Justification complète de modèles unidirectionnels et découplés

pour la propagation d'ondes internes en océanographie

Vincent Duchêne

IRMAR, Univ. Rennes 1 – UMR 6625

GdT Analyse non linéaire et EDP
14 Mai 2013

Korteweg-de Vries equation

$$\partial_t \eta + \partial_x \eta + \varepsilon \left(\frac{1}{6} \partial_x^3 \eta + \frac{3}{2} \eta \partial_x \eta \right) = 0.$$

The early study

- John Scott Russell (1834)
- Boussinesq (1871), Rayleigh (1876), Korteweg & de Vries (1895).

Korteweg-de Vries equation

$$\partial_t \eta + \partial_x \eta + \varepsilon \left(\frac{1}{6} \partial_x^3 \eta + \frac{3}{2} \eta \partial_x \eta \right) = 0.$$

The early study

- John Scott Russell (1834)
- Boussinesq (1871), Rayleigh (1876), Korteweg & de Vries (1895).

Korteweg-de Vries equation

$$\partial_t \eta + \partial_x \eta + \varepsilon \left(\frac{1}{6}\partial_x^3 \eta + \frac{3}{2}\eta\partial_x \eta\right) = 0.$$

The early study

- John Scott Russell (1834)
- Boussinesq (1871), Rayleigh (1876), Korteweg & de Vries (1895).

Many, many studies

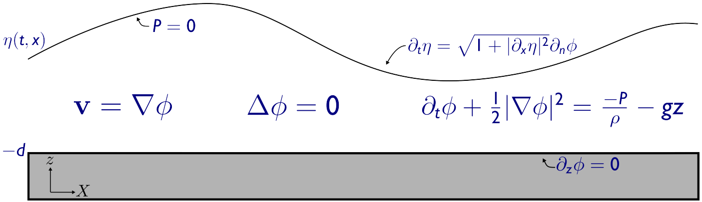
- Integrability, inverse scattering
- Stability of solitons
- Global well-posedness
- *etc.*

Complete, rigorous justification

- Craig (1985)
- Kano & Nishida (1986)
- Schneider & Wayne (2000)
- Bona, Colin & Lannes (2005)

The full Euler system

One layer of homogeneous, incompressible, irrotational, inviscid fluid, under the only influence of gravity.



The system is uniquely described by η and $\psi \equiv \phi_{|\text{surface}}$.

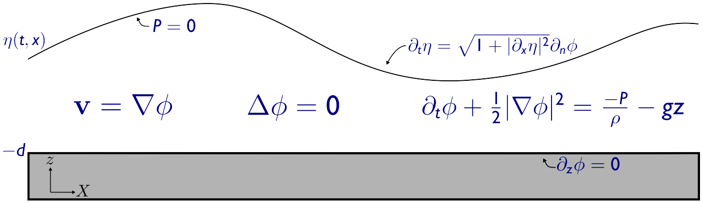
Definition : Dirichlet-Neumann operator

The following operator is well-defined

$$\mathcal{G}[\eta]\psi \equiv \sqrt{1+|\partial_x\eta|^2}\partial_n\phi_{|\text{surface}}.$$

The full Euler system

One layer of homogeneous, incompressible, irrotational, inviscid fluid, under the only influence of gravity.



The system is uniquely described by η and $\psi \equiv \phi_{|\text{surface}}$.

Definition : Dirichlet-Neumann operator

The following operator is well-defined

$$\mathcal{G}[\eta]\psi \equiv \sqrt{1+|\partial_x\eta|^2}\partial_n\phi_{|\text{surface}}.$$

The full Euler system

The system is uniquely described by η and $\psi \equiv \phi_{|\text{surface}}$.

Definition : Dirichlet-Neumann operator

The following operator is well-defined

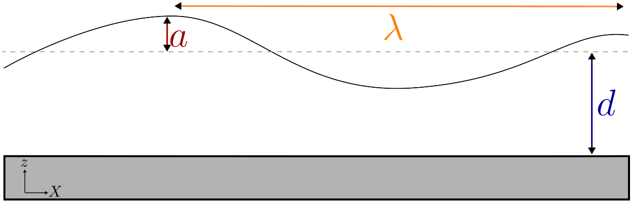
$$\mathcal{G}[\eta]\psi \equiv \sqrt{1+|\partial_x \eta|^2}\partial_n \phi_{|\text{surface}}.$$

The full Euler system

$$\begin{cases} \partial_t \zeta \ - \ \mathcal{G}[\eta]\psi = 0, \\ \partial_t \partial_x \psi \ + \ g\partial_x \eta \ + \ \frac{1}{2}\partial_x(|\partial_x \psi|^2) \ = \ \partial_x \mathcal{N}. \end{cases} \qquad (\Sigma)$$

with $\mathcal{N} \ \equiv \ \dfrac{\big(\mathcal{G}[\eta]\psi + (\partial_x \eta)(\partial_x \psi)\big)^2}{2(1+|\partial_x \eta|^2)}$.

Asymptotic models

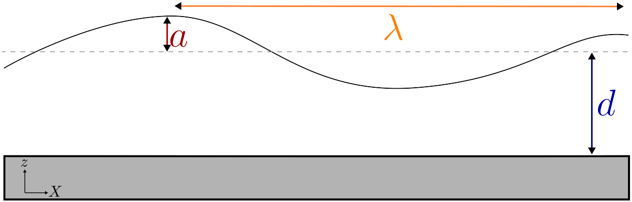


Define the dimensionless parameters $\epsilon \equiv \frac{a}{d}$ and $\mu \equiv \frac{d^2}{\lambda^2}$.

The full Euler system

$$\begin{cases} \partial_t \zeta \; - \; \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi = 0, \\ \partial_t\partial_x\psi \; + \; \partial_x\eta \; + \; \epsilon\frac{1}{2}\partial_x(|\partial_x\psi|^2) \; = \; \mu\epsilon\partial_x\mathcal{N}^{\mu,\epsilon}. \end{cases} \qquad (\Sigma)$$

Asymptotic models



Define the dimensionless parameters $\epsilon \equiv \frac{a}{d}$ and $\mu \equiv \frac{d^2}{\lambda^2}$.

The full Euler system

$$\left\{ \begin{array}{l} \partial_t \zeta \; - \; \frac{1}{\mu} \mathcal{G}^{\mu}[\epsilon \eta] \psi = 0, \\ \\ \partial_t \partial_x \psi \; + \; \partial_x \eta \; + \; \epsilon \frac{1}{2} \partial_x (|\partial_x \psi|^2) \; = \; \mu \epsilon \partial_x \mathcal{N}^{\mu,\epsilon}. \end{array} \right. \qquad (\Sigma)$$

Asymptotic models

The full Euler system

$$\begin{cases} \partial_t \zeta \ - \ \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi = 0, \\ \partial_t\partial_x\psi \ + \ \partial_x\eta \ + \ \epsilon\frac{1}{2}\partial_x(|\partial_x\psi|^2) \ = \ \mu\epsilon\partial_x\mathcal{N}^{\mu,\epsilon}. \end{cases} \qquad (\Sigma)$$

Asymptotic expansion

Let $s \geq 0$, $\zeta, \ \partial_x\psi \in \ H^{s+5}(\mathbb{R})$. Then

$$\left| \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi \ + \ \partial_x\big((1+\epsilon\eta)\psi\big) \ - \ \mu\partial_x\big(\mathcal{T}\psi\big) \right|_{H^s} \ \leq \ \mu^2 C$$

Complete justification

- Consistency
- Well-posedness
- Convergence

Asymptotic models

The full Euler system

$$\begin{cases} \partial_t \zeta \; - \; \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi = 0, \\ \partial_t\partial_x\psi \; + \; \partial_x\eta \; + \; \epsilon\frac{1}{2}\partial_x(|\partial_x\psi|^2) \; = \; \mu\epsilon\partial_x\mathcal{N}^{\mu,\epsilon}. \end{cases} \qquad (\Sigma)$$

Asymptotic expansion

Let $s \geq 0$, $\zeta, \; \partial_x\psi \in \; H^{s+5}(\mathbb{R})$. Then

$$\left| \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi \; + \; \partial_x\big((1+\epsilon\eta)\psi\big) \; - \; \mu\partial_x\big(\mathcal{T}\psi\big) \right|_{H^s} \; \leq \; \mu^2 C$$

Complete justification

- Consistency
- Well-posedness
- Convergence

Asymptotic models

The full Euler system

$$\begin{cases} \partial_t \zeta \; - \; \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi = 0, \\ \partial_t\partial_x\psi \; + \; \partial_x\eta \; + \; \epsilon\frac{1}{2}\partial_x(|\partial_x\psi|^2) \; = \; \mu\epsilon\partial_x\mathcal{N}^{\mu,\epsilon}. \end{cases} \qquad (\Sigma)$$

Asymptotic expansion

Let $s \geq 0$, $\zeta, \; \partial_x\psi \in \; H^{s+5}(\mathbb{R})$. Then

$$\left| \frac{1}{\mu}\mathcal{G}^{\mu}[\epsilon\eta]\psi \; + \; \partial_x\big((1+\epsilon\eta)\psi\big) \; - \; \mu\partial_x\big(\mathcal{T}\psi\big) \right|_{H^s} \; \leq \; \mu^2 C$$

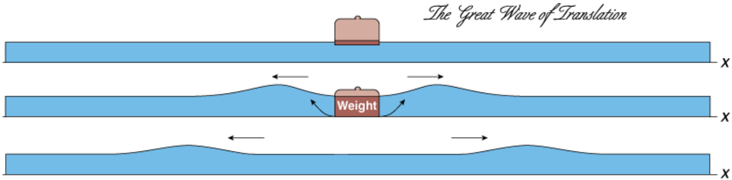
Complete justification

- Consistency
- Well-posedness
- Convergence

[Alvarez-Samaniego&Lannes (2008)]

The Cauchy problem for (Σ) is well-posed in Sobolev spaces for $t \in [0, T/\epsilon]$, uniformly w.r.t. ϵ, μ.
Stability : Consistency $\Longrightarrow$ Convergence.

Unidirectional vs decoupled approximation



Unidirectional KdV approximation

The unidirectional approximate solution is defined as

$$\partial_t \eta \ + \ \partial_x \eta \ + \ \frac{3}{2}\epsilon \ \eta \partial_x \eta \ + \ \mu \ \frac{1}{6} \ \partial_x^3 \eta \ = \ 0 \ ,$$

$$\partial_x \psi \ = \ F[\eta] \ = \ \eta \ + \ \epsilon \ \frac{3}{4} \ \eta^2 \ + \ \mu \ \frac{1}{6} \ \partial_x^2 \eta \ .$$

It satisfies (Σ) up to $\mathcal{O}(\epsilon^2 + \mu^2)$ terms, thus is an approximate solution with accuracy $\mathcal{O}\big((\epsilon^2 + \mu^2)t\big)$

Unidirectional vs decoupled approximation

Unidirectional KdV approximation

The unidirectional approximate solution is defined as

$$\partial_t \eta \;+\; \partial_x \eta \;+\; \frac{3}{2}\epsilon\;\eta \partial_x \eta \;+\; \mu\;\frac{1}{6}\;\partial_x^3 \eta \;=\; 0\;,$$
$$\partial_x \psi \;=\; F[\eta] \;=\; \eta \;+\; \epsilon\;\frac{3}{4}\;\eta^2 \;+\; \mu\;\frac{1}{6}\;\partial_x^2 \eta\;.$$

It is an approximate solution with accuracy $\mathcal{O}\big((\epsilon^2+\mu^2)t\big)$

Decoupled KdV approximation

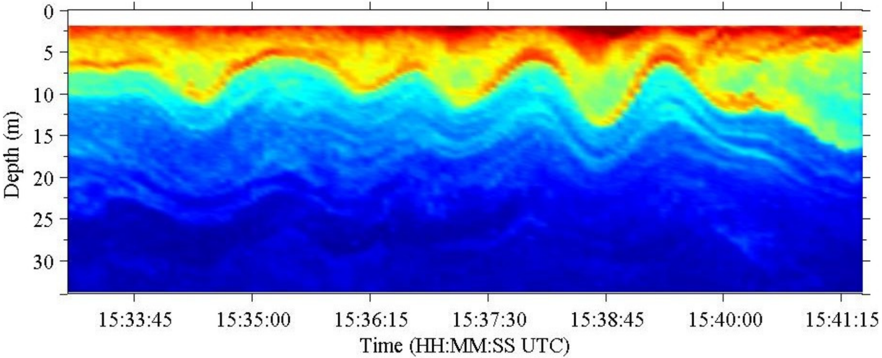
The decoupled approximate solution is defined as
$\eta_{\rm dec}(t,x) \;=\; \eta_+(t,x) \;+\; \eta_-(t,x)$, where $\eta_\pm$ satisfies

$$\partial_t \eta_\pm \;\pm\; \partial_x \eta_+ \;\pm\; \epsilon\frac{3}{2}\eta_\pm \partial_x \eta_\pm \;\pm\; \mu\frac{1}{6}\partial_x^3 \eta_\pm \;=\; 0.$$

It is an approximate solution with accuracy $\mathcal{O}\big((\epsilon+\mu)\min(t,t^{1/2})\big)$,
and $\mathcal{O}\big((\epsilon+\mu)\min(t,1)\big)$ if the initial data is sufficiently localized.

Internal gravity waves

Stratification, due to variation of salinity and temperature.



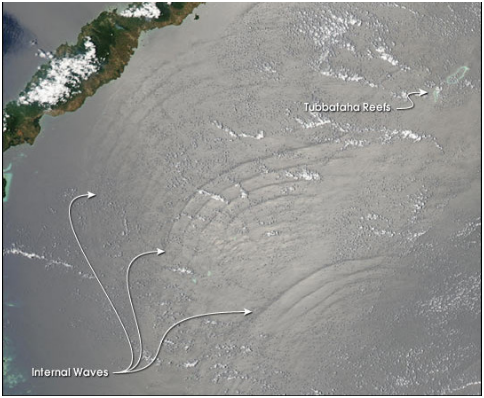
Other pictures

Figure : St. Lawrence Estuary [1]

1. Credits : St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) `http://myweb.dal.ca/kelley/SLEIWEX/index.php`

Internal gravity waves

Stratification, due to variation of salinity and temperature.



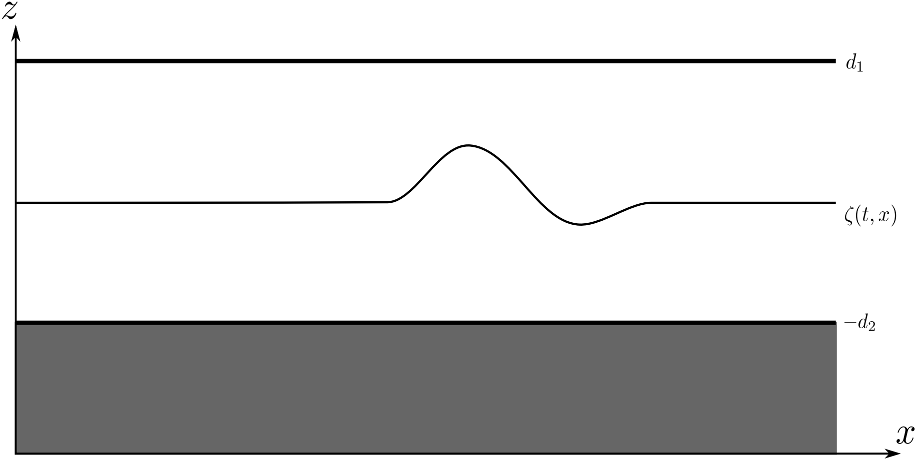
The large picture

Figure : Sulu Sea. April 8, 2003 [1]

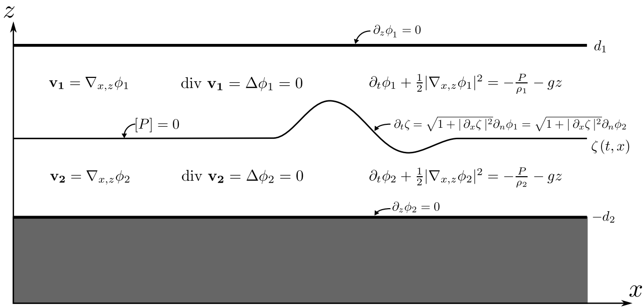
1. Credits : NASA's Earth Observatory (Picture of the Day July 1, 2003) http://earthobservatory.nasa.gov/IOTD/view.php?id=3586

Internal gravity waves

Framework : two layers of irrotational, immiscible, homogeneous, inviscid, incompressible fluids, with *free interface and rigid lid*.

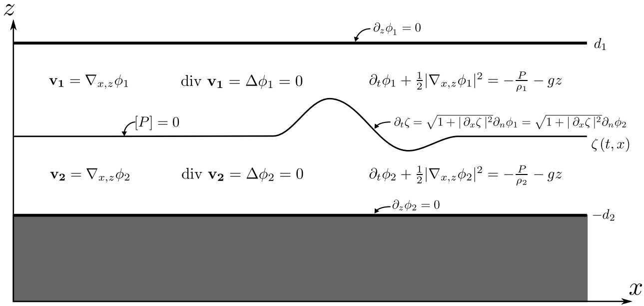


Full Euler system



- Horizontal dimension $d = 1$, flat bottom, rigid lid.
- Irrotational, incompressible, inviscid, immiscible fluids.
- Fluids at rest at infinity, no surface tension.

Full Euler system

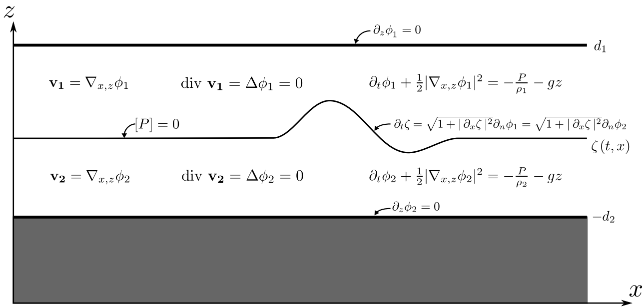


Dirichlet-Neumann operators

Define $\psi \equiv \phi_{1|\text{interface}}$ and

$$G[\zeta]\psi \equiv \sqrt{1+|\partial_x\zeta|^2}\partial_n\phi_{1|\text{interface}},$$
$$H[\zeta]\psi \equiv \phi_{2|\text{interface}}.$$

Full Euler system

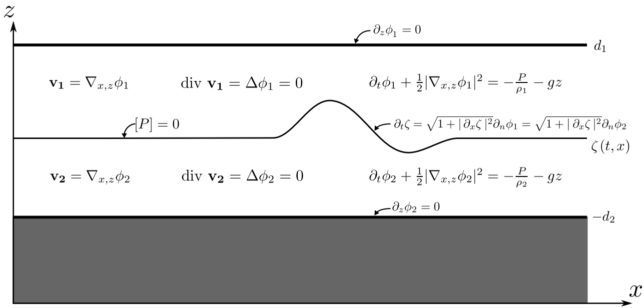


Full Euler system

$$\begin{cases} \partial_t\zeta \ - \ \dfrac{1}{\mu} G\psi \ = \ 0, \\ \partial_t\Big(\partial_x\psi - \gamma\partial_x H\psi\Big) + (\gamma+\delta)\partial_x\zeta + \dfrac{\epsilon}{2}\partial_x\Big(|\partial_x\psi|^2 - \gamma|\partial_x H\psi|^2\Big) \ = \ \mu\epsilon\partial_x N, \end{cases}$$

with $\epsilon \equiv \dfrac{a}{d_1}$, $\mu \equiv \dfrac{{d_1}^2}{\lambda^2}$, $\gamma \equiv \dfrac{\rho_1}{\rho_2}$, $\delta \equiv \dfrac{d_1}{d_2}$.

Full Euler system



State of the art (for well-posedness) :

- *Ill-posedness without surface tension :* Ebin (1988), Iguchi&Tanaka&Tani (1997), Kamotski-Lebeau (2005)
- *"Very" local WP with surface tension :* Ambrose (2003), Ambrose&Masmoudi (2007), Shatah&Zeng (2008)
- *Stability criterion :* Lannes (2013). *No stability result.*

Coupled asymptotic models

The dimensionless full Euler system

$$\begin{cases} \partial_t \zeta \; - \; \frac{1}{\mu} G\psi \; = \; 0, \\ \partial_t \Big(\partial_x \psi - \gamma \partial_x H \psi \Big) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \Big(|\partial_x \psi|^2 - \gamma |\partial_x H\psi|^2 \Big) \; = \; \mu \epsilon \partial_x N, \end{cases} \quad \text{(fE)}$$

Expansion of the Dirichlet-Neumann operators

Let $s \geq 0$, $\zeta, \; \partial_x \psi, \; \in \; H^{s+n}(\mathbb{R})$. Then one has

$$\Big| \frac{1}{\mu} G[\epsilon\zeta]\psi \; - \; \partial_x\big((1-\epsilon\zeta)\partial_x\psi\big) \; + \; \mu \partial_x \big(T_G \psi \big) \Big|_{H^s} \; \leq \; \mu^2 C$$

$$\Big| \partial_x \big(H[\epsilon\zeta]\psi \big) \; - \; \frac{1/\delta + \epsilon\zeta}{1-\epsilon\zeta} \partial_x \psi \; - \; \mu \partial_x \big(T_H \psi \big) \Big|_{H^s} \; \leq \; \mu^2 C,$$

Proof.

$G[\epsilon\zeta]\psi \equiv \sqrt{1+|\epsilon\partial_x\zeta|^2}\partial_n \phi_{1|\text{interface}} \quad \Longrightarrow G[\epsilon\zeta]\psi \equiv -\mathcal{G}^\mu[-\epsilon\zeta]\psi$

$H[\epsilon\zeta]\psi \equiv \phi_{2|\text{interface}} \quad \Longrightarrow H[\epsilon\zeta]\psi \equiv \big\{ \mathcal{G}^{\mu,\delta}[\epsilon\zeta] \big\}^{-1} \big(-\mathcal{G}^\mu[-\epsilon\zeta]\psi \big).$

Coupled asymptotic models

The dimensionless full Euler system

$$\begin{cases} \partial_t \zeta \ - \ \dfrac{1}{\mu} G\psi \ = \ 0, \\ \partial_t \Big(\partial_x \psi - \gamma \partial_x H\psi \Big) + (\gamma + \delta)\partial_x \zeta + \dfrac{\epsilon}{2} \partial_x \Big(|\partial_x \psi|^2 - \gamma |\partial_x H\psi|^2 \Big) \ = \ \mu\epsilon \partial_x N, \end{cases} \tag{fE}$$

Expansion of the Dirichlet-Neumann operators

Let $s \geq 0$, $\zeta, \ \partial_x \psi, \ \in \ H^{s+n}(\mathbb{R})$. Then one has

$$\Big| \frac{1}{\mu} G[\epsilon\zeta]\psi \ - \ \partial_x((1-\epsilon\zeta)\partial_x\psi) \ + \ \mu\partial_x\big(T_G\psi\big) \Big|_{H^s} \ \leq \ \mu^2 C$$

$$\Big| \partial_x\big(H[\epsilon\zeta]\psi\big) \ - \ \frac{1/\delta + \epsilon\zeta}{1-\epsilon\zeta}\partial_x\psi \ - \ \mu\partial_x\big(T_H\psi\big) \Big|_{H^s} \ \leq \ \mu^2 C,$$

Proof.

$G[\epsilon\zeta]\psi \equiv \sqrt{1+|\epsilon\partial_x\zeta|^2}\partial_n\phi_{1|\text{interface}}$ $\Longrightarrow G[\epsilon\zeta]\psi \equiv -\mathcal{G}^{\mu}[-\epsilon\zeta]\psi$

$H[\epsilon\zeta]\psi \equiv \phi_{2|\text{interface}}$ $\Longrightarrow H[\epsilon\zeta]\psi \equiv \big\{\mathcal{G}^{\mu,\delta}[\epsilon\zeta]\big\}^{-1}\big(-\mathcal{G}^{\mu}[-\epsilon\zeta]\psi\big).$

Coupled asymptotic models

The Green-Naghdi equation

$$\begin{cases} \partial_t \zeta \;+\; \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \;=\; 0, \\[2ex] \partial_t \Big(v \;+\; \mu \mathcal{Q}[\zeta] v \Big) \;+\; (\gamma + \delta) \partial_x \zeta \;+\; \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \;=\; \mu \epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \tag{GN}$$

with $h_1 = 1 - \epsilon\zeta$ and $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$v \;\equiv\; \frac{1}{h_2(t,x)} \int_{-\frac{1}{\delta}}^{\epsilon\zeta(t,x)} \partial_x \phi_2(t,x,z)\, dz \;-\; \gamma \frac{1}{h_1(t,x)} \int_{\epsilon\zeta(t,x)}^{1} \partial_x \phi_1(t,x,z)\, dz.$$

$$\mathcal{Q}[\zeta] V \;\equiv\; \tfrac{-1}{3 h_1 h_2} \Big(h_1 \partial_x \Big({h_2}^3 \partial_x \big(\tfrac{h_1 \; V}{h_1 + \gamma h_2} \big) \Big) \;+\; \gamma h_2 \partial_x \Big({h_1}^3 \partial_x \big(\tfrac{h_2 \; V}{h_1 + \gamma h_2} \big) \Big) \Big),$$

$$\mathcal{R}[\zeta] V \;\equiv\; \tfrac{1}{2} \Big(\Big(h_2 \partial_x \big(\tfrac{h_1 \; V}{h_1 + \gamma h_2} \big) \Big)^2 \;-\; \gamma \Big(h_1 \partial_x \big(\tfrac{h_2 \; V}{h_1 + \gamma h_2} \big) \Big)^2 \Big) + \tfrac{1}{3} \tfrac{V}{h_1 + \gamma h_2} \; \Big(\tfrac{h_1}{h_2} \partial_x \Big({h_2}^3 \partial_x \big(\tfrac{h_1 \; V}{h_1 + \gamma h_2} \big) \Big) \;-\; \gamma \tfrac{h_2}{h_1} \partial_x \Big({h_1}^3 \partial_x \big(\tfrac{h_2 \; V}{h_1 + \gamma h_2} \big) \Big) \Big).$$

Coupled asymptotic models

The Green-Naghdi equation

$$\begin{cases} \partial_t\zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2}v\Big) \ = \ 0, \\ \partial_t\Big(v \ + \ \mu\mathcal{Q}[\zeta]v\Big) \ + \ (\gamma+\delta)\partial_x\zeta \ + \ \dfrac{\epsilon}{2}\partial_x\Big(\dfrac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}|v|^2\Big) \ = \ \mu\epsilon\partial_x\big(\mathcal{R}[\zeta]v\big), \end{cases} \tag{GN}$$

with $h_1 = 1-\epsilon\zeta$ and $h_2 = \frac{1}{\delta}+\epsilon\zeta$ and

$$v \ \equiv \ \frac{1}{h_2(t,x)}\int_{-\frac{1}{\delta}}^{\epsilon\zeta(t,x)} \partial_x\phi_2(t,x,z)\,dz \ - \ \gamma\frac{1}{h_1(t,x)}\int_{\epsilon\zeta(t,x)}^{1}\partial_x\phi_1(t,x,z)\,dz.$$

$$\mathcal{Q}[\zeta]V \ \equiv \ \tfrac{-1}{3h_1h_2}\Big(h_1\partial_x\Big({h_2}^3\partial_x\big(\tfrac{h_1\ V}{h_1+\gamma h_2}\big)\Big) \ + \ \gamma h_2\partial_x\Big({h_1}^3\partial_x\big(\tfrac{h_2\ V}{h_1+\gamma h_2}\big)\Big)\Big),$$

$$\mathcal{R}[\zeta]V \ \equiv \ \tfrac{1}{2}\Big(\Big(h_2\partial_x\big(\tfrac{h_1\ V}{h_1+\gamma h_2}\big)\Big)^2 \ - \ \gamma\Big(h_1\partial_x\big(\tfrac{h_2\ V}{h_1+\gamma h_2}\big)\Big)^2\Big) + \tfrac{1}{3}\tfrac{V}{h_1+\gamma h_2}\ \Big(\tfrac{h_1}{h_2}\partial_x\Big({h_2}^3\partial_x\big(\tfrac{h_1\ V}{h_1+\gamma h_2}\big)\Big) \ - \ \gamma\tfrac{h_2}{h_1}\partial_x\Big({h_1}^3\partial_x\big(\tfrac{h_2\ V}{h_1+\gamma h_2}\big)\Big)\Big).$$

Coupled asymptotic models

The Green-Naghdi equation

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \partial_t \Big(v \ + \ \mu \mathcal{Q}[\zeta] v \Big) \ + \ (\gamma + \delta) \partial_x \zeta \ + \ \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \ = \ \mu \epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \tag{GN}$$

Consistency

The full Euler system is consistent with the Green-Naghdi model, with precision $\mathcal{O}(\mu^2)$.

Open Questions :

- Well-posedness ?
- Convergence of solutions towards solutions of the full Euler system ?

The Camassa-Holm regime

The Green-Naghdi equation

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \partial_t \Big(v \ + \ \mu \mathcal{Q}[\zeta] v \Big) \ + \ (\gamma+\delta)\partial_x \zeta \ + \ \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1+\gamma h_2)^2} |v|^2 \Big) \ = \ \mu\epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \qquad \text{(GN)}$$

Formal simplification under the *long wave regime* : $\epsilon = \mathcal{O}(\mu)$.

$$\begin{cases} \partial_t \zeta \ + \ \frac{1}{\delta+\gamma} \partial_x v + \epsilon \frac{\delta^2 - \gamma}{(\gamma+\delta)^2} \partial_x (\zeta v) \ = \ 0, \\ \partial_t v \ - \ \mu \frac{1+\gamma\delta}{3\delta(\delta+\gamma)} \partial_x^2 \partial_t v \ + \ (\gamma+\delta)\partial_x \zeta \ + \ \frac{\epsilon}{2} \frac{\delta^2-\gamma}{(\gamma+\delta)^2} \partial_x \left(|v|^2 \right) \ = \ 0. \end{cases} \qquad \text{(B)}$$

$\Longrightarrow$ This motivates the *Camassa-Holm regime* : $\epsilon = \mathcal{O}(\sqrt{\mu})$.

We will neglect $\mathcal{O}(\mu\epsilon^2)$ terms.

The Camassa-Holm regime

The Green-Naghdi equation

$$\begin{cases} \partial_t \zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2} v\Big) \ = \ 0, \\ \partial_t\Big(v \ + \ \mu \mathcal{Q}[\zeta]v\Big) \ + \ (\gamma+\delta)\partial_x\zeta \ + \ \dfrac{\epsilon}{2}\partial_x\Big(\dfrac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}|v|^2\Big) \ = \ \mu\epsilon\partial_x\big(\mathcal{R}[\zeta]v\big), \end{cases} \tag{GN}$$

Formal simplification under the *long wave regime* : $\epsilon = \mathcal{O}(\mu)$.

$$\begin{cases} \partial_t \zeta \ + \ \frac{1}{\delta+\gamma}\partial_x v + \epsilon\frac{\delta^2-\gamma}{(\gamma+\delta)^2}\partial_x(\zeta v) \ = \ 0, \\ \partial_t v \ - \ \mu\frac{1+\gamma\delta}{3\delta(\delta+\gamma)}\partial_x^2\partial_t v \ + \ (\gamma+\delta)\partial_x\zeta \ + \ \frac{\epsilon}{2}\frac{\delta^2-\gamma}{(\gamma+\delta)^2}\partial_x\left(|v|^2\right) \ = \ 0. \end{cases} \tag{B}$$

$\Longrightarrow$ This motivates the *Camassa-Holm regime* : $\epsilon = \mathcal{O}(\sqrt{\mu})$.

We will neglect $\mathcal{O}(\mu\epsilon^2)$ terms.

Coupled models

The Green-Naghdi equations

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \\ \partial_t \Big(v + \mu \mathcal{Q}[\zeta] v \Big) \ + \ (\gamma + \delta) \partial_x \zeta \ + \ \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \ = \ \mu \epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \tag{GN}$$

with $h_1 = 1 - \epsilon\zeta$, $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$\mathcal{Q}[\zeta]V \equiv \frac{-1}{3 h_1 h_2} \Big(h_1 \partial_x \Big({h_2}^3 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big) \ + \ \gamma h_2 \partial_x \Big({h_1}^3 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big) \Big),$$

$$\mathcal{R}[\zeta]V \equiv \frac{1}{2} \Big(\Big(h_2 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big)^2 \ - \ \gamma \Big(h_1 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big)^2 \Big) + \frac{1}{3} \frac{V}{h_1 + \gamma h_2} \ \Big(\frac{h_1}{h_2} \partial_x \Big({h_2}^3 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big) \ - \ \gamma \frac{h_2}{h_1} \partial_x \Big({h_1}^3 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big) \Big).$$

Coupled models

The Serre equations

$$\begin{cases} \partial_t\zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2}v\Big) \ = \ 0, \\ \\ \partial_t\Big(v+\mu\mathcal{Q}[\zeta]v\Big) \ + \ (\gamma+\delta)\partial_x\zeta \ + \ \dfrac{\epsilon}{2}\partial_x\Big(\dfrac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}|v|^2\Big) \ = \ \mu\epsilon\partial_x\big(\mathcal{R}[\zeta]v\big), \end{cases} \tag{S}$$

with $h_1 = 1-\epsilon\zeta$, $h_2 = \frac{1}{\delta}+\epsilon\zeta$ and

$$\mathcal{Q}[\zeta]V \ \equiv \ -a\,\partial_x^2 V + \epsilon\left(b\ V\partial_x^2\zeta + c\ (\partial_x\zeta)(\partial_x V) + d\ \partial_x(\zeta\partial_x V)\right) + \mathcal{O}(\epsilon^2),$$
$$\mathcal{R}[\zeta]V \ \equiv \ e\ (\partial_x V)^2 \ + \ f\ V\partial_x^2 V \ + \ \mathcal{O}(\epsilon)$$

Coupled models

The Serre equations

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \\ \partial_t \Big(v + \mu \mathcal{Q}[\zeta] v \Big) \ + \ (\gamma + \delta) \partial_x \zeta \ + \ \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \ = \ \mu \epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \tag{S}$$

with $h_1 = 1 - \epsilon\zeta$, $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$\mathcal{Q}[\zeta]V \ \equiv \ -a\, \partial_x^2 V + \epsilon \big(b\ V \partial_x^2 \zeta + c\ (\partial_x \zeta)(\partial_x V) + d\ \partial_x(\zeta \partial_x V) \big) + \mathcal{O}(\epsilon^2),$$
$$\mathcal{R}[\zeta]V \ \equiv \ e\ (\partial_x V)^2 \ + \ f\ V \partial_x^2 V \ + \ \mathcal{O}(\epsilon)$$

Introduce

$$\mathcal{S}[\epsilon\zeta]V \ = \ (1 + \kappa_1 \epsilon \zeta) V \ - \ \mu\ a\ \partial_x \Big((1 + \kappa_2 \epsilon \zeta) \partial_x V \Big).$$

One has

$$(1 + \kappa_1 \epsilon \zeta) \partial_t \Big(v + \mu \mathcal{Q}[\epsilon\zeta] v \Big) = \mathcal{S}[\epsilon\zeta] \partial_t v + \mu \epsilon \partial_x (\mathcal{R}'[\epsilon\zeta] v) + \mathcal{O}(\mu^2 + \mu \epsilon^2)$$

Coupled models

The Serre equations

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \partial_t \Big(v + \mu \mathcal{Q}[\zeta] v \Big) \ + \ (\gamma + \delta) \partial_x \zeta \ + \ \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \ = \ \mu \epsilon \partial_x \big(\mathcal{R}[\zeta] v \big), \end{cases} \tag{S}$$

with $h_1 = 1 - \epsilon\zeta$, $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$\mathcal{Q}[\zeta]V \ \equiv \ -a\, \partial_x^2 V + \epsilon \left(b\ V \partial_x^2 \zeta + c\ (\partial_x \zeta)(\partial_x V) + d\ \partial_x(\zeta \partial_x V) \right) + \mathcal{O}(\epsilon^2),$$
$$\mathcal{R}[\zeta]V \ \equiv \ e\ (\partial_x V)^2 \ + \ f\ V \partial_x^2 V \ + \ \mathcal{O}(\epsilon)$$

Introduce

$$\mathcal{S}[\epsilon\zeta]V \ = \ (1 + \kappa_1 \epsilon\zeta) V \ - \ \mu\ a\ \partial_x \Big((1 + \kappa_2 \epsilon \zeta) \partial_x V \Big).$$

One has

$$(1 + \kappa_1 \epsilon\zeta) \partial_t \Big(v + \mu \mathcal{Q}[\epsilon\zeta] v \Big) = \mathcal{S}[\epsilon\zeta] (\partial_t v + \epsilon \sigma v \partial_x v) + \mu\epsilon \partial_x (\mathcal{R}'' v) + \mathcal{O}(\mu^2 + \mu\epsilon^2)$$

Coupled models

The modified Serre equations **[VD, Israwi, Talhouk]**

$$\begin{cases} \partial_t \zeta \; + \; \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \; = \; 0, \\ \\ \mathcal{S}[\epsilon\zeta](\partial_t v + \epsilon \sigma v \partial_x v) \; + \; (\gamma+\delta)(1+\kappa_1 \epsilon\zeta)\partial_x \zeta \; + \; \frac{\epsilon}{2}\partial_x \Big(\big(\frac{{h_1}^2 - \gamma {h_2}^2}{(h_1+\gamma h_2)^2} - \sigma \big) |v|^2 \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \; \mu\epsilon \, \varsigma \, \partial_x \big((\partial_x v)^2 \big), \end{cases} \tag{S'}$$

with $h_1 = 1 - \epsilon\zeta$, $h_2 = \frac{1}{\delta} + \epsilon\zeta$, and

$$\mathcal{S}[\epsilon\zeta] V \; = \; (1 + \kappa_1 \epsilon\zeta) V \; - \; \mu \, a \, \partial_x \Big((1 + \kappa_2 \epsilon\zeta) \partial_x V \Big).$$

The system is

- Consistent with full Euler system at precision $\mathcal{O}(\mu^2 + \mu\epsilon^2)$;
- Control of energy $E^s(\zeta, v) \approx |\zeta|_{H^s}^2 + |v|_{H^s}^2 + \mu |v|_{H^{s+1}}^2$;
- Well-posed over times $\mathcal{O}(1/\epsilon)$;
- Stable (so that consistency $\Longrightarrow$ convergence).

Coupled models

The modified Serre equations **[VD, Israwi, Talhouk]**

$$\begin{cases} \partial_t \zeta \; + \; \partial_x\Big(\frac{h_1 h_2}{h_1+\gamma h_2} v\Big) \; = \; 0, \\ \mathcal{S}[\epsilon\zeta](\partial_t v + \epsilon\sigma v \partial_x v) \; + \; (\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta \; + \; \frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2} - \sigma\big)|v|^2\Big) \\ \qquad = \; \mu\epsilon \, \varsigma \, \partial_x\big((\partial_x v)^2\big), \end{cases} \tag{S'}$$

with $h_1 = 1-\epsilon\zeta$, $h_2 = \frac{1}{\delta} + \epsilon\zeta$, and

$$\mathcal{S}[\epsilon\zeta]V \; = \; (1+\kappa_1\epsilon\zeta)V \; - \; \mu \, a \, \partial_x\Big((1+\kappa_2\epsilon\zeta)\partial_x V\Big).$$

The system is

- Consistent with full Euler system at precision $\mathcal{O}(\mu^2+\mu\epsilon^2)$;
- Control of energy $E^s(\zeta,v) \approx |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|v|^2_{H^{s+1}}$;
- Well-posed over times $\mathcal{O}(1/\epsilon)$;
- Stable (so that consistency $\Longrightarrow$ convergence).

1. Introduction
2. Internal waves
3. Coupled models
   - Construction
   - Full justification
4. Scalar models
   - Unidirectional approximation
   - Decoupled approximation

The unidirectional model

Seek an approximate solution under the form

$$\partial_t\zeta \;+\; a\,\partial_x\zeta \;+\; \epsilon\, b\,\zeta\partial_x\zeta \;+\; \mu\, c\,\partial_x^3\zeta$$
$$+\; \epsilon^2 d\; u^2\partial_x u \;+\; \epsilon^3 e\; u^3\partial_x u \;+\; \mu\epsilon\partial_x\big(f\; u\partial_x^2 u \;+\; g\;(\partial_x u)^2\big) \;=\; 0\,,$$
$$v \;=\; F[\zeta] \;=\; \alpha\,\zeta \;+\; \epsilon\,\beta\,\zeta^2 \;+\; \mu\,\nu\,\partial_x^2\zeta + \cdots\,,$$

with precision (consistency) $\mathcal{O}(\mu^2+\epsilon^4)$.

Unidirectional scalar approximation **[VD (2013)]**

If the initial data satisfies $v(0,x) \;=\; F[\zeta(0,x)]$, then let $U_{\rm uni}=(v,\zeta)$ be defined by $v(t,x) \;=\; F[\zeta(t,x)]$ and

$$\partial_t\zeta \;+\; \partial_x\zeta \;+\; \varepsilon\frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta\partial_x\zeta \;+\; \varepsilon\frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^3\zeta$$
$$+\; \epsilon^2\alpha_2 u^2\partial_x u \;+\; \epsilon^3\alpha_3 u^3\partial_x u \;+\; \mu\epsilon\partial_x\big(\kappa u\partial_x^2 u \;+\; \iota(\partial_x u)^2\big) \;=\; 0.$$

Then $U_{\rm uni}$ is an approximate solution, with accuracy $\mathcal{O}\Big((\mu^2+\epsilon^4)t\Big)$.

The unidirectional model

Seek an approximate solution under the form

$$\partial_t\zeta + a\,\partial_x\zeta + \epsilon\, b\,\zeta\partial_x\zeta + \mu\, c\,\partial_x^3\zeta + \epsilon^2 d\, u^2\partial_x u + \epsilon^3 e\, u^3\partial_x u + \mu\epsilon\partial_x\big(f\, u\partial_x^2 u + g\,(\partial_x u)^2\big) = 0\,,$$

$$v = F[\zeta] = \alpha\,\zeta + \epsilon\,\beta\,\zeta^2 + \mu\,\nu\,\partial_x^2\zeta + \cdots\,,$$

with precision (consistency) $\mathcal{O}(\mu^2+\epsilon^4)$.

Unidirectional scalar approximation [VD (2013)]

If the initial data satisfies $v(0,x) = F[\zeta(0,x)]$, then let $U_{\rm uni} = (v,\zeta)$ be defined by $v(t,x) = F[\zeta(t,x)]$ and

$$\partial_t\zeta + \partial_x\zeta + \varepsilon\frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta\partial_x\zeta + \varepsilon\frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^3\zeta + \epsilon^2\alpha_2 u^2\partial_x u + \epsilon^3\alpha_3 u^3\partial_x u + \mu\epsilon\partial_x\big(\kappa u\partial_x^2 u + \iota(\partial_x u)^2\big) = 0.$$

Then $U_{\rm uni}$ is an approximate solution, with accuracy $\mathcal{O}\Big((\mu^2+\epsilon^4)t\Big)$.

The unidirectional model

Unidirectional scalar approximation **[VD (2013)]**

If the initial data satisfies $v(0,x) = F[\zeta(0,x)]$, then let $U_{\rm uni} = (v,\zeta)$ be defined by $v(t,x) = F[\zeta(t,x)]$ and

$$\partial_t\zeta + \partial_x\zeta + \varepsilon\frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta\partial_x\zeta + \varepsilon\frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^3\zeta$$
$$+ \epsilon^2\alpha_2 u^2\partial_x u + \epsilon^3\alpha_3 u^3\partial_x u + \mu\epsilon\partial_x\big(\kappa u\partial_x^2 u + \iota(\partial_x u)^2\big) = 0.$$

Then $U_{\rm uni}$ is an approximate solution, with accuracy $\mathcal{O}\Big((\mu^2+\epsilon^4)t\Big)$.

Proof. 1. $U_{\rm uni}$, $U_{\rm Serre}$, $U_{\rm Euler}$ are uniquely defined over time $\mathcal{O}(1/\epsilon)$.

2. $U_{\rm uni}$ satisfies the Serre system up to $\mathcal{O}\big(\mu^2+\epsilon^4\big)$ terms.

$$\Longrightarrow \big|U_{\rm uni} - U_{\rm Serre}\big|_{H^s} \lesssim (\mu^2+\epsilon^4)t.$$

3. $U_{\rm Euler}$ satisfies the Serre system up to $\mathcal{O}\big(\mu^2+\mu\epsilon^2\big)$ terms.

$$\Longrightarrow \big|U_{\rm Euler} - U_{\rm Serre}\big|_{H^s} \lesssim (\mu^2+\mu\epsilon^2)t.$$

Decomposition of the flow

Is it true that after a certain time, any perturbation will decompose into two waves, each one satisfying (approximately) $v = F[\zeta]$ *?*

Decomposition of the flow

Is it true that after a certain time, any perturbation will decompose into two waves, each one satisfying (approximately) $v = F[\zeta]$ *?*

Numerically, yes.

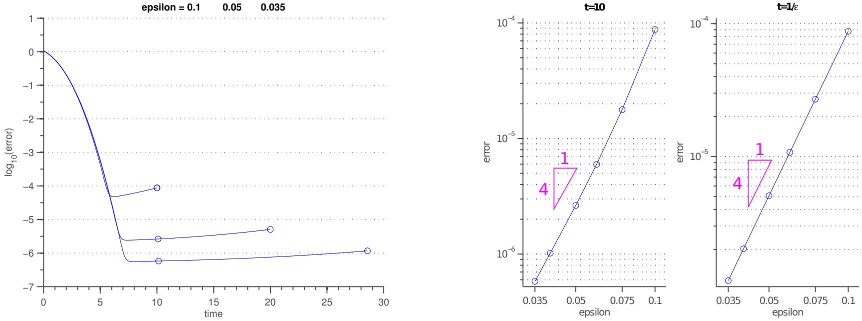


Figure : Camassa-Holm regime : $\epsilon^2 = \mu$, localized initial data.

Decoupled models : Strategy

We seek a decoupled approximate solution of the modified Serre system (S') :

$$\begin{cases} \partial_t \zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2} v\Big) \ = \ 0, \\ \mathcal{S}[\zeta](\partial_t v + \epsilon\sigma v \partial_x v) + (\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}-\sigma\big)|v|^2\Big) \\ \qquad\qquad\qquad\qquad = \ \mu\epsilon \ \varsigma \ \partial_x\big((\partial_x v)^2\big), \end{cases}$$

1. First order : $\partial_t U + \Sigma_0 \partial_x U = 0$
   $\longrightarrow$ Decomposition of the flow : $U = \sum u_i \mathbf{e}_i$, $\partial_t u_i + c_i \partial_x u_i = 0$
2. $U_{\rm app} = \sum u_i(\varepsilon t, t, x)\mathbf{e}_i + \varepsilon U^c[u_i](\varepsilon t, t, x)$. $\varepsilon = \max\{\epsilon(\delta^2-\gamma), \epsilon^2, \mu\}$
   $\longrightarrow$ Equation on u_i, then U^c, for maximal consistency
3. Control of the secular growth of U^c.
   $\longrightarrow$ Consistency result.
   $\longrightarrow$ Convergence result.

Decoupled models : Strategy

We seek a decoupled approximate solution of the modified Serre system (S') :

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \mathcal{S}[\zeta](\partial_t v + \epsilon\sigma v \partial_x v) + (\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2 - \gamma {h_2}^2}{(h_1+\gamma h_2)^2} - \sigma \big) |v|^2 \Big) \\ \qquad = \ \mu\epsilon \ \varsigma \ \partial_x\big((\partial_x v)^2 \big), \end{cases}$$

1. First order : $\partial_t U + \Sigma_0 \partial_x U = 0$
   $\longrightarrow$ Decomposition of the flow : $U = \sum u_i \mathbf{e}_i$, $\partial_t u_i + c_i \partial_x u_i = 0$
2. $U_{\rm app} = \sum u_i(\varepsilon t, t, x)\mathbf{e}_i + \varepsilon U^c[u_i](\varepsilon t, t, x)$. $\varepsilon = \max\{\epsilon(\delta^2 - \gamma), \epsilon^2, \mu\}$
   $\longrightarrow$ Equation on u_i, then U^c, for maximal consistency
3. Control of the secular growth of U^c.
   $\longrightarrow$ Consistency result.
   $\longrightarrow$ Convergence result.

Decoupled models : Strategy

We seek a decoupled approximate solution of the modified Serre system (S') :

$$\begin{cases} \partial_t\zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2}v\Big) \ = \ 0, \\ \mathcal{S}[\zeta](\partial_t v+\epsilon\sigma v\partial_x v)+(\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta+\frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}-\sigma\big)|v|^2\Big) \\ \qquad = \ \mu\epsilon\,\varsigma\,\partial_x\big((\partial_x v)^2\big), \end{cases}$$

1. First order : $\partial_t U+\Sigma_0\partial_x U=0$
   $\longrightarrow$ Decomposition of the flow : $U=\sum u_i\mathbf{e}_i$, $\partial_t u_i+c_i\partial_x u_i=0$
2. $U_{\rm app}=\sum u_i(\varepsilon t,t,x)\mathbf{e}_i+\varepsilon U^c[u_i](\varepsilon t,t,x)$. $\varepsilon=\max\{\epsilon(\delta^2-\gamma),\epsilon^2,\mu\}$
   $\longrightarrow$ Equation on u_i, then U^c, for maximal consistency
3. Control of the secular growth of U^c.
   $\longrightarrow$ Consistency result.
   $\longrightarrow$ Convergence result.

Decoupled models : Strategy

We seek a decoupled approximate solution of the modified Serre system (S') :

$$\begin{cases} \partial_t \zeta \; + \; \partial_x \Big(\frac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \; = \; 0, \\ \\ \mathcal{S}[\zeta](\partial_t v + \epsilon \sigma v \partial_x v) + (\gamma + \delta)(1 + \kappa_1 \epsilon \zeta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \Big(\big(\frac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} - \sigma \big) |v|^2 \Big) \\ \qquad\qquad = \; \mu \epsilon \; \varsigma \; \partial_x \big((\partial_x v)^2 \big), \end{cases}$$

1. First order : $\partial_t U + \Sigma_0 \partial_x U = 0$
   $\longrightarrow$ Decomposition of the flow : $U = \sum u_i \mathbf{e}_i$, $\partial_t u_i + c_i \partial_x u_i = 0$
2. $U_{\rm app} = \sum u_i(\varepsilon t, t, x) \mathbf{e}_i + \varepsilon U^c[u_i](\varepsilon t, t, x)$. $\varepsilon = \max\{\epsilon(\delta^2 - \gamma), \epsilon^2, \mu\}$
   $\longrightarrow$ Equation on u_i, then U^c, for maximal consistency
3. Control of the secular growth of U^c.
   $\longrightarrow$ Consistency result.
   $\longrightarrow$ Convergence result.

More details

At first order : $$\partial_t U_0 + \begin{pmatrix} 0 & \frac{1}{\gamma+\delta} \\ \gamma+\delta & 0 \end{pmatrix} \partial_x U_0 = 0.$$

$$U_0 = P \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \text{ with } \partial_t \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0.$$

$$\Longrightarrow U_0 = u_+ \mathbf{e}_+ + u_- \mathbf{e}_- \qquad \text{with} \qquad (\partial_t \pm \partial_x) u_\pm(t,x) = 0. \tag{1}$$

At next order : We seek an approximation of the form

$$U_{\rm dec}(t,x) \equiv u_+(\varepsilon t, x-t)\mathbf{e}_+ + u_-(\varepsilon t, x+t)\mathbf{e}_- + \varepsilon u_+^c[u_\pm]\mathbf{e}_+ + \varepsilon u_-^c[u_\pm]\mathbf{e}_+.$$

$$\partial_\tau \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \partial_t \begin{pmatrix} u_+^c \\ u_-^c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u_+^c \\ u_-^c \end{pmatrix} = \begin{pmatrix} f_+(u_+,u_-) \\ f_-(u_+,u_-) \end{pmatrix}.$$

$$\partial_\tau u_+ = f_+(u_+,0) \qquad (\partial_t + \partial_x) u_+^c = f_+(u_+,u_-) - f_+(u_+,0), \tag{2}$$

$$\partial_\tau u_- = f_-(0,u_-) \qquad (\partial_t - \partial_x) u_-^c = f_-(u_+,u_-) - f_-(0,u_-). \tag{3}$$

More details

At first order : $$\partial_t U_0 + \begin{pmatrix} 0 & \frac{1}{\gamma+\delta} \\ \gamma+\delta & 0 \end{pmatrix} \partial_x U_0 = 0.$$

$$U_0 = P\begin{pmatrix} u_+ \\ u_- \end{pmatrix} \text{ with } \partial_t \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0.$$

$$\Longrightarrow U_0 = u_+\mathbf{e}_+ + u_-\mathbf{e}_- \quad \text{with} \quad (\partial_t \pm \partial_x)u_\pm(t,x) = 0. \tag{1}$$

At next order : We seek an approximation of the form

$$U_{\rm dec}(t,x) \equiv u_+(\varepsilon t, x-t)\mathbf{e}_+ + u_-(\varepsilon t, x+t)\mathbf{e}_- + \varepsilon u_+^c[u_\pm]\mathbf{e}_+ + \varepsilon u_-^c[u_\pm]\mathbf{e}_+.$$

$$\partial_\tau \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \partial_t \begin{pmatrix} u_+^c \\ u_-^c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u_+^c \\ u_-^c \end{pmatrix} = \begin{pmatrix} f_+(u_+,u_-) \\ f_-(u_+,u_-) \end{pmatrix}.$$

$$\partial_\tau u_+ = f_+(u_+,0) \qquad (\partial_t + \partial_x)u_+^c = f_+(u_+,u_-) - f_+(u_+,0), \tag{2}$$
$$\partial_\tau u_- = f_-(0,u_-) \qquad (\partial_t - \partial_x)u_-^c = f_-(u_+,u_-) - f_-(0,u_-). \tag{3}$$

Rigorous justification

Well-posedness

Let $U(t=0) \in H^{s+n}$, $s > 1/2$. Then there exists a unique strong solution $u_i(\tau, t, x)$, uniformly bounded in $C^1([0,T] \times \mathbb{R}; H^{s+n})$.

The residual U^c is explicit, and $U^c \in C^1([0,T] \times \mathbb{R}; H^s)$.

Secular growth of the residual

$$\forall (\tau, t) \in [0,T] \times \mathbb{R}, \quad \big| U^c(\tau, t, \cdot) \big|_{H^s} \leq C_0 \sqrt{t}.$$

Moreover, if $(1+x^2)U(t=0) \in H^{s+n}$, then one has the uniform estimate

$$\big| U^c(\tau, t, \cdot) \big|_{H^s} \leq C_0,$$

Key Lemma [Lannes (2005)] : If $u_{|t=0} = 0$ and

- $\partial_t u + c_1 \partial_x u = \partial_x f(x - c_2 t) \Longrightarrow |u|_{H^s} \leq \min\left\{t, \frac{2}{|c_1 - c_2|}\right\} \big|f\big|_{H^{s+1}}$.
- $\partial_t u + c_1 \partial_x u = g\big(f_1(x - c_1 t), f_2(x - c_2 t)\big)$
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, \sqrt{t}\right\} \big|f_1\big|_{H^s} \big|f_2\big|_{H^s}$.
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, 1\right\} \big|(1+x^2) f_1\big|_{H^s} \big|(1+x^2) f_2\big|_{H^s}$.

Rigorous justification

Well-posedness

Let $U(t=0) \in H^{s+n}$, $s > 1/2$. Then there exists a unique strong solution $u_i(\tau, t, x)$, uniformly bounded in $C^1([0,T] \times \mathbb{R}; H^{s+n})$.

The residual U^c is explicit, and $U^c \in C^1([0,T] \times \mathbb{R}; H^s)$.

Secular growth of the residual

$$\forall (\tau, t) \in [0,T] \times \mathbb{R}, \quad \big|U^c(\tau, t, \cdot)\big|_{H^s} \leq C_0 \sqrt{t}.$$

Moreover, if $(1+x^2)U(t=0) \in H^{s+n}$, then one has the uniform estimate

$$\big|U^c(\tau, t, \cdot)\big|_{H^s} \leq C_0,$$

Key Lemma [Lannes (2005)] : If $u_{|t=0} = 0$ and

- $\partial_t u + c_1 \partial_x u = \partial_x f(x - c_2 t) \Longrightarrow |u|_{H^s} \leq \min\left\{t, \frac{2}{|c_1 - c_2|}\right\} \big|f\big|_{H^{s+1}}.$
- $\partial_t u + c_1 \partial_x u = g\big(f_1(x - c_1 t), f_2(x - c_2 t)\big)$
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, \sqrt{t}\right\} \big|f_1\big|_{H^s} \big|f_2\big|_{H^s}.$
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, 1\right\} \big|(1+x^2) f_1\big|_{H^s} \big|(1+x^2) f_2\big|_{H^s}.$

Rigorous justification

Well-posedness+persistence

Let $U(t=0) \in H^{s+n}$, $s > 1/2$. Then there exists a unique strong solution $u_i(\tau, t, x)$, uniformly bounded in $C^1([0,T] \times \mathbb{R}; H^{s+n})$.
If $(1+x^2)U(t=0,\cdot) \in H^{s+n}$, then $(1+x^2)u_i(\tau,\cdot) \in H^{s+n}$.
The residual U^c is explicit, and $U^c \in C^1([0,T] \times \mathbb{R}; H^s)$.

Secular growth of the residual

$$\forall (\tau, t) \in [0,T] \times \mathbb{R}, \quad \big|U^c(\tau, t, \cdot)\big|_{H^s} \leq C_0 \sqrt{t}.$$

Moreover, if $(1+x^2)U(t=0) \in H^{s+n}$, then one has the uniform estimate

$$\big|U^c(\tau, t, \cdot)\big|_{H^s} \leq C_0,$$

Key Lemma [Lannes (2005)] : If $u_{|t=0} = 0$ and

- $\partial_t u + c_1 \partial_x u = \partial_x f(x - c_2 t) \Longrightarrow |u|_{H^s} \leq \min\left\{t, \frac{2}{|c_1 - c_2|}\right\}\big|f\big|_{H^{s+1}}$.
- $\partial_t u + c_1 \partial_x u = g\big(f_1(x - c_1 t), f_2(x - c_2 t)\big)$
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, \sqrt{t}\right\}\big|f_1\big|_{H^s}\big|f_2\big|_{H^s}$.
  $\Longrightarrow |u|_{H^s} \lesssim \min\left\{t, 1\right\}\big|(1+x^2)f_1\big|_{H^s}\big|(1+x^2)f_2\big|_{H^s}$.

Rigorous justification

Well-posedness+persistence

Secular growth of the residual

Consistency

$\sum u_i(\varepsilon t, t, x)\mathbf{e}_i + \varepsilon U^c(\varepsilon t, t, x)$ satisfies the Serre model, with precision $\mathcal{O}\big(\varepsilon^2(1+\sqrt{t})\big)$ (and $\mathcal{O}(\varepsilon^2)$ if $(1+x^2)U(t=0) \in H^{s+n}$).
(recall : $\varepsilon = \max\{\epsilon(\delta^2-\gamma), \epsilon^2, \mu\}$).

Convergence [VD (2013)]

The difference between the solution of the full Euler system and the decoupled model is of size $\mathcal{O}\big(\varepsilon \times \min\{t, \sqrt{t}\}\big)$, and of size $\mathcal{O}\big(\varepsilon \times \min\{t, 1\}\big)$ if the initial data is localized in space.

Rigorous justification

Well-posedness+persistence

Secular growth of the residual

Consistency

$\sum u_i(\varepsilon t, t, x)\mathbf{e}_i + \varepsilon U^c(\varepsilon t, t, x)$ satisfies the Serre model, with precision $\mathcal{O}\big(\varepsilon^2(1+\sqrt{t})\big)$ (and $\mathcal{O}(\varepsilon^2)$ if $(1+x^2)U(t=0) \in H^{s+n}$).
(recall : $\varepsilon = \max\{\epsilon(\delta^2-\gamma), \epsilon^2, \mu\}$).

Convergence **[VD (2013)]**

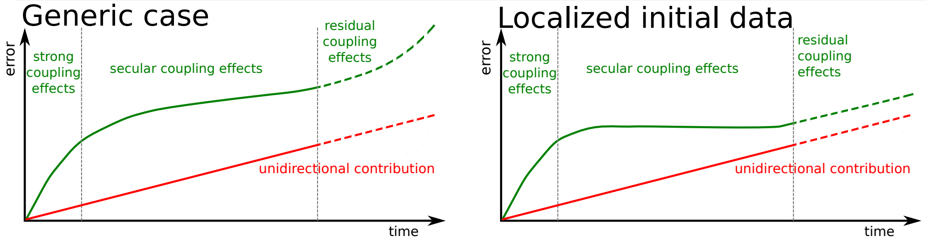
The difference between the solution of the full Euler system and the decoupled model is of size $\mathcal{O}\big(\varepsilon \times \min\{t, \sqrt{t}\}\big)$, and of size $\mathcal{O}\big(\varepsilon \times \min\{t, 1\}\big)$ if the initial data is localized in space.

The decoupled approximations

The decoupled approximations

$U_{\rm dec} = (\zeta, v) \ = \ \big(\ \zeta_+ + \zeta_- \ , \ (\gamma+\delta)(\zeta_+ - \zeta_-)\ \big),$ with

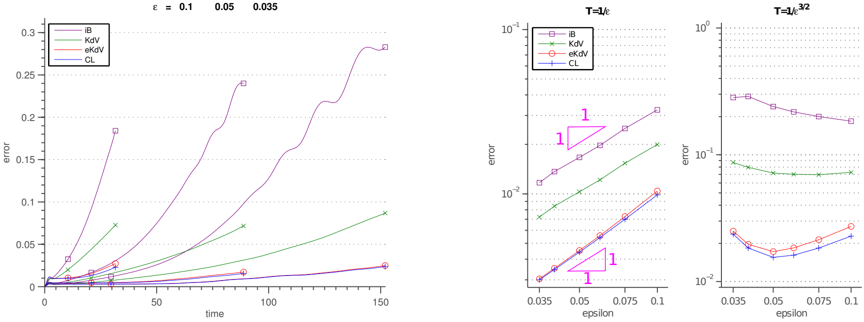
$$\partial_t \zeta_\pm \ \pm\ \partial_x \zeta_\pm \ \pm\ \epsilon \frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta_\pm \partial_x \zeta_\pm \ \pm\ \mu \frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^3 \zeta_\pm$$
$$\pm\ \epsilon^2 \alpha_2 \zeta_\pm^2 \partial_x \zeta_\pm \ +\ \epsilon^3 \alpha_3 \zeta_\pm^3 \partial_x \zeta_\pm \ \pm\ \mu\epsilon \partial_x \big(\kappa \zeta_\pm \partial_x^2 \zeta_\pm \ +\ \iota (\partial_x \zeta_\pm)^2 \big)$$
$$= \ 0. \quad \text{(CL)}$$



Error in the Camassa-Holm regime

In the non-critical case $\delta^2 - \gamma \neq 0$, the inviscid Burgers' equation is as precise as any higher order decoupled model.

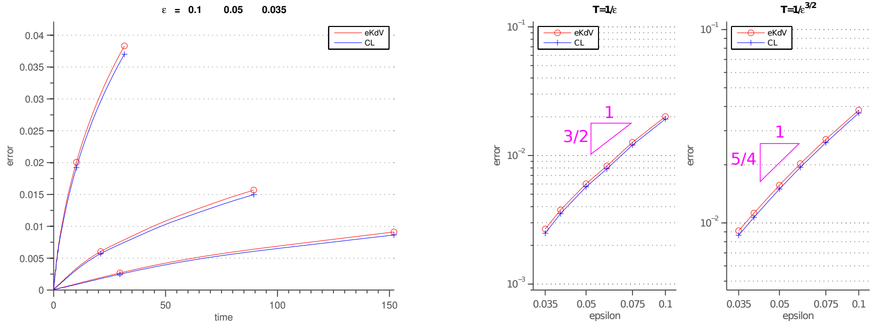
$$\partial_t \zeta_\pm \ \pm \ \partial_x \zeta_\pm \ \pm \ \epsilon \frac{3}{2} \frac{\delta^2 - \gamma}{\gamma + \delta} \zeta_\pm \partial_x \zeta_\pm \ = \ 0 \ . \qquad \text{(iB)}$$



Error in the Camassa-Holm regime

In the critical case $\delta^2 = \gamma$, if the initial data is not localized in space, (eKdV) is as precise as (CL) ;

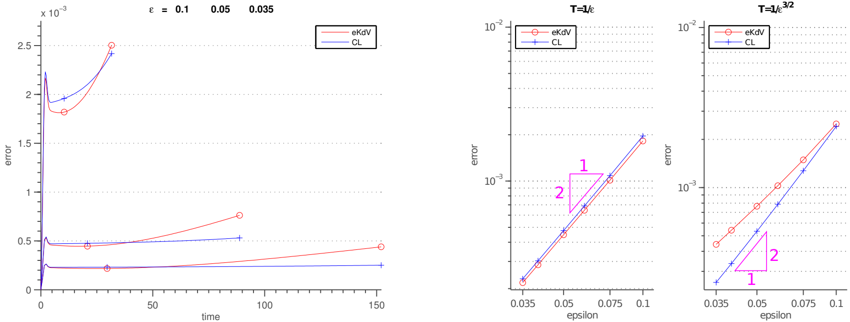
$$\partial_t \zeta_\pm \pm \partial_x \zeta_\pm \pm \epsilon^2 \alpha_2 \zeta_\pm^2 \partial_x \zeta_\pm \pm \mu \frac{1}{6} \frac{1+\gamma\delta}{\delta(\gamma+\delta)} \partial_x^3 \zeta_\pm = 0. \qquad \text{(eKdV)}$$



Error in the Camassa-Holm regime

In the critical case $\delta^2 = \gamma$, if the initial data is localized in space, then (CL) is the most precise decoupled model for very large times

$$\partial_t\zeta_\pm \pm \partial_x\zeta_\pm \pm \epsilon\frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta_\pm\partial_x\zeta_\pm \pm \mu\frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^3\zeta_\pm$$
$$\pm \epsilon^2\alpha_2\zeta_\pm^2\partial_x\zeta_\pm + \epsilon^3\alpha_3\zeta_\pm^3\partial_x\zeta_\pm \pm \mu\epsilon\partial_x\big(\kappa\zeta_\pm\partial_x^2\zeta_\pm + \iota(\partial_x\zeta_\pm)^2\big)$$
$$= 0. \quad \text{(CL)}$$



Thank you for your attention !