On the well-posedness of the Green-Naghdi system

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State of the art

Formal derivation

[Serre'53, Su&Gardner'69, Green&Naghdi'76, Miles&Salmon'85...]
[Bonneton&Lannes'09]

Rigorous justification

- Consistency [Lannes'13]
- Well-posedness of WW and GN
- Stability [Iguchi'09, Lannes'13]

Existence and uniqueness of a strong solution of the Cauchy problem, in the Sobolev setting and uniformly with respect to $\mu \ll 1$.

- Water-waves system [Alvarez-Samaniego&Lannes'08, Iguchi'09, Lannes'13]
- $d = 1$ Green-Naghdi system [Li'02, Israwi'11]
- modified $d = 2$ Green-Naghdi system [Israwi'10]
- original $d = 2$ Green-Naghdi system [Alvarez-Samaniego&Lannes'08] using a Nash-Moser iterative scheme.
  $\leadsto U|_{t=0} \in X^{d/2+40+s}$ yields $U \in C([0,T]; X^{d/2+6+s})$, $s > 0$.

Lack of energy estimates

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0 \\ \mathcal{T}_\mu[h]\partial_t \mathbf{u} + \mathcal{T}_\mu[h](\mathbf{u}\cdot\nabla)\mathbf{u} + h\nabla\zeta \\ \qquad + \mu\frac{2}{3}\nabla\big(h^3(\partial_1\mathbf{u})\cdot(\partial_2\mathbf{u}^\perp) + h^3(\nabla\cdot\mathbf{u})^2\big) = 0 \end{cases} \qquad \text{(GN)}$$

with

$$\mathcal{T}_\mu[h]\mathbf{u} = h\mathbf{u} - \mu\frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{u}).$$

- By blue terms, we control

$$\big|\mathbf{u}\big|_{X^n}^2 \overset{\text{def}}{=} \big|\mathbf{u}\big|_{H^n}^2 + \mu\big|\nabla\cdot\mathbf{u}\big|_{H^n}^2.$$

- green terms are controled by $\big|\mathbf{u}\big|_{X^n}^2$.
- red terms are controled by $\big|\mathbf{u}\big|_{H^n}^2 + \mu\big|\mathbf{u}\big|_{H^{n+1}}^2$.

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Hamiltonian formulation (1/2)

Zakharov/Craig-Sulem formulation of the water-waves system:

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H} \\ \delta_\psi \mathcal{H} \end{pmatrix}. \tag{WW}$$

with $\psi = \phi|_{z=1+\zeta}$ (ϕ is the velocity potential)

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + \frac{1}{2\mu}\int_{\mathbb{R}^d}\int_0^{1+\zeta} |\nabla^\mu \phi|^2 = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + \frac{1}{2\mu}\int_{\mathbb{R}^d} \psi G^\mu[h]\psi.$$

Hamiltonian formulation of the Green-Naghdi system [Camassa&Holm&Levermore'96,Matsuno'16]

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H}_{\rm GN} \\ \delta_\psi \mathcal{H}_{\rm GN} \end{pmatrix}. \tag{GN}$$

with

$$\mathcal{H}_{\rm GN} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 - \psi\nabla\cdot\big(h\mathcal{T}_\mu[h]^{-1}(h\nabla\psi)\big).$$

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Zakharov/Craig-Sulem formulation of the water-waves system:

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$$\mathcal{H}_{\rm GN} \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 - \psi\nabla\cdot\big(h\mathcal{T}_\mu[h]^{-1}(h\nabla\psi)\big).$$

Hamiltonian formulation (2/2)

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0 \\ \mathcal{T}_\mu[h]\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) + h\nabla\zeta = -\mu\frac{2}{3}\nabla\big(h^3(\partial_1\mathbf{u})\cdot(\partial_2\mathbf{u}^\perp) + h^3(\nabla\cdot\mathbf{u})^2\big) \end{cases}$$

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathcal{T}_\mu^{-1}(h\nabla\psi)) = 0 \\ \partial_t\psi + \zeta + \frac{1}{2}|\mathcal{T}_\mu^{-1}(h\nabla\psi)|^2 = \mu\big(\frac{\mathcal{T}_\mu^{-1}(h\nabla\psi)}{3h}\cdot\nabla(h^3\nabla\cdot\mathcal{T}_\mu^{-1}(h\nabla\psi)) \\ \qquad +\frac{1}{2}h^2(\nabla\cdot\mathcal{T}_\mu^{-1}(h\nabla\psi))^2\big) \end{cases}$$

Hamiltonian formulation (2/2)

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$$\Updownarrow \qquad h\mathbf{v} = \mathcal{T}_\mu[h]\mathbf{u} \stackrel{\text{def}}{=} h\mathbf{u} - \mu\frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{u})$$

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0 \\ \big(\partial_t + \mathbf{u}^\perp \mathrm{curl}\big)\mathbf{v} + \nabla\zeta + \frac{1}{2}\nabla(|\mathbf{u}|^2) = \mu\nabla\big(\frac{\mathbf{u}}{3h}\cdot\nabla(h^3\nabla\cdot\mathbf{u}) + \frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2\big) \end{cases}$$

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$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0 \\ \partial_t\psi + \zeta + \frac{1}{2}|\mathbf{u}|^2 = \mu\big(\frac{\mathbf{u}}{3h}\cdot\nabla(h^3\nabla\cdot\mathbf{u}) + \frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2\big) \end{cases}$$

Main results

Well-posedness

Let $N \geq 4$ and $\zeta_0 \in H^N$ and $\mathbf{u}_0 \in X^N$ be such that

$$h_0 = 1 + \zeta_0 > 0 \quad ; \quad \mathrm{curl}\left(h_0^{-1}\mathcal{T}_\mu[h_0]\mathbf{u}_0\right) = 0. \tag{H}$$

Then there exist $T > 0$ and a unique $(\zeta, \mathbf{u}) \in C([0,T]; H^N \times X^N)$ satisfying (H), strong solution to (GN) and $(\zeta, \mathbf{u})|_{t=0} = (\zeta_0, \mathbf{u}_0)$. Moreover, one can restrict $T^{-1} \lesssim \left|\zeta_0\right|_{H^4} + \left|\mathbf{u}_0\right|_{X^4}$ and set C such that

$$\forall t \in [0,T], \qquad \left|\zeta\right|_{H^N}(t) + \left|\mathbf{u}_0\right|_{X^N}(t) \leq C\left|\zeta_0\right|_{H^N} + C\left|\mathbf{u}_0\right|_{X^N}.$$

+ continuity of the flow map.

Rigorous justification is a consequence:

For any initial data satisfying the assumptions of [Lannes '13], there exist

- a unique solution to the water-waves system,
- a unique solution to the Green-Naghdi system,

and the difference is of order $\mathcal{O}(\mu^2)$ in $C^0([0,T]; H^N \times X^N)$.

Preparation

Recall $\big|\mathbf{u}\big|_{X^n}^2 \stackrel{\text{def}}{=} \big|\mathbf{u}\big|_{H^n}^2 + \mu\big|\nabla\cdot\mathbf{u}\big|_{H^n}^2$ and $\mathcal{T}_\mu[h]\mathbf{u} = h\mathbf{u} - \mu\frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{u})$.

Invertibility

Let $h \in L^\infty$ be such that $h \geq h_0 > 0$. Then $\mathcal{T}_\mu[h] : X^0 \to (X^0)'$ is linear, bounded, symmetric, coercive. It is a topological isomorphism and

$$\forall \mathbf{v} \in (X^0)', \quad \big|\mathcal{T}_\mu[h]^{-1}\mathbf{v}\big|_{X^0} \leq C(h_0^{-1})\big|\mathbf{v}\big|_{(X^0)'}.$$

Differentiability

Let $h \in H^3$ such that $h \geq h_0 > 0$, and $\mathbf{v} \in Y^n$. Then $\mathcal{T}_\mu[h]^{-1}\mathbf{v} \in X^n$ and

$$\big|\mathcal{T}_\mu[h]^{-1}\mathbf{v}\big|_{X^n} \leq C(h_0^{-1}, \big|h\big|_{H^{3\vee n}})\big|\mathbf{v}\big|_{Y^n}.$$

Linearization

Let $|\alpha| \geq 1$ and $\zeta \in H^{3\vee|\alpha|-1}$ such that $1+\zeta \geq h_0 > 0$, and $\mathbf{v} \in Y^{3\vee|\alpha|-1}$. Then one has

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Linearization

Let $|\alpha|\geq 1$ and $\zeta\in H^{3\vee|\alpha|-1}$ such that $1+\zeta\geq h_0>0$, and $\mathbf{v}\in Y^{3\vee|\alpha|-1}$. Then one has

$$\Big|\partial^\alpha\big(\mathcal{T}_\mu[h]^{-1}\mathbf{v}\big) - \mathcal{T}_\mu[h]^{-1}\partial^\alpha\mathbf{v} + \mathcal{T}_\mu[h]^{-1}\big\{\mathrm{d}_h\mathcal{T}_\mu[h](\partial^\alpha\zeta,\mathcal{T}_\mu[h]^{-1}\mathbf{v})\big\}\Big|_{X^0} \leq C(h_0^{-1},\big|\zeta\big|_{H^{3\vee|\alpha|-1}},\big|\mathbf{v}\big|_{Y^{3\vee|\alpha|-1}})$$

with $\mathrm{d}_h\mathcal{T}_\mu[h]f \stackrel{\text{def}}{=} f\mathbf{u} - \mu\nabla(h^2 f\nabla\cdot\mathbf{u})$.

Quasi-linearization

Differentiate the GN system

$$\begin{cases} \partial_t \zeta + \nabla\cdot(h\mathbf{u}) = 0 \\ \partial_t \psi + \zeta + \frac{1}{2}|\mathbf{u}|^2 = \mu\big(\frac{\mathbf{u}}{3h}\cdot\nabla(h^3\nabla\cdot\mathbf{u}) + \frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2\big) \end{cases}$$

where $\mathbf{u} = \mathcal{T}_\mu[h]^{-1}(h\nabla\psi)$, and withdraw all zero-th order terms.

At first order in terms of μ, $\mathbf{u} \approx \nabla\psi$ and we find

$$\begin{cases} \partial_t\partial^\alpha\zeta + \nabla\cdot(\mathbf{u}\partial^\alpha\zeta) + \nabla\cdot(h\partial^\alpha\nabla\psi) = r^1_{(\alpha)} \\ \partial_t\partial^\alpha\psi + \partial^\alpha\zeta + \mathbf{u}\cdot\partial^\alpha\nabla\psi = r^2_{(\alpha)} \end{cases}$$

And zero-th order means $r^1_{(\alpha)} \in L^2$ and $\nabla r^2_{(\alpha)} \in L^2$.

Energy estimates are obtained by testing against $\big(\partial^\alpha\zeta, \nabla\cdot(h\partial^\alpha\nabla\psi)\big)^\top$.

$$\rightsquigarrow \text{control of } \big|\partial^\alpha\zeta\big|^2_{L^2} + \big|\partial^\alpha\nabla\psi\big|^2_{L^2}.$$

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Quasi-linearization of the GN system

Let α be a non-zero multi-index and $\zeta \in H^{4\vee|\alpha|}$ be such that $h \geq h_0 > 0$, and $\nabla\psi \in Y^{4\vee|\alpha|}$, satisfying (GN). Denote

$$\zeta_{(\alpha)} \stackrel{\text{def}}{=} \partial^\alpha \zeta \quad ; \quad \psi_{(\alpha)} \stackrel{\text{def}}{=} \partial^\alpha \psi + \mu\, w \partial^\alpha \zeta \quad \text{where} \quad w \stackrel{\text{def}}{=} h\nabla\cdot\mathbf{u}.$$

Then $\zeta_{(\alpha)}, \psi_{(\alpha)}$ satisfy

$$\begin{cases} \partial_t \zeta_{(\alpha)} + \nabla\cdot(\mathbf{u}\zeta_{(\alpha)}) + \nabla\cdot(h\mathbf{u}_{(\alpha)}) = r^1_{(\alpha)} \\ \partial_t \psi_{(\alpha)} + \zeta_{(\alpha)} + \mathbf{u}\cdot\nabla\psi_{(\alpha)} = r^2_{(\alpha)} \end{cases} \tag{Q}$$

where we denote

$$\mathbf{u} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi\} \quad \text{and} \quad \mathbf{u}_{(\alpha)} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\},$$

and $r^1_{(\alpha)}, r^2_{(\alpha)}$ satisfy the estimates

$$\big|r^1_{(\alpha)}\big|_{L^2} + \big|\nabla r^2_{(\alpha)}\big|_{Y^0} \leq \mathbf{R}\ \left(\big|\zeta\big|_{H^{|\alpha|}} + \big|\nabla\psi\big|_{Y^{|\alpha|}}\right)$$

with $\mathbf{R} = C(\mu, h_0^{-1}, \big|\nabla\zeta\big|_{H^3}, \big|\nabla\psi\big|_{Y^4})$.

Quasi-linearization of the GN system

Let α be a non-zero multi-index and $\zeta \in H^{4\vee|\alpha|}$ be such that $h \geq h_0 > 0$, and $\nabla\psi \in Y^{4\vee|\alpha|}$, satisfying (GN). Denote

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Then $\zeta_{(\alpha)}, \psi_{(\alpha)}$ satisfy

$$\begin{cases} \partial_t \zeta_{(\alpha)} + \nabla\cdot(\mathbf{u}\zeta_{(\alpha)}) + \nabla\cdot(h\mathbf{u}_{(\alpha)}) = r^1_{(\alpha)} \\ \partial_t \psi_{(\alpha)} + \zeta_{(\alpha)} + \mathbf{u}\cdot\nabla\psi_{(\alpha)} = r^2_{(\alpha)} \end{cases} \tag{Q}$$

Remark: One recovers the quasilinear structure of the water-wave system in [Lannes '13], up to two differences.

- $\mathbf{u}$ replaces $U = (\nabla_X \phi)|_{z=\zeta}$ and μw replaces $(\partial_z \phi)|_{z=\zeta}$;
- There is no Rayleigh-Taylor criterion, $(-\partial_z P)|_{z=\zeta} > 0$, or

$$\mathfrak{a} \stackrel{\text{def}}{=} 1 - \mu\partial_t w - \mu \mathbf{u}\cdot\nabla w > 0.$$

since $\mu(\partial_t w - \mathbf{u}\cdot\nabla w)\zeta_{(\alpha)} \in L^2 \Longrightarrow \sqrt{\mu}\nabla(\partial_t w - \mathbf{u}\cdot\nabla w)\zeta_{(\alpha)} \in Y^0$.

A priori energy estimates

Let $\zeta_{(\alpha)}, \psi_{(\alpha)}$ satisfy

$$\begin{cases} \partial_t \zeta_{(\alpha)} + \nabla\cdot(\mathbf{u}\zeta_{(\alpha)}) + \nabla\cdot(h\mathbf{u}_{(\alpha)}) = r^1_{(\alpha)} \\ \partial_t \psi_{(\alpha)} + \zeta_{(\alpha)} + \mathbf{u}\cdot\nabla\psi_{(\alpha)} = r^2_{(\alpha)} \end{cases} \qquad \text{(Q)}$$

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$$\mathbf{u} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi\} \quad \text{and} \quad \mathbf{u}_{(\alpha)} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\},$$

Test against $\big(\zeta_{(\alpha)}, \nabla\cdot(h\mathbf{u}_{(\alpha)})\big)^\top$.

- Remainder terms are zero-th order. Green terms compensate.
- Blue term is estimated by skew-symmetry:
  $\big(\nabla\cdot(\mathbf{u}\zeta_{(\alpha)}), \zeta_{(\alpha)}\big) \leq \frac{1}{2}\big|\nabla\mathbf{u}\big|_{L^\infty}\big|\zeta_{(\alpha)}\big|^2_{L^2} \lesssim \big|\mathbf{u}\big|_{H^3}\big|\zeta_{(\alpha)}\big|^2_{L^2}$.
- Red term satisfies a similar estimate

$$\big(\mathbf{u}\cdot\nabla\psi_{(\alpha)}, \nabla\cdot(h\mathcal{T}_\mu[h]^{-1}h\nabla\psi_{(\alpha)})\big)_{L^2} \leq C\big(h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\mathbf{u}\big|_{H^3}\big)\big|\nabla\psi_{(\alpha)}\big|^2_{Y^0}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \leq \mathbf{C}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) + \mathbf{R}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)})^{1/2}$$

with $\mathbf{C}, \mathbf{R} = C(\mu, h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\nabla\psi\big|_{Y^4})$, and

$\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \stackrel{\text{def}}{=} \big(\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \big(\nabla\psi_{(\alpha)}, \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\}\big)_{L^2} \approx \big|\zeta_{(\alpha)}\big|^2_{L^2} + \big|\nabla\psi_{(\alpha)}\big|^2_{Y^0}$.

A priori energy estimates

Let $\zeta_{(\alpha)}, \psi_{(\alpha)}$ satisfy

$$\begin{cases} \partial_t \zeta_{(\alpha)} + \nabla\cdot(\mathbf{u}\zeta_{(\alpha)}) + \nabla\cdot(h\mathbf{u}_{(\alpha)}) = r^1_{(\alpha)} \\ \partial_t \psi_{(\alpha)} + \zeta_{(\alpha)} + \mathbf{u}\cdot\nabla\psi_{(\alpha)} = r^2_{(\alpha)} \end{cases} \qquad \text{(Q)}$$

where we denote

$$\mathbf{u} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi\} \quad \text{and} \quad \mathbf{u}_{(\alpha)} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\},$$

Test against $\big(\zeta_{(\alpha)}, \nabla\cdot(h\mathbf{u}_{(\alpha)})\big)^\top$.

- Remainder terms are zero-th order. Green terms compensate.
- Blue term is estimated by skew-symmetry:
  $\big(\nabla\cdot(\mathbf{u}\zeta_{(\alpha)}), \zeta_{(\alpha)}\big) \leq \frac{1}{2}\big|\nabla\mathbf{u}\big|_{L^\infty}\big|\zeta_{(\alpha)}\big|_{L^2}^2 \lesssim \big|\mathbf{u}\big|_{H^3}\big|\zeta_{(\alpha)}\big|_{L^2}^2.$
- Red term satisfies a similar estimate

$$\big(\mathbf{u}\cdot\nabla\psi_{(\alpha)}, \nabla\cdot(h\mathcal{T}_\mu[h]^{-1}h\nabla\psi_{(\alpha)})\big)_{L^2} \leq C(h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\mathbf{u}\big|_{H^3})\big|\nabla\psi_{(\alpha)}\big|_{Y^0}^2.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \leq \mathbf{C}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) + \mathbf{R}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)})^{1/2}$$

with $\mathbf{C}, \mathbf{R} = C(\mu, h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\nabla\psi\big|_{Y^4})$, and

$$\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \stackrel{\text{def}}{=} \big(\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \big(\nabla\psi_{(\alpha)}, \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\}\big)_{L^2} \approx \big|\zeta_{(\alpha)}\big|_{L^2}^2 + \big|\nabla\psi_{(\alpha)}\big|_{Y^0}^2.$$

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$$\mathbf{u} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi\} \quad \text{and} \quad \mathbf{u}_{(\alpha)} \stackrel{\text{def}}{=} \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\},$$

Test against $\big(\zeta_{(\alpha)}, \nabla \cdot (h\mathbf{u}_{(\alpha)})\big)^\top$.

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- Red term satisfies a similar estimate

$$\big(\mathbf{u} \cdot \nabla\psi_{(\alpha)}, \nabla \cdot (h\mathcal{T}_\mu[h]^{-1}h\nabla\psi_{(\alpha)})\big)_{L^2} \leq C(h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\mathbf{u}\big|_{H^3})\big|\nabla\psi_{(\alpha)}\big|^2_{Y^0}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \leq \mathbf{C}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) + \mathbf{R}\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)})^{1/2}$$

with $\mathbf{C}, \mathbf{R} = C(\mu, h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\nabla\psi\big|_{Y^4})$, and

$\mathcal{E}(\zeta_{(\alpha)}, \psi_{(\alpha)}) \stackrel{\text{def}}{=} \big(\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2} + \big(\nabla\psi_{(\alpha)}, \mathcal{T}_\mu[h]^{-1}\{h\nabla\psi_{(\alpha)}\}\big)_{L^2} \approx \big|\zeta_{(\alpha)}\big|^2_{L^2} + \big|\nabla\psi_{(\alpha)}\big|^2_{Y^0}.$

Well-posedness

We have the uniform *a priori* energy estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta_{(\alpha)},\psi_{(\alpha)}) \leq \mathbf{C}\mathcal{E}(\zeta_{(\alpha)},\psi_{(\alpha)}) + \mathbf{R}\mathcal{E}(\zeta_{(\alpha)},\psi_{(\alpha)})^{1/2}$$

with $\mathbf{C},\mathbf{R} = C(\mu, h_0^{-1}, \big|\zeta\big|_{H^4}, \big|\nabla\psi\big|_{Y^4})$; and, for any $N \geq 4$

$$\sum_{|\alpha|=0}^{N} \mathcal{E}(\zeta_{(\alpha)},\psi_{(\alpha)}) \approx \big|\zeta\big|_{H^N}^2 + \big|\nabla\psi\big|_{Y^N}^2.$$

Well-posedness

Let $N \geq 4$ and $\zeta_0 \in H^N$ be such that $h_0 = 1 + \zeta_0 > 0$ and $\nabla\psi_0 \in Y^N$.
Then there exist $T > 0$ and a unique $(\zeta, \nabla\psi) \in C([0,T]; H^N \times Y^N)$, strong solution to (GN) and $(\zeta,\psi)|_{t=0} = (\zeta_0,\psi_0)$.
Moreover, one can restrict $T^{-1} \lesssim \big|\zeta_0\big|_{H^4} + \big|\nabla\psi_0\big|_{Y^4}$ and set C such that

$$\forall t \in [0,T], \qquad \big|\zeta\big|_{H^N}(t) + \big|\nabla\psi\big|_{Y^N}(t) \leq C\big|\zeta_0\big|_{Y^N} + C\big|\nabla\psi_0\big|_{Y^N}.$$

$+$ continuity of the flow map.

Well-posedness with non-trivial bathymetry

Let $N \geq 4$ and $\zeta_0 \in H^N$, $b \in \dot{H}^{N+2}$ and $\mathbf{u}_0 \in X^N$ be such that

$$h_0 = 1 + \zeta_0 > 0 \quad ; \quad \operatorname{curl}\left(h_0^{-1}\mathcal{T}_\mu[h_0, b]\mathbf{u}_0\right) = 0. \tag{H}$$

Then there exist $T > 0$ and a unique $(\zeta, \mathbf{u}) \in C([0,T]; H^N \times X^N)$ satisfying (H), strong solution to (GN) and $(\zeta, \mathbf{u})|_{t=0} = (\zeta_0, \mathbf{u}_0)$. Moreover, one can restrict $T^{-1} \lesssim \big|\zeta_0\big|_{H^4} + \big|\mathbf{u}_0\big|_{X^4} + \big|\nabla b\big|_{H^{N+1}}$ and set C such that

$$\forall t \in [0,T], \qquad \big|\zeta\big|_{H^N}(t) + \big|\mathbf{u}\big|_{X^N}(t) \leq C\big|\zeta_0\big|_{H^N} + C\big|\mathbf{u}_0\big|_{X^N}.$$

$+$ continuity of the flow map.

Problem: Large time behavior of small data: we would like

$$\big|\zeta_0\big|_{H^4} + \big|\mathbf{u}_0\big|_{X^4} = \mathcal{O}(\epsilon) \quad \text{and} \quad T^{-1} = \mathcal{O}(\epsilon).$$

Formally, we obtain the *Great Lake equations* [Camassa&Holm&Levermore '96],

$$\nabla \cdot \big((1+b)\mathbf{u}\big) = 0, \quad + \quad \text{evolution equation on } \mathbf{u}\text{, involving a “pressure”}$$

whose unique solution [Oliver et al. '97] is zero(!) $\rightsquigarrow$ only acoustic component.

Saint-Venant system

$$\begin{cases} \partial_t \zeta + \frac{1}{\epsilon}\nabla\cdot\big((1+\epsilon\zeta-b)\mathbf{u}\big) = 0 \\ \partial_t \mathbf{u} + \frac{1}{\epsilon}\nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0 \end{cases} \tag{SV}$$

Large time well-posedness [Bresch&Métivier '10]

- Use *time derivatives*: $\zeta_N \overset{\text{def}}{=} (\epsilon\partial_t)^N\zeta, \mathbf{u}_N \overset{\text{def}}{=} (\epsilon\partial_t)^N\mathbf{u}$ satisfies

$$\begin{cases} \partial_t \zeta_N + \frac{1}{\epsilon}\nabla\cdot(h\mathbf{u}_N) + \mathbf{u}\cdot\nabla\zeta_N = \mathcal{O}(1) \\ \partial_t \mathbf{u}_N + \frac{1}{\epsilon}\nabla\zeta_N + (\mathbf{u}\cdot\nabla)\mathbf{u}_N = \mathcal{O}(1) \end{cases} \tag{Q}$$

- Use the equation to deduce space-control from time-control:

$$\big|\zeta\big|_{H^N}^2 + \big|\mathbf{u}\big|_{H^N}^2 \lesssim \sum_{n=0}^{N}\big|(\epsilon\partial_t)^n\zeta\big|_{L^2}^2 + \big|(\epsilon\partial_t)^n\mathbf{u}\big|_{L^2}^2.$$

Note: we could use directly a differential operator:

$$\zeta_N \overset{\text{def}}{=} (\nabla\cdot(1-b)\nabla)^N\zeta, \qquad \mathbf{u}_N \overset{\text{def}}{=} (\nabla(1-b)\nabla\cdot)^N\mathbf{u}$$

Water-waves and Green-Naghdi systems

Water-waves system

The technique based on time derivatives works up to technical difficulties [Mésognon-Gireau '16].

One could "in principle" [Mélinand&Mésognon-Gireau] use the Dirichlet-Neumann operator $\frac{1}{\mu}G^{\mu}[0,b]$ as "space derivatives"

$$\frac{1}{\mu}G^{\mu}[0,0] = \frac{1}{\sqrt{\mu}}|D|\tanh(\sqrt{\mu}|D|)$$

is of order one.

Green-Naghdi system

No luck with the corresponding operator, $\nabla\cdot\mathcal{T}_{\mu}[1-b,b]^{-1}\nabla$, because

$$\nabla\cdot\mathcal{T}_{\mu}[1,0]^{-1}\nabla = \nabla\cdot(1-\mu\frac{1}{3}\nabla\nabla\cdot)^{-1}\nabla$$

is of order zero.

Water-waves and Green-Naghdi systems

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is of order zero.

Possible ideas:

- Change the model by adding (small) higher order terms on ζ [Mésognon-Gireau'16]
  $\rightsquigarrow$ but it breaks the structure
- Lower the order of the operators through Fourier multipliers [D&Israwi&Talhouk'16]
  $\rightsquigarrow$ unsuitable for numerical simulations
- Work with well-prepared initial data
  $\rightsquigarrow$ inconsistent with non-dimensionalization

Thank you for your attention