Large-time asymptotic stability of traveling wave solutions to scalar balance laws

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CNRS and Univ. Rennes 1

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$$\partial_t u + \partial_x\big(f(u)\big) = g(u). \qquad (\star)$$

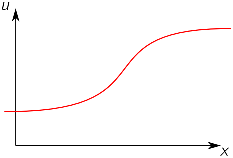
with $u : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^1 \to \mathbb{R}^1$ and $f, g \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$.

Travelling waves : $u(t,x) = \underline{U}(x - \sigma t)$, $\sigma \in \mathbb{R}$.

Constants

$\underline{U} \equiv u_\star, \quad g(u_\star) = 0.$

Fronts

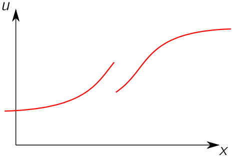


$(f'(\underline{U}) - \sigma)\underline{U}' = g(\underline{U}).$

Shocks

$\underline{U}(x) \in \{u_l, u_r\}, \ \sigma = \frac{f(u_l)-f(u_r)}{u_l-u_r}.$

Composites



Possibly periodic.

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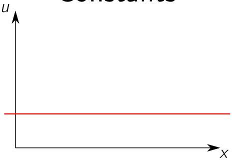

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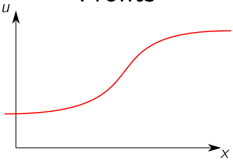

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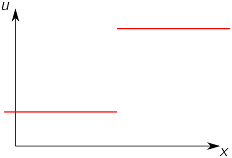

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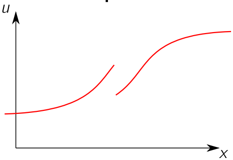
Composites



Possibly periodic.

Some litterature

State of the art.

- Periodic setting [Fan&Hale '93, Lyberopoulos '94, Sinestrari '95&'97]
- Constant near infinity [Sinstrari '96][Mascia&Sinestrari '97]
- Riemann initial data [Sinestrari '97][Mascia '98 & '00]
- Monotone initial data [Mascia '98]

Result: Convergence in $L^\infty(\mathbb{R})$ into a succession of traveling waves.
Tool: generalized characteristics of Dafermos, comparison principles.

Our result.

- Initial data in a neighborhood of traveling waves.
- Distinguish "stable" equilibria with sharp decay rate.
- Stronger topology: $W^{k,\infty}(\mathbb{R})$, $k \geq 1$.
- Use spectral analysis, resolvent estimates and semigroup theory.

in the spirit of [Kapitula&Promislow'13][Johnson,Noble,Rodrigues&Zumbrun'14] (among many others) for viscous conservation laws.

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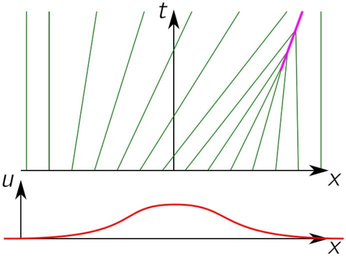
Conservation laws vs Balance laws

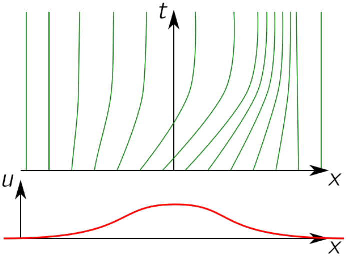
- Conservation laws $\partial_t u + u\partial_x u = 0.$



Generation of shocks for any smooth and decaying initial data.

Algebraic decay at best.

- Balance laws $\partial_t u + u\partial_x u = -u.$



No shock provided $\partial_x u(t=0,\cdot) \geq -1.$

Exponential decay.

Some more litterature

Other works.

[Hanouzet&Natalini '03][Yong '04][Ruggeri&Serre '04]
[Bianchini,Hanouzet&Natalini '07][Kawashima&Yong '04&'09]
[Xu&Kawashima '14] (and many others, e.g. [Bianchini&Natalini])

Result: Global existence of classical solutions (and asymptotic stability) of constant states for partially dissipative multi-dimensional systems.
Tool: modified energy method, Shizuta-Kawashima condition, null forms...

Our hope.

- Maybe provide sharper result (loss of decay or regularity)?
- Offer "black-box" results starting from spectral considerations.
- Deal with more general asymptotic states, and in particular shocks. See [Yang&Zumbrun '19] for inviscid shallow water system with inclination.

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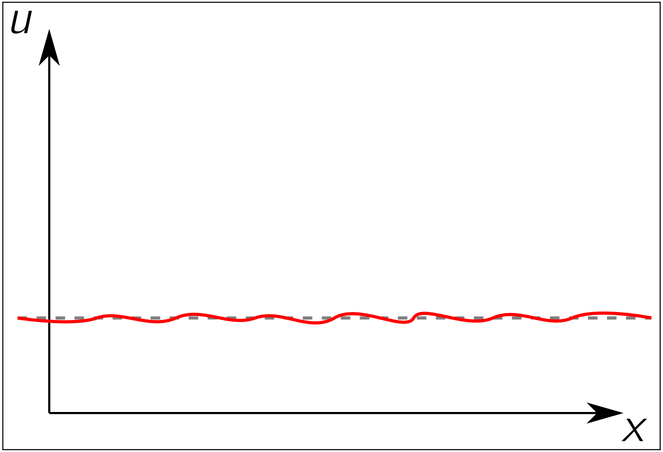
But let's go back to the unidimensional scalar case...

Asymptotic stability of constant states

$$\partial_t u + \partial_x\big(f(u)\big) = g(u) \qquad (\star)$$



$$u(t=0,x) \approx \underline{U}(x) = u_\star, \qquad g(u_\star) = 0.$$

Main result

Asymptotic stability of constants

Let $f, g \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$, $u_\star \in \mathbb{R}$ such that

$$g(u_\star) = 0\,, \quad g'(u_\star) < 0.$$

For any $C_0 > 1$, there exists $\epsilon > 0$ such that for any $v_0 \in BUC^1(\mathbb{R})$ with

$$\left\|v_0\right\|_{W^{1,\infty}} \leq \epsilon,$$

the classical solution to $(\star)$ and $u(t=0) = u_\star + v_0$ is global in time and satisfies for any $t \geq 0$,

$$\left\|u - u_\star\right\|_{L^\infty} \leq \left\|v_0\right\|_{L^\infty} C_0 e^{g'(u_\star)t}, \quad \left\|\partial_x u\right\|_{L^\infty} \leq \left\|\partial_x v_0\right\|_{L^\infty} C_0 e^{g'(u_\star)t}.$$

Rmk: if $f''(u_\star) \neq 0$, we may assume only

$$\left\|v_0\right\|_{L^\infty} + \left\|(\mathrm{sgn}(f''(u_\star))\partial_x v_0)_-\right\|_{L^\infty} \leq \epsilon.$$

This covers initial discontinuities generating rarefaction waves.

Sketch of the proof (1/2)

Denoting $u = u_\star + v$, the solution satisfies

$$\partial_t v + f'(u_\star + v)\partial_x v - g'(u_\star)v = g(u_\star + v) - g(u_\star) - g'(u_\star)v.$$

Hence we study the linear operator

$$L_{a,b} \stackrel{\text{def}}{=} -a(\cdot)\partial_x + b$$

with $a \in BUC^1(\mathbb{R})$, $a(x) > 0$ and $b \in \mathbb{R}$. $\rightsquigarrow$ solve $\partial_t v = L_{a,b} v$. We have

- $L_{a,b}$ is closed, densely defined on $BUC^0(\mathbb{R})$ with domain $BUC^1(\mathbb{R})$.
- $b + i\mathbb{R} \in \mathrm{Spec}(L_{a,b})$.
- If $\Re(\lambda) > b$, then

$$(\lambda - L_{a,b})^{-1} v = \int_{-\infty}^{\cdot} e^{\int_y^{\cdot} \frac{b-\lambda}{a(z)} \mathrm{d}z} \frac{v(y)}{a(y)} \mathrm{d}y.$$

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In particular,

$$\left\|(\lambda - L_{a,b})^{-1} v\right\|_{L^\infty} \leq \frac{1}{\Re(\lambda) - b} \left\|v\right\|_{L^\infty}$$

and if $\Re(\lambda) > b - \inf(a')$ and $v \in W^{1,\infty}(\mathbb{R})$, then

$$\left\|\partial_x(\lambda - L_{a,b})^{-1} v\right\|_{L^\infty} \leq \frac{1}{\Re(\lambda) - b + \inf(a')} \left\|\partial_x v\right\|_{L^\infty}$$

(Hille-Yosida) $\Longrightarrow$ $L_{a,b}$ generates a $\mathcal{C}^0$ semi-group $\mathcal{S}_{a,b}$

Sketch of the proof (2/2)

Denoting $u = u_\star + v$, the solution satisfies

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There remains to

1. "unfreeze time": consider $a \in \mathcal{C}^0(\mathbb{R}^+, BUC^1(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+, L^\infty(\mathbb{R}))$ and

$$\mathcal{U}_{a,b}(s,t) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathcal{S}_{a(t_n^n,\cdot),b}(\delta t) \cdots \mathcal{S}_{a(t_0^n,\cdot),b}(\delta t)$$

with $\delta t = \frac{t-s}{n}$, $t_k^n = s + k\delta t$ [Pazy '83]. $\rightsquigarrow$ solve $\partial_t v + a\partial_x v - bv = 0$.

2. "go nonlinear": let $\mathcal{U}_v \stackrel{\text{def}}{=} \mathcal{U}_{f'(u_\star+v),g'(u_\star)}$ and use Duhamel's formula

$$v(t,\cdot) = \mathcal{U}_v(0,t)v_0 + \int_0^t \mathcal{U}_v(s,t)\big(g(u_\star+v) - g(u_\star) - g'(u_\star)v\big)(s,\cdot)\mathrm{d}s.$$

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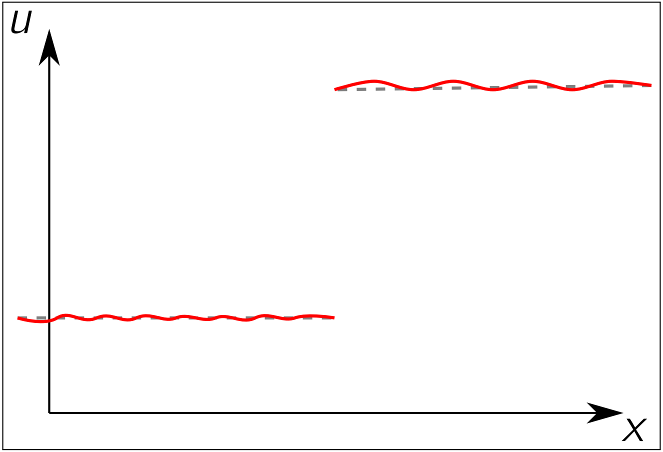
Asymptotic stability of shocks

$$\partial_t u + \partial_x \big(f(u)\big) = g(u) \qquad (\star)$$



$$u(t=0,x) \approx \underline{U}(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}, \qquad g(u_l) = g(u_r) = 0.$$

Main result

Orbital asymptotic stability of strictly entropic shocks

Let $f, g \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$, $u_l, u_r \in \mathbb{R}$ such that

$$g(u_l) = g(u_r) = 0\,, \quad g'(u_l), g'(u_r) < 0\,.$$

For any $C_0 > 1$, there exist $\epsilon, C > 0$ such that for any $v_0 \in BUC^1(\mathbb{R}^\star)$ with

$$\|v_0\|_{W^{1,\infty}(\mathbb{R}^\star)} \leq \epsilon,$$

there exist $\psi \in \mathcal{C}^2(\mathbb{R}^+)$, $v \in BUC^1(\mathbb{R}^+ \times \mathbb{R}^\star)$ with $u(t=0,\cdot) = \underline{U} + v_0$ and $u(t,\cdot) = \underline{U}(\cdot + \psi(t)) + v(t, \cdot + \psi(t))$ is a weak solution to $(\star)$, and

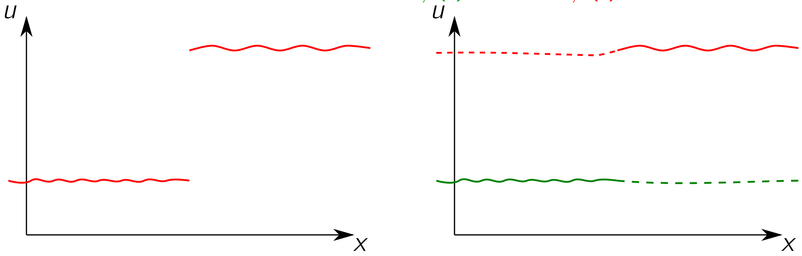
$$\|v\|_{L^\infty(\mathbb{R}^\pm)} \leq \|v_0\|_{L^\infty(\mathbb{R}^\pm)} C_0 e^{\omega_\pm t}, \quad \|\partial_x v\|_{L^\infty(\mathbb{R}^\pm)} \leq \|\partial_x v_0\|_{L^\infty(\mathbb{R}^\pm)} C_0 e^{\omega_\pm t}$$

$$|\psi'(t) - \tfrac{f(u_l)-f(u_r)}{u_l-u_r}| \leq C\|v_0\|_{L^\infty(\mathbb{R}^\star)} e^{\omega t}$$

with $\omega_- \stackrel{\text{def}}{=} g'(u_l)$, $\omega_+ \stackrel{\text{def}}{=} g'(u_r)$, $\omega \stackrel{\text{def}}{=} \max\{g'(u_l), g'(u_r)\}$.

Sketch of the proof

1. Extend the initial data: $\rightsquigarrow u_l + v_{0,l}(\cdot)$, $u_r + v_{0,r}(\cdot)$



2. Use previous result $\rightsquigarrow v_l(t,\cdot)$, $v_r(t,\cdot)$, exponentially decaying.
3. Set ψ through Rankine-Hugoniot: $\psi(0) = 0$ and

$$\psi'(t) = \frac{f(u_l + v_l) - f(u_r + v_r)}{(u_l + v_l) - (u_r + v_r)}(t, \psi(t))$$

4. Define

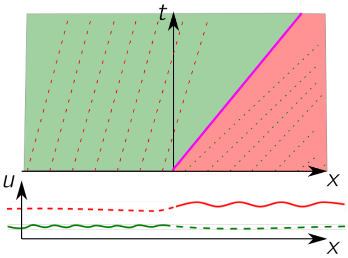
$$u(t,x) = \begin{cases} u_l + v_l(t,x) & \text{if } x < \psi(t), \\ u_r + v_r(t,x) & \text{if } x > \psi(t). \end{cases}$$

Comments on the proof

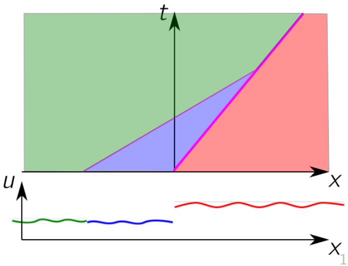
1. $v_l(t,\cdot)$ and $v_r(t,\cdot)$ are not uniquely defined, but $u(t,\cdot)$ is unique as soon as

$$f'(u_l) < \sigma < f'(u_r).$$

If the shock is strictly entropic (Oleinik condition), then u is the unique entropy solution by [Kružkov '70].

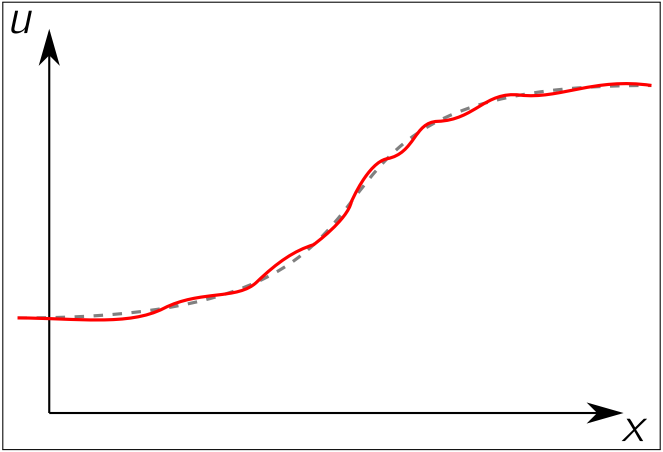


2. The same strategy allows to consider perturbations with small shocks.



Asymptotic stability of fronts

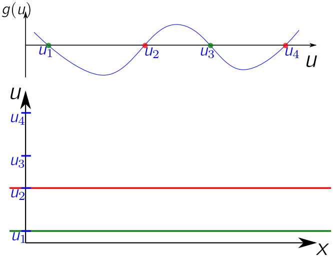
$$\partial_t u + \partial_x\big(f(u)\big) = g(u) \qquad (\star)$$



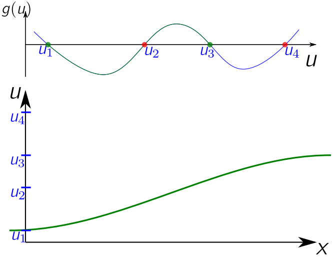
$$u(t=0,x) \approx \underline{U}(x), \qquad (f'(\underline{U}) - \sigma)\underline{U}' = g(\underline{U}).$$

Classification of fronts

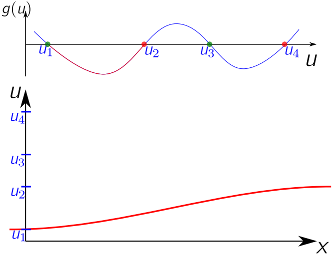
$$(f'(\underline{U}) - \sigma)\underline{U}' = g(\underline{U}).$$



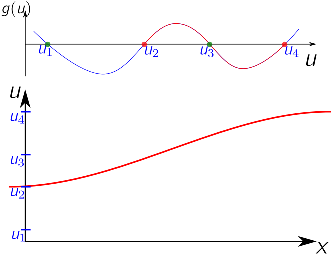
$\sigma \in \mathbb{R}$ irrelevant.



$\sigma = f'(u_2)$



$\sigma \notin f'([u_1, u_2])$



$\sigma = f'(u_3)$

Main result

Asymptotic stability of (some) fronts

Let $f, g \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$, $u_l, u_\star, u_r \in \mathbb{R}$ consecutive zeros of g such that

$$g'(u_l), g'(u_r) < 0\,, \quad g'(u_\star) > 0\,, \quad f''(u_\star) \neq 0, f'(u_\star) \notin f'([u_l, u_r]\backslash\{u_\star\}).$$

Denote $\underline{U} : \mathbb{R} \to (u_l, u_r)$ the (strictly monotonic) solution to

$$(f'(\underline{U}) - f'(u_\star))\underline{U}' = g(\underline{U}) \quad \text{and} \quad \underline{U}\,|_{x=0} = u_\star.$$

There exists $\|\cdot\|_\star \approx \|\cdot\|_{W^{1,\infty}}$ such that for any $C_0 > 1$ and $\omega \in (\max(g'(u_l), g'(u_r), -g'(u_\star)), 0)$, there exists $\epsilon > 0$ and $C > 0$ such that for any $v_0 \in BUC^1(\mathbb{R})$ with

$$\|v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \epsilon,$$

the classical solution to $(\star)$ and $u(t=0) = \underline{U} + v_0$ is global in time and

$$\forall t \geq 0, \quad \|u - \underline{U}(\cdot - f'(u_\star)t - \psi_0)\|_\star \leq \|v_0\|_\star C_0 e^{\omega t}$$

where $\psi_0 \in \mathbb{R}$ with $|\psi_0| \leq C\|v_0\|_{L^\infty}$.

Sketch of the proof

Linearizing around $u(t,x) = \underline{U}(x - \sigma t)$ yields the linear operator

$$L_a \stackrel{\mathrm{def}}{=} -a\partial_x + a\frac{\underline{U}''}{\underline{U}'}, \qquad a \approx f'(\underline{U}) - f'(u_\star) = \frac{g(\underline{U})}{\underline{U}'}$$

Using weights, we can focus on the behavior

- as $x \to \pm\infty$: $L_{\pm\infty} = \alpha_\pm \partial_x + \beta_\pm$, $\alpha_\pm \neq 0$, $\beta_\pm = \lim_{x\to\pm\infty} \left(a\frac{\underline{U}''}{\underline{U}'}\right)(x)$.

$$\beta_\pm + i\mathbb{R} \subset \mathrm{Sp}(L_{\pm\infty}) \quad ; \quad \beta_- \approx g'(u_l),\ \beta_+ \approx g'(u_r).$$

- at $x = x_\star$ such that $a(x_\star) = 0$: $L_\star = x(\alpha_\star \partial_x + \beta_\star)$, $\alpha_\star = a'(x_\star)$.

$$\{0, \alpha_\star, 2\alpha_\star, \cdots\} \subset \mathrm{Sp}(L_\star) \quad ; \quad \alpha_\star \approx -g'(u_\star).$$

Sketch of the proof

Linearizing around $u(t,x) = \underline{U}(x - \sigma t)$ yields the linear operator

$$L_a \stackrel{\text{def}}{=} -a\partial_x + a\frac{\underline{U}''}{\underline{U}'}, \qquad a \approx f'(\underline{U}) - f'(u_\star) = \frac{g(\underline{U})}{\underline{U}'}$$

Let $a \in X^1_\star = \{v \in BUC^1(\mathbb{R}) \;:\; a(x_\star) = 0,\; a'(x_\star) \neq 0\}$. For any $w \in L^1(\mathbb{R})$, there exists w_1, w_2 such that for any $\lambda \in \mathbb{C}$ such that

$$\Re\lambda > \inf\left(a' + a(w - \frac{\underline{U}''}{\underline{U}'})\right) \stackrel{\text{def}}{=}: \omega_{a,w},$$

one has $(\lambda - L_a) : X^2_\star \to X^1_\star$ is invertible and

$$\left\|(\lambda - L_a)^{-1} v\right\|_\star \leq \frac{1}{\Re(\lambda) - \omega_{a,w}} \left\|v\right\|_\star$$

with $\left\|v\right\|_\star \stackrel{\text{def}}{=} \max\left(\left\|w_1\left(v' - \frac{\underline{U}''}{\underline{U}'} v\right)\right\|_{L^\infty}, \left\|w_2\, v\right\|_{L^\infty}\right) \approx \left\|v\right\|_{W^{1,\infty}}.$

Sketch of the proof

Linearizing around $u(t,x) = \underline{U}(x - \sigma t)$ yields the linear operator

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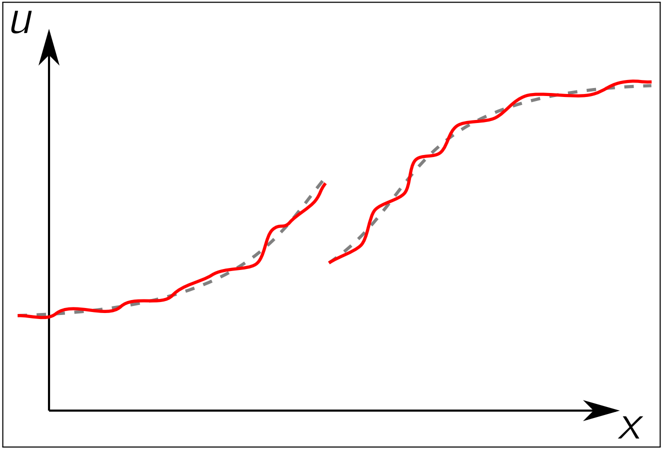
For any $\omega > \max(g'(u_l), g'(u_r), -g'(u_\star))$, there exists $\epsilon > 0$ such that if $\left\|f'(\underline{U}) - f'(u_\star) - a\right\|_{W^{1,\infty}} \leq \epsilon$, there exists $w \in L^1(\mathbb{R})$ such that $\omega_{a,w} = \omega$.

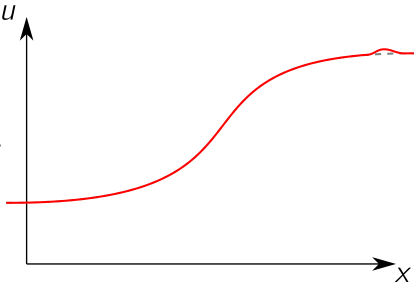
Asymptotic stability of composite waves

$$\partial_t u + \partial_x\big(f(u)\big) = g(u) \qquad (\star)$$

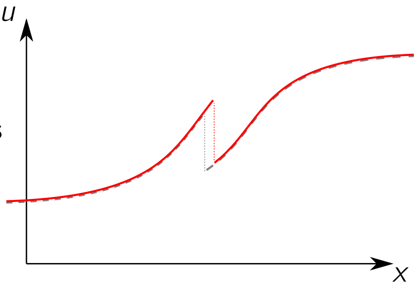


Instability mechanisms

- If $\lim_{x\to\infty} \underline{U} = u_\star$ with $g(u_\star) = 0$ and $g'(u_\star) > 0$, then $\underline{U}$ is linearly and nonlinearly unstable.



- If $\frac{[g(\underline{U})]}{[\underline{U}]} > 0$ at $x_\star \in \mathbb{R}$, then $\underline{U}$ is linearly and nonlinearly unstable.



Conclusion

We studied the stability of piecewise regular traveling waves solutions to

$$\partial_t u + \partial_x (f(u)) = g(u). \qquad (\star)$$

- Constants: asymptotically stable if $g'(u_\star) < 0$, linearly and nonlinearly unstable if $g'(u_\star) > 0$.
- (entropic) shocks: asymptotically stable if $g'(u_l) < 0, g'(u_r) < 0$, linearly and nonlinearly unstable if $g'(u_l) > 0$ or $g'(u_r) > 0$.
- Fronts: generically, only fronts connecting stable equilibria through a sonic point $(g(u_\star) = 0, g'(u_\star) > 0)$ are stable.
- Composites: the only stable traveling waves with an entropic discontinuity are piecewise-constant shocks or satisfy $\frac{[g(\underline{U})]}{[\underline{U}]} < 0$.

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Thank you for your attention !

A few words on the multidimensional setting

$$\partial_t u + \nabla \cdot \big(\mathbf{f}(u)\big) = g(u). \qquad (\star)$$

with $u : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ and $\mathbf{f} \in \mathcal{C}^2(\mathbb{R};\mathbb{R}^d)$, $g \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$.

- **Stability of constants**

  $$L_{\mathbf{a},b} = -\mathbf{a}\cdot\nabla + b, \qquad b = g'(u_\star),\ \mathbf{a} \approx \mathbf{f}'(u_\star + v).$$

  If $\Re\lambda > b$, then

  $$\big((\lambda - L_{\mathbf{a},b})^{-1} v\big)(x) = \int_{-\infty}^{0} e^{\int_s^0 (b(X(\sigma;x))-\lambda)\,\mathrm{d}\sigma}\, v(X(s;x))\,\mathrm{d}s$$

  where $X(0;x) = x$ and $\partial_s X(s;x) = \mathbf{a}(X(s;x))$.

  (to compare with $\big((\lambda - L_{a,b})^{-1} v\big)(x) = \int_{-\infty}^{x} e^{\int_y^x \frac{b-\lambda}{a(z)}\,\mathrm{d}z} \frac{v(y)}{a(y)}\,\mathrm{d}y$ when $d=1$)

- **Stability of plane Riemann shocks**

  The extension-gluing procedure is available, with the position of the discontinuity, $\{x = \psi(t,\mathbf{y}),\ \mathbf{y} \in \mathbb{R}^{d-1}\}$ being determined by

  $$\partial_t \psi(t,\cdot) + \left(\frac{\mathbf{f}_\perp(u_+) - \mathbf{f}_\perp(u_-)}{u_+ - u_-}\right)(t,\psi(t,\cdot)) \cdot \nabla_{\mathbf{y}} \psi(t,\cdot) = \left(\frac{\mathbf{f}_\parallel(u_+) - \mathbf{f}_\parallel(u_-)}{u_+ - u_-}\right)(t,\psi(t,\cdot)).$$