Kelvin-Helmholtz instabilities in shallow water

Propagation of large amplitude, long wavelength, internal waves

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Internal gravity waves

Stratification, due to variation of salinity and temperature.

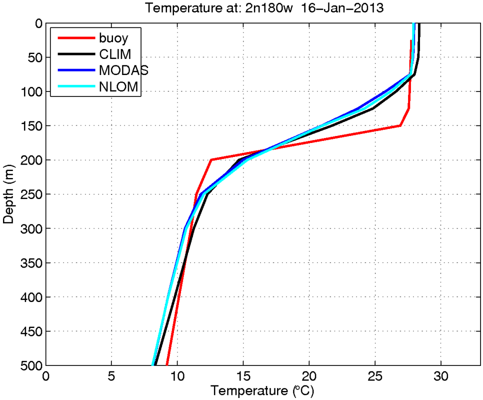

Figure : Temperature vs depth [1]

1. Credits : Naval Research Laboratory http://www7320.nrlssc.navy.mil/global_nlom/

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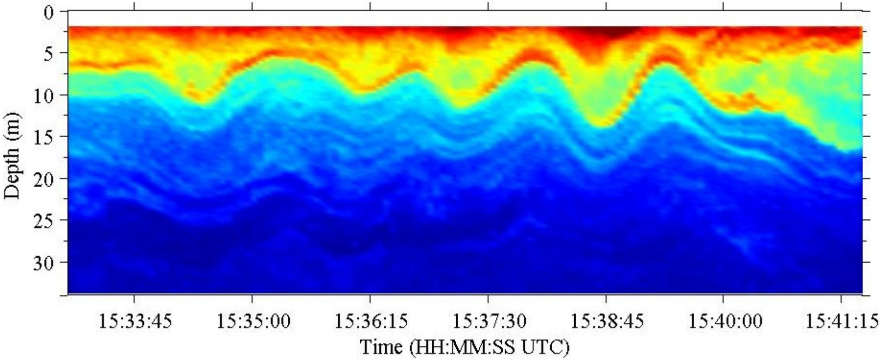

Figure : St. Lawrence Estuary [1]

1. Credits : St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) http://myweb.dal.ca/kelley/SLEIWEX/index.php

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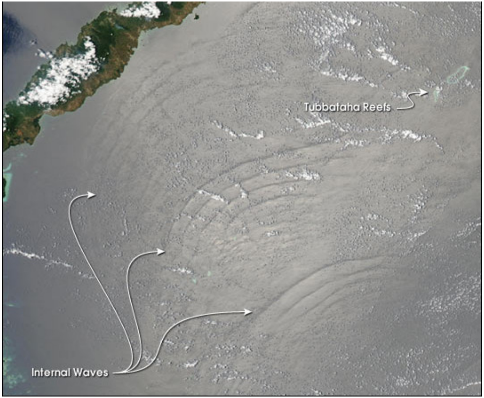
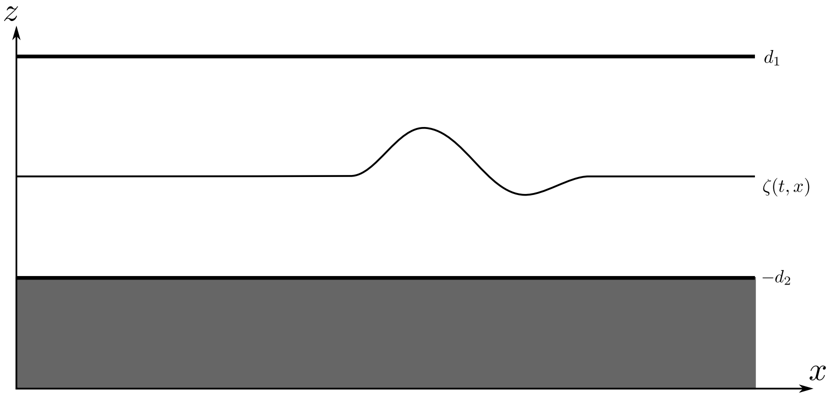


Figure : Sulu Sea. April 8, 2003 [1]

1. Credits : NASA's Earth Observatory (Picture of the Day July 1, 2003) http://earthobservatory.nasa.gov/IOTD/view.php?id=3586

The full Euler system



- Horizontal dimension $d = 1$, flat bottom, rigid lid.
- Irrotational, incompressible, inviscid, immiscible fluids.
- Fluids at rest at infinity, (small) surface tension.

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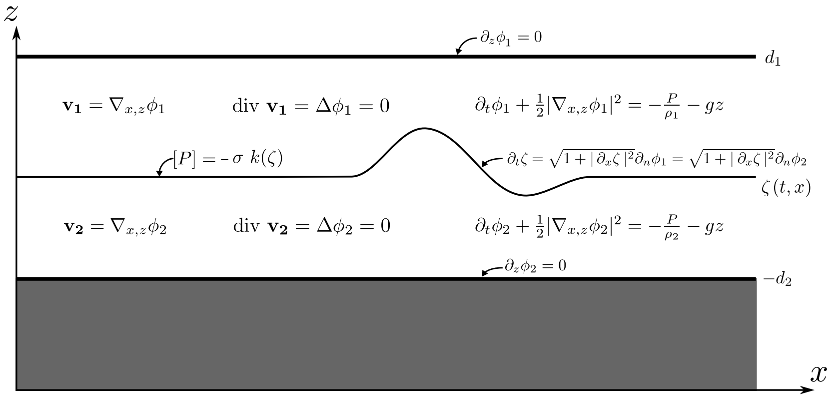

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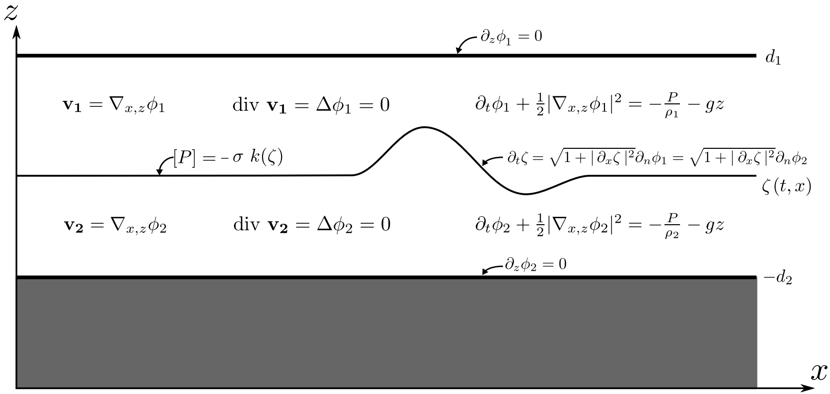

The system can be rewritten as two coupled evolution equations in

$$\zeta \quad \text{and} \quad \psi \equiv \phi_{1|\text{interface}}.$$

using Dirichlet-Neumann operators.

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Full Euler system (Zakharov's formulation)

$$\begin{cases} \partial_t \zeta = -\partial_x \dfrac{\delta \mathcal{H}}{\delta v}, \\ \partial_t v = -\partial_x \dfrac{\delta \mathcal{H}}{\delta \zeta}, \end{cases} \qquad v = \partial_x \left(\rho_2 \phi_{2|\text{interface}} - \rho_1 \phi_{1|\text{interface}} \right)$$

with $\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} g(\rho_2 - \rho_1)\zeta^2 dx + \frac{\rho_2}{2}\int_{\mathbb{R}}\int_{-d_2}^{\zeta} |\nabla \phi_2|^2 dzdx + \frac{\rho_1}{2}\int_{\mathbb{R}}\int_{\zeta}^{d_1} |\nabla \phi_1|^2 dzdx$
$+\sigma \int_{\mathbb{R}} \left(\sqrt{1 + |\partial_x \zeta|^2} - 1 \right).$

This system is ill-posed without surface tension.

[Ebin '88 ; Iguchi,Tanaka&Tani '97 ; Kamotski&Lebeau '05]

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Kelvin-Helmholtz instabilities

Linearize the system around $\zeta = 0, v = v_0$, constant.

Linearized system [Lannes&Ming]

$$\begin{cases} \partial_t \zeta + c_{v_0}(D)\partial_x \zeta + b(D)\partial_x v = 0, \\ \partial_t v + a_{v_0}(D)\partial_x \zeta + c_{v_0}(D)\partial_x v = 0, \end{cases} \quad \text{(L)}$$

with $b > 0$ and

$$a_{v_0}(D) = g(\rho_2 - \rho_1) - |v_0|^2 \frac{\rho_1 \rho_2}{\rho_2 \tanh(d_1|D|) + \rho_1 \tanh(d_2|D|)} |D| - \sigma \partial_x^2$$

There are growing modes, $e^{i(kx - \omega(k)t)}$ with $\Im(\omega(k)) \neq 0$ iff $a_{v_0}(k) < 0$, i.e.

$$|v_0|^2 < \left(\frac{\tanh(d_1|k|)}{\rho_1 |k|} + \frac{\tanh(d_2|k|)}{\rho_2 |k|} \right) \left(g(\rho_2 - \rho_1) + \sigma |k|^2 \right).$$

- Modes are stable for $|k|$ small iff $|v_0|^2 < \frac{\rho_2 d_1 + \rho_1 d_2}{\rho_1 \rho_2} g(\rho_2 - \rho_1)$.
- There are always unstable modes if $\sigma = 0$ and $\rho_1, |v_0| \neq 0$.
- All modes are stable if $\sigma \neq 0$ and $\rho_1 \rho_2 |v_0|^2$ is sufficiently small.

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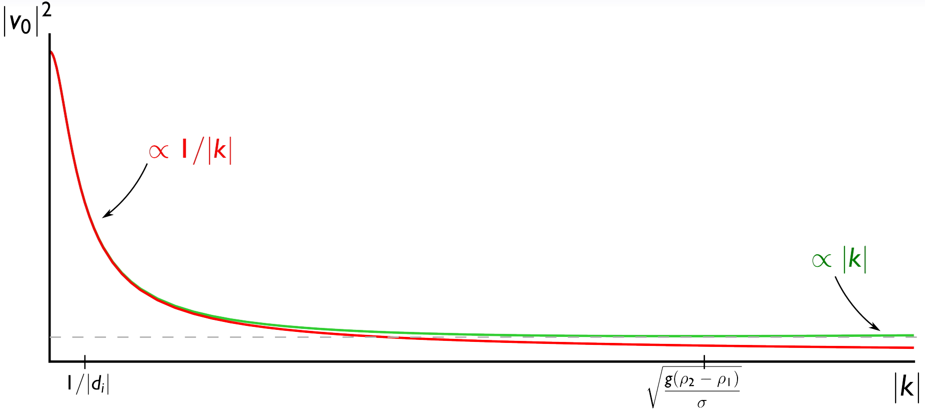

Figure : Stability domains, with surface tension (green) and without (red)

When surface tension is present, all the modes are stable provided

$$|v_0|^2 < |v_{\min}|^2 \approx 2\frac{\rho_1+\rho_2}{\rho_1\rho_2}\sqrt{\sigma g(\rho_2-\rho_1)}.$$

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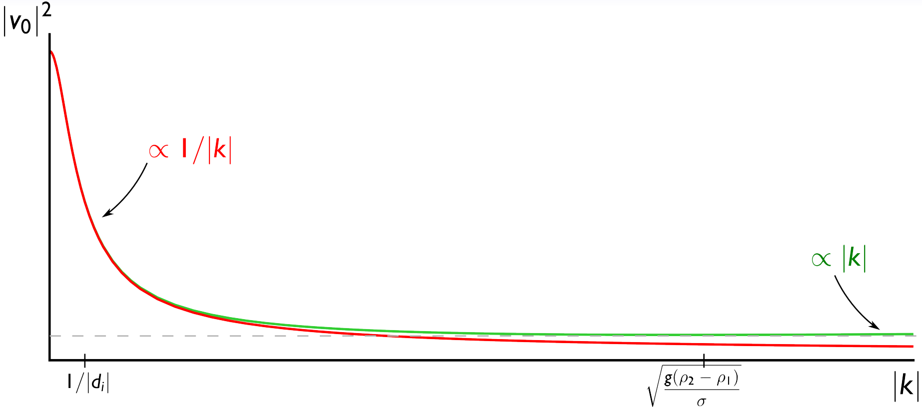

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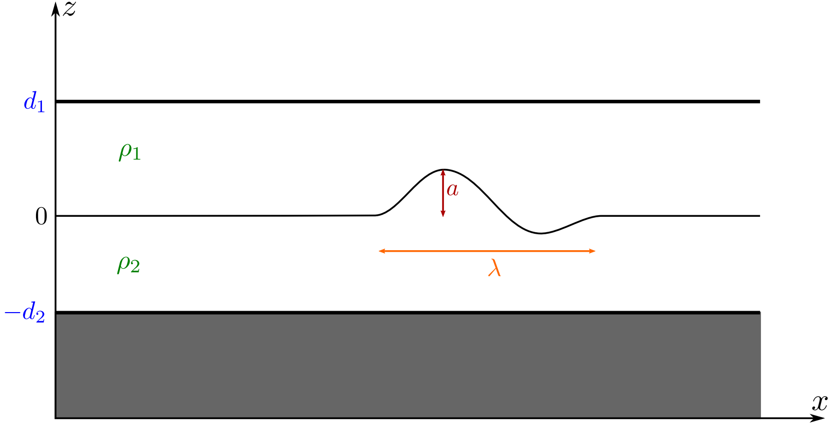
Nonlinear generalization of this criterion : [Lannes '13]

Asymptotic models : construction

Asymptotic models may be constructed from asymptotic expansions of the Dirichlet-Neumann operators, w. r. t. given dimensionless parameters.



$$\epsilon \overset{\text{def}}{=} \frac{a}{d_1}, \quad \boxed{\mu \overset{\text{def}}{=} \frac{{d_1}^2}{\lambda^2}}, \quad \gamma \overset{\text{def}}{=} \frac{\rho_1}{\rho_2}, \quad \delta \overset{\text{def}}{=} \frac{d_1}{d_2}, \quad \text{Bo}^{-1} \overset{\text{def}}{=} \frac{\sigma}{g(\rho_2 - \rho_1)\lambda^2}.$$

Asymptotic models : examples

Precision $\mathcal{O}(\mu)$

Shallow water (a.k.a Saint-Venant) system (+ surface tension)

$$\begin{cases} \partial_t \zeta \ + \ \partial_x w \ = \ 0, \\ \partial_t \Big(\frac{h_1+\gamma h_2}{h_1 h_2} w \Big) + (\gamma+\delta)\partial_x \zeta + \frac{\epsilon}{2}\partial_x \Big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1 h_2)^2} |w|^2 \Big) = \frac{\gamma+\delta}{\mathrm{Bo}} \partial_x^2 \Big(\frac{\partial_x \zeta}{1+\mu\epsilon^2 |\partial_x \zeta|^2} \Big), \end{cases}$$

with $h_1 = 1 - \epsilon\zeta$ and $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$\frac{h_1+\gamma h_2}{h_1 h_2} w \stackrel{\text{def}}{=} \frac{1}{h_2(t,x)} \int_{-\frac{1}{\delta}}^{\epsilon\zeta(t,x)} \partial_x \phi_2(t,x,z)\, \mathrm{d}z - \frac{\gamma}{h_1(t,x)} \int_{\epsilon\zeta(t,x)}^{1} \partial_x \phi_1(t,x,z)\, \mathrm{d}z.$$

Asymptotic models : examples

Precision $\mathcal{O}(\mu^2)$

Green-Naghdi system [Miyata'85 ; Mal'tseva'89 ; Choi&Camassa'99]

$$\begin{cases} \partial_t\zeta \ + \ \partial_x w \ = \ 0, \\ \partial_t\Big(\frac{h_1+\gamma h_2}{h_1 h_2}w + \mu\mathcal{Q}[\zeta]w\Big) + (\gamma+\delta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\Big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1h_2)^2}|w|^2\Big) = \mu\epsilon\partial_x\big(\mathcal{R}[\zeta,w]\big) \\ \qquad\qquad + \frac{\gamma+\delta}{\mathrm{Bo}}\partial_x^2\Big(\frac{\partial_x\zeta}{1+\mu\epsilon^2|\partial_x\zeta|^2}\Big), \end{cases}$$

with $h_1 = 1-\epsilon\zeta$ and $h_2 = \frac{1}{\delta}+\epsilon\zeta$ and

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$$\mathcal{Q}[\zeta]w \overset{\text{def}}{=} -\frac{1}{3}\Big(h_2^{-1}\partial_x\big({h_2}^3\partial_x(h_2^{-1}w)\big) \ + \ \gamma h_1^{-1}\partial_x\big({h_1}^3\partial_x(h_1^{-1}w)\big)\Big),$$

$$\mathcal{R}[\zeta,w] \overset{\text{def}}{=} \frac{1}{2}\Big(\big(h_2\partial_x(h_2^{-1}w)\big)^2 \ - \ \gamma\big(h_1\partial_x(h_1^{-1}w)\big)^2\Big) + \frac{1}{3}w\ \Big(h_2^{-2}\partial_x\big({h_2}^3\partial_x(h_2^{-1}w)\big) \ - \ \gamma h_1^{-2}\partial_x\big({h_1}^3\partial_x(h_1^{-1}w)\big)\Big).$$

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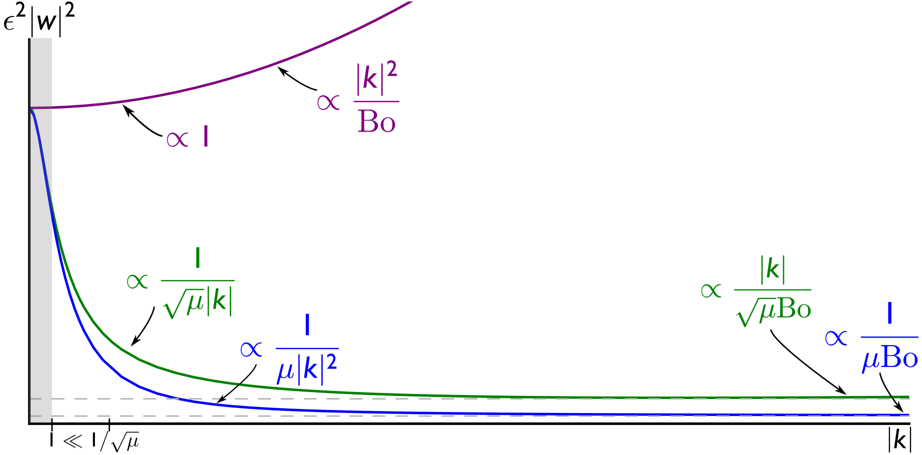

Figure : Stability domains for full Euler (green), Saint-Venant (purple) and Green-Naghdi (blue)

$$\epsilon^2|w^{FE}_{\min}|^2 \propto \frac{1}{\sqrt{\mu}\,\mathrm{Bo}} \qquad \text{vs} \qquad \epsilon^2|w^{GN}_{\min}|^2 \propto \frac{1}{\mu\,\mathrm{Bo}}.$$

Shallow water models and KH instabilities

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A : The classical GN model follows the same behavior [Choi&Camassa '99]
However, whereas Saint-Venant underestimates, Green-Naghdi overestimates KH instabilities [Jo&Choi '02 ; Lannes&Ming]

Qn : Can we do better ?
A : [Nguyen&Dias '08 ; Choi,Barros&Jo '09 ; Cotter,Holm&Percival '10 ; Boonkasame&Milewski'14 ; Lannes&Ming]

Strategy :

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Construction of our model

The original GN system

$$\begin{cases} \partial_t \zeta \ + \ \partial_x w \ = \ 0, \\ \partial_t \Big(\frac{h_1+\gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}[\zeta] w \Big) + (\gamma+\delta)\partial_x \zeta + \frac{\epsilon}{2}\partial_x \Big(\frac{{h_1}^2 - \gamma {h_2}^2}{(h_1 h_2)^2} |w|^2 \Big) = \mu\epsilon \partial_x \big(\mathcal{R}[\zeta, w] \big) \\ \qquad\qquad + \frac{\gamma+\delta}{\mathrm{Bo}} \partial_x^2 \Big(\frac{\partial_x \zeta}{1+\mu\epsilon^2 |\partial_x \zeta|^2} \Big), \end{cases}$$

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Construction of our model

The GN system with improved (and nonlocal !) frequency dispersion

$$\begin{cases} \partial_t\zeta \;+\; \partial_x w \;=\; 0, \\ \partial_t\Big(\frac{h_1+\gamma h_2}{h_1 h_2}w + \mu\mathcal{Q}^{\mathsf{F}}[\zeta]w\Big) + (\gamma+\delta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\Big(\frac{{h_1}^2-\gamma{h_2}^2}{(h_1h_2)^2}|w|^2\Big) = \mu\epsilon\partial_x\big(\mathcal{R}^{\mathsf{F}}[\zeta,w]\big) \\ \qquad\qquad + \frac{\gamma+\delta}{\mathrm{Bo}}\partial_x^2\Big(\frac{\partial_x\zeta}{1+\mu\epsilon^2|\partial_x\zeta|^2}\Big), \end{cases}$$

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where $\mathsf{F}_i^{\mu} = \mathsf{F}_i(\sqrt{\mu}|D|)$.

Notation (Fourier multiplier) : $\widehat{\mathsf{F}_i^{\mu}u}(\xi) = \mathsf{F}_i(\sqrt{\mu}|\xi|)\widehat{u}(\xi)$. $\qquad \partial_x = iD$.

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where $\mathsf{F}_i^\mu = \mathsf{F}_i(\sqrt{\mu}|D|)$.

Examples : $\mathsf{F}_i^\mu = \frac{1}{\sqrt{1+\mu\theta_i |D|^2}}$ or $\mathsf{F}_i^\mu = \sqrt{\frac{3}{\delta^{-i}\sqrt{\mu}|D| \tanh(\delta^{-i}\sqrt{\mu}|D|)} - \frac{3}{\delta^{-2i}\mu |D|^2}}$.

Promotion...

Our systems

1. can be tuned to reproduce (formally) the formation of the Kelvin-Helmholtz instabilities, or to suppress them ;
2. preserves natural quantities :

$$\mathcal{I} \stackrel{\rm def}{=} \int_{-\infty}^{\infty} \zeta \, {\rm d}x, \qquad \mathcal{V}_i \stackrel{\rm def}{=} \int_{-\infty}^{\infty} h_i^{-1} w + \mu \mathcal{Q}_i^{\rm F}[\zeta] w \, {\rm d}x,$$

$$\mathcal{E} \stackrel{\rm def}{=} \int_{-\infty}^{\infty} (\gamma+\delta)\zeta^2 + \frac{2(\gamma+\delta)}{\mu\epsilon^2 \,{\rm Bo}}\big(\sqrt{1+\mu\epsilon^2|\partial_x\zeta|^2}-1\big) + \frac{h_1+\gamma h_2}{h_1 h_2}|w|^2 + \mu\frac{\gamma}{3}h_1^3\big(\partial_x {\rm F}_1^\mu \frac{w}{h_1}\big)^2 + \mu\frac{1}{3}h_2^3\big(\partial_x {\rm F}_2^\mu \frac{w}{h_2}\big)^2 \, {\rm d}x$$

3. possesses symmetry groups

$$x \rightsquigarrow x+\alpha \ , \quad t \rightsquigarrow t+\alpha \ , \quad (x,u_1,u_2) \rightsquigarrow (x+\alpha t, u_1+\alpha, u_2+\alpha)$$

4. enjoys a Hamiltonian structure.

Numerical scheme

Spatial discretization : spectral method

- $u(t,x) \approx \sum_j u(t,x_j) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h}$ with $x_j = jh$.
- $\mathsf{F}(D)u \approx \sum_j u(t,x_j)\mathsf{F}(D) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h} = M_{\mathsf{F}}\big(u(t,x_j)\big)_j$.

$\rightsquigarrow$ exponential accuracy for smooth functions
$2^9 = 512$ points gives machine precision (10^{-18})

Time evolution : High order Runge-Kutta scheme (Matlab's `ode45`)

- Reasonable CFL condition $\Delta t \leq Ch$
- High order scheme = high accuracy (here, typically 10^{-10})

Without surface tension With surface tension

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▸ Without surface tension ▸ With surface tension

Strategy (ideas...)

The system may be rewritten as

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{b} \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \mathfrak{a} & \mathfrak{c} \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ w \end{pmatrix} = \text{“l.o.t”}$$

where

$$\mathfrak{a}\bullet \stackrel{\text{def}}{=} \big(a_0(\epsilon\zeta) - \epsilon^2 \tilde{a}_0(\epsilon\zeta)|w|^2\big)\bullet - \frac{1}{\mathrm{Bo}}\partial_x\big(a_1(\epsilon\zeta)\partial_x \bullet\big) + \mu\partial_x \mathsf{F}_i^\mu\big(\tilde{a}_1(\epsilon\zeta)w^2\partial_x \mathsf{F}_i^\mu \bullet\big)$$

$$\mathfrak{b}\bullet \stackrel{\text{def}}{=} b_0(\epsilon\zeta)\bullet - \mu\partial_x \mathsf{F}_i^\mu\big(b_1(\epsilon\zeta)\partial_x \mathsf{F}_i^\mu \bullet\big)$$

$$\mathfrak{c}\bullet \stackrel{\text{def}}{=} c_0(\epsilon\zeta)\bullet - \mu\partial_x \mathsf{F}_i^\mu\big(c_1(\epsilon\zeta)\partial_x \mathsf{F}_i^\mu \bullet\big)$$

The symmetrizer defines a natural energy :

$$\big|(\zeta, w)\big|_{X^0}^2 \stackrel{\text{def}}{=} \big(\,\mathfrak{a}\zeta\,,\,\zeta\,\big)_{L^2} + \big(\,\mathfrak{b}w\,,\,w\,\big)_{L^2}.$$

Hyperbolicity conditions

If $h_1 = 1 - \epsilon\zeta > h_0$ and $h_2 = \delta^{-1} + \epsilon\zeta > h_0$, then $a_0, a_1, b_0, b_1 > 0$.

Strategy (ideas...)

The system may be rewritten as

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{b} \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \mathfrak{a} & \mathfrak{c} \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ w \end{pmatrix} = \text{“l.o.t”}$$

where

$$\mathfrak{a}\bullet \stackrel{\text{def}}{=} \big(a_0(\epsilon\zeta) - \epsilon^2 \tilde{a}_0(\epsilon\zeta)|w|^2\big)\bullet - \frac{1}{\mathrm{Bo}}\partial_x\big(a_1(\epsilon\zeta)\partial_x \bullet\big) + \mu\partial_x \mathsf{F}_i^{\mu}\big(\tilde{a}_1(\epsilon\zeta)w^2\partial_x \mathsf{F}_i^{\mu}\bullet\big)$$

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If $h_1 = 1 - \epsilon\zeta > h_0$ and $h_2 = \delta^{-1} + \epsilon\zeta > h_0$, then $a_0, a_1, b_0, b_1 > 0$.
If $\mathsf{F_i}(\xi) \lesssim |\xi|^{-\sigma}$ and $\epsilon^2|w|^2(1 + (\mu\,\text{Bo})^{1-\sigma})$ sufficiently small, then

$$\big|(\zeta, w)\big|_{X^0}^2 \approx \big|\zeta\big|_{L^2}^2 + \frac{1}{\text{Bo}}\big|\partial_x\zeta\big|_{L^2}^2 + \big|w\big|_{L^2}^2 + \mu\big|\partial_x\mathsf{F}_i^\mu w\big|_{L^2}^2.$$

A priori estimates

$$\begin{pmatrix} \mathfrak{a} & 0 \\ 0 & \mathfrak{b} \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 0 & \mathfrak{a} \\ \mathfrak{a} & \mathfrak{c} \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ w \end{pmatrix} = \text{“l.o.t”}$$

Usual strategy : multiply the system with $\Lambda^s = (1+|D|^2)^{s/2}$ and use commutator estimates to control X^s norm, $s > 1/2 + 1$:

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Here, we cannot estimate

$$\rightsquigarrow \big([\Lambda^s, \mathfrak{a}]\partial_x w, \Lambda^s\zeta\big)_{L^2} \text{ and/or } \big([\Lambda^s, \mathfrak{a}]\partial_t \zeta, \Lambda^s\zeta\big)_{L^2}.$$

Instead, work with

$$\big|(\zeta, w)\big|_{E^N} \approx \sum \big|\partial^\alpha\zeta\big|_{L^2}^2 + \frac{1}{\mathrm{Bo}}\big|\partial_x\partial^\alpha\zeta\big|_{L^2}^2 + \big|\partial^\alpha w\big|_{L^2}^2 + \mu\big|\partial_x \mathsf{F}_i^\mu \partial^\alpha w\big|_{L^2}^2.$$

where the sum is over $|\alpha| \leq N$, α multi-indices in space *and time*.

Differentiate α times, extract leading order system for $(\partial^\alpha\zeta, \partial^\alpha w)$

$$\Longrightarrow \text{control of } \big|(\partial^\alpha\zeta, \partial^\alpha w)\big|_{X^0}.$$

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Well-posedness

Well-posedness of the Green-Naghdi models

Assume that $(\zeta^0, w^0) \in E^N$ with N large enough (time-derivatives given by the system) ;

$$\big|\epsilon\zeta^0\big|_{L^\infty} < \min\Big\{1, \frac{1}{\delta}\Big\}, \quad \epsilon^2 \sum_{|\alpha|\leq 1}\Big(\big|\partial^\alpha w^0\big|^2_{L^\infty} + (\mu\,\mathrm{Bo})^{1-\sigma}\sum_{|j|\leq M}\big|(\sqrt{\mu}\partial_x \mathsf{F}_i^\mu)^j\partial^\alpha w^0\big|^2_{L^\infty}\Big) < C(\epsilon\zeta^0).$$

Then there exists C, T and a unique strong solution (ζ, w) to the Green-Naghdi system for $t \in [0, T)$, and

$$\forall t < T, \quad \big|(\zeta, w)\big|_{E^N}(t) \leq C\big|(\zeta^0, w^0)\big|_{E^N}\exp(Ct),$$

and the hyperbolicity conditions remain satisfied.

Remarks :

- This result allows to fully justify the models w.r.t. the full Euler system [Lannes '13]
- C, T^{-1} are uniformly bounded with respect to $\gamma \in [0,1], \epsilon \in [0,1], \mu \in [0, \mu_{\max}]$, and also $\sigma \in \mathbb{R}^+$.
- The time domain is $[0, T/\epsilon)$ if a stronger hyperbolicity condition is satisfied.
- If $\sigma \geq 1$ or $\mu = 0$, the result is also valid without surface tension ($\mathrm{Bo}^{-1} = 0$), and

$$\big|(\zeta, w)\big|_{E^N} = \big|\zeta\big|^2_{H^N} + \big|w\big|^2_{H^N}.$$

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Thank you for your attention !