On WW2
Propagation of deep water waves

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Joint work with Benjamin Melinand (Paris Dauphine)
The model

\[
\begin{aligned}
\partial_t \zeta - |D| |\psi| + \epsilon |D|(\zeta |D| \psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) &= 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D| \psi)^2) &= 0,
\end{aligned}
\]

(WW2)

where \(|D| = (-\Delta_x)^{1/2}, x \in \mathbb{R}^d, d \in \{1, 2\}.

- (WW2) is a model for water waves in infinite depth, assuming small steepness, \(\epsilon \ll 1\).
- (WW2) enjoys a Hamiltonian structure. In particular, it preserves

\[
\int \zeta \, dx, \quad \int \zeta \nabla \psi \, dx,
\]

\[
\mathcal{H}_1(\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi |D| \psi + \epsilon \zeta (|\nabla \psi|^2 - (|D| \psi)^2) \, dx.
\]

- (WW2) belongs to a hierarchy of models [Craig&Sulem '93] based on the converging asymptotic expansion

\[
\mathcal{H}(\zeta, \psi) \overset{\text{def}}{=} \mathcal{H}_1(\zeta, \psi) + \frac{1}{2} \int_{\mathbb{R}^d} \epsilon^2 \psi G_2[\zeta, \zeta] \psi + \epsilon^3 \psi G_3[\zeta, \zeta, \zeta] \psi + \ldots
\]
Numerical instabilities

Numerical integration of the systems in the hierarchy are easily and efficiently implemented using Fourier spectral methods (as done in e.g. [Guyenne&Nicholls '07-08])

In the computations [...] it was observed that spurious oscillations can develop in the wave profile, due to the onset of an instability related to the growth of numerical errors at high wavenumbers. [...] Similar high-wavenumber instabilities were observed by other authors [...] who used smoothing techniques to circumvent this difficulty.
Numerical instabilities

Numerical integration of the systems in the hierarchy are easily and efficiently implemented using Fourier spectral methods (as done in e.g. [Guyenne&Nicholls ’07-08])

In the computations [...] it was observed that spurious oscillations can develop in the wave profile, due to the onset of an instability related to the growth of numerical errors at high wavenumbers. [...] Similar high-wavenumber instabilities were observed by other authors [...] who used smoothing techniques to circumvent this difficulty.

\[ \epsilon = \frac{1}{4}, \, N = 2^9, \, L = 4\pi, \, dt = 10^{-3} \]
Proposed instability mechanism

[Ambrose, Bona & Nicholls '14] suggest that

\[
\begin{align*}
\partial_t \zeta - |D|\psi + \epsilon |D| (\zeta |D|\psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) &= 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D|\psi)^2) &= 0,
\end{align*}
\]  
(WW2)

[and also (WW3)] is ill-posed in Sobolev spaces, based on

- tailored numerical experiments;
- the toy model

\[
\partial_t \psi + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D|\psi)^2) = 0.
\]  
(toy)

[Ambrose, Bona & Nicholls '14]

For all \( s \in [0, 3) \), the Cauchy problem associated with (toy) is ill-posed\(^a\) in \( H^s(\mathbb{T}) \).

\(^a\)there exists a sequence \((\psi_n)_{n \in \mathbb{N}}\) of smooth solutions to (toy) defined on \( t \in [0, T_n) \) and such that \( |\psi_n(0; \cdot)|_{H^s} \downarrow 0 \), \( T_n \downarrow 0 \) as \( n \to \infty \) and \( |\psi_n(t, \cdot)|_{L^2} \to \infty \) as \( t \nearrow T_n \).
Outline

1. Context

2. Instabilities
   - Quasi-linearization
   - Toy model
   - Numerics

3. Rectification
   - Construction
   - Justification
Quasi-linearization

\[
\begin{align*}
\partial_t \zeta - |D|\psi + \epsilon |D|(|\zeta|D|\psi|) + \epsilon \nabla \cdot (\zeta \nabla \psi) &= 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D|\psi)^2) &= 0,
\end{align*}
\]

(WW2)

Compensation Lemma [Saut&Xu '12]

Let \( d \in \{1, 2\}, \ t_0 > d/2 \). For all \( r \leq 1 \) and \( s \geq t_0 + r \),

\[
|D|(f|D|g) + \nabla \cdot (f\nabla g)|_{H^s} \lesssim |\nabla f|_{H^s} |\nabla g|_{H^{s-r}}.
\]

Proof \((d = 1)\). Denote \( a = |D|(f|D|g) + \partial_x (f \partial_x g) \). For \( \xi \geq 0 \),

\[
\hat{a}(\xi) = \int_\mathbb{R} (|\xi||\xi - \eta| - \xi(\xi - \eta)) \hat{f}(\eta) \hat{g}(\xi - \eta) \, d\eta = 2 \int_\mathbb{R} \xi(\eta - \xi) \hat{f}(\eta) \hat{g}(\xi - \eta) \, d\eta.
\]

Since \(|\xi| \leq |\eta| \) and \(|\eta - \xi| \leq |\eta| \), one has for all \( s \geq 0 \) and \( r' \geq 0 \)

\[
\langle \xi \rangle^s |\hat{a}(\xi)| \leq 2 \int_\mathbb{R} \langle \eta \rangle^{s+r'} |\eta| |\hat{f}(\eta)| |\xi - \eta|^{-r'} |\xi - \eta| |\hat{g}(\xi - \eta)| \, d\eta.
\]

By Young's inequality and since \( \langle \cdot \rangle^{-t_0} \in L^2(\mathbb{R}) \), with \( r' = 0 \),

\[
|D|(f|D|g) + \nabla \cdot (f\nabla g)|_{H^s} \lesssim |\partial_x f|_{H^s} |\partial_x g|_{H^{t_0}}.
\]

This shows the result, without restriction on \( r \) when \( d = 1 \).
Quasi-linearization

\[
\begin{align*}
\partial_t \zeta - |D|\psi + \epsilon |D|(\zeta |D|\psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) &= 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D|\psi)^2) &= 0.
\end{align*}
\]

(WW2)

The principle part of the first equation is

\[
\partial_t \dot{\zeta} - |D|\dot{\psi} + \epsilon |D|(\dot{\zeta} |D|\dot{\psi}) + \epsilon \nabla \cdot (\dot{\zeta} \nabla \psi) = 0.
\]

The principle part of the second equation is

\[
\partial_t \dot{\psi} + \dot{\zeta} + \epsilon (\nabla \psi) \cdot (\nabla \dot{\psi}) - \epsilon |D|\psi (|D|\dot{\psi}) = 0.
\]

One recognizes Alinhac’s good unknown: \( \tilde{\psi} \equiv \dot{\psi} - \epsilon (|D|\psi)\dot{\zeta} \):

\[
\begin{align*}
\partial_t \dot{\zeta} - |D|\tilde{\psi} + \epsilon \nabla \cdot (\dot{\zeta} \nabla \tilde{\psi}) &= 0, \\
\partial_t \tilde{\psi} + a[\zeta, \psi] \dot{\zeta} + \epsilon (\nabla \psi \cdot \nabla \tilde{\psi}) &= 0,
\end{align*}
\]

(Q-WW2)

with

\[
a[\zeta, \psi] f \equiv \underbrace{f - \epsilon (|D|\zeta) f}_{a(\zeta) f} - \epsilon^2 |D|\psi |D| \left\{ |D|\psi f \right\}.
\]
Quasi-linearization

\[
\begin{cases}
\partial_t \zeta - |D| \dot{\psi} + \epsilon |D| (\zeta |D| \dot{\psi}) + \epsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} \left( |\nabla \psi|^2 - (|D| \psi)^2 \right) = 0.
\end{cases}
\]

(WW2)

The principle part of the first equation is

\[
\partial_t \dot{\zeta} - |D| \dot{\psi} + \epsilon |D| (\dot{\zeta} |D| \dot{\psi}) + \epsilon \nabla \cdot (\dot{\zeta} \nabla \psi) = 0.
\]

The principle part of the second equation is

\[
\partial_t \dot{\psi} + \dot{\zeta} + \epsilon (\nabla \psi) \cdot (\nabla \dot{\psi}) - \epsilon (|D| \psi)(|D| \dot{\psi}) = 0.
\]

One recognizes Alinhac’s good unknown: \( \tilde{\psi} \equiv \psi - \epsilon (|D| \psi) \dot{\zeta} \):

\[
\begin{cases}
\partial_t \tilde{\zeta} - |D| \tilde{\psi} + \epsilon \nabla \cdot (\tilde{\zeta} \nabla \tilde{\psi}) = 0, \\
\partial_t \tilde{\psi} + a[\zeta, \psi] \dot{\zeta} + \epsilon (\nabla \psi \cdot \nabla \tilde{\psi}) = 0,
\end{cases}
\]

(Q-WW2)

with

\[
a[\zeta, \psi] f \equiv \underbrace{f - \epsilon (|D| \zeta) f}_a(\zeta) f - \epsilon^2 \underbrace{|D| \psi |D| \{(|D| \psi) f\}}_a(|D| \psi) f.
\]
We mimic

\[
\begin{aligned}
\partial_t \dot{\zeta} - |D|\dot{\psi} + \epsilon \nabla \cdot (\dot{\zeta} \nabla \psi) &= 0, \\
\partial_t \dot{\psi} + a[\zeta, \psi] \dot{\zeta} + \epsilon (\nabla \psi \cdot \nabla \dot{\psi}) &= 0,
\end{aligned}
\]

(Q-WW2)

\[
a[\zeta, \psi] f \overset{\text{def}}{=} f - \epsilon(|D|\zeta)f - \epsilon^2 \left( |D|\psi \left\{|D|\psi f\right\} \right).
\]

with

\[
\begin{aligned}
\partial_t \zeta - |D|\psi &= 0, \\
\partial_t \psi + (1 - \alpha[\psi]|D|)\zeta &= 0,
\end{aligned}
\]

\[
\alpha[\psi] \overset{\text{def}}{=} \epsilon^2 \int (|D|\psi)^2 \, d\mathbf{x}. \quad \text{(toy)}
\]
**Toy model**

\[
\begin{aligned}
\partial_t \zeta - |D| \psi &= 0, \\
\partial_t \psi + (1 - \alpha[\psi]|D|) \zeta &= 0,
\end{aligned}
\]

\[\alpha[\psi] \overset{\text{def}}{=} \epsilon^2 \int (|D| \psi)^2 \, dx. \quad \text{\text{(toy)}}\]

**Ill-posedness in** $H^\infty(\mathbb{T}^d)$ [D-Melinand]

For all $\epsilon > 0$, there exists $(\zeta_n, \psi_n)_{n \in \mathbb{N}}$ smooth solutions to (toy) defined on $[0, T_n)$ with

\[\forall s \in \mathbb{R}, \quad \left| \zeta_n|_{t=0} \right|_{H^s(\mathbb{T}^d)} + \left| \psi_n|_{t=0} \right|_{H^s(\mathbb{T}^d)} \downarrow 0 \quad \text{and} \quad T_n \downarrow 0 \quad (n \to \infty),\]

and

\[\forall s' \in \mathbb{R}, \quad \left| \psi^n(t, \cdot) \right|_{H^{s'}(\mathbb{T}^d)} \to \infty \quad (t \nearrow T_n).\]

**Proof.** We put $\zeta_n|_{t=0} = 0$ and $\psi_n|_{t=0} = b_n \cos(k_0 \cdot x) + c_n \cos(k_n \cdot x)$ where

\[k_0 \neq 0, \quad |k_n| \nearrow \infty, \quad b_n = |k_n|^{-1/4}, \quad c_n = \exp(-|k_n|^{1/4}).\]
Toy model

\[
\begin{aligned}
\partial_t \zeta - |D| \psi &= 0, \\
\partial_t \psi + (1 - \alpha[\psi]|D|) \zeta &= 0,
\end{aligned}
\]

\[
\alpha[\psi] \overset{\text{def}}{=} \epsilon^2 \int (|D| \psi)^2 \, dx. \quad \text{(toy)}
\]

Ill-posedness in $H^\infty(\mathbb{T}^d)$ [D-Melinand]

Proof. We put $\zeta_n|_{t=0} = 0$ and $\psi_n|_{t=0} = b_n \cos(k_0 \cdot x) + c_n \cos(k_n \cdot x)$ where

\[
k_0 \neq 0, \quad |k_n| \searrow \infty, \quad b_n = |k_n|^{-1/4}, \quad c_n = \exp(-|k_n|^{1/4}).
\]

By studying the system of ODEs on Fourier coefficients, we observe successively

- **low-high instabilities:** $\alpha[\psi]|k_0| < 1$ but $\alpha[\psi_0]|k_n| > 2 \Rightarrow c_n(t) \geq \frac{c_n}{8} e^{k_n^{1/2} t}$,
- **high-high instabilities:** $c_n(t) \gtrsim |k_n|^{-1} \Rightarrow \alpha[\psi_n]|k_n| > 2 \Rightarrow c_n(t) \geq \frac{c_n}{8} e^{k_n^{1/2} t}$,
- **high-high blow-up:** $c_n(t) \gtrsim 1 \Rightarrow \frac{d}{dt} \hat{\psi}_n^{k_n} \geq \frac{1}{4} |k_n|^{1/2} \hat{\psi}_n^{k_n}(t)^2 \Rightarrow \text{blow-up}.$
Numerical validation

Numerical integration of (WW2) initial data

\[ \zeta(t = 0, x) = 0 \quad \text{et} \quad \psi'(t = 0, x) = \left( \sin(x) + \frac{\sin(Kx)}{K^2} \right) \exp(-|x|^2). \]

Figure: Time integration with \( \epsilon = 1/5, \ K = 400. \)
Numerical validation

Numerical integration of (WW2) initial data

\[ \zeta(t=0,x) = 0 \quad \text{et} \quad \psi'(t=0,x) = \left( \sin(x) + \frac{\sin(Kx)}{K^2} \right) \exp(-|x|^2). \]

Figure: Blow-up time \( T^* \) depending on \( K \) and \( \epsilon \).

The toy model predicts \( T^* \propto (\epsilon K)^{-1} \) if \( \epsilon^2 K \gg 1 \).
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Quasi-linearization

\[
\begin{aligned}
&\partial_t \zeta - |D|\psi + \epsilon |D|(\zeta |D|\psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\
&\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D|\psi)^2) = 0.
\end{aligned}
\]

(WW2)

The principle part of the first equation is

\[
\partial_t \dot{\zeta} - |D|\dot{\psi} + \epsilon |D|(\dot{\zeta} |D|\psi) + \epsilon \nabla \cdot (\dot{\zeta} \nabla \psi) = 0.
\]

The principle part of the second equation is

\[
\partial_t \dot{\psi} + \dot{\zeta} + \epsilon (\nabla \psi) \cdot (\nabla \dot{\psi}) - \epsilon (|D|\psi)(|D|\dot{\psi}) = 0.
\]

One recognizes Alinhac’s good unknown: $\check{\psi} \overset{\text{def}}{=} \psi - \epsilon (|D|\psi)\dot{\zeta}$:

\[
\begin{aligned}
&\partial_t \check{\zeta} - |D|\check{\psi} + \epsilon \nabla \cdot (\check{\zeta} \nabla \psi) = 0, \\
&\partial_t \check{\psi} + a[\zeta, \psi] \check{\zeta} + \epsilon (\nabla \psi \cdot \nabla \check{\psi}) = 0,
\end{aligned}
\]

(Q-WW2) with

\[
a[\zeta, \psi] f \overset{\text{def}}{=} f - \epsilon (|D|\zeta) f - \epsilon^2 (|D|\psi |D| \{ |D|\psi \} f).
\]
Quasi-linearization

\[
\begin{align*}
\partial_t \zeta - |D| \psi + \epsilon |D|((J \zeta)|D| \psi) + \epsilon \nabla \cdot ((J \zeta) \nabla \psi) &= 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} (|\nabla \psi|^2 - (|D| \psi)^2) &= 0.
\end{align*}
\]

(WW2)

The principle part of the first equation is

\[
\partial_t \dot{\zeta} - |D| \dot{\psi} + \epsilon |D|((J \dot{\zeta})|D| \dot{\psi}) + \epsilon \nabla \cdot ((J \dot{\zeta}) \nabla \psi) = 0.
\]

The principle part of the second equation is

\[
\partial_t \dot{\psi} + \dot{\zeta} + \epsilon (\nabla \psi) \cdot (\nabla \dot{\psi}) = 0.
\]

One recognizes Alinhac’s good unknown: \( \tilde{\psi} \overset{\text{def}}{=} \dot{\psi} - \epsilon (|D| \psi)(J \dot{\zeta}) \):

\[
\begin{align*}
\partial_t \tilde{\psi} - |D| \ddot{\psi} + \epsilon \nabla \cdot ((J \ddot{\zeta}) \nabla \psi) &= 0, \\
\partial_t \ddot{\psi} + a_{J}[\zeta, \psi] \dot{\psi} + \epsilon (\nabla \psi \cdot \nabla \ddot{\psi}) &= 0,
\end{align*}
\]

(Q-WW2)

with

\[
a_{J}[\zeta, \psi] f \overset{\text{def}}{=} f - \epsilon (|D| \zeta) f - \epsilon^2 (|D| \psi) |D| \{(|D| \psi) J f \}.
\]
The regularized system

By plugging $J = J(D)$ self-adjoint,

$$\begin{cases}
\partial_t \zeta - |D| \psi + \epsilon |D|((J \zeta)|D| \psi) + \epsilon \nabla \cdot ((J \zeta) \nabla \psi) = 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} J (|\nabla \psi|^2 - (|D| \psi)^2) = 0,
\end{cases}$$

enjoys a canonical Hamiltonian structure, with

$$\mathcal{H}_1^J(\zeta, \psi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi |D| \psi + \epsilon (J \zeta) (|\nabla \psi|^2 - (|D| \psi)^2) \, dx.$$

Well-posedness [D-Melinand]

Let $J = J(D)$ with $J \lesssim \langle \cdot \rangle^{-1}$. Let $s \geq d/2 + 1/2$. For all $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$, there exists a unique maximal solution $(\zeta, \psi) \in C((-T_*, T^*); H^s(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d))$ to (RWW2), $(\zeta, \psi)|_{t=0} = (\zeta_0, \psi_0)$. Moreover, if $J \lesssim \langle \cdot \rangle^{-m}$, $m > d/2 + 3/2$ and $\epsilon$ small enough, then $T_* = T^* = +\infty$.


For $\epsilon$ small enough, $\mathcal{H}_1^J(\zeta, \psi) \approx |\zeta|^2_{L^2} + ||D|^{1/2} \psi||^2_{L^2}$ is preserved.
The cost of regularizing

**Consistency [D-Melinand]**

If $J = J_0(\delta D)$ with $J_0 \in L^\infty(\mathbb{R}^d)$ and $\cdot |^{-\ell}(1 - J_0) \in L^\infty(\mathbb{R}^d)$, then for all $\delta > 0$, $s > d/2$ and $(\zeta, \psi) \in C([0, T]; H^{\max(s+\ell+1, s+2)}(\mathbb{R}^d) \times H^{\max(s+\ell+\frac{3}{2})}(\mathbb{R}^d))$ solution to (RWW2), $(\zeta, \psi)$ satisfies (WW) up to remainder terms

\[
| R_0 |_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon^2 \left( |\zeta|_{H^{s+2}} + |D|^{1/2} |\psi|_{H^{s+1}} \right),
\]

\[
| R_J |_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon |D^\ell| \left( |\zeta|_{H^{s+\ell+1}} + |\nabla \psi|_{H^{s+\ell+\frac{1}{2}}} \right).
\]

**Proof.**

For $R_0$, [Alvarez-Samaniego&Lannes'08].

For $R_J$, $|f - Jf|_{H^s} \leq C_\ell \delta^\ell |f|_{H^s}$, with $C_\ell = \left| | \cdot |^{-\ell}(1 - J_0) \right|_{L^\infty}$. 

□
The cost of regularizing

**Consistency [D-Melinand]**

If \( J = J_0(\delta D) \) with \( J_0 \in L^\infty(\mathbb{R}^d) \) and \( \cdot \mid^{-\ell}(1 - J_0) \in L^\infty(\mathbb{R}^d) \), then for all \( \delta > 0, s > d/2 \) and \( (\zeta, \psi) \in C([0, T]; H^{\max(s+\ell+1, s+2)}(\mathbb{R}^d) \times H^{\max(s+\ell+3/2)}(\mathbb{R}^d)) \) solution to (RWW2), \( (\zeta, \psi) \) satisfies (WW) up to remainder terms

\[
|R_0|_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon^2 \left( |\zeta|_{H^{s+2}} + |D|^{1/2} |\psi|_{H^{s+1}} \right), \\
|R_J|_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon \delta^\ell \left( |\zeta|_{H^{s+\ell+1}} + |\nabla \psi|_{H^{s+\ell+\frac{1}{2}}} \right).
\]

Error between (RWW2) and (WW): smooth i.d. (left) and non-smooth i.d. (right).
The gain of regularizing

Large time well-posedness [D-Melinand]

Let \( J_0 = J_0(|D|) \) with \( \langle \cdot \rangle^{-1} J_0 \in L^\infty, \langle \cdot \rangle \nabla J \in L^\infty \). Let \( s > d/2 + 2, s \in \mathbb{N}, C > 1 \) and \( M > 0 \). There exists \( T_0 > 0 \) such that for all \( \epsilon > 0 \), for all \( (\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s + \frac{1}{2}}(\mathbb{R}^d) \) with

\[
0 < \epsilon M_0 \overset{\text{def}}{=} \epsilon \left( |\zeta_0|_{H^s} + |D|^{1/2} \psi_0|_{H^s} \right) \leq M,
\]

and for all \( \delta \geq \epsilon M_0 \), one has: for all \( J = J_0(\delta D) \), there exists a unique \( (\zeta, \psi) \in C([0, T_0/(\epsilon M_0)]; H^s \times H^{s + \frac{1}{2}}) \) solution to (RWW2), \( (\zeta, \psi)|_{t=0} = (\zeta_0, \psi_0) \), with

\[
\sup_{t \in [-T_0/(\epsilon M_0), T_0/(\epsilon M_0)]} \left( |\zeta(t, \cdot)|_{H^s}^2 + |D|^{1/2} \psi(t, \cdot)|_{H^s}^2 \right) \leq C \left( |\zeta_0|_{H^s}^2 + |D|^{1/2} \psi_0|_{H^s}^2 \right).
\]

Proof.

If \( \epsilon \gtrsim 1 \), Duhamel formula \Rightarrow \( T_0 \approx \delta \).

If \( \epsilon \ll 1 \), energy method \Rightarrow \( T_0 \approx \min(\{1, \delta/\epsilon\}) \).
The gain of regularizing

Large time well-posedness [D-Melinand]

Let $J_0 = J_0(|D|)$ with $\langle \cdot \rangle^{-1} J_0 \in L^\infty$, $\langle \cdot \rangle \nabla J \in L^\infty$. Let $s > d/2 + 2$, $s \in \mathbb{N}$, $C > 1$ and $M > 0$. There exists $T_0 > 0$ such that for all $\epsilon > 0$, for all $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$ with

$$0 < \epsilon M_0 \overset{\text{def}}{=} \epsilon (|\zeta_0|_{H^s} + |D|^{1/2} \psi_0|_{H^s}) \leq M,$$

and for all $\delta \geq \epsilon M_0$, one has: for all $J = J_0(\delta D)$, there exists a unique $(\zeta, \psi) \in C([0, T_0/(\epsilon M_0)]; H^s \times H^{s+\frac{1}{2}})$ solution to (RWW2), $(\zeta, \psi)|_{t=0} = (\zeta_0, \psi_0)$, with

$$\sup_{t \in [-T_0/(\epsilon M_0), T_0/(\epsilon M_0)]} (|\zeta(t, \cdot)|_{H^s}^2 + |D|^{1/2} \psi(t, \cdot)|_{H^s}^2) \leq C (|\zeta_0|_{H^s}^2 + |D|^{1/2} \psi_0|_{H^s}^2).$$

Proof.

If $\epsilon \gtrsim 1$, Duhamel formula $\Rightarrow T_0 \approx \delta$.

If $\epsilon \ll 1$, energy method $\Rightarrow T_0 \approx \min\{1, \delta/\epsilon\}$. 


Energy method

\[
\begin{cases}
\partial_t \zeta - |D|\psi + \epsilon |D|(J\zeta)|D|\psi + \epsilon \nabla \cdot ((J\zeta)\nabla \psi) = 0, \\
\partial_t \psi + \zeta + \frac{\epsilon}{2} J \left(|\nabla \psi|^2 - (|D|\psi)^2\right) = 0.
\end{cases}
\]  

(RWW2)

The principle part of the first equation is

\[
\partial_t \dot{\zeta} - |D|\dot{\psi} + \epsilon |D|(J\dot{\zeta})|D|\psi + \epsilon \nabla \cdot ((J\dot{\zeta})\nabla \psi) = 0.
\]

The principle part of the second equation is

\[
\partial_t \dot{\psi} + \dot{\zeta} + \epsilon (\nabla \psi) \cdot J(\nabla \dot{\psi}) - \epsilon (|D|\psi)J(|D|\dot{\psi}) = 0.
\]

One recognizes Alinhac’s good unknown: \(\tilde{\psi} \df \dot{\psi} - \epsilon (|D|\psi)(J\dot{\zeta})\):

\[
\begin{cases}
\partial_t \tilde{\zeta} - |D|\tilde{\psi} + \epsilon \nabla \cdot ((J\tilde{\zeta})\nabla \psi) = 0, \\
\partial_t \tilde{\psi} + a_J[\zeta, \psi] \tilde{\zeta} + \epsilon \nabla \psi \cdot J \nabla \tilde{\psi} = 0,
\end{cases}
\]  

(Q-RWW2)

with

\[
a_J[\zeta, \psi] f \df f - \epsilon(|D|\zeta)Jf - \epsilon^2(|D|\psi)J^2|D|\left\{(|D|\psi)f\right\}.
\]

and one has

\[
\min(\epsilon, \epsilon^2 \delta^{-1}) \ll 1 \implies (a_J[\zeta, \psi] f, f)_{L^2} \geq \frac{1}{2} |f|_{L^2}^2.
\]
Numerical validation

In numerical simulations, we observe a dichotomy:

- If $\delta > \delta_{\text{crit.}}$, then large time stability.
- If $\delta < \delta_{\text{crit.}}$, then rapid blow-up.

Figure: Critical value $\delta_{\text{crit.}}$, depending on $\epsilon$. 
Conclusion and perspectives

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We exhibited the instability mechanism induced by (WW2), and proposed a “rectification” which (for $\delta$ well-chosen)

- does not harm the precision (in the sense of consistency) of the model;
- restores large time well-posedness (and hence convergence);
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- Results are proved in deep but finite depth, not in shallow water.
- We have not proved ill-posedness.
- We would like to extend the analysis to (WW$N$) with $N$ arbitrary.
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Thank you for your attention