Trapping waves

Bifurcation of discrete eigenvalues from the edges of the continuous spectrum of the Schrödinger operator

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(joint work with M.I. Weinstein and I. Vukicevic)

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MOTIVATION

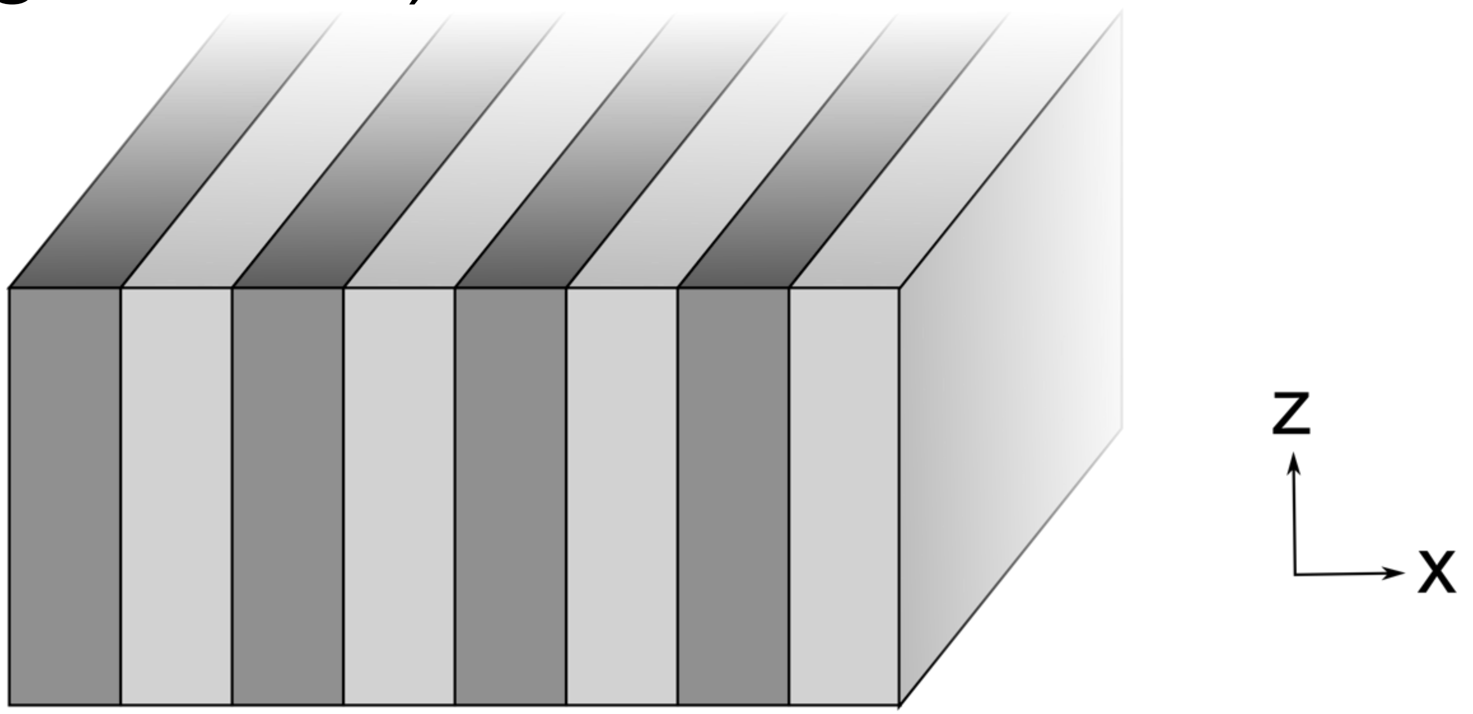
Maxwell's equations (no charge, current)

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$$

$$\nabla \times (\mathbf{B}/\mu) = -\varepsilon \partial_t \mathbf{E}$$

$$\varepsilon = \varepsilon(\boldsymbol{x}) \qquad \mu = \mu_0$$



Transverse electric mode (TE) $\mathbf{E} = (E_x, E_y, 0)$ $\mathbf{B} = (0, 0, B_z)$

Harmonic solutions $\mathbf{E} = \mathbf{U}(x, z)e^{i\omega t}$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \omega^2 \mu_0 \varepsilon \mathbf{E}$$

Scalar approximation $\rightsquigarrow$ Helmholtz equation

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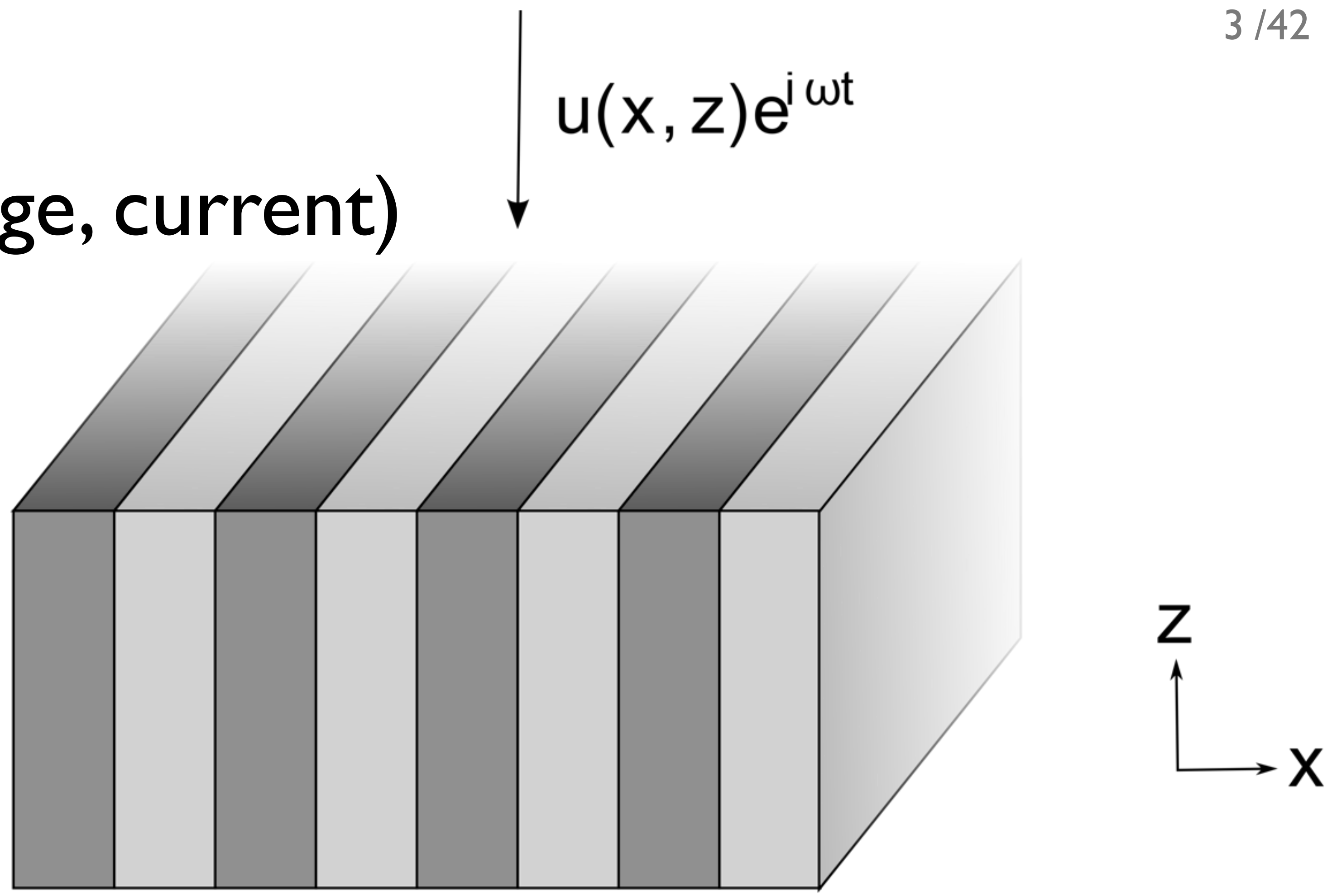

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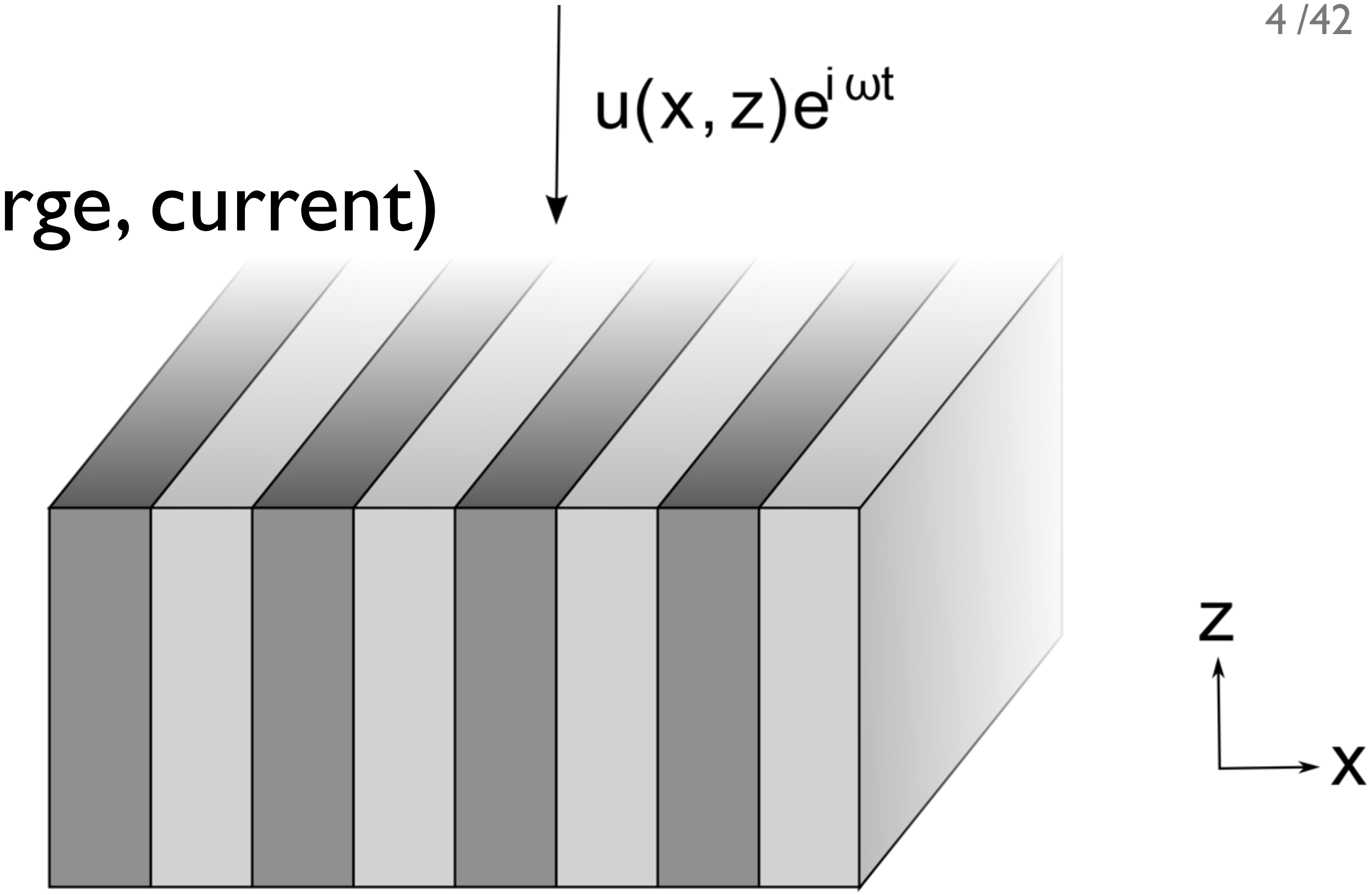

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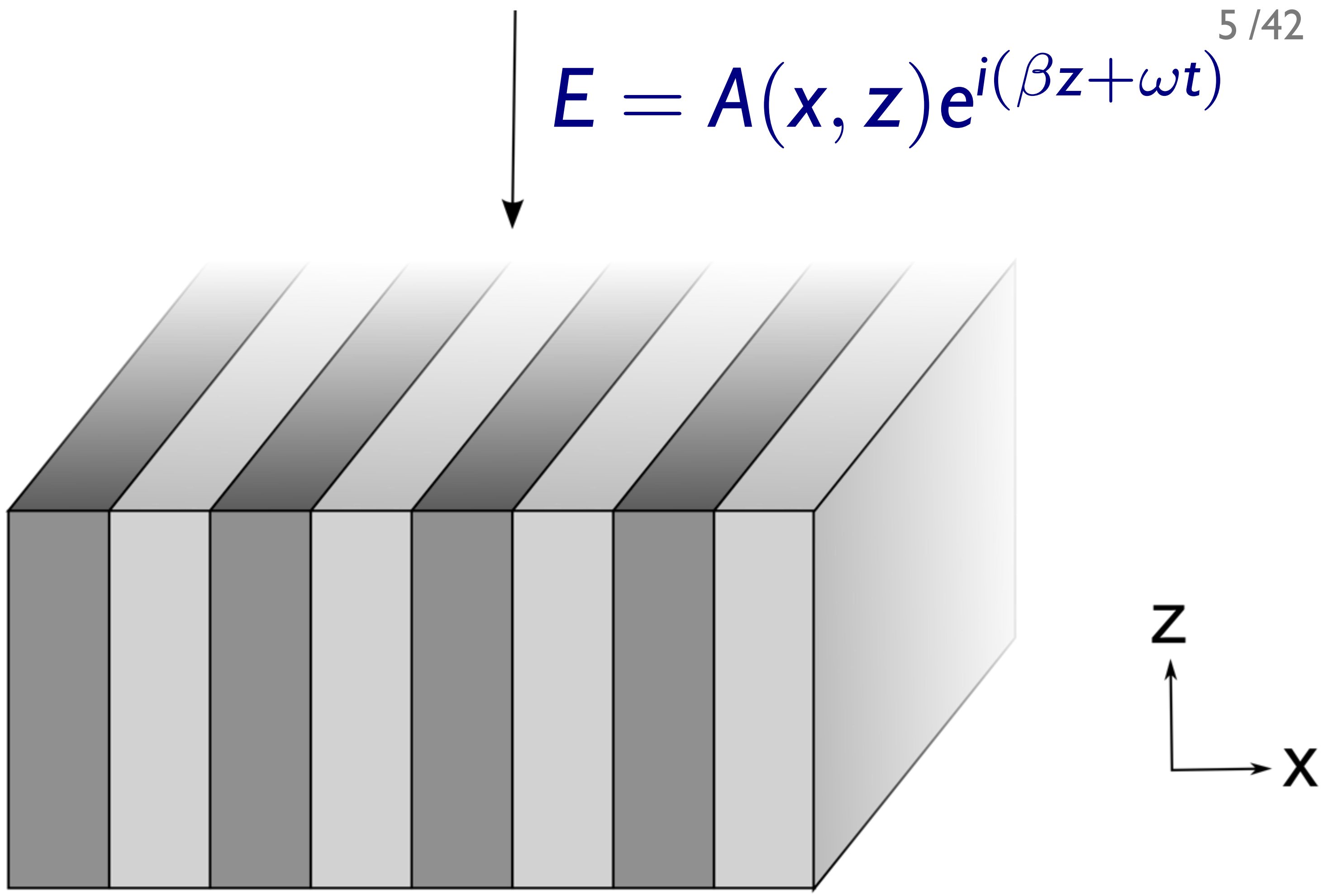

Paraxial wave solutions

$$E = A(x,z)e^{i(\beta z + \omega t)}$$

$$\partial_x^2 A + \partial_z^2 A + 2i\beta \partial_z A + (\omega^2 \mu_0 \varepsilon - \beta^2)A = 0$$

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We seek localised (in x) propagating waves, so discrete eigenvalues of the Schrödinger operator $H_V \equiv (-\partial_x^2 + V)$

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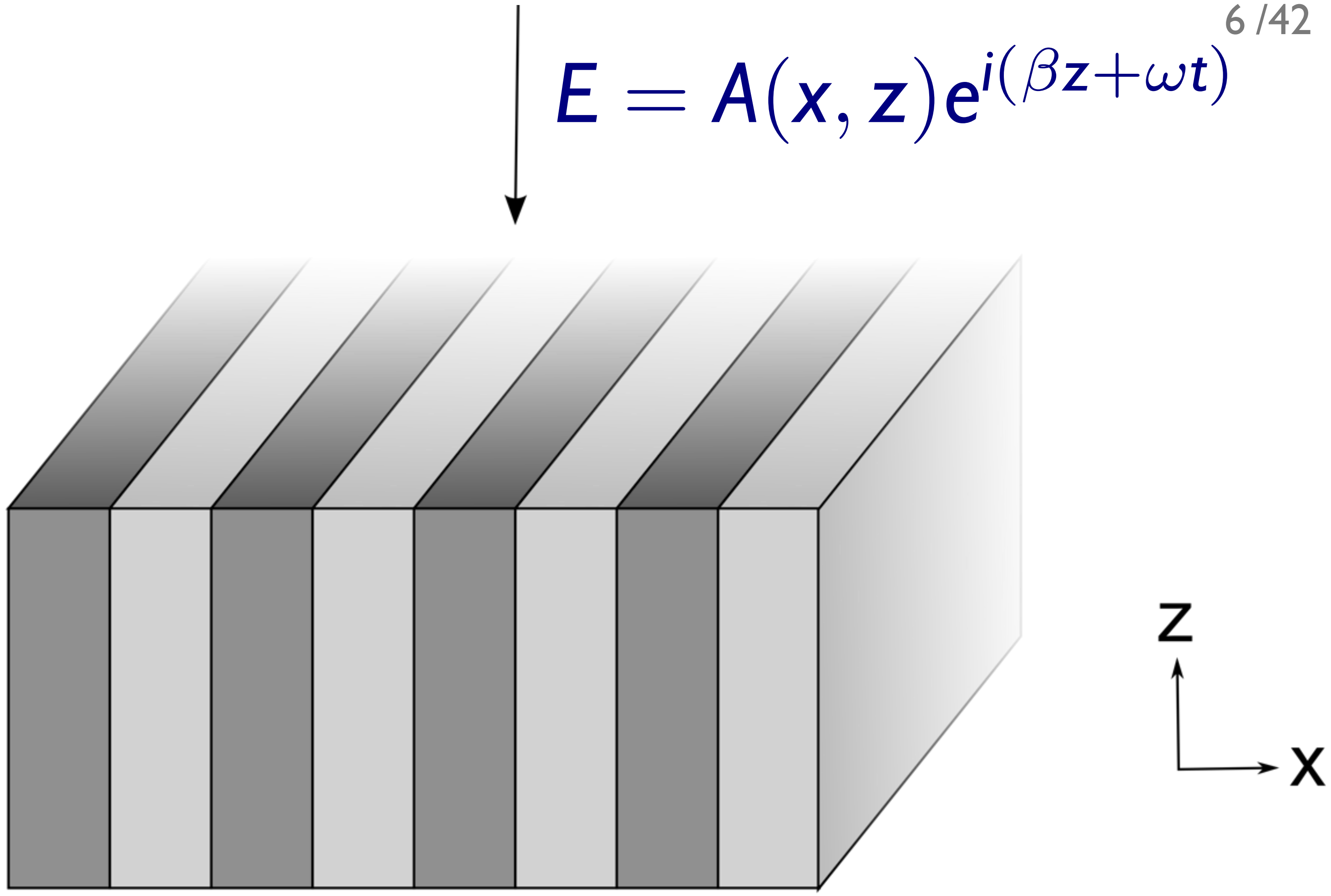

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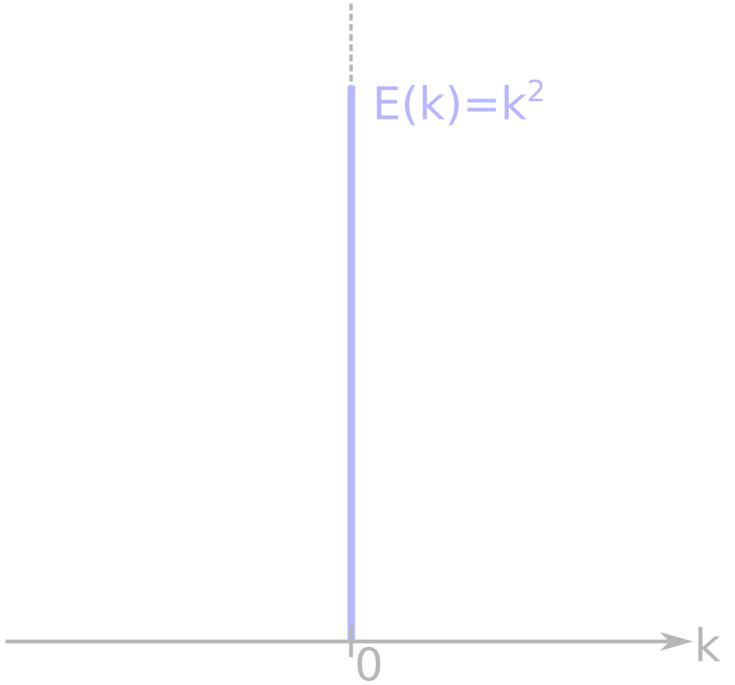
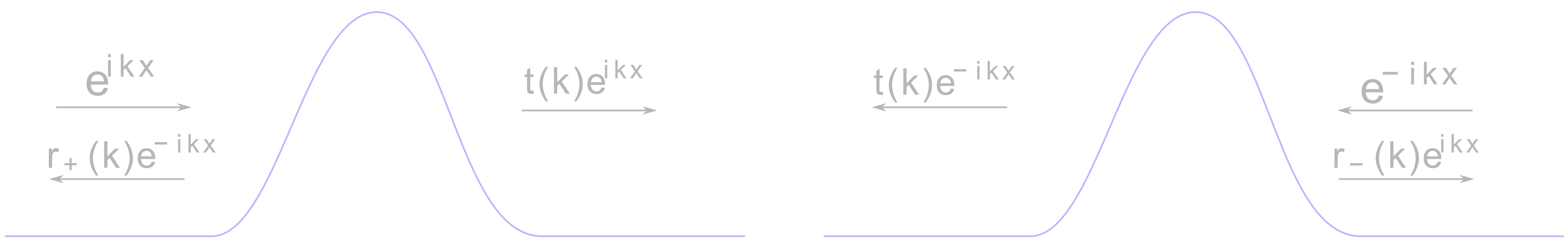
Scattering on the line

We study the eigenvalue problem

$$\left(-\frac{d^2}{dx^2} + V \right) \psi = E \psi$$





For any $k \in \mathbb{R}$, $|t(k)|^2 + |r_\pm(k)|^2 = 1$, and $E(k) = k^2 > 0$

If $|t(\kappa)| = \infty$ then $\kappa = i\theta, \theta > 0,\ E(\kappa) = -\theta^2 < 0$
is a discrete eigenvalue (bound state)

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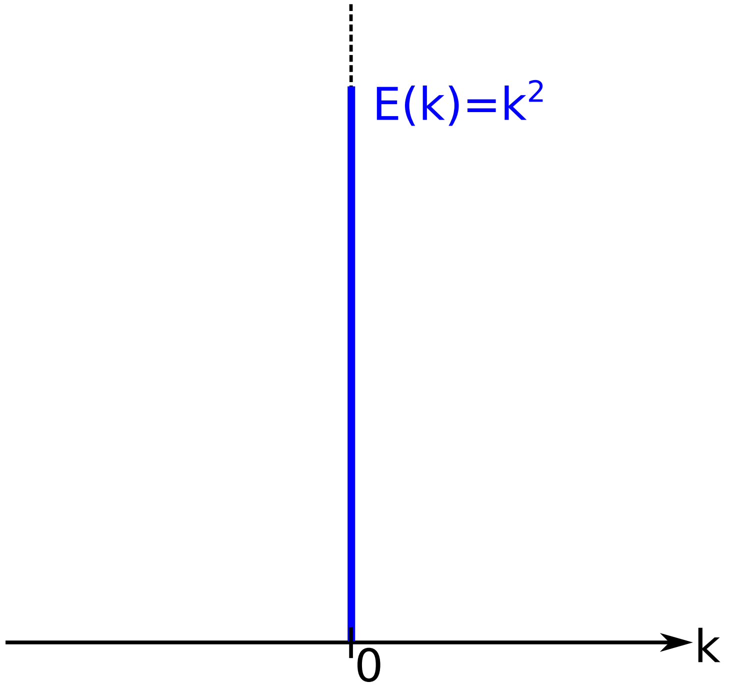

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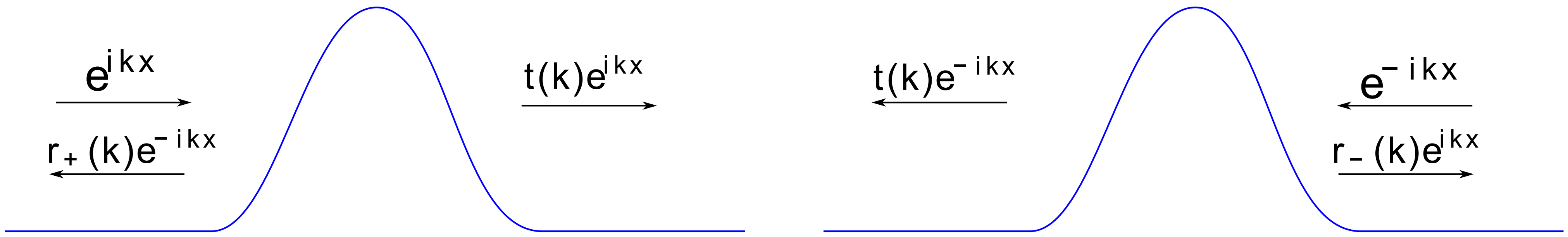

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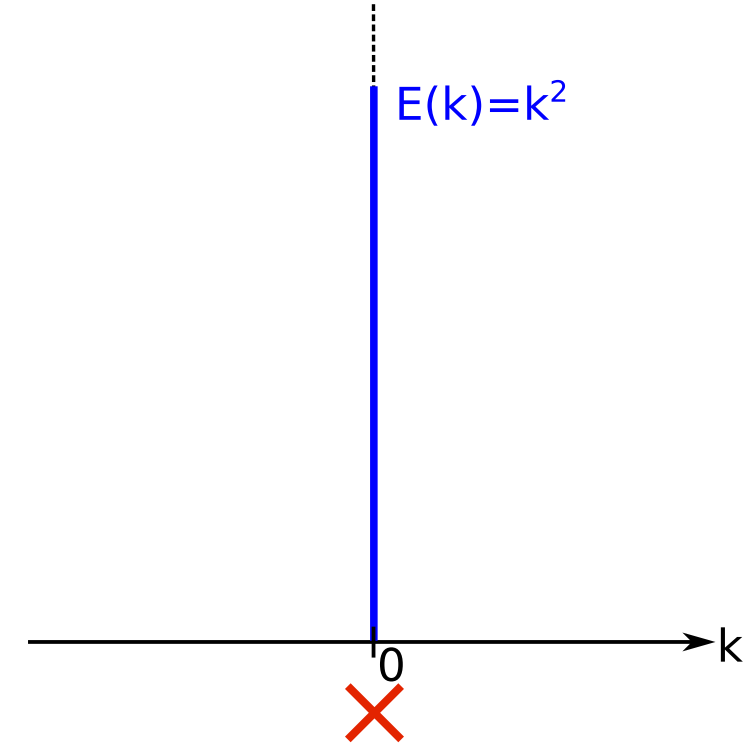

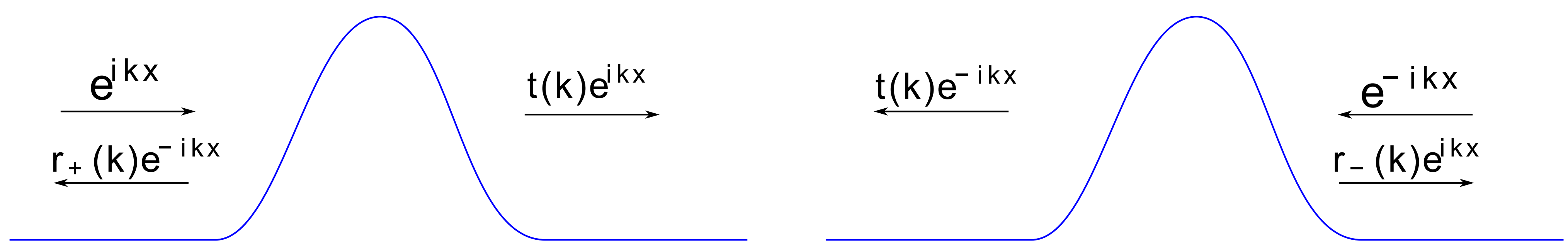

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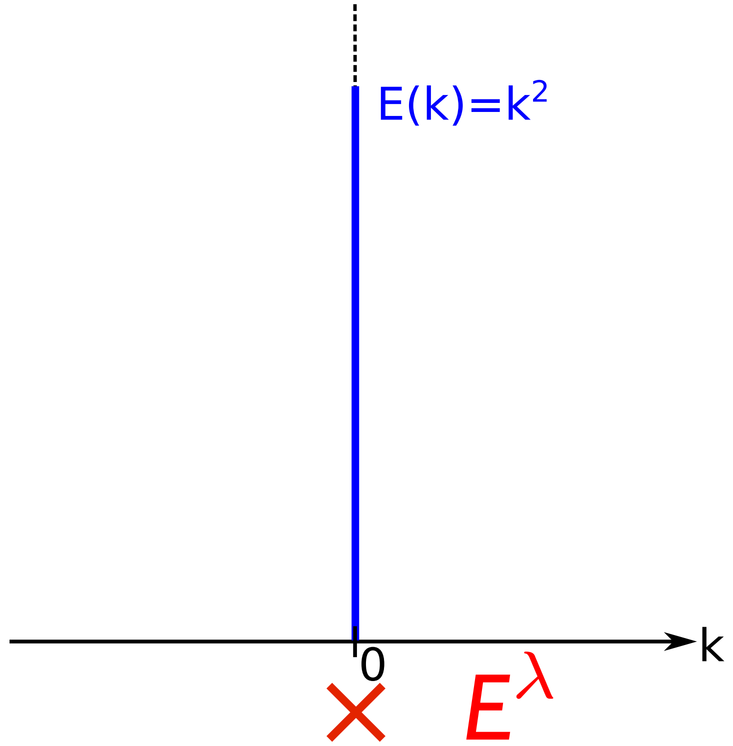
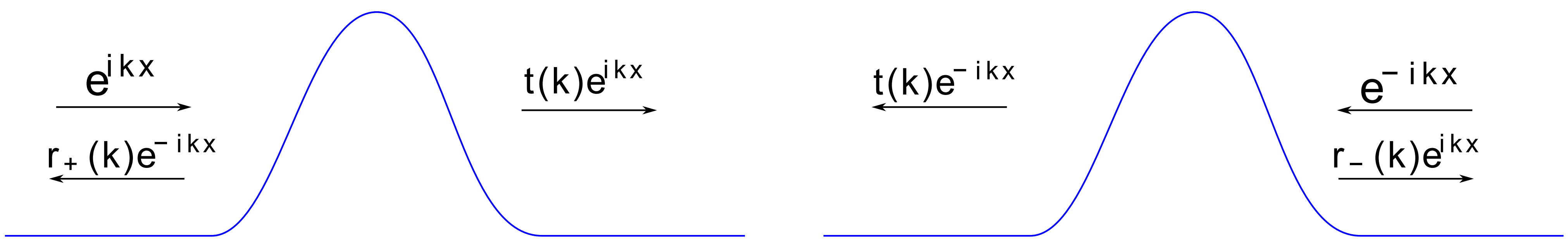
$$\left(-\frac{d^2}{dx^2} + \lambda V \right) \psi = E^{\lambda} \psi$$





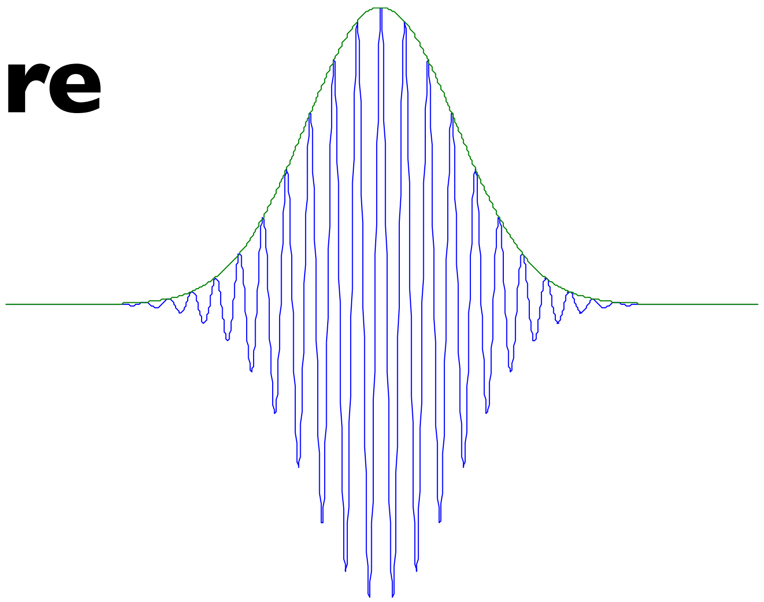
Th'm (Simon '76): Assume $(1+|x|^2)V \in L^1$ and $\int V \leq 0$

Then there exists a bound state $E^{\lambda} = -\lambda^2\theta^2$ and

$$\theta = -\frac{1}{2}\int V - \lambda\frac{1}{4}\iint V(x)|x-y|V(y) + o(\lambda)$$

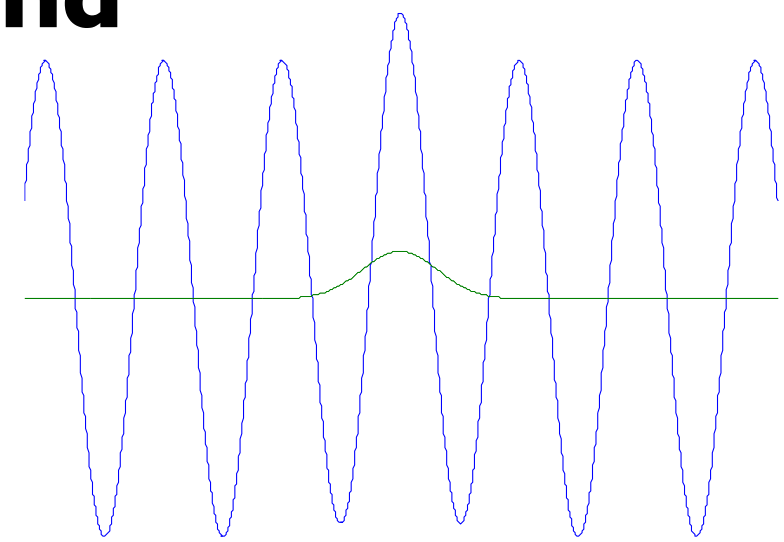
Outline of the talk

The case of a localized microstructure



- Homogenization
- The effective mass
- Consequences
- Sketch of the proof

The case of an oscillatory background



- Floquet-Bloch mode
- Introduction of a localized defect
- Sketch of the proof

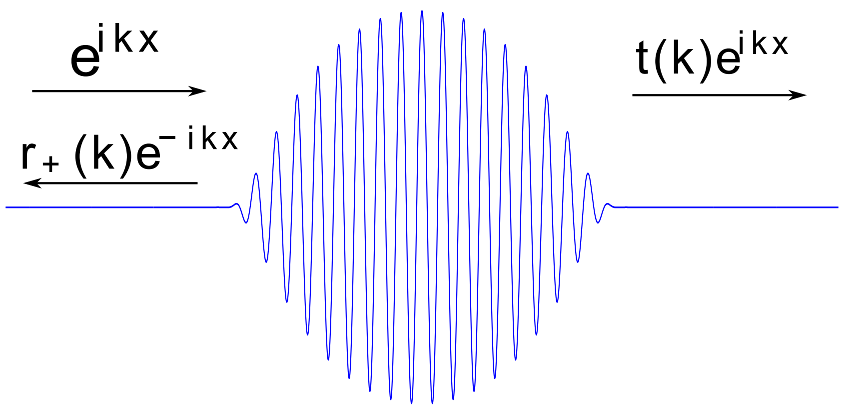
Perspectives

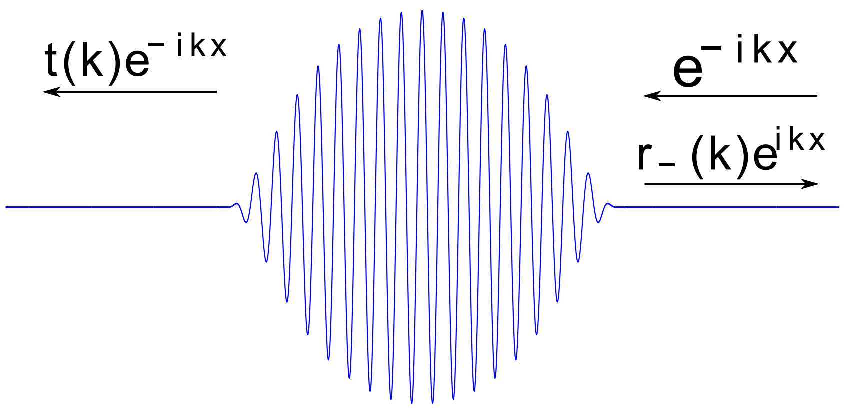
A localized microstructure

We study the eigenvalue problem

$$\left(-\frac{d^2}{dx^2} + q_\epsilon \right) \psi^\epsilon = E^\epsilon \psi^\epsilon \qquad \text{(E)}$$

where $q_\epsilon(x) = q(x, x/\epsilon)$, with $x \mapsto q(x,\cdot)$ localized and $y \mapsto q(\cdot, y)$ is 1-periodic





We seek the distorted plane waves of (E) under the form

$$e_+^{q_\epsilon}(x) \equiv F^\epsilon(x, x/\epsilon) \equiv F_0(x, x/\epsilon) + \epsilon F_1(x, x/\epsilon) + \epsilon^2 F_2(x, x/\epsilon) + \ldots$$

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Homogenization

We study the eigenvalue problem

$$\left(- \left(\frac{\partial}{\partial x} + \frac{1}{\epsilon} \frac{\partial}{\partial y} \right)^2 + q(x,y) - k^2 \right) F^\epsilon(x,y) = 0, \qquad \text{(E')}$$

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One obtains:

- $F_0(x,y) = F_0^{(h)}(x)$ with $(-\frac{d^2}{dx^2} + q_{av} - k^2) F_0^{(h)} = 0$
  $q_{av}(x) \equiv \int_0^1 q(x,y)\, dy$
- $F_1(x,y) \equiv 0$
- $F_2(x,y) = F_2^{(h)}(x) + F_2^{(p)}(x,y)$

$$t^{q_\epsilon}(k) = t_{av}(k) + \epsilon^2 t_2(k) + \mathcal{O}(\epsilon^3)$$

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A non-uniform expansion

The potential $V = 0$ is an *exceptional* potential! $t^0(k) \equiv 1$

Generically, $t^V(k) \to 0 \; (k \to 0)$

Thus if $\int_0^1 q(\cdot, y)\, dy = 0$, the expansion is not uniform.

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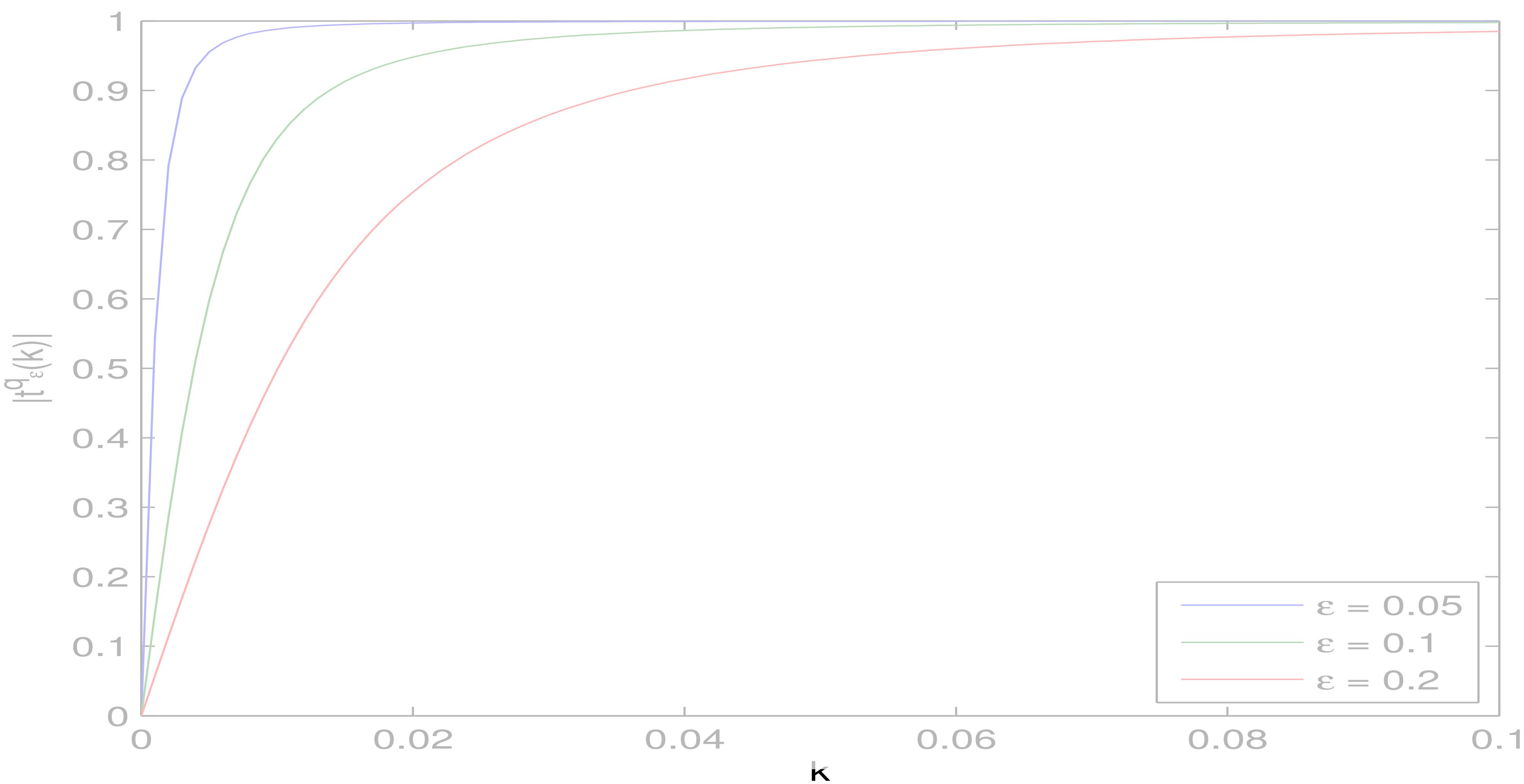

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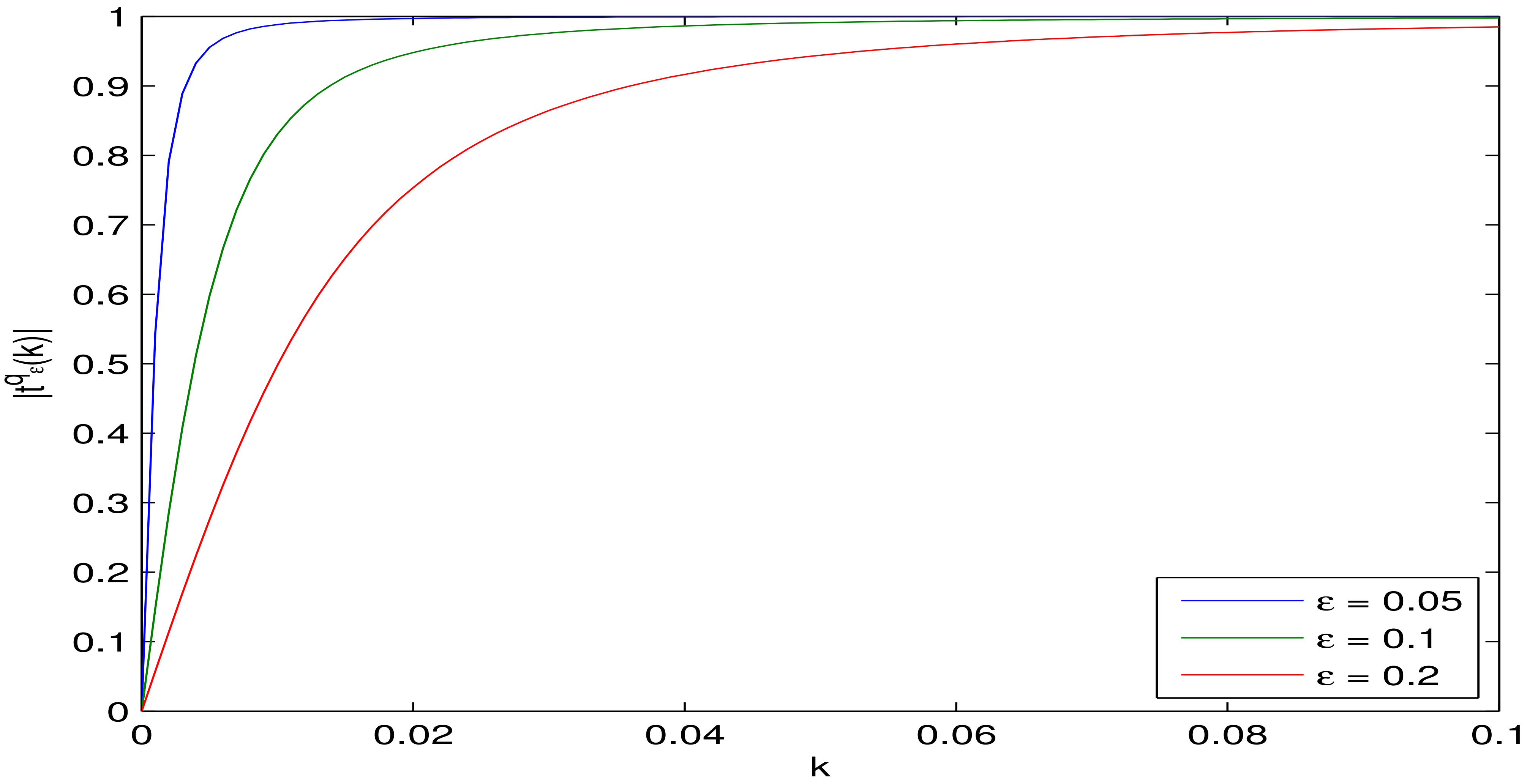

Main result

Th'm (VD, M.I. Weinstein, I. Vukicevic '12)
Let $q_\epsilon(x) = \sum_j q_j(x) e^{2i\pi j \frac{x}{\epsilon}}$ be smooth, exponentially decaying. Then there exists $\epsilon_0 > 0$ and K a complex neighborhood of 0 s.t.

$$\forall (\epsilon, k) \in [0, \epsilon_0) \times K \qquad \left| \frac{k}{t^{\sigma_{\rm eff}}(k)} - \frac{k}{t^{q_\epsilon}(k)} \right| \leq \epsilon^3 \, C(K, \|V\|).$$

where $\sigma_{\rm eff}(x) \equiv -\epsilon^2 \, \Lambda_{\rm eff}(x) \equiv -\frac{\epsilon^2}{(2\pi)^2} \sum_{j \neq 0} \frac{|q_j(x)|^2}{j^2}$

Corollaries. Assume moreover that $q_{\rm av}(x) = q_0(x) = 0$

i) Uniformly in $k \in \mathbb{R}$ $\quad \sup_{k \in \mathbb{R}} |t^{\sigma_{\rm eff}}(k) - t^{q_\epsilon}(k)| = \mathcal{O}(\epsilon)$.

ii) Scaled limit $k = \epsilon^2 \kappa$, $\kappa \neq i \frac{\int \Lambda_{\rm eff}}{2}$ $\quad \lim_{\epsilon \to 0} t^{q_\epsilon}(\epsilon^2 \kappa) = \frac{\kappa}{\kappa - i \frac{\int \Lambda_{\rm eff}}{2}}$.

iii) Bound state $E^\epsilon = k_\epsilon^2$ $\quad k_\epsilon = i \frac{\epsilon^2}{2} \int \Lambda_{\rm eff} + \mathcal{O}(\epsilon^3)$.

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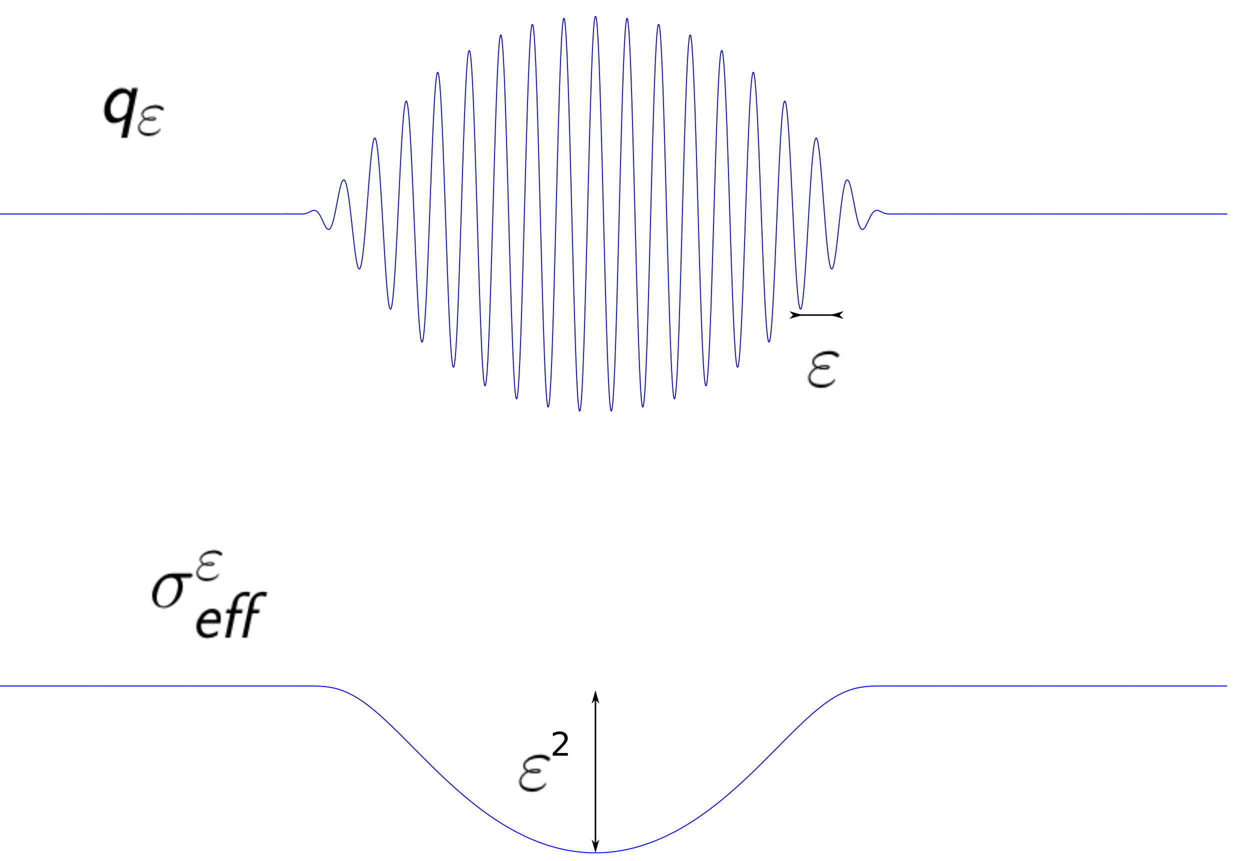

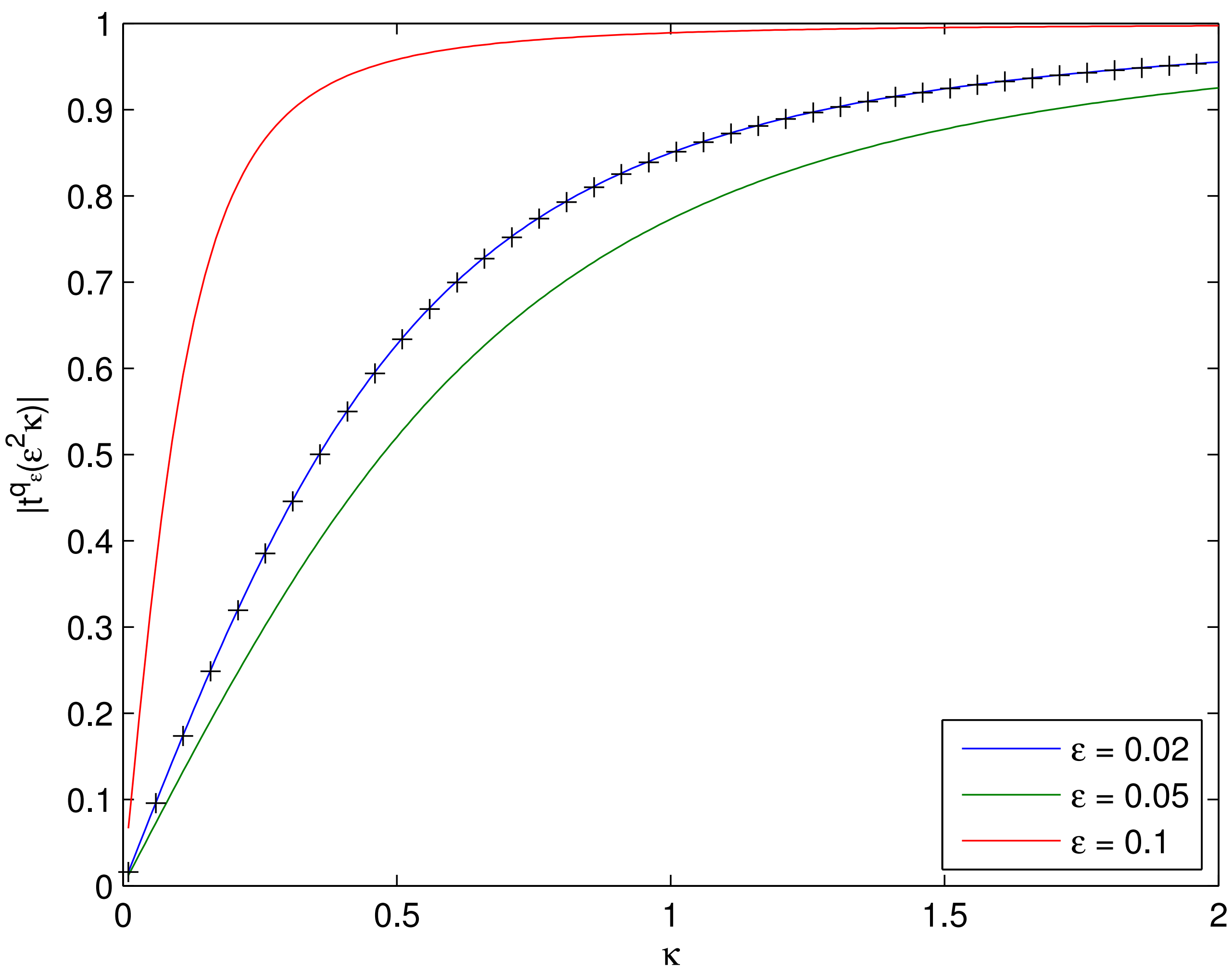

Summary

The eigenvalue problem

$$\left(-\frac{d^2}{dx^2} + q_\epsilon \right) \psi^\epsilon = E^\epsilon \psi^\epsilon$$

with $q_\epsilon(x) = \sum_{j\neq 0} q_j(x) e^{2i\pi j \frac{x}{\epsilon}}$ is approximated by

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$$\left(-\frac{d^2}{dx^2} - \epsilon^2 \Lambda_{\mathrm{eff}} \right) \psi^\epsilon = E^\epsilon_{\mathrm{eff}} \psi^\epsilon$$

with $\Lambda_{\mathrm{eff}}(x) \equiv \frac{1}{(2\pi)^2} \sum_{j\neq 0} \frac{|q_j(x)|^2}{j^2}$ is approximated by

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$$\left(-\frac{d^2}{dx^2} - \epsilon^2 \int \Lambda_{\mathrm{eff}} \times \delta \right) \psi^\epsilon = E^\epsilon_\delta \psi^\epsilon$$

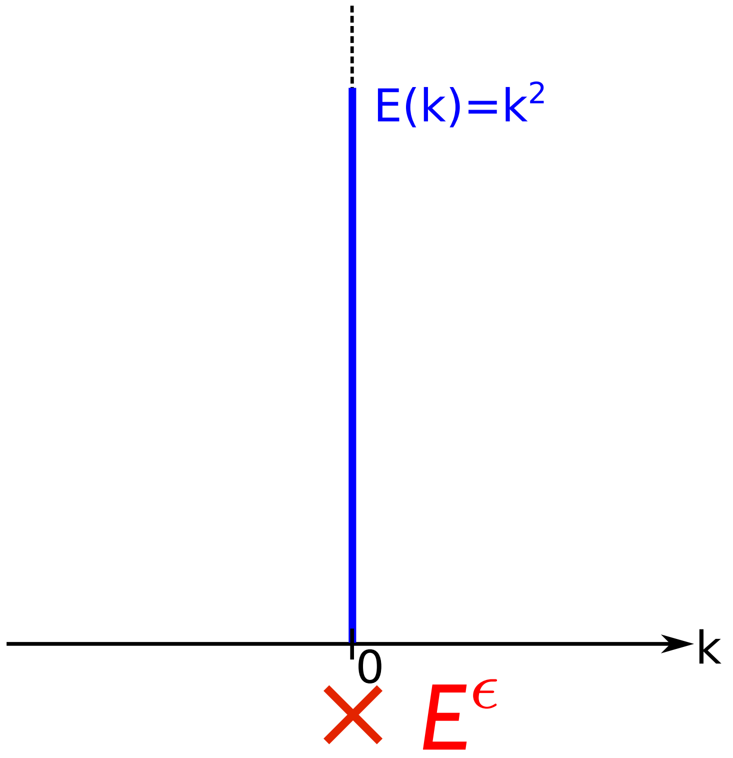
Summary

If $\epsilon > 0$ is sufficiently small

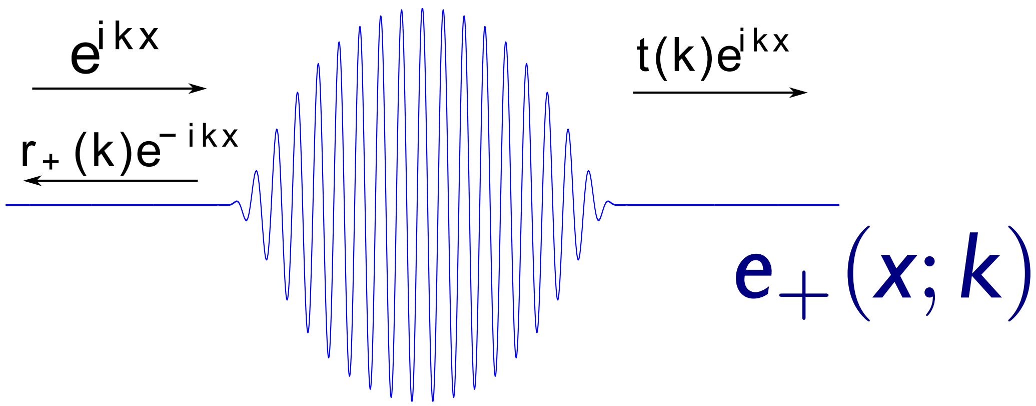
$$\left(-\frac{d^2}{dx^2} + q_\epsilon \right) \psi^\epsilon = E^\epsilon \psi^\epsilon$$

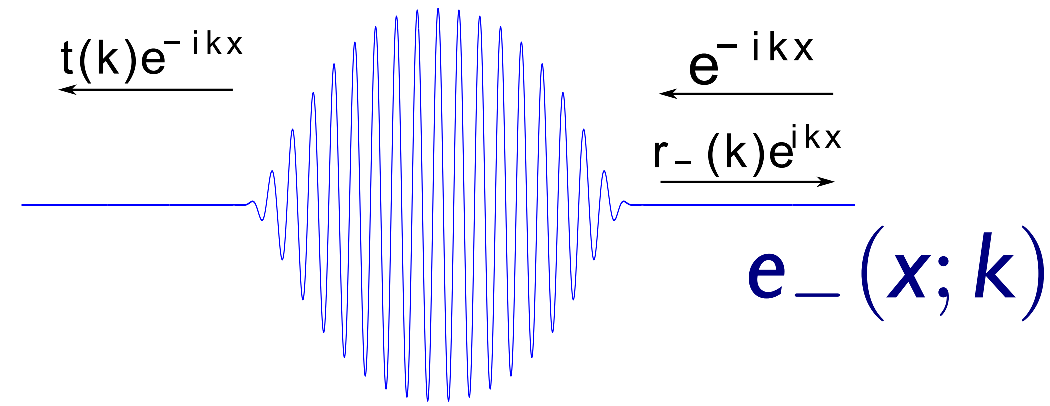
with $q_\epsilon(x) = \sum_{j \neq 0} q_j(x) e^{2i\pi j \frac{x}{\epsilon}}$ has a unique eigenvalue:

$$E^\epsilon = k_\epsilon^2 \qquad\qquad k_\epsilon = i\frac{\epsilon^2}{2}\int \Lambda_{\text{eff}} + \mathcal{O}(\epsilon^3).$$



Sketch of the proof





Define the Jost solutions $f_\pm(x;k) \; = \; \frac{1}{t(k)} e_\pm(x;k)$

The Jost solutions are the unique solutions to Volterra equation

$$f_+^V(x;k) \; = \; e^{ikx} \; + \; \int_x^\infty \frac{e^{ik(y-x)} - e^{ik(x-y)}}{2ik} V(y) f_+^V(y) dy.$$

One has

$$\frac{k}{t^V(k)} \; = \; k - \frac{1}{2i} I^V(k), \qquad I^V(k) \; \equiv \; \int_{-\infty}^{\infty} V(x) e^{-ikx} f_+^V(x;k) dx.$$

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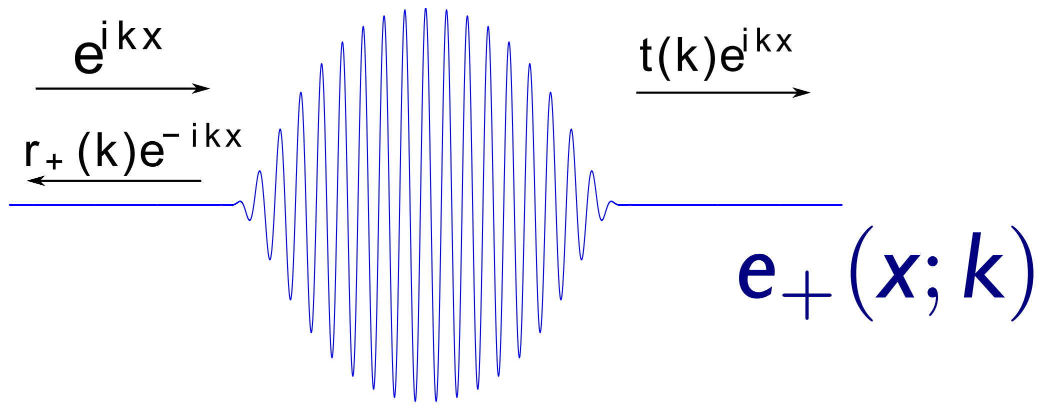

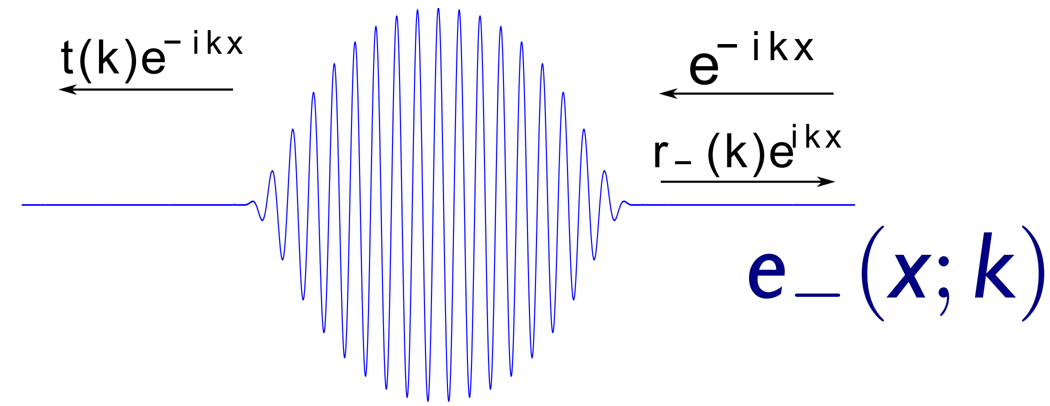

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More generally,

$$\frac{k}{t^V(k)} = \frac{k}{t^W(k)} - \frac{1}{2i} I^{[V,W]}(k),$$

$$I^{[V,W]}(k) \equiv \int f_-^W(\cdot;k)(V-W) f_+^V(\cdot;k)$$

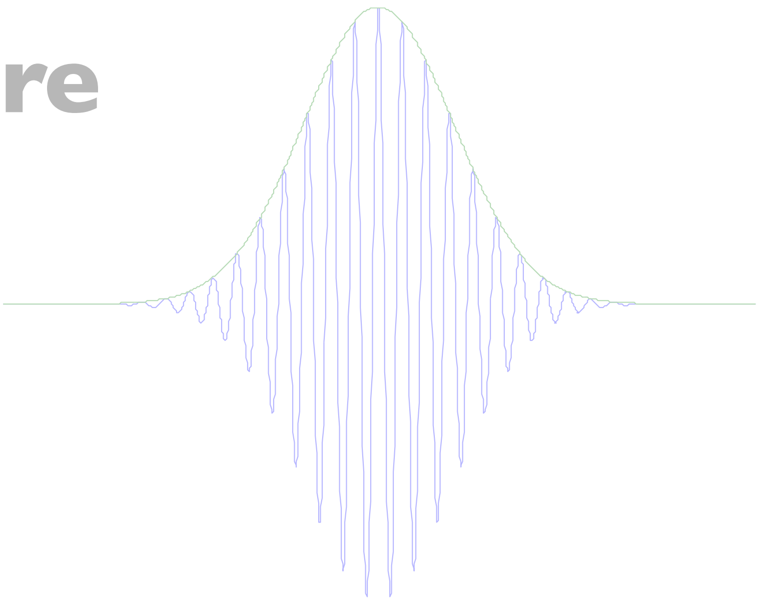
Integration by part, with well-chosen potentials yields

$$\forall (\epsilon, k) \in [0, \epsilon_0) \times K \qquad \left| \frac{k}{t^{\sigma_{\text{eff}}}(k)} - \frac{k}{t^{q_\epsilon}(k)} \right| \leq \epsilon^3 \, C(K, \|V\|).$$

Outline of the talk

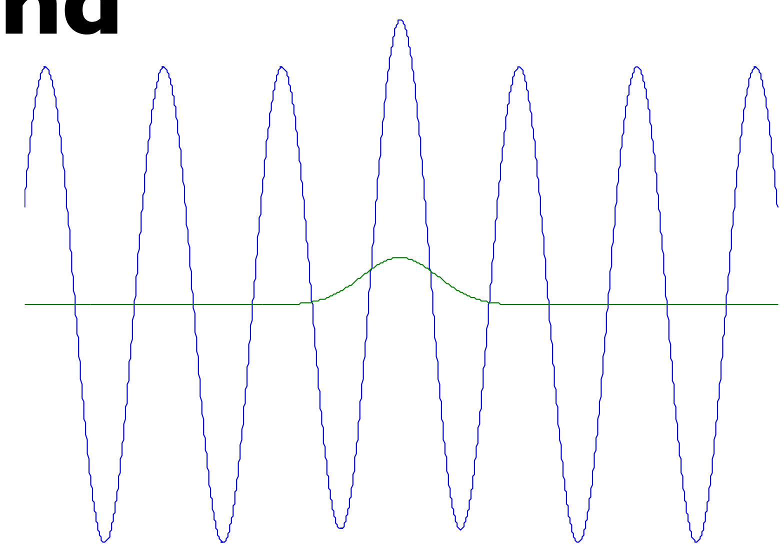
The case of a localized microstructure

- Homogenization
- The effective mass
- Consequences
- Sketch of the proof



The case of an oscillatory background

- Floquet-Bloch mode
- Introduction of a localized defect
- Sketch of the proof



Perspectives

Floquet-Bloch states

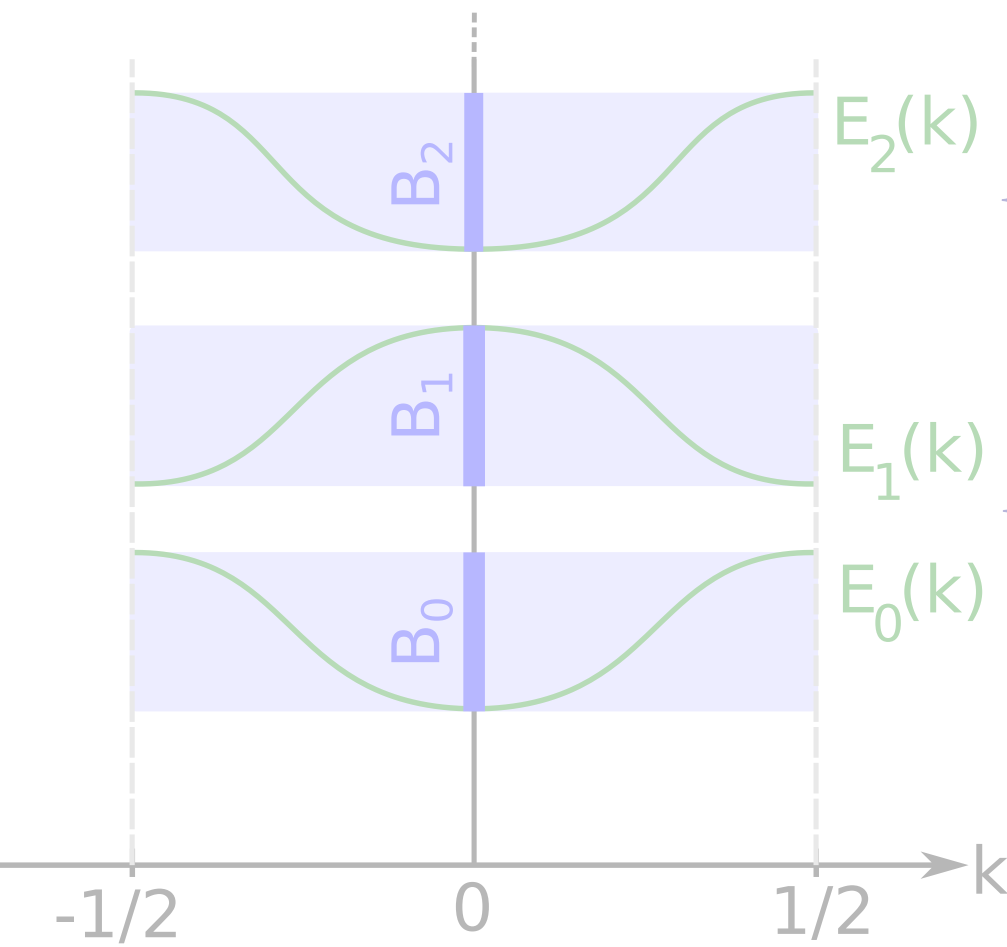
Pseudo-periodic eigenvalue problem $Q(x+1) = Q(x)$

$$\left(-\frac{d^2}{dx^2} + Q \right) \psi = E(k)\, \psi \qquad \psi(x+1;k) = e^{2\pi i\, k}\psi(x;k)$$

Or, equivalently, $\psi(x;k) \equiv e^{2\pi i\, kx} p_b(x;k)$

$$\left(-\left(\frac{d}{dx} + 2\pi i\, k\right)^2 + Q \right) p = E(k)\, p \qquad p(x+1;k) = p(x;k)$$

For any $k \in (-\frac{1}{2}, \frac{1}{2}]$ there exists $E_0(k),\ E_1(k), \ldots$



$\{p_b(x;k)\}_{b\in\mathbb{N}}$ orthonormal set in $L^2_{\mathrm{per}}([0,1])$

$\{u_b(x;k) \equiv e^{2\pi i\, kx} p_b(x;k)\}_{b\in\mathbb{N}}$

Floquet-Bloch states,

complete in $L^2(\mathbb{R})$

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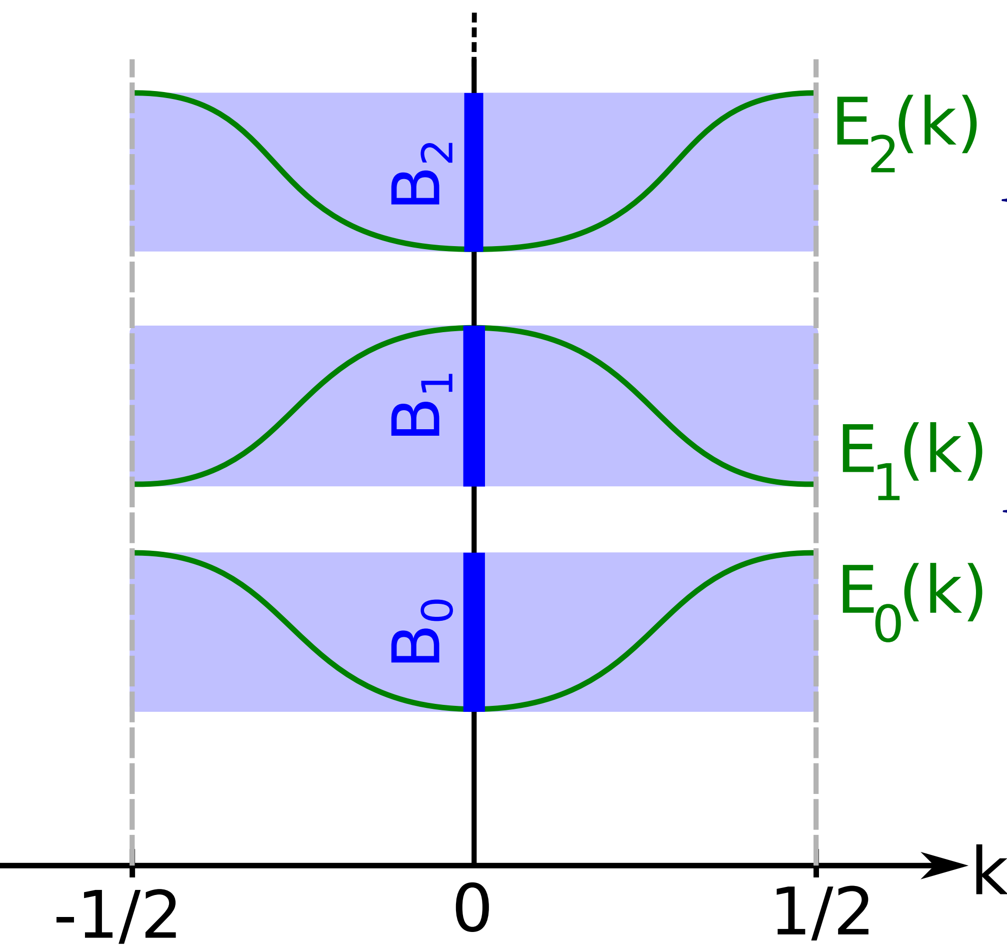

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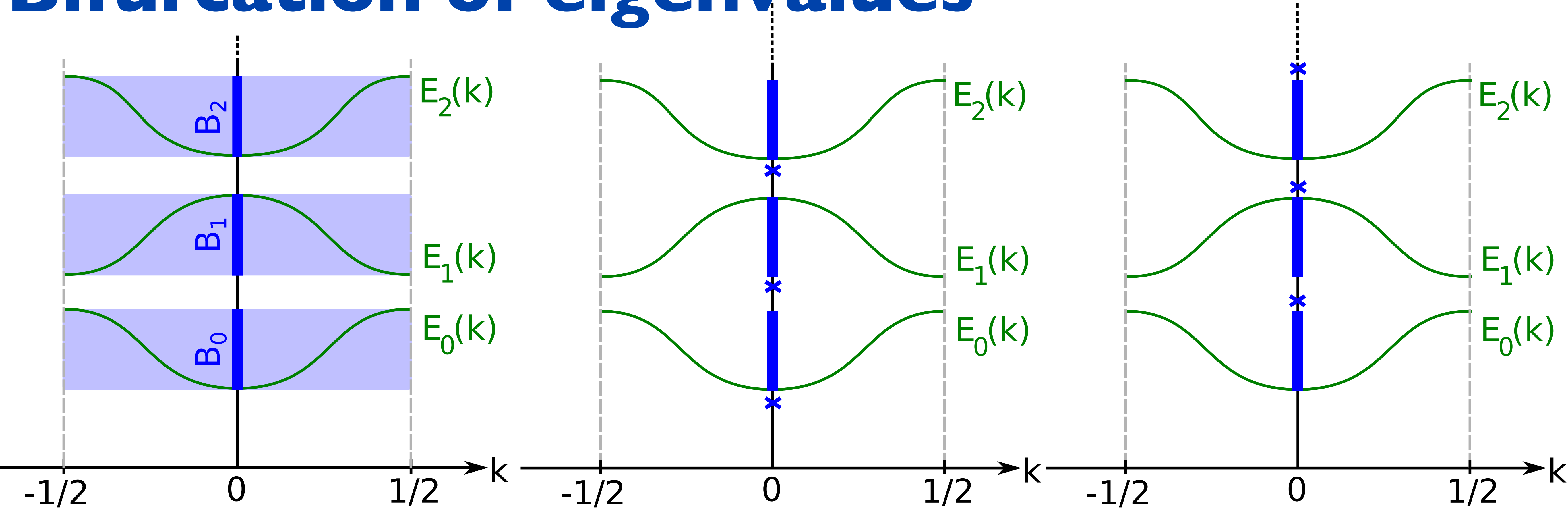
Bifurcation of eigenvalues



Th'm (VD, M.I. Weinstein, I. Vukicevic '12)

Let $E_* = E_b(k_*)$ be a spectral band edge, and assume

$$\partial_k^2 E_{b_*}(k_*) > 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |u_{b_*}(x; k_*)|^2 V(x)\, dx < 0$$

$$\text{or} \quad \partial_k^2 E_{b_*}(k_*) < 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |u_{b_*}(x; k_*)|^2 V(x)\, dx > 0$$

Then there exists an eigenpair $(E^\lambda, \psi^\lambda)$ of the problem

$$\left(-\frac{d^2}{dx^2} + Q + \lambda V \right) \psi^\lambda = E^\lambda\, \psi^\lambda$$

The effective problem

Th'm (VD, M.I. Weinstein, I. Vukicevic '12)

Let $Q \in L^\infty$ 1 -periodic and $V \in L^\infty$, $(1+|\cdot|)V \in L^1$

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Then there exists an eigenpair $(E^\lambda, \psi^\lambda)$ of the problem

$$\left(-\frac{d^2}{dx^2} + Q + \lambda V \right) \psi^\lambda = E^\lambda \psi^\lambda$$

Moreover, one has

$$E^\lambda - E_* = \lambda^2 E_0 + \mathcal{O}(\lambda^3) \quad \text{and} \quad \psi^\lambda \approx u_{b_*}(x;k_*) e^{-\lambda \alpha_0 |x|}$$

where $(E_0, \psi_0 \equiv e^{-\alpha_0|x|})$ is the solution of the effective problem

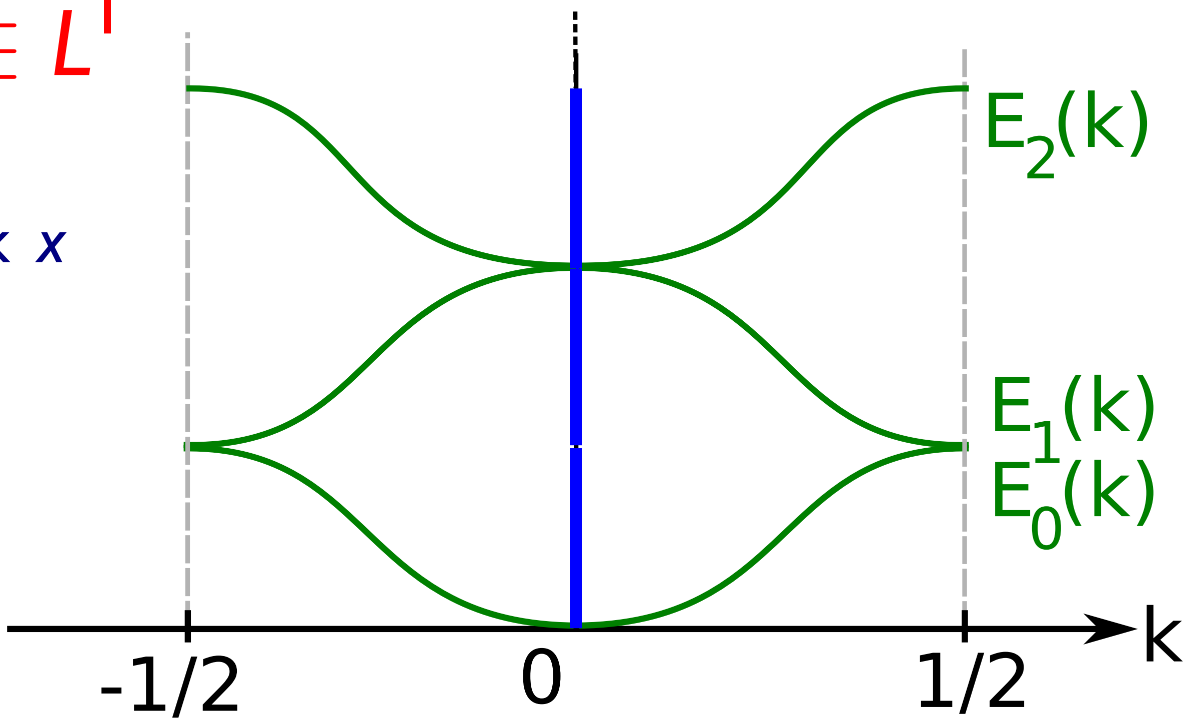
$$\left(-\frac{\partial_k E_{b_*}(k_*)}{8\pi^2} \frac{d^2}{dy^2} + \delta(y) \times \int_{-\infty}^{\infty} |u_{b_*}(x;k_*)|^2 V(x)\, dx \right) \psi_0 = E_0 \psi_0$$

The Case Q=0

Th'm (VD, M.I. Weinstein, I. Vukicevic '12)

If $Q = 0$ and $V \in L^\infty$, $(1 + |\cdot|)V \in L^1$

$E_0(k) = 4\pi^2 k^2$ and $u_b(x;k) = e^{2i\pi\ k\ x}$



Then there exists an eigenpair $(E^\lambda, \psi^\lambda)$ of the problem

$$\left(-\frac{d^2}{dx^2} + \lambda V \right) \psi^\lambda = E^\lambda \psi^\lambda$$

Moreover, one has $E = \lambda^2 E_0 + \mathcal{O}(\lambda^3)$ and $\psi^\lambda \approx e^{-\lambda \alpha_0 |x|}$

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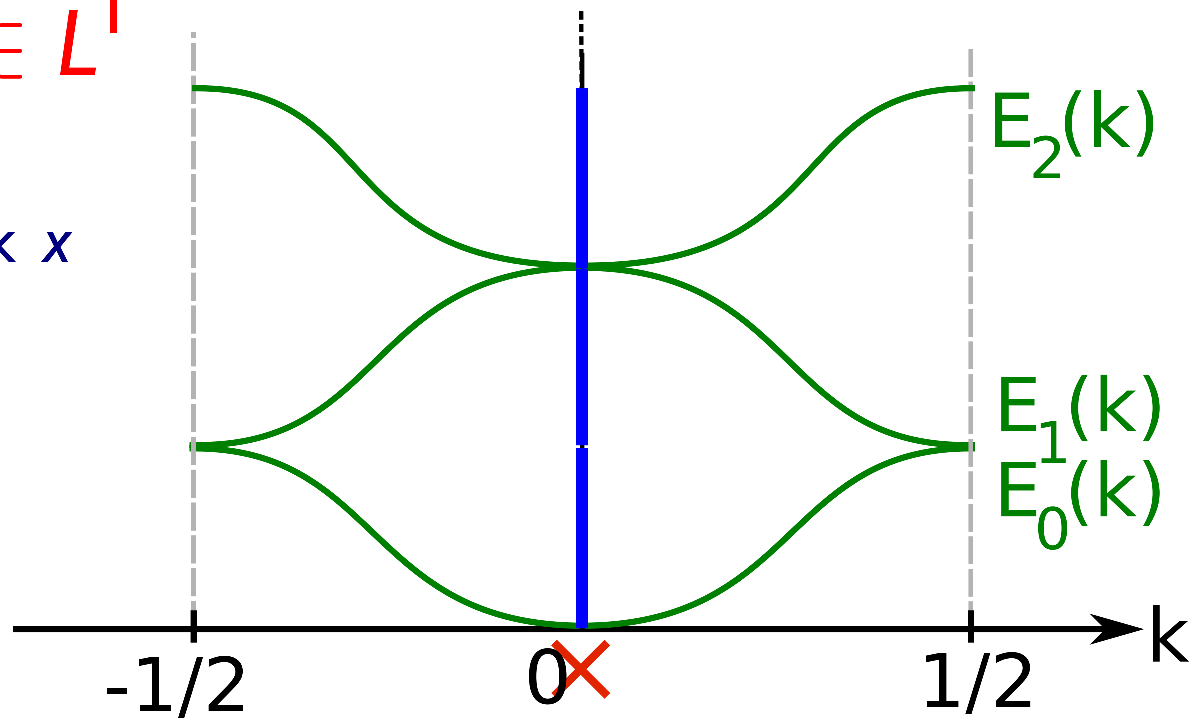

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Sketch of the proof (Q=0)

We study the eigenvalue problem $\left(-\frac{d^2}{dx^2} + \lambda V \right) \psi^\lambda = E^\lambda \psi^\lambda$

Equivalently, $\left(-4\pi^2\xi^2 - E^\lambda \right) \widehat{\psi}^\lambda(\xi) + \lambda \left(\widehat{V} \star \widehat{\psi}^\lambda \right)(\xi) = 0$

Near and far frequency decomposition: $1 = \chi(|\xi| < \lambda^r) + \chi(|\xi| > \lambda^r)$

$$\widehat{\psi}^\lambda \equiv \chi(|\xi| > \lambda^r)\widehat{\psi}^\lambda + \chi(|\xi| < \lambda^r)\widehat{\psi}^\lambda \equiv \widehat{\psi}_{\mathrm{far}} + \widehat{\psi}_{\mathrm{near}}$$

$$(4\pi^2\xi^2 - E^\lambda)\widehat{\psi}_{\mathrm{far}}(\xi) + \lambda\chi(|\xi| > \lambda^r) \int_{-\infty}^{\infty} \widehat{V}(\xi - \zeta)(\widehat{\psi}_{\mathrm{far}} + \widehat{\psi}_{\mathrm{near}})(\zeta) = 0$$

$$(4\pi^2\xi^2 - E^\lambda)\widehat{\psi}_{\mathrm{near}}(\xi) + \lambda\chi(|\xi| < \lambda^r) \int_{-\infty}^{\infty} \widehat{V}(\xi - \zeta)(\widehat{\psi}_{\mathrm{far}} + \widehat{\psi}_{\mathrm{near}})(\zeta) = 0$$

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Sketch of the proof (Q=0)

Far frequency equation $\widehat{\psi}_{\rm far} \equiv \chi(\xi > \lambda^r)\widehat{\psi}_{\rm far}$

$$(4\pi^2\xi^2 - E^\lambda)\widehat{\psi}_{\rm far}(\xi) \; + \; \lambda\chi(|\xi| > \lambda^r)\int_{-\infty}^{\infty}\widehat{V}(\xi-\zeta)(\widehat{\psi}_{\rm far}+\widehat{\psi}_{\rm near})(\zeta) \; = \; 0$$

$$\widehat{\psi}_{\rm far}(\xi) \; + \; \lambda\mathcal{T}^\lambda\widehat{\psi}_{\rm far} \; + \; \lambda\mathcal{T}^\lambda\widehat{\psi}_{\rm near} \; = \; 0$$

So $\widehat{\psi}_{\rm far}(\xi) \; = \; -\lambda\left(I+\lambda\mathcal{T}^\lambda\right)^{-1}\mathcal{T}^\lambda\widehat{\psi}_{\rm near}$ is uniquely determined.

Near frequency equation

$$(4\pi^2\xi^2 - E^\lambda)\widehat{\psi}_{\rm near}(\xi) \; + \; \lambda\chi(|\xi| < \lambda^r)\int_{-\infty}^{\infty}\widehat{V}(\xi-\zeta)(\widehat{\psi}_{\rm far}+\widehat{\psi}_{\rm near})(\zeta) \; = \; 0$$

is a closed equation on $\widehat{\psi}_{\rm near} \equiv \chi(\xi < \lambda^r)\widehat{\psi}_{\rm near}$

Sketch of the proof (Q=0)

Far frequency equation $\widehat{\psi}_{\text{far}} \equiv \chi(\xi > \lambda^r)\widehat{\psi}_{\text{far}}$

$$(4\pi^2\xi^2 - E^\lambda)\widehat{\psi}_{\text{far}}(\xi) \;+\; \lambda\chi(|\xi| > \lambda^r)\int_{-\infty}^{\infty}\widehat{V}(\xi-\zeta)(\widehat{\psi}_{\text{far}}+\widehat{\psi}_{\text{near}})(\zeta) \;=\; 0$$

$$\widehat{\psi}_{\text{far}}(\xi) \;+\; \lambda\mathcal{T}^\lambda\widehat{\psi}_{\text{far}} \;+\; \lambda\mathcal{T}^\lambda\widehat{\psi}_{\text{near}} \;=\; 0$$

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Sketch of the proof (Q=0)

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After a rescaling: $E^\lambda = -\lambda^2\theta^2$ $\quad \xi = \lambda\xi'$ $\quad \widehat{\psi}_{\rm near} = \frac{1}{\lambda}\widehat{\phi}^\lambda(\frac{\xi}{\lambda})$

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with $R = \mathcal{O}(\lambda^r, \lambda^{1-r})$

Perturbation of $(4\pi^2\xi'^2 + \theta^2)\widehat{\phi}^\lambda(\xi') \; + \; \widehat{V}(0)\int_{-\infty}^{\infty}\widehat{\phi}^\lambda \; = \; 0$

$$\left(-\frac{d^2}{dy^2} + \theta^2\right)\phi^\lambda(y) \; + \; \widehat{V}(0) \times \delta(y)\phi^\lambda \; = \; 0$$

Sketch of the proof (Q=0)

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Near frequency equation

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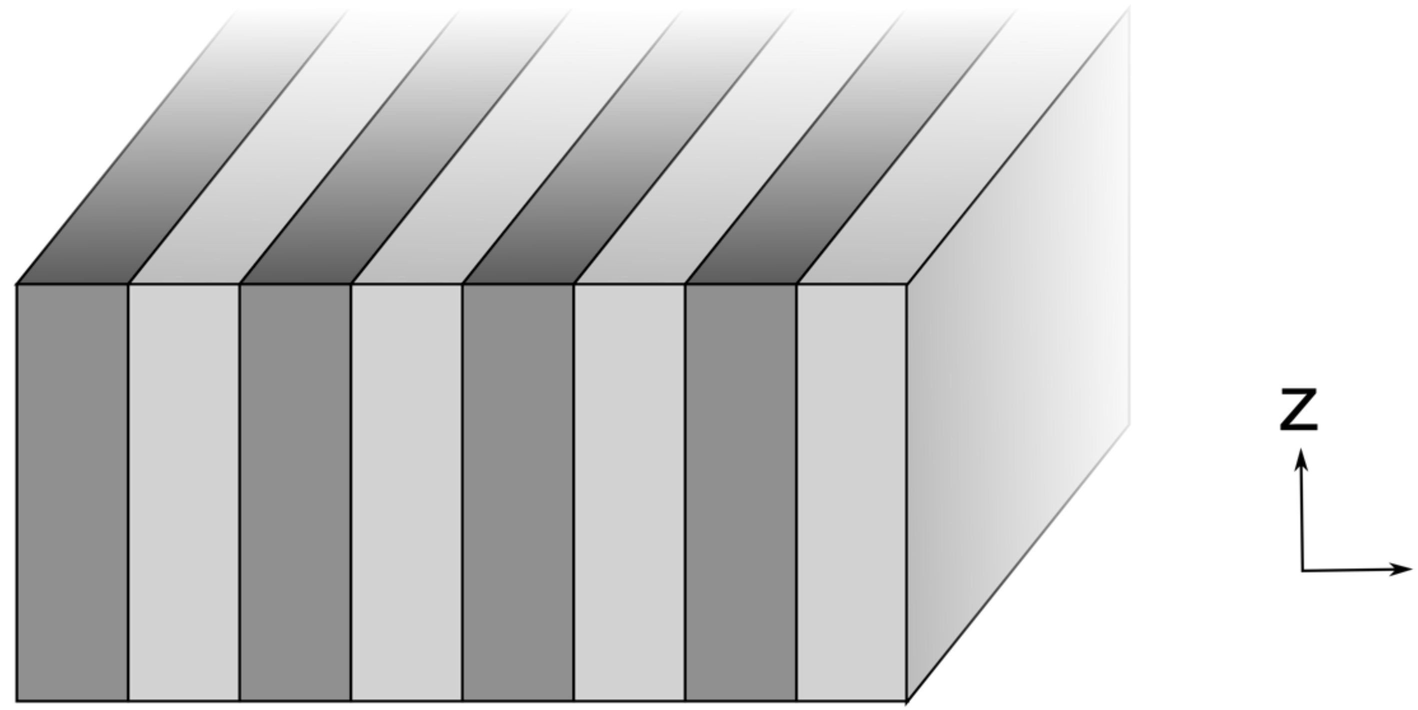
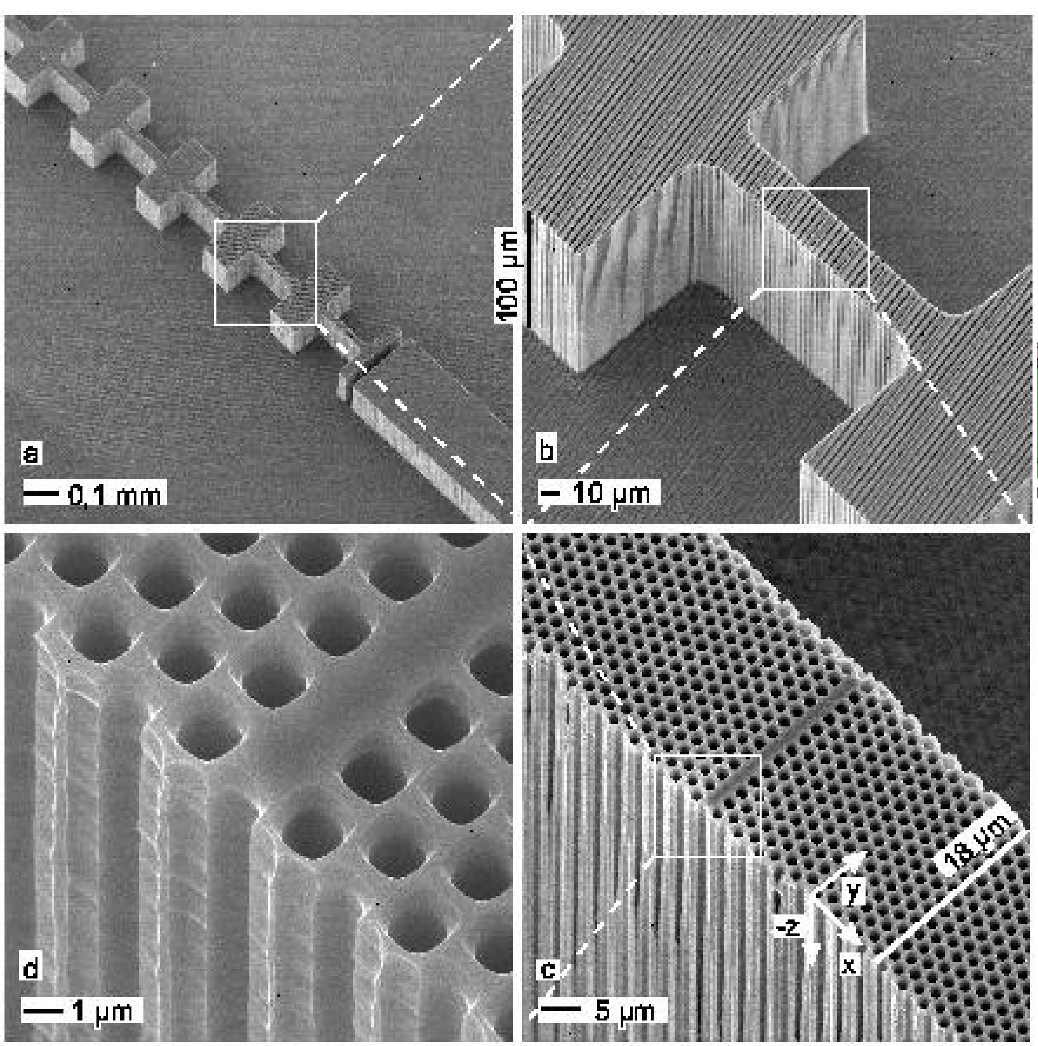
Perspectives

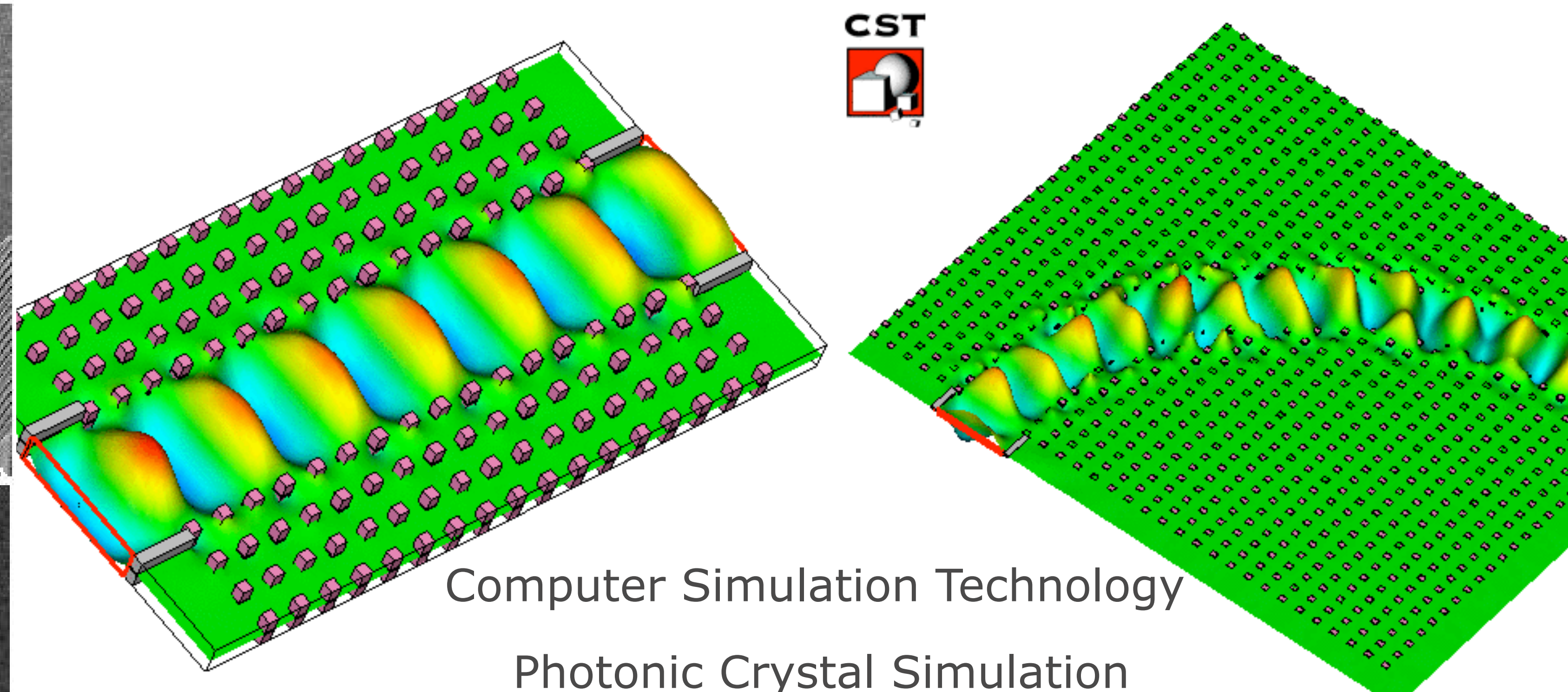
$$E = A(x, z)e^{i(\beta z + \omega t)}$$

Our work indicates the existence of a solution of Maxwell's equations with u(x,z), localized in x, for a careful choice of $\beta(\omega, \mu_0, \epsilon_0)$





Institut für Theoretische Festkörperphysik (Photonics Group), Karlsruhe

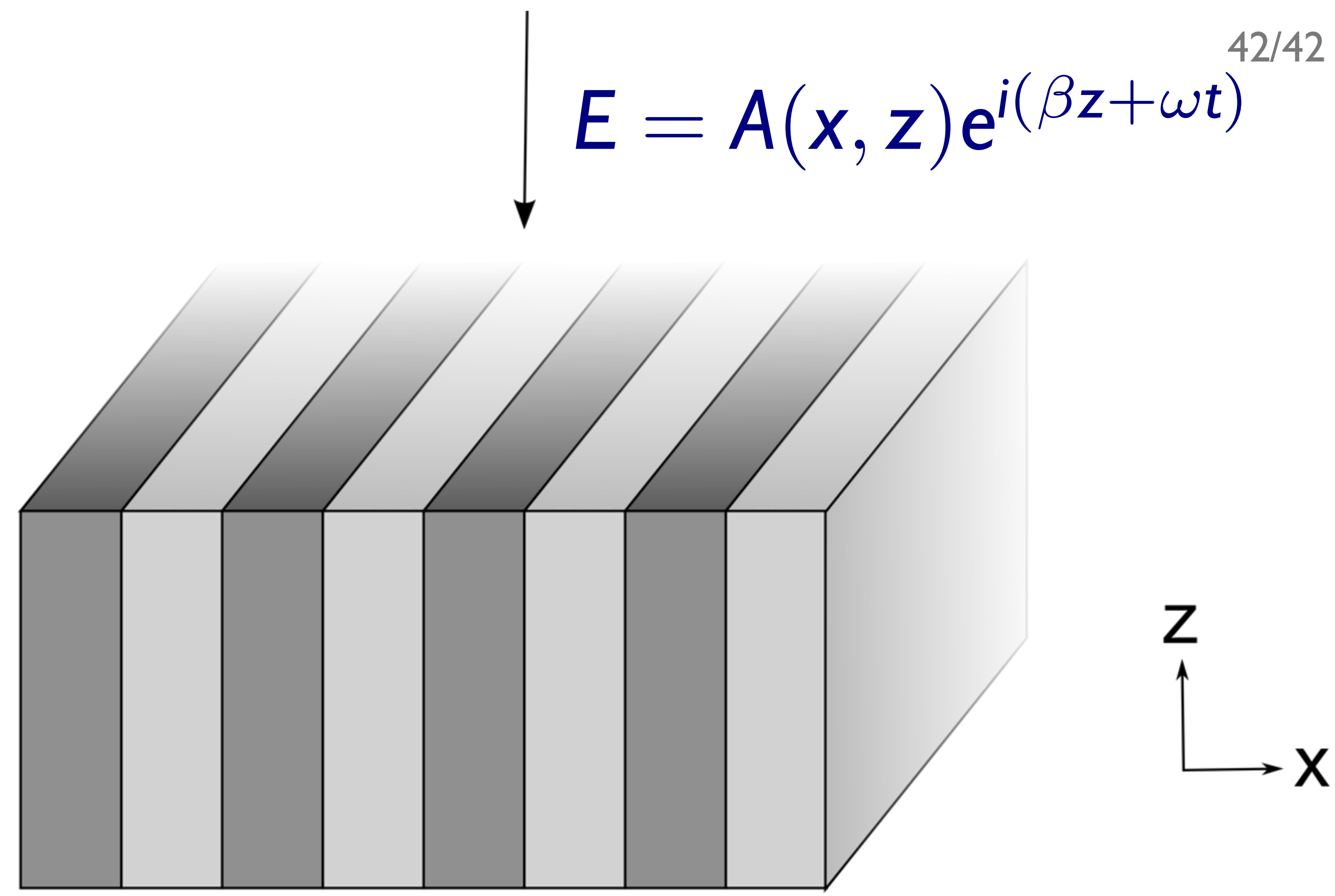


Computer Simulation Technology

Photonic Crystal Simulation

http://updates.cst.com/Content/Applications/Article/Photonic+Crystal+Simulation

Perspectives



Our work indicates the existence of a solution of Maxwell's equations with u(x,z), localized in x, for a careful choice of $\beta(\omega, \mu_0, \epsilon_0)$

- Numerical evidence
- $\left(-\frac{d}{dx}a(x)\frac{d}{dx} + V(x) \right) \psi(x) = E\,\psi(x)$ (TM mode)
- Multi-dimensional case (2-dimensional structures)
- full Helmholtz equation $-\nabla^2\mathbf{E} = \omega^2\mu_0\varepsilon\mathbf{E}$
- non-small defect, *etc.*