, z), z = \epsilon \zeta (t, x)\}, \\
\partial_z \Phi &= 0 & \text{on } \{(x, z), z = -1\}.
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\]

Water waves \( = (\text{Hyperbolic}) \times (\text{Elliptic}) \).
Our journey starts with ripples

We set $\epsilon = 0$

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\mathcal{H}(\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi \mathcal{G}^\mu [\epsilon \zeta] \psi \, dx
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where the Dirichlet-to-Neumann operator $\mathcal{G}^\mu [\epsilon \zeta]$ is defined by

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\mathcal{H}(\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi \mathcal{G}_\mu[0] \psi \, dx
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where the Dirichlet-to-Neumann operator \( \mathcal{G}_\mu[0] \) is defined by

\[
\mathcal{G}_\mu[0] \psi = (\frac{1}{\mu} \partial_z \Phi)|_{z=0} = \frac{1}{\sqrt{|\mu|}} |D| \tanh(\sqrt{|\mu|} |D|) \psi
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Zakharov/Craig-Sulem formulation [Zakharov '68, Craig&Sulem '93]

\[
\begin{align*}
\partial_t \zeta - \frac{1}{\sqrt{\mu}} |D| \tanh(\sqrt{\mu} |D|) \psi &= 0, \\
\partial_t \psi + \zeta &= 0,
\end{align*}
\]

(Airy)

with

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\mathcal{H}(\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \frac{1}{\sqrt{\mu}} \psi |D| \tanh(\sqrt{\mu} |D|) \psi \, dx
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where the Dirichlet-to-Neumann operator $G^{\mu}[0]$ is defined by

\[
G^{\mu}[0]\psi = (\frac{1}{\mu} \partial_z \Phi)|_{z=0} = \frac{1}{\sqrt{\mu}} |D| \tanh(\sqrt{\mu} |D|) \psi
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with $\Phi$ solution to

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\begin{cases}
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\end{cases}
\]
Lessons from the modal analysis

\[ \begin{align*}
\partial_t \zeta - \frac{1}{\sqrt{\mu}} |D| \tanh(\sqrt{\mu} |D|) \psi &= 0, \\
\partial_t \psi + \zeta &= 0.
\end{align*} \] (Airy)

Dispersion relation (for plane waves \( \propto e^{ik \cdot x - i \omega t} \))

\[ c^2 \overset{\text{def}}{=} \frac{\omega^2}{|k|^2} = \frac{\tanh(\sqrt{\mu} |k|)}{\sqrt{\mu} |k|}. \]

Some approximations (valid when \( \sqrt{\mu} |k| \ll 1 \))

1. \( c^2 = 1 \) \( \checkmark \) Non-dispersive. Relative error less than 10% for \( \sqrt{\mu} |k| < 0.055 \).
2. \( c^2 = 1 - \frac{\mu}{3} |k|^2 \)
3. \( c^2 = \frac{1}{1 + \frac{\mu}{3} |k|^2} \)
4. \( c^2 = 1 - \frac{\mu}{3} |k|^2 + \frac{2\mu^2}{15} |k|^4 + \ldots \)
5. \( c^2 = \frac{1}{1 + \frac{\mu}{3} |k|^2 - \frac{\mu^2}{45} |k|^4 + \ldots} \)
6. \( c^2 = \frac{1}{1 + \frac{\mu}{3} |k|^2 - \frac{\mu^2}{45} |k|^4 + \ldots} \)

\( \checkmark \) Padé approximant. Uniformly convergent.

\( \times \) Series do not converge for \( \sqrt{\mu} |k| > \frac{\pi}{2} \).

\( \times \) Series do not converge for \( \sqrt{\mu} |k| > \pi \).
1 About the title

2 Water waves and ripples

3 Shallow water models
   - Derivation
   - Justification
   - Numerical simulation

4 Higher order models
   - A unified framework
   - Interfacial waves
Switching back on the nonlinearity

Recall the **Dirichlet-to-Neumann operator** $G^\mu[\epsilon \zeta] \psi$ is defined by

$$G^\mu[\epsilon \zeta] \psi = \left( \frac{1}{\mu} \partial_z \Phi - \epsilon \nabla \zeta \cdot \nabla_x \Phi \right) |_{z=\epsilon \zeta}$$

with $\Phi$ solution to

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\end{cases}$$

An equivalent formulation is

$$G^\mu[\epsilon \zeta] \psi = -\nabla \cdot \left( \int_{-1}^{\epsilon \zeta(t, \cdot)} \nabla_x \Phi(\cdot, z) \, dz \right)$$

with $\Phi$ solution to

$$\Phi + \mu \ell[\epsilon \zeta] \Phi = \psi, \quad \ell[\epsilon \zeta] \Phi(\cdot, z) \overset{\text{def}}{=} -\int_z^{\epsilon \zeta} \int_{-1}^{z'} \Delta_x \Phi(\cdot, z'') \, dz'' \, dz'.$$

We infer approximate formula at any order $O(\mu^N)$:

$$\Phi = \psi + O(\mu), \quad \Phi = \psi - \mu \ell[\epsilon \zeta] \psi + O(\mu^2), \quad \ldots$$
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\]

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\[
\Phi = \psi + \mathcal{O}(\mu), \quad \Phi = \psi - \mu \ell[\epsilon \zeta] \psi + \mathcal{O}(\mu^2), \ldots
\]

This yields approximations to the Dirichlet-to-Neumann operator:

\[
\checkmark \quad G^\mu[\epsilon \zeta] \psi = -\nabla \cdot ((1 + \epsilon \zeta) \nabla \psi) + \mathcal{O}(\mu),
\]

\[
\times \quad G^\mu[\epsilon \zeta] \psi = -\nabla \cdot (h \nabla \psi) + \mu \nabla \cdot (h T[h] \nabla \psi) + \mathcal{O}(\mu^2),
\]

\[
\checkmark \quad G^\mu[\epsilon \zeta] \psi = -\nabla \cdot \left( h (1 + \mu T[h])^{-1} \nabla \psi \right) + \mathcal{O}(\mu^2),
\]

with \( h = 1 + \epsilon \zeta \) and \( T[h] u = \frac{-1}{3h} \nabla (h^3 \nabla \cdot u) \).
Approximations to the Dirichlet-to-Neumann operator

For any sufficiently regular $\zeta$ such that

$$\forall x \in \mathbb{R}^d, \quad h(x) \overset{\text{def}}{=} 1 + \epsilon \zeta(x) \geq h_* > 0,$$

one has for any $k \in \mathbb{N}$, $\epsilon \geq 0$ and $\mu \in (0, 1]$,

**✓** $|G^\mu [\epsilon \zeta] \psi + \nabla \cdot ((1 + \epsilon \zeta) \nabla \psi)|_{H^k} \leq C_{k+4} \mu |\nabla \psi|_{H^{k+3}},$

**✗** $|G^\mu [\epsilon \zeta] \psi + \nabla \cdot (h \nabla \psi) - \mu \nabla \cdot (h T[h] \nabla \psi)|_{H^k} \leq C_{k+6} \mu^2 |\nabla \psi|_{H^{k+5}},$

**✓** $|G^\mu [\epsilon \zeta] \psi + \nabla \cdot \left( h (\text{Id} + \mu T[h])^{-1} \nabla \psi \right)|_{H^k} \leq C_{k+6} \mu^2 |\nabla \psi|_{H^{k+5}},$

with $C_k = C(k, h_*^{-1}, |\epsilon \zeta|_{H^n})$ and $T[h]u \overset{\text{def}}{=} \frac{-1}{3h} \nabla (h^3 \nabla \cdot u)$.

Plugging these approximations in the water waves equations yields...
Historical shallow-water models

The Saint-Venant system

\[
\begin{aligned}
\partial_t \zeta + \nabla \cdot ((1 + \epsilon \zeta)u) &= 0, \\
\partial_t u + \nabla \zeta + \epsilon (u \cdot \nabla)u &= 0,
\end{aligned}
\]  

(SV)

with \( u = \nabla \psi \) (or \( u = \bar{u} \overset{\text{def}}{=} \frac{1}{1+\epsilon \zeta} \int_{-1}^{\epsilon \zeta} \nabla_x \Phi(\cdot, z) \, dz \)).

The Green–Naghdi system

\[
\begin{aligned}
\partial_t \zeta + \nabla \cdot (hu) &= 0, \\
(\text{Id} + \mu \mathcal{T}[h]) \partial_t u + \nabla \zeta + \epsilon (u \cdot \nabla)u + \mu \epsilon Q[h, u] &= 0,
\end{aligned}
\]  

(GN)

where \( h \overset{\text{def}}{=} 1 + \epsilon \zeta \), \( Q[h, u] \overset{\text{def}}{=} -\frac{1}{3h} \nabla \left( h^3 ((u \cdot \nabla)(\nabla \cdot u) - (\nabla \cdot u)^2) \right) \), and \( \mathcal{T}[h]u \overset{\text{def}}{=} -\frac{1}{3h} \nabla (h^3 \nabla \cdot u) \) with \( u = (\text{Id} + \mu \mathcal{T}[h])^{-1} \nabla \psi \) (or \( u = \bar{u} \)).
Historical shallow-water models

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\begin{align*}
\frac{\partial}{\partial t} \zeta + \nabla \cdot ((1 + \epsilon \zeta)u) &= 0, \\
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(SV)

with \( u = \nabla \psi \) (or \( u = \overline{u} \overset{\text{def}}{=} \frac{1}{1 + \epsilon \zeta} \int_{-1}^{\epsilon \zeta} \nabla_x \Phi(\cdot, z) \, dz \)).

A special case of \textit{compressible} Euler equations. Finite-time singularity formation. Used when the problem features dry zones, discontinuities (dam-break), etc.

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where \( h \overset{\text{def}}{=} 1 + \epsilon \zeta \), \( Q[h, u] \overset{\text{def}}{=} \frac{-1}{3h} \nabla \left( h^3 ((u \cdot \nabla)(\nabla \cdot u) - (\nabla \cdot u)^2) \right) \), and

\[
\mathcal{T}[h]u \overset{\text{def}}{=} \frac{-1}{3h} \nabla (h^3 \nabla \cdot u) \text{ with } u = (\text{Id} + \mu \mathcal{T}[h])^{-1} \nabla \psi \text{ (or } u = \overline{u}).
\]

Explicit family of solitary waves. Globally well-posed?

A \textit{lot} of activity around (GN) recently.
The full justification of a model typically stems from the combination of

1. **Consistency**
   Regular solutions to the water waves system satisfy approximately the model

2. **Well-posedness**
   Existence and control of solutions on a relevant time interval

3. **Stability**
   Control of the difference between exact and approximate solutions of the model

\[ \Rightarrow \text{Control of } \epsilon, \text{ the difference between the solution to the water waves system and the corresponding solution to the model.} \]

**The Saint-Venant system** is a quasilinear hyperbolic symmetrizable system.

\[ \Rightarrow [\text{Friedrichs, Garding, Kato '50s}] \ \text{WP and Stability in } H^s(\mathbb{R}^d)^{1+d}, \ s > 1 + d/2 \]

\[ |\epsilon_{SV}|_{H^k} \lesssim \mu \ t, \quad t \lesssim 1/\epsilon. \]
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Fully rigorous justification

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\[ \leadsto \text{Control of } \epsilon, \text{ the difference between the solution to the water waves system and the corresponding solution to the model.} \]

**The Green–Naghdī system** is a “quasilinear hyperbolic symmetrizable system”.

\[ \leadsto [\text{Li '06, Fujiwara & Iguchi '15}] \] WP and Stability in \( H^s(\mathbb{R}^d) \times X^s, s > 1 + d/2 \)

\[ X^s \overset{\text{def}}{=} \left\{ u : \| u \|^2_{X^s} = \| u \|^2_{H^s} + \mu \| \nabla \cdot u \|^2_{H^s} < \infty \right\}. \]

\[ \| \epsilon_{\text{GN}} \|_{H^k \times X^k} \lesssim \mu^2 t, \quad t \lesssim 1/\epsilon. \]
Hyperbolic reformulation

Recall The Green–Naghdi system

\[
\begin{align*}
\partial_t \zeta + \nabla \cdot (hu) &= 0, \\
(\text{Id} + \mu \mathcal{T}[h]) \partial_t u + \nabla \zeta + \epsilon (u \cdot \nabla)u + \mu \epsilon Q[h, u] &= 0,
\end{align*}
\]

where \( h \overset{\text{def}}{=} 1 + \epsilon \zeta \), \( \mathcal{T}[h]u = -\frac{1}{3h} \nabla (h^3 \nabla \cdot u) \) and
\[
Q[h, u] \overset{\text{def}}{=} -\frac{1}{3h} \nabla \left( h^3 ((u \cdot \nabla)(\nabla \cdot u) - (\nabla \cdot u)^2) \right).
\]

In numerical simulations, we need to solve at each timestep (for \( u \))

\[
(\text{Id} + \mu \mathcal{T}[h]) u = v.
\]

The Green–Naghdi system can be written as

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\partial_t u + \nabla \zeta + \epsilon (u \cdot \nabla)u + \frac{\mu}{3h} \nabla (hq) &= 0, \\
\frac{q}{h} &= \partial_t v + \epsilon u \cdot \nabla v, \quad v = \partial_t \zeta + \epsilon u \cdot \nabla \zeta = -h \nabla \cdot u
\end{align*}
\]

3 evolution equations + constraint. \( \rightsquigarrow \) relaxation methods.
Hyperbolic reformulation

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\]  

(\text{GN})

3 evolution equations + constraint. \(\sim\) relaxation methods.

[Favrie&Gavrilyuk '17] proposed

\[
\begin{aligned}
\partial_t \zeta + \nabla \cdot ((1 + \epsilon \zeta)u) &= 0, \\
\partial_t u + \nabla \zeta + \epsilon (u \cdot \nabla)u - \frac{\lambda \mu}{3h} \nabla \left( \frac{1+\epsilon \eta}{1+\epsilon \zeta} (\eta - \zeta) \right) &= 0, \\
\partial_t w + \epsilon u \cdot \nabla w &= -\frac{\lambda}{h^2} (\eta - \zeta), \\
\partial_t \eta + \epsilon u \cdot \nabla \eta &= w.
\end{aligned}
\]  

(FG)

Quasilinear system of balance laws, singular limit \(\lambda \gg 1\) and \(\mu \ll 1\).

[VD '19]: rigorous justification for well-prepared initial data and \(\lambda \gtrsim \mu^{-1}\).

\[
|\epsilon_{FG}|_{H^k \times X^k} \lesssim (\mu^2 + \mu \lambda^{-1}) t, \quad t \lesssim 1/\epsilon.
\]
Toy models

Order 1

\[
\begin{align*}
\partial_t h &= 0, \\
\partial_t u + \frac{1}{\epsilon} h \partial_x u &= 0
\end{align*}
\]

Order 0

\[
\begin{align*}
\partial_t h &= 0, \\
\partial_t u + \frac{i}{\epsilon} h u &= 0
\end{align*}
\]
1. About the title
2. Water waves and ripples
3. Shallow water models
   - Derivation
   - Justification
   - Numerical simulation
4. Higher order models
   - A unified framework
   - Interfacial waves
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A first method

Recall we (and [Lagrange, Boussinesq, Rayleigh]) had an expansion of the Dirichlet-to-Neumann operator

$$G^\mu[\epsilon \zeta]\psi = -\mu \nabla \cdot \left( \int_{-1}^{\epsilon \zeta(t, \cdot)} \nabla_x \Phi(\cdot, z) \, dz \right)$$

with $\Phi$ solution to

$$\Phi + \mu \ell[\epsilon \zeta] \Phi = \psi, \quad \ell[\epsilon \zeta] \Phi(\cdot, z) \overset{\text{def}}{=} -\int_{z}^{\epsilon \zeta} \int_{-1}^{z'} \Delta_x \Phi(\cdot, z'') \, dz'' \, dz'.$$

$$\Phi = \sum_{k=0}^{N} (-\mu \ell[\epsilon \zeta])^k \psi + O(\mu^{N+1}).$$

This yields to extended Green–Naghd diluted systems [Matsuno ’15,’16].

× The loss of derivatives is $2N + p$ for some $p$. No hope of convergence when $N \to \infty$, by the modal analysis.
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\( \times \) The loss of derivatives is \( 2N + p \) for some \( p \). No hope of convergence when \( N \to \infty \), by the modal analysis.
A second method

We have another expansion of the Dirichlet-to-Neumann operator

\[ G^\mu[\epsilon \zeta]\psi = \sum_{k=0}^{N} \epsilon^k d^k G^\mu[0](\zeta, \ldots, \zeta)\psi + \mathcal{O}(\epsilon^{N+1}). \]

Plugging the truncated expansion into the Hamiltonian yields a hierarchy of models [Craig&Sulem '93] and also [Dommermuth&Yue '87, West et al. '87].

This is known as the high-order spectral method.

✓ The series converge (shape-analyticity of \( G^\mu \))

✗ Each of the models could be ill-posed [Ambrose,Bona&Nicholls '14]

✓ Well-posedness can be restored without any cost [VD&Melinand]
We have another expansion of the Dirichlet-to-Neumann operator

$$G^\mu[\epsilon \zeta] \psi = \sum_{k=0}^{N} \epsilon^k \frac{d^k G^\mu[0]}{d \zeta^k} (\zeta, \ldots, \zeta) \psi + O(\epsilon^{N+1}).$$

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The how-to guide to all (?) other methods

1. Select a variational formulation of the Laplace problem

\[
\begin{aligned}
\mu \Delta_x \Phi + \partial_z^2 \Phi &= 0 & \text{in } \{(x, z), \ -1 < z < \epsilon \zeta(t, x)\}, \\
\Phi &= \psi & \text{on } \{(x, z), \ z = \epsilon \zeta(t, x)\}, \\
\partial_z \Phi &= 0 & \text{on } \{(x, z), \ z = -1\}.
\end{aligned}
\]

2. Select a vertical distribution \( \{\Psi_i(x, z, \epsilon \zeta)\}_i \) and define the “finite-dimensional” vector space

\[
V = \left\{ \Phi, \ \Phi(t, x, z) = \sum_{i=0}^{N} \phi_i(x, t) \Psi_i(x, z, \epsilon \zeta(t, x)) \right\}.
\]

3. Define \( \Phi^{\text{app}}_N \) as the Galerkin approximation of the variational problem.

4. Plug in the D2N operator, then the Hamiltonian.

5. Use Hamilton’s equations and enjoy.
An example

\[ V = \left\{ \Phi, \, \Phi(t, x, z) = \sum_{i=0}^{N} \phi_i(x, t) \psi_i(x, z, \epsilon \zeta(t, x)) \right\}. \]

Setting \( \psi_i(x, z, \epsilon \zeta(t, x)) = (z + 1)^{2i} \) (motivated by the [Boussinesq, Rayleigh] shallow-water expansion) yields the Isobe–Kakinuma model

\[
\begin{align*}
\partial_t \zeta + \sum_{i=0}^{N} \nabla \cdot \left( \frac{h^{2i+1}_{2i+1}}{h^{2i}_{2i+1}} \nabla \phi_i \right) &= 0, \\
\partial_t \psi + \zeta + \epsilon \left( \sum_{i=0}^{N} h^{2i} \phi_i \right) \left( \sum_{j=0}^{N} \nabla \cdot \left( \frac{h^{2j+1}_{2j+1}}{h^{2j}_{2j+1}} \nabla \phi_i \right) \right) &+ \frac{\epsilon}{2} \left( | \sum_{i=0}^{N} h^{2i} \nabla \phi_i |^2 + \frac{1}{\mu} \left( \sum_{i=0}^{N} 2i h^{2i-1} \phi_i \right)^2 \right) &= 0,
\end{align*}
\]

with \( h = 1 + \epsilon \zeta \) and \( \{ \phi_i \}_{i \in \{0, 1, \ldots, N\}} \) solution to

\[
\begin{align*}
\sum_{j=0}^{N} \left( -\frac{2i}{(2j+1)(2i+2j+1)} h^{2i+2j+1} \Delta \phi_j - \frac{1}{\mu} \frac{4ij}{2i+2j-1} h^{2i+2j-1} \phi_j \right) &= 0, \\
\sum_{i=0}^{N} h^{2i} \phi_i &= \psi,
\end{align*}
\]

\forall i \in \{1, \ldots, N\}.\]
An example

the Isobe–Kakinuma model

\[
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\partial_t \zeta + \sum_{i=0}^{N} \nabla \cdot \left( \frac{h^{2i+1}}{2i+1} \nabla \phi_i \right) &= 0, \\
\partial_t \psi + \zeta + \epsilon \left( \sum_{i=0}^{N} 2ih^{2i} \phi_i \right) \left( \sum_{j=0}^{N} \nabla \cdot \left( \frac{h^{2j+1}}{2j+1} \nabla \phi_i \right) \right) &+ \frac{\epsilon}{2} \left( | \sum_{i=0}^{N} h^{2i} \nabla \phi_i |^2 + \frac{1}{\mu} \left( \sum_{i=0}^{N} 2ih^{2i-1} \phi_i \right)^2 \right) &= 0, \\
\end{aligned}
\]

(IK)

with \( h = 1 + \epsilon \zeta \) and \( \{ \phi_i \}_{i \in \{0,1,\ldots,N\}} \) solution to

\[
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\forall i \in \{1, \ldots, N\}, \\
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\]

Isobe–Kakinuma \( = ( \text{Hyperbolic} ) \times ( \text{Elliptic} ) \).
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\forall i \in \{1, \ldots, N\}, \\
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\end{align*}
\]

Other models with similar features can be derived with other choices for \( \{\Psi_i(x, z, \epsilon \zeta)\}_{i} \) [Athanassoulis&Belibassakis '99][Lynett&Liu '04][Klopman,vanGroesen&Dingemans '10]. Yet only the Isobe–Kakinuma model benefits from [Iguchi '18].

✓ full justification as a model of order \( \mathcal{O}(\mu^{1+2N}) \) (recall Padé approximants).
A brief introduction to interfacial waves

Waves at the interface between two homogeneous layers is a natural generalization of the water waves framework. New phenomena arise.

- Role of the density contrast
  \(~\xrightarrow{}~\) Boussinesq approximation, rigid-lid framework \cite{VD14,VD16}
- Kelvin–Helmholtz instabilities (KH)
  \(~\xrightarrow{}~\) ill-posedness (!)

Do shallow water models predict the propagation of sharp interfaces?

- ✓ The hydrostatic model (which extends the Saint-Venant model) 
  *tames* KH. \cite{GuyenneLannesSaut10,BreschRenardy11} (WP when \(h_1, h_2 > 0\) and \(\gamma \epsilon |u_1 - u_2|^2 < a_0\) with some explicit \(a_0(h_1, h_2) > 0\)).
- ✗ The Miyata–Choi–Camassa model (which extends Green–Naghdi) 
  *overestimates* KH. \cite{JoChoi02,LannesMing15} (modal analysis).
- ✓ The Kakinuma model (which extends the Isobe–Kakinuma model) 
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What we have done

- Rigorously justified standard and widely used models for water waves \((Saint-Venant, Green–Naghdi)\);
- Rigorously justified a hyperbolic relaxation of the Green–Naghdi system \((Favrie–Gavrilyuk)\);
- Formally derived a class of high-order models, and rigorously justified a family \((Isobe–Kakinuma)\);
- Ventured into the world of interfacial waves \((Choi–Camassa, Kakinuma)\).

What we have not done

- Compared high-order models and their limit towards the water waves system;
- Entered the world of non-potential and/or continuously stratified flows;
- Said anything on solutions besides local-in-time existence (existence and stability of solitary waves, global existence vs finite-time singularity);
- Used models for practical problems (bottom reconstruction, fluid-structure interaction, etc.).
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Thank you for your attention