The hydrostatic approximation for stratified fluids

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Overview

The initial set of equations

Motivated by stratified flows, we study the inhomogeneous, incompressible Euler equations with gravity force:

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \rho\big(\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w\partial_z w\big) + \partial_z P + g\,\rho = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \\ \text{boundary conditions.} \end{cases} \tag{E}$$

The velocity field is denoted $(\mathbf{u}, w) : \Omega \to \mathbb{R}^d \times \mathbb{R}$, the density $\rho : \Omega \to \mathbb{R}^+_\star$.
The pressure $P : \Omega \to \mathbb{R}$ satisfies the elliptic equations

$$-\nabla_{\mathbf{x}} \cdot \big(\tfrac{1}{\rho}\nabla_{\mathbf{x}} P\big) - \partial_z\big(\tfrac{1}{\rho}\partial_z P\big) = \nabla_{\mathbf{x}} \cdot \big((\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \partial_z\big(g\rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w\partial_z w\big).$$

After rescaling $\mathbf{x} \leftarrow L_{\mathbf{x}}\mathbf{x}$, $z \leftarrow L_z z$, $(\cdots)$, the system becomes ...

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$$-\nabla_{\mathbf{x}}\cdot\big(\tfrac{1}{\rho}\nabla_{\mathbf{x}} P\big) - \tfrac{L_{\mathbf{x}}^2}{L_z^2}\partial_z\big(\tfrac{1}{\rho}\partial_z P\big) = \nabla_{\mathbf{x}}\cdot\big((\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + w\partial_z\mathbf{u}\big) + \partial_z\big(\tfrac{L_{\mathbf{x}}^2}{L_z^2}\rho + \mathbf{u}\cdot\nabla_{\mathbf{x}} w + w\partial_z w\big).$$

After rescaling $\mathbf{x} \leftarrow L_{\mathbf{x}}\mathbf{x}$, $z \leftarrow L_z z$, $(\cdots)$, the system becomes ...

The hydrostatic approximation

Formally, the Euler equations

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z (\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w \partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \frac{L_z^2}{L_{\mathbf{x}}^2} \rho\big(\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w \partial_z w\big) + \partial_z P + \rho = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \\ \text{boundary conditions.} \end{cases} \tag{E}$$

in the hydrostatic limit $\mu \stackrel{\text{def}}{=} \frac{L_z^2}{L_{\mathbf{x}}^2} \to 0$, read

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z (\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w \partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions.} \end{cases} \tag{H}$$

$$\Longrightarrow \quad P = P\big|_{z=z_{\rm surf}} + \int_z^{z_{\rm surf}} \rho \, \mathrm{d}z', \quad w = w\big|_{z=z_{\rm bot}} - \int_{z_{\rm bot}}^z \nabla_{\mathbf{x}} \cdot \mathbf{u} \, \mathrm{d}z'.$$

(partial and biased) state of the art on the hydrostatic equations

Homogeneous case: $\rho \equiv 1$.

- Spectral stability of the linearized system about shear flows, under the Rayleigh criterion $\underline{\mathbf{u}}''(z) \neq 0$. [Rayleigh (1880)][Arnold][Drazin&Reid]
- Ill-posedness of the linearized system about certain shear flows. [Renardy '09]
- Well-posedness of the (nonlinear) system in Sobolev spaces. [Masmoudi&Wong '12] (after [Grenier '99], [Brenier '03])

Stably stratified case: $\partial_z \rho < 0$.

- Spectral stability of the linearized system about shear flows, under the Miles and Howard criterion $|\underline{\mathbf{u}}'(z)|^2 \leq \frac{1}{4}\frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}$. [Miles '61][Howard '61]
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The linearized system about shear flows

Let us linearize the system

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z (\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w \partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (periodic).} \end{cases} \tag{H}$$

about shear solutions $(\rho, \mathbf{u})(t, \mathbf{x}, z) = (\underline{\rho}(z), \underline{\mathbf{u}}(z))$. We get

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho + \underline{\rho}'(z) w = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}})\mathbf{u} + \underline{\mathbf{u}}'(z) w + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \qquad \left(\Longrightarrow P = \int_z^0 \rho \, \mathrm{d}z' \stackrel{\text{def}}{=} \mathsf{L}\rho\right) \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \qquad \left(\Longrightarrow w = -\int_{-1}^z \nabla_{\mathbf{x}} \cdot \mathbf{u} \, \mathrm{d}z'\right) \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

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The linearized system enjoys a partial symmetric structure

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho - \underline{\rho}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}}) \mathbf{u} - \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} \mathrm{L}\rho = 0, \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\big(\rho \,,\, \tfrac{-1}{\underline{\rho}'(z)} \rho \big)_{L^2_{\mathbf{x},z}} + \big(\mathbf{u} \,,\, \underline{\rho}(z) \mathbf{u} \big)_{L^2_{\mathbf{x},z}} \Big) = \big(\mathbf{u} \,,\, \underline{\rho}(z) \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} \big)_{L^2_{\mathbf{x},z}}.$$

$\rightsquigarrow$ no (obvious) control of the energy.

Conclusion

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho - \underline{\rho}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}}) \mathbf{u} - \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} \mathrm{L} \rho = 0, \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

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No (obvious) control of the energy
in the presence of shear velocity, $\underline{\mathbf{u}}'(z) \neq 0$,
even under the Miles&Howard criterion.
The stable stratification, $\underline{\rho}'(z) < 0$, helps (a bit).

Possible ways around:

- Work in the analytic framework. [Kukavica, Temam, Vicol & Ziane '11]
- Add (horizontal) viscosity. [Cao, Li & Titi '16]
- Do both. [Paicu, Zhang& Zhang '20]

Eddy viscosity and diffusivity

The system studied in [Cao, Li & Titi '16] is

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot \rho + w\partial_z \rho = \kappa \Delta_{\mathbf{x}} \rho, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = \nu \Delta_{\mathbf{x}} \mathbf{u}, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (periodic).} \end{cases} \tag{H}$$

The horizontal viscosity ($\nu > 0$) and diffusivity ($\kappa > 0$) approximate effective isopycnal viscosity and diffusivity:
[Gent & McWilliams '90] propose

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}})\mathbf{u} + (w + w_\star)\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions.} \end{cases} \tag{H}$$

with

$$\mathbf{u}_\star = \kappa \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) , \quad w_\star = -\kappa \nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) .$$

Main result

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}})\mathbf{u} + (w + w_\star)\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (free surface).} \end{cases} \tag{H}$$

[VD & R. Bianchini]

For sufficiently regular data satisfying the (strict) stable stratification assumption

$$-\partial_z \rho|_{t=0} \geq \alpha > 0$$

and for any $\kappa > 0$, there exists a unique (classical) solution of (H) on the time interval $[0, T]$ with

$$T^{-1} = C\left(1 + \kappa^{-1}\left(\left|\underline{\mathbf{u}}'\right|_{L^2_z}^2 + M_0^2\right)\right),$$

where M_0 is the size of the initial deviation from the shear flow equilibrium $(\underline{\rho}(z), \underline{\mathbf{u}}(z))$, and C depends only on M_0, α and the size of $(\underline{\rho}(z), \underline{\mathbf{u}}(z))$.

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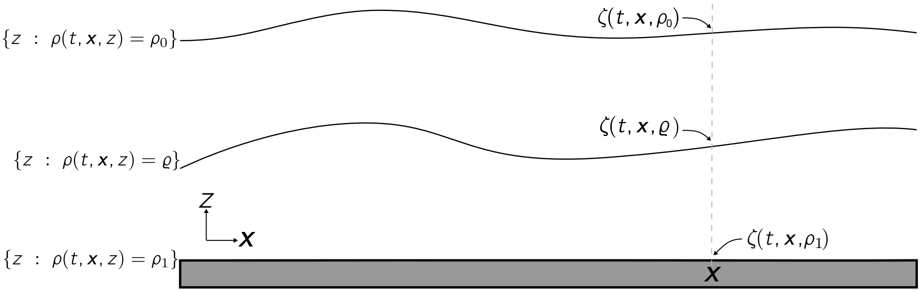
Key points.

- We do not use viscosity, nor analytic data;
- The destabilizing role of the shear velocity is apparent;
- Stable stratification is used in a crucial way.

Isopycnal coordinates

We define the variable $h(t, \mathbf{x}, \varrho) > 0$ from the density $\rho(t, \mathbf{x}, z)$ through

$$h \stackrel{\text{def}}{=} -\partial_\varrho \zeta, \qquad \rho(t, \mathbf{x}, \zeta(t, \mathbf{x}, \varrho)) = \varrho, \quad \zeta(t, \mathbf{x}, \rho(t, \mathbf{x}, z)) = z.$$



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The system

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \rho + (w + w_\star) \partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}})\mathbf{u} + (w + w_\star)\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (free surface).} \end{cases} \tag{H}$$

reads in isopycnal coordinates

$$\begin{cases} \partial_t h + \nabla_{\mathbf{x}} \cdot (h\mathbf{u}) = \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + ((\mathbf{u} + \kappa \frac{-\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}})\mathbf{u}\Big) + \nabla_{\mathbf{x}} \psi = 0, \end{cases} \tag{H}$$

where

$$\psi(t, \mathbf{x}, \varrho) \stackrel{\text{def}}{=} \rho_0 \int_{\rho_0}^{\rho_1} h(t, \mathbf{x}, \varrho')\, \mathrm{d}\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, \mathbf{x}, \varrho'')\, \mathrm{d}\varrho''\, \mathrm{d}\varrho',$$

The system in isopycnal coordinates

$$\begin{cases} \partial_t h + \nabla_{\mathbf{x}} \cdot (h\mathbf{u}) = \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + \big((\mathbf{u} + \kappa \frac{-\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}}\big)\mathbf{u}\Big) + \nabla_{\mathbf{x}} \psi = 0. \end{cases} \tag{H}$$

Key points:

- The advection in the variable z has disappeared (semi-Lagrangian coordinates).
- The domain is flattened.
- The system is easily discretized (multilayer framework).
- The eddy viscosity of [Gent & McWilliams '90] is nice and simple.
- The system enjoys a partial symmetric structure analogous as the one exhibited in the Eulerian coordinates, but the symmetry defect involves a derivative on h (not $\mathbf{u}$).

$\rightsquigarrow$ The proof of the results is based on the energy method on (H), using product, commutator, composition estimates in anisotropic Sobolev spaces.

The hydrostatic limit

The standard procedure to prove convergence (with a convergence rate, and loss of derivatives) of regular solutions of the Euler system towards corresponding solutions of the hydrostatic system as $\mu \stackrel{\text{def}}{=} \frac{L_z^2}{L_x^2} \to 0$, rely on with

- the existence and control of solutions to the limit hydrostatic system;
- a consistency result on the latter solutions;
- the (small-time) existence of solutions to the Euler equations,
- uniform stability estimates on the linearized Euler equations.

Remark: in [Desjardins, Lannes & Saut '20], the authors show the well-posedness on small (that is non-uniform) time of the Euler system (without diffusivity, and rigid-lid assumption).

The non-hydrostatic problem

In isopycnal coordinates, the Euler equations read

$$\begin{cases} \partial_t h + \nabla_{\mathbf{x}} \cdot (h\mathbf{u}) = \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + \big((\mathbf{u} - \kappa\frac{\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}}\big)\mathbf{u}\Big) + \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}}\zeta}{h}\partial_\varrho P = 0, \\ \mu\varrho\Big(\partial_t w + (\mathbf{u} - \kappa\frac{\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}} w\Big) - \frac{\partial_\varrho P}{h} + \varrho = 0, \\ -h\nabla_{\mathbf{x}} \cdot \mathbf{u} - \nabla_{\mathbf{x}}\zeta \cdot \partial_\varrho \mathbf{u} + \partial_\varrho w = 0, \qquad \text{(incomp.)} \\ P\big|_{\varrho=\rho_0} = 0, \qquad w\big|_{\varrho=\rho_1} = 0, \qquad \text{(bound. cond.)} \end{cases} \tag{E}$$

where $\zeta(\cdot,\varrho) \stackrel{\text{def}}{=} \int_\varrho^{\rho_1} h(\cdot,\varrho')\, \mathrm{d}\varrho'$.

The pressure reconstruction system becomes

$$\begin{cases} \dfrac{1}{\mu}\begin{pmatrix} \sqrt{\mu}\nabla_{\mathbf{x}} \\ \partial_\varrho \end{pmatrix} \cdot \left(\begin{pmatrix} \frac{h}{\varrho}\,\mathrm{Id} & \frac{\sqrt{\mu}\nabla_{\mathbf{x}}\zeta}{\varrho} \\ \frac{\sqrt{\mu}\nabla_{\mathbf{x}}^\top\zeta}{\varrho} & \frac{1+\mu|\nabla_{\mathbf{x}}\zeta|^2}{\varrho h} \end{pmatrix} \begin{pmatrix} \sqrt{\mu}\nabla_{\mathbf{x}} \\ \partial_\varrho \end{pmatrix} P \right) = \mathrm{RHS} \\ P\big|_{\varrho=\rho_0} = 0, \qquad (\partial_\varrho P)\big|_{\varrho=\rho_1} = \rho_1 h\big|_{\varrho=\rho_1}. \end{cases}$$

where RHS is long, ugly, but nice.

Symmetric structure

Decomposing $P = P_{\rm hydro} + \tilde{P}$, (with $P_{\rm hydro}(\cdot, \varrho) := \int_{\rho_0}^{\varrho} \varrho' h(\cdot, \varrho')\, \mathrm{d}\varrho'$) we have

$$\begin{cases} \partial_t h + \nabla_{\mathbf{x}} \cdot (h\mathbf{u}) = \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + \big((\mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}}\big)\mathbf{u}\Big) + \nabla_{\mathbf{x}} \psi_{\rm hyd.} + \nabla_{\mathbf{x}} \tilde{P} + \frac{\nabla_{\mathbf{x}} \zeta}{h} \partial_\varrho \tilde{P} = 0, \\ \mu\varrho\Big(\partial_t w + (\mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}} w\Big) - \frac{\partial_\varrho \tilde{P}}{h} = 0, \\ -h\nabla_{\mathbf{x}} \cdot \mathbf{u} - \nabla_{\mathbf{x}} \zeta \cdot \partial_\varrho \mathbf{u} + \partial_\varrho w = 0, \qquad \text{(incomp.)} \\ \tilde{P}\big|_{\varrho=\rho_0} = 0, \qquad w\big|_{\varrho=\rho_1} = 0, \qquad \text{(bound. cond.)} \end{cases} \tag{E}$$

where $(\sqrt{\mu}\nabla_{\mathbf{x}} \tilde{P}, \partial_\varrho \tilde{P})$ is of size $\mathcal{O}(1)$ without loss of derivatives, $\mathcal{O}(\mu)$ with loss.

We observe a symmetric structure superposed to the one of the hydrostatic system, which provides the desired estimates.

[VD & R. Bianchini]

For any sufficiently regular solution satisfying the stable stratification assumption $h \geq \alpha > 0$, and for any $\kappa > 0$, we have existence and uniqueness of solutions to the Euler equation and strong convergence towards the hydrostatic solution as $\mu \to 0$.

Thoughts to go

- The well-posedness of the non-homogeneous hydrostatic equations, without diffusivity or viscosity, is an open problem.
- Stable stratification helps, but seemingly (?) not enough.
- Isopycnal coordinates are quite interesting for numerical <u>and</u> theoretical analyses.

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Thank you for your attention