On the inviscid and non-diffusive primitive equations with Gent and McWilliams parametrization

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Joint work with Roberta Bianchini (Rome) and Mahieddine Adim (Rennes)

Opening remark

The stability of shear flow solutions to the hydrostatic Euler equations with stable density stratification is an open problem.

Outline

1. Motivation
2. An open problem
3. Isopycnal coordinates
4. Results

The initial set of equations

We are interested in the stability of shear flows as solutions to the primitive equations with Redi-Gent-McWilliams parametrization

$$\begin{cases} \partial_t S + (\mathbf{U}_3 \cdot \nabla_3) S = \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{R}} \nabla_3 S\big) + \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{GM}} \nabla_3 S\big), \\ \partial_t \theta + (\mathbf{U}_3 \cdot \nabla_3) \theta = \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{R}} \nabla_3 \theta\big) + \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{GM}} \nabla_3 \theta\big), \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{U}_3 \cdot \nabla_3)\mathbf{u}\big) + \nabla_{\mathbf{x}} P + f\mathbf{k} \times \mathbf{u} = \nabla_3 \mathbb{S}(\nabla_3 \mathbf{U}_3), \\ \nabla_3 \cdot \mathbf{U}_3 = 0, \\ \rho = \rho(\theta, S, P), \\ \partial_z P + g\rho = 0, \\ \text{boundary conditions.} \end{cases} \tag{P}$$

where $\nabla_3 = (\nabla_{\mathbf{x}}, \partial_z)$, $\mathbf{U}_3 = (\mathbf{u}, w)$ and $\mathbb{K}_{\mathrm{R}}$ and $\mathbb{K}_{\mathrm{GM}}$ are defined by

$$\mathbb{K}_{\mathrm{R}} := \frac{K_I}{1+|\mathbf{L}|^2} \begin{pmatrix} 1+L_y^2 & -L_x L_y & L_x \\ -L_x L_y & 1+L_x^2 & L_y \\ L_x & L_y & |\mathbf{L}|^2 \end{pmatrix}, \quad \mathbb{K}_{\mathrm{GM}} := \kappa \begin{pmatrix} 0 & 0 & -L_x \\ 0 & 0 & -L_y \\ L_x & L_y & 0 \end{pmatrix},$$

with $\kappa, K_I > 0$ and $\mathbf{L} := (L_x, L_y) = \frac{-\nabla_{\mathbf{x}} \rho}{\partial_z \rho}$.

Assumptions

$$
\begin{cases}
\partial_t S + (\mathbf{U}_3 \cdot \nabla_3) S = \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{R}} \nabla_3 S\big) + \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{GM}} \nabla_3 S\big), \\
\partial_t \theta + (\mathbf{U}_3 \cdot \nabla_3) \theta = \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{R}} \nabla_3 \theta\big) + \nabla_3 \cdot \big(\mathbb{K}_{\mathrm{GM}} \nabla_3 \theta\big), \\
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\end{cases} \tag{P}
$$

We will make quite a few (additional) assumptions...

1. The earth is flat, infinite, and covered by an ocean (!)
2. Viscosity and friction effects are negligible (!!)
3. The equation of state is linear function of S, θ (!!!)
4. The density is stably stratified, $\partial_z \rho < 0$ (!!!!)
5. Coriolis force is negligible, $f = 0$ (!?!?)

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Our set of equations

We notice the identities

$$\nabla_3 \cdot (\mathbb{K}_{\mathrm{R}} \nabla_3 \rho) = 0, \quad \nabla_3 \cdot (\mathbb{K}_{\mathrm{GM}} \nabla_3 \rho) = -\mathbf{U}_3^\star \cdot \nabla_3 \rho$$

with

$$\mathbf{U}_3^\star = \kappa \left(\partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right), -\nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \right).$$

Since the equation of state is a linear function of tracers, we have a closed system

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \rho + (w + w_\star) \partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + (w + w_\star) \partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -g\rho, \qquad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions.} \end{cases} \qquad (\mathrm{H}_\kappa)$$

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Qn: stability of shear flows $(\rho, \mathbf{u})(t, \mathbf{x}, z) = (\underline{\rho}(z), \underline{\mathbf{u}}(z))$?

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Conclusion

Key observations

1. Without thickness diffusivity, the stability of the system is widely open.
2. The Gent and McWilliams contribution act as a thickness diffusivity, and provides stability of shear flows w.r.t. (high-frequency) perturbations.
3. Isopycnal coordinates provide interesting insights on the structure of the system in idealized frameworks (stable stratification, flat bottom...)
4. Gent and McWilliams contribution (in isopycnal coordinates) have a structure which has been mathematically studied in other contexts.

Main results

1. (with R. Bianchini) Well-posedness of the initial value problem and control of solutions on a relevant time interval.
2. (with R. Bianchini) Rigorous justification of the hydrostatic assumption in the shallow water limit.
3. (with M. Adim and R. Bianchini) Connection between the (continuously stratified) hydrostatic system and the multilayer shallow water system.

Outline

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Some stability results (when $\kappa = 0$, dimension 2)

Homogeneous case: $\rho \equiv 1$**.**

- Spectral stability of the linearized system about shear flows, under the Rayleigh criterion $\underline{\mathbf{u}}''(z) \neq 0$. [Rayleigh (1880)]
- Lyapunov stability under the Rayleigh criterion. [Arnold '65]
- Well-posedness of the (nonlinear) system in Sobolev spaces under the Rayleigh criterion. [Grenier '99], [Brenier '03], [Masmoudi&Wong '12].

Inhomogeneous case: $\partial_z \rho \neq 0$**.**

- Spectral stability of the linearized system about shear flows, under the Miles and Howard criterion $\frac{1}{4}|\underline{\mathbf{u}}'(z)|^2 \leq \frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}$. [Miles '61][Howard '61]
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Open problem

Well-posedness of the initial-value problem for the (linear or nonlinear) hydrostatic system in the presence of stable density stratification?

Possible source of instabilities

The hydrostatic limit of the Euler equations is singular:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} \rho + w \partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + w \partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -g\rho, \qquad \Longleftrightarrow \qquad P = P|_{z=z_{\rm surf}} + g \int_{\cdot}^{z_{\rm surf}} \rho \, \mathrm{d}z \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \qquad \Longleftrightarrow \qquad w = w|_{z=z_{\rm bot}} - \int_{z_{\rm bot}}^{\cdot} \nabla_{\mathbf{x}} \cdot \mathbf{u} \, \mathrm{d}z \\ \text{boundary conditions.} \end{cases} \tag{H}$$

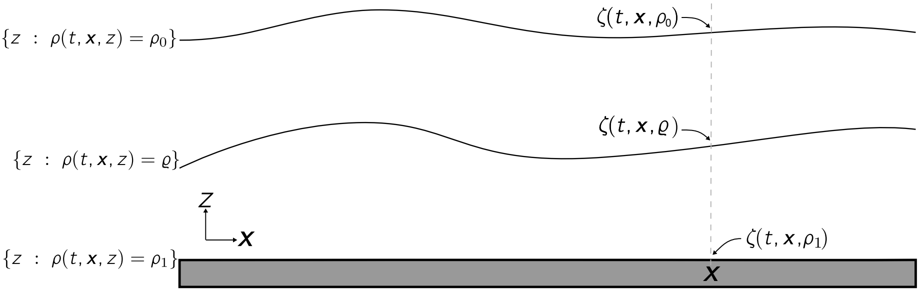
- Singular contributions from $w\partial_z \rho$ and $\nabla_{\mathbf{x}} P$ compensate in energy estimates if $\rho > 0$, $\partial_z \rho < 0$. Stable stratification helps.
- There is no obvious way to deal with the contribution $w \partial_z \mathbf{u}$. Shear velocity hurts.
- In the mathematical literature, authors either
  1. Use the analytic framework. e.g. [Kukavica, Temam, Vicol & Ziane '11]
  2. Add regularization (viscosity, diffusivity). e.g. [Cao, Li & Titi '07+...]
  3. Relax the hydrostatic approximation [Desjardins, Lannes & Saut '21]

Outline

Isopycnal coordinates

We consider stratified flows and define the variable $h(t, \mathbf{x}, r) > 0$ through

$$\rho(t, \mathbf{x}, \zeta(t, \mathbf{x}, r)) = r, \quad \zeta(t, \mathbf{x}, \rho(t, \mathbf{x}, z)) = z, \qquad h \stackrel{\text{def}}{=} -\partial_r \zeta.$$



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$$\rho(t,\mathbf{x},\zeta(t,\mathbf{x},r))=r,\quad \zeta(t,\mathbf{x},\rho(t,\mathbf{x},z))=z,\qquad h\stackrel{\text{def}}{=}-\partial_r\zeta.$$

The system

$$\begin{cases}\partial_t\rho+(\mathbf{u}+\mathbf{u}_\star)\cdot\nabla_{\mathbf{x}}\rho+(w+w_\star)\partial_z\rho=0,\\ \rho\big(\partial_t\mathbf{u}+(\mathbf{u}+\mathbf{u}_\star)\cdot\nabla_{\mathbf{x}}\mathbf{u}+(w+w_\star)\partial_z\mathbf{u}\big)+\nabla_{\mathbf{x}}P=0,\\ \partial_z P=-g\rho,\quad \partial_z w=-\nabla_{\mathbf{x}}\cdot\mathbf{u},\\ \text{boundary conditions (free surface).}\end{cases}\tag{H_κ}$$

reads in isopycnal coordinates

$$\begin{cases}\partial_t h+\nabla_{\mathbf{x}}\cdot\big(h(\mathbf{u}+\mathbf{u}_\star)\big)=0,\\ \partial_t\mathbf{u}+((\mathbf{u}+\mathbf{u}_\star)\cdot\nabla_{\mathbf{x}})\mathbf{u}+\nabla_{\mathbf{x}}\big(\mathcal{M}h\big)=0,\end{cases}\tag{H_κ}$$

where $\mathbf{u}_\star\stackrel{\text{def}}{=}\kappa\frac{-\nabla_{\mathbf{x}}h}{h}$ and

$$\big(\mathcal{M}h\big)(t,\mathbf{x},r)\stackrel{\text{def}}{=}g\int_{\rho_0}^{\rho_1}\frac{\min(r,r')}{r}h(t,\mathbf{x},r')\,\mathrm{d}r'$$

The system in isopycnal coordinates

The isopycnal formulation of the hydrostatic system is

$$\begin{cases} \partial_t h + \mathbf{u}\cdot\nabla_{\mathbf{x}} h + h\nabla_{\mathbf{x}}\cdot\mathbf{u} = \kappa\Delta_{\mathbf{x}} h, \\ \partial_t \mathbf{u} + \big((\mathbf{u} + \kappa\frac{-\nabla_{\mathbf{x}} h}{h})\cdot\nabla_{\mathbf{x}}\big)\mathbf{u} + \nabla_{\mathbf{x}}(\mathcal{M}h) = 0, \end{cases} \qquad (\mathrm{H}_\kappa)$$

where $\mathcal{M}h(t,\mathbf{x},r) \stackrel{\text{def}}{=} g\int_{\rho_0}^{\rho_1} \frac{\min(r,r')}{r} h(t,\mathbf{x},r')\,\mathrm{d}r'$.

Remarks:

- The parameterization of Gent & McWilliams is nice and simple.
- The advection in the variable z (or r) has disappeared.
- The domain is flattened.
- The system is easily discretized (multilayer framework) [Adim]
- The structure of the system is clarified: in horizontal dimension $d = 1$,

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} 0 & h \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & u + \kappa\frac{-\partial_x h}{h} \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = \kappa \begin{pmatrix} \partial_x^2 h \\ 0 \end{pmatrix}.$$

Structure of the system

The isopycnal formulation of the hydrostatic system (in horizontal dimension $d = 1$) is

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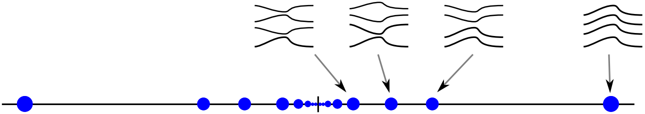
1. The first two terms generate wave propagation



2. Advection contribution mixes modes when $\partial_r u \neq 0$.
3. The Gent&McWilliams contribution provides (partial)[1] diffusion.

[1] On a side note, if we define $v = u - \kappa \frac{\partial_x h}{h}$, then the system becomes

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & h \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = \kappa \begin{pmatrix} 0 \\ \frac{1}{h}\partial_x(h\partial_x v) \end{pmatrix},$$

and the Gent&McWilliams contribution acts as an effective "degenerate" viscosity, proposed in [Gent '93] and studied by Bresch, Desjardins, Vasseur... when $\mathcal{M} = h$.

Outline

Some results

[R. Bianchini & VD]

Giver a stably stratified shear flow $(\underline{\rho}(r), \underline{\mathbf{u}}(r))$ and for sufficiently regular data satisfying the assumption

$$h|_{t=0} \geq h_0 > 0$$

and for any $\kappa \in (0, 1]$, there exists a unique (classical) solution which we control on the time interval $[0, T]$ with

$$T^{-1} = C\left(1 + \kappa^{-1}\left(\left|\underline{\mathbf{u}}'\right|_{L^2_r}^2 + M_0^2\right)\right),$$

where M_0 is the size of the initial deviation from the shear flow equilibrium $(\underline{\rho}(r), \underline{\mathbf{u}}(r))$, and C depends only on M_0, h_0 and the size of $(\underline{\rho}(r), \underline{\mathbf{u}}(r))$.

[R. Bianchini & VD]

As long as this solution is controlled, we have strong convergence of the corresponding solutions to the incompressible Euler equations towards this solution in the limit of shallow water aspect ratio $L_z/L_{\mathbf{x}} \ll 1$.

Tool: the energy method. Use partial symmetric structure, and use parabolic regularization only when necessary.

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Other results

[M. Adim]

As long as the solution of the continuously stratified hydrostatic system

$$\begin{cases} \partial_t h + \nabla_{\mathbf{x}} \cdot (h\mathbf{u}) = \kappa \Delta_{\mathbf{x}} h, \\ \partial_t \mathbf{u} + \left(\left(\mathbf{u} + \kappa \frac{-\nabla_{\mathbf{x}} h}{h}\right) \cdot \nabla_{\mathbf{x}}\right)\mathbf{u} + \nabla_{\mathbf{x}}(\mathcal{M} h) = 0, \end{cases}$$

$$(\mathcal{M}h)(t, \mathbf{x}, r) \stackrel{\text{def}}{=} g \int_{\rho_0}^{\rho_1} \frac{\min(r, r')}{r} h(t, \mathbf{x}, r')\, \mathrm{d}r'$$

is controlled, we can approximate it through solutions to the multilayer shallow water system

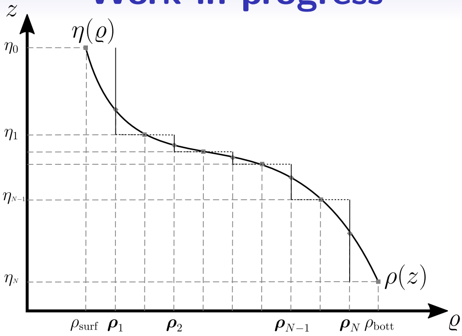
$$\begin{cases} \partial_t h_i + \nabla_{\mathbf{x}} \cdot (h_i \mathbf{u}_i) = \kappa \Delta_{\mathbf{x}} h_i, \\ \partial_t \mathbf{u}_i + \left(\left(\mathbf{u}_i + \kappa \frac{-\nabla_{\mathbf{x}} h_i}{h_i}\right) \cdot \nabla_{\mathbf{x}}\right)\mathbf{u}_i + \nabla_{\mathbf{x}}(\mathbb{M} h)_i = 0, \end{cases}$$

$$(\mathbb{M}h)_i(t, \mathbf{x}) \stackrel{\text{def}}{=} g \sum_{j=1}^{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} h_j(t, \mathbf{x})$$

with strong convergence as the number of layers go to ∞.

Tool: Mimic the method of the continuously stratified case, by introducing suitable discrete analogues of ∂_r, integration by parts, trace at the surface...

Work in progress



[M. Adim, R. Bianchini & VD]

As long as the solution to the bilayer shallow water system is controlled, and for any fixed $\kappa > 0$, we have strong convergence of solutions to the "continuously" stratified hydrostatic system in the limit of sharp stratification and columnar motion.

Tool: stability estimate for the "continuously" stratified hydrostatic system in the relevant topology (measuring the distance in L^1_r and in L^∞_r except in a small domain around the density discontinuity).

Conclusion

Key observations

1. Without thickness diffusivity, the stability of the system is widely open.
2. The Gent and McWilliams contribution act as a thickness diffusivity, and provides stability of shear flows w.r.t. (high-frequency) perturbations.
3. Isopycnal coordinates provide interesting insights on the structure of the system in idealized frameworks (stable stratification, flat bottom...)
4. Gent and McWilliams contribution (in isopycnal coordinates) have a structure which has been mathematically studied in other contexts.

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Thank you for your attention