JOURNAL OF MATHEMATICAL PHYSICS **52**, 013505 (2011)

Wave operator bounds for one-dimensional Schrödinger operators with singular potentials and applications

Vincent Duchêne,[1,a)] Jeremy L. Marzuola,[2,b)] and Michael I. Weinstein[3,c)]
[1]*Equipe EDP, DMA - Ecole Normale Superieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France*
[2]*Department of Mathematics, University of North Carolina-Chapel Hill, Phillips Hall, Chapel Hill, North Carolina 27599, USA*
[3]*Department of Applied Physics and Applied Mathematics, Columbia University, 200 S. W. Mudd, 500 W. 120th St., New York City, New York 10027, USA*

(Received 28 May 2010; accepted 18 November 2010; published online 7 January 2011)

Boundedness of wave operators for Schrödinger operators in one space dimension for a class of singular potentials, admitting finitely many Dirac delta distributions, is proved. Applications are presented to, for example, dispersive estimates and commutator bounds. © *2011 American Institute of Physics*. [doi:10.1063/1.3525977]

I. INTRODUCTION

Wave operators provide a means for converting operator bounds for a "free" dynamics generated by a constant coefficient Hamiltonian, $H_0 = -\Delta$, to analogous operator bounds about "interacting" dynamics associated with a variable coefficient Hamiltonian, $H = -\Delta + V$, on its continuous spectral subspace. Indeed, let $W_\pm$ and $W_\pm^*$ denote wave operators associated with the free and interacting Hamiltonians H_0 and H (defined by (2.1) and (2.2)). Then, we have

$$W_\pm W_\pm^* = P_c, \quad W_\pm^* W_\pm = Id, \tag{1.1}$$

$$f(H)P_c = W_\pm f(H_0) W_\pm^*, \quad f(H_0) = W_\pm^* f(H) W_\pm, \quad f \text{ Borel on } \mathbb{R}\,. \tag{1.2}$$

It follows that bounds on $f(H)P_c$, acting between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$, can be derived from bounds on $f(H_0)$ between these spaces if the wave operators $W_\pm$ are bounded between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$ for $k_j \geq 0$ and $p \geq 1$. Here, $W^{k,p}(\mathbb{R}^d)$, $k \geq 1$, $p \geq 1$, denotes the Sobolev space of functions having derivatives up to order k in $L^p(\mathbb{R}^d)$.

Boundedness of wave operators in $W^{k,p}(\mathbb{R}^d)$, under smoothness and decay assumptions on $V(x)$, was proved by Yajima[27] in dimensions $d \geq 2$. Weder[26] proved boundedness in dimension one; also see the article of D'Ancona and Fanelli.[2] In Ref. 26 it is assumed that $V \in L^1_\gamma(\mathbb{R})$, the space of all complex-valued measurable functions ϕ defined on $\mathbb{R}$ such that

$$\|\phi\|_{L^1_\gamma} = \int |\phi(x)|(1+|x|)^\gamma dx < \infty. \tag{1.3}$$

For V in a class of generic potentials, the assumption is $\gamma > 3/2$, and otherwise it is assumed $\gamma > 5/2$. Wave operator bounds can be used to establish dispersive estimates, namely,

$$\begin{aligned} &\left\| e^{-iHt} P_c(H) f \right\|_{L^p(\mathbb{R}^d)} \\ &\quad = \left\| W_\pm e^{-iH_0 t} W_\pm^* f \right\|_{L^p(\mathbb{R}^d)} \leq C\, |t|^{-\frac{d}{2}-\frac{d}{p}}\, \|f\|_{L^q(\mathbb{R}^d)}, \quad p^{-1}+q^{-1}=1, \quad p \geq 1. \end{aligned}$$

Applications of wave operator bounds for singular potentials appear in Refs. 4, 15, and 19. Schrödinger operators with singular potentials arise in several mathematical models, which have

a)Electronic mail: vincent.duchene@ens.fr.
b)Electronic mail: marzuola@math.unc.edu.
c)Author to whom correspondence should be addressed. Electronic mail: miw2103@columbia.edu.

recently been extensively investigated. For example, see Refs. 6–8, 11–13, 15, 17, and 19, where Dirac delta function potentials are considered. Boundedness of wave operators in $W^{1,2}(\mathbb{R})$ for singular potentials is used implicitly in Refs. 8 and 15, but this property appears not to have been addressed previously. This gap in the literature is addressed in the present work. Another motivation for the present work is the study of scattering for highly oscillatory potentials, containing local singularities, in the homogenization limit.[4] In this work, bounds on $(m^2 + H)^{-1} P_c(H)(m^2 - \partial_x^2)$, where $H = -\partial_x^2 + V(x)$ is a Schrödinger operator with a singular (distribution) part to the potential $V(x)$, are required; see Sec. VIII.

This article is devoted to an extension of the one-dimensional results[26] to the case of singular potentials. Specifically, our results apply to Hamiltonians of the form

$$H = -\partial_x^2 + V(x),$$

where $V(x)$ satisfies the following.

Hypotheses (V):

$$V(x) = V_{sing}(x) + V_{reg}(x), \tag{1.4}$$

$$V_{sing}(x) = \sum_{j=0}^{N-1} q_j\, \delta(x - y_j), \quad q_j, y_j \in \mathbb{R}, \quad y_j < y_{j+1}, \quad q_j \neq 0, \tag{1.5}$$

$$\|V_{reg}\|_{L^1_{\frac{3}{2}+}(\mathbb{R})} \equiv \int_{\mathbb{R}} (1 + |s|)^{\frac{3}{2}+} |V_{reg}(s)|\, ds < \infty. \tag{1.6}$$

The paper is structured as follows. In Sec. II we state our main result, Theorem 1, concerning boundedness of the wave operators. In Sec. III the strategy of proof is outlined. Section IV summarizes facts about Jost solutions, distorted plane waves, reflection and transmission coefficients, etc. In Sec. V we state a general result, Theorem 3, from which Theorem 1 follows. The proof of Theorem 3 is given in Sec. VI, and the completion of Theorem 1 is given in Sec. VII. Finally, in Sec. VIII we present examples (multidelta function potentials) and applications to dispersive estimates, commutator bounds, and well-posedness.

II. MAIN RESULTS

We first define and review properties of the wave operators. For basic results on wave operators, see, for example, Refs. 1, 21, and 22.

Introduce the self-adjoint operators $H_0 = -\Delta$ and $H = -\Delta + V$. Here, V is a real-valued potential, satisfying assumptions given below; see Sec. V. Let $P_c = P_c(H)$ denote the continuous spectral projection associated with H. The wave operators, $W_\pm$, and their adjoints, $W_\pm^*$, are defined by

$$W_\pm \equiv s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}, \tag{2.1}$$

$$W_\pm^* \equiv s - \lim_{t \to \pm\infty} e^{itH_0} e^{-itH} P_c. \tag{2.2}$$

The wave operators satisfy the properties (1.1) and (1.2). The notion of wave operators is intimately related to the idea of distorted Fourier bases, which are discussed in detail in Refs. 1, 14, and 20. In one dimension, this is directly related to the Jost solutions, studied for general self-adjoint Schrödinger operators in Refs. 3 and 20 and for a certain class of non-self-adjoint operators in Ref. 16.

Theorem 3 of Sec. V, combined with the calculations of Sec. VII, implies the following.

Theorem 1: *Consider the Schrödinger operator with a potential, $V(x)$, satisfying* **Hypotheses (V)**. *Then $W_\pm$ and $W_\pm^*$, originally defined on $W^{1,p} \cap L^2$, $1 \le p \le \infty$, have extensions to bounded operators on $W^{1,p}$, $1 < p < \infty$. Moreover, there are constants C_p such that*

$$\|W_\pm f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})}, \|W_\pm^* f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})}, f \in W^{1,p}(\mathbb{R}), 1 < p < \infty. \tag{2.3}$$

Remark 2.1: *In general, the wave operators are not bounded on L^1. The constraint $p > 1$ is due to the Hilbert transform, $\mathcal{H}$ not being bounded on L^1; see Ref. 26.*

III. STRATEGY OF PROOF

We use the approach for wave operators on $\mathbb{R}$ initiated by Weder in Ref. 26. The heart of the matter concerns the detailed low and high frequency behaviors of the Jost solutions, worked out by Deift and Trubowitz,[3] or a consequence of their methods. The idea is to split the wave operators into high and low frequency components,

$$W_\pm = W_{\pm,high} + W_{\pm,low}.$$

For the high frequency component, we prove for $\phi \in \mathcal{S}$,

$$W_{\pm,high}\phi = \sum_j S_{A_j}\phi, \quad \text{where} \quad S_A\phi \equiv \int_{-\infty}^{\infty} A(x,y)\phi(y)dy.$$

For each $A = A_j$, we use the criterion (Young's inequality[5]) for L^p, $1 \le p \le \infty$, boundedness,

$$C_A \equiv \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\, dy + \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\, dx < \infty$$
$$\Rightarrow \|S_A\phi\|_{L^p} \le C_A \|\phi\|_{L^p},$$

to prove

$$\left\| W_{\pm,high}\phi \right\|_{W^{1,p}} \le C_p \|\phi\|_{W^{1,p}}, \quad 1 < p < \infty. \tag{3.1}$$

For the low frequency components, we have

$$W_{\pm,low} \sim \mathcal{H} + \sum_j S_{A_j},$$

where S_{A_j} is as above and $\mathcal{H}$ denotes the Hilbert transform

$$(\mathcal{H}\phi)(x) = \frac{1}{\pi}\,\text{P.V.} \int \frac{\phi(x-y)}{y}dy = \int_{-\infty}^{\infty} e^{ikx}\,(-i\,\text{sgn}(k))\,\hat{\phi}(k)dk. \tag{3.2}$$

Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform on $\mathbb{R}$ and its inverse, defined by

$$\hat{\phi}(k) \equiv \mathcal{F}\phi(k) = \frac{1}{2\pi}\int e^{-ikx}\phi(x)dx, \quad \check{\Phi}(x) \equiv \mathcal{F}^{-1}\Phi(x) = \int e^{ikx}\Phi(k)dk. \tag{3.3}$$

Thus, for low frequencies, boundedness

$$\left\| W_{\pm,low}\phi \right\|_{W^{1,p}} \le C_p \|\phi\|_{W^{1,p}}, \quad 1 < p < \infty, \tag{3.4}$$

reduces to the boundedness properties of the Hilbert transform.[23]

Theorem 2: $\mathcal{H} : W^{s,p} \to W^{s,p}$, for $1 < p < \infty$ and $s \ge 0$, with $\|\mathcal{H}\phi\|_{W^{s,p}(\mathbb{R})} \le K_p \|\phi\|_{W^{s,p}(\mathbb{R})}$.

Estimates (3.1) and (3.4) then imply the theorem. The proof of (3.1) and (3.4) is given in Sec. VI. We now develop some background for implementing the strategy.

IV. BACKGROUND SPECTRAL THEORY OF $H = -\partial_x^2 + V$

A. Distorted plane waves, $e_\pm(x;k)$

Consider the operator $H = -\partial_x^2 + V(x)$, defined as a self-adjoint operator on $L^2(\mathbb{R})$. Denote by P_d and P_c the discrete and continuous spectrum projections. P_d and P_c are orthogonal projections with $P_c = Id - P_d$.

Denote by R_0 the outgoing "free" resolvent operator $R_0(k) = (-\partial_x^2 - k^2)^{-1}$ with kernel

$$R_0(k)(x, y) = -(2ik)^{-1} \exp(ik|x - y|),$$

and finally introduce the *distorted plane waves*, $e_\pm(x; k)$.

Definition 4.1: The functions $u = e_\pm(x; k)$ are the unique solutions to $(H - k^2)u = 0$ satisfying

$$e_\pm(x; k) \;=\; e^{\pm ikx} \;+\; \text{outgoing}(x), \tag{4.1}$$

where a function U is said to be outgoing as $|x| \to \infty$ if

$$(\partial_x \;\mp\; ik)U \;\to\; 0, \;\; x \to \pm\infty.$$

Thus, $e_\pm(x; k)$ is given by the integral equation,

$$e_\pm(x; k) = e^{\pm ikx} - R_0(k)V e_\pm(x; k), \tag{4.2}$$

or equivalently by

$$e_\pm(x; k) = e^{\pm ikx} - R_V(k)V e^{\pm ikx}. \tag{4.3}$$

The continuous spectral projection, P_c, is given by

$$P_c f(x) = \frac{1}{2\pi} \int \int_0^\infty \left(e_+(x, k)\, \overline{e_+(y, k)} + e_-(x, k)\, \overline{e_-(y, k)}\right) f(y)\, dk\, dy, \tag{4.4}$$

see, for example, Ref. 25.

We write

$$P_c f \equiv F_+^* F_+ f,$$

where it follows from (4.4) that

$$F_+ f \equiv \int_{\mathbb{R}} \overline{\Psi_+(y, k)}\, f(y)\, dy, \quad F_+^* f \;\equiv\; \int_{\mathbb{R}} \Psi_+(y, k)\, f(k)\, dk, \;\; \text{and} \tag{4.5}$$

$$\Psi_+(y, k) = \frac{1}{\sqrt{2\pi}} \begin{cases} e_+(x; k), & k \geq 0 \\ e_-(x; -k), & k < 0 \end{cases}. \tag{4.6}$$

We also define $\Psi_-(x, k) = \overline{\Psi_+(x, -k)}$. Recall $W_\pm = F_\pm^* \mathcal{F}$.[22]

B. Jost solutions

To make direct use of the arguments in Refs. 3 and 26, we express the results of the Sec. IV A in terms of *Jost solutions*, commonly introduced for one-dimensional Schrödinger operators.

Given the Schrödinger equation

$$-\frac{d^2}{dx^2}u + Vu = k^2 u, \;\; k \in \mathbb{C}, \tag{4.7}$$

we define the Jost solutions, $f_j(x, k)$, $j = 1, 2$, $\text{Im}k \geq 0$, to be the unique solutions of (4.7) satisfying the conditions,

$$\begin{aligned} f_1(x, k) - e^{ikx} &\to 0, \quad x \to \infty, \;\; \text{and} \\ f_2(x, k) - e^{-ikx} &\to 0, \quad x \to -\infty. \end{aligned} \tag{4.8}$$

The Jost solutions are linearly independent solutions of (4.7) for $k \neq 0$. Therefore, there are unique functions $T(k)$, $R_j(k)$, $j = 1, 2$, such that for $k \in \mathbb{R} \setminus 0$,

$$f_2(x, k) = \frac{R_1(k)}{T(k)}\, f_1(x, k) + \frac{1}{T(k)}\, f_1(x, -k), \tag{4.9}$$

$$f_1(x, k) = \frac{R_2(k)}{T(k)}\, f_2(x, k) + \frac{1}{T(k)}\, f_2(x, -k). \tag{4.10}$$

For a potential, V, with compact support within $(-r, r)$, $R_j(k)$, and $T(k)$ are defined via the solutions,

$$e_+(x;k) = T(k) f_1(x;k) = \begin{cases} e^{ikx} + R_2(k)e^{-ikx}, & x < -r \\ T(k)e^{ikx}, & x > r \end{cases}, \tag{4.11}$$

$$e_-(x;k) = T(k) f_2(x;k) = \begin{cases} T(k)e^{-ikx}, & x < -r \\ e^{-ikx} + R_1(k)e^{ikx}, & x > r \end{cases}. \tag{4.12}$$

Generically,

$$T(k) = \alpha k + o(k), \quad 1 + R_j(k) = \alpha_j k + o(k), \quad j = 1, 2, \quad k \to 0. \tag{4.13}$$

$T(k)$ is called the transmission coefficient associated with H. $R_1(k)$ is the right to left reflection coefficient, and $R_2(k)$ is the left to right reflection coefficient.

Finally, it is convenient to denote by $m_j(x, k)$, $j = 1, 2$,

$$m_1(x, k) = e^{-ikx} f_1(x, k), \qquad \text{and} \qquad m_2(x, k) = e^{ikx} f_2(x, k). \tag{4.14}$$

It follows from (4.1), (4.8), and (4.9) that

$$\Psi_+(x, k) = \frac{1}{\sqrt{2\pi}} \begin{cases} T(k)\, e^{ikx}\, m_1(x, k), & k \geq 0 \\ T(-k)\, e^{ikx}\, m_2(x, -k), & k < 0 \end{cases}, \tag{4.15}$$

where $m_1(x, k) - 1 \to 0$ as $x \to \infty$ and $m_2(x, k) - 1 \to 0$ as $x \to -\infty$. The detailed smoothness and decay properties, in x and k, of $m_j(x;k) - 1$ are required in estimates. These are given in Sec. VII.

V. STATEMENT OF THE CENTRAL THEOREM

Our central result, from which Theorem 1 follows, is as follows.

Theorem 3: *Let $H = -\partial_x^2 + V(x)$ be self-adjoint on $L^2(\mathbb{R})$ for which the transmission and reflection coefficients (see (4.9)) satisfy the bounds*

$$|R_1(k)| + |R_2(k)| + |T(k) - 1| \leq \frac{C}{\langle k \rangle}, \tag{5.1}$$

$$|\partial_k R_1(k)| + |\partial_k R_2(k)| + |\partial_k T(k)| = \mathcal{O}\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \tag{5.2}$$

Let S_1 and S_2 be defined by

$$\left(S_j \Phi\right)(x) \equiv \int_{\mathbb{R}} R_j(x, y)\, \Phi(y)\, dy, \quad \text{where} \tag{5.3}$$

$$R_j(x, y) \equiv \int_{\mathbb{R}} e^{ikx} \left(m_j(x, k) - 1\right) e^{-iky}\, dk. \tag{5.4}$$

and assume, for $1 < p < \infty$, that S_1 is bounded on $W^{1,p}(\mathbb{R}_+)$ and S_2 is bounded on $W^{1,p}(\mathbb{R}_-)$.

Then $W_\pm$ and $W_\pm^*$ originally defined on $W^{1,p} \cap L^2$, $1 \leq p \leq \infty$, extend to bounded operators on $W^{1,p}$, $1 < p < \infty$. Furthermore, there are constants C_p such that

$$\|W_\pm f\|_{W^{1,p}} \leq C_p \|f\|_{W^{1,p}}, \ \|W_\pm^* f\|_{W^{1,p}} \leq C_p \|f\|_{W^{1,p}}, \ f \in W^{1,p} \cap L^2, \ 1 < p < \infty. \tag{5.5}$$

Remark 5.1: *Deift and Trubowitz[3] establish the hypotheses of the theorem for any potential $V(x)$, for which $(1 + |x|)^{\frac{3}{2}+} |V(x)| \in L^1(\mathbb{R})$ (see Sec. VII. We show in Sec. VII B that their proof also applies to a potential of the type in Hypothesis* **(V)**, *$V = V_{sing} + V_{reg}$, where V_{sing} has a finite set of Dirac masses within an interval $(-A, A)$, and such that $(1 + |x|)^{\frac{3}{2}+} \left|V_{reg}(x)\right| \in L^1(\mathbb{R})$.*

Remark 5.2: *In fact, less restrictive bounds on V_{reg}, as developed in Ref. 3, would suffice. However, for simplicity we will follow the work of Weder[26] as it makes some computations more explicit.*

VI. PROOF OF CENTRAL THEOREM 3

We follow the strategy described in Sec. III. Theorem 1 will follow from Theorem 3 by verifying the hypotheses of Theorem 3 for $V = V_{sing} + V_{reg}$. This verification is computed in Sec. VII.

Let $\chi(x \geq 1) \in C^\infty(\mathbb{R})$ denote nondecreasing cut-off functions such that

$$\chi(x \geq 1) = \begin{cases} 0, & x \leq \frac{1}{2} \\ 1, & x \geq 1 \end{cases}. \tag{6.1}$$

To localize in frequency space, introduce $\psi\,(|k| \leq k_0) \in C_0^\infty(\mathbb{R})$ be a compactly supported cut-off function, depending on a parameter, k_0, to be chosen, such that

$$\psi\,(|k| \leq k_0) = \begin{cases} 1, & |k| \leq k_0 \\ 0, & |k| \geq 2k_0 \end{cases}. \tag{6.2}$$

We decompose any $\phi \in L^2(\mathbb{R})$ into its low and high frequency parts,

$$\phi(x) \;=\; \phi_{low}(x) \;+\; \phi_{high}(x), \quad \text{where using} \quad D \equiv -i\,\partial_x, \tag{6.3}$$

$$\phi_{low}(x) \;\equiv\; \psi(|D| \leq k_0)\phi(x) \;\equiv\; \int_{\mathbb{R}} e^{ikx}\psi\,(|k| \leq k_0)\,\hat{\phi}(k)\,dk, \tag{6.4}$$

$$\phi_{high}(x) \;\equiv\; (1 \;-\; \psi\,(|D| \leq k_0))\;\phi(x) \;\equiv\; \int_{\mathbb{R}} e^{ikx}\,(1 - \psi\,(|k| \leq k_0))\,\hat{\phi}(k)\,dk\,. \tag{6.5}$$

A. Bounds on $W_+\phi_{low}$

For $x \geq 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_1(x,k)$, and for $x \leq 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_2(x,k)$. Since the cases $x \geq 0$ and $x \leq 0$ are very similar, we only carry out this calculation in detail for $x \geq 0$. We have, using the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
W_+\phi_{low} \;&=\; F_+^*\mathcal{F}\;\psi(|D| \leq k_0)\phi \\
&= \int_0^\infty e^{ikx}\;T(k)\,m_1(x,k)\;\psi\,(|k| \leq k_0)\;\hat{\phi}(k)\,dk + \int_{-\infty}^0 e^{ikx}\;T(-k)\,m_2(x,-k)\;\psi\,(|k| \leq k_0)\;\hat{\phi}(k)\,dk \\
&= \int_0^\infty e^{ikx}\;T(k)\,m_1(x,k)\;\psi\,(|k| \leq k_0)\;\hat{\phi}(k)\,dk \\
&\quad + \int_{-\infty}^0 e^{ikx}\;[R_1(-k)e^{-2ikx}\;m_1(x,-k) + m_1(x,k)]\;\psi\,(|k| \leq k_0)\,\hat{\phi}(k)\,dk \\
&= \int_0^\infty e^{ikx}\;m_1(x,k)\;[T(k) + R_1(k)P]\;\psi\,(|k| \leq k_0)\;\hat{\phi}(k)\,dk + \int_{-\infty}^0 e^{ikx}\;m_1(x,k)\;\hat{\phi}(k)\,dk, \quad x \geq 0,
\end{aligned}$$

where we have applied (4.5) and (4.15).

We continue by using that $\int_0^\infty\;[\ldots]\;dk = \frac{1}{2}\int_{-\infty}^\infty\,(1 + \mathrm{sgn}(k))\;[\ldots]\;dk$, we have

$$\begin{aligned}
W_+\phi_{low} \;&=\; \frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\;e^{ikx}(m_1(x,k) - 1)T(k)\;\psi\,(|k| \leq k_0)\,\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\;e^{ikx}(m_1(x,k) - 1)R_1(k)P\;\psi\,(|k| \leq k_0)\,\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty (1 - \mathrm{sgn}(k))\;e^{ikx}(m_1(x,k) - 1)\;\psi\,(|k| \leq k_0)\,\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\;e^{ikx}T(k)\;\psi\,(|k| \leq k_0)\,\hat{\phi}(k)dk
\end{aligned}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}(1+\mathrm{sgn}(k))\,e^{ikx}R_1(k)P\;\psi\,(|k|\le k_0)\,\hat{\phi}(k)dk$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}(1-\mathrm{sgn}(k))\,e^{ikx}\;\psi\,(|k|\le k_0)\,\hat{\phi}(k)dk,\quad x\ge 0. \tag{6.6}$$

For $x \le 0$, an analogous representation holds with $m_1(x,k)$ replaced by $m_2(x,k)$.

We now show that $W_{+,1,low}$ is a bounded operator on $W^{1,p}(\mathbb{R}_+)$. Each term in the first three lines of (6.6) is of the form

$$\phi \;\mapsto\; S_1 \circ (I \pm i\,\mathcal{H}) \circ \Psi(D)\,\phi\,, \tag{6.7}$$

and each term in the last three lines is of the form

$$\phi \;\mapsto\; (I \pm i\,\mathcal{H}) \circ \Psi(D)\,\phi\,, \tag{6.8}$$

where S_1 is defined in (5.3) and (5.4), $\mathcal{H}$ denotes the Hilbert transform (3.2), and

$$\Psi(D) = \mathcal{F}^{-1}\,\hat{\Psi}(k)\,\mathcal{F} \quad \text{and}$$
$$\hat{\Psi}(k) = T(k)\;\psi(|k|\le k_0) \text{ or } R_1(k)\;P\;\psi(|k|\le k_0) \text{ or } \psi(|k|\le k_0).$$

(For $x \le 0$, the argument is parallel with S_1 replaced by S_2.)

By hypotheses on $T(k)$ and $R(k)$, $\hat{\Psi}(k)$ is a multiplier on $W^{1,p}(\mathbb{R})$ for $1 < p < \infty$.[23] The Hilbert transform is bounded (Theorem 2), so that the boundedness of the operators in (6.7) and (6.8) on $W^{1,p}$ for $1 < p < \infty$ follows from the boundedness of S_j, which holds by hypothesis. Therefore, one has

$$\|W_+\phi_{low}\|_{W^{1,p}(\mathbb{R})} \;\le\; C\;\|\phi\|_{W^{1,p}(\mathbb{R})}, \tag{6.9}$$

and this completes the low frequency analysis.

B. High frequencies

We have, using (4.9) and the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
W_+\phi_{high} &= F_+^*\mathcal{F}\,(1-\psi(|D|\le k_0))\,\phi\\
&= \int_0^{\infty} T(k)e^{ikx}m_1(x,k)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&\quad+\int_{-\infty}^{0} T(-k)e^{ikx}m_2(x,-k)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&= \int_0^{\infty} T(k)e^{ikx}m_1(x,k)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&\quad+\int_{-\infty}^{0} e^{ikx}[R_1(-k)e^{-2ikx}m_1(x,-k)+m_1(x,k)](1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&= \int_0^{\infty} e^{ikx}m_1(x,k)[T(k)+R_1(k)P](1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk+\int_{-\infty}^{0} e^{ikx}m_1(x,k)\hat{\phi}(k)dk.
\end{aligned}$$

For $x \ge 0$, we rewrite this expression as

$$\begin{aligned}
W_+\phi_{high} &= \frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\;(1+\mathrm{sgn}(k))\;(m_1(x,k)-1)T(k)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&\quad+\frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\;(1+\mathrm{sgn}(k))(m_1(x,k)-1)R_1(k)P(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk\\
&\quad+\frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\;(1-\mathrm{sgn}(k))(m_1(x,k)-1)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk
\end{aligned}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\,(1+\mathrm{sgn}(k))T(k)(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk$$
$$+\frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\,(1+\mathrm{sgn}(k))R_1(k)P(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk$$
$$+\frac{1}{2}\int_{-\infty}^{\infty} e^{ikx}\,(1-\mathrm{sgn}(k))(1-\psi\,(|k|\le k_0))\hat{\phi}(k)dk,\quad x\ge 0.$$

An analogous expression, with $m_1(x,k)$ replaced by $m_2(x,k)$, is used for $x\le 0$. We now proceed to show that each term is bounded on $W^{1,p}(\mathbb{R}_+)$, $p\ge 1$.

Each summand in this decomposition of $W_+\phi_{high}$ is of the form

$$\phi\mapsto S_j\circ\rho(D)\,\phi,\qquad \text{or}\quad \phi\mapsto\rho(D)\phi. \tag{6.10}$$

where $\rho(D)=\mathcal{F}^{-1}\hat{\rho}(k)\mathcal{F}$. Here, $S_j,\ j=1,2$, defined in (5.3) and (5.4), is bounded on $W^{1,p}(\mathbb{R}_+)$ for $1<p<\infty$ by hypothesis. Moreover, $\rho(k)$ is a multiplier on $W^{1,p}(\mathbb{R})$ for $1<p<\infty$ due to hypotheses on $R(k)$, $T(k)-1$, $\partial_k R(k)$ and $\partial_k T(k)$, and the fact that $1-\psi(|k|\le k_0)$ is smooth, asymptotically constant as $k\to\infty$, and vanishing in a neighborhood of 0. It follows that

$$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R}_+)}\le C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\,\|\phi\|_{W^{1,p}(\mathbb{R}_+)}. \tag{6.11}$$

An estimate analogous to (6.11), similarly proved using a representation of $W_+\phi_{high}(x)$ for $x\le 0$, in terms of S_2, also holds. Thus,

$$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R})}\le C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\,\|\phi\|_{W^{1,p}(\mathbb{R})}. \tag{6.12}$$

The decomposition (6.3) and the bounds (6.9) and (6.12) imply the result. This completes the proof of the central result, Theorem 3.

VII. COMPLETION OF THE PROOF OF THEOREM 1

This section is devoted to the completion of the proof of Theorem 1, as a consequence of Theorem 3. The hypotheses of Theorem 3 are satisfied for potentials $V\in L^1_{\frac{3}{2}+}(\mathbb{R})$ by results given in Ref. 3. We briefly recall the argument below, and then we generalize it to potentials of the form (1.4), $V=V_{sing}+V_{reg}$, in Sec. VII B.

A. The case of regular potentials

From the relation $m_1(x,k)=e^{-ikx}\,f_1(x,k)$, $k\in\mathbb{C}$, we have that $m_1(x,k)$ is the unique solution of

$$\frac{d^2}{dx^2}m_1+2ik\,\frac{d}{dx}m_1=V\,m_1,\ \text{and}\ m_1(x;k)\to 1,\ \text{as}\ x\to\infty. \tag{7.1}$$

Consequently, we have

$$m_1(x,k)=1+\int_x^{\infty}D_k(y-x)V(y)m_1(y,k)dy,\quad\text{where}\ \ D_k(x)\equiv\int_0^x e^{2iky}dy. \tag{7.2}$$

Indeed, for $V\in L^1_{\frac{3}{2}+}(\mathbb{R})$, the iterates of the Volterra integral are bounded by $\frac{\gamma(x)^n}{n!}$, with

$$\gamma(x)\equiv\int_x^{\infty}(t-x)|V(t)|\,dt.$$

Summing on n, we find that the majoring series converges and $m_1(x,k)$ satisfies the bound

$$|m_1(x,k)|\ \le\ e^{\gamma(x)}\gamma(x).$$

By a careful analysis for $x \to -\infty$, one has the improved estimate

$$|m_1(x,k)| \;\le\; C(1+\max(-x,0))\int_x^\infty (1+|t|)|V(t)|\,dt. \tag{7.3}$$

As a consequence, the function $m_1(x,k)-1$ is in the Hardy space, and therefore there exists $B_1 \in L^2(\mathbb{R}\times\mathbb{R}_+)$ such that

$$m_1(x,k) = 1 + \int_0^\infty B_1(x,y)e^{2iky}dy. \tag{7.4}$$

Now, the function $B_1(x,y)$ is equivalently defined with

$$B_1(x,y) \;\equiv\; \int_{x+y}^\infty V(t)\,dt \;+\; \int_0^y \int_{x+y-z}^\infty V(t)B_1(t,z)\,dt\,dz, \tag{7.5}$$

$$= \sum_{n=0}^\infty K_n(x,y), \tag{7.6}$$

where K_n is defined by induction with

$$K_0(x,y) = \int_{x+y}^\infty V(t)dt, \qquad K_{n+1}(x,y) = \int_0^y \int_{x+y-z}^\infty V(t)K_n(t,z)dtdz.$$

It is then easy to prove by induction that

$$|K_n(x,y)| \;\le\; \frac{\gamma^n(x)}{n!}\eta(x+y), \qquad \text{with } \eta(x) \;\equiv\; \int_x^\infty |V(t)|\,dt.$$

This allows us to confirm that the sum in (7.6) is well-defined and satisfies (7.5), plus the estimates

$$|B_1(x,y)| \le e^{\gamma(x)}\eta(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \le e^{\gamma(x)}\gamma(x). \tag{7.7}$$

From (7.7), taking the x-derivative of (7.5), we have

$$|\partial_x B_1(x,y)| \le\; C\, e^{\gamma_1(x)}\left(V(x+y) \;+\; \int_{x+y}^\infty |V(t)|dt\right),\; x\in\mathbb{R},\; y>0. \tag{7.8}$$

The construction above and (7.4) with the estimates (7.7) and (7.8) are sufficient to prove.

Lemma 7.1: *S_1 is bounded on $W^{1,p}(\mathbb{R}_+)$ and S_2 is bounded on $W^{1,p}(\mathbb{R}_-)$ for $1<p<\infty$.*

Proof: We focus on the bound for S_1 on $W^{1,p}(\mathbb{R}_+)$. The bound for S_2 on $W^{1,p}(\mathbb{R}_-)$ is similar. To prove boundedness of S_1 and ∂S_1 on L^p, we use the operator

$$S_R\Phi(x) \;=\; \int_{\mathbb{R}} R(x,y)\,\Phi(y)\,dy,$$

which is bounded on L^p with estimate

$$\|S_R\Phi\|_{L^p} \;\le\; C_R\;\|\Phi\|_{L^p}\,,\;\; 1\le p\le\infty \tag{7.9}$$

if

$$C_R \equiv \sup_{x\ge 0}\int_{\mathbb{R}} |R(x,y)|\,dy \;+\; \sup_{y\ge 0}\int_{\mathbb{R}} |R(x,y)|\,dx \;<\; \infty. \tag{7.10}$$

Using the representation formula (7.4), we have

$$R_j(x,y) \;\equiv\; \int_{\mathbb{R}} e^{ik(x-y)}\int_0^\infty e^{2ikz}\,B_1(x,z)\,dk\,dz \;=\; B_1\left(x,\frac{y-x}{2}\right).$$

Thus, the operator S_1 simplifies to

$$(S_1\Phi)(x) = \int_x^\infty B_1\left(x,\frac{y-x}{2}\right)\Phi(y)dy \;=\; \int_0^\infty B_1\left(x,\frac{\zeta}{2}\right)\,\Phi(\zeta-x)\,d\zeta, \quad x\ge 0.$$

Since we must estimate S_1 on $W^{1,p}$, we also compute

$$\partial_x (S_1\Phi)(x) = \int_0^\infty B_1\left(x, \frac{\zeta}{2}\right)(-\partial_\zeta)\Phi(\zeta - x)\, d\zeta + \int_0^\infty \partial_x B_1\left(x, \frac{\zeta}{2}\right)\Phi(\zeta - x)\, d\zeta$$
$$= \int_x^\infty B_1\left(x, \frac{y-x}{2}\right)(-\partial_y)\Phi(y)\, dy + \int_x^\infty \partial_x B_1\left(x, \frac{y-x}{2}\right)\Phi(y)\, dy, \quad x \geq 0.$$

Note that by (7.7) and (7.8), we have for large enough x that

$$|B_1(x,z)| \lesssim \int_{x+z}^\infty |V(s)|ds \text{ and } |\partial_x B_1(x,z)| \lesssim |V(x)| + \int_{x+z}^\infty |V(s)|ds. \tag{7.11}$$

Therefore,

$$\sup_{x\geq 0}\int \mathbf{1}_{y\geq x}\left|B_1\left(x, \frac{y-x}{2}\right)\right| dy + \sup_{y\geq 0}\int \mathbf{1}_{y\geq x}\left|B_1\left(x, \frac{y-x}{2}\right)\right| dx$$
$$\leq 2\sup_{x\geq 0}\int_0^\infty \int_{\frac{x+y}{2}}^\infty |V(s)|ds\, dy$$
$$\leq 2\int_0^\infty \left(1 + \frac{x+y}{2}\right)^{-\frac{3}{2}-} \int_{\frac{x+y}{2}}^\infty (1+s)^{\frac{3}{2}+}|V(s)|ds$$
$$\leq const \times \|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}.$$

A similar bound applies to the kernel $\mathbf{1}_{x\geq y}\partial_x B_1\left(x, \frac{y-x}{2}\right)$. Thus, we have

$$\|S_1\Phi\|_{W^{1,p}(\mathbb{R}_+)} \equiv \|S_1\Phi\|_{L^p(\mathbb{R}_+)} + \|\partial_x (S_1\Phi)\|_{L^p(\mathbb{R}_+)} \leq C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\|\Phi\|_{W^{1,p}(\mathbb{R}_+)}.$$

Applying similar arguments with S_1 replaced by S_2 for $x \leq 0$ yields boundedness of S_2 on $W^{1,p}$. □

Lemma 7.2:

$$|R_j(k)|,\ |T(k)-1| \leq \frac{C}{\langle k\rangle} \quad \forall k \in \mathbb{R}, \qquad |\partial_k T(k)|,\ |\partial_k R_1(k)|,\ |\partial_k R_2(k)| \leq \frac{C}{|k|} \quad \text{as } |k| \to \infty.$$

Proof of Lemma 7.2 for $V = V_{reg}$:

We follow again the method of Ref. 3. From (7.2), one has

$$m_1(x,k) = 1 + \frac{1}{2ik}\int_0^\infty (e^{2ik(y-x)} - 1)V(y)m_1(y,k)\, dy$$
$$= e^{-2ikx}\left(\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_1(y,k)V(y)\, dy\right)$$
$$+ \left(1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\, dy\right) + o(1), \quad x \to -\infty.$$

Moreover, one has from (4.8) and (4.9)

$$m_1(x,k) = e^{-2ikx}\frac{R_2(k)}{T(k)} + \frac{1}{T(k)} + o(1), \quad x \to -\infty.$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

$$\frac{1}{T(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\, dy, \tag{7.12}$$
$$\frac{R_2(k)}{T(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_1(y,k)V(y)\, dy, \tag{7.13}$$
$$\frac{R_1(k)}{T(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_2(y,k)V(y)\, dy. \tag{7.14}$$

These integral representations, together with the $L^1_{\frac{3}{2}+}$ decay assumption on the potential and the uniform bounds (7.3), immediately lead to

$$|R_j(k)|,\ |T(k)-1| \le \frac{C}{\langle k\rangle}.$$

Now, differentiating (7.4) with respect to k leads to the uniform estimate

$$|\ \partial_k m_1(x,k)\ |\ \le\ C\langle k\rangle\langle x\rangle,$$

so that (7.12) yields

$$|\partial_k T(k)| \le \frac{C}{|k|}, \qquad \text{as} \qquad |k|\to\infty.$$

The equivalent estimates for $R_1(k)$, $R_2(k)$ follow similarly from (7.13) and (7.14), and Lemma 7.2 holds.

B. The case of potentials with a singular component

In this section we prove that one can generalize the construction above for generalized potentials, satisfying Hypothesis **(V)**, with equivalent estimates, so that Lemmas 7.1 and 7.2 hold. As a consequence, the hypotheses of Theorem 3 are satisfied and Theorem 1 is proved.

*Proof of Lemma 7.1 for V satisfying Hypotheses (**V**):*

We prove the desired estimates for $m_1(x,k)$, $x\ge 0$, and similar results apply to $m_2(x,k)$, $x\le 0$. Let us define the function $B_1(x,y)$ with

$$B_1(x,y) \equiv \int_{x+y}^{\infty} V_{reg}(t)\,dt\ +\ \sum_{l=0}^{N-1} c_l\mathbf{1}(x_l-(x+y))\ +\ \int_0^y\int_{x+y-z}^{\infty} V_{reg}(t)B_1(t,z)\,dt\,dz$$

$$+\int_0^y\sum_{l=0}^{N-1} c_l B_1(x_l,z)\mathbf{1}(x_l-(x+y-z))\,dz, \tag{7.15}$$

$$=\sum_{n=0}^{\infty} K_n(x,y), \tag{7.16}$$

with $\mathbf{1}$ the classical symmetric Heaviside function defined such that

$$\mathbf{1}(x)=\begin{cases}1\ ,\ x>0\\ \frac{1}{2}\ ,\ x=0\,,\\ 0\ ,\ x<0\end{cases}$$

and K_n defined by induction, with

$$K_0(x,y)=\int_{x+y}^{\infty} V_{reg}(t)dt\ +\ \sum_{l=0}^{N-1} c_l\mathbf{1}(x_l-(x+y)),$$

$$K_{n+1}(x,y)=\int_0^y\int_{x+y-z}^{\infty} V_{reg}(t)K_n(t,z)dtdz\ +\ \sum_{l=0}^{N-1} c_l\int_0^y K_n(x_l,z)\mathbf{1}(x_l-(x+y-z))\,dz.$$

Following the proof of Lemma 3 of Ref. 3, it is easy to show the following pointwise bound by induction:

$$|K_n(x,y)|\ \le\ \frac{\gamma_1^n(x)}{n!}\eta_1(x+y),$$

with γ_1 and η_1 defined as

$$\gamma_1(x)\ \equiv\ \int_x^{\infty}(t-x)|V_{reg}(t)|\,dt\ +\ \sum_{l=0}^{N-1}|c_l|(x_l-x)\mathbf{1}(x_l-x),$$

$$\eta_1(x) \equiv \int_x^\infty |V_{reg}(t)|\, dt + \sum_{l=0}^{N-1} |c_l| \mathbf{1}(x_l - x).$$

This allows us to confirm that the sum in (7.16) is well-defined and satisfies (7.15), plus the estimates

$$|B_1(x,y)| \le e^{\gamma_1(x)} \eta_1(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \le e^{\gamma(x)} \gamma(x) \tag{7.17}$$

and, differentiating with respect to x,

$$|\partial_x B_1(x,y)| \le C\, e^{\gamma_1(x)} \left(V(x+y) + \int_{x+y}^\infty |V(t)| dt \right), \; x \in \mathbb{R}, \; y > 0. \tag{7.18}$$

Abusing notation (see justification below), we define

$$m_1(x,k) \equiv 1 + \int_0^\infty B_1(x,y) e^{2iky}\, dy, \tag{7.19}$$

it is easy to deduce from (7.15) that

$$\begin{aligned}
\partial_x m_1(x,k) &= \int_0^\infty \left(\partial_x B_1(x,y) - \partial_y B_1(x,y)\right) e^{2iky}\, dy + \int_0^\infty \partial_y B_1(x,y) e^{2iky}\, dy \\
&= -\int_0^\infty \int_x^\infty V_{reg}(t) B_1(t,y)\, dt\, e^{2iky}\, dy - \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y) \mathbf{1}(x_l - x) e^{2iky}\, dy \\
&\quad - \int_0^\infty 2ik B_1(x,y) e^{2iky}\, dy - B_1(x,0).
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
\partial_x^2 m_1(x,k) &= \int_0^\infty V_{reg}(x) B_1(x,y) e^{2iky}\, dy + \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y) \delta(x_l - x) e^{2iky}\, dy \\
&\quad - \int_0^\infty 2ik \partial_x B_1(x,y) e^{2iky}\, dy + V_{reg}(x) + \sum_{l=0}^{N-1} c_l \delta(x_l - x).
\end{aligned} \tag{7.21}$$

Therefore, $m_1(x,k)$ is the unique function satisfying

$$\frac{d^2}{dx^2} m_1 + 2ik \frac{d}{dx} m_1 = \sum_l c_l \delta(x - x_l) + V_{reg}\, m_1, \; k \in \mathbb{C},$$

with $m_1(x;k) \to 1$ as $x \to \infty$. Equation (7.19) is thus justified.

Finally, Lemma 7.1 follows from (7.19), with the estimates (7.17) and (7.18).

Proof of Lemma 7.2: We follow again the method of Ref. 3. The generalization of (7.2) to potentials satisfying Hypotheses **(V)** is

$$\begin{aligned}
m_1(x,k) &= 1 + \frac{1}{2ik} \int_0^\infty (e^{2ik(y-x)} - 1) V_{reg}(y) m_1(y,k)\, dy \\
&\quad + \frac{1}{2ik} \sum_{l=0}^{N-1} c_l (e^{2ik(x_l - x)} - 1) m_1(x_l,k) \mathbf{1}(x_l - x) \\
&= \frac{e^{-2ikx}}{2ik} \left(\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k) V_{reg}(y)\, dy + \sum_{l=0}^{N-1} c_l e^{2ikx_l} m_1(x_l,k) \right) \\
&\quad + 1 - \frac{1}{2ik} \left(\int_{-\infty}^{+\infty} m_1(y,k) V_{reg}(y)\, dy + \sum_{l=0}^{N-1} c_l m_1(x_l,k) \right) + o(1)(x \to -\infty).
\end{aligned}$$

Moreover, one has from (4.8) and (4.9)

$$m_1(x,k) \;=\; e^{-2ikx}\frac{R_2(k)}{T(k)} \;+\; \frac{1}{T(k)} \;+\; o(1)(x \to -\infty).$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

$$\frac{1}{T(k)} \;=\; 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V_{reg}(y)\,dy \;-\; \frac{1}{2ik}\sum_{l=0}^{N-1} c_l m_1(x_l,k), \tag{7.22}$$

$$\frac{R_2(k)}{T(k)} \;=\; \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k)V_{reg}(y)\,dy \;+\; \frac{1}{2ik}\sum_{l=0}^{N-1} c_l e^{2ikx_l} m_1(x_l,k), \tag{7.23}$$

$$\frac{R_1(k)}{T(k)} \;=\; \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_2(y,k)V_{reg}(y)\,dy \;+\; \frac{1}{2ik}\sum_{l=0}^{N-1} c_l e^{2ikx_l} m_2(x_l,k). \tag{7.24}$$

The identity (7.19), with the estimates (7.17), guarantees the uniform bounds

$$|m_1(x,k)| \le C\langle x\rangle, \qquad |\partial_k m_1(x,k)| \;\le\; C\langle k\rangle\langle x\rangle.$$

Therefore the $L^1_{\frac{3}{2}+}$ decay assumption on the potential V_{reg} immediately leads to

$$|R_j(k)|,\ |T(k)-1| \le \frac{C}{\langle k\rangle}.$$

Now, differentiating (7.22) with respect to k yields

$$|\partial_k T(k)| \le \frac{C}{|k|}, \qquad \text{as} \qquad |k| \to \infty.$$

The equivalent estimates for $R_1(k)$, $R_2(k)$ follow similarly from (7.23) and (7.24), and Lemma 7.2 holds.

VIII. EXAMPLES AND APPLICATIONS

A. $V(x)$ = a sum of Dirac delta masses

In this section we directly verify the hypotheses of Theorem 3 for the case of a potential, which is the sum of Dirac delta functions, thereby establishing the applicability of our main results to this case.

We follow the analysis from Refs. 11 and 25, see also Refs. 9 and 10 for specific examples. Seek solutions of the form

$$\left(H_{\vec{q},\vec{y}} - \frac{1}{2}k^2\right) e_\pm(x,k) = 0, \tag{8.1}$$

where $H_{\vec{q},\vec{y}} = \sum_{j=0}^{N-1} q_j\delta(x-y_j)$ when $\vec{q} = (q_0,\ldots,q_{N-1})$, $\vec{y} = (y_0,\ldots,y_{N-1})$, and where $e_\pm(x,k)$ represent the distorted Fourier basis functions as defined in (4.1). Thus,

$$e_+(x,k) = \begin{cases} e^{ikx} + B_0 e^{-ikx}, & \text{for } x < y_0 \\ A_1 e^{ikx} + B_1 e^{-ikx}, & \text{for } y_0 < x < y_1 \\ \vdots & \\ A_N e^{ikx}, & \text{for } x > y_{N-1} \end{cases}, \tag{8.2}$$

where we have taken $A_0 = 1$ and $B_N = 0$. With this choice of notation, we have, referring to (4.11) and (4.12), $A_N = T$, the transmission coefficient, and $B_0 = R_1$, the reflection coefficient for the "incoming" plane wave e^{ikx} from $-\infty$. Then, we have the following system of equations implied by continuity and jump conditions at the points $\{y_j\}$ for $j = 0,\ldots,N-1$:

$$
\begin{aligned}
e^{iky_0} + B_0 e^{-iky_0} &= A_1 e^{ikx_0} + B_1 e^{-iky_0} \\
ik\left[A_1 e^{iky_0} - B_1 e^{-iky_0} - e^{iky_0} + B_0 e^{-iky_0}\right] &= 2q_0\left[A_1 e^{iky_0} + B_1 e^{-iky_0}\right] \\
&\vdots \\
A_{N-1} e^{iky_{N-1}} + B_{N-1} e^{-iky_{N-1}} &= A_N e^{iky_{N-1}} \\
ik\left[A_N e^{iky_{N-1}} - A_{N-1} e^{iky_0} + B_{N-1} e^{-iky_0}\right] &= 2q_{N-1}\left[A_N e^{iky_{N-1}}\right].
\end{aligned}
$$

Note, the above system guarantees unitarity or that

$$|B_0|^2 + |A_N|^2 = 1. \tag{8.3}$$

We can define similarly

$$e_-(x,k) = \begin{cases} D_0 e^{-ikx}, & \text{for } x < y_0 \\ C_1 e^{ikx} + D_1 e^{-ikx}, & \text{for } y_0 < x < y_1 \\ \vdots & \\ C_N e^{ikx} + e^{-ikx}, & \text{for } x > y_{N-1} \end{cases}, \tag{8.4}$$

where now the incoming wave is e^{-ikx} from ∞, and the scattering matrix is determined by the transmission coefficients $D_0 = T$ and the reflection coefficient $C_N = R_2$ for the "incoming" plane wave e^{-ikx} from ∞.

1. Bounds on m_1, m_2:

In addition, for general singular potentials with compact support, we have

$$m_1(x,k) = e^{-ikx} f_1(x,k) = \begin{cases} e^{-ikx}\frac{e_+(x,k)}{T(k)}, & \text{for } x < y_{N-1} \\ 1, & \text{for } x > y_{N-1} \end{cases},$$

$$m_2(x,k) = e^{ikx} f_2(x,k) = \begin{cases} e^{ikx}\frac{e_-(x,k)}{T(k)}, & \text{for } x > y_0 \\ 1, & \text{for } x < y_0 \end{cases}.$$

Hence, there exists constants $C^1_\alpha(y_{N-1})$ and $C^2_\alpha(y_0)$ such that

$$|\partial_k^\alpha m_1(x,k)| \le C^1_\alpha(y_{N-1}), \text{ for } y_{N-1} > x \ge 0, \tag{8.5}$$

$$|\partial_k^\alpha m_2(x,k)| \le C^2_\alpha(y_0), \text{ for } y_0 < x \le 0. \tag{8.6}$$

As a result, we can see that an arbitrary collection of δ functions satisfies the required estimate for the proof of Lemma 7.1.

We conclude this subsection with explicit computations of the transmission and reflection coefficients for single and double δ well potentials.

2. Single δ potential ($H_q = -q\delta(x)$):

By setting up the appropriate equations, we have

$$R_1 = r_q = \frac{q}{ik - q}, \tag{8.7}$$

$$T = t_q = \frac{ik}{ik - q}, \tag{8.8}$$

where r_q and t_q are the reflection and transmission coefficients for H_q, respectively. We must show that the bounds from (5.1) hold; however, such bounds follow clearly for (8.8) and (8.7).

3. Double δ potential ($H_{q,L} = -q(\delta(x+L) + \delta(x-L))$):

Setting up the appropriate equations, we have

$$R_1 = r_{q,L} = \left(\frac{q(ik-q)e^{2ikL} + q(ik+q)e^{-2ikL}}{q^2 e^{2ikL} - (ik+q)^2 e^{-2ikL}} \right) e^{-2ikL}, \tag{8.9}$$

$$T = t_{q,L} = \left(\frac{k^2}{q^2 e^{2ikL} - (ik+q)^2 e^{-2ikL}} \right) e^{-2ikL}, \tag{8.10}$$

where $r_{q,L}$ and $t_{q,L}$ are the reflection and transmission coefficients for $H_{q,L}$, respectively.

Again, we must verify Lemma (7.2); hence, we must prove, for instance,

$$|\partial_k t_{q,L}(k)| \leq C(1+|k|)^{-1},$$

provided $qL \neq 1/2$. Indeed, we have

$$\partial_k t_{q,L}(k) = \frac{2k(k^2 - 2ikq + q^2(e^{4ikL}-1)) - 2ik^2(2Lq^2 e^{4ikL} - (ik+q))}{(k^2 - 2ikq + q^2(e^{4ikL}-1))^2},$$

which satisfies

$$|\partial_k t_{q,L}(k)| \sim \mathcal{O}(|k|^{-1}),$$

as $k \to \infty$ and

$$|\partial_k t_{q,L}(k)| \sim \mathcal{O}\left(\frac{1}{4q^2 L - 2q} \right),$$

as $k \to 0$. A similar computation holds for $r_{q,L}$.

B. Commutator/resolvent-type bounds

In Ref. 4, where homogenization of high contrast oscillatory structures with defects is studied, bounds on $(H_0+1)^{-1}(H_{\vec{q},\vec{y}}+1)$ are required to estimate a Lipmann Schwinger equation. We have by our main theorem that

$$(H_0+1)^{-1}(H_{\vec{q},\vec{y}}+1)P_c = (H_0+1)^{-1} W_+ (H_0+1) W_+^* : L^2 \to L^2.$$

C. Dispersive and Strichartz estimates in H^1 for δ-Schrödinger

We may represent

$$e^{-itH} P_c f = \frac{1}{2\pi} \int \int_0^\infty e^{-\frac{itk^2}{2}} \left(e_+(x,k)\, \overline{e_+(x,k)} + e_-(x,k)\, \overline{e_-(x,k)} \right) f(y) dk dy. \tag{8.11}$$

From here, we may use direct computations to arrive at Strichartz estimates and apply Weder's results on wave operators since the potentials are all in L^1 with compact support.

Using the properties of wave operators, we have

$$\|e^{iHt} P_c f\|_{L^p} = \|W_\pm e^{itH_0} W_\pm^* f\|_{L^p} \tag{8.12}$$

and using standard dispersive estimates for the linear Schrödinger operator (see, for instance, Ref. 24 for a concise overview) we arrive at

$$\|e^{iHt} P_c f\|_{L^p} \leq C_p t^{-(\frac{1}{2}-\frac{1}{p})} \|f\|_{W^{1,p}}. \tag{8.13}$$

Define a Strichartz pair (q, r) to be admissible if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r} \tag{8.14}$$

with $2 \leq r < \infty$. Then, we arrive at the celebrated Strichartz estimates

$$\|e^{iHt} P_c u_0\|_{L^q W^{1,r}} \lesssim \|u_0\|_{W^{1,2}} \tag{8.15}$$

and

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^q W^{1,r}} \lesssim \| f(x,t) \|_{L_t^{\tilde{q}} W_x^{1,\tilde{r}}} \tag{8.16}$$

using duality techniques and once again the boundedness of the wave operators.

As a side note, using positive commutators and well crafted local smoothing spaces, from Ref. 18 we have the Strichartz estimate

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^\infty L^2} \lesssim \| f(x,t) \|_{L_t^{\tilde{p}} L_x^{\tilde{q}}}. \tag{8.17}$$

Now, by boundedness of wave operators on $W^{1,p}$ spaces for singular potentials, as proved in Theorem 3, we have the following useful relation:

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^\infty H^1} \lesssim \| f(x,t) \|_{L_t^{\tilde{p}} W_x^{1,\tilde{q}}}, \tag{8.18}$$

where $(\tilde{p}, \tilde{q})$ is a dual Strichartz pair without going through the dispersive estimates first.

D. Local well-posedness in H^1 for δ-NLS/Gross-Pitaevskii equation

Consider the nonlinear Schrödinger (NLS)/Gross-Pitaevskii equation, with a potential consisting of a finite set of Dirac delta functions:

$$\begin{cases} i\partial_t u + H_{\vec{q},\vec{y}} u - |u|^{2\sigma} u = 0 \\ \quad u(x,0) = u_0(x) \in H^1 \end{cases},$$

for $0 < \sigma < \infty$. We seek a solution in the following sense:

$$u = \Lambda[u],$$

where

$$\Lambda[u](t) = e^{-iH_{\vec{q},\vec{y}}t} u_0 - i \int_0^t e^{-iH_{\vec{q},\vec{y}}(t-s)} |u|^{2\sigma} u(s) ds. \tag{8.19}$$

We claim that local well-posedness can be established via the contraction mapping principle in the space $C^0([0,T); H^1(\mathbb{R}))$ for T sufficiently small. To prove the necessary boundedness and contraction estimates, it is natural to apply the operator $(I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c$, which commutes with the group $e^{-iH_{\vec{q},\vec{y}}t}$ to (8.19). Then, estimates follow in a straightforward way, using that $H^1(\mathbb{R})$ is an algebra, provided the space

$$\mathcal{H}^1(\mathbb{R}) = \left\{ f : (I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c f \in L^2(\mathbb{R}) \right\} \tag{8.20}$$

is equivalent to the classical Sobolev space H^1. This follows from the relations

$$(I+H)^{\frac{1}{2}} P_c = W(I - \partial_x^2)^{\frac{1}{2}} W^*, \quad W^*(I+H)^{\frac{1}{2}} W = (I - \partial_x^2)^{\frac{1}{2}}$$

and our results on the boundedness of wave operators associated with $H_{\vec{q},\vec{y}}$ on H^1.

E. Long time dynamics for *NLS* with a double δ well potential

In Ref. 19, the long time dynamics of solutions to the nonlinear Schrödinger/Gross-Pitaevskii equation,

$$i\partial_t u = (-\Delta + V(x))u \; + \; gK\left[|u|^2\right] u, \tag{8.21}$$

where V is a symmetric, double well potential, are studied. In particular, under appropriate spectral assumptions on the operator $H = -\partial_x^2 + V(x)$, in a neighborhood of a symmetry breaking bifurcation point, there are different classes of oscillating solutions (8.21) which *shadow* periodic orbits of a finite dimensional reduction on very long, but finite, time scales. These solutions correspond to states with mass concentrations oscillating between the two wells of a symmetric potential well. The

proof requires dispersive/Strichartz-type estimates. The results of this paper imply that the results from Ref. 19 extend to (8.21) for the case of singular potentials, such as

$$V(x) = -q[\delta(x - L) + \delta(x + L)].$$

ACKNOWLEDGMENTS

J.L.M. was supported, in part, by a U.S. National Science Foundation Postdoctoral Fellowship in the Department of Applied Physics and Applied Mathematics (APAM) at Columbia University. M.I.W was supported, in part, by U.S. NSF Grant DMS-07-07850. J.L.M and M.I.W wish to acknowledge the hospitality of the Courant Institute of Mathematical Sciences, where MIW was on sabbatical during the preparation of this manuscript. V.D. was supported, in part, by Agence Nationale de la Recherche Grant ANR-08-BLAN-0301-01. V.D. would like to thank APAM for its hospitality during the Spring of 2008, when this work was initiated.

[1] Agmon, S., "Spectral properties of Schrödinger operators and scattering theory," Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) **2**, 151 (1975).
[2] D'Ancona, D. and Fanelli, L., "L^p-boundedness of the wave operator for the one dimensional Schrödinger operator," Commun. Math. Sci. **268**, 415 (2006).
[3] Deift, P. and Trubowitz, E., "Inverse scattering on the line," Commun. Pure Appl. Math. **32**, 121 (1979).
[4] Duchêne, V. and Weinstein, M. I., "Scattering, homogenization and interface effects for oscillatory potentials with strong singularities," e-print arXiv:1010.2694 (2010).
[5] Folland, G. B., *Introduction to Partial Differential Equations* (Princeton University, Princeton, NJ, 1995).
[6] Fukuizumi, R. and Jeanjean, L., "Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential," Discrete Contin. Dyn. Syst. **21**, 121 (2008).
[7] Fukuizumi, R., Ohta, R., and Ozawa, T., "Nonlinear Schrödinger equation with a point defect," Ann. Inst. Henri. Poincaré Anal. Non Linéaire **25**, 837 (2008).
[8] Goodman, R. H., Holmes, P. J., and Weinstein, M. I., "Strong NLS soliton-defect interactions," Phys. D (Nonlinear Phenomena) **192**, 215 (2004).
[9] Griffiths, D. J. and Steinke, C. A., "Waves in locally periodic media," Am. J. Phys. **69**, 137 (2001).
[10] Griffiths, D. J. and Taussig, N. F., "Scattering from a locally periodic potential," Am. J. Phys. **60**, 883 (1992).
[11] Holmer, J., Marzuola, J., and Zworski, M., "Fast soliton scattering by delta impurities," Commun. Math. Phys. **274**, 187 (2007).
[12] Holmer, J., Marzuola, J., and Zworski, M., "Soliton splitting by external delta potentials," J. Nonlinear Sci. **17**, 349 (2007).
[13] Holmer, J. and Zworski, M., "Slow soliton interaction with delta impurities," J. Mod. Dyn. **1**, 689 (2007).
[14] Hörmander, H., *The Analysis of Linear Partial Differential Operators. II: Differential Operators with Constant Coefficients*, Classics in Mathematics (Springer-Verlag, Berlin, 2005).
[15] Jackson, R. K. and Weinstein, M. I., "Geometric analysis of bifurcation and symmetry breaking in a Gross-Pitaevskii equation," J. Stat. Phys. **116**, 881 (2004).
[16] Krieger, J. and Schlag, W., "Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension," J. Am. Math. Soc. **19**, 815 (2006).
[17] Le Coz, S., Fukuizumi, R., Fibich, G., Ksherim, B., and Sivan, Y., "Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential," Physica D **237**, 1103 (2008).
[18] Marzuola, J., Metcalfe, J., and Tataru, D., "Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations," J. Funct. Anal. **255**, 1497 (2008).
[19] Marzuola, J. L. and Weinstein, M. I., "Long time dynamics near the symmetry breaking bifurcation for nonlinear Schrödinger/Gross-Pitaevskii Equations," Discrete Contin. Dyn. Syst. **28** (in press).
[20] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. IV. Analysis of Operators* (Academic, New York, 1978).
[21] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. III. Scattering Theory* (Academic, New York, 1979).
[22] Schechter, M., *Operator Methods in Quantum Mechanics* (North-Holland Publishing Co., New York, 1981).
[23] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series Vol. 30 (Princeton University, Princeton, NJ, 1970).
[24] Sulem, C. and Sulem, P.-L., *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse*, Applied Mathematical Sciences Vol. 139 (Springer-Verlag, New York, 1999).
[25] Tang, S. H. and Zworski, M., "Potential scattering on the real line," Lecture Notes (unpublished). See: http://math.berkeley.edu/~zworski/tz1.pdf.
[26] Weder, R., "The $W_{k,p}$-continuity of the Schrödinger wave operators on the line," Commun. Math. Phys. **208**, 507 (1999).
[27] Yajima, K., "The $W^{k,p}$-continuity of wave operators for Schrödinger operators," J. Math. Soc. Jpn. **47**, 551 (1995).