Boussinesq-Whitham "full-dispersion" systems as asymptotic models for water waves

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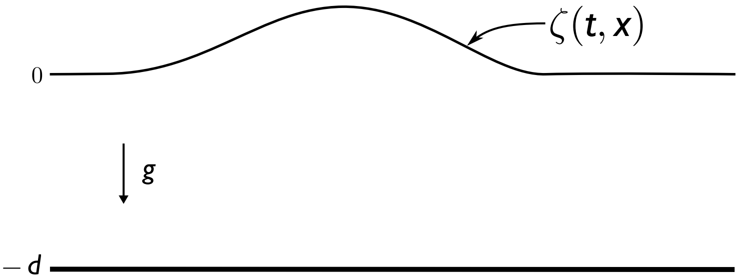
Models for free-surface flows



[Feynman] "[water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...]; they have all the complications that waves can have."

Standard models include: Saint-Venant, Boussinesq, Serre–Green–Naghdi, Matsuno, Korteweg–de Vries, Benjamin–Bona–Mahony, Camassa–Holm, Kawahara, Whitham, Kadomtsev–Petviashvili, Benney-Roskes, NLS...

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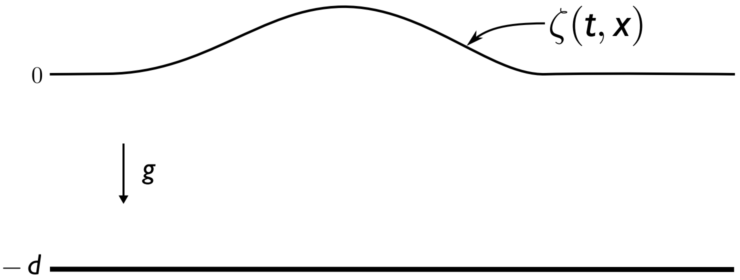

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Models for free-surface flows: comparison

- **Korteweg–de Vries equation**

$$\partial_t\zeta + \sqrt{gd}\,\partial_x\big(\zeta + \tfrac{3}{4d}\zeta^2 + \tfrac{d^2}{6}\partial_x^2\zeta\big) = 0.$$

✓ rigorously justified for long waves
✓ Hamiltonian structure (conserved quantities)
✗ dimension $n = 1$, unidirectional
✗ no wavebreaking, solitary waves of arbitrary amplitude

- **Whitham equation**

$$\partial_t\zeta + \sqrt{gd}\,\partial_x\Big(\sqrt{\frac{\tanh(d|D|)}{d|D|}}\zeta + \tfrac{3}{4d}\zeta^2\Big) = 0.$$

✓ rigorously justified for long waves[1]
✓ Hamiltonian structure (conserved quantities)
✗ dimension $n = 1$, unidirectional
✓ wavebreaking, solitary waves with maximal height[2]

[1] [Klein,Linares,Pilod,Saut '18]
[2] respectively [Hur '17] and [Ehrnström&Wahlén '19]

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- **Boussinesq systems**

  For given $a, b, c \in \mathbb{R}$,

$$\begin{cases} \partial_t\zeta - ad^2\Delta\partial_t\zeta + \nabla\cdot\big(d\mathbf{u} + bd^3\Delta\mathbf{u} + \zeta\mathbf{u}\big) = 0, \\ \partial_t\mathbf{u} - cd^2\Delta\partial_t\mathbf{u} + g\nabla\zeta + g(\frac{1}{3} - a - b - c)d^2\Delta\nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0}. \end{cases}$$

  ✓ rigorously justified for long waves (if well-posed [Saut&Xu '12, '15])
  ✓ Hamiltonian structure (conserved quantities) (if $a = c$)
  ✓ dimension $n \geq 1$, bidirectional
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- **Boussinesq-Whitham systems**

  Aim: Exhibit at least one system with the following properties:
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Models for free-surface flows: comparison

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Boussinesq-Whitham systems

Surveys by [Klein,Linares,Pilod,Saut'18][Carter'18][Dinvay,Dutykh,Kalisch'19]

1. [Aceves-Sánchez,Minzoni,Payanotaros '13], [Moldabayev,Kalisch,Dutykh '15]

$$\begin{cases} \partial_t\zeta + \frac{\tanh(d|D|)}{d|D|} d\nabla\cdot\mathbf{u} + \nabla\cdot(\zeta\mathbf{u}) = 0 \\ \partial_t\mathbf{u} + g\nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0} \end{cases}$$

✗ existence of solutions requires either positive deformation [Pei,Wang'19] or surface tension [Kalisch,Pilod'19]

2. [Hur,Pandey'19]

$$\begin{cases} \partial_t\zeta + d\nabla\cdot\mathbf{u} + \nabla\cdot(\zeta\mathbf{u}) = 0 \\ \partial_t\mathbf{u} + \frac{\tanh(d|D|)}{d|D|} g\nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0} \end{cases}$$

✗ Wrong Hamiltonian. Well-posedness not known.

3. [Dinvay,Dutykh,Kalisch'19]

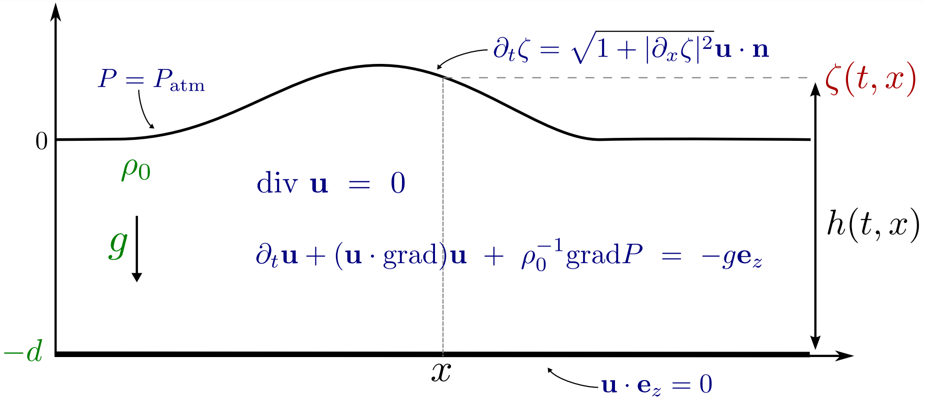
$$\begin{cases} \partial_t\zeta + d\nabla\cdot\mathbf{u} + \frac{\tanh(d|D|)}{d|D|}\nabla\cdot(\zeta\mathbf{u}) = 0 \\ \partial_t\mathbf{u} + \frac{\tanh(d|D|)}{d|D|} g\nabla\zeta + \frac{\tanh(d|D|)}{d|D|}(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0} \end{cases}$$

✓ Hamiltonian, good well-posedness properties [Dinvay,Selberg,Tesfahun].

Aim of this talk: show that this model (among others) is a long wave asymptotic model with improved precision with respect to Boussinesq.

1. Motivation
2. Consistency
   - the water-waves system
   - long wave expansions
   - Boussinesq-Whitham models
3. Full justification
   - from consistency to convergence
   - large-time well-posedness theory
4. Conclusion

The water-waves system



- The domain is an infinite layer with a free surface.
- The fluid is incompressible, the only external force is gravity.
- Particles of fluid cannot cross the surface or bottom.
- Surface tension, viscosity, atm. pressure are not taken into account.
- Irrotational motion: $\mathbf{u} = \text{grad}\phi$.

Irrotational motion

Potential flows

In the irrotational setting $\mathbf{u} = \mathrm{grad}\phi$, the fluid velocity is determined by

$$\zeta(t,x) \text{ and } \psi(t,x) \stackrel{\text{def}}{=} \phi(t,x,\zeta(t,x))$$

after solving

$$\begin{cases} \Delta\phi = 0 & \text{in } \{(x,z)\ \ -d < z < \zeta(t,x)\}, \\ \partial_z\phi = 0 & \text{on } \{(x,z)\ z = -d\}, \\ \phi = \psi & \text{on } \{(x,z)\ z = \zeta(t,x)\}. \end{cases}$$

Zakharov/Craig-Sulem formulation [Zakharov '68, Craig&Sulem '93]

The full Euler equations are equivalent to the following system

$$\begin{cases} \partial_t\zeta - \frac{\delta\mathcal{H}}{\delta\psi} = 0, \\ \partial_t\psi + \frac{\delta\mathcal{H}}{\delta\zeta} = 0, \end{cases}$$

where

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}} g\zeta^2 + \psi G[\zeta]\psi \,\mathrm{d}x, \qquad G[\zeta]\psi = (\partial_z\phi - \nabla\zeta\cdot\nabla\phi)|_{z=\zeta}.$$

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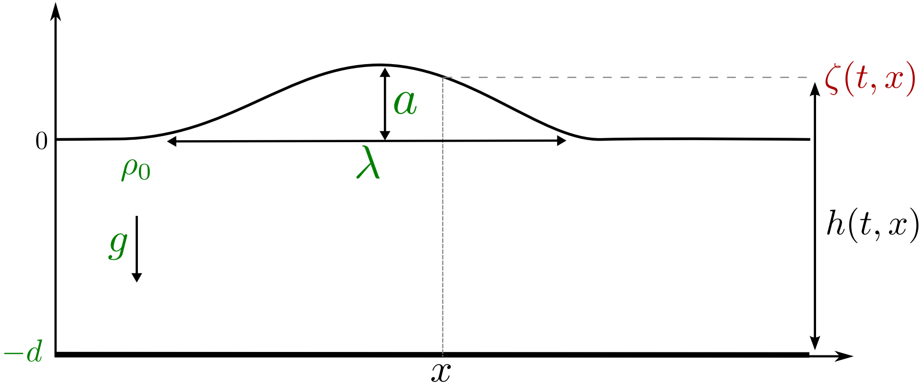
Asymptotic models

We seek simplified models with valid predictions in a given regime.



Through rescaling, we work with dimensionless variables of size ≈ 1, and dimensionless parameters appear in the system :

$$\epsilon \stackrel{\text{def}}{=} a/d \qquad ; \qquad \mu \stackrel{\text{def}}{=} d^2/\lambda^2.$$

Shallow-water: $\mu \ll 1$. Weakly nonlinear: $\epsilon \ll 1$. Long waves: $\epsilon, \mu \ll 1$.

The Dirichlet-to-Neumann operator

We want provide approximate expressions for (as $\epsilon, \mu \ll 1$) to

$$G^{\mu}[\epsilon\zeta]\psi = (\partial_z\phi - \mu\epsilon\nabla\zeta\cdot\nabla\phi)|_{z=\epsilon\zeta}$$

where ϕ satisfies the (scaled) Laplace problem

$$\begin{cases} \partial_z^2\phi + \mu\Delta_x\phi = 0 & \text{in } \{(x,z)\ -1<z<\epsilon\zeta(t,x)\}, \\ \partial_z\phi = 0 & \text{on } \{(x,z)\ z=-1\}, \\ \phi = \psi & \text{on } \{(x,z)\ z=\epsilon\zeta(t,x)\}. \end{cases}$$

Strategy [Lannes '13]

- Flatten the domain with $\Sigma : (X,z)\in\mathbb{R}^n\times(0,1)\mapsto(X,\sigma(X,z))$ where $\sigma(X,z)=(1+\epsilon\zeta)z+\epsilon\zeta \quad \rightsquigarrow \tilde\phi(X,z)\overset{\text{def}}{=}\phi(X,\sigma(X,z))$.
- Construct $\tilde\phi^{\text{app}}$ solving approximately the equations on the flat strip.
- Elliptic estimates $\rightsquigarrow$ control of $\tilde\phi-\tilde\phi^{\text{app}}$.
- Use in the identity

$$\frac{1}{\mu}G^{\mu}[\epsilon\zeta]\psi = \nabla\cdot\left(\int_{-1}^{0}\partial_z\tilde\phi(\nabla\sigma)-\nabla_x\tilde\phi(\partial_z\sigma)\mathrm{d}z\right).$$

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For instance,

- $\tilde\phi^{\rm app}(\cdot,z) = \psi \quad \rightsquigarrow \frac{1}{\mu} G^\mu[\epsilon\zeta]\psi = -\nabla\cdot\big((1+\epsilon\zeta)\nabla\psi\big) + \mathcal{O}(\mu)$

  $\rightsquigarrow$ Saint-Venant system is consistent with precision $\mathcal{O}(\mu)$
- $\tilde\phi^{\rm app}(\cdot,z) = \psi - \frac12(1+\epsilon\zeta)^2(z^2+2z)\Delta\psi \quad \rightsquigarrow \frac{1}{\mu} G^\mu[\epsilon\zeta]\psi = [\cdots] + \mathcal{O}(\mu^2)$

  $\rightsquigarrow$ Green-Naghdi system is consistent with precision $\mathcal{O}(\mu^2)$
- $\tilde\phi^{\rm app}(\cdot,z) = \frac{\cosh(\sqrt{\mu}|D|(z+1))}{\cosh(\sqrt{\mu}|D|)}\psi \quad \rightsquigarrow \frac{1}{\mu} G^\mu[\epsilon\zeta]\psi = [\cdots] + \mathcal{O}(\epsilon\mu)$

  $\rightsquigarrow$ Boussinesq-Whitham systems are consistent with precision $\mathcal{O}(\epsilon\mu)$

[Emerald]

Let $s \in \mathbb{N}$, $h, M > 0$. There exists $C > 0$ such that for any $\zeta \in H^{s+4}(\mathbb{R}^n)$ such that

$$1 + \epsilon\zeta \geq h \quad \text{and} \quad \epsilon\big\|\zeta\big\|_{H^{s+4}} \leq M,$$

and for any $\psi \in L^2_{\mathrm{loc}}(\mathbb{R})$ such that $\nabla\psi \in H^{s+3}(\mathbb{R}^n)$ we have

$$\Big\|\frac{1}{\mu}G^{\mu}[\epsilon\zeta]\psi + \nabla\cdot\big((1+\epsilon\zeta)\mathrm{F}^{\mu}\nabla\psi\big)\Big\|_{H^s} \leq \epsilon\mu\, C\,\big\|\zeta\big\|_{H^{s+4}}\big\|\nabla\psi\big\|_{H^{s+3}}$$

where $\mathrm{F}^{\mu} = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$.

[Emerald]

$$\Big\| \frac{1}{\mu} G^{\mu}[\epsilon\zeta]\psi + \nabla\cdot\big((1+\epsilon\zeta)\mathsf{F}^{\mu}\nabla\psi\big)\Big\|_{H^s} \leq \epsilon\mu\, C\, \big\|\zeta\big\|_{H^{s+4}}\big\|\nabla\psi\big\|_{H^{s+3}}$$

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Consistency [Emerald]

Let $s\in\mathbb{N}$, $h, M > 0$. There exists $C>0$ such that for any ζ,ψ strong solution to (WW) such that for any $t\in[0,T)$,

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we have

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big((1+\epsilon\zeta)\mathsf{F}^{\mu}\nabla\psi\big) = r_1 \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 = r_2, \end{cases}$$

with

$$\big\|(r_1,r_2)\big\|_{H^s} \leq \epsilon\mu\, C\, \Big(\big\|\zeta\big\|_{H^{s+4}} + \big\|\nabla\psi\big\|_{H^{s+3}}\Big)^2.$$

✗ Not Hamiltonian (?). Not well-posed (?)

Hamiltonian systems

To construct Hamiltonian systems, we use the approximation

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}} \zeta^2 + \psi\frac{1}{\mu}G^{\mu}[\epsilon\zeta]\psi \, \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \zeta^2 + \nabla\psi\cdot \mathsf{F}^{\mu}\nabla\psi + \epsilon\zeta|\nabla\psi|^2 \mathrm{d}x + \mathcal{O}(\epsilon\mu)$$

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Well-posed Hamiltonian systems

To construct well-posed Hamiltonian systems, we regularize

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where ${\rm F}_2^\mu = f_2(\sqrt{\mu}|D|)$, with $f_2(0)=1$, $f_2'(0)=0$ and f_2 of order $\sigma<0$.

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Well-posed Hamiltonian systems

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- large-time well-posedness theory

4 Conclusion

From consistency to convergence

Large-time well-posedness [Álvarez-Samaniego,Lannes '08][Iguchi '09]

Let $h_0, M_0 > 0$ and $N \geq N_0 \geq 6$, $2 \leq P \leq N$. There exists $C, T > 0$ such that for any $\epsilon, \mu \in (0,1]$ and any $\zeta_0, \nabla\psi_0 \in H^N(\mathbb{R}^n)$ such that

$$1 + \epsilon\zeta_0 \geq h_0 \quad \text{and} \quad \big\|\zeta_0\big\|_{H^{N_0}} + \big\|\nabla\zeta_0\big\|_{H^{N_0}} \leq M_0,$$

there exists a unique solution to (WW) with initial data (ζ_0, ψ_0) and satisfying

$$\forall t \in (0, T/\epsilon), \qquad \big\|(\zeta, \nabla\psi)\big\|_{L^\infty(0,t;H^{N-P})} \leq C\big\|(\zeta_0, \nabla\psi_0)\big\|_{H^N}.$$

Rmk: There is no "loss of derivative". C, T are uniform w.r.t $\epsilon, \mu \in (0,1]$.

Stability [Álvarez-Samaniego,Lannes '08][Iguchi '09]

Let $h_0, M_0 > 0$, $N \geq N_0 \geq 6$, $3 \leq P \leq N$. There exists $C, T > 0$ such that for any $\epsilon, \mu \in (0,1]$ and any $(\zeta^{\rm app}, \psi^{\rm app})$ solution to (WW) with remainder term R and

$$\sup_{t\in(0,T/\epsilon)} 1 + \epsilon\zeta^{\rm app}(t,\cdot) \geq h_0 \quad \text{and} \quad \big\|(\zeta^{\rm app}, \nabla\psi^{\rm app})\big\|_{L^\infty(0,T/\epsilon;H^N)} \leq M_0,$$

the difference with respect to the exact solution to (WW) with same initial data

$$\forall t \in (0, T/\epsilon), \qquad \big\|(\zeta - \zeta^{\rm app}, \nabla\psi - \nabla\psi^{\rm app})\big\|_{L^\infty(0,t;H^{N-P})} \leq C\, t\,\big\|R\big\|_{L^\infty(0,t;H^N)}.$$

From consistency to convergence

Large-time well-posedness [Álvarez-Samaniego,Lannes '08][Iguchi '09]

Let $h_0, M_0 > 0$ and $N \geq N_0 \geq 6$, $2 \leq P \leq N$. There exists $C, T > 0$ such that for any $\epsilon, \mu \in (0,1]$ and any $\zeta_0, \nabla\psi_0 \in H^N(\mathbb{R}^n)$ such that

$$1 + \epsilon\zeta_0 \geq h_0 \quad \text{and} \quad \big\|\zeta_0\big\|_{H^{N_0}} + \big\|\nabla\zeta_0\big\|_{H^{N_0}} \leq M_0,$$

there exists a unique solution to (WW) with initial data (ζ_0, ψ_0) and satisfying

$$\forall t \in (0, T/\epsilon), \qquad \big\|(\zeta, \nabla\psi)\big\|_{L^\infty(0,t;H^{N-P})} \leq C\big\|(\zeta_0, \nabla\psi_0)\big\|_{H^N}.$$

Rmk: There is no "loss of derivative". C, T are uniform w.r.t $\epsilon, \mu \in (0,1]$.

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Well-posedness: framework

We consider the initial-value problem for

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(\mathsf{F}^\mu \nabla \psi + \epsilon \mathsf{F}_2^\mu (\zeta \, \mathsf{F}_2^\mu \nabla \psi) \big) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2} |\mathsf{F}_2^\mu \nabla \psi|^2 = 0, \end{cases}$$

where $\mathsf{F}^\mu = f(\sqrt{\mu}|D|)$ and $\mathsf{F}_2^\mu = f_2(\sqrt{\mu}|D|)$, with f, f_2 of order $\sigma \leq 0$ and

$$\boxed{f_2^2 \ \leq \ f \ \lesssim \ 1}$$

For instance, $\mathsf{F}^\mu = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and $\mathsf{F}_2^\mu = (\mathsf{F}^\mu)^\alpha$ with $\alpha \geq 1/2$.

Qn: Existence and uniqueness of a strong solution to the Cauchy problem, in the Sobolev setting and uniformly with respect to $\epsilon, \mu \ll 1$.

- The system is of quasilinear type unless $|f_2| \lesssim f$ (i.e. $\alpha \in [1/2, 1)$).
- The weak dispersion ($\mu \ll 1$) forbids the use of dispersive techniques.

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Well-posedness: energy estimates

We repeatedly use product and commutator estimates in Sobolev spaces, and in particular

$$\big\|[\mathsf{F}^{\mu}\nabla, f]g\big\|_{L^2} + \big\|[\mathsf{F}_2^{\mu}\nabla, f]g\big\|_{L^2} \lesssim \big\|\nabla f\big\|_{H^{k_\star}}\big\|g\big\|_{L^2} \qquad (k_\star > n/2)$$

uniformly with respect to $\mu \in (0,1]$, and using f, f_2 of order $\sigma \leq 0$.

Quasilinearisation: for $k > 1 + n/2$, using $|f_2| \lesssim 1$,

$$\begin{cases} \partial_t \zeta^{(k)} + \nabla\cdot\big(\mathsf{F}^{\mu}\nabla\psi^{(k)} + \epsilon\mathsf{F}_2^{\mu}(\zeta\,\mathsf{F}_2^{\mu}\nabla\psi^{(k)}) + \epsilon\mathsf{F}_2^{\mu}(\zeta^{(k)}\,\mathsf{F}_2^{\mu}\nabla\psi)\big) = r_1^k, \\ \partial_t\nabla\psi^{(k)} + \nabla\zeta^{(k)} + \epsilon(\mathsf{F}_2^{\mu}\nabla\psi\cdot\nabla)(\mathsf{F}_2^{\mu}\nabla\psi^{(k)}) = r_2^k, \end{cases}$$

with

$$\big\|r_1^k\big\|_{L^2} + \big\|r_2^k\big\|_{L^2} \leq \epsilon\; C(\big\|\zeta\big\|_{H^k}, \big\|\mathsf{F}_2^{\mu}\nabla\psi\big\|_{H^k}).$$

Energy space: provided $1+\epsilon\zeta \in L^\infty(\mathbb{R}^n)$ and $1+\epsilon\zeta \geq h_0 > 0$, one has

$$\mathfrak{S}^{\mu}\bullet \stackrel{\text{def}}{=} \mathsf{F}^{\mu}\bullet + \epsilon\mathsf{F}_2^{\mu}(\zeta\,\mathsf{F}_2^{\mu}\bullet) \;\approx\; \mathsf{F}^{\mu}$$

using $f_2^2 \leq f$ (in the setting $f_2^2 \leq Cf$, we would require $1 + C\epsilon\zeta \geq h_0 > 0$)

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Energy estimate: testing the first and second equations with $\zeta^{(k)}$ and $\mathfrak{S}^\mu \nabla \psi^{(k)}$ respectively, we obtain the following *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\big\| \zeta^{(k)} \big\|_{L^2}^2 + \big\| \sqrt{\mathsf{F}^\mu} \nabla \psi^{(k)} \big\|_{L^2}^2 \Big) \lesssim \epsilon\ C(\|\zeta\|_{H^k}, \|\sqrt{\mathsf{F}^\mu} \nabla \psi\|_{H^k}).$$

Well-posedness: result

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(\mathsf{F}^\mu\nabla\psi + \epsilon\mathsf{F}_2^\mu(\zeta\,\mathsf{F}_2^\mu\nabla\psi)\big) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\mathsf{F}_2^\mu\nabla\psi|^2 = 0, \end{cases}$$

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large-time well-posedness

Let $M_0, h_0 > 0$ and $k > 1 + n/2$. There exists $C, T > 0$ such that for any $\epsilon, \mu \in (0,1)$ and any $\zeta^0, \sqrt{\mathsf{F}^\mu}\nabla\psi^0 \in H^k(\mathbb{R}^n)$ such that

$$1 + \epsilon\zeta^0 \geq h_0 > 0 \quad \text{and} \quad \big\|\zeta^0\big\|_{H^k} + \big\|\sqrt{\mathsf{F}^\mu}\nabla\psi^0\big\|_{H^k} \leq M_0,$$

there exists (ζ, ψ) unique strong solution to $(\star)$ with initial data ζ^0, ψ^0 on the time interval $[0, T/\epsilon]$ and one has

$$\forall t \in [0, T/\epsilon], \quad \big(\|\zeta\|_{H^k} + \big\|\sqrt{\mathsf{F}^\mu}\nabla\psi\big\|_{H^k}\big)(t) \leq C\big(\|\zeta^0\|_{H^k} + \big\|\sqrt{\mathsf{F}^\mu}\nabla\psi^0\big\|_{H^k}\big).$$

1. Motivation
2. Consistency
   - the water-waves system
   - long wave expansions
   - Boussinesq-Whitham models
3. Full justification
   - from consistency to convergence
   - large-time well-posedness theory
4. Conclusion

Comparison with Boussinesq systems

Recall: Hamilton's equations with

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}} \zeta^2 + \nabla\psi \cdot \mathsf{F}^\mu \nabla\psi + \epsilon\zeta|\mathsf{F}_2^\mu \nabla\psi|^2$$

yields

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- If $\mathsf{F}^\mu = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and $\mathsf{F}_2^\mu = \mathrm{Id} + \mathcal{O}(\mu)$, then $(\star)$ has precision $\mathcal{O}(\epsilon\mu)$.
  $\rightharpoonup$ If $\mathsf{F}_2^\mu = \mathsf{F}^\mu$, we recover the system in [Dinvay,Dutykh,Kalisch]
- If $\mathsf{F}^\mu = \mathrm{Id} + \frac{\mu}{3}\Delta + \mathcal{O}(\mu^2)$ and $\mathsf{F}_2^\mu = \mathrm{Id} + \mathcal{O}(\mu)$, $(\star)$ has precision $\mathcal{O}(\mu^2 + \epsilon\mu)$.
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From uniform estimates stem bounds between exact and approximate solutions as

$$\text{error} = \mathcal{O}(\text{precision} \times t) \qquad \text{for } t = \mathcal{O}(1/\epsilon).$$

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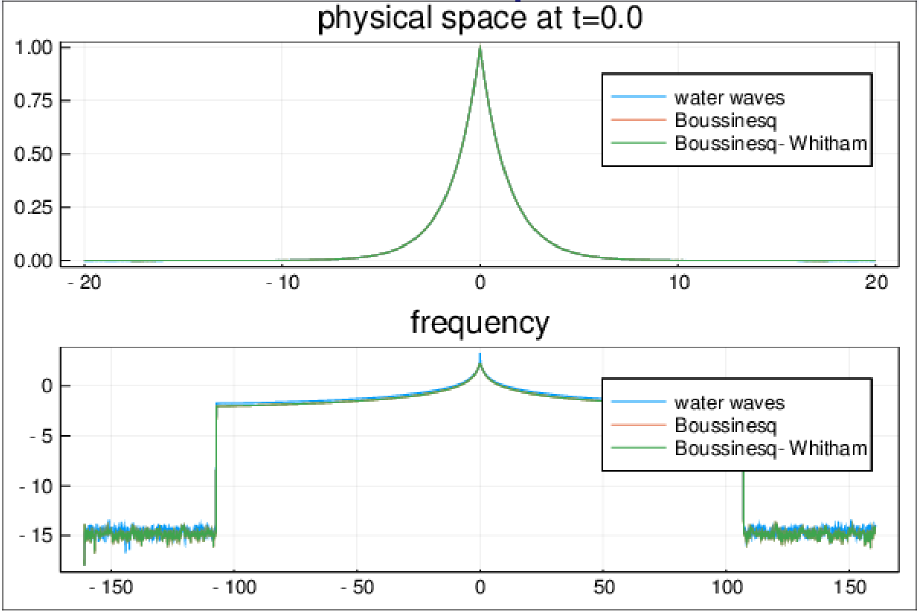
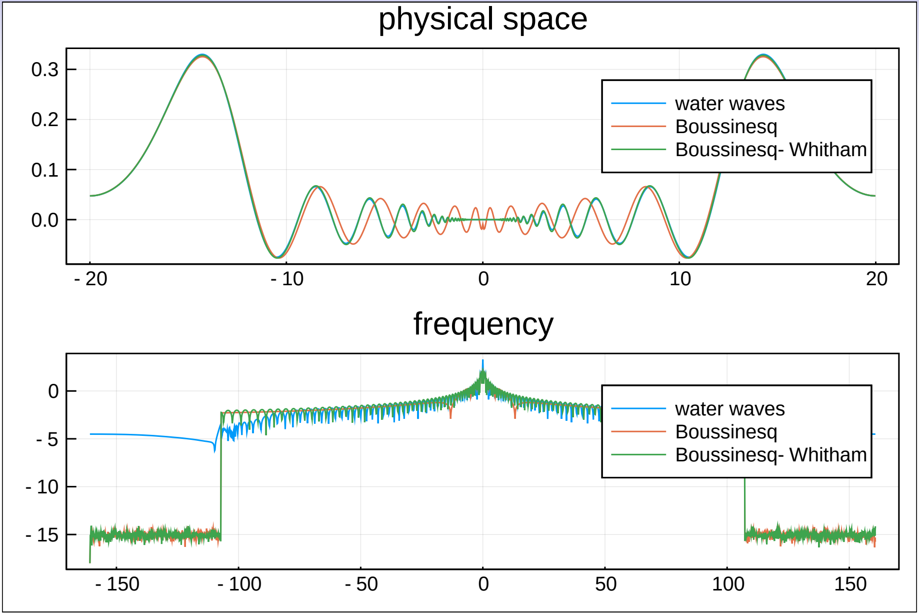
A numerical experiment





Thank you for your attention