A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part II: Justification as a shallow water approximation

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Abstract

We consider the Kakinuma model for the motion of interfacial gravity waves. The Kakinuma model is a system of Euler–Lagrange equations for an approximate Lagrangian, which is obtained by approximating the velocity potentials in the Lagrangian of the full model. Structures of the Kakinuma model and the well-posedness of its initial value problem were analyzed in the companion paper [14]. In this present paper, we show that the Kakinuma model is a higher order shallow water approximation to the full model for interfacial gravity waves with an error of order $O(\delta_1^{4N+2}+\delta_2^{4N+2})$ in the sense of consistency, where δ_1 and δ_2 are shallowness parameters, which are the ratios of the mean depths of the upper and the lower layers to the typical horizontal wavelength, respectively, and N is, roughly speaking, the size of the Kakinuma model and can be taken an arbitrarily large number. Moreover, under a hypothesis of the existence of the solution to the full model with a uniform bound, a rigorous justification of the Kakinuma model is proved by giving an error estimate between the solution to the Kakinuma model and that of the full model. An error estimate between the Hamiltonian of the Kakinuma model and that of the full model is also provided.

1 Introduction

We will consider the motion of the interfacial gravity waves at the interface between two layers of immiscible fluids in $(n+1)$-dimensional Euclidean space. Let t be the time, $\boldsymbol{x}=(x_1,\dots,x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the layers are infinite in the horizontal directions, bounded from above by a flat rigid-lid, and from below by a time-independent variable topography. The interface, the rigid-lid, and the bottom are represented as $z=\zeta(\boldsymbol{x},t)$, $z=h_1$, and $z=-h_2+b(\boldsymbol{x})$, respectively, where $\zeta=\zeta(\boldsymbol{x},t)$ is the elevation of the interface, h_1 and h_2 are mean depths of the upper and lower layers, and $b=b(\boldsymbol{x})$ represents the bottom topography. See Figure 1.1. We assume that the fluids in the upper and the lower layers are both incompressible and inviscid fluids with constant densities ρ_1 and ρ_2, respectively, and that the flows are both irrotational. Then, the motion of the fluids is described by the velocity potentials $\Phi_1(\boldsymbol{x},z,t)$ and $\Phi_2(\boldsymbol{x},z,t)$ and the pressures $P_1(\boldsymbol{x},z,t)$ and $P_2(\boldsymbol{x},z,t)$ in the upper and the lower layers. We recall the governing equations, referred as the full model for interfacial gravity waves, in Section 2 below. Generalizing the work of J. C. Luke [30], these equations can be obtained as the Euler–Lagrange equations associated with the Lagrangian density $\mathscr{L}(\Phi_1,\Phi_2,\zeta)$ given by the vertical integral of the pressure in both water regions. Building on this variational structure, T. Kakinuma [23, 24, 25] proposed and studied numerically the model obtained as the Euler–Lagrange equations for an approximated Lagrangian density, $\mathscr{L}(\Phi_1^{\rm app},\Phi_2^{\rm app},\zeta)$, where

$$\Phi_\ell^{\rm app}(\boldsymbol{x},z,t)=\sum_{i=0}^{N_\ell} Z_{\ell,i}(z;\tilde h_\ell(\boldsymbol{x}))\phi_{\ell,i}(\boldsymbol{x},t) \tag{1.1}$$

for $\ell=1,2$, and $\{Z_{1,i}\}$ and $\{Z_{2,i}\}$ are appropriate function systems in the vertical coordinate z and may depend on $\tilde h_1(\boldsymbol{x})$ and $\tilde h_2(\boldsymbol{x})$, respectively, which are the depths of the upper and

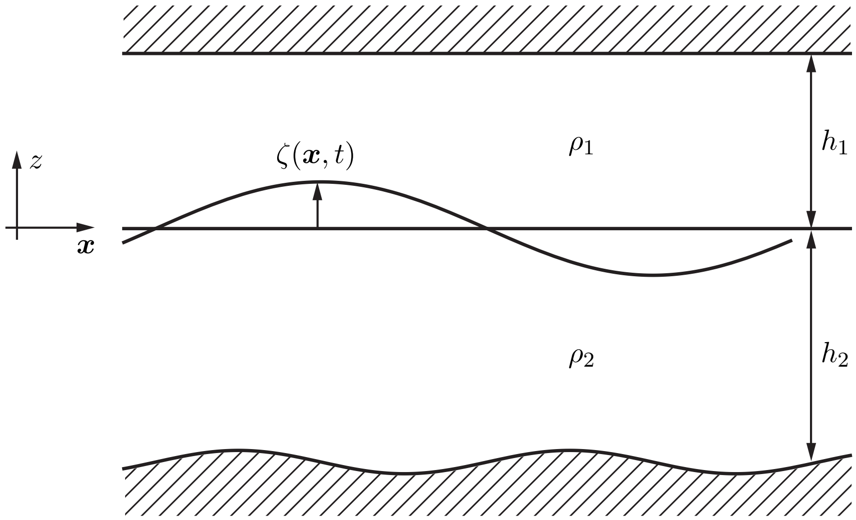

Figure 1.1: Internal gravity waves

the lower layers in the rest state, whereas $\boldsymbol{\phi}_\ell = (\phi_{\ell,0}, \phi_{\ell,1}, \dots, \phi_{\ell,N_\ell})^{\mathrm{T}}$, $\ell = 1,2$, are unknown variables. This yields a coupled system of equations for $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, and ζ, depending on the function systems $\{Z_{1,i}\}$ and $\{Z_{2,i}\}$, which we named Kakinuma model. Note that in our setting of the problem we have $\tilde{h}_1(\boldsymbol{x}) = h_1$ and $\tilde{h}_2(\boldsymbol{x}) = h_2 - b(\boldsymbol{x})$. In this work we study the Kakinuma model obtained when the approximate velocity potentials are defined by

$$(1.2)\qquad \begin{cases} \Phi_1^{\mathrm{app}}(\boldsymbol{x}, z, t) := \displaystyle\sum_{i=0}^{N} (-z + h_1)^{2i} \phi_{1,i}(\boldsymbol{x}, t), \\ \Phi_2^{\mathrm{app}}(\boldsymbol{x}, z, t) := \displaystyle\sum_{i=0}^{N^*} (z + h_2 - b(\boldsymbol{x}))^{p_i} \phi_{2,i}(\boldsymbol{x}, t), \end{cases}$$

where N, N^*, and $p_0, p_1, \dots, p_{N^*}$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_{N^*}$. Specifically, we show that the Kakinuma model obtained through the approximated potentials (1.2) with

(H1) $N^* = N$ and $p_i = 2i$ $(i = 0, 1, \dots, N)$ in the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$,

(H2) $N^* = 2N$ and $p_i = i$ $(i = 0, 1, \dots, 2N)$ in the case with general bottom topographies,

provides a higher order shallow water approximation to the full model for interfacial gravity waves in the strongly nonlinear regime. The choice of the function systems as well as N, N^*, and $p_0, p_1, \dots, p_{N^*}$ is discussed and motivated later on.

Comparison with surface gravity waves The Kakinuma model is an extension to interfacial gravity waves of the so-called Isobe–Kakinuma model for surface gravity waves, that is, water waves, in which Luke's Lagrangian density $\mathscr{L}_{\mathrm{Luke}}(\Phi, \zeta)$, where ζ is the surface elevation and Φ is the velocity potential of the water, is approximated by a density $\mathscr{L}^{\mathrm{app}}(\boldsymbol{\phi}, \zeta) = \mathscr{L}_{\mathrm{Luke}}(\Phi^{\mathrm{app}}, \zeta)$, where

$$(1.3)\qquad \Phi^{\mathrm{app}}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N} Z_i(z; b(\boldsymbol{x})) \phi_i(\boldsymbol{x}, t).$$

The Isobe–Kakinuma model was first proposed by M. Isobe [21, 22] and then applied by T. Kakinuma to simulate numerically the water waves. Recently, this model was analyzed from a mathematical point of view when the function system $\{Z_i\}$ is a set of polynomials in z: $Z_i(z; b(\boldsymbol{x})) = (z + h - b(\boldsymbol{x}))^{p_i}$ with integers p_i satisfying $0 = p_0 < p_1 < \cdots < p_N$. The initial value problem was analyzed by Y. Murakami and T. Iguchi [34] in a special case and by R. Nemoto and T. Iguchi [35] in the general case. The hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model in the sense that the operator acting on time derivatives of the unknowns has a non-trivial kernel. As a consequence, one needs to impose some compatibility conditions on the initial data for the existence of the solution. Under these compatibility conditions, the non-cavitation condition, and a Rayleigh–Taylor type condition $-\partial_z P^{\mathrm{app}} \geq c_0 > 0$ on the water surface, where P^{app} is an approximate pressure in the Isobe–Kakinuma model calculated from Bernoulli's equation, they showed the well-posedness of the initial value problem in Sobolev spaces locally in time. Moreover, T. Iguchi [19, 20] showed that under the choice of the function system

$$Z_i(z; b(\boldsymbol{x})) = \begin{cases} (z+h)^{2i} & \text{in the case of the flat bottom,} \\ (z+h-b(\boldsymbol{x}))^i & \text{in the case of a variable bottom,} \end{cases} \tag{1.4}$$

the Isobe–Kakinuma model is a higher order shallow water approximation for the water wave problem in the strongly nonlinear regime. Furthermore, V. Duchêne and T. Iguchi [13] showed that the Isobe–Kakinuma model also enjoys a Hamiltonian structure analogous to the one exhibited by V. E. Zakharov [43] on the full water wave problem and that the Hamiltonian of the Isobe–Kakinuma model is a higher order shallow water approximation to the one of the full water wave problem.

Our aim in the present paper and the companion paper [14] is to extend these results on surface gravity waves to the framework of interfacial gravity waves. With respect to surface gravity waves, our interfacial gravity waves framework brings two additional difficulties. The first one is that, due to the rigid-lid assumption, the full system for interfacial gravity waves described in Section 2 features only one evolution equation for the two velocity potentials, and a constraint associated with the fixed fluid domain. From a physical perspective, the unknown velocity potential at the interface may be interpreted as a Lagrange multiplier associated with the constraint. A second important difference between water waves and interfacial gravity waves is that the latter suffer from Kelvin–Helmholtz instabilities. As a consequence the initial value problem of the full model for interfacial gravity waves is ill-posed in Sobolev spaces; see T. Iguchi, N. Tanaka, and A. Tani [37], V. Kamotski and G. Lebeau [26]. This raises the question of the validity of *any* model for interfacial gravity waves. A partial answer is offered by the work of D. Lannes [27], which proves the existence and uniqueness of solutions over large time intervals in the presence of interfacial tension. While interfacial tension effects is not expected to be the relevant regularization mechanism for the propagation of waves between, for instance, fresh and salted water, the key observation is that physical systems allow the propagation of waves with large amplitude and long wavelengths provided that some mechanism tames Kelvin–Helmholtz instabilities acting on the high-frequency component of the flow. This description is consistent with the fact that the initial value problem of the bi-layer shallow water system for the propagation of interfacial gravity waves in the hydrostatic framework is well-posed in Sobolev spaces under some hyperbolicity condition describing the absence of low-frequency Kelvin–Helmholtz instabilities, as proved by D. Bresch and M. Renardy [5]. Let us mention however that such a property is not automatic for higher order shallow water models. Specifically, we note that the Miyata–Choi–Camassa model derived by M. Miyata [33] and W. Choi and R. Camassa [8] and which can be regarded as a two-layer generalization of the

Green–Naghdi equations for water waves turns out to overestimate Kelvin–Helmholtz estimates with respect to the full model; see D. Lannes and M. Ming [29].

In [14], we analyzed the initial value problem of the Kakinuma model when the approximated velocity potentials are defined by (1.2). We found that the Kakinuma model has a stability regime which can be expressed as

$$-\partial_z(P_2^{\rm app} - P_1^{\rm app}) - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}|\nabla\Phi_2^{\rm app} - \nabla\Phi_1^{\rm app}|^2 \geq c_0 > 0 \tag{1.5}$$

on the interface, where $H_1 := h_1 - \zeta$ and $H_2 := h_2 + \zeta - b$ are the depths of the upper and the lower layers, $P_1^{\rm app}$ and $P_2^{\rm app}$ are approximate pressures of the fluids in the upper and the lower layers, α_1 and α_2 are positive constants depending only on N and on $p_0, p_1, \ldots, p_{N^*}$, respectively. This is a generalization of the aforementioned Rayleigh–Taylor type condition for the Isobe–Kakinuma model. It is worth noticing that, consistently with the expectation that the Kakinuma model is a higher order model for the full system for interfacial gravity waves and that the latter suffers from Kelvin–Helmholtz instabilities, the constants α_1 and α_2 converge to 0 as N and N^* go to infinity so that the stability condition becomes more and more stringent as N and N^* grow. When $N = N^* = 0$, the Kakinuma model coincides with the aforementioned bi-layer shallow water system, and the stability regime coincides with the hyperbolic domain exhibited in [5]. Moreover, when the motion of the fluids together with the motion of the interface is in the rest state, the above stability condition is reduced to the well-known stable stratification condition

$$(\rho_2 - \rho_1)g > 0. \tag{1.6}$$

In [14], we showed that under the stability condition (1.5), the non-cavitation assumptions

$$H_1 \geq c_0 > 0, \qquad H_2 \geq c_0 > 0, \tag{1.7}$$

and intrinsic compatibility conditions on the initial data, the initial value problem for the Kakinuma model is well-posed in Sobolev spaces locally in time. We also showed in [14] that the Kakinuma model enjoys a Hamiltonian structure analogous to the one exhibited by T. B. Benjamin and T. J. Bridges [3] on the full model for interfacial gravity waves.

Comparison with other higher order models The Isobe–Kakinuma and the Kakinuma models belong to higher order models for the water waves and for the full interfacial gravity waves, respectively. By this we mean a family of systems of equations parametrized by nonnegative integers describing the order of the system within the family, that is N for the Isobe–Kakinuma model,and whose solutions are expected to approach solutions to the full system as the order increases. Several such models have been introduced in the literature, mostly in the water waves framework, and we will restrict the discussion to water waves in this paragraph.

Based on a Taylor expansion of the Dirichlet-to-Neumann operator at stake in the water waves system with respect to the shape of the domain D. G. Dommermuth and D. K. Yue[10], B. J. West et al. [41] and W. Craig and C. Sulem [9] have proposed the so-called high order spectral (HOS) models. While these models have been successfully employed in efficient numerical schemes (see recent accounts by J. Wilkening and V. Vasan [42], D. P. Nicholls [36] and P. Guyenne [16]), the equations feature Fourier multipliers which prevent their direct use in situations involving non-trivial geometries such as horizontal boundaries. Moreover, the rigorous justification of HOS models is challenged by well-posedness issues; see the discussion in D. M. Ambrose, J. L. Bona, and D. P. Nicholls [1], and V. Duchêne and B. Melinand [15].

A second class of higher order models originate from formal shallow water expansions put forward by J. Boussinesq [4] and J. W. S. Rayleigh [39]. A systematic derivation procedure has been described by K. O. Friedrichs in the appendix to [40]. Recently, these higher order shallow water models have been described and discussed by Y. Matsuno in [31, 32] and W. Choi in [6, 7]. The derivation procedure displays formula for approximate velocity potentials under the form (1.3)–(1.4) (in particular, only even powers appear in the flat bottom case), with the important difference that the functions ϕ_i $(i = 0, \ldots, N)$ are prescribed through explicit recursion relations. An important consequence of this derivation is that the resulting systems of equations involve only standard differential operators. However the order of the differential operators at stake augments with the order of the system, which renders such models impractical for numerical simulations.

By contrast, the Isobe–Kakinuma model features only differential operators of order at most two acting on the variables ϕ_i $(i = 0, \ldots, N)$ which are unknowns of the system. Notice that the size of the system augments with its order, N. However the degrees of freedom do not augment with the order since, as mentioned above, some compatibility conditions must be satisfied. In fact all quantities are uniquely determined by two scalar functions which represent the canonical variables in the Hamiltonian formulation of the water waves system. Let us mention that function systems different from (1.4) have been considered by G. A. Athanassoulis and K. A. Belibassakis [2], G. Klopman, B. van Groesen, and M. W. Dingemans [17] and C. E. Papoutsellis and G. A. Athanassoulis [38] (see also references therein). While the systems obtained in these works have a similar nature, they are all different. We let the reader refer to V. Duchêne [11, Chapter D] for an extended discussion and comparison of these models.

The choice of the function systems in (1.4) is motivated by the aforementioned Friedrichs expansion and is essential in the analysis of T. Iguchi [19, 20] proving that the Isobe–Kakinuma model is a higher order shallow water approximation for the water wave problem in the strongly nonlinear regime. We note that one may modify (1.4) by putting all odd and even terms $(z+h)^i$ for $i = 0, 1, \ldots$ in the case of the flat bottom. However, in that case one needs to use the terms up to order $2N$ to keep the same precision of the approximation. Therefore, such a choice increases the number of unkonwns and equations by N so that it is undesirable for practical application. In other words, one can save memories in numerical simulations by using only even terms in the case of the flat bottom. On the contrary, if we put only odd terms $(z + h - b(\boldsymbol{x}))^{2i}$ for $i = 0, 1, 2, \ldots$ in the case of a non-flat bottom, then the corresponding Isobe–Kakinuma model does not give any good approximation even if we take N a sufficiently large number, because the corresponding approximate velocity potential Φ^{app} cannot approximate the boundary condition on the bottom so well due to the lack of odd order terms $(z + h - b(\boldsymbol{x}))^{2i+1}$ for $i = 0, 1, 2, \ldots$.

Following this discussion, the choice of the function systems (1.2) with (H1) or (H2) in our interfacial waves framework is very natural. In particular, the rigid-lid is assumed to be flat so that we do not need to use odd order terms $(-z + h_1)^{2i+1}$ for $i = 0, 1, 2, \ldots$, in the approximate velocity potential Φ_1^{app} to obtain a good approximation, because Φ_1^{app} can approximate the boundary condition on the rigid-lid without such terms.

Description of the results In the present paper we show that the Kakinuma model obtained through the approximated potentials (1.2) with

(H1) $N^* = N$ and $p_i = 2i$ $(i = 0, 1, \ldots, N)$ in the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$,

(H2) $N^* = 2N$ and $p_i = i$ $(i = 0, 1, \ldots, 2N)$ in the case with general bottom topographies,

provides a higher order shallow water approximation to the full model for interfacial gravity waves in the strongly nonlinear regime. Our results apply to the dimensionless Kakinuma model obtained after suitable rescaling. The system of equations then depend on the positive dimensionless parameters δ_1 and δ_2 which are shallowness parameters related to the upper and the lower layers, respectively, that is, $\delta_\ell = \frac{h_\ell}{\lambda}$ $(\ell = 1, 2)$ with the typical horizontal wavelength λ. The shallow water regime is described through the smallness of the parameters δ_1 and δ_2. What is more, our results are uniform with respect to parameters satisfying either $\rho_2 \lesssim \rho_1 < \rho_2$, or $\rho_1 \ll \rho_2$ and $h_2 \lesssim h_1$. We notice that the rigid-lid framework is expected to be invalid in the regime $\rho_1 \ll \rho_2$ and $h_1 \ll h_2$ which is excluded in this paper; see V. Duchêne [12].

Our first result extends the result of [14] on the well-posedness of the initial value problem by showing that solutions to the dimensionless Kakinuma model are defined on a time interval which does not vanish for arbitrarily small values of δ_1 and δ_2.

Theorem 1.1 (Long-time well-posedness)**.** *Under the (dimensionless) stability condition* (1.5)*, the (dimensionless) non-cavitation assumptions* (1.7)*, and intrinsic compatibility conditions on the initial data, the initial value problem for the Kakinuma model is well-posed in Sobolev spaces on a time interval which is independent of* $\delta_1 \in (0, 1]$ *and* $\delta_2 \in (0, 1]$*.*

While the non-cavitation assumption and the stability condition are automatically satisfied for small initial data and small bottom topography b, an arrangement of nontrivial initial data satisfying the compatibility conditions with suitable bounds is a non-trivial issue, and demands a specific analysis.

Proposition 1.2. *Initial data satisfying the compatibility conditions and necessary bounds in Theorem 1.1 are uniquely determined (up to an additive constant) by sufficiently regular initial data for the canonical variables of the Hamiltonian structure.*

Then, we show that under the special choice of the indices $p_0, p_1, \ldots, p_{N^*}$ as in (H1) or (H2), the dimensionless Kakinuma model is consistent with the full model for interfacial gravity waves with an error of order $O(\delta_1^{4N+2} + \delta_2^{4N+2})$.

Theorem 1.3 (Consistency)**.** *Assume (H1) or (H2). The solutions to the dimensionless Kakinuma model constructed in Theorem 1.1 produce functions that satisfy approximately the dimensionless full interfacial gravity waves system up to error terms of size* $O(\delta_1^{4N+2} + \delta_2^{4N+2})$*.*

Conversely, solutions to the dimensionless full interfacial gravity waves system satisfying suitable uniform bounds produce through Proposition 1.2 functions that satisfy approximately the dimensionless Kakinuma model up to error terms of size $O(\delta_1^{4N+2} + \delta_2^{4N+2})$*.*

In the last result we assume the existence of a solution to the full model with a uniform bound since for general initial data in Sobolev spaces, one cannot expect to construct a solution to the initial value problem, due to the ill-posedness of the problem discussed previously. The same issue arises for the full justification of the Kakinuma model.

Theorem 1.4 (Full justification)**.** *Assuming the existence of a solution to the dimensionless full interfacial gravity waves system with a uniform bound and satisfying initially the (dimensionless) stability condition* (1.5) *and (dimensionless) non-cavitation assumptions* (1.7)*, then the Kakinuma model with (H1) or (H2) and appropriate initial data produces an approximate solution with the error estimate*

$$|\zeta^{\mathrm{K}}(\boldsymbol{x}, t) - \zeta^{\mathrm{IW}}(\boldsymbol{x}, t)| \lesssim \delta_1^{4N+2} + \delta_2^{4N+2}$$

on some time interval independent of $\delta_1 \in (0, 1]$ *and* $\delta_2 \in (0, 1]$*, where* ζ^{K} *and* ζ^{IW} *are solutions to the dimensionless Kakinuma model and to the full model, respectively.*

In our last main result, we show that the Hamiltonian structure of the Kakinuma model is a shallow water approximation of the Hamiltonian structure of the full interfacial gravity waves model.

Theorem 1.5 (Hamiltonians)**.** *Assume (H1) or (H2). Under appropriate assumptions on the canonical variables* (ζ, ϕ)*, we have*

$$|\mathscr{H}^{\mathrm{K}}(\zeta,\phi) - \mathscr{H}^{\mathrm{IW}}(\zeta,\phi)| \lesssim \delta_1^{4N+2} + \delta_2^{4N+2},$$

where $\mathscr{H}^{\mathrm{K}}$ *and* $\mathscr{H}^{\mathrm{IW}}$ *are the Hamiltonians of the dimensionless Kakinuma model and of the dimensionless full interfacial gravity waves model, respectively.*

Remark 1.6. *The precise statements of our main results are displayed in Section 3. Specifically, Theorem 1.1 corresponds to Theorem 3.1, Proposition 1.2 corresponds to Proposition 3.4, Theorem 1.3 corresponds to Theorem 3.5 and Theorem 3.6 (see also Remark 3.8), Theorem 1.4 corresponds to Theorem 3.9, and Theorem 1.5 corresponds to Theorem 3.10.*

Structures of the Kakinuma model In order to obtain our main results, we exploit several structures of the Kakinuma model. The Kakinuma model can be written compactly as

$$\begin{cases} \boldsymbol{l}_1(H_1)\partial_t\zeta + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ \boldsymbol{l}_2(H_2)\partial_t\zeta - L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ \rho_1\big\{\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \frac{1}{2}\big(|\boldsymbol{u}_1|^2 + w_1^2\big)\big\} \\ \quad -\rho_2\big\{\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 + \frac{1}{2}\big(|\boldsymbol{u}_2|^2 + w_2^2\big)\big\} + (\rho_1-\rho_2)g\zeta = 0, \end{cases} \tag{1.8}$$

where we denote $\boldsymbol{\phi}_1 := (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 := (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$, put $\boldsymbol{l}_1(H_1) := (1, H_1^2, H_1^4, \ldots, H_1^{2N})^{\mathrm{T}}$, $\boldsymbol{l}_2(H_2) := (1, H_2^{p_1}, H_2^{p_2}, \ldots, H_2^{p_{N^*}})^{\mathrm{T}}$, and the linear operators L_ℓ, and functions $\boldsymbol{u}_\ell$ and w_ℓ for $\ell = 1,2$ are defined (after non-dimensionalization) in Section 3. Here we recognize the fact that the hypersurface $t=0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Kakinuma model, since the system of evolution equations is overdetermined for the variable ζ, and underdetermined for the variables $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. As a consequence, solutions to the Kakinuma model must satisfy some compatibility conditions. Introducing linear operators $\mathcal{L}_{1,i}$ $(i = 0,\ldots,N)$ acting on $\boldsymbol{\varphi}_1 = (\varphi_{1,0},\ldots,\varphi_{1,N})^{\mathrm{T}}$ and $\mathcal{L}_{2,i}$ $(i = 0,\ldots,N^*)$ acting on $\boldsymbol{\varphi}_2 = (\varphi_{2,0},\ldots,\varphi_{2,N^*})^{\mathrm{T}}$ by

$$\begin{cases} \mathcal{L}_{1,0}(H_1)\boldsymbol{\varphi}_1 := \displaystyle\sum_{j=0}^{N} L_{1,0j}(H_1)\varphi_{1,j}, \\ \mathcal{L}_{1,i}(H_1)\boldsymbol{\varphi}_1 := \displaystyle\sum_{j=0}^{N}(L_{1,ij}(H_1)\varphi_{1,j} - H_1^{2i}L_{1,0j}(H_1)\varphi_{1,j}) \quad \text{for} \quad i = 1,2,\ldots,N, \\ \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\varphi}_2 := \displaystyle\sum_{j=0}^{N^*} L_{2,0j}(H_2,b)\varphi_{2,j}, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\varphi}_2 := \displaystyle\sum_{j=0}^{N^*}(L_{2,ij}(H_2,b)\varphi_{2,j} - H_2^{p_i}L_{2,0j}(H_2,b)\varphi_{2,j}) \quad \text{for} \quad i = 1,2,\ldots,N^*, \end{cases}$$

the necessary conditions can be written simply as

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i = 1,2,\ldots,N, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i = 1,2,\ldots,N^*, \\ \mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2 = 0. \end{cases} \tag{1.9}$$

The first two vectorial identities are analogous to the compatibility conditions of the Isobe–Kakinuma model for water waves, while the last identity is specific to the bi-layer framework and is related to the continuity of the normal component of the velocity at the interface.

A first key ingredient of the analysis is the fact that for sufficiently regular functions ζ, b and ϕ_1 (respectively ϕ_2), there exists a unique solution $\boldsymbol{\phi}_1$ (respectively $\boldsymbol{\phi}_2$) to the problems

$$
\begin{cases} \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 = \phi_1, \quad \mathcal{L}_{1,i}(H_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i = 1,2,\dots,N, \\ \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 = \phi_2, \quad \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i = 1,2,\dots,N^* \end{cases} \tag{1.10}
$$

satisfying suitable elliptic estimates. What is more, the well-defined linear operators

$$
\begin{aligned}
&\Lambda_1^{(N)}(\zeta)\colon \phi_1 \mapsto \mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1, \\
&\Lambda_2^{(N^*)}(\zeta,b)\colon \phi_2 \mapsto \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2,
\end{aligned}
$$

are found to approximate the corresponding Dirichlet-to-Neumann maps $\Lambda_1(\zeta)$ and $\Lambda_2(\zeta,b)$ defined by

$$
\begin{aligned}
\Lambda_1(\zeta)\phi_1 &:= \big(-\partial_z\Phi_1 + \nabla\Phi_1\cdot\nabla\zeta\big)\big|_{z=\zeta(\boldsymbol{x},t)}, \\
\Lambda_2(\zeta,b)\phi_2 &:= \big(\partial_z\Phi_2 - \nabla\Phi_2\cdot\nabla\zeta\big)\big|_{z=\zeta(\boldsymbol{x},t)},
\end{aligned}
$$

where Φ_1 and Φ_2 are the unique solutions to Laplace's equations

$$
\begin{cases} \Delta\Phi_1 + \partial_z^2\Phi_1 = 0 & \text{in} \quad \Omega_1(t), \\ \Phi_1 = \phi_1 & \text{on} \quad \Gamma(t), \\ \partial_z\Phi_1 = 0 & \text{on} \quad \Sigma_1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in} \quad \Omega_2(t), \\ \Phi_2 = \phi_2 & \text{on} \quad \Gamma(t), \\ \nabla\Phi_2\cdot\nabla b - \partial_z\Phi_2 = 0 & \text{on} \quad \Sigma_2, \end{cases}
$$

where we denote the upper layer, the lower layer, the interface, the rigid-lid, and the bottom at time t by $\Omega_1(t)$, $\Omega_2(t)$, $\Gamma(t)$, Σ_1, and Σ_2, respectively. Specifically it is proved that, under the special choice of the indices $p_0, p_1, \dots, p_{N^*}$ in (H1) or (H2) and after suitable rescaling, the difference between the dimensionless operators is of size $O(\delta_1^{4N+2} + \delta_2^{4N+2})$. This analysis, which follows directly from the corresponding analysis for surface waves developed in [20] and scaling arguments, provides the key argument in the proof of the consistency result described in Theorem 1.3.

In order to study the Kakinuma model, we also need to analyze the full elliptic system

$$
\begin{cases} \mathcal{L}_{1,i}(H_1)\boldsymbol{\phi}_1 = f_{1,i} \quad \text{for} \quad i = 1,2,\dots,N, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\phi}_2 = f_{2,i} \quad \text{for} \quad i = 1,2,\dots,N^*, \\ \mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2 = \nabla\cdot\boldsymbol{f}_3, \\ -\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 + \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 = f_4, \end{cases} \tag{1.11}
$$

for sufficiently regular functions ζ, b and $\boldsymbol{f}_1 = (f_{1,1},\dots f_{1,N})^{\mathrm{T}}, \boldsymbol{f}_2 = (f_{2,1},\dots,f_{2,N^*})^{\mathrm{T}}, \boldsymbol{f}_3, f_4$. The ellipticity of the problem relies on the coercivity of the corresponding operators $L_1(H_1)$ and $L_2(H_2)$. The solvability of (1.11) is essential in several directions. Firstly, it provides an alternative consistency result, where solutions to the full interfacial gravity waves system produce approximate solutions to the Kakinuma model but satisfying exactly and not approximately the necessary conditions (1.9). In turn, this provides a crucial ingredient to the full justification of the Kakinuma model described in Theorem 1.4. Furthermore, the arrangement of initial data satisfying the compatibility conditions as stated in Proposition 1.2 amounts to solving (1.11) with $\boldsymbol{f}_1=\mathbf{0}$, $\boldsymbol{f}_2=\mathbf{0}$, $\boldsymbol{f}_3=\mathbf{0}$ and $f_4=\phi$. Similarly, our result on the Hamiltonians $\mathscr{H}^{\mathrm{K}}$ and $\mathscr{H}^{\mathrm{IW}}$ described in Theorem 1.5 relies on a comparison of solutions to (1.11) with $\boldsymbol{f}_1=\mathbf{0}$, $\boldsymbol{f}_2=\mathbf{0}$, $\boldsymbol{f}_3=\mathbf{0}$ and $f_4=\phi$ and solutions to

$$\begin{cases}\Delta\Phi_1+\partial_z^2\Phi_1=0 & \text{in} \quad \Omega_1(t),\\ \Delta\Phi_2+\partial_z^2\Phi_2=0 & \text{in} \quad \Omega_2(t),\\ \partial_z\Phi_1=0 & \text{on} \quad \Sigma_1,\\ \nabla\Phi_2\cdot\nabla b-\partial_z\Phi_2=0 & \text{on} \quad \Sigma_2,\\ (\nabla\Phi_1\cdot\nabla\zeta-\partial_z\Phi_1)-(\nabla\Phi_2\cdot\nabla\zeta-\partial_z\Phi_2)=0 & \text{on} \quad \Gamma(t),\\ \rho_2\Phi_2-\rho_1\Phi_1=\phi & \text{on} \quad \Gamma(t),\end{cases}$$

thus extending to the interfacial gravity waves framework the analysis in [13]. Finally, the solvability of (1.11) allows to determine and control time derivatives $\partial_t\boldsymbol{\phi}_1$ and $\partial_t\boldsymbol{\phi}_2$ of sufficiently regular solutions to the Kakinuma model (1.8) by using the equations obtained when differentiating with respect to time the compatibility conditions (1.9) combined with the last equation of (1.8). This is a crucial ingredient for the analysis of the initial value problem.

Another crucial ingredient for the analysis of the initial value problem concerns uniform energy estimates on the linearized Kakinuma system. To this end, we write the linearized system under the form

$$\mathscr{A}_1(\partial_t+\boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}+\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}=\dot{\boldsymbol{F}}, \tag{1.12}$$

where $\dot{\boldsymbol{U}}:=(\dot{\zeta},\dot{\boldsymbol{\phi}}_1,\dot{\boldsymbol{\phi}}_2)^{\mathrm{T}}$ is the deviation from the reference state $\boldsymbol{U}:=(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)^{\mathrm{T}}$, $\boldsymbol{u}$ is a suitable velocity which is a convex combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ whose weights depend on ρ_ℓ, H_ℓ as well as α_ℓ ($\ell=1,2$) the positive constants mentioned previously, $\dot{\boldsymbol{F}}$ represent lower order terms and $\mathscr{A}_1:=\mathscr{A}_1(\boldsymbol{U})$ is a skew-symmetric matrix and $\mathscr{A}_0^{\mathrm{mod}}:=\mathscr{A}_0^{\mathrm{mod}}(\boldsymbol{U})$ is a linear operator symmetric in L^2. The energy function associated to (1.12) is given by $(\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}},\dot{\boldsymbol{U}})_{L^2}$, and we prove that

$$(\mathscr{A}_0^{\mathrm{mod}}\dot{U},\dot{U})_{L^2}\simeq\mathscr{E}(\dot{\boldsymbol{U}}):=\|\dot{\zeta}\|_{L^2}^2+\sum_{\ell=1,2}\rho_\ell(\|\nabla\dot{\boldsymbol{\phi}}_\ell\|_{L^2}^2+\|\dot{\boldsymbol{\phi}}_\ell'\|_{L^2}^2)$$

under the non-cavitation assumption (1.7) and the stability condition (1.5). Because the structure of (1.12) is not standard, the control of the energy function is obtained by testing (1.12) with the time derivatives, $\partial_t\dot{\boldsymbol{U}}$. This, together with suitable product and commutator estimates in Sobolev spaces, provides the *a priori* control of the energy function for solutions to the Kakinuma model and their derivatives, and we show that this control is uniform in the shallow water regime after suitable rescaling. Since the construction and uniqueness of a solution was obtained in the companion paper [14], the uniform estimates provide the proof of the long-time well-posedness of the initial value problem for the Kakinuma model result stated in Theorem 1.1. Furthermore, using the aforementioned consistency result, we prove that the difference between solutions to the full interfacial gravity waves system and corresponding solutions to the Kakinuma model satisfy an identity analogous to (1.12), and hence infer a control of the energy function of the difference and its derivatives, which yields the full justification of the Kakinuma model stated in Theorem 1.4.

Outline The contents of this paper are as follows. In Section 2 we first recall the basic equations governing the interfacial gravity waves and write down the Kakinuma model that we are going to analyze in this paper, and then rewrite them in a nondimensional form by introducing several nondimensional parameters. Hamiltonians of the full model and of the Kakinuma model in the nondimensional variables are also provided. In Section 3 we first introduce some differential operators, which enable us to write the Kakinuma model simply in the form (1.8), and then we present the precise statements of our main results in this paper. In Section 4 we first recall results in the framework of surface waves related to the consistency of the Isobe–Kakinuma model, and then prove Theorems 3.5 and 3.6 concerning the consistency of the Kakinuma model by a simple scaling argument. In Section 5 we first derive an elliptic estimate related to the compatibility conditions for the Kakinuma model, which explains how to prepare the initial data, as stated in Proposition 3.4. Then we give uniform a priori bounds on regular solutions to the Kakinuma model, especially, a priori bounds of time derivatives. In Section 6 we provide uniform energy estimates for the solution to the Kakinuma model and prove Theorem 3.1, which ensures the existence of the solution to the initial value problem for the Kakinuma model on a time interval independent of parameters, especially, δ_1 and δ_2, under the stability condition, the non-cavitation assumptions, and intrinsic compatibility conditions on the initial data, together with a uniform bound of the solution. In Section 7 we first give a supplementary estimate on an approximation of the Dirichlet-to-Neumann map, and then revisit the consistency of the Kakinuma model. We prove Proposition 7.6, which is another version of the consistency given in Theorem 3.6, where we adopt a different construction of an approximate solution to the Kakinuma model from the solution to the full model. Then, by making use of the well-posedness of the initial value problem for the Kakinuma model we prove Theorem 3.9 which provides a conditional rigorous justification of the Kakinuma model, that is, assuming the existence of a solution to the full model with a uniform bound we derive an error estimate between a corresponding solution to the Kakinuma model and that of the full model. Finally, in Section 8 we prove Theorem 3.10 which gives an error estimate between the Hamiltonian of the Kakinuma model and that of the full model. For the convenience of the reader, the structure of the paper and proofs dependencies are sketched in Figure 1.2.

Notation We denote by $W^{m,p}$ the L^p Sobolev space of order m on $\mathbf{R}^n$ and $H^m = W^{m,2}$. We put $\mathring{H}^m = \{\phi\,;\, \nabla\phi \in H^{m-1}\}$. The norm of a Banach space B is denoted by $\|\cdot\|_B$. The L^2-inner product is denoted by $(\cdot,\cdot)_{L^2}$. We put $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$, and $\partial_z = \frac{\partial}{\partial z}$. $[P,Q] = PQ - QP$ denotes the commutator and $[P;u,v] = P(uv) - (Pu)v - u(Pv)$ denotes the symmetric commutator. For a matrix A we denote by A^{T} the transpose of A. O denotes a zero matrix. For a vector $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\mathrm{T}}$ we denote the last N components by $\boldsymbol{\phi}' = (\phi_1, \dots, \phi_N)^{\mathrm{T}}$. $f \lesssim g$ means that there exists a non-essential positive constant C such that $f \leq Cg$ holds. $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$ hold.

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2 The basic equations and the Kakinuma model

2.1 Equations with physical variables

We first recall the equations governing potential flows for two layers of immiscible, incompressible, homogeneous, and inviscid fluids, and then write down the Kakinuma model at stake in this work. In the following, we denote the upper layer, the lower layer, the interface, the rigid-lid,

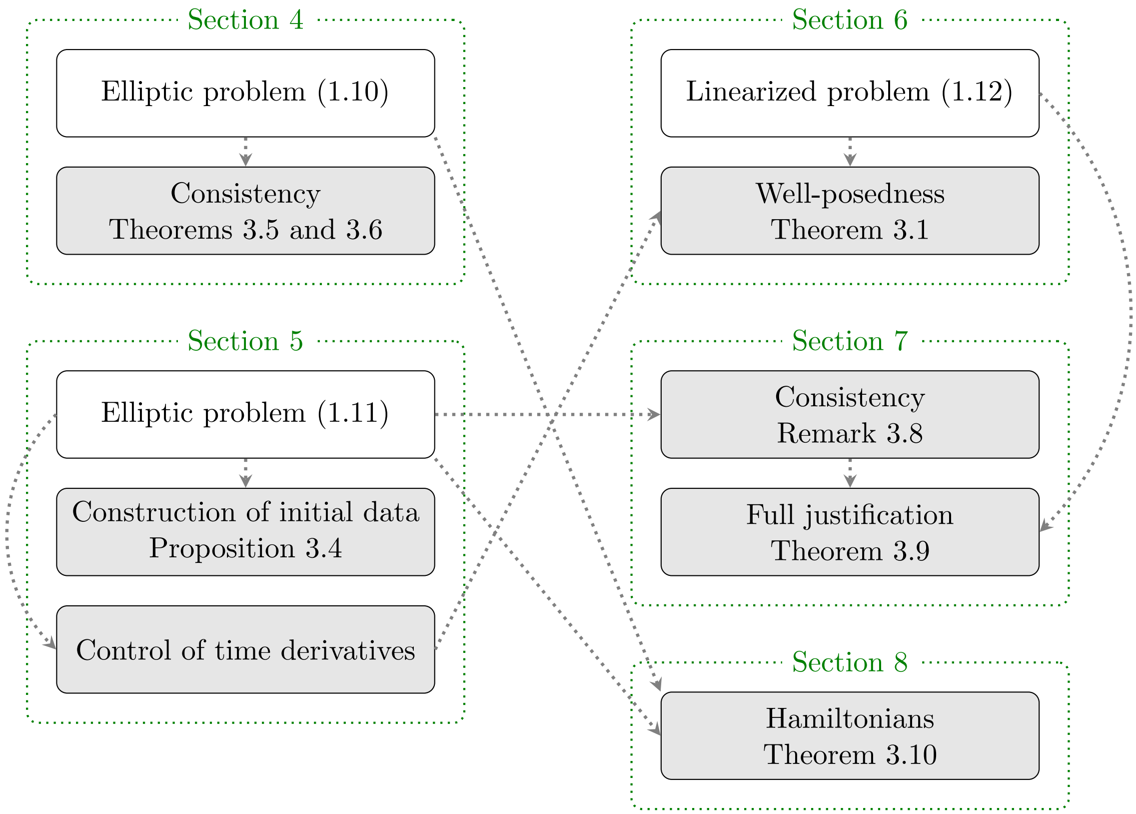

Figure 1.2: Articulation of the proofs.

and the bottom at time t by $\Omega_1(t)$, $\Omega_2(t)$, $\Gamma(t)$, Σ_1, and Σ_2, respectively. The velocity potentials $\Phi_1(\boldsymbol{x}, z, t)$ and $\Phi_2(\boldsymbol{x}, z, t)$ in the upper and lower layers, respectively, satisfy Laplace's equations

$$\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 \quad \text{in} \quad \Omega_1(t), \tag{2.1}$$

$$\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 \quad \text{in} \quad \Omega_2(t), \tag{2.2}$$

where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian with respect to the horizontal space variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. Bernoulli's laws of each layers have the form

$$\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2) + gz\right) + P_1 = 0 \quad \text{in} \quad \Omega_1(t), \tag{2.3}$$

$$\rho_2\left(\partial_t\Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2) + gz\right) + P_2 = 0 \quad \text{in} \quad \Omega_2(t), \tag{2.4}$$

where $\nabla = (\partial_1, \ldots, \partial_n)$, the positive constant g is the acceleration due to gravity, and $P_1(\boldsymbol{x}, z, t)$ and $P_2(\boldsymbol{x}, z, t)$ are pressures in the upper and lower layers, respectively. The dynamical boundary condition on the interface is given by

$$P_1 = P_2 \quad \text{on} \quad \Gamma(t). \tag{2.5}$$

The kinematic boundary conditions on the interface, the rigid-lid, and the bottom are given by

$$\partial_t\zeta + \nabla\Phi_1\cdot\nabla\zeta - \partial_z\Phi_1 = 0 \quad \text{on} \quad \Gamma(t), \tag{2.6}$$

$$\partial_t\zeta + \nabla\Phi_2\cdot\nabla\zeta - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Gamma(t), \tag{2.7}$$

$$\partial_z\Phi_1 = 0 \quad \text{on} \quad \Sigma_1, \tag{2.8}$$

$$\nabla\Phi_2\cdot\nabla b - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Sigma_2. \tag{2.9}$$

These are the basic equations for interfacial gravity waves. It follows from Bernoulli's laws (2.3)–(2.4) and the dynamical boundary condition (2.5) that

$$\begin{aligned}\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2)\right)& \\ - \rho_2\left(\partial_t\Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2)\right) &= (\rho_2-\rho_1)g\zeta \quad \text{on} \quad \Gamma(t).\end{aligned} \tag{2.10}$$

We will always assume the stable stratification condition $(\rho_2-\rho_1)g > 0$. As in the case of surface water waves, the basic equations have a variational structure and the corresponding Luke's Lagrangian is given, up to terms which do not contribute to the variation of the Lagrangian, by the vertical integral of the pressure in the water regions. After using Bernoulli's laws (2.3)–(2.4) we can find the Lagrangian density

$$\begin{aligned}\mathscr{L}(\Phi_1,\Phi_2,\zeta) = -\rho_1\int_\zeta^{h_1}&\left(\partial_t\Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2)\right)\mathrm{d}z \\ &- \rho_2\int_{-h_2+b}^{\zeta}\left(\partial_t\Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2)\right)\mathrm{d}z - \frac{1}{2}(\rho_2-\rho_1)g\zeta^2.\end{aligned} \tag{2.11}$$

In fact, one checks readily that (2.1)–(2.2) and (2.6)–(2.10) are Euler–Lagrange equations associated with the action function

$$\mathscr{J}(\Phi_1,\Phi_2,\zeta) := \int_{t_0}^{t_1}\int_{\mathbf{R}^n}\mathscr{L}(\Phi_1,\Phi_2,\zeta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

We proceed to the Kakinuma model. Let N and N^* be nonnegative integers. In view of the analysis for the Isobe–Kakinuma model for surface water waves, we approximate the velocity potentials Φ_1 and Φ_2 in the Lagrangian by

$$\begin{cases}\Phi_1^{\mathrm{app}}(\boldsymbol{x},z,t) = \displaystyle\sum_{i=0}^{N}(-z+h_1)^{2i}\phi_{1,i}(\boldsymbol{x},t), \\ \Phi_2^{\mathrm{app}}(\boldsymbol{x},z,t) = \displaystyle\sum_{i=0}^{N^*}(z+h_2-b(\boldsymbol{x}))^{p_i}\phi_{2,i}(\boldsymbol{x},t),\end{cases} \tag{2.12}$$

where $p_0,p_1,\ldots,p_{N^*}$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_{N^*}$. Plugging (2.12) into the Lagrangian density (2.11), we obtain an approximate Lagrangian density

$$\mathscr{L}^{\mathrm{app}}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\zeta) := \mathscr{L}(\Phi_1^{\mathrm{app}},\Phi_2^{\mathrm{app}},\zeta),$$

where $\boldsymbol{\phi}_1 := (\phi_{1,0}, \phi_{1,1}, \dots, \phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2 := (\phi_{2,0}, \phi_{2,1}, \dots, \phi_{2,N^*})^{\mathrm{T}}$. The corresponding Euler–Lagrange equation is the Kakinuma model, which has the form

(2.13)

$$\begin{cases} H_1^{2i}\partial_t\zeta - \displaystyle\sum_{j=0}^{N}\biggl\{\nabla\cdot\biggl(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\biggr) - \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,j}\biggr\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \dots, N, \\ H_2^{p_i}\partial_t\zeta + \displaystyle\sum_{j=0}^{N^*}\biggl\{\nabla\cdot\biggl(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\biggr) \\ \qquad\qquad + \dfrac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\phi_{2,j} - \dfrac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,j}\biggr\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \dots, N^*, \\ \rho_1\biggl\{\displaystyle\sum_{j=0}^{N}H_1^{2j}\partial_t\phi_{1,j} + g\zeta + \frac{1}{2}\biggl(\biggl|\sum_{j=0}^{N}H_1^{2j}\nabla\phi_{1,j}\biggr|^2 + \biggl(\sum_{j=0}^{N}2jH_1^{2j-1}\phi_{1,j}\biggr)^2\biggr)\biggr\} \\ \quad -\rho_2\biggl\{\displaystyle\sum_{j=0}^{N^*}H_2^{p_j}\partial_t\phi_{2,j} + g\zeta \\ \qquad + \dfrac{1}{2}\biggl(\biggl|\displaystyle\sum_{j=0}^{N^*}(H_2^{p_j}\nabla\phi_{2,j} - p_jH_2^{p_j-1}\phi_{2,j}\nabla b)\biggr|^2 + \biggl(\sum_{j=0}^{N^*}p_jH_2^{p_j-1}\phi_{2,j}\biggr)^2\biggr)\biggr\} = 0, \end{cases}$$

where H_1 and H_2 are depths of the upper and the lower layers, that is,

$$H_1(t,\boldsymbol{x}) := h_1 - \zeta(\boldsymbol{x},t), \qquad H_2(\boldsymbol{x},t) := h_2 + \zeta(\boldsymbol{x},t) - b(\boldsymbol{x}).$$

In (2.13), we used the notational convention $0/0 = 0$. More precisely, this convention was used so as to dictate $p_0/(p_0+p_0) = 0$ and $p_0p_1/(p_0+p_1-1) = 0$ in the case $p_1 = 1$. We recall also that $p_0 = 0$ is always assumed.

2.2 The dimensionless equations

In order to rigorously validate the Kakinuma model (2.13) as a higher order shallow water approximation of the full model for interfacial gravity waves (2.1)–(2.9), we first introduce nondimensional parameters and then non-dimensionalize the equations, through a convenient rescaling of variables. Let λ be a typical horizontal wavelength. Following D. Lannes [27], we introduce a nondimensional parameter δ by

$$\delta := \frac{h}{\lambda} \qquad \text{with} \quad h := \frac{h_1h_2}{\underline{\rho}_1h_2 + \underline{\rho}_2h_1},$$

where $\underline{\rho}_1$ and $\underline{\rho}_2$ are relative densities. We also need to use relative depths $\underline{h}_1$ and $\underline{h}_2$ of the layers. These nondimensional parameters are defined by

$$\underline{\rho}_\ell := \frac{\rho_\ell}{\rho_1+\rho_2}, \qquad \underline{h}_\ell := \frac{h_\ell}{h} \qquad (\ell = 1, 2),$$

which satisfy the relations

$$\underline{\rho}_1 + \underline{\rho}_2 = 1, \qquad \frac{\underline{\rho}_1}{\underline{h}_1} + \frac{\underline{\rho}_2}{\underline{h}_2} = 1. \tag{2.14}$$

Note also that $\min\{h_1, h_2\} \le h \le \max\{h_1, h_2\}$. It follows from the second relation in (2.14) that

$$1 < \min\left\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\right\} \le 2. \tag{2.15}$$

Here, we note that the standard shallowness parameters $\delta_1 := \frac{h_1}{\lambda}$ and $\delta_2 := \frac{h_2}{\lambda}$ relative to the upper and the lower layers, respectively, are related to the above parameters by $\delta_\ell = \underline{h}_\ell \delta$ for $\ell = 1, 2$. In many results of this paper, we restrict our consideration to the parameter regime

$$\underline{h}_1^{-1} + \underline{h}_2^{-1} \lesssim 1. \tag{2.16}$$

To understand this restriction, it is convenient to use nondimensional parameters $\gamma := \frac{\rho_1}{\rho_2}$ and $\theta := \frac{h_1}{h_2}$. In terms of these parameters, $\underline{h}_\ell^{-1}$ $(\ell = 1, 2)$ can be represented as

$$\underline{h}_1^{-1} = \frac{\gamma + 1}{\gamma + \theta}, \qquad \underline{h}_2^{-1} = \frac{\gamma^{-1} + 1}{\gamma^{-1} + \theta^{-1}}.$$

Therefore, the only cases that (2.16) excludes are the case $\gamma, \theta \ll 1$ and the case $\gamma, \theta \gg 1$. Since we shall also assume the stable stratification condition $(\rho_2 - \rho_1)g > 0$, we can describe the two regimes considered in this paper as

(i) $\gamma \simeq 1$, i.e., $\rho_1 \simeq \rho_2$,

(ii) $\gamma \ll 1$ and $\theta \gtrsim 1$, i.e., $\rho_1 \ll \rho_2$ and $h_2 \lesssim h_1$.

Introducing $c_{\mathrm{SW}} := \sqrt{(\underline{\rho}_2 - \underline{\rho}_1)gh}$ the speed of infinitely long and small interfacial gravity waves, we rescale the independent and the dependent variables by

$$\boldsymbol{x} = \lambda \tilde{\boldsymbol{x}}, \quad z = h\tilde{z}, \quad t = \frac{\lambda}{c_{\mathrm{SW}}}\tilde{t}, \quad \zeta = h\tilde{\zeta}, \quad b = h\tilde{b}, \quad \Phi_\ell = \lambda c_{\mathrm{SW}} \tilde{\Phi}_\ell \quad (\ell = 1, 2).$$

Plugging these into the full model (2.1)–(2.2) and (2.6)–(2.10) and dropping the tilde sign in the notation we obtain

$$\begin{cases} \Delta\Phi_1 + \delta^{-2}\partial_z^2\Phi_1 = 0 & \text{in} \quad \Omega_1(t), \\ \Delta\Phi_2 + \delta^{-2}\partial_z^2\Phi_2 = 0 & \text{in} \quad \Omega_2(t), \\ \partial_t\zeta + \nabla\Phi_1 \cdot \nabla\zeta - \delta^{-2}\partial_z\Phi_1 = 0 & \text{on} \quad \Gamma(t), \\ \partial_t\zeta + \nabla\Phi_2 \cdot \nabla\zeta - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Gamma(t), \\ \partial_z\Phi_1 = 0 & \text{on} \quad \Sigma_1, \\ \nabla\Phi_2 \cdot \nabla b - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Sigma_2, \\ \underline{\rho}_1\Big(\partial_t\Phi_1 + \frac12|\nabla\Phi_1|^2 + \frac12\delta^{-2}(\partial_z\Phi_1)^2\Big) & \\ \quad -\underline{\rho}_2\Big(\partial_t\Phi_2 + \frac12|\nabla\Phi_2|^2 + \frac12\delta^{-2}(\partial_z\Phi_2)^2\Big) - \zeta = 0 & \text{on} \quad \Gamma(t), \end{cases}$$

where in this scaling the upper layer $\Omega_1(t)$, the lower layer $\Omega_2(t)$, the interface $\Gamma(t)$, the rigid-lid Σ_1, and the bottom Σ_2 are written as

$$\begin{aligned} &\Omega_1(t) = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, \zeta(\boldsymbol{x}, t) < z < \underline{h}_1\}, \\ &\Omega_2(t) = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, -\underline{h}_2 + b(\boldsymbol{x}) < z < \zeta(\boldsymbol{x}, t)\}, \\ &\Gamma(t) = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = \zeta(\boldsymbol{x}, t)\}, \\ &\Sigma_1 = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = \underline{h}_1\}, \\ &\Sigma_2 = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = -\underline{h}_2 + b(\boldsymbol{x})\}. \end{aligned}$$

Denoting

$$\phi_\ell(\boldsymbol{x},t) := \Phi_\ell(\boldsymbol{x},\zeta(\boldsymbol{x},t),t) \qquad (\ell=1,2)$$

and using the chain rule, the above system can be written in a more compact and closed form as

$$(2.17)\qquad \begin{cases} \partial_t\zeta + \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 = 0, \\ \partial_t\zeta - \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 = 0, \\ \underline{\rho}_1\left(\partial_t\phi_1 + \dfrac{1}{2}|\nabla\phi_1|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 - \nabla\zeta\cdot\nabla\phi_1)^2}{1+\delta^2|\nabla\zeta|^2}\right) \\ \quad -\underline{\rho}_2\left(\partial_t\phi_2 + \dfrac{1}{2}|\nabla\phi_2|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 + \nabla\zeta\cdot\nabla\phi_2)^2}{1+\delta^2|\nabla\zeta|^2}\right) - \zeta = 0, \end{cases}$$

where $\Lambda_1(\zeta,\delta,\underline{h}_1)$ and $\Lambda_2(\zeta,b,\delta,\underline{h}_2)$ are the Dirichlet-to-Neumann maps for Laplace's equations. More precisely, these are defined by

$$\begin{aligned} \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 &:= \big(-\delta^{-2}\partial_z\Phi_1 + \nabla\Phi_1\cdot\nabla\zeta\big)\big|_{z=\zeta(\boldsymbol{x},t)}, \\ \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 &:= \big(\delta^{-2}\partial_z\Phi_2 - \nabla\Phi_2\cdot\nabla\zeta\big)\big|_{z=\zeta(\boldsymbol{x},t)}, \end{aligned}$$

where Φ_1 and Φ_2 are unique solutions to the boundary value problems

$$\begin{cases} \Delta\Phi_1 + \delta^{-2}\partial_z^2\Phi_1 = 0 & \text{in} \quad \Omega_1(t), \\ \Phi_1 = \phi_1 & \text{on} \quad \Gamma(t), \\ \partial_z\Phi_1 = 0 & \text{on} \quad \Sigma_1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\Phi_2 + \delta^{-2}\partial_z^2\Phi_2 = 0 & \text{in} \quad \Omega_2(t), \\ \Phi_2 = \phi_2 & \text{on} \quad \Gamma(t), \\ \nabla\Phi_2\cdot\nabla b - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Sigma_2. \end{cases}$$

As for the Kakinuma model, we introduce additionally the rescaled variables

$$\phi_{1,i} := \frac{\lambda c_{\mathrm{SW}}}{h_1^{2i}}\tilde{\phi}_{1,i}, \qquad \phi_{2,i} := \frac{\lambda c_{\mathrm{SW}}}{h_2^{p_i}}\tilde{\phi}_{2,i},$$

where we recall that $p_0, p_1, \dots, p_{N^*}$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_{N^*}$ appearing in the approximation (2.12). Plugging these and the previous scaling into the Kakinuma model (2.13) and dropping the tilde sign in the notation we obtain the Kakinuma model in the nondimensional form, which is written as

(2.18)

$$\begin{cases} H_1^{2i}\partial_t\zeta - \underline{h}_1\displaystyle\sum_{j=0}^{N}\left\{\nabla\cdot\left(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\right) - \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}(\underline{h}_1\delta)^{-2}\phi_{1,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0,1,\dots,N, \\ H_2^{p_i}\partial_t\zeta + \underline{h}_2\displaystyle\sum_{j=0}^{N^*}\left\{\nabla\cdot\left(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\underline{h}_2^{-1}\nabla b\right)\right. \\ \qquad \left. + \dfrac{p_i}{p_i+p_j}H_2^{p_i+p_j}\underline{h}_2^{-1}\nabla b\cdot\nabla\phi_{2,j} - \dfrac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}((\underline{h}_2\delta)^{-2} + \underline{h}_2^{-2}|\nabla b|^2)\phi_{2,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0,1,\dots,N^*, \\ \underline{\rho}_1\left\{\displaystyle\sum_{j=0}^{N}H_1^{2j}\partial_t\phi_{1,j} + \frac{1}{2}\left(\left|\sum_{j=0}^{N}H_1^{2j}\nabla\phi_{1,j}\right|^2 + (\underline{h}_1\delta)^{-2}\left(\sum_{j=0}^{N}2jH_1^{2j-1}\phi_{1,j}\right)^2\right)\right\} \\ \qquad -\underline{\rho}_2\left\{\displaystyle\sum_{j=0}^{N^*}H_2^{2j}\partial_t\phi_{2,j} + \frac{1}{2}\left(\left|\sum_{j=0}^{N^*}(H_2^{p_j}\nabla\phi_{2,j} - p_jH_2^{p_j-1}\phi_{2,j}\underline{h}_2^{-1}\nabla b)\right|^2\right.\right. \\ \qquad\qquad\qquad\qquad \left.\left. + (\underline{h}_2\delta)^{-2}\left(\displaystyle\sum_{j=0}^{N^*}p_jH_2^{p_j-1}\phi_{2,j}\right)^2\right)\right\} - \zeta = 0, \end{cases}$$

where we used the notational convention $0/0=0$, and

$$H_1(\boldsymbol{x},t) := 1-\underline{h}_1^{-1}\zeta(\boldsymbol{x},t), \qquad H_2(\boldsymbol{x},t) := 1+\underline{h}_2^{-1}\zeta(\boldsymbol{x},t)-\underline{h}_2^{-1}b(\boldsymbol{x}). \tag{2.19}$$

We impose the initial conditions to the Kakinuma model of the form

$$(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2) = (\zeta_{(0)},\boldsymbol{\phi}_{1(0)},\boldsymbol{\phi}_{2(0)}) \quad \text{at} \quad t=0. \tag{2.20}$$

2.3 Hamiltonian structures

T. B. Benjamin and T. J. Bridges [3] found that the full model for interfacial gravity waves can be written in Hamilton's canonical form

$$\partial_t\zeta = \frac{\delta\mathscr{H}^{\mathrm{IW}}}{\delta\phi}, \qquad \partial_t\phi = -\frac{\delta\mathscr{H}^{\mathrm{IW}}}{\delta\zeta},$$

where the canonical variable ϕ is defined by

$$\phi = \underline{\rho}_2\phi_2 - \underline{\rho}_1\phi_1 \tag{2.21}$$

and the Hamiltonian $\mathscr{H}^{\mathrm{IW}}$ is the total energy $\mathscr{E}$ written in terms of the canonical variables (ζ,ϕ). Specifically, $\mathscr{E}$ is the sum of the kinetic energies of the fluids in the upper and the lower layers and the potential energy due to the gravity defined as

$$\begin{aligned}\mathscr{E} &:= \sum_{\ell=1,2}\iint_{\Omega_\ell(t)}\frac{1}{2}\underline{\rho}_\ell\big(|\nabla\Phi_\ell(\boldsymbol{x},z,t)|^2+\delta^{-2}(\partial_z\Phi_\ell(\boldsymbol{x},z,t))^2\big)\mathrm{d}\boldsymbol{x}\mathrm{d}z + \int_{\mathbf{R}^n}\frac{1}{2}\zeta(\boldsymbol{x},t)^2\mathrm{d}\boldsymbol{x}\\ &= \sum_{\ell=1,2}\frac{1}{2}\underline{\rho}_\ell(\Lambda_\ell(\zeta)\phi_\ell(t),\phi_\ell(t))_{L^2} + \frac{1}{2}\|\zeta(t)\|_{L^2}^2.\end{aligned}$$

Here and in what follows, we denote simply $\Lambda_1(\zeta)=\Lambda_1(\zeta,\delta,\underline{h}_1)$ and $\Lambda_2(\zeta)=\Lambda_2(\zeta,b,\delta,\underline{h}_2)$. It follows from the kinematic boundary conditions on the interface that $\Lambda_1(\zeta)\phi_1+\Lambda_2(\zeta)\phi_2=0$, so that ϕ_1 and ϕ_2 can be written in terms of the canonical variables (ζ,ϕ) as

$$\begin{cases}\phi_1 = -(\underline{\rho}_1\Lambda_2(\zeta)+\underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_2(\zeta)\phi,\\ \phi_2 = (\underline{\rho}_1\Lambda_2(\zeta)+\underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_1(\zeta)\phi.\end{cases}$$

Therefore, the Hamiltonian $\mathscr{H}^{\mathrm{IW}}(\zeta,\phi)$ of the full model for interfacial gravity waves is given explicitly by

$$\mathscr{H}^{\mathrm{IW}}(\zeta,\phi) = \frac{1}{2}((\underline{\rho}_1\Lambda_2(\zeta)+\underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_1(\zeta)\phi,\Lambda_2(\zeta)\phi)_{L^2} + \frac{1}{2}\|\zeta\|_{L^2}^2. \tag{2.22}$$

As was shown in the companion paper [14], the Kakinuma model (2.18) also enjoys a Hamiltonian structure analogous to that of the full model for interfacial gravity waves. The canonical variables are the elevation of the interface ζ and ϕ defined by

$$\begin{aligned}\phi(\boldsymbol{x},t) &:= \underline{\rho}_2\Phi_2^{\mathrm{app}}(\boldsymbol{x},\zeta(\boldsymbol{x},t),t) - \underline{\rho}_1\Phi_1^{\mathrm{app}}(\boldsymbol{x},\zeta(\boldsymbol{x},t),t)\\ &= \underline{\rho}_2\sum_{i=0}^{N^*}H_2(\boldsymbol{x},t)^{p_i}\phi_{2,i}(\boldsymbol{x},t) - \underline{\rho}_1\sum_{i=0}^{N}H_1(\boldsymbol{x},t)^{2i}\phi_{1,i}(\boldsymbol{x},t),\end{aligned} \tag{2.23}$$

where Φ_ℓ^{app} ($\ell = 1,2$) are nondimensional versions of the approximate velocity potentials, which are defined by

$$
\tag{2.24}
\begin{cases}
\Phi_1^{\mathrm{app}}(\boldsymbol{x}, z, t) := \displaystyle\sum_{i=0}^{N} (1 - \underline{h}_1^{-1} z)^{2i} \phi_{1,i}(\boldsymbol{x}, t), \\
\Phi_2^{\mathrm{app}}(\boldsymbol{x}, z, t) := \displaystyle\sum_{i=0}^{N^*} (1 + \underline{h}_2^{-1}(z - b(\boldsymbol{x})))^{p_i} \phi_{2,i}(\boldsymbol{x}, t),
\end{cases}
$$

and H_ℓ ($\ell = 1,2$) are depths of the upper and lower layers defined by (2.19). We note that if the canonical variables (ζ, ϕ) are given, then the Kakinuma model (2.18) determines $\boldsymbol{\phi}_1 = (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2 = (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$, which are unique up to an additive constant of the form $(\mathcal{C}\underline{\rho}_1, \mathcal{C}\underline{\rho}_2)$ to $(\phi_{1,0}, \phi_{2,0})$. For details, we refer to [14, Subsection 8.1] and Lemma 5.1 in Section 5. Then, the Hamiltonian $\mathscr{H}^{\mathrm{K}}(\zeta, \phi)$ of the Kakinuma model is given by

$$
\tag{2.25}
\mathscr{H}^{\mathrm{K}}(\zeta, \phi) := \sum_{\ell=1,2} \iint_{\Omega_\ell} \frac{1}{2} \underline{\rho}_\ell \big(|\nabla \Phi_\ell^{\mathrm{app}}(\boldsymbol{x}, z, t)|^2 + \delta^{-2} (\partial_z \Phi_\ell^{\mathrm{app}}(\boldsymbol{x}, z, t))^2 \big) \mathrm{d}\boldsymbol{x} \mathrm{d}z + \int_{\mathbf{R}^n} \frac{1}{2} \zeta(\boldsymbol{x}, t)^2 \mathrm{d}\boldsymbol{x}.
$$

3 Statements of the main results

Before stating the main results in this paper, let us introduce some notations which allow in particular to rewrite (2.18) in a compact form. We introduce second order differential operators $L_{1,ij} = L_{1,ij}(H_1, \delta, \underline{h}_1)$ ($i, j = 0, 1, \ldots, N$) and $L_{2,ij} = L_{2,ij}(H_2, b, \delta, \underline{h}_2)$ ($i, j = 0, 1, \ldots, N^*$) by

$$
\tag{3.1}
L_{1,ij} \varphi_{1,j} := -\nabla \cdot \left(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla \varphi_{1,j} \right) + \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} (\underline{h}_1 \delta)^{-2} \varphi_{1,j},
$$

$$
\tag{3.2}
\begin{aligned}
L_{2,ij} \varphi_{2,j} := & -\nabla \cdot \left(\frac{1}{p_i + p_j + 1} H_2^{p_i + p_j + 1} \nabla \varphi_{2,j} - \frac{p_j}{p_i + p_j} H_2^{p_i + p_j} \varphi_{2,j} \underline{h}_2^{-1} \nabla b \right) \\
& - \frac{p_i}{p_i + p_j} H_2^{p_i + p_j} \underline{h}_2^{-1} \nabla b \cdot \nabla \varphi_{2,j} \\
& + \frac{p_i p_j}{p_i + p_j - 1} H_2^{p_i + p_j - 1} ((\underline{h}_2 \delta)^{-2} + \underline{h}_2^{-2} |\nabla b|^2) \varphi_{2,j},
\end{aligned}
$$

where we use the notational convention $0/0 = 0$. Notice that we have $(L_{\ell,ij})^* = L_{\ell,ji}$ for $\ell = 1,2$, where $(L_{\ell,ij})^*$ is the adjoint operator of $L_{\ell,ij}$ in $L^2(\mathbf{R}^n)$. We put $\boldsymbol{\phi}_1 := (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 := (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$, and

$$
\tag{3.3}
\begin{cases}
\boldsymbol{l}_1(H_1) := (1, H_1^2, H_1^4, \ldots, H_1^{2N})^{\mathrm{T}}, \\
\boldsymbol{l}_1'(H_1) := (0, 2H_1, \ldots, 2N H_1^{2N-1})^{\mathrm{T}}, \\
\boldsymbol{l}_1''(H_1) := (0, 2, \ldots, 2N(2N-1) H_1^{2N-2})^{\mathrm{T}}, \\
\boldsymbol{l}_2(H_2) := (1, H_2^{p_1}, H_2^{p_2}, \ldots, H_2^{p_{N^*}})^{\mathrm{T}}, \\
\boldsymbol{l}_2'(H_2) := (0, p_1 H_2^{p_1 - 1}, \ldots, p_{N^*} H_2^{p_{N^*}})^{\mathrm{T}}, \\
\boldsymbol{l}_2''(H_2) := (0, p_1(p_1 - 1) H_2^{p_1 - 2}, \ldots, p_{N^*}(p_{N^*} - 1) H_2^{p_{N^*}})^{\mathrm{T}},
\end{cases}
$$

and define $\boldsymbol{u}_\ell$ and w_ℓ for $\ell = 1,2$, which represent approximately the horizontal and the vertical components of the velocity field on the interface from the water region $\Omega_\ell(t)$, by

$$
\tag{3.4}
\begin{cases}
\boldsymbol{u}_1 := (\boldsymbol{l}_1(H_1) \otimes \nabla)^{\mathrm{T}} \boldsymbol{\phi}_1, & w_1 := -\boldsymbol{l}_1'(H_1) \cdot \boldsymbol{\phi}_1, \\
\boldsymbol{u}_2 := (\boldsymbol{l}_2(H_2) \otimes \nabla)^{\mathrm{T}} \boldsymbol{\phi}_2 - (\boldsymbol{l}_2'(H_2) \cdot \boldsymbol{\phi}_2) \underline{h}_2^{-1} \nabla b, & w_2 := \boldsymbol{l}_2'(H_2) \cdot \boldsymbol{\phi}_2.
\end{cases}
$$

Then, denoting $L_1 := (L_{1,ij})_{0\le i,j\le N}$ and $L_2 := (L_{2,ij})_{0\le i,j\le N^*}$ we can write the Kakinuma model (2.18) more compactly as

$$
\begin{cases}
\boldsymbol{l}_1(H_1)\partial_t\zeta + \underline{h}_1 L_1(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\
\boldsymbol{l}_2(H_2)\partial_t\zeta - \underline{h}_2 L_2(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\
\underline{\rho}_1\bigl\{\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \frac{1}{2}\bigl(|\boldsymbol{u}_1|^2 + (\underline{h}_1\delta)^{-2}w_1^2\bigr)\bigr\} \\
\quad -\underline{\rho}_2\bigl\{\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 + \frac{1}{2}\bigl(|\boldsymbol{u}_2|^2 + (\underline{h}_2\delta)^{-2}w_2^2\bigr)\bigr\} - \zeta = 0.
\end{cases} \tag{3.5}
$$

By eliminating $\partial_t\zeta$ from the first two vectorial identities in (3.5), we obtain $N+N^*+1$ scalar relations which are necessary conditions for the existence of solutions to the Kakinuma model, as stated below. Introducing linear operators $\mathcal{L}_{1,i} := \mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)$ $(i=0,\dots,N)$ acting on $\boldsymbol{\varphi}_1 = (\varphi_{1,0},\dots,\varphi_{1,N})^{\mathrm{T}}$ and $\mathcal{L}_{2,i} := \mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)$ $(i=0,\dots,N^*)$ acting on $\boldsymbol{\varphi}_2 = (\varphi_{2,0},\dots,\varphi_{2,N^*})^{\mathrm{T}}$ by

$$
\begin{cases}
\mathcal{L}_{1,0}\boldsymbol{\varphi}_1 := \displaystyle\sum_{j=0}^{N} L_{1,0j}\varphi_{1,j}, \\
\mathcal{L}_{1,i}\boldsymbol{\varphi}_1 := \displaystyle\sum_{j=0}^{N}(L_{1,ij}\varphi_{1,j} - H_1^{2i}L_{1,0j}\varphi_{1,j}) \quad \text{for} \quad i=1,2,\dots,N, \\
\mathcal{L}_{2,0}\boldsymbol{\varphi}_2 := \displaystyle\sum_{j=0}^{N^*} L_{2,0j}\varphi_{2,j}, \\
\mathcal{L}_{2,i}\boldsymbol{\varphi}_2 := \displaystyle\sum_{j=0}^{N^*}(L_{2,ij}\varphi_{2,j} - H_2^{p_i}L_{2,0j}\varphi_{2,j}) \quad \text{for} \quad i=1,2,\dots,N^*,
\end{cases} \tag{3.6}
$$

the necessary conditions can be written simply as

$$
\begin{cases}
\mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i=1,2,\dots,N, \\
\mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i=1,2,\dots,N^*, \\
\underline{h}_1\mathcal{L}_{1,0}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 + \underline{h}_2\mathcal{L}_{2,0}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0.
\end{cases} \tag{3.7}
$$

Hereafter, these necessary conditions will be referred to as the compatibility conditions. Notice that under these compatibility conditions we have for $\ell = 1,2$

$$
L_\ell\boldsymbol{\phi}_\ell = \boldsymbol{l}_\ell\mathcal{L}_{\ell,0}\boldsymbol{\phi}_\ell, \tag{3.8}
$$

where $\boldsymbol{l}_\ell = \boldsymbol{l}_\ell(H_\ell)$ and similar simplifications of notations will be used in the following without any comments. In connection with the stability condition (1.5), we introduce a function

$$
\begin{aligned}
a := 1 &+ \underline{\rho}_1\underline{h}_1^{-1}\{\boldsymbol{l}_1'(H_1)\cdot(\partial_t + \boldsymbol{u}_1\cdot\nabla)\boldsymbol{\phi}_1 - (\underline{h}_1\delta)^{-2}w_1\boldsymbol{l}_1''(H_1)\cdot\boldsymbol{\phi}_1\} \\
&+ \underline{\rho}_2\underline{h}_2^{-1}\{\boldsymbol{l}_2'(H_2)\cdot(\partial_t + \boldsymbol{u}_2\cdot\nabla)\boldsymbol{\phi}_2 + \bigl((\underline{h}_2\delta)^{-2}w_2 - \underline{h}_2^{-1}\nabla b\cdot\boldsymbol{u}_2\bigr)\boldsymbol{l}_2''(H_2)\cdot\boldsymbol{\phi}_2\},
\end{aligned} \tag{3.9}
$$

which corresponds to $-(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}))|_{\Gamma(t)}$ in the stability condition.

Our first main result in this paper is the existence of the solution to the initial value problem (2.18)–(2.20) for the Kakinuma model on a time interval independent of parameters, especially, the shallowness parameters $\delta_1 = \underline{h}_1\delta$ and $\delta_2 = \underline{h}_2\delta$ together with a uniform bound of the solution. For simplicity, we denote $H_{\ell(0)} := H_\ell|_{t=0}$, $\boldsymbol{u}_{\ell(0)} := \boldsymbol{u}_\ell|_{t=0}$ for $\ell = 1,2$, and $a_{(0)} := a|_{t=0}$, which can be written in terms of the initial data according to the initial condition (2.20). Although the function a includes the terms $(\partial_t\boldsymbol{\phi}_\ell')|_{t=0}$ for $\ell = 1,2$, where $\boldsymbol{\phi}_1' = (\phi_{1,1},\dots,\phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2' = (\phi_{2,1},\dots,\phi_{2,N^*})^{\mathrm{T}}$, and the hypersurface $t=0$ is characteristic for the Kakinuma model, we can uniquely determine them in terms of the initial data. For details, we refer to Remark 5.3.

Theorem 3.1. *Let $c_0, M_0, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exist a time $T > 0$ and a constant $M > 0$ such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (2.14), *$\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, as well as the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, if the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b satisfy*

$$
\begin{cases} \|\zeta_{(0)}\|_{H^m}^2 + \displaystyle\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \big(\|\nabla \boldsymbol{\phi}_{\ell(0)}\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}'_{\ell(0)}\|_{H^m}^2 \big) \leq M_0, \\ \underline{h}_2^{-1} \big(\|b\|_{W^{m+1,\infty}} + (\underline{h}_2\delta) \|b\|_{W^{m+2,\infty}} \big) \leq M_0, \end{cases} \tag{3.10}
$$

the non-cavitation assumption

$$
H_{1(0)}(\boldsymbol{x}) \geq c_0, \quad H_{2(0)}(\boldsymbol{x}) \geq c_0 \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \tag{3.11}
$$

the stability condition

$$
a_{(0)}(\boldsymbol{x}) - \frac{\underline{\rho}_1 \underline{\rho}_2}{\underline{\rho}_1 \underline{h}_2 H_{2(0)}(\boldsymbol{x}) \alpha_2 + \underline{\rho}_2 \underline{h}_1 H_{1(0)}(\boldsymbol{x}) \alpha_1} |\boldsymbol{u}_{1(0)}(\boldsymbol{x}) - \boldsymbol{u}_{2(0)}(\boldsymbol{x})|^2 \geq c_0 \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \tag{3.12}
$$

with positive constants α_1 and α_2 defined by (3.16), *and the compatibility conditions*

$$
\begin{cases} \mathcal{L}_{1,i}(H_{1(0)}, \delta, \underline{h}_1) \boldsymbol{\phi}_{1(0)} = 0 \quad \text{for} \quad i = 1, 2, \dots, N, \\ \mathcal{L}_{2,i}(H_{2(0)}, b, \delta, \underline{h}_2) \boldsymbol{\phi}_{2(0)} = 0 \quad \text{for} \quad i = 1, 2, \dots, N^*, \\ \underline{h}_1 \mathcal{L}_{1,0}(H_{1(0)}, \delta, \underline{h}_1) \boldsymbol{\phi}_{1(0)} + \underline{h}_2 \mathcal{L}_{2,0}(H_{2(0)}, b, \delta, \underline{h}_2) \boldsymbol{\phi}_{2(0)} = 0, \end{cases} \tag{3.13}
$$

then the initial value problem (2.18)–(2.20) *has a unique solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ on the time interval $[0, T]$ satisfying*

$$
\begin{cases} \zeta, \nabla \phi_{1,0}, \nabla \phi_{2,0} \in C([0,T]; H^m) \cap C^1([0,T]; H^{m-1}), \\ \boldsymbol{\phi}'_1, \boldsymbol{\phi}'_2 \in C([0,T]; H^{m+1}) \cap C^1([0,T]; H^m), \end{cases}
$$

where we recall the notation $\boldsymbol{\phi}'_1 = (\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}'_2 = (\phi_{2,1}, \phi_{2,2}, \dots, \phi_{2,N^*})^{\mathrm{T}}$. Moreover, the solution satisfies the uniform bound

$$
\|\zeta(t)\|_{H^m}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \big(\|\nabla \boldsymbol{\phi}_\ell(t)\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}'_\ell(t)\|_{H^m}^2 \big) \leq M \tag{3.14}
$$

for $t \in [0, T]$ together with

$$
\begin{cases} a(\boldsymbol{x}, t) - \dfrac{\underline{\rho}_1 \underline{\rho}_2}{\underline{\rho}_1 \underline{h}_2 H_2(\boldsymbol{x}, t) \alpha_2 + \underline{\rho}_2 \underline{h}_1 H_1(\boldsymbol{x}, t) \alpha_1} |\boldsymbol{u}_1(\boldsymbol{x}, t) - \boldsymbol{u}_2(\boldsymbol{x}, t)|^2 \geq c_0/2, \\ H_1(\boldsymbol{x}, t) \geq c_0/2, \quad H_2(\boldsymbol{x}, t) \geq c_0/2 \qquad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, t \in [0, T]. \end{cases} \tag{3.15}
$$

Remark 3.2. *The constants α_1 and α_2 are defined by*

$$
\alpha_\ell := \frac{\det A_{\ell,0}}{\det \tilde{A}_{\ell,0}}, \qquad \tilde{A}_{\ell,0} := \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}, \tag{3.16}
$$

for $\ell = 1, 2$, where $\mathbf{1} := (1, \dots, 1)^{\mathrm{T}}$ and the matrices $A_{1,0}$ and $A_{2,0}$ are defined by

$$
\begin{cases} A_{1,0} := \left(\dfrac{1}{2(i+j)+1} \right)_{0 \leq i,j \leq N}, \\ A_{2,0} := \left(\dfrac{1}{p_i + p_j + 1} \right)_{0 \leq i,j \leq N^*}. \end{cases}
$$

Hence, α_1 and α_2 are positive constants depending only on N and the nonnegative integers $0 = p_0 < p_1 < \dots < p_{N^*}$, respectively, and go to 0 as $N, N^* \to \infty$.

Remark 3.3. It is easy to check that the non-cavitation assumption (3.11) and the stability condition (3.12) are automatically satisfied for small initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and small bottom topography b, whereas an arrangement of nontrivial initial data satisfying the compatibility conditions (3.13) together with the uniform bound (3.10) is a non-trivial issue. To this end, we use the canonical variable ϕ defined by (2.23), which can be written as

$$\phi = \underline{\rho}_2 \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\phi}_2 - \underline{\rho}_1 \boldsymbol{l}_1(H_1) \cdot \boldsymbol{\phi}_1. \tag{3.17}$$

Given the initial data $(\zeta_{(0)}, \phi_{(0)})$ for the canonical variables (ζ, ϕ), and the bottom topography b, the necessary conditions (3.7) and the above relation (3.17) determine the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ for the Kakinuma model (2.18)–(2.20) satisfying the compatibility conditions (3.13) and the uniform bound (3.10), which is unique up to an additive constant of the form $(\mathcal{C}\underline{\rho}_2, \mathcal{C}\underline{\rho}_1)$ to $(\phi_{1,0(0)}, \phi_{2,0(0)})$. In fact, we have the following proposition, which is a simple corollary of Lemma 5.1 given in Section 5.

Proposition 3.4. *Let c_0, M_0 be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (2.14) *and $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if the initial data $(\zeta_{(0)}, \phi_{(0)}) \in H^m \times \mathring{H}^m$ of the canonical variables, the bottom topography $b \in W^{m,\infty}$, and initial depths $H_{1(0)} := 1 - \underline{h}_1^{-1}\zeta_{(0)}$ and $H_{2(0)} := 1 + \underline{h}_2^{-1}\zeta_{(0)} - \underline{h}_2^{-1}b$ satisfy*

$$\begin{cases} \underline{h}_1^{-1}\|\zeta_{(0)}\|_{H^m} + \underline{h}_2^{-1}\|\zeta_{(0)}\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m,\infty}} \leq M_0, \\ H_{1(0)}(\boldsymbol{x}) \geq c_0, \quad H_{2(0)}(\boldsymbol{x}) \geq c_0 \quad \mathit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

then there exist initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ satisfying the compatibility conditions (3.13) *as well as $\phi_{(0)} = \underline{\rho}_2 \boldsymbol{l}_2(H_{2(0)}) \cdot \boldsymbol{\phi}_{2(0)} - \underline{\rho}_1 \boldsymbol{l}_1(H_{1(0)}) \cdot \boldsymbol{\phi}_{1(0)}$. Moreover, we have*

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \big(\|\nabla \boldsymbol{\phi}_{\ell(0)}\|_{H^{m-1}}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_{\ell(0)}'\|_{H^{m-1}}^2 \big) \leq C \|\nabla \phi_{(0)}\|_{H^{m-1}}^2.$$

The next theorem shows that the Kakinuma model (2.18) is consistent with the full model for interfacial gravity waves (2.17) at order $O((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})$ under the special choice of the indices $p_0, p_1, \ldots, p_{N^*}$ as

(H1) $N^* = N$ and $p_i = 2i$ $(i = 0, 1, \ldots, N)$ in the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$,

(H2) $N^* = 2N$ and $p_i = i$ $(i = 0, 1, \ldots, 2N)$ in the case with general bottom topographies.

Theorem 3.5. *Let c, M be positive constants and m an integer such that $m \geq 4(N+1)$ and $m > \frac{n}{2}+1$. We assume* (H1) *or* (H2). *There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$ and for any solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model* (2.18) *on a time interval $[0, T]$ with a bottom topography $b \in W^{m+1,\infty}$ satisfying*

$$\begin{cases} \underline{h}_1^{-1}\|\zeta(t)\|_{H^m} + \underline{h}_2^{-1}\|\zeta(t)\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m+1,\infty}} \leq M, \\ H_1(\boldsymbol{x}, t) \geq c, \quad H_2(\boldsymbol{x}, t) \geq c \quad \mathit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, t \in [0, T], \end{cases} \tag{3.18}$$

if we define $\phi_\ell := \boldsymbol{l}_\ell(H_\ell) \cdot \boldsymbol{\phi}_\ell$ for $\ell = 1, 2$, then (ζ, ϕ_1, ϕ_2) satisfy approximately the full model for *interfacial gravity waves as*

$$\begin{cases} \partial_t\zeta + \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 = \mathfrak{r}_1, \\ \partial_t\zeta - \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 = \mathfrak{r}_2, \\ \underline{\rho}_1\left(\partial_t\phi_1 + \dfrac{1}{2}|\nabla\phi_1|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 - \nabla\zeta\cdot\nabla\phi_1)^2}{1+\delta^2|\nabla\zeta|^2}\right) \\ \quad -\underline{\rho}_2\left(\partial_t\phi_2 + \dfrac{1}{2}|\nabla\phi_2|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 + \nabla\zeta\cdot\nabla\phi_2)^2}{1+\delta^2|\nabla\zeta|^2}\right) - \zeta = \mathfrak{r}_0. \end{cases}$$

Here, the errors $(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_0)$ *satisfy*

$$\begin{cases} \|\mathfrak{r}_\ell(t)\|_{H^{m-4(N+1)}} \leq C\underline{h}_\ell(\underline{h}_\ell\delta)^{4N+2}\|\nabla\phi_\ell(t)\|_{H^{m-1}} \quad (\ell=1,2), \\ \|\mathfrak{r}_0(t)\|_{H^{m-4(N+1)}} \leq C\displaystyle\sum_{\ell=1,2}\underline{\rho}_\ell(\underline{h}_\ell\delta)^{4N+2}\|\nabla\phi_\ell(t)\|_{H^{m-1}}^2 \end{cases}$$

for $t\in[0,T]$.

Particularly, we see that under the special choice of indices (H1) or (H2) the solutions to the Kakinuma model (2.18)–(2.20) constructed in Theorem 3.1 satisfy approximately the full model for interfacial gravity waves (2.17) with the choice $\phi_\ell = \boldsymbol{l}_\ell(H_\ell)\cdot\boldsymbol{\phi}_\ell$ $(\ell=1,2)$ and that the error is of order $O((\underline{h}_1\delta)^{4N+2}+(\underline{h}_2\delta)^{4N+2})$.

Conversely, the next theorem shows that the full model for interfacial gravity waves is consistent with the Kakinuma model at order $O((\underline{h}_1\delta)^{4N+2}+(\underline{h}_2\delta)^{4N+2})$ under the special choice of indices (H1) or (H2).

Theorem 3.6. *Let* c, M *be positive constants and* m *an integer such that* $m\geq 4(N+1)$ *and* $m>\frac{n}{2}+1$. *We assume* (H1) *or* (H2). *There exists a positive constant* C *such that for any positive parameters* $\underline{\rho}_1,\underline{\rho}_2,\underline{h}_1,\underline{h}_2,\delta$ *satisfying* $\underline{h}_1\delta,\underline{h}_2\delta\leq 1$ *and for any solution* (ζ,ϕ_1,ϕ_2) *to the full model for interfacial gravity waves* (2.17) *on a time interval* $[0,T]$ *with a bottom topography* $b\in W^{m+1,\infty}$ *satisfying* (3.18), *if we define* H_1 *and* H_2 *as in* (2.19) *and* $\boldsymbol{\phi}_1$ *and* $\boldsymbol{\phi}_2$ *as the unique solutions to the problems*

$$\begin{cases} \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 = \phi_1, \quad \mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i=1,2,\dots,N, \\ \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 = \phi_2, \quad \mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i=1,2,\dots,N^*, \end{cases} \tag{3.19}$$

then $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ *satisfy approximately the Kakinuma model as*

$$\begin{cases} \boldsymbol{l}_1(H_1)\underline{h}_1^{-1}\partial_t\zeta + L_1(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = \tilde{\mathfrak{r}}_1, \\ \boldsymbol{l}_2(H_2)\underline{h}_2^{-1}\partial_t\zeta - L_2(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = \tilde{\mathfrak{r}}_2, \\ \underline{\rho}_1\big\{\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \frac{1}{2}\big(|\boldsymbol{u}_1|^2 + (\underline{h}_1\delta)^{-2}w_1^2\big)\big\} \\ \quad -\underline{\rho}_2\big\{\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 + \frac{1}{2}\big(|\boldsymbol{u}_2|^2 + (\underline{h}_2\delta)^{-2}w_2^2\big)\big\} - \zeta = \tilde{\mathfrak{r}}_0. \end{cases}$$

Here, the errors $(\tilde{\mathfrak{r}}_1,\tilde{\mathfrak{r}}_2,\tilde{\mathfrak{r}}_0)$ *satisfy*

$$\begin{cases} \|\tilde{\mathfrak{r}}_\ell(t)\|_{H^{m-4(N+1)}} \leq C(\underline{h}_\ell\delta)^{4N+2}\|\nabla\phi_\ell(t)\|_{H^{m-1}} \quad (\ell=1,2), \\ \|\tilde{\mathfrak{r}}_0(t)\|_{H^{m-4(N+1)}} \leq C\displaystyle\sum_{\ell=1,2}\underline{\rho}_\ell(\underline{h}_\ell\delta)^{4N+2}\|\nabla\phi_\ell(t)\|_{H^{m-1}}^2 \end{cases} \tag{3.20}$$

for $t\in[0,T]$.

Remark 3.7. The unique existence of the solutions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ to the problems (3.19) is guaranteed by Lemma 4.4 below under an additional assumption $\phi_1(\cdot,t), \phi_2(\cdot,t) \in \mathring{H}^m$. Lemma 4.4 is essentially a simple corollary of [20, Lemma 3.4].

Remark 3.8. In order to define the approximate solution $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model (2.18) from the solution (ζ, ϕ_1, ϕ_2) to the full model, we can use, in place of (3.19), the following system of equations

$$
\begin{cases}
\mathcal{L}_{1,i}(H_1, \delta, \underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i = 1, 2, \dots, N, \\
\mathcal{L}_{2,i}(H_1, b, \delta, \underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i = 1, 2, \dots, N^*, \\
\underline{h}_1 \mathcal{L}_{1,0}(H_1, \delta, \underline{h}_1)\boldsymbol{\phi}_1 + \underline{h}_2 \mathcal{L}_{2,0}(H_2, b, \delta, \underline{h}_2)\boldsymbol{\phi}_2 = 0, \\
\underline{\rho}_2 \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\phi}_2 - \underline{\rho}_1 \boldsymbol{l}_1(H_1) \cdot \boldsymbol{\phi}_1 = \phi,
\end{cases}
\tag{3.21}
$$

where $\phi = \underline{\rho}_2 \phi_2 - \underline{\rho}_1 \phi_1$ is the canonical variable for the full model for interfacial gravity waves. The above system is nothing but the compatibility conditions (3.7) together with the definition (3.17) of the canonical variable for the Kakinuma model. The existence of the approximate solution $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ is guaranteed by Lemma 5.1 given in Section 5. Then, we have similar error estimates to (3.20). For details, we refer to Proposition 7.6.

The above Theorems 3.5 and 3.6 concern essentially the approximation of the equations. To give a rigorous justification of the Kakinuma model (2.18) as a higher order shallow water approximation to the full model for interfacial gravity waves (2.17), one needs to give an error estimate between solutions to the Kakinuma model and that to the full model. However, we cannot expect to construct general solutions to the initial value problem for the full model for interfacial gravity waves because the initial value problem is ill-posed. Nevertheless, if we assume the existence of a solution to the full model with a uniform bound with respect to the shallowness parameters $\delta_1 = \underline{h}_1 \delta$ and $\delta_2 = \underline{h}_2 \delta$, then we can give an error estimate with respect to a solution to the Kakinuma model by making use of the well-posedness of the initial value problem for the Kakinuma model as we can see in the following theorem.

Theorem 3.9. *Let $c, M, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 4(N+1)$. We assume* (H1) *or* (H2)*. Then, there exist a time $T > 0$ and a constant $C > 0$ such that the following holds true. Let $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ be positive parameters satisfying the natural restrictions* (2.14)*, $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, and let $b \in W^{m+2,\infty}$ such that $\underline{h}_2^{-1}\|b\|_{W^{m+2,\infty}} \leq M$. Suppose that the full model for interfacial gravity waves* (2.17) *possesses a solution $(\zeta^{\mathrm{IW}}, \phi_1^{\mathrm{IW}}, \phi_2^{\mathrm{IW}}) \in C([0, T^{\mathrm{IW}}]; H^{m+1} \times \mathring{H}^{m+1} \times \mathring{H}^{m+1})$ satisfying a uniform bound*

$$
\begin{cases}
\|\zeta^{\mathrm{IW}}(t)\|_{H^{m+1}}^2 + \displaystyle\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\nabla \phi_\ell^{\mathrm{IW}}(t)\|_{H^m}^2 \leq M, \\
H_1^{\mathrm{IW}}(\boldsymbol{x}, t) \geq c, \quad H_2^{\mathrm{IW}}(\boldsymbol{x}, t) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, t \in [0, T^{\mathrm{IW}}],
\end{cases}
$$

where we denote $H_1^{\mathrm{IW}} := 1 - \underline{h}_1^{-1}\zeta^{\mathrm{IW}}$ and $H_2^{\mathrm{IW}} := 1 + \underline{h}_2^{-1}\zeta^{\mathrm{IW}} - \underline{h}_2^{-1}b$. Let $\zeta_{(0)} := \zeta^{\mathrm{IW}}|_{t=0}$ and $\phi_{(0)} := (\underline{\rho}_2\phi_2^{\mathrm{IW}} - \underline{\rho}_1\phi_1^{\mathrm{IW}})|_{t=0}$ be the initial data for the canonical variables, and let $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ be the initial data to the Kakinuma model constructed from $(\zeta_{(0)}, \phi_{(0)})$ by Proposition 3.4*. Assume moreover that the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ satisfy the stability condition* (3.12)*, let $(\zeta^{\mathrm{K}}, \boldsymbol{\phi}_1^{\mathrm{K}}, \boldsymbol{\phi}_2^{\mathrm{K}})$ be the solution to the initial value problem for the Kakinuma model* (2.18)*–*(2.20) *on the time interval $[0, T]$ whose unique existence is guaranteed by Theorem* 3.1*, and put*

$\phi_\ell^{\rm K} = \boldsymbol{l}_\ell(H_\ell) \cdot \boldsymbol{\phi}_\ell^{\rm K}$ *for* $\ell = 1,2$. *Then, we have the error bound*

$$\|\zeta^{\rm K}(t) - \zeta^{\rm IW}(t)\|_{H^{m-4(N+1)}} + \sum_{\ell=1,2} \sqrt{\underline{\rho}_\ell \underline{h}_\ell} \|\nabla\phi_\ell^{\rm K}(t) - \nabla\phi_\ell^{\rm IW}(t)\|_{H^{m-(4N+5)}} \\ \leq C((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})$$

for $0 \leq t \leq \min\{T, T^{\rm IW}\}$.

The next theorem is the final main result in this paper and states the consistency of the Hamiltonian $\mathscr{H}^{\rm K}(\zeta,\phi)$ of the Kakinuma model with respect to the Hamiltonian $\mathscr{H}^{\rm IW}(\zeta,\phi)$ of the full model for interfacial gravity waves. We recall that these Hamiltonians are defined in (2.25) and (2.22), respectively.

Theorem 3.10. *Let* $c, M, \underline{h}_{\min}$ *be positive constants and* m *an integer such that* $m > \frac{n}{2}+1$ *and* $m \geq 4(N+1)$. *We assume* (H1) *or* (H2). *There exists a positive constant* C *such that for any positive parameters* $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ *satisfying the natural restrictions* (2.14), $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, *and the condition* $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, *and for any* $(\zeta,\phi) \in H^m \times \mathring{H}^{4(N+1)}$ *and* $b \in W^{m+1,\infty}$ *satisfying*

$$\begin{cases} \underline{h}_1^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m+1,\infty}} \leq M, \\ H_1(\boldsymbol{x}) \geq c, \quad H_2(\boldsymbol{x}) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

with H_1 *and* H_2 *defined by* (2.19), *we have*

$$|\mathscr{H}^{\rm K}(\zeta,\phi) - \mathscr{H}^{\rm IW}(\zeta,\phi)| \leq C\|\nabla\phi\|_{H^{4N+3}}\|\nabla\phi\|_{L^2}((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2}).$$

4 Consistency of the Kakinuma model; proof of Theorems 3.5 and 3.6

In this section we show that under the special choice of the indices $p_0, p_1, \dots, p_{N^*}$ as

(H1) $N^* = N$ and $p_i = 2i$ $(i = 0,1,\dots,N)$ in the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$,

(H2) $N^* = 2N$ and $p_i = i$ $(i = 0,1,\dots,2N)$ in the case with general bottom topographies,

the Kakinuma model (2.18)is a higher order model to the full model for interfacial gravity waves (2.17) in the limit $\delta_1 = \underline{h}_1\delta \to 0$, $\delta_2 = \underline{h}_2\delta \to 0$, in the sense of consistency. Specifically, we prove Theorems 3.5 and 3.6. Our proof relies essentially on results obtained in the framework of surface waves in [20], which are recalled in Subsection 4.1. The extension to the framework of interfacial waves and the completion of the proof are provided in Subsection 4.2.

4.1 Results in the framework of surface waves

In this subsection, we consider the case of surface waves where the water surface and the bottom of the water are represented as $z = \zeta(\boldsymbol{x})$ and $z = -1 + b(\boldsymbol{x})$, respectively. Here, the time t is fixed arbitrarily, so that we omit the dependence of t in notations. Let $H(\boldsymbol{x}) = 1 + \zeta(\boldsymbol{x}) - b(\boldsymbol{x})$ be the water depth. For a nonnegative integer N, let N^* and $p_0, p_1, \dots, p_{N^*}$ be nonnegative integers satisfying the condition (H1) or (H2). Put

(4.1) $$\boldsymbol{l}(H) := (1, H^{p_1}, \dots, H^{p_{N^*}})^{\rm T}$$

and define $L_{ij} = L_{ij}(H, b, \delta)$ $(i, j = 0, 1, \dots, N^*)$ by

$$
\begin{aligned}
L_{ij}\varphi_j := &-\nabla\cdot\left(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\varphi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\varphi_j\nabla b\right) \\
&- \frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\varphi_j + \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\varphi_j,
\end{aligned}
\tag{4.2}
$$

where we use the notational convention $0/0 = 0$. Introduce linear operators $\mathcal{L}_i = \mathcal{L}_i(H, b, \delta)$ $(i = 0, 1, \dots, N^*)$ acting on $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_{N^*})^{\mathrm{T}}$ by

$$
\begin{cases}
\mathcal{L}_0\boldsymbol{\varphi} := \displaystyle\sum_{j=0}^{N^*} L_{0j}\varphi_j, \\
\mathcal{L}_i\boldsymbol{\varphi} := \displaystyle\sum_{j=0}^{N^*}(L_{ij}\varphi_j - H^{p_i}L_{0j}\varphi_j) \quad \text{for} \quad i = 1, 2, \dots, N^*.
\end{cases}
\tag{4.3}
$$

The following Lemma has been proved in [20, Lemmas 3.2 and 3.4].

Lemma 4.1. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exists a positive constant C such that if $\zeta \in H^m$, $b \in W^{m,\infty}$, and $H = 1+\zeta-b$ satisfy*

$$
\begin{cases}
\|\zeta\|_{H^m} + \|b\|_{W^{m,\infty}} \le M, \\
H(\boldsymbol{x}) \ge c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n,
\end{cases}
\tag{4.4}
$$

then for any $k = \pm 0, \dots, \pm(m-1)$, any $\delta \in (0, 1]$, and any $\phi \in \mathring{H}^{k+1}$ there exists a unique solution $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_{N^*}) = (\phi_0, \boldsymbol{\phi}') \in \mathring{H}^{k+1} \times (H^{k+1})^{N^*}$ to the problem

$$
\begin{cases}
\mathcal{L}_i(H, b, \delta)\boldsymbol{\phi} = 0 \quad \text{for} \quad i = 1, 2, \dots, N^*, \\
\boldsymbol{l}(H)\cdot\boldsymbol{\phi} = \phi.
\end{cases}
\tag{4.5}
$$

Moreover, the solution satisfies $\|\nabla\boldsymbol{\phi}\|_{H^k} + \delta^{-1}\|\boldsymbol{\phi}'\|_{H^k} \le C\|\nabla\phi\|_{H^k}$.

As a corollary of this lemma, under the assumptions of Lemma 4.1

$$
\Lambda^{(N)}(\zeta, b, \delta) \colon \phi \mapsto \mathcal{L}_0(H, b, \delta)\boldsymbol{\phi},
$$

where $\boldsymbol{\phi}$ is the unique solution to (4.5), is defined as a bounded linear operator from $\mathring{H}^{k+1}$ to H^{k-1} for any $k = \pm 0, \dots, \pm(m-1)$. A key result is that the operator $\Lambda^{(N)}(\zeta, b, \delta)$ provide good approximations in the shallow water regime $\delta \ll 1$ to the corresponding Dirichlet-to-Neumann map $\Lambda(\zeta, b, \delta)$, which is defined by

$$
\Lambda(\zeta, b, \delta)\phi := \left(\delta^{-2}\partial_z\Phi - \nabla\zeta\cdot\nabla\Phi\right)\big|_{z=\zeta},
\tag{4.6}
$$

where Φ is the unique solution to the boundary value problem

$$
\begin{cases}
\Delta\Phi + \delta^{-2}\partial_z^2\Phi = 0 & \text{in} \quad -1+b(\boldsymbol{x}) < z < \zeta(\boldsymbol{x}), \\
\Phi = \phi & \text{on} \quad z = \zeta(\boldsymbol{x}), \\
\nabla b\cdot\nabla\Phi - \delta^{-2}\partial_z\Phi = 0 & \text{on} \quad z = -1+b(\boldsymbol{x}).
\end{cases}
\tag{4.7}
$$

More precisely, we have the following Lemma.

Lemma 4.2. *Let c, M be positive constants and m, j integers such that $m > \frac{n}{2}+1$, $m \geq 2(j+1)$, and $1 \leq j \leq 2N+1$. We assume* (H1) *or* (H2). *There exists a positive constant C such that if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, and $H = 1+\zeta-b$ satisfy*

$$\begin{cases} \|\zeta\|_{H^m} + \|b\|_{W^{m+1,\infty}} \leq M, \\ H(\boldsymbol{x}) \geq c \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases} \tag{4.8}$$

then for any $\phi \in \mathring{H}^{k+2(j+1)}$ with $0 \leq k \leq m-2(j+1)$ and any $\delta \in (0,1]$ we have

$$\|\Lambda^{(N)}(\zeta, b, \delta)\phi - \Lambda(\zeta, b, \delta)\phi\|_{H^k} \leq C\delta^{2j}\|\nabla\phi\|_{H^{k+2j+1}}.$$

Proof. We observe that the bound on $\mathfrak{r}_1 := \Lambda^{(N)}(\zeta, b, \delta)\phi - \Lambda(\zeta, b, \delta)\phi$ in the case $j = 2N+1$ and $k = m-4(N+1)$ is given in [20, Theorem 2.2] and proved in [20, Sections 8.1 and 8.2]. The proof is also valid in the case $1 \leq j \leq 2N+1$ and $0 \leq k \leq m-2(j+1)$. □

The above estimate allows us to obtain the desired consistency result on the equations describing the conservation of mass. We need a similar estimate for the contributions of Bernoulli's equation. To this end, we denote

$$B(\phi; \zeta, b, \delta) := \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2\frac{(\Lambda(\zeta, b, \delta)\phi + \nabla\zeta\cdot\nabla\phi)^2}{1+\delta^2|\nabla\zeta|^2} \tag{4.9}$$

and

$$B^{(N)}(\phi; \zeta, b, \delta) := \frac{1}{2}\big(|\boldsymbol{u}|^2 + \delta^{-2}w^2\big) - w\Lambda^{(N)}(\zeta, b, \delta)\phi \tag{4.10}$$

with

$$\begin{cases} \boldsymbol{u} := (\boldsymbol{l}(H)\otimes\nabla)^{\mathrm{T}}\boldsymbol{\phi} - (\boldsymbol{l}'(H)\cdot\boldsymbol{\phi})\nabla b, \\ w := \boldsymbol{l}'(H)\cdot\boldsymbol{\phi}, \end{cases}$$

where $\boldsymbol{l}'(H) := (0, p_1H^{p_1-1}, \ldots, p_{N^*}H^{p_{N^*}-1})^{\mathrm{T}}$ and $\boldsymbol{\phi} := (\phi_0, \phi_1, \ldots, \phi_{N^*})^{\mathrm{T}}$ is the solution to (4.5), whose unique existence is guaranteed by Lemma 4.1. Then, the following lemma shows that $B^{(N)}(\phi; \zeta, b, \delta)$ is a higher order approximation to $B(\phi; \zeta, b, \delta)$ in the shallow water regime $\delta \ll 1$.

Lemma 4.3. *Let c, M be positive constants and m an integer such that $m \geq 4(N+1)$ and $m > \frac{n}{2}+1$. We assume* (H1) *or* (H2). *There exists a positive constant C such that if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, and $H = 1+\zeta-b$ satisfy* (4.8), *then for any $\phi \in \mathring{H}^m$ and any $\delta \in (0,1]$ we have*

$$\|B^{(N)}(\phi; \zeta, b, \delta) - B(\phi; \zeta, b, \delta)\|_{H^{m-4(N+1)}} \leq C\delta^{4N+2}\|\nabla\phi\|_{H^{m-1}}^2.$$

Proof. Notice first that differentiating $\phi = \boldsymbol{l}(H)\cdot\boldsymbol{\phi}$ we have $\nabla\phi = \boldsymbol{u} + w\nabla\zeta$, so that

$$\begin{aligned} B^{(N)}(\phi; \zeta, b, \delta) =& \frac{1}{2}\big(|\nabla\phi|^2 + \delta^{-2}w^2(1+\delta^2|\nabla\zeta|^2)\big) - w\big(\nabla\zeta\cdot\nabla\phi + \Lambda^{(N)}(\zeta, b, \delta)\phi\big) \\ =& \frac{1}{2}\big(|\nabla\phi|^2 + \delta^{-2}w^2(1+\delta^2|\nabla\zeta|^2)\big) - w\big(\Lambda(\zeta, b, \delta)\phi + \nabla\zeta\cdot\nabla\phi\big) \\ &+ w\big(\Lambda(\zeta, b, \delta)\phi - \Lambda^{(N)}(\zeta, b, \delta)\phi\big). \end{aligned}$$

If we introduce a residual $\mathfrak{r}$ by

$$\mathfrak{r} = (\delta^{-2}\partial_z\Phi^{\mathrm{app}} - \nabla\zeta\cdot\nabla\Phi^{\mathrm{app}})|_{z=\zeta} - (\delta^{-2}\partial_z\Phi - \nabla\zeta\cdot\nabla\Phi)|_{z=\zeta},$$

where Φ is the solution to the boundary value problem (4.7) and $\Phi^{\rm app}$ is an approximate velocity potential defined by

$$\Phi^{\rm app}(\boldsymbol{x},z) = \sum_{i=0}^{N^*}(z+1-b(\boldsymbol{x}))^{p_i}\phi_i(\boldsymbol{x}),$$

then we have $\mathfrak{r} = \delta^{-2}w - \nabla\zeta\cdot\boldsymbol{u} - \Lambda(\zeta,b,\delta)\phi = \delta^{-2}w(1+\delta^2|\nabla\zeta|^2) - \nabla\zeta\cdot\nabla\phi - \Lambda(\zeta,b,\delta)\phi$. Therefore, we obtain

$$B^{(N)}(\phi;\zeta,b,\delta) - B(\phi;\zeta,b,\delta) = \frac{1}{2}\delta^2\frac{\mathfrak{r}^2}{1+\delta^2|\nabla\zeta|^2} + w\bigl(\Lambda(\zeta,b,\delta)\phi - \Lambda^{(N)}(\zeta,b,\delta)\phi\bigr).$$

The desired estimate for the second term readily follows from Lemma 4.1 and Lemma 4.2. As for the first term, in view of $m > \frac{n}{2}$ we can use a calculus inequality $\|\mathfrak{r}^2\|_{H^k} \lesssim \|\mathfrak{r}\|_{H^{(m+k)/2}}^2$ for $k\in\{0,1,\dots,m\}$. Particularly, we have $\|\mathfrak{r}^2\|_{H^{m-4(N+1)}} \lesssim \|\mathfrak{r}\|_{H^{m-2(N+1)}}^2$. The last term can be evaluated by estimates in [20, Sections 8.1 and 8.2]. $\square$

4.2 Results in the framework of interfacial waves

In this section, we prove Theorems 3.5 and 3.6. To this end, we first rewrite the Kakinuma model (2.18) using a formulation which allows a direct comparison with the full model for interfacial gravity waves (2.17), thanks to the following Lemma.

Lemma 4.4. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exists a positive constant C such that for any positive parameters $\underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \le 1$, if $\zeta\in H^m$, $b\in W^{m,\infty}$, $H_1 = 1-\underline{h}_1^{-1}\zeta$, and $H_2 = 1+\underline{h}_2^{-1}\zeta - \underline{h}_2^{-1}b$ satisfy*

$$\begin{cases} \underline{h}_1^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m,\infty}} \le M, \\ H_1(\boldsymbol{x}) \ge c, \quad H_2(\boldsymbol{x}) \ge c \quad \text{for} \quad \boldsymbol{x}\in\mathbf{R}^n, \end{cases} \tag{4.11}$$

then for any $k = 0,\pm1,\dots,\pm(m-1)$ and any $\phi_1,\phi_2 \in \mathring{H}^{k+1}$ there exists a unique solution $\boldsymbol{\phi}_1 = (\phi_{1,0},\boldsymbol{\phi}_1') \in \mathring{H}^{k+1}\times(H^{k+1})^N$, $\boldsymbol{\phi}_2 = (\phi_{2,0},\boldsymbol{\phi}_2') \in \mathring{H}^{k+1}\times(H^{k+1})^{N^*}$ to the problem

$$\begin{cases} \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 = \phi_1, \quad \mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i=1,2,\dots,N, \\ \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 = \phi_2, \quad \mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i=1,2,\dots,N^*. \end{cases} \tag{4.12}$$

Moreover, the solution satisfies $\|\nabla\boldsymbol{\phi}_\ell\|_{H^k} + (\underline{h}_\ell\delta)^{-1}\|\boldsymbol{\phi}_\ell'\|_{H^k} \le C\|\nabla\phi_\ell\|_{H^k}$ for $\ell=1,2$.

Proof. Notice that we have identities

$$L_{1,ij}(H_1,\delta,\underline{h}_1) = L_{ij}(H_1,0,\underline{h}_1\delta), \quad L_{2,ij}(H_2,b,\delta,\underline{h}_2) = L_{ij}(H_2,\underline{h}_2^{-1}b,\underline{h}_2\delta)$$

with suitable choices of indices $\{p_i\}$. Hence, Lemma 4.1 gives the desired result. $\square$

As a corollary of this lemma, under the assumptions of Lemma 4.4

$$\Lambda_1^{(N)}(\zeta,\delta,\underline{h}_1)\colon \phi_1 \mapsto \mathcal{L}_{1,0}(H_1,\underline{h}_1,\delta)\boldsymbol{\phi}_1,$$
$$\Lambda_2^{(N)}(\zeta,b,\delta,\underline{h}_2)\colon \phi_2 \mapsto \mathcal{L}_{2,0}(H_2,b,\underline{h}_2,\delta)\boldsymbol{\phi}_2,$$

where $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ is the unique solution to (4.12), are defined as bounded linear operators from $\mathring{H}^{k+1}$ to H^{k-1} for any $k = \pm 0, \ldots, \pm(m-1)$. Using these definitions and noting the relations (3.8) and $\boldsymbol{l}_\ell(H_\ell) \cdot \partial_t \boldsymbol{\phi}_\ell = \partial_t(\boldsymbol{l}_\ell(H_\ell) \cdot \boldsymbol{\phi}_\ell) - w_\ell \underline{h}_\ell^{-1} \partial_t \zeta$, we can transform the Kakinuma model (2.18) equivalently as

$$
\begin{cases}
\partial_t \zeta + \underline{h}_1 \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1 = 0, \\
\partial_t \zeta - \underline{h}_2 \Lambda_1^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2 = 0, \\
\underline{\rho}_1 \bigl\{ \partial_t \phi_1 + \frac{1}{2}\bigl(|\boldsymbol{u}_1|^2 + (\underline{h}_1\delta)^{-2} w_1^2\bigr) + w_1 \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1 \bigr\} \\
\quad - \underline{\rho}_2 \bigl\{ \partial_t \phi_2 + \frac{1}{2}\bigl(|\boldsymbol{u}_2|^2 + (\underline{h}_2\delta)^{-2} w_2^2\bigr) - w_2 \Lambda_2^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2 \bigr\} - \zeta = 0,
\end{cases}
\tag{4.13}
$$

where we recall that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, w_1, and w_2 are uniquely determined from ϕ_1 and ϕ_2 by (3.4), wherein $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are defined as the solutions to (4.12).

We further introduce notations, which are contributions of Bernoulli's equation and interfacial versions of B and $B^{(N)}$ defined by (4.9) and (4.10). We denote

$$
\begin{cases}
B_1(\phi_1; \zeta, \delta, \underline{h}_1) := \dfrac{1}{2}|\nabla \phi_1|^2 - \dfrac{1}{2}\delta^2 \dfrac{(\Lambda_1(\zeta, \delta, \underline{h}_1)\phi_1 - \nabla\zeta \cdot \nabla\phi_1)^2}{1 + \delta^2 |\nabla \zeta|^2}, \\
B_2(\phi_2; \zeta, b, \delta, \underline{h}_2) := \dfrac{1}{2}|\nabla \phi_2|^2 - \dfrac{1}{2}\delta^2 \dfrac{(\Lambda_2(\zeta, b, \delta, \underline{h}_2)\phi_2 + \nabla\zeta \cdot \nabla\phi_2)^2}{1 + \delta^2 |\nabla \zeta|^2},
\end{cases}
$$

and

$$
\begin{cases}
B_1^{(N)}(\phi_1; \zeta, \delta, \underline{h}_1) := \dfrac{1}{2}\bigl(|\boldsymbol{u}_1|^2 + (\underline{h}_1\delta)^{-2} w_1^2\bigr) + w_1 \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1, \\
B_2^{(N)}(\phi_2; \zeta, b, \delta, \underline{h}_2) := \dfrac{1}{2}\bigl(|\boldsymbol{u}_2|^2 + (\underline{h}_2\delta)^{-2} w_2^2\bigr) - w_2 \Lambda_2^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2.
\end{cases}
$$

Then, the full model for interfacial gravity waves (2.17) and the Kakinuma model (4.13) can be written simply as

$$
\begin{cases}
\partial_t \zeta + \Lambda_1(\zeta, \delta, \underline{h}_1)\phi_1 = 0, \\
\partial_t \zeta - \Lambda_2(\zeta, b, \delta, \underline{h}_2)\phi_2 = 0, \\
\underline{\rho}_1 \bigl(\partial_t \phi_1 + B_1(\phi_1; \zeta, \delta, \underline{h}_1) \bigr) - \underline{\rho}_2 \bigl(\partial_t \phi_2 + B_2(\phi_2; \zeta, b, \delta, \underline{h}_2) \bigr) - \zeta = 0,
\end{cases}
$$

and

$$
\begin{cases}
\partial_t \zeta + \underline{h}_1 \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1 = 0, \\
\partial_t \zeta - \underline{h}_2 \Lambda_1^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2 = 0, \\
\underline{\rho}_1 \bigl(\partial_t \phi_1 + B_1^{(N)}(\phi_1; \zeta, \delta, \underline{h}_1) \bigr) - \underline{\rho}_2 \bigl(\partial_t \phi_2 + B_2^{(N)}(\phi_2; \zeta, b, \delta, \underline{h}_2) \bigr) - \zeta = 0,
\end{cases}
$$

respectively. The following lemmas show that $\underline{h}_1 \Lambda_1^{(N)}$, $\underline{h}_2 \Lambda_2^{(N)}$, $B_1^{(N)}$, and $B_2^{(N)}$ are higher order approximations in the shallow water regime $\delta_1 = \underline{h}_1 \delta \ll 1$ and $\delta_2 = \underline{h}_2 \delta \ll 1$ to Λ_1, Λ_2, B_1, and B_2, respectively.

Lemma 4.5. *Let c, M be positive constants and m, j integers such that $m > \frac{n}{2} + 1$, $m \geq 2(j+1)$, and $1 \leq j \leq 2N+1$. We assume* (H1) *or* (H2). *There exists a positive constant C such that for any positive parameters $\underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, $H_1 = 1 - \underline{h}_1^{-1}\zeta$, and $H_2 = 1 + \underline{h}_2^{-1}\zeta - \underline{h}_2^{-1} b$ satisfy*

$$
\begin{cases}
\underline{h}_1^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m+1,\infty}} \leq M, \\
H_1(\boldsymbol{x}) \geq c, \quad H_2(\boldsymbol{x}) \geq c \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n,
\end{cases}
\tag{4.14}
$$

then for any $\phi_1, \phi_2 \in \mathring{H}^{k+2(j+1)}$ *with* $0 \le k \le m - 2(j+1)$ *we have*

$$\begin{cases} \|\underline{h}_1 \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1 - \Lambda_1(\zeta, \delta, \underline{h}_1)\phi_1\|_{H^k} \le C\underline{h}_1(\underline{h}_1\delta)^{2j}\|\nabla\phi_1\|_{H^{k+2j+1}}, \\ \|\underline{h}_2 \Lambda_2^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2 - \Lambda_2(\zeta, b, \delta, \underline{h}_2)\phi_2\|_{H^k} \le C\underline{h}_2(\underline{h}_2\delta)^{2j}\|\nabla\phi_2\|_{H^{k+2j+1}}. \end{cases}$$

Proof. By simple scaling arguments, we have

$$\begin{cases} \Lambda_1(\zeta, \delta, \underline{h}_1) = \underline{h}_1 \Lambda(-\underline{h}_1^{-1}\zeta, 0, \underline{h}_1\delta), \\ \Lambda_2(\zeta, b, \delta, \underline{h}_2) = \underline{h}_2 \Lambda(\underline{h}_2^{-1}\zeta, \underline{h}_2^{-1}b, \underline{h}_2\delta), \\ \Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1) = \Lambda^{(N)}(-\underline{h}_1^{-1}\zeta, 0, \underline{h}_1\delta), \\ \Lambda_2^{(N)}(\zeta, b, \delta, \underline{h}_2) = \Lambda^{(N)}(\underline{h}_2^{-1}\zeta, \underline{h}_2^{-1}b, \underline{h}_2\delta). \end{cases} \tag{4.15}$$

Therefore, the results follow from Lemma 4.2. □

Lemma 4.6. *Let* c, M *be positive constants and* m *an integer such that* $m \ge 4(N+1)$ *and* $m > \frac{n}{2} + 1$. *We assume* (H1) *or* (H2). *There exists a positive constant* C *such that for any positive parameters* $\underline{h}_1, \underline{h}_2, \delta$ *satisfying* $\underline{h}_1\delta, \underline{h}_2\delta \le 1$, *if* $\zeta \in H^m$, $b \in W^{m+1,\infty}$, $H_1 = 1 - \underline{h}_1^{-1}\zeta$, *and* $H_2 = 1 + \underline{h}_2^{-1}\zeta - \underline{h}_2^{-1}b$ *satisfy* (4.14), *then for any* $\phi_1, \phi_2 \in \mathring{H}^m$ *we have*

$$\begin{cases} \|B_1^{(N)}(\phi_1; \zeta, \delta, \underline{h}_1) - B_1(\phi_1; \zeta, \delta, \underline{h}_1)\|_{H^{m-4(N+1)}} \le C\|\nabla\phi_1\|_{H^{m-1}}^2 (\underline{h}_1\delta)^{4N+2}, \\ \|B_2^{(N)}(\phi_2; \zeta, b, \delta, \underline{h}_2) - B_2(\phi_2; \zeta, b, \delta, \underline{h}_2)\|_{H^{m-4(N+1)}} \le C\|\nabla\phi_2\|_{H^{m-1}}^2 (\underline{h}_2\delta)^{4N+2}. \end{cases}$$

Proof. By simple scaling arguments, we have

$$\begin{cases} B_1(\phi_1; \zeta, \delta, \underline{h}_1) = B(\phi_1; -\underline{h}_1^{-1}\zeta, 0, \underline{h}_1\delta), \\ B_2(\phi_2; \zeta, b, \delta, \underline{h}_1) = B(\phi_2; \underline{h}_2^{-1}\zeta, \underline{h}_2^{-1}b, \underline{h}_2\delta), \\ B_1^{(N)}(\phi_1; \zeta, \delta, \underline{h}_1) = B^{(N)}(\phi_1; -\underline{h}_1^{-1}\zeta, 0, \underline{h}_1\delta), \\ B_2^{(N)}(\phi_2; \zeta, b, \delta, \underline{h}_1) = B^{(N)}(\phi_2; \underline{h}_2^{-1}\zeta, \underline{h}_2^{-1}b, \underline{h}_2\delta). \end{cases}$$

Therefore, the results follow from Lemma 4.3. □

We can now prove Theorems 3.5 and 3.6. In view of (3.8) the errors $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_0)$ and $(\tilde{\mathfrak{r}}_1, \tilde{\mathfrak{r}}_2, \tilde{\mathfrak{r}}_0)$ can be written explicitly as

$$\begin{cases} \mathfrak{r}_1 = \Lambda_1(\zeta, \delta, \underline{h}_1)\phi_1 - \underline{h}_1\Lambda_1^{(N)}(\zeta, \delta, \underline{h}_1)\phi_1, \\ \mathfrak{r}_2 = \underline{h}_2\Lambda_2^{(N)}(\zeta, b, \delta, \underline{h}_2)\phi_2 - \Lambda_2(\zeta, b, \delta, \underline{h}_2)\phi_2, \\ \mathfrak{r}_0 = \frac{1}{2}\underline{\rho}_1\big(B_1(\phi_1; \zeta, \delta, \underline{h}_1) - B_1^{(N)}(\phi_1; \zeta, \delta, \underline{h}_1)\big) \\ \qquad -\frac{1}{2}\underline{\rho}_2\big(B_2(\phi_2; \zeta, b, \delta, \underline{h}_2) - B_2^{(N)}(\phi_2; \zeta, b, \delta, \underline{h}_2)\big), \\ \tilde{\mathfrak{r}}_1 = -\underline{h}_1^{-1}\boldsymbol{l}_1(H_1)\mathfrak{r}_1, \quad \tilde{\mathfrak{r}}_2 = -\underline{h}_2^{-1}\boldsymbol{l}_2(H_2)\mathfrak{r}_2, \quad \tilde{\mathfrak{r}}_0 = -\mathfrak{r}_0. \end{cases}$$

Therefore, the theorems are simple corollaries of the above Lemmas 4.5 and 4.6.

5 Elliptic estimates and time derivatives

In this section we derive useful uniform *a priori* bounds on regular solutions to the Kakinuma model (2.18). Firstly, due to the fact that the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Kakinuma model, we need the following key elliptic estimate in order to be able to estimate time derivatives of the solution. Let us recall that the operators $\mathcal{L}_{1,i}$ for $i = 0, 1, \ldots, N$ and $\mathcal{L}_{2,i}$ for $i = 0, 1, \ldots, N^*$ are defined by (3.6), and the vectors $\boldsymbol{l}_1(H_1)$ and $\boldsymbol{l}_2(H_2)$ are defined by (3.3). We recall the convention that for a vector $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_N)^{\mathrm{T}}$ we denote the last N components by $\boldsymbol{\phi}' = (\phi_1, \ldots, \phi_N)^{\mathrm{T}}$.

Lemma 5.1. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if $\zeta \in H^m$, $b \in W^{m,\infty}$, $H_1 = 1 - \underline{h}_1^{-1}\zeta$, and $H_2 = 1 + \underline{h}_2^{-1}\zeta - \underline{h}_2^{-1}b$ satisfy (4.11), then for any $\boldsymbol{f}_1' = (f_{1,1}, \dots, f_{1,N})^{\mathrm{T}} \in (H^k)^N$, $\boldsymbol{f}_2' = (f_{2,1}, \dots, f_{2,N^*})^{\mathrm{T}} \in (H^k)^{N^*}$, $\boldsymbol{f}_3 \in (H^k)^n$, and $f_4 \in \mathring{H}^{k+1}$ with $k \in \{0, 1, \dots, m-1\}$, there exists a solution $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ to*

$$
\begin{cases}
\mathcal{L}_{1,i}(H_1, \delta, \underline{h}_1)\boldsymbol{\varphi}_1 = f_{1,i} \quad \text{for} \quad i = 1, 2, \dots, N, \\
\mathcal{L}_{2,i}(H_2, b, \delta, \underline{h}_2)\boldsymbol{\varphi}_2 = f_{2,i} \quad \text{for} \quad i = 1, 2, \dots, N^*, \\
\underline{h}_1\mathcal{L}_{1,0}(H_1\delta, \underline{h}_1)\boldsymbol{\varphi}_1 + \underline{h}_2\mathcal{L}_{2,0}(H_2, b, \delta, \underline{h}_2)\boldsymbol{\varphi}_2 = \nabla \cdot \boldsymbol{f}_3, \\
-\underline{\rho}_1\boldsymbol{l}_1(H_1) \cdot \boldsymbol{\varphi}_1 + \underline{\rho}_2\boldsymbol{l}_2(H_2) \cdot \boldsymbol{\varphi}_2 = f_4,
\end{cases}
\tag{5.1}
$$

satisfying

$$
\begin{aligned}
&\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \big(\|\nabla\boldsymbol{\varphi}_\ell\|_{H^k}^2 + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\varphi}_\ell'\|_{H^k}^2\big) \\
&\leq C\biggl(\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \min\{\|\boldsymbol{f}_\ell'\|_{H^{k-1}}^2, (\underline{h}_\ell\delta)^2\|\boldsymbol{f}_\ell'\|_{H^k}^2\} \\
&\qquad + \min\biggl\{\frac{\underline{\rho}_1}{\underline{h}_1}, \frac{\underline{\rho}_2}{\underline{h}_2}\biggr\}\|\boldsymbol{f}_3\|_{H^k}^2 + \min\biggl\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\biggr\}\|\nabla f_4\|_{H^k}^2\biggr).
\end{aligned}
$$

Moreover, the solution is unique up to an additive constant of the form $(\mathcal{C}\underline{\rho}_2, \mathcal{C}\underline{\rho}_1)$ to $(\varphi_{1,0}, \varphi_{2,0})$.

Proof. The existence and uniqueness up to an additive constant of the solution has been given in the companion paper [14, Lemma 6.4]. We focus here on the derivation of uniform estimates. By direct rescaling within the proof of [14, Lemma 6.1], we infer that

$$
(L_\ell\boldsymbol{\varphi}_\ell, \boldsymbol{\varphi}_\ell)_{L^2} \simeq \|\nabla\boldsymbol{\varphi}_\ell\|_{L^2}^2 + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\varphi}_\ell'\|_{L^2}^2
$$

for $\ell = 1, 2$. We note the identities

$$
\begin{cases}
L_1\boldsymbol{\varphi}_1 = \boldsymbol{l}_1\mathcal{L}_{1,0}\boldsymbol{\varphi}_1 + (0, \mathcal{L}_{1,1}\boldsymbol{\varphi}_1, \dots, \mathcal{L}_{1,N}\boldsymbol{\varphi}_1)^{\mathrm{T}}, \\
L_2\boldsymbol{\varphi}_2 = \boldsymbol{l}_2\mathcal{L}_{2,0}\boldsymbol{\varphi}_2 + (0, \mathcal{L}_{2,1}\boldsymbol{\varphi}_2, \dots, \mathcal{L}_{2,N^*}\boldsymbol{\varphi}_2)^{\mathrm{T}},
\end{cases}
$$

so that for the solution $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ to (5.1) we have

$$
\begin{aligned}
\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (L_\ell\boldsymbol{\varphi}_\ell, \boldsymbol{\varphi}_\ell)_{L^2} &= \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\mathcal{L}_{\ell,0}\boldsymbol{\varphi}_\ell, \boldsymbol{l}_\ell \cdot \boldsymbol{\varphi}_\ell)_{L^2} + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\boldsymbol{f}_\ell', \boldsymbol{\varphi}_\ell')_{L^2} \\
&=: I_1 + I_2.
\end{aligned}
\tag{5.2}
$$

Therefore, it is sufficient to evaluate I_1 and I_2. As for the term I_2 we have

$$
\begin{aligned}
|(\boldsymbol{f}_\ell', \boldsymbol{\varphi}_\ell')_{L^2}| &\leq \min\{\|\boldsymbol{f}_\ell'\|_{H^{-1}}\|\boldsymbol{\varphi}_\ell'\|_{H^1}, \|\boldsymbol{f}_\ell'\|_{L^2}\|\boldsymbol{\varphi}_\ell'\|_{L^2}\} \\
&\leq \min\{\|\boldsymbol{f}_\ell'\|_{H^{-1}}, (\underline{h}_\ell\delta)\|\boldsymbol{f}_\ell'\|_{L^2}\}(\|\nabla\boldsymbol{\varphi}_\ell\|_{L^2} + (\underline{h}_\ell\delta)^{-1}\|\boldsymbol{\varphi}_\ell'\|_{L^2}).
\end{aligned}
$$

As for the term I_1, we note the trivial identities

$$
\begin{aligned}
&\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\mathcal{L}_{\ell,0}\boldsymbol{\varphi}_\ell, \boldsymbol{l}_\ell \cdot \boldsymbol{\varphi}_\ell)_{L^2} \\
&= \begin{cases}
(\underline{h}_1\mathcal{L}_{1,0}\boldsymbol{\varphi}_1 + \underline{h}_2\mathcal{L}_{2,0}\boldsymbol{\varphi}_2, \underline{\rho}_1\boldsymbol{l}_1 \cdot \boldsymbol{\varphi}_1)_{L^2} + (\underline{h}_2\mathcal{L}_{2,0}\boldsymbol{\varphi}_2, \underline{\rho}_2\boldsymbol{l}_2 \cdot \boldsymbol{\varphi}_2 - \underline{\rho}_1\boldsymbol{l}_1 \cdot \boldsymbol{\varphi}_1)_{L^2}, \\
(\underline{h}_1\mathcal{L}_{1,0}\boldsymbol{\varphi}_1 + \underline{h}_2\mathcal{L}_{2,0}\boldsymbol{\varphi}_2, \underline{\rho}_2\boldsymbol{l}_2 \cdot \boldsymbol{\varphi}_2)_{L^2} + (\underline{h}_1\mathcal{L}_{1,0}\boldsymbol{\varphi}_1, \underline{\rho}_1\boldsymbol{l}_1 \cdot \boldsymbol{\varphi}_1 - \underline{\rho}_2\boldsymbol{l}_2 \cdot \boldsymbol{\varphi}_2)_{L^2}.
\end{cases}
\end{aligned}
$$

Therefore, the term I_1 in (5.2) can be expressed in two ways as

$$I_1 = \begin{cases} \underline{\rho}_1(\nabla\cdot\boldsymbol{f}_3, \boldsymbol{l}_1\cdot\boldsymbol{\varphi}_1)_{L^2} + \underline{h}_2(\mathcal{L}_{2,0}\boldsymbol{\varphi}_2, f_4)_{L^2}, \\ \underline{\rho}_2(\nabla\cdot\boldsymbol{f}_3, \boldsymbol{l}_2\cdot\boldsymbol{\varphi}_2)_{L^2} - \underline{h}_1(\mathcal{L}_{1,0}\boldsymbol{\varphi}_1, f_4)_{L^2}. \end{cases}$$

By the linearity of (5.1) it is sufficient to evaluate it in the case $f_4 = 0$ and in the case $\boldsymbol{f}_3 = \boldsymbol{0}$, separately. In the case $f_4 = 0$, we evaluate it as

$$\begin{aligned} |I_1| &\le \min\{\underline{\rho}_1\|\boldsymbol{f}_3\|_{L^2}\|\nabla(\boldsymbol{l}_1\cdot\boldsymbol{\varphi}_1)\|_{L^2}, \underline{\rho}_2\|\boldsymbol{f}_3\|_{L^2}\|\nabla(\boldsymbol{l}_2\cdot\boldsymbol{\varphi}_2)\|_{L^2}\} \\ &= \min\left\{\sqrt{\frac{\underline{\rho}_1}{\underline{h}_1}}\|\boldsymbol{f}_3\|_{L^2}\sqrt{\underline{\rho}_1\underline{h}_1}\|\nabla(\boldsymbol{l}_1\cdot\boldsymbol{\varphi}_1)\|_{L^2}, \sqrt{\frac{\underline{\rho}_2}{\underline{h}_2}}\|\boldsymbol{f}_3\|_{L^2}\sqrt{\underline{\rho}_2\underline{h}_2}\|\nabla(\boldsymbol{l}_2\cdot\boldsymbol{\varphi}_2)\|_{L^2}\right\} \\ &\lesssim \min\left\{\sqrt{\frac{\underline{\rho}_1}{\underline{h}_1}}, \sqrt{\frac{\underline{\rho}_2}{\underline{h}_2}}\right\}\|\boldsymbol{f}_3\|_{L^2}\sum_{\ell=1,2}\sqrt{\underline{\rho}_\ell\underline{h}_\ell}(\|\nabla\boldsymbol{\varphi}_\ell\|_{L^2} + \|\boldsymbol{\varphi}_\ell'\|_{L^2}). \end{aligned}$$

In the case $\boldsymbol{f}_3 = \boldsymbol{0}$ we evaluate it as

$$\begin{aligned} |I_1| &\lesssim \min\{\underline{h}_1\|\nabla\boldsymbol{\varphi}_1\|_{L^2}\|\nabla f_4\|_{L^2}, \underline{h}_2(\|\nabla\boldsymbol{\varphi}_2\|_{L^2} + \|\boldsymbol{\varphi}_2'\|_{L^2})\|\nabla f_4\|_{L^2}\} \\ &= \min\left\{\sqrt{\frac{\underline{h}_1}{\underline{\rho}_1}}\|\nabla f_4\|_{L^2}\sqrt{\underline{\rho}_1\underline{h}_1}\|\nabla\boldsymbol{\varphi}_1\|_{L^2}, \sqrt{\frac{\underline{h}_2}{\underline{\rho}_2}}\|\nabla f_4\|_{L^2}\sqrt{\underline{\rho}_2\underline{h}_2}(\|\nabla\boldsymbol{\varphi}_1\|_{L^2} + \|\boldsymbol{\varphi}_2'\|_{L^2})\right\} \\ &\le \min\left\{\sqrt{\frac{\underline{h}_1}{\underline{\rho}_1}}, \sqrt{\frac{\underline{h}_2}{\underline{\rho}_2}}\right\}\|\nabla f_4\|_{L^2}\sum_{\ell=1,2}\sqrt{\underline{\rho}_\ell\underline{h}_\ell}(\|\nabla\boldsymbol{\varphi}_\ell\|_{L^2} + \|\boldsymbol{\varphi}_\ell'\|_{L^2}). \end{aligned}$$

From the above estimates we deduce immediately the desired inequality for $k = 0$.

In order to obtain the desired inequality on derivatives, we let $k \in \{1, 2, \dots, m-1\}$ and β be a multi-index such that $1 \le |\beta| \le k$. Applying the differential operator ∂^β to (5.1), we have

$$\begin{cases} \mathcal{L}_{1,i}\partial^\beta\boldsymbol{\varphi}_1 = \partial^\beta f_{1,i} + f_{1,i,\beta} \quad \text{for} \quad i = 1, 2, \dots, N, \\ \mathcal{L}_{2,i}\partial^\beta\boldsymbol{\varphi}_2 = \partial^\beta f_{2,i} + f_{2,i,\beta} \quad \text{for} \quad i = 1, 2, \dots, N^*, \\ \underline{h}_1\mathcal{L}_{1,0}\partial^\beta\boldsymbol{\varphi}_1 + \underline{h}_2\mathcal{L}_{2,0}\partial^\beta\boldsymbol{\varphi}_2 = \nabla\cdot(\partial^\beta\boldsymbol{f}_3 + \underline{h}_1\boldsymbol{f}_{3,1,\beta} + \underline{h}_2\boldsymbol{f}_{3,2,\beta}), \\ -\underline{\rho}_1\boldsymbol{l}_1\cdot\partial^\beta\boldsymbol{\varphi}_1 + \underline{\rho}_2\boldsymbol{l}_2\cdot\partial^\beta\boldsymbol{\varphi}_2 = \partial^\beta f_4 + \underline{\rho}_1 f_{4,1,\beta} + \underline{\rho}_2 f_{4,2,\beta}, \end{cases}$$

where

$$\begin{cases} f_{1,i,\beta} := -[\partial^\beta, \mathcal{L}_{1,i}(H_1, \delta, \underline{h}_1)]\boldsymbol{\varphi}_1 \quad \text{for} \quad i = 1, 2, \dots, N, \\ f_{2,i,\beta} := -[\partial^\beta, \mathcal{L}_{2,i}(H_2, b, \delta, \underline{h}_2)]\boldsymbol{\varphi}_2 \quad \text{for} \quad i = 1, 2, \dots, N^*, \\ \nabla\cdot\boldsymbol{f}_{3,1,\beta} := -[\partial^\beta, \mathcal{L}_{1,0}(H_1, \delta, \underline{h}_1)]\boldsymbol{\varphi}_1, \\ \nabla\cdot\boldsymbol{f}_{3,2,\beta} := -[\partial^\beta, \mathcal{L}_{2,0}(H_2, b, \delta, \underline{h}_2)]\boldsymbol{\varphi}_2, \\ f_{4,1,\beta} := [\partial^\beta, \boldsymbol{l}_1(H_1)]\cdot\boldsymbol{\varphi}_1, \\ f_{4,2,\beta} := -[\partial^\beta, \boldsymbol{l}_2(H_2)]\cdot\boldsymbol{\varphi}_2. \end{cases}$$

We put $\boldsymbol{f}_{1,\beta} = (0, f_{1,1,\beta}, \dots, f_{1,N,\beta})$ and $\boldsymbol{f}_{2,\beta} = (0, f_{2,1,\beta}, \dots, f_{2,N^*,\beta})$. Then, with a suitable decomposition $\boldsymbol{f}_{\ell,\beta} = \boldsymbol{f}_{\ell,\beta}^{\mathrm{high}} + \boldsymbol{f}_{\ell,\beta}^{\mathrm{low}}$ for $\ell = 1, 2$, we see that

$$\|\boldsymbol{f}_{\ell,\beta}^{\mathrm{high}}\|_{H^{-1}} + (\underline{h}_\ell\delta)\|\boldsymbol{f}_{\ell,\beta}^{\mathrm{low}}\|_{L^2} + \|\boldsymbol{f}_{3,\ell,\beta}\|_{L^2} + \|\nabla f_{4,\ell}\|_{L^2} \lesssim \|\nabla\boldsymbol{\varphi}_\ell\|_{H^{k-1}} + (\underline{h}_\ell\delta)^{-1}\|\boldsymbol{\varphi}_\ell'\|_{H^{k-1}}$$

for $\ell = 1, 2$. Therefore, in view of the linearity of (5.1) the desired inequality for $k \ge 1$ follows by induction on k. $\square$

From the above elliptic estimates we deduce the following bounds on time derivatives of regular solutions to the Kakinuma model (2.18). We introduce a mathematical energy $E_m(t)$ for a solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model by

$$E_m(t) := \|\zeta(t)\|_{H^m}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \boldsymbol{\phi}_\ell(t)\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_\ell'(t)\|_{H^m}^2), \tag{5.3}$$

where $\boldsymbol{\phi}_1' = (\phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2' = (\phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$.

Lemma 5.2. *Let $c, M_1, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C_1 such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (2.14)*, $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, if a regular solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model* (2.18) *with bottom topography $b \in W^{m+1,\infty}$ satisfy*

$$\begin{cases} E_m(t) + \underline{h}_2^{-1} \|b\|_{W^{m+1,\infty}} \leq M_1, \\ H_1(\boldsymbol{x}, t) \geq c, \quad H_2(\boldsymbol{x}, t) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, 0 \leq t \leq T, \end{cases}$$

then we have

$$\begin{aligned} &\|\partial_t \zeta(t)\|_{H^{m-1}}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \partial_t \boldsymbol{\phi}_\ell(t)\|_{H^{m-1}}^2 + (\underline{h}_\ell \delta)^{-2} \|\partial_t \boldsymbol{\phi}_\ell'(t)\|_{H^{m-1}}^2) \\ &\quad + \|\partial_t^2 \zeta(t)\|_{H^{m-2}}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \partial_t^2 \boldsymbol{\phi}_\ell(t)\|_{H^{m-2}}^2 + (\underline{h}_\ell \delta)^{-2} \|\partial_t^2 \boldsymbol{\phi}_\ell'(t)\|_{H^{m-2}}^2) \leq C_1 E_m(t) \end{aligned} \tag{5.4}$$

for $0 \leq t \leq T$.

Proof. First, we recall that the Kakinuma model (2.18) can be written compactly as (3.5). It follows from the first component of the first two equations in (3.5) that $\partial_t \zeta$ can be written in two ways as $\partial_t \zeta = -\underline{h}_1 \mathcal{L}_{1,0} \boldsymbol{\phi}_1 = \underline{h}_2 \mathcal{L}_{2,0} \boldsymbol{\phi}_2$, so that

$$\begin{aligned} \|\partial_t \zeta\|_{H^{m-1}}^2 &= \min\{\underline{h}_1^2 \|\mathcal{L}_{1,0}\boldsymbol{\phi}_1\|_{H^{m-1}}^2, \underline{h}_2^2 \|\mathcal{L}_{2,0}\boldsymbol{\phi}_2\|_{H^{m-1}}^2\} \\ &\lesssim \min\{\underline{h}_1^2 \|\nabla \boldsymbol{\phi}_1\|_{H^m}^2, \underline{h}_2^2 (\|\nabla \boldsymbol{\phi}_2\|_{H^m}^2 + \|\boldsymbol{\phi}_2'\|_{H^m}^2)\} \\ &\leq \min\left\{ \frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2} \right\} E_m \leq 2E_m, \end{aligned}$$

where we used (2.15).

As for the estimate of $(\partial_t \boldsymbol{\phi}_1, \partial_t \boldsymbol{\phi}_2)$, we differentiate the compatibility conditions (3.7) with respect to time and use the last equation in (3.5). Then, we have

$$\begin{cases} \mathcal{L}_{1,i} \partial_t \boldsymbol{\phi}_1 = f_{1,i} \quad \text{for} \quad i = 1, 2, \ldots, N, \\ \mathcal{L}_{2,i} \partial_t \boldsymbol{\phi}_2 = f_{2,i} \quad \text{for} \quad i = 1, 2, \ldots, N^*, \\ \underline{h}_1 \mathcal{L}_{1,0} \partial_t \boldsymbol{\phi}_1 + \underline{h}_2 \mathcal{L}_{2,0} \partial_t \boldsymbol{\phi}_2 = \nabla \cdot \boldsymbol{f}_3, \\ -\underline{\rho}_1 \boldsymbol{l}_1 \cdot \partial_t \boldsymbol{\phi}_1 + \underline{\rho}_2 \boldsymbol{l}_2 \cdot \partial_t \boldsymbol{\phi}_2 = f_4, \end{cases} \tag{5.5}$$

where

$$\begin{cases} f_{1,i} := -[\partial_t, \mathcal{L}_{1,i}(H_1, \delta, \underline{h}_1)] \boldsymbol{\phi}_1 \quad \text{for} \quad i = 1, 2, \ldots, N, \\ f_{2,i} := -[\partial_t, \mathcal{L}_{2,i}(H_2, b, \delta, \underline{h}_2)] \boldsymbol{\phi}_2 \quad \text{for} \quad i = 1, 2, \ldots, N^*, \\ \boldsymbol{f}_3 := (\boldsymbol{u}_2 - \boldsymbol{u}_1) \partial_t \zeta, \\ f_4 := \frac{1}{2} \underline{\rho}_1 \big(|\boldsymbol{u}_1|^2 + (\underline{h}_1 \delta)^{-2} w_1^2 \big) - \frac{1}{2} \underline{\rho}_2 \big(|\boldsymbol{u}_2|^2 + (\underline{h}_2 \delta)^{-2} w_2^2 \big) - \zeta. \end{cases} \tag{5.6}$$

Therefore, by Lemma 5.1 we have

$$
\begin{aligned}
(5.7) \qquad & \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \partial_t \boldsymbol{\phi}_\ell\|_{H^{m-1}}^2 + (\underline{h}_\ell \delta)^{-2} \|\partial_t \boldsymbol{\phi}_\ell'\|_{H^{m-1}}^2) \\
& \lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell \delta)^2 \|\boldsymbol{f}_\ell'\|_{H^{m-1}}^2 + \min\left\{ \frac{\underline{\rho}_1}{\underline{h}_1}, \frac{\underline{\rho}_2}{\underline{h}_2} \right\} \|\boldsymbol{f}_3\|_{H^{m-1}}^2 + \|f_4\|_{H^m}^2,
\end{aligned}
$$

where $\boldsymbol{f}_1' = (f_{1,1}, \dots, f_{1,N})^{\mathrm{T}}$, $\boldsymbol{f}_2' = (f_{2,1}, \dots, f_{2,N^*})^{\mathrm{T}}$, and we used (2.15). We proceed to evaluate the right-hand side. By writing down the operators $\mathcal{L}_{\ell,i}$ explicitly, we see that the operators do not include any derivatives of H_ℓ. Therefore, we can write $f_{\ell,i}$ as

$$
f_{1,i} = \left(\left(\frac{\partial}{\partial H_1} \mathcal{L}_{1,i} \right) \boldsymbol{\phi}_1 \right) \underline{h}_1^{-1} \partial_t \zeta, \quad f_{2,i} = -\left(\left(\frac{\partial}{\partial H_2} \mathcal{L}_{2,i} \right) \boldsymbol{\phi}_2 \right) \underline{h}_2^{-1} \partial_t \zeta.
$$

We note also that the differential operators $\frac{\partial}{\partial H_\ell} \mathcal{L}_{\ell,i}$ have a similar structure as $\mathcal{L}_{\ell,i}$. Therefore,

$$
\begin{aligned}
\underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell \delta)^2 \|\boldsymbol{f}_\ell'\|_{H^{m-1}}^2 &\lesssim \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell \delta)^2 (\|\nabla \boldsymbol{\phi}_\ell\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-4} \|\boldsymbol{\phi}_\ell'\|_{H^{m-1}}^2) \|\underline{h}_\ell^{-1} \partial_t \zeta\|_{H^{m-1}}^2 \\
&\lesssim E_m^2 \quad \text{for} \quad \ell = 1, 2,
\end{aligned}
$$

where, here and henceforth, we utilize fully our restriction $\underline{h}_1^{-1}, \underline{h}_2^{-1} \lesssim 1$. In view of the definition (3.4) of $\boldsymbol{u}_1, \boldsymbol{u}_2, w_1$, and w_2, we see easily that

$$
(5.8) \qquad \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\boldsymbol{u}_\ell\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|w_\ell\|_{H^m}^2) \lesssim E_m.
$$

We evaluate the term on $\boldsymbol{f}_3$ as

$$
\begin{aligned}
\min\left\{ \frac{\underline{\rho}_1}{\underline{h}_1}, \frac{\underline{\rho}_2}{\underline{h}_2} \right\} \|\boldsymbol{f}_3\|_{H^{m-1}}^2 &\lesssim \sum_{\ell=1,2} \frac{\underline{\rho}_\ell}{\underline{h}_\ell} \|\boldsymbol{u}_\ell \partial_t \zeta\|_{H^{m-1}}^2 \\
&\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\boldsymbol{u}_\ell\|_{H^{m-1}}^2 \|\underline{h}_\ell^{-1} \partial_t \zeta\|_{H^{m-1}}^2 \\
&\lesssim E_m^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|f_4\|_{H^m}^2 &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell^2 (\|\boldsymbol{u}_\ell\|_{H^m}^2 + (\underline{h}_1 \delta)^{-2} \|w_\ell\|_{H^m}^2)^2 + \|\zeta\|_{H^m}^2 \\
&\lesssim \sum_{\ell=1,2} \underline{h}_\ell^{-2} \{ \underline{\rho}_\ell \underline{h}_\ell (\|\boldsymbol{u}_\ell\|_{H^m}^2 + (\underline{h}_1 \delta)^{-2} \|w_\ell\|_{H^m}^2) \}^2 + \|\zeta\|_{H^m}^2 \\
&\lesssim E_m^2 + E_m.
\end{aligned}
$$

Plugging in (5.7) the above estimates, we obtain the desired estimate for $(\partial_t \boldsymbol{\phi}_1, \partial_t \boldsymbol{\phi}_2)$.

Finally, the estimate of $\partial_t^2 \zeta$ can be obtained by differentiating $\partial_t \zeta = -\underline{h}_1 \mathcal{L}_{1,0} \boldsymbol{\phi}_1 = \underline{h}_2 \mathcal{L}_{2,0} \boldsymbol{\phi}_2$ with respect to time. Then, the estimate of $(\partial_t^2 \boldsymbol{\phi}_1, \partial_t^2 \boldsymbol{\phi}_2)$ can be obtained by differentiating (5.5) with respect to time once more and applying Lemma 5.1. □

Remark 5.3. In view of the above arguments, we see easily that for the Kakinuma model (2.18), $(\partial_t \boldsymbol{\phi}_1, \partial_t \boldsymbol{\phi}_2)|_{t=0}$ can be determined from the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b, although the hypersurface $t = 0$ is characteristic for the model. They are unique up to an additive constant of the form $(\mathcal{C}\underline{\rho}_2, \mathcal{C}\underline{\rho}_1)$ to $(\partial_t \phi_{1,0}, \partial_t \phi_{2,0})|_{t=0}$. Particularly, $(\partial_t \boldsymbol{\phi}_1', \partial_t \boldsymbol{\phi}_2')|_{t=0}$ and hence $a|_{t=0}$ with the function a given in (3.9) can be uniquely determined from the data.

6 Uniform energy estimates; proof of Theorem 3.1

In this section we provide uniform energy estimates for solutions to the Kakinuma model. Consequently, we prove Theorem 3.1. We recall that the Kakinuma model (2.18) can be written compactly as

$$
\begin{cases}
\boldsymbol{l}_1(H_1)\partial_t\zeta + \underline{h}_1 L_1(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = \mathbf{0}, \\
\boldsymbol{l}_2(H_2)\partial_t\zeta - \underline{h}_2 L_2(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = \mathbf{0}, \\
\underline{\rho}_1\big\{\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \frac{1}{2}\big(|\boldsymbol{u}_1|^2 + (\underline{h}_1\delta)^{-2}w_1^2\big)\big\} \\
\quad -\underline{\rho}_2\big\{\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 + \frac{1}{2}\big(|\boldsymbol{u}_2|^2 + (\underline{h}_2\delta)^{-2}w_2^2\big)\big\} - \zeta = 0,
\end{cases}
\tag{6.1}
$$

where we recall that $H_1 := 1 - \underline{h}_1^{-1}\zeta$, $H_2 := 1 + \underline{h}_2^{-1}\zeta - \underline{h}_2^{-1}b$, $\boldsymbol{\phi}_1 := (\phi_{1,0},\phi_{1,1},\dots,\phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 := (\phi_{2,0},\phi_{2,1},\dots,\phi_{2,N^*})^{\mathrm{T}}$, and $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, L_1, L_2, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, w_1, w_2 are defined in Section 3.

6.1 Analysis of linearized equations

Before deriving linearized equations to the Kakinuma model (6.1), we introduce some more notations. For $\ell = 1,2$, the coefficient matrices of the principal part and the singular part with respect to the small parameter $\delta_\ell = \underline{h}_\ell\delta$ of the operator L_ℓ are denoted by $A_\ell(H_\ell)$ and $C_\ell(H_\ell)$, respectively, that is,

$$
\begin{cases}
A_1(H_1) := \left(\dfrac{1}{2(i+j)+1}H_1^{2(i+j)+1}\right)_{0\le i,j\le N}, \\
A_2(H_2) := \left(\dfrac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\right)_{0\le i,j\le N^*},
\end{cases}
\tag{6.2}
$$

and

$$
\begin{cases}
C_1(H_1) := \left(\dfrac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\right)_{0\le i,j\le N}, \\
C_2(H_2) := \left(\dfrac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}\right)_{0\le i.j\le N^*}.
\end{cases}
\tag{6.3}
$$

We put also

$$
\begin{cases}
B_2(H_2) := \left(\dfrac{p_j}{p_i+p_j}H_2^{p_i+p_j}\right)_{0\le i,j\le N^*}, \\
\tilde{B}_2(H_2) := B_2(H_2) - B_2(H_2)^{\mathrm{T}}, \\
\tilde{C}_2(H_2,\underline{h}_2^{-1}b) := |\underline{h}_2^{-1}\nabla b|^2 C_2(H_2) + \underline{h}_2^{-1}(\Delta b)B_2(H_2).
\end{cases}
\tag{6.4}
$$

In the above expressions, we used the notational convention $0/0 = 0$. Then, the operators L_1 and L_2 can also be written as

$$
\begin{cases}
L_1\boldsymbol{\phi}_1 = -A_1\Delta\boldsymbol{\phi}_1 - \boldsymbol{l}_1(\boldsymbol{u}_1\cdot\nabla H_1) + (\underline{h}_1\delta)^{-2}C_1\boldsymbol{\phi}_1, \\
L_2\boldsymbol{\phi}_2 = -A_2\Delta\boldsymbol{\phi}_2 - \boldsymbol{l}_2(\boldsymbol{u}_2\cdot\nabla H_2) + (\underline{h}_2\delta)^{-2}C_2\boldsymbol{\phi}_2 + \tilde{B}_2(\underline{h}_2^{-1}\nabla b\cdot\nabla)\boldsymbol{\phi}_2 + \tilde{C}_2\boldsymbol{\phi}_2.
\end{cases}
\tag{6.5}
$$

For $\ell = 1,2$, we decompose the operator L_ℓ as $L_\ell = L_\ell^{\mathrm{pr}} + L_\ell^{\mathrm{low}}$, where

$$
L_\ell^{\mathrm{pr}}(H_\ell)\boldsymbol{\varphi}_\ell := -\sum_{l=1}^{n}\partial_l(A_\ell(H_\ell)\partial_l\boldsymbol{\varphi}_\ell) + (\underline{h}_\ell\delta)^{-2}C_\ell(H_\ell)\boldsymbol{\varphi}_\ell.
\tag{6.6}
$$

We now linearize the Kakinuma model (6.1) around an arbitrary flow $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and denote the variation by $(\dot{\zeta}, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)$. After neglecting lower order terms, the linearized equations have the form

$$
\begin{cases} \boldsymbol{l}_1(H_1)(\partial_t + \boldsymbol{u}_1\cdot\nabla)\dot{\zeta} + \underline{h}_1 L_1^{\mathrm{pr}}(H_1,\delta,\underline{h}_1)\dot{\boldsymbol{\phi}}_1 = \dot{\boldsymbol{f}}_1, \\ \boldsymbol{l}_2(H_2)(\partial_t + \boldsymbol{u}_2\cdot\nabla)\dot{\zeta} - \underline{h}_2 L_2^{\mathrm{pr}}(H_2,\delta,\underline{h}_2)\dot{\boldsymbol{\phi}}_2 = \dot{\boldsymbol{f}}_2, \\ \underline{\rho}_1\boldsymbol{l}_1(H_1)\cdot(\partial_t + \boldsymbol{u}_1\cdot\nabla)\dot{\boldsymbol{\phi}}_1 - \underline{\rho}_2\boldsymbol{l}_2(H_2)\cdot(\partial_t + \boldsymbol{u}_1\cdot\nabla)\dot{\boldsymbol{\phi}}_2 - a\dot{\zeta} = \dot{f}_0, \end{cases} \tag{6.7}
$$

where the function a is defined by (3.9). In order to derive a good symmetric structure of the equations, following the companion paper [14] we introduce

$$
\theta_1 := \frac{\underline{\rho}_2\underline{h}_1 H_1\alpha_1}{\underline{\rho}_1\underline{h}_2 H_2\alpha_2 + \underline{\rho}_2\underline{h}_1 H_1\alpha_1}, \qquad \theta_2 := \frac{\underline{\rho}_1\underline{h}_2 H_2\alpha_2}{\underline{\rho}_1\underline{h}_2 H_2\alpha_2 + \underline{\rho}_2\underline{h}_1 H_1\alpha_1}, \tag{6.8}
$$

where

$$
\alpha_\ell := \frac{\det A_{\ell,0}}{\det \tilde{A}_{\ell,0}}, \qquad \tilde{A}_{\ell,0} := \begin{pmatrix} 0 & \boldsymbol{1}^{\mathrm{T}} \\ -\boldsymbol{1} & A_{\ell,0} \end{pmatrix}, \qquad A_{\ell,0} := A_\ell(1) \tag{6.9}
$$

for $\ell = 1, 2$ and $\boldsymbol{1} := (1,\ldots,1)^{\mathrm{T}}$. Then, we have $\theta_1 + \theta_2 = 1$. We recall that α_1 and α_2 are positive constants depending only on N and the nonnegative integers $0 = p_0 < p_1 < \ldots < p_{N^*}$, respectively, and go to 0 as $N, N^* \to \infty$. We also introduce

$$
\boldsymbol{u} := \theta_2\boldsymbol{u}_1 + \theta_1\boldsymbol{u}_2, \qquad \boldsymbol{v} := \boldsymbol{u}_2 - \boldsymbol{u}_1.
$$

Then, we have $\boldsymbol{u}_1 = \boldsymbol{u} - \theta_1\boldsymbol{v}$ and $\boldsymbol{u}_2 = \boldsymbol{u} + \theta_2\boldsymbol{v}$. Plugging these into the linearized equations (6.7), we can write them in a matrix form as

$$
\mathscr{A}_1(\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}} + \mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}} = \dot{\boldsymbol{F}}, \tag{6.10}
$$

where

$$
\dot{\boldsymbol{U}} := \begin{pmatrix} \dot{\zeta} \\ \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix}, \qquad \dot{\boldsymbol{F}} := \begin{pmatrix} \dot{f}_0 \\ \underline{\rho}_1(\dot{\boldsymbol{f}}_1 - (\nabla\cdot(\theta_1\boldsymbol{l}_1\otimes\boldsymbol{v}))\dot{\zeta} \\ \underline{\rho}_2(\dot{\boldsymbol{f}}_2 - (\nabla\cdot(\theta_2\boldsymbol{l}_2\otimes\boldsymbol{v}))\dot{\zeta} \end{pmatrix},
$$

and

$$
\mathscr{A}_1 := \begin{pmatrix} 0 & -\underline{\rho}_1\boldsymbol{l}_1^{\mathrm{T}} & \underline{\rho}_2\boldsymbol{l}_2^{\mathrm{T}} \\ \underline{\rho}_1\boldsymbol{l}_1 & O & O \\ -\underline{\rho}_2\boldsymbol{l}_2 & O & O \end{pmatrix},
$$

$$
\mathscr{A}_0^{\mathrm{mod}} := \begin{pmatrix} a & \underline{\rho}_1\theta_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) & \underline{\rho}_2\theta_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) \\ (\boldsymbol{v}\cdot\nabla)^*(\underline{\rho}_1\theta_1\boldsymbol{l}_1\,\cdot) & \underline{\rho}_1\underline{h}_1 L_1^{\mathrm{pr}} & O \\ (\boldsymbol{v}\cdot\nabla)^*(\underline{\rho}_2\theta_1\boldsymbol{l}_2\,\cdot) & O & \underline{\rho}_2\underline{h}_2 L_2^{\mathrm{pr}} \end{pmatrix}.
$$

Here, $(\boldsymbol{v}\cdot\nabla)^*$ denotes the adjoint operator of $\boldsymbol{v}\cdot\nabla$ in L^2, that is, $(\boldsymbol{v}\cdot\nabla)^* f = -\nabla\cdot(f\boldsymbol{v})$. We note that $\mathscr{A}_1$ is a skew-symmetric matrix and $\mathscr{A}_0^{\mathrm{mod}}$ is symmetric in L^2. Therefore, the corresponding energy function is given by $(\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2}$. We put

$$
\mathscr{E}(\dot{\boldsymbol{U}}) := \|\dot{\zeta}\|_{L^2}^2 + \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla\dot{\boldsymbol{\phi}}_\ell\|_{L^2}^2 + (\underline{h}_\ell\delta)^{-2}\|\dot{\boldsymbol{\phi}}_\ell'\|_{L^2}^2). \tag{6.11}
$$

The following lemma shows that $(\mathscr{A}_0^{\mathrm{mod}}\dot{U}, \dot{U})_{L^2} \simeq \mathscr{E}(\dot{\boldsymbol{U}})$ under the non-cavitation assumption and the stability condition, stated respectively as (3.11) and (3.12) in Theorem 3.1.

Lemma 6.1. *Let $c, M, \underline{h}_{\min}$ be positive constants. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the condition $\underline{h}_{\min} \le \underline{h}_1, \underline{h}_2$, if $H_1, H_2, \boldsymbol{u}_1, \boldsymbol{u}_2$, and the function a satisfy*

$$
\begin{cases} \displaystyle\sum_{\ell=1,2} \big(\|H_\ell\|_{L^\infty} + \sqrt{\underline{\rho}_\ell \underline{h}_\ell}\|\boldsymbol{u}_\ell\|_{L^\infty}\big) + \|a\|_{L^\infty} \le M, \\ a(\boldsymbol{x}) - \dfrac{\underline{\rho}_1\underline{\rho}_2}{\underline{\rho}_1\underline{h}_2 H_2(\boldsymbol{x})\alpha_2 + \underline{\rho}_2\underline{h}_1 H_1(\boldsymbol{x})\alpha_1}|\boldsymbol{u}_2(\boldsymbol{x}) - \boldsymbol{u}_1(\boldsymbol{x})|^2 \ge c, \\ H_1(\boldsymbol{x}) \ge c, \quad H_2(\boldsymbol{x}) \ge c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases} \tag{6.12}
$$

then for any $\dot{\boldsymbol{U}} = (\dot\zeta, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)^{\mathrm{T}} \in L^2 \times (\mathring{H}^1 \times (H^1)^N) \times (\mathring{H}^1 \times (H^1)^{N^*})$ we have

$$
C^{-1}\mathscr{E}(\dot{\boldsymbol{U}}) \le (\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \le C\mathscr{E}(\dot{\boldsymbol{U}}).
$$

Proof. This lemma can be shown along with the proof of [14, Lemma 7.4]. For the sake of completeness, we sketch the proof. We first note that

$$
\begin{aligned}
(\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} &= (a\dot\zeta, \dot\zeta)_{L^2} + \sum_{\ell=1,2}\{\underline{\rho}_\ell\underline{h}_\ell(L_\ell^{\mathrm{pr}}\dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2} + 2\underline{\rho}_\ell(\theta_\ell\boldsymbol{l}_\ell\cdot(\boldsymbol{v}\cdot\nabla)\dot{\boldsymbol{\phi}}_\ell, \dot\zeta)_{L^2}\} \\
&= (a\dot\zeta, \dot\zeta)_{L^2} + \sum_{\ell=1,2}\bigg\{\underline{\rho}_\ell\underline{h}_\ell\bigg(\sum_{l=1}^n (A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell, \partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2} + (\underline{h}_\ell\delta)^{-2}(C_\ell\dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2}\bigg) \\
&\qquad + 2\underline{\rho}_\ell(\theta_\ell\boldsymbol{v}\cdot(\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell, \dot\zeta)_{L^2}\bigg\},
\end{aligned}
$$

where we used the identity $\boldsymbol{a}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{\varphi} = \boldsymbol{v}\cdot(\boldsymbol{a}\otimes\nabla)^{\mathrm{T}}\boldsymbol{\varphi}$. On the other hand, we can put

$$
\begin{pmatrix} q_\ell(H_\ell) & \boldsymbol{q}_\ell(H_\ell)^{\mathrm{T}} \\ -\boldsymbol{q}_\ell(H_\ell) & Q_\ell(H_\ell) \end{pmatrix} := \begin{pmatrix} 0 & \boldsymbol{l}_\ell(H_\ell)^{\mathrm{T}} \\ -\boldsymbol{l}_\ell(H_\ell) & A_\ell(H_\ell) \end{pmatrix}^{-1}
$$

for $\ell = 1, 2$. Then, we see that $q_\ell(H_\ell) = H_\ell\alpha_\ell$ and that $Q_\ell(H_\ell)$ is nonnegative. Moreover, the identity

$$
A_\ell(H_\ell)\boldsymbol{\varphi}_\ell\cdot\boldsymbol{\varphi}_\ell = q_\ell(H_\ell)(\boldsymbol{l}_\ell(H_\ell)\cdot\boldsymbol{\varphi}_\ell)^2 + Q_\ell(H_\ell)A_\ell(H_\ell)\boldsymbol{\varphi}_\ell\cdot A_\ell(H_\ell)\boldsymbol{\varphi}_\ell \tag{6.13}
$$

holds for any $\boldsymbol{\varphi}_\ell$. Therefore,

$$
\begin{aligned}
\sum_{l=1}^n (A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell, \partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2} &= \sum_{l=1}^n\{(q_\ell\boldsymbol{l}_\ell\cdot\partial_l\dot{\boldsymbol{\phi}}_\ell, \boldsymbol{l}_\ell\cdot\partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2} + (Q_\ell A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell, A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2}\} \\
&= (H_\ell\alpha_\ell(\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell, (\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell)_{L^2} + \sum_{l=1}^n (Q_\ell A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell, A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2},
\end{aligned}
$$

so that

$$
\begin{aligned}
(\mathscr{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} &= (a\dot\zeta, \dot\zeta)_{L^2} + \sum_{\ell=1,2}\{\underline{\rho}_\ell\underline{h}_\ell(H_\ell\alpha_\ell(\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell, (\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell)_{L^2} \\
&\qquad + 2\underline{\rho}_\ell(\theta_\ell\boldsymbol{v}\cdot(\boldsymbol{l}_\ell\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_\ell, \dot\zeta)_{L^2}\} \\
&\quad + \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\bigg\{\sum_{l=1}^n (Q_\ell A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell, A_\ell\partial_l\dot{\boldsymbol{\phi}}_\ell)_{L^2} + (\underline{h}_\ell\delta)^{-2}(C_\ell\dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2}\bigg\} \\
&=: I_1 + I_2.
\end{aligned}
$$

We proceed to evaluate I_1.

$$
\begin{aligned}
I_1 &\geq \int_{\mathbf{R}^n} \Big\{ a\dot{\zeta}^2 + \sum_{\ell=1,2} \big(\underline{\rho}_\ell \underline{h}_\ell H_\ell \alpha_\ell |(\boldsymbol{l}_\ell \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_\ell|^2 - 2\underline{\rho}_\ell \theta_\ell |\boldsymbol{v}| |(\boldsymbol{l}_\ell \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_\ell| |\dot{\zeta}| \big) \Big\} \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbf{R}^n} \mathfrak{A}_0 \begin{pmatrix} \dot{\zeta} \\ \sqrt{\underline{\rho}_1 \underline{h}_1} |(\boldsymbol{l}_1 \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_1| \\ \sqrt{\underline{\rho}_2 \underline{h}_2} |(\boldsymbol{l}_2 \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_2| \end{pmatrix} \cdot \begin{pmatrix} \dot{\zeta} \\ \sqrt{\underline{\rho}_1 \underline{h}_1} |(\boldsymbol{l}_1 \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_1| \\ \sqrt{\underline{\rho}_2 \underline{h}_2} |(\boldsymbol{l}_2 \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_2| \end{pmatrix} \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

where the matrix $\mathfrak{A}_0$ is given by

$$
\mathfrak{A}_0 = \begin{pmatrix} a & -\sqrt{\underline{\rho}_1/\underline{h}_1}\theta_1|\boldsymbol{v}| & -\sqrt{\underline{\rho}_2/\underline{h}_2}\theta_2|\boldsymbol{v}| \\ -\sqrt{\underline{\rho}_1/\underline{h}_1}\theta_1|\boldsymbol{v}| & H_1\alpha_1 & 0 \\ -\sqrt{\underline{\rho}_2/\underline{h}_2}\theta_2|\boldsymbol{v}| & 0 & H_2\alpha_2 \end{pmatrix}.
$$

Here, we see that

$$
\det \mathfrak{A}_0 = H_1 H_2 \alpha_1 \alpha_2 \bigg(a - \frac{\underline{\rho}_1 \underline{\rho}_2}{\underline{\rho}_1 \underline{h}_2 H_2 \alpha_2 + \underline{\rho}_2 \underline{h}_1 H_1 \alpha_1} |\boldsymbol{v}|^2 \bigg) \geq c^3 \alpha_1 \alpha_2 > 0,
$$

so that $\mathfrak{A}_0$ is positive definite by Sylvester's criterion. Moreover, $\operatorname{tr}\mathfrak{A}_0 \leq \max\{1, \alpha_1, \alpha_2\}\, M \lesssim 1$ and the minimal eigenvalue of the matrix $\mathfrak{A}_0$ is bounded from below by $4\det\mathfrak{A}_0/(\operatorname{tr}\mathfrak{A}_0)^2 \gtrsim 1$. Therefore, we obtain

$$
I_1 \gtrsim \int_{\mathbf{R}^n} \bigg(\dot{\zeta}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell H_\ell \alpha_\ell |(\boldsymbol{l}_\ell \otimes \nabla)^{\mathrm{T}} \dot{\boldsymbol{\phi}}_\ell|^2 \bigg) \mathrm{d}\boldsymbol{x}.
$$

As for I_2, it is easy to see that $(C_\ell \dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2} \simeq \|\dot{\boldsymbol{\phi}}_\ell'\|_{L^2}^2$ for $\ell = 1, 2$. Summarizing the above estimates and using the decomposition (6.13) again, we obtain $(\mathscr{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \gtrsim \mathscr{E}(\dot{\boldsymbol{U}})$.

In order to obtain the estimate of $(\mathscr{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2}$ from above, it is sufficient to show that each element of the matrix $\mathfrak{A}_0$ is uniformly bounded. Since $\theta_1 + \theta_2 = 1$, we have

$$
\begin{cases} \sqrt{\underline{\rho}_1/\underline{h}_1}\theta_1|\boldsymbol{v}| \leq \underline{h}_1^{-1}\sqrt{\underline{\rho}_1\underline{h}_1}|\boldsymbol{u}_1| + \sqrt{\underline{\rho}_1/\underline{h}_1}\theta_1|\boldsymbol{u}_2|, \\ \sqrt{\underline{\rho}_2/\underline{h}_2}\theta_2|\boldsymbol{v}| \leq \sqrt{\underline{\rho}_2/\underline{h}_2}\theta_2|\boldsymbol{u}_1| + \underline{h}_2^{-1}\sqrt{\underline{\rho}_2\underline{h}_2}|\boldsymbol{u}_2|. \end{cases}
$$

Here, we see that

$$
\begin{aligned}
\sqrt{\underline{\rho}_1/\underline{h}_1}\theta_1|\boldsymbol{u}_2| &= \frac{1}{\underline{h}_2}\sqrt{\frac{H_1\alpha_1}{H_2\alpha_2}} \frac{\sqrt{(\underline{\rho}_1\underline{h}_2H_2\alpha_2)(\underline{\rho}_2\underline{h}_1H_1\alpha_1)}}{\underline{\rho}_1\underline{h}_2H_2\alpha_2 + \underline{\rho}_2\underline{h}_1H_1\alpha_1} \sqrt{\underline{\rho}_2\underline{h}_2}|\boldsymbol{u}_2| \\
&\leq \frac{1}{2\underline{h}_2}\sqrt{\frac{H_1\alpha_1}{H_2\alpha_2}}\sqrt{\underline{\rho}_2\underline{h}_2}|\boldsymbol{u}_2| \\
&\leq \frac{1}{2\underline{h}_{\min}\sqrt{\frac{M\alpha_1}{c\alpha_2}}M} \lesssim 1.
\end{aligned}
$$

Similarly, we have $\sqrt{\underline{\rho}_2/\underline{h}_2}\theta_2|\boldsymbol{u}_1| \lesssim 1$. Therefore, we obtain $(\mathscr{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \lesssim \mathscr{E}(\dot{\boldsymbol{U}})$. $\square$

In the following Lemma we provide uniform energy estimates for regular solutions to the linearized Kakinuma model (6.7).

Proposition 6.2. *Let $c, M, M_1, \underline{h}_{\rm min}$ be positive constants. There exist positive constants $C = C(c, M, \underline{h}_{\rm min})$ and $C_1 = C_1(c, M, M_1, \underline{h}_{\rm min})$ such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (2.14) *and the condition $\underline{h}_{\rm min} \leq \underline{h}_1, \underline{h}_2$, if $H_1, H_2, \boldsymbol{u}_1, \boldsymbol{u}_2$, and the function a satisfy* (6.12) *and*

$$\sum_{\ell=1,2} \bigl(\|\partial_t H_\ell\|_{L^\infty} + \|\nabla H_\ell\|_{L^\infty} + \underline{\rho}_\ell \underline{h}_\ell (\|\partial_t \boldsymbol{u}_\ell\|_{L^\infty}^2 + \|\nabla \boldsymbol{u}_\ell\|_{L^\infty}^2) \bigr) + \|\partial_t a\|_{L^\infty} + \|\nabla a\|_{L^\infty} \leq M_1,$$

then for any regular solution $\dot{\boldsymbol{U}} = (\dot{\zeta}, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)^{\rm T}$ to the linearized Kakinuma model (6.7) *we have*

$$\begin{aligned}\mathscr{E}(\dot{\boldsymbol{U}}(t)) \leq C{\rm e}^{C_1 t}\mathscr{E}(\dot{\boldsymbol{U}}(0)) + C_1 \int_0^t {\rm e}^{C_1(t-\tau)} \biggl\{ &\|\dot{f}_0(\tau)\|_{H^1}(\|\partial_t \dot{\zeta}(\tau)\|_{H^{-1}} + \|\dot{\zeta}(\tau)\|_{L^2}) \\ &+ \sum_{\ell=1,2} \underline{\rho}_\ell (\|\dot{\boldsymbol{f}}_\ell(\tau)\|_{L^2} + \|\dot{\zeta}(\tau)\|_{L^2})\|(\partial_t \dot{\boldsymbol{\phi}}_\ell(\tau), \nabla \dot{\boldsymbol{\phi}}_\ell(\tau))\|_{L^2} \biggr\} {\rm d}\tau.\end{aligned}$$

Proof. We deduce from (6.10) that

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}(\mathscr{A}_0^{\rm mod}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \\ &= ([\partial_t, \mathscr{A}_0^{\rm mod}]\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} + 2(\mathscr{A}_0^{\rm mod}\partial_t \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \\ &= ([\partial_t, \mathscr{A}_0^{\rm mod}]\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} + 2((\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}, \mathscr{A}_0^{\rm mod}\dot{\boldsymbol{U}})_{L^2} - 2((\boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}, \mathscr{A}_0^{\rm mod}\dot{\boldsymbol{U}})_{L^2} \\ &= ([\partial_t, \mathscr{A}_0^{\rm mod}]\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} - 2((\boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}, \mathscr{A}_0^{\rm mod}\dot{\boldsymbol{U}})_{L^2} + 2((\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}, \dot{\boldsymbol{F}})_{L^2} \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where we used the fact that $\mathscr{A}_0^{\rm mod}$ is a symmetric operator in L^2 and that $\mathscr{A}_1$ is a skew-symmetric matrix. As for I_1, we have

$$\begin{aligned}I_1 = ((\partial_t a)\dot{\zeta}, \dot{\zeta})_{L^2} + \sum_{\ell=1,2} \biggl\{ &\underline{\rho}_\ell \underline{h}_\ell \biggl(\sum_{l=1}^n ((\partial_t A_\ell)\partial_l \dot{\boldsymbol{\phi}}_\ell, \partial_l \dot{\boldsymbol{\phi}}_\ell)_{L^2} + (\underline{h}_\ell \delta)^{-2}((\partial_t C_\ell)\dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2} \biggr) \\ &+ 2\underline{\rho}_\ell ([\partial_t, \theta_\ell \boldsymbol{l}_\ell^{\rm T}(\boldsymbol{v}\cdot\nabla)]\dot{\boldsymbol{\phi}}_\ell, \dot{\zeta})_{L^2} \biggr\}.\end{aligned}$$

Here, as in the proof of Lemma 6.1 we have $\sqrt{\underline{\rho}_\ell / \underline{h}_\ell}\,\theta_\ell(|\boldsymbol{v}| + |\partial_t \boldsymbol{v}|) \lesssim 1$ for $\ell = 1, 2$. In view of the relations $\partial_t \theta_1 = -\partial_t \theta_2 = \theta_1\theta_2(H_1^{-1}\partial_t H_1 - H_2^{-1}\partial_t H_2)$, we have $|\partial_t \theta_\ell| \lesssim \theta_1\theta_2$ for $\ell = 1, 2$. Therefore, we obtain $|I_1| \lesssim \mathscr{E}(\dot{\boldsymbol{U}})$. As for I_2, by integration by parts we have

$$\begin{aligned}I_2 &= ((\nabla\cdot(a\boldsymbol{u}))\dot{\zeta}, \dot{\zeta})_{L^2} \\ &\quad - \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \biggl\{ \sum_{l=1}^n \bigl\{ 2(A_\ell \partial_l \dot{\boldsymbol{\phi}}_\ell, ((\partial_l \boldsymbol{u})\cdot\nabla)\dot{\boldsymbol{\phi}}_\ell)_{L^2} + (((\boldsymbol{u}\cdot\nabla)^* A_\ell)\partial_l \dot{\boldsymbol{\phi}}_\ell, \partial_\ell \dot{\boldsymbol{\phi}}_\ell)_{L^2} \bigr\} \\ &\qquad\qquad + (\underline{h}_\ell \delta)^{-2}(((\boldsymbol{u}\cdot\nabla)^* C_\ell)\dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{\phi}}_\ell)_{L^2} \biggr\} \\ &\quad + 2\sum_{\ell=1,2} \underline{\rho}_\ell \bigl\{ ((\nabla\cdot\boldsymbol{u})\dot{\zeta}, \theta_\ell \boldsymbol{l}_\ell^{\rm T}(\boldsymbol{v}\cdot\nabla)\dot{\boldsymbol{\phi}}_\ell)_{L^2} + (\dot{\zeta}, [\boldsymbol{u}\cdot\nabla, \theta_\ell \boldsymbol{l}_\ell^{\rm T}(\boldsymbol{v}\cdot\nabla)]\dot{\boldsymbol{\phi}}_\ell)_{L^2}.\end{aligned}$$

By using (2.14), we see that

$$\theta_1 \simeq \frac{\underline{\rho}_2 \underline{h}_1}{\underline{\rho}_1 \underline{h}_2 + \underline{\rho}_2 \underline{h}_1} = \frac{\underline{\rho}_2}{\underline{h}_2}, \qquad \theta_2 \simeq \frac{\underline{\rho}_1 \underline{h}_2}{\underline{\rho}_1 \underline{h}_2 + \underline{\rho}_2 \underline{h}_1} = \frac{\underline{\rho}_1}{\underline{h}_1}.$$

Therefore, we have $|\boldsymbol{u}| \leq \theta_2 |\boldsymbol{u}_1| + \theta_1 |\boldsymbol{u}_2| \lesssim 1$. In view of $|\nabla \theta_\ell| \lesssim \theta_1 \theta_2$ for $\ell = 1, 2$, we have also $|\nabla \boldsymbol{u}| \lesssim 1$ and $\sqrt{\rho_\ell / \underline{h}_\ell} \theta_l |\nabla \boldsymbol{v}| \lesssim 1$ for $\ell = 1, 2$. Hence, we obtain $|I_2| \lesssim \mathscr{E}(\dot{\boldsymbol{U}})$. Finally, as for I_3, we have

$$\begin{aligned} I_3 &= 2(\partial_t \dot{\zeta}, \dot{f}_0)_{L^2} - 2(\dot{\zeta}, \nabla \cdot (\boldsymbol{u} \dot{f}_0))_{L^2} \\ &\quad + 2 \sum_{\ell=1,2} \underline{\rho}_\ell ((\partial_t + \boldsymbol{u} \cdot \nabla) \dot{\boldsymbol{\phi}}_\ell, \dot{\boldsymbol{f}}_\ell - (\nabla \cdot (\theta_\ell \boldsymbol{l}_\ell \otimes \boldsymbol{v})) \dot{\zeta})_{L^2} \\ &\lesssim \| \dot{f}_0 \|_{H^1} (\| \partial_t \dot{\zeta} \|_{H^{-1}} + \| \dot{\zeta} \|_{L^2}) + \sum_{\ell=1,2} \underline{\rho}_\ell (\| \dot{\boldsymbol{f}}_\ell \|_{L^2} + \| \dot{\zeta} \|_{L^2}) \| (\partial_t \dot{\boldsymbol{\phi}}_\ell, \nabla \dot{\boldsymbol{\phi}}_\ell) \|_{L^2}. \end{aligned}$$

Summarizing the above estimates we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} (\mathscr{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} &\lesssim \mathscr{E}(\dot{\boldsymbol{U}}) + \| \dot{f}_0 \|_{H^1} (\| \partial_t \dot{\zeta} \|_{H^{-1}} + \| \dot{\zeta} \|_{L^2}) \\ &\qquad + \sum_{\ell=1,2} \underline{\rho}_\ell (\| \dot{\boldsymbol{f}}_\ell \|_{L^2} + \| \dot{\zeta} \|_{L^2}) \| (\partial_t \dot{\boldsymbol{\phi}}_\ell, \nabla \dot{\boldsymbol{\phi}}_\ell) \|_{L^2}. \end{aligned}$$

This together with Lemma 6.1 and Gronwall's inequality gives the desired estimate. □

6.2 Energy estimates

In this subsection, we will complete the proof of Theorem 3.1. The existence and the uniqueness of the solution to the initial value problem for the Kakinuma model (6.1) has already been established in the companion paper [14], so that it is sufficient to derive the uniform bound (3.14) of the solution for some time interval $[0, T]$ independent of parameters. The following lemma can be shown in the same way as the proof of [20, Lemma 4.2].

Lemma 6.3. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C such that for any positive parameters $\underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1 \delta, \underline{h}_2 \delta \leq 1$, if $\zeta \in H^{m-1}$, $b \in W^{m,\infty}$, $H_1 = 1 - \underline{h}_1^{-1} \zeta$, and $H_2 = 1 + \underline{h}_2^{-1} \zeta - \underline{h}_2^{-1} b$ satisfy*

$$\begin{cases} \underline{h}_1^{-1} \| \zeta \|_{H^{m-1}} + \underline{h}_2^{-1} \| \zeta \|_{H^{m-1}} + \underline{h}_2^{-1} \| b \|_{W^{m,\infty}} \leq M, \\ H_1(\boldsymbol{x}) \geq c, \quad H_2(\boldsymbol{x}) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

and if $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ satisfy

$$\begin{cases} \mathcal{L}_{1,i}(H_1, \delta, \underline{h}_1) \boldsymbol{\varphi}_1 = f_{1,i} \quad \text{for} \quad i = 1, 2, \ldots, N, \\ \mathcal{L}_{2,i}(H_2, b, \delta, \underline{h}_2) \boldsymbol{\varphi}_2 = f_{2,i} \quad \text{for} \quad i = 1, 2, \ldots, N^*, \end{cases}$$

then for any $k = 0, \pm 1, \ldots, \pm (m-1)$ we have

$$(\underline{h}_\ell \delta)^{-2} \| \boldsymbol{\varphi}_\ell' \|_{H^k} \leq C (\| \nabla \boldsymbol{\varphi}_\ell \|_{H^{k+1}} + \| \boldsymbol{\varphi}_\ell' \|_{H^{k+1}} + \| \boldsymbol{f}_\ell' \|_{H^k}) \qquad (\ell = 1, 2).$$

The next lemma gives an energy estimate of the solution to the Kakinuma model (6.1) under appropriate assumptions on the solution. We recall that the mathematical energy function $E_m(t)$ is defined by (5.3).

Lemma 6.4. *Let $c, M, M_1, \underline{h}_{\min}$ be positive constants. There exist two positive constants $C = C(c, M, \underline{h}_{\min})$ and $C_1 = C_1(c, M, M_1, \underline{h}_{\min})$ such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions (2.14), $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, if a regular solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model (6.1) with a bottom topography b satisfies (6.12), $\underline{h}_2^{-1}\big(\|b\|_{W^{m+1,\infty}} + (\underline{h}_2\delta)\|b\|_{W^{m+2,\infty}}\big) \leq M_1$, and $E_m(t) \leq M_1$ for some time interval $[0,T]$, then we have $E_m(t) \leq C\mathrm{e}^{C_1 t}E_m(0)$ for $0 \leq t \leq T$.*

Proof. Let β be a multi-index such that $1 \leq |\beta| \leq m$. Applying ∂^β to the Kakinuma model (6.1), after a tedious but straightforward calculation, we obtain

$$
\begin{cases}
\boldsymbol{l}_1(H_1)(\partial_t + \boldsymbol{u}_1\cdot\nabla)\partial^\beta\zeta + \underline{h}_1 L_1^{\mathrm{pr}}(H_1,\delta,\underline{h}_1)\partial^\beta\boldsymbol{\phi}_1 = \boldsymbol{f}_{1,\beta}, \\
\boldsymbol{l}_2(H_2)(\partial_t + \boldsymbol{u}_2\cdot\nabla)\partial^\beta\zeta - \underline{h}_2 L_2^{\mathrm{pr}}(H_2,\delta,\underline{h}_2)\partial^\beta\boldsymbol{\phi}_2 = \boldsymbol{f}_{2,\beta}, \\
\underline{\rho}_1\boldsymbol{l}_1(H_1)\cdot(\partial_t + \boldsymbol{u}_1\cdot\nabla)\partial^\beta\boldsymbol{\phi}_1 - \underline{\rho}_2\boldsymbol{l}_2(H_2)\cdot(\partial_t + \boldsymbol{u}_2\cdot\nabla)\partial^\beta\boldsymbol{\phi}_2 - a\partial^\beta\zeta = f_{0,\beta},
\end{cases}
\tag{6.14}
$$

where L_1^{pr} and L_2^{pr} are defined by (6.6), the function a by (3.9), and

$$
\begin{aligned}
\boldsymbol{f}_{1,\beta} &:= -[\partial^\beta, \boldsymbol{l}_1(H_1)]\partial_t\zeta + \underline{h}_1\big\{[\partial^\beta, A_1(H_1)]\Delta\boldsymbol{\phi}_1 - (\boldsymbol{l}_1(H_1)\otimes\boldsymbol{l}_1(H_1))(\nabla H_1\cdot\nabla)\partial^\beta\boldsymbol{\phi}_1 \\
&\quad + [\partial^\beta, \boldsymbol{l}_1(H_1)\otimes\boldsymbol{u}_1]\nabla H_1 - (\underline{h}_1\delta)^{-2}[\partial^\beta, C_1(H_1)]\boldsymbol{\phi}_1\big\},
\end{aligned}
\tag{6.15}
$$

$$
\begin{aligned}
\boldsymbol{f}_{2,\beta} &:= -[\partial^\beta, \boldsymbol{l}_2(H_2)]\partial_t\zeta - \underline{h}_2\big\{[\partial^\beta, A_2(H_2)]\Delta\boldsymbol{\phi}_2 - (\boldsymbol{l}_2(H_2)\otimes\boldsymbol{l}_2(H_2))(\nabla H_2\cdot\nabla)\partial^\beta\boldsymbol{\phi}_2 \\
&\quad + [\partial^\beta, \boldsymbol{l}_2(H_2)\otimes\boldsymbol{u}_2]\nabla H_2 - (\underline{h}_2\delta)^{-2}[\partial^\beta, C_2(H_2)]\boldsymbol{\phi}_2 \\
&\quad - \boldsymbol{l}_2(H_2)(\boldsymbol{u}_2\cdot\partial^\beta(\underline{h}_2^{-1}\nabla b)) - \partial^\beta\big(\tilde{B}_2(H_2)(\underline{h}_2^{-1}\nabla b\cdot\nabla)\boldsymbol{\phi}_2 + \tilde{C}_2(H_2,\underline{h}_2^{-1}b)\boldsymbol{\phi}_2\big)\big\},
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
f_{0,\beta} &:= -\underline{\rho}_1\big\{\big([\partial^\beta, \boldsymbol{l}_1(H_1)] - \boldsymbol{l}_1'(H_1)(\partial^\beta H_1)\big)^{\mathrm{T}}\partial_t\boldsymbol{\phi}_1 \\
&\quad + \tfrac{1}{2}[\partial^\beta; \boldsymbol{u}_1, \boldsymbol{u}_1] + \tfrac{1}{2}(\underline{h}_1\delta)^{-2}[\partial^\beta; w_1, w_1] \\
&\quad + \boldsymbol{u}_1\cdot\big(\big([\partial^\beta, \boldsymbol{l}_1(H_1)] - \boldsymbol{l}_1'(H_1)(\partial^\beta H_1)\big)\otimes\nabla\big)^{\mathrm{T}}\boldsymbol{\phi}_1 \\
&\quad - (\underline{h}_1\delta)^{-2}w_1\big(\big([\partial^\beta, \boldsymbol{l}_1'(H_1)] - \boldsymbol{l}_1''(H_1)(\partial^\beta H_1)\big)^{\mathrm{T}}\boldsymbol{\phi}_1 + \boldsymbol{l}_1'(H_1)\cdot\partial^\beta\boldsymbol{\phi}_1\big)\big\} \\
&\quad + \underline{\rho}_2\big\{\big([\partial^\beta, \boldsymbol{l}_2(H_2)] - \boldsymbol{l}_2'(H_2)(\partial^\beta H_2) - \boldsymbol{l}_2'(H_2)(\partial^\beta(\underline{h}_2^{-1}b))\big)^{\mathrm{T}}\partial_t\boldsymbol{\phi}_2 \\
&\quad + \tfrac{1}{2}[\partial^\beta; \boldsymbol{u}_2, \boldsymbol{u}_2] + \tfrac{1}{2}(\underline{h}_2\delta)^{-2}[\partial^\beta; w_2, w_2] \\
&\quad + \boldsymbol{u}_2\cdot\big(\big([\partial^\beta, \boldsymbol{l}_2(H_2)] - \boldsymbol{l}_2'(H_2)(\partial^\beta H_2) - \boldsymbol{l}_2'(H_2)(\partial^\beta(\underline{h}_2^{-1}b))\big)\otimes\nabla\big)^{\mathrm{T}}\boldsymbol{\phi}_2 \\
&\quad - \boldsymbol{u}_2\cdot[\partial^\beta, \underline{h}_2^{-1}\nabla b\otimes\boldsymbol{\phi}_2]\boldsymbol{l}_2'(H_2) \\
&\quad - (\boldsymbol{u}_2\cdot\underline{h}_2^{-1}\nabla b)\boldsymbol{\phi}_2\cdot\big(\partial^\beta\boldsymbol{l}_2'(H_2) - \boldsymbol{l}_2''(H_2)(\partial^\beta H_2) - \boldsymbol{l}_2''(H_2)(\partial^\beta(\underline{h}_2^{-1}b))\big) \\
&\quad + (\underline{h}_2\delta)^{-2}w_2\big(\big([\partial^\beta, \boldsymbol{l}_2'(H_2)] - \boldsymbol{l}_2''(H_2)(\partial^\beta H_2) - \boldsymbol{l}_2''(H_2)(\partial^\beta(\underline{h}_2^{-1}b))\big)^{\mathrm{T}}\boldsymbol{\phi}_2 \\
&\qquad + \boldsymbol{l}_2'(H_2)\cdot\partial^\beta\boldsymbol{\phi}_2\big)\big\}.
\end{aligned}
\tag{6.17}
$$

Here, $[\partial^\beta; u, v] = \partial^\beta(uv) - (\partial^\beta u)v - u(\partial^\beta v)$ is the symmetric commutator. For vector valued functions, it is defined by $[\partial^\beta; \boldsymbol{u}, \boldsymbol{v}] = \partial^\beta(\boldsymbol{u}\cdot\boldsymbol{v}) - (\partial^\beta\boldsymbol{u})\cdot\boldsymbol{v} - \boldsymbol{u}\cdot(\partial^\beta\boldsymbol{v})$.

On the other hand, by Lemma 5.2 we have the estimate (5.4) for time derivatives of the solution. Particularly, we have

$$
\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\big(\|\partial_t\boldsymbol{u}_\ell\|_{H^{m-1}}^2 + (\underline{h}_\ell\delta)^{-2}\|\partial_t w_\ell\|_{H^{m-1}}^2 + \|\partial_t\boldsymbol{\phi}_\ell'\|_{H^m}^2 + \|\partial_t^2\boldsymbol{\phi}_\ell'\|_{H^{m-1}}^2\big) \lesssim E_m.
\tag{6.18}
$$

Note that we have also the estimate (5.8) for the velocities $(\boldsymbol{u}_\ell, w_\ell)$ $(\ell = 1,2)$. Moreover, it follows from Lemma 6.3 that $\underline{\rho}_\ell\underline{h}_\ell(\underline{h}_\ell\delta)^{-4}\|\boldsymbol{\phi}_\ell'\|_{H^{m-1}}^2 \lesssim E_m$ for $\ell = 1,2$. In view of the definition (3.9) of the function a, it is not difficult to check the estimate $\|a-1\|_{H^m}^2 + \|\partial_t a\|_{H^{m-1}}^2 \lesssim E_m$. Therefore, by the Sobolev imbedding theorem we see that all the assumptions in Proposition 6.2 are satisfied, so that for the solution $\boldsymbol{U} = (\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)^{\mathrm{T}}$ we have

$$\mathscr{E}(\partial^\beta \boldsymbol{U}(t)) \le C\mathrm{e}^{C_1 t}\mathscr{E}(\partial^\beta \boldsymbol{U}(0)) + C_1 \int_0^t \mathrm{e}^{C_1(t-\tau)}\mathscr{F}_\beta(\tau)\mathrm{d}\tau,$$

where

$$\begin{aligned}\mathscr{F}_\beta &= \|f_{0,\beta}\|_{H^1}(\|\partial_t\partial^\beta\zeta\|_{H^{-1}} + \|\partial^\beta\zeta\|_{L^2}) \\ &\quad + \sum_{\ell=1,2} \underline{\rho}_\ell(\|\boldsymbol{f}_{\ell,\beta}\|_{L^2} + \|\partial^\beta\zeta\|_{L^2})\|(\partial_t\partial^\beta\boldsymbol{\phi}_\ell, \nabla\partial^\beta\boldsymbol{\phi}_\ell)\|_{L^2}.\end{aligned}$$

In view of the estimates (5.4), (5.8), and (6.18) together with

$$\|([\partial^\beta, \boldsymbol{l}_\ell(H_\ell)] - \boldsymbol{l}_\ell'(H_1)(\partial^\beta H_\ell))^{\mathrm{T}}\boldsymbol{\varphi}_\ell\|_{H^1} \lesssim \|\boldsymbol{\varphi}_\ell'\|_{H^m}$$

for $\ell = 1,2$, we obtain $\mathscr{F}_\beta \lesssim E_m$. We note that the multi-index β is assumed to satisfy $1 \le |\beta| \le m$. As for the case $\beta = 0$, in view of $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\boldsymbol{U}(t)) \lesssim E_m(t)$ we infer the inequality $\mathscr{E}(\boldsymbol{U}(t)) \le \mathscr{E}(\boldsymbol{U}(0)) + C_1\int_0^t E_m(\tau)\mathrm{d}\tau$. Summarizing the above estimates we obtain

$$E_m(t) \le C\mathrm{e}^{C_1 t}E_m(0) + C_1\int_0^t \mathrm{e}^{C_1(t-\tau)}E_m(\tau)\mathrm{d}\tau$$

with constants $C = C(c, M, \underline{h}_{\min})$ and $C_1 = C_1(c, M, M_1, \underline{h}_{\min})$. Therefore, Gronwall's inequality gives the desired estimate. □

Now, we are ready to prove Theorem 3.1. Suppose that the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b satisfy (3.10)–(3.13). Let C_0 be a positive constant such that

$$\sum_{\ell=1,2}(\|H_{\ell(0)}\|_{L^\infty} + \underline{\rho}_\ell\underline{h}_\ell\|\boldsymbol{u}_{\ell(0)}\|_{L^\infty}^2) + \|a_{(0)}\|_{L^\infty} \le C_0.$$

Such a constant C_0 exists as a constant depending on $c_0, M_0, \underline{h}_{\min}$, and m. We will show that the solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ satisfies (3.14), (3.15), and

$$\sum_{\ell=1,2}(\|H_\ell(t)\|_{L^\infty} + \underline{\rho}_\ell\underline{h}_\ell\|\boldsymbol{u}_\ell(t)\|_{L^\infty}^2) + \|a(t)\|_{L^\infty} \le 2C_0 \tag{6.19}$$

for $0 \le t \le T$ with a constant M and a time T which will be determined below. We note that (3.14) is equivalent to $E_m(t) \le M$. To this end, we assume that the solution satisfies (3.14), (3.15), and (6.19) for $0 \le t \le T$. In the following, the constant depending on $c_0, C_0, \underline{h}_{\min}, m$ but not on M is denoted by C and the constant depending also on M by C_1. These constants may change from line to line. Then, it follows from Lemma 6.4 that $E_m(t) \le C\mathrm{e}^{C_1 t}M_0$ for $0 \le t \le T$. Therefore, if we chose $M = 2CM_0$ and if T is so small that $T \le C_1^{-1}\log 2$, then (3.14) holds in fact for $0 \le t \le T$. It remains to show (3.15) and (6.19). As before, we can check

$$\begin{cases} \sum_{\ell=1,2}\bigl(\|\partial_t H_\ell(t)\|_{L^\infty} + \sqrt{\underline{\rho}_\ell\underline{h}_\ell}\|\partial_t\boldsymbol{u}_\ell(t)\|_{L^\infty}\bigr) + \|\partial_t a(t)\|_{L^\infty} \le C_1, \\ \|\partial_t\biggl(a(t) - \dfrac{\underline{\rho}_1\underline{\rho}_2}{\underline{\rho}_1 H_2(t)\alpha_2 + \underline{\rho}_2 H_1(t)\alpha_1}|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2\biggr)\|_{L^\infty} \le C_1. \end{cases}$$

Therefore, if T is so small that $T \le (2C_1)^{-1}c_0$ and $T \le ((2C_0^{1/2}+1)C_1)^{-1}C_0$, then the lower bound (3.15) and the upper bound (6.19) hold in fact for $0 \le t \le T$. This completes the proof of Theorem 3.1.

7 Approximation of solutions; proof of Theorem 3.9

In this section we prove Theorem 3.9, which gives a rigorous justification of the Kakinuma model (2.18) as a higher order shallow water approximation to the full model for interfacial gravity waves (2.17) under the hypothesis of the existence of the solution to the full model with uniform bounds.

7.1 Supplementary estimate for the Dirichlet-to-Neumann map

In this subsection, we give a supplementary estimate to Lemma 4.2 for the Dirichlet-to-Neumann map $\Lambda(\zeta, b, \delta)$ defined by (4.6) appearing in the framework of surface waves. We recall the map $\Lambda^{(N)}(\zeta, b, \delta) \colon \phi \mapsto \mathcal{L}_0(H, b, \delta)\boldsymbol{\phi}$, where $\mathcal{L}_0(H, b, \delta)$ is defined by (4.3) and $\boldsymbol{\phi}$ is the unique solution to (4.5). In this section we omit the dependence of t in notations.

Lemma 7.1. *Let c, M be positive constants and m, j integers such that $m > \frac{n}{2}+1$, $m \geq 2(j+1)$, and $1 \leq j \leq 2N+1$. We assume* (H1) *or* (H2)*. There exists a positive constant C such that if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, and $H = 1 + \zeta - b$ satisfy* (4.8)*, then for any $\phi \in \mathring{H}^{k+2(j+1)}$ with $0 \leq k \leq m - 2(j+1)$ and any $\delta \in (0, 1]$ we have*

$$\|(-\Delta)^{-\frac{1}{2}}(\Lambda^{(N)}(\zeta, b, \delta)\phi - \Lambda(\zeta, b, \delta)\phi)\|_{H^k} \leq C\delta^{2j}\|\nabla\phi\|_{H^{k+2j+1}}.$$

Proof. This lemma can be proved in a similar way to the proof of Lemma 4.2 with a slight modification. For the completeness, we sketch the proof. By the duality $(H^k)^* = H^{-k}$ and the symmetry of the operator $(-\Delta)^{-\frac{1}{2}}$, it is sufficient to show the estimate

$$|((\Lambda - \Lambda^{(N)})\phi, \psi)_{L^2}| \lesssim \delta^{2j}\|\nabla\phi\|_{H^{k+2j+1}}\|\nabla\psi\|_{H^{-k}}$$

for any $\phi \in \mathring{H}^{k+2(j+1)}$ and any $\psi \in H^{1-k}$. We decompose it as

$$\begin{aligned}((\Lambda - \Lambda^{(N)})\phi, \psi)_{L^2} &= ((\Lambda - \Lambda^{(2N+2)})\phi, \psi)_{L^2} + ((\Lambda^{(2N+2)} - \Lambda^{(N)})\phi, \psi)_{L^2} \\ &=: I_1 + I_2\end{aligned}$$

and evaluate the two components of the right-hand side separately.

We recall the definitions (4.1) of the (N^*+1) vector-valued function $\boldsymbol{l}(H)$ and (4.3) of the operator $\mathcal{L}_i(H, b, \delta)$, which acts on (N^*+1) vector-valued functions. These depend on N, so that we denote them by $\boldsymbol{l}^{(N)}(H)$ and $\mathcal{L}_i^{(N)}(H, b, \delta)$, respectively, in the following argument. Let Φ be the solution to the boundary value problem (4.7) and let $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_{N^*})$, $\tilde{\boldsymbol{\phi}} = (\tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_{2N^*+2})$, and $\boldsymbol{\psi} = (\psi_0, \psi_1, \ldots, \psi_{2N^*+2})$ be the solutions to the problems

$$\begin{cases}\mathcal{L}_i^{(N)}(H, b, \delta)\boldsymbol{\phi} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N^*, \\ \boldsymbol{l}^{(N)}(H) \cdot \boldsymbol{\phi} = \phi,\end{cases}$$

$$\begin{cases}\mathcal{L}_i^{(2N+2)}(H, b, \delta)\tilde{\boldsymbol{\phi}} = 0 \quad \text{for} \quad i = 1, 2, \ldots, 2N^*+2, \\ \boldsymbol{l}^{(2N+2)}(H) \cdot \tilde{\boldsymbol{\phi}} = \phi,\end{cases}$$

and

$$\begin{cases}\mathcal{L}_i^{(2N+2)}(H, b, \delta)\boldsymbol{\psi} = 0 \quad \text{for} \quad i = 1, 2, \ldots, 2N^*+2, \\ \boldsymbol{l}^{(2N+2)}(H) \cdot \boldsymbol{\psi} = \psi,\end{cases}$$

respectively. Put

$$
\begin{cases} \tilde{\Phi}^{\mathrm{app}}(\boldsymbol{x}, z) := \displaystyle\sum_{i=0}^{2N^*+2} (z+1-b(x))^{p_i} \tilde{\phi}_i(\boldsymbol{x}), \\ \Psi(\boldsymbol{x}, z) := \displaystyle\sum_{i=0}^{2N^*+2} (z+1-b(x))^{p_i} \psi_i(\boldsymbol{x}), \end{cases} \tag{7.1}
$$

and $\Phi^{\mathrm{res}} := \Phi - \tilde{\Phi}^{\mathrm{app}}$. We note that $\tilde{\Phi}^{\mathrm{app}}$ is a higher order approximation of the velocity potential Φ and that it satisfies the boundary value problem (4.7) approximately in the sense that

$$
\begin{cases} \Delta \tilde{\Phi}^{\mathrm{app}} + \delta^{-2} \partial_z^2 \tilde{\Phi}^{\mathrm{app}} = R & \text{in} \quad -1+b(\boldsymbol{x}) < z < \zeta(\boldsymbol{x}), \\ \tilde{\Phi}^{\mathrm{app}} = \phi & \text{on} \quad z = \zeta(\boldsymbol{x}), \\ \nabla b \cdot \nabla \tilde{\Phi}^{\mathrm{app}} - \delta^{-2} \partial_z \tilde{\Phi}^{\mathrm{app}} = r_B & \text{on} \quad z = -1 + b(\boldsymbol{x}), \end{cases}
$$

where the residual R can be written in the form

$$
R(\boldsymbol{x}, z) = \sum_{i=0}^{2N^*+2} (z+1-b(\boldsymbol{x}))^{p_i} r_i(\boldsymbol{x}).
$$

Estimates for the residuals $(r_0, r_1, \ldots, r_{2N^*+2})$ and r_B were given in [20, Lemmas 6.4 and 6.9]. In fact, we have $\|(r_0, r_1, \ldots, r_{2N^*+2})\|_{H^k} + \|r_B\|_{H^k} \lesssim \delta^{2j} \|\nabla\phi\|_{H^{k+2j+1}}$ for $-m \leq k \leq m-2(j+1)$ and $0 \leq j \leq 2N+1$.

Now, with a slight modification from the strategy in [20], we use the identity

$$
I_1 = \int_\Omega I_\delta \nabla_X \Phi^{\mathrm{res}} \cdot I_\delta \nabla_X \Psi \mathrm{d}X,
$$

where we denote $\Omega := \{X = (\boldsymbol{x}, z)\,;\, -1 + b(\boldsymbol{x}) < z < \zeta(\boldsymbol{x})\}$, $I_\delta := \mathrm{diag}(1, \ldots, 1, \delta^{-1})$, and $\nabla_X := (\nabla, \partial_z) = (\partial_1, \ldots, \partial_n, \partial_z)$. Indeed, we have on one hand

$$
(\Lambda\phi, \psi)_{L^2} = \int_\Omega I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi \mathrm{d}X
$$

as a consequence of (4.7), $\Psi(\boldsymbol{x}, \zeta(\boldsymbol{x})) = \psi(\boldsymbol{x})$, and Green's identity, and on the other hand

$$
\begin{aligned}
(\Lambda^{(2N+2)}\phi, \psi)_{L^2} &= (\mathcal{L}_0^{(2N+2)} \tilde{\boldsymbol{\phi}}, \boldsymbol{l}^{(2N+2)} \cdot \boldsymbol{\psi})_{L^2} = \sum_{i=0}^{2N^*+2} (H^{p_i} \mathcal{L}_0^{(2N+2)} \tilde{\boldsymbol{\phi}}, \psi_i)_{L^2} \\
&= \sum_{i,j=0}^{2N^*+2} (L_{ij} \tilde{\phi}_j, \psi_i)_{L^2} = \int_\Omega I_\delta \nabla_X \tilde{\Phi}^{\mathrm{app}} \cdot I_\delta \nabla_X \Psi \mathrm{d}X,
\end{aligned}
$$

where the last identity follows from the expressions (4.2) and (7.1).

To evaluate I_1, it is convenient to transform the water region Ω into a simple flat domain $\Omega_0 = \mathbf{R}^n \times (-1, 0)$ by using a diffeomorphism which simply stretches the vertical direction $\Theta(\boldsymbol{x}, z) = (\boldsymbol{x}, \theta(\boldsymbol{x}, z)) \colon \Omega_0 \to \Omega$, where $\theta(\boldsymbol{x}, z) = \zeta(\boldsymbol{x})(z+1) + (1 - b(\boldsymbol{x}))z$. Put $\tilde{\Phi}^{\mathrm{res}} = \Phi^{\mathrm{res}} \circ \Theta$ and $\tilde{\Psi} = \Psi \circ \Theta$. Then, the above integral is transformed into

$$
I_1 = \int_{\Omega_0} \mathcal{P} I_\delta \nabla_X \tilde{\Phi}^{\mathrm{res}} \cdot I_\delta \nabla_X \tilde{\Psi} \mathrm{d}X,
$$

where

$$\mathcal{P} = \det\left(\frac{\partial\Theta}{\partial X}\right) I_\delta^{-1}\left(\frac{\partial\Theta}{\partial X}\right)^{-1} I_\delta^2\left(\left(\frac{\partial\Theta}{\partial X}\right)^{-1}\right)^{\mathrm{T}} I_\delta^{-1}.$$

Therefore, under the restriction $|k| \le m-1$ and using the hypothesis (4.8), we have

$$|I_1| \lesssim \|J^k I_\delta \nabla_X \tilde{\Phi}^{\mathrm{res}}\|_{L^2(\Omega_0)} \|J^{-k} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)},$$

where $J = (1-\Delta)^{\frac{1}{2}}$. Moreover, $\tilde{\Phi}^{\mathrm{res}}$ satisfies the boundary value problem

$$\begin{cases} \nabla_X \cdot I_\delta \mathcal{P} I_\delta \nabla_X \tilde{\Phi}^{\mathrm{res}} = -\tilde{R} & \text{in} \quad \Omega_0, \\ \tilde{\Phi}^{\mathrm{res}} = 0 & \text{on} \quad z = 0, \\ \boldsymbol{e}_z \cdot I_\delta \mathcal{P} I_\delta \nabla_X \tilde{\Phi}^{\mathrm{res}} = -r_B & \text{on} \quad z = -1, \end{cases}$$

where $\tilde{R} = R \circ \Theta = \sum_{i=0}^{2N^*+2} (z+1)^{p_i} H^{p_i} r_j$ and $\boldsymbol{e}_z = (0,\dots,0,1)^{\mathrm{T}}$. By applying the standard theory of elliptic partial differential equations to the above problem, for $0 \le k \le m-1$ we have

$$\begin{aligned} \|J^k I_\delta \nabla_X \tilde{\Phi}^{\mathrm{res}}\|_{L^2(\Omega_0)} &\lesssim \delta(\|J^k \tilde{R}\|_{L^2(\Omega_0)} + \|r_B\|_{H^k}) \\ &\lesssim \delta(\|(r_0, r_1, \dots, r_{2N^*+2})\|_{H^k} + \|r_B\|_{H^k}). \end{aligned}$$

Moreover, in view of $\tilde{\Psi} = \sum_{i=0}^{2N^*+2} (z+1)^{p_i} H^{p_i} \psi_j$ and by Lemma 4.1, we have

$$\begin{aligned} \|J^{-k} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)} &\lesssim \|\nabla \boldsymbol{\psi}\|_{H^{-k}} + \delta^{-1}\|\boldsymbol{\psi}'\|_{H^{-k}} \\ &\lesssim \|\nabla \psi\|_{H^{-k}} \end{aligned}$$

for $|k| \le m-1$. Summarizing the above estimates we have $|I_1| \lesssim \delta^{2j+1} \|\nabla\phi\|_{H^{k+2j+1}} \|\nabla\psi\|_{H^{-k}}$ for $0 \le k \le m - 2(j+1)$ and $0 \le j \le 2N+1$.

As for the term I_2, the evaluation is exactly the same as in [20]. In fact, the identities

$$\begin{aligned} I_2 &= \sum_{i,j=0}^{2N^*+2} (L_{ij}\tilde{\phi}_j, \psi_i)_{L^2} - \sum_{j=0}^{N^*} (L_{0j}\phi_j, \psi)_{L^2} \\ &= \sum_{j=0}^{N^*} \sum_{i=N^*+1}^{2N^*+2} ((L_{ij} - H^{p_i} L_{0j})\varphi_j, \psi_i)_{L^2} - \sum_{i,j=N^*+1}^{2N^*+2} ((L_{ij} - H^{p_i} L_{0j})\tilde{\phi}_j, \psi_i)_{L^2} \end{aligned}$$

were shown in [20, Equation (7.7)], where $\boldsymbol{\varphi} := (\varphi_0, \varphi_1, \dots, \varphi_{N^*})$ was defined by $\varphi_i := \phi_i - \tilde{\phi}_i$ for $i = 0, 1, \dots, N^*$. Now, we decompose $j = j_1 + j_2$ such that $1 \le j_1 \le N+1$ and $0 \le j_2 \le N$. Then, by [20, Lemmas 5.2, 5.4, 6.2 and 6.7] we see that

$$\begin{aligned} |I_2| &\lesssim \{\|\boldsymbol{\varphi}\|_{H^{k+2j_1+1}} + \|(\tilde{\phi}_{N^*+1}, \dots, \tilde{\phi}_{2N^*+2})\|_{H^{k+2j_1+1}} \\ &\qquad + \delta^{-2}(\|\boldsymbol{\varphi}\|_{H^{k+2j_1-1}} + \|(\tilde{\phi}_{N^*+1}, \dots, \tilde{\phi}_{2N^*+2})\|_{H^{k+2j_1-1}})\} \|(\psi_{N^*+1}, \dots, \psi_{2N^*+2})\|_{H^{-(k+2j_1-1)}} \\ &\lesssim \delta^{2(j_1+j_2)} \|\nabla\phi\|_{H^{k+2(j_1+j_2)}} \|\nabla\psi\|_{H^{-k}} \end{aligned}$$

if $\max\{|k|, |k+2j_1-2|, |k+2j_1+1|, |k+2(j_1+j_2)|\} \le m-1$ and $\max\{|k|, |k+1|, |k+2j_1-1|\} \le m$. These conditions are satisfied under the restriction $-m+1 \le k \le m-2(j+1)$.

To summarize, we obtain as desired $|((\Lambda - \Lambda^{(N)})\phi, \psi)_{L^2}| \lesssim \delta^{2j} \|\nabla\phi\|_{H^{k+2j+1}} \|\nabla\psi\|_{H^{-k}}$ for $0 \le k \le m-2(j+1)$ and $1 \le j \le 2N+1$. The proof is complete. $\square$

This lemma and the scaling relations (4.15) imply immediately the following lemma.

Lemma 7.2. *Let c, M be positive constants and m, j integers such that $m > \frac{n}{2}+1$, $m \geq 2(j+1)$, and $1 \leq j \leq 2N+1$. We assume* (H1) *or* (H2). *There exists a positive constant C such that for any positive parameters $\underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, $H_1 = 1-\underline{h}_1^{-1}\zeta$, and $H_2 = 1+\underline{h}_2^{-1}\zeta-\underline{h}_2^{-1}b$ satisfy* (4.14), *then for any $\phi_1, \phi_2 \in \mathring{H}^{k+2(j+1)}$ with $0 \leq k \leq m-2(j+1)$ we have*

$$\begin{cases} \|(-\Delta)^{-\frac{1}{2}}(\underline{h}_1\Lambda_1^{(N)}(\zeta,\delta,\underline{h}_1)\phi_1 - \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1)\|_{H^k} \leq C\underline{h}_1(\underline{h}_1\delta)^{2j}\|\nabla\phi_1\|_{H^{k+2j+1}}, \\ \|(-\Delta)^{-\frac{1}{2}}(\underline{h}_2\Lambda_2^{(N)}(\zeta,b,\delta,\underline{h}_2)\phi_2 - \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2)\|_{H^k} \leq C\underline{h}_2(\underline{h}_2\delta)^{2j}\|\nabla\phi_2\|_{H^{k+2j+1}}. \end{cases}$$

We recall also the estimate for the Dirichlet-to-Neumann map $\Lambda(\zeta, b, \delta)$ itself. The following lemma is now standard. For sharper estimates, we refer to T. Iguchi [18] and D. Lannes [28].

Lemma 7.3. *Let c, M be positive constants m an integer such that $m > \frac{n}{2}+2$. There exists a positive constant C such that if $\zeta \in H^m$, $b \in W^{m,\infty}$, and $H = 1+\zeta-b$ satisfy* (4.4), *then for any $\phi \in \mathring{H}^{k+1}$ with $|k| \leq m-1$ and any $\delta \in (0,1]$ we have $\|\Lambda(\zeta,b,\delta)\phi\|_{H^{k-1}} \leq C\|\nabla\phi\|_{H^k}$.*

This lemma and the scaling relations (4.15) imply immediately the following lemma.

Lemma 7.4. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2}+2$. There exists a positive constant C such that for any positive parameters $\underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if $\zeta \in H^m$, $b \in W^{m,\infty}$, $H_1 = 1-\underline{h}_1^{-1}\zeta$, and $H_2 = 1+\underline{h}_2^{-1}\zeta-\underline{h}_2^{-1}b$ satisfy* (4.11), *then for any $\phi_1, \phi_2 \in \mathring{H}^{k+1}$ with $|k| \leq m-1$ we have*

$$\begin{cases} \|\Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1\|_{H^{k-1}} \leq C\underline{h}_1\|\nabla\phi_1\|_{H^k}, \\ \|\Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2\|_{H^{k-1}} \leq C\underline{h}_2\|\nabla\phi_2\|_{H^k}. \end{cases}$$

7.2 Consistency of the Kakinuma model revisited

As we mentioned in Remark 3.8, the approximate solution to the Kakinuma model (2.18) made from the solution (ζ, ϕ_1, ϕ_2) to the full model can be constructed as a solution to (3.21), that is,

$$\begin{cases} \mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)\tilde{\boldsymbol{\phi}}_1 = 0 \quad \text{for} \quad i=1,2,\ldots,N, \\ \mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)\tilde{\boldsymbol{\phi}}_2 = 0 \quad \text{for} \quad i=1,2,\ldots,N^*, \\ \underline{h}_1\mathcal{L}_{1,0}(H_1,\delta,\underline{h}_1)\tilde{\boldsymbol{\phi}}_1 + \underline{h}_2\mathcal{L}_{2,0}(H_2,b,\delta,\underline{h}_2)\tilde{\boldsymbol{\phi}}_2 = 0, \\ \underline{\rho}_2\boldsymbol{l}_2(H_2)\cdot\tilde{\boldsymbol{\phi}}_2 - \underline{\rho}_1\boldsymbol{l}_1(H_1)\cdot\tilde{\boldsymbol{\phi}}_1 = \underline{\rho}_2\phi_2 - \underline{\rho}_2\phi_1, \end{cases} \tag{7.2}$$

in place of (3.19), that is,

$$\begin{cases} \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 = \phi_1, \quad \mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i=1,2,\ldots,N, \\ \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 = \phi_2, \quad \mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i=1,2,\ldots,N^*. \end{cases} \tag{7.3}$$

To show this fact, we need to guarantee that the difference between these two solutions is of order $O((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})$. The following lemma gives such an estimate.

Lemma 7.5. *Let c, M be positive constants and m an integer such that $m > \frac{n}{2}+1$ and $m \geq 4(N+1)$. We assume* (H1) *or* (H2). *There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, if $\zeta \in H^m$, $b \in W^{m+1,\infty}$, $H_1 = 1-\underline{h}_1^{-1}\zeta$, and $H_2 = 1+\underline{h}_2^{-1}\zeta-\underline{h}_2^{-1}b$ satisfy* (4.14), *then for any $\phi_1, \phi_2 \in \mathring{H}^{k+4(N+1)}$ with*

$0 \le k \le m-4(N+1)$ *satisfying the compatibility condition* $\Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 + \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 = 0$ *the solution* $(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ *to* (7.3) *and the solution* $(\tilde{\boldsymbol{\phi}}_1,\tilde{\boldsymbol{\phi}}_2)$ *to* (7.2) *satisfy*

$$\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla(\tilde{\boldsymbol{\phi}}_\ell-\boldsymbol{\phi}_\ell)\|_{H^k}^2+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell'-\boldsymbol{\phi}_\ell'\|_{H^k}^2+(\underline{h}_\ell\delta)^{-4}\|\tilde{\boldsymbol{\phi}}_\ell'-\boldsymbol{\phi}_\ell'\|_{H^{k-1}}^2)$$
$$\le C\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\underline{h}_\ell\delta)^{2(4N+2)}\|\nabla\phi_\ell\|_{H^{k+4N+3}}^2.$$

Proof. For simplicity, we write $\mathcal{L}_{1,i}=\mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)$, $\boldsymbol{l}_1=\boldsymbol{l}_1(H_1)$, and so on. We recall that $\Lambda_1^{(N)}\colon \phi_1\mapsto\mathcal{L}_{1,0}\boldsymbol{\phi}_1$ and $\Lambda_2^{(N)}\colon \phi_2\mapsto\mathcal{L}_{2,0}\boldsymbol{\phi}_2$. Notice that $\tilde{\boldsymbol{\phi}}_\ell-\boldsymbol{\phi}_\ell$ for $\ell=1,2$ satisfy

$$\begin{cases}\mathcal{L}_{1,i}(\tilde{\boldsymbol{\phi}}_1-\boldsymbol{\phi}_1)=0 \quad \text{for} \quad i=1,2,\dots,N,\\ \mathcal{L}_{2,i}(\tilde{\boldsymbol{\phi}}_2-\boldsymbol{\phi}_2)=0 \quad \text{for} \quad i=1,2,\dots,N^*,\\ \underline{h}_1\mathcal{L}_{1,0}(\tilde{\boldsymbol{\phi}}_1-\boldsymbol{\phi}_1)+\underline{h}_2\mathcal{L}_{2,0}(\tilde{\boldsymbol{\phi}}_2-\boldsymbol{\phi}_2)=(\Lambda_1-\underline{h}_1\Lambda_1^{(N)})\phi_1+(\Lambda_2-\underline{h}_2\Lambda_2^{(N)})\phi_2,\\ \underline{\rho}_2\boldsymbol{l}_2\cdot(\tilde{\boldsymbol{\phi}}_2-\boldsymbol{\phi}_2)-\underline{\rho}_1\boldsymbol{l}_1\cdot(\tilde{\boldsymbol{\phi}}_1-\boldsymbol{\phi}_1)=0.\end{cases}$$

Since the right-hand side of the third equation can be written as $\nabla\cdot\boldsymbol{f}_3$ with

$$\boldsymbol{f}_3=-\nabla(-\Delta)^{-1}\big((\Lambda_1-\underline{h}_1\Lambda_1^{(N)})\phi_1-(\Lambda_2-\underline{h}_2\Lambda_2^{(N)})\phi_2\big),$$

by Lemmas 5.1 and 7.2 we obtain

$$\begin{aligned}\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla(\tilde{\boldsymbol{\phi}}_\ell-\boldsymbol{\phi}_\ell)\|_{H^k}^2+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell'-\boldsymbol{\phi}_\ell'\|_{H^k}^2)&\\ &\lesssim\min\left\{\frac{\underline{\rho}_1}{\underline{h}_1},\frac{\underline{\rho}_2}{\underline{h}_2}\right\}\|\boldsymbol{f}_3\|_{H^k}^2\\ &\lesssim\sum_{l=1,2}\frac{\underline{\rho}_\ell}{\underline{h}_\ell}\|(-\Delta)^{-\frac12}(\Lambda_\ell-\underline{h}_\ell\Lambda_\ell^{(N)})\phi_\ell\|_{H^k}^2\\ &\lesssim\sum_{l=1,2}\underline{\rho}_\ell\underline{h}_\ell(\underline{h}_\ell\delta)^{2(4N+2)}\|\nabla\phi_\ell\|_{H^{k+4N+3}}^2.\end{aligned}$$

Moreover, it follows from Lemma 6.3 that

$$(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell'-\boldsymbol{\phi}_\ell'\|_{H^{k-1}}\lesssim\|\nabla(\tilde{\boldsymbol{\phi}}_\ell-\boldsymbol{\phi}_\ell)\|_{H^k}+(\underline{h}_\ell\delta)^{-1}\|\tilde{\boldsymbol{\phi}}_\ell'-\boldsymbol{\phi}_\ell'\|_{H^k}$$

for $\ell=1,2$. This completes the proof. □

The following proposition gives another version of Theorem 3.6 for the consistency of the Kakinuma model.

Proposition 7.6. *Let c,M be positive constants and m an integer such that $m\ge 4N+4$ and $m>\frac{n}{2}+2$. We assume* (H1) *or* (H2)*. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1,\underline{\rho}_2,\underline{h}_1,\underline{h}_2,\delta$ satisfying $\underline{h}_1\delta,\underline{h}_2\delta\le 1$, and for any solution (ζ,ϕ_1,ϕ_2) to the full model for interfacial gravity waves* (2.17) *on a time interval $[0,T]$ satisfying* (3.18)*, if we define H_1 and H_2 as in* (2.19) *and $(\tilde{\boldsymbol{\phi}}_1,\tilde{\boldsymbol{\phi}}_2)$ as a solution to* (7.2)*, then $(\zeta,\tilde{\boldsymbol{\phi}}_1,\tilde{\boldsymbol{\phi}}_2)$ satisfy approximately the Kakinuma model as*

$$\begin{cases}\boldsymbol{l}_1(H_1)\underline{h}_1^{-1}\partial_t\zeta+L_1(H_1,\delta,\underline{h}_1)\tilde{\boldsymbol{\phi}}_1=\mathfrak{r}_1,\\ \boldsymbol{l}_2(H_2)\underline{h}_2^{-1}\partial_t\zeta-L_2(H_2,b,\delta,\underline{h}_2)\tilde{\boldsymbol{\phi}}_2=\mathfrak{r}_2,\\ \underline{\rho}_1\big\{\boldsymbol{l}_1(H_1)\cdot\partial_t\tilde{\boldsymbol{\phi}}_1+\frac12\big(|\tilde{\boldsymbol{u}}_1|^2+(\underline{h}_1\delta)^{-2}\tilde{w}_1^2\big)\big\}\\ \quad-\underline{\rho}_2\big\{\boldsymbol{l}_2(H_2)\cdot\partial_t\tilde{\boldsymbol{\phi}}_2+\frac12\big(|\tilde{\boldsymbol{u}}_2|^2+(\underline{h}_2\delta)^{-2}\tilde{w}_2^2\big)\big\}-\zeta=\mathfrak{r}_0,\end{cases}\tag{7.4}$$

where $\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{w}_1, \tilde{w}_2$ are defined by (3.4) with $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ replaced by $(\tilde{\boldsymbol{\phi}}_1, \tilde{\boldsymbol{\phi}}_2)$, and the errors $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_0)$ satisfy

$$\begin{cases} \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\mathfrak{r}_\ell(t)\|^2_{H^{m-(4N+5)}} \le C \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell \delta)^{2(4N+2)} \|\nabla \phi_\ell(t)\|^2_{H^{m-1}}, \\ \|\mathfrak{r}_0(t)\|_{H^{m-4(N+1)}} \le C\big((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2}\big)(\underline{h}_1^{-1} + \underline{h}_2^{-1}) \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\nabla \phi_\ell(t)\|^2_{H^{m-1}}, \end{cases} \tag{7.5}$$

for $t \in [0,T]$.

Proof. Let $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ be the unique solutions to (7.3), and $(\tilde{\mathfrak{r}}_1, \tilde{\mathfrak{r}}_2, \tilde{\mathfrak{r}}_0)$ the errors in Theorem 3.6. Then, the errors $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_0)$ in the proposition can be written as

$$\begin{cases} \mathfrak{r}_1 = \tilde{\mathfrak{r}}_1 - L_1(H_1, \delta, \underline{h}_1)(\tilde{\boldsymbol{\phi}}_1 - \boldsymbol{\phi}_1), \\ \mathfrak{r}_2 = \tilde{\mathfrak{r}}_2 + L_2(H_2, b, \delta, \underline{h}_2)(\tilde{\boldsymbol{\phi}}_2 - \boldsymbol{\phi}_2), \\ \mathfrak{r}_0 = \tilde{\mathfrak{r}}_0 + \underline{\rho}_1\{\underline{h}_1^{-1}(\partial_t\zeta)(\tilde{w}_1 - w_1) - \frac{1}{2}\big((\tilde{\boldsymbol{u}}_1 + \boldsymbol{u}_1)\cdot(\tilde{\boldsymbol{u}}_1 - \boldsymbol{u}_1) + (\underline{h}_1\delta)^{-2}(\tilde{w}_1 + w_1)(\tilde{w}_1 - w_1)\big)\} \\ \qquad -\underline{\rho}_2\{\underline{h}_2^{-1}(\partial_t\zeta)(\tilde{w}_2 - w_2) - \frac{1}{2}\big((\tilde{\boldsymbol{u}}_2 + \boldsymbol{u}_2)\cdot(\tilde{\boldsymbol{u}}_2 - \boldsymbol{u}_2) + (\underline{h}_2\delta)^{-2}(\tilde{w}_2 + w_2)(\tilde{w}_2 - w_2)\big)\}. \end{cases}$$

Therefore, we have

$$\|\mathfrak{r}_\ell - \tilde{\mathfrak{r}}_\ell\|_{H^k} \lesssim \|\nabla(\tilde{\boldsymbol{\phi}}_\ell - \boldsymbol{\phi}_\ell)\|_{H^{k+1}} + \|\tilde{\boldsymbol{\phi}}_\ell' - \boldsymbol{\phi}_\ell'\|_{H^{k+1}} + (\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell' - \boldsymbol{\phi}_\ell'\|_{H^k}$$

for $-m \le k \le m-1$ and $\ell = 1,2$. Applying this estimate with $k = m-(4N+5)$ and the estimate in Lemma 7.5 with $k = m-4(N+1)$ and using the result in Theorem 3.6, we obtain the first estimate in (7.5). Since $m-2 > \frac{n}{2}$, we have

$$\begin{aligned} \|\mathfrak{r}_0 - \tilde{\mathfrak{r}}_0\|_{H^k} \lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \{&(\|\tilde{\boldsymbol{u}}_\ell\|_{H^{m-2}} + \|\boldsymbol{u}_\ell\|_{H^{m-2}})\|\tilde{\boldsymbol{u}}_\ell - \boldsymbol{u}_\ell\|_{H^k} \\ &+ \big(\underline{h}_\ell^{-1}\|\partial_t\zeta\|_{H^{m-2}} + (\underline{h}_\ell\delta)^{-2}(\|\tilde{w}_\ell\|_{H^{m-2}} + \|w_\ell\|_{H^{m-2}})\big)\|\tilde{w}_\ell - w_\ell\|_{H^k}\} \end{aligned}$$

for $|k| \le m-2$. Here, it follows from Lemmas 4.4, 5.1, and 7.5 that

$$\begin{aligned} \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\boldsymbol{u}_\ell\|^2_{H^{m-1}} + (\underline{h}_\ell\delta)^{-2}\|w_\ell\|^2_{H^{m-1}}) &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla\boldsymbol{\phi}_\ell\|^2_{H^{m-1}} + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\phi}_\ell'\|^2_{H^{m-1}}) \\ &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\nabla\phi_\ell\|^2_{H^{m-1}}, \end{aligned}$$

$$\begin{aligned} \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\tilde{\boldsymbol{u}}_\ell\|^2_{H^{m-1}} + (\underline{h}_\ell\delta)^{-2}\|\tilde{w}_\ell\|^2_{H^{m-1}}) &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla\tilde{\boldsymbol{\phi}}_\ell\|^2_{H^{m-1}} + (\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell'\|^2_{H^{m-1}}) \\ &\lesssim \min\left\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\right\}\|\nabla(\underline{\rho}_2\phi_2 - \underline{\rho}_1\phi_1)\|^2_{H^{m-1}} \\ &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\nabla\phi_\ell\|^2_{H^{m-1}}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\tilde{\boldsymbol{u}}_\ell - \boldsymbol{u}_\ell\|^2_{H^k} + (\underline{h}_\ell\delta)^{-2}\|\tilde{w}_\ell - w_\ell\|^2_{H^k}) \\ &\qquad \lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla(\tilde{\boldsymbol{\phi}}_\ell - \boldsymbol{\phi}_\ell)\|^2_{H^k} + (\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell' - \boldsymbol{\phi}_\ell'\|^2_{H^k}) \\ &\qquad \lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{2(4N+2)}\|\nabla\phi_\ell\|^2_{H^{k+4N+3}} \end{aligned}$$

for $0 \le k \le m-4(N+1)$. Moreover, it follows from Lemma 7.4 that $\|\partial_t\zeta\|_{H^{m-2}} = \|\Lambda_\ell\phi_\ell\|_{H^{m-2}} \lesssim \underline{h}_\ell\|\nabla\phi_\ell\|_{H^{m-1}}$ for $\ell = 1,2$. Summarizing the above estimates and using the result in Theorem 3.6, we easily obtain the second estimate in (7.5). The proof is complete. □

7.3 Completion of the proof of Theorem 3.9

Now we are ready to prove Theorem 3.9. Let $(\zeta^{\mathrm{IW}}, \phi_1^{\mathrm{IW}}, \phi_2^{\mathrm{IW}})$ be the solution to the full model for interfacial gravity waves (2.17) with uniform bound stated in the theorem, and define $\phi^{\mathrm{IW}} := \underline{\rho}_2\phi_2^{\mathrm{IW}} - \underline{\rho}_1\phi_1^{\mathrm{IW}}$, which is a canonical variable of the full model. We first ensure a uniform bound on the time derivative of the canonical variables $(\zeta^{\mathrm{IW}}, \phi^{\mathrm{IW}})$. It follows from the first and the second equations in (2.17) that $\partial_t\zeta^{\mathrm{IW}} = -\Lambda_1^{\mathrm{IW}}\phi_1^{\mathrm{IW}} = \Lambda_2^{\mathrm{IW}}\phi_2^{\mathrm{IW}}$, where $\Lambda_1^{\mathrm{IW}} = \Lambda_1(\zeta^{\mathrm{IW}}, \delta, \underline{h}_1)$ and $\Lambda_2^{\mathrm{IW}} = \Lambda_2(\zeta^{\mathrm{IW}}, b, \delta, \underline{h}_2)$. Similar notations will be used in the following without any comment. Therefore, by Lemma 7.4 we have

$$\begin{aligned}
\|\partial_t\zeta^{\mathrm{IW}}\|_{H^{m-1}}^2 &= \min\{\|\Lambda_1^{\mathrm{IW}}\phi_1^{\mathrm{IW}}\|_{H^{m-1}}^2, \|\Lambda_2^{\mathrm{IW}}\phi_2^{\mathrm{IW}}\|_{H^{m-1}}^2\} \\
&\lesssim \min\{\underline{h}_1^2\|\nabla\phi_1^{\mathrm{IW}}\|_{H^m}^2, \underline{h}_2^2\|\nabla\phi_2^{\mathrm{IW}}\|_{H^m}^2\} \\
&\lesssim \min\left\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\right\} \sum_{\ell=1,2} \underline{\rho}_\ell\underline{h}_\ell\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^m}^2 \\
&\le 2\sum_{\ell=1,2} \underline{\rho}_\ell\underline{h}_\ell\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^m}^2,
\end{aligned}$$

where we used (2.15). It follows from the third equation in (2.17) that

$$\begin{aligned}
\partial_t\phi^{\mathrm{IW}} &= \underline{\rho}_2\partial_t\phi_2^{\mathrm{IW}} - \underline{\rho}_1\partial_t\phi_1^{\mathrm{IW}} \\
&= \frac{1}{2}\underline{\rho}_1\left(|\nabla\phi_1^{\mathrm{IW}}|^2 - \delta^2\frac{(\Lambda_1^{\mathrm{IW}}\phi_1^{\mathrm{IW}} - \nabla\zeta^{\mathrm{IW}}\cdot\nabla\phi_1^{\mathrm{IW}})^2}{1+\delta^2|\nabla\zeta^{\mathrm{IW}}|^2}\right) \\
&\quad - \frac{1}{2}\underline{\rho}_2\left(|\nabla\phi_2^{\mathrm{IW}}|^2 - \delta^2\frac{(\Lambda_2^{\mathrm{IW}}\phi_2^{\mathrm{IW}} + \nabla\zeta^{\mathrm{IW}}\cdot\nabla\phi_2^{\mathrm{IW}})^2}{1+\delta^2|\nabla\zeta^{\mathrm{IW}}|^2}\right) - \zeta^{\mathrm{IW}}.
\end{aligned}$$

Here, we note that in view of the conditions $\underline{h}_1\delta, \underline{h}_2\delta \le 1$ and $\underline{h}_1^{-1}, \underline{h}_2^{-1} \lesssim 1$ we have $\delta \lesssim 1$. Therefore, by Lemma 7.4 we have

$$\begin{aligned}
\|\partial_t\phi^{\mathrm{IW}}\|_{H^{m-1}} &\lesssim \|\zeta^{\mathrm{IW}}\|_{H^{m-1}} + \sum_{\ell=1,2}\underline{\rho}_\ell\{\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^{m-1}}^2 + \delta^2(\underline{h}_\ell^2\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^m}^2 + \|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^{m-1}}^2)\} \\
&\lesssim \|\zeta^{\mathrm{IW}}\|_{H^{m-1}} + \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^m}^2.
\end{aligned}$$

Hence, we obtain $\|\partial_t\zeta^{\mathrm{IW}}\|_{H^{m-1}} + \|\partial_t\phi^{\mathrm{IW}}\|_{H^{m-1}} \lesssim 1$.

Let $(\tilde{\boldsymbol{\phi}}_1^{\mathrm{IW}}, \tilde{\boldsymbol{\phi}}_2^{\mathrm{IW}})$ be the solution to (7.2) with $(\zeta, \phi) = (\zeta^{\mathrm{IW}}, \phi^{\mathrm{IW}})$. Then, Proposition 7.6 states that $(\zeta^{\mathrm{IW}}, \tilde{\boldsymbol{\phi}}_1^{\mathrm{IW}}, \tilde{\boldsymbol{\phi}}_2^{\mathrm{IW}})$ satisfy approximately the Kakinuma model as (7.4) and the errors $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_0)$ satisfy (7.5). Moreover, it follows from Lemma 5.1 that

$$\begin{aligned}
\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}}\|_{H^m}^2 + (\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}\prime}\|_{H^m}^2) &\lesssim \min\left\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\right\}\|\nabla\phi^{\mathrm{IW}}\|_{H^m}^2 \\
&\lesssim \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\|\nabla\phi_\ell^{\mathrm{IW}}\|_{H^m}^2 \lesssim 1,
\end{aligned}$$

which yields

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\tilde{\boldsymbol{u}}_\ell^{\rm IW}\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\tilde{w}_\ell^{\rm IW}\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-4} \|\tilde{\boldsymbol{\phi}}_\ell^{\rm IW}{}'\|_{H^{m-1}}^2) \lesssim 1,$$

where $\tilde{\boldsymbol{u}}_1^{\rm IW}, \tilde{\boldsymbol{u}}_2^{\rm IW}, \tilde{w}_1^{\rm IW}, \tilde{w}_2^{\rm IW}$ are defined by (3.4) with $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ replaced by $(\tilde{\boldsymbol{\phi}}_1^{\rm IW}, \tilde{\boldsymbol{\phi}}_2^{\rm IW})$, and we used Lemma 6.3. We proceed to evaluate $(\partial_t \tilde{\boldsymbol{\phi}}_1^{\rm IW}, \partial_t \tilde{\boldsymbol{\phi}}_2^{\rm IW})$. To this end, we derive equations for these time derivatives by differentiating (7.2) with respect to t. The procedure is almost the same as in the proof of Lemma 5.2. The only difference is the last equation in (5.5), especially, the expression of f_4. In this case, f_4 has the form

$$f_4 = \partial_t \phi^{\rm IW} + \underline{\rho}_1 \tilde{w}_1^{\rm IW} \underline{h}_1^{-1} \partial_t \zeta^{\rm IW} - \underline{\rho}_2 \tilde{w}_2^{\rm IW} \underline{h}_2^{-1} \partial_t \zeta^{\rm IW},$$

so that $\|f_4\|_{H^{m-1}} \lesssim 1$. Therefore, we obtain

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \partial_t \tilde{\boldsymbol{\phi}}_\ell^{\rm IW}\|_{H^{m-2}}^2 + (\underline{h}_\ell \delta)^{-2} \|\partial_t \tilde{\boldsymbol{\phi}}_\ell^{\rm IW}{}'\|_{H^{m-2}}^2) \lesssim 1.$$

Let $(\zeta^{\rm K}, \boldsymbol{\phi}_1^{\rm K}, \boldsymbol{\phi}_2^{\rm K})$ be the solution to the initial value problem for the Kakinuma model (2.18)–(2.20) stated in the theorem, whose unique existence is guaranteed by Theorem 3.1 and Proposition 3.4. Note also that the solution satisfies the uniform bound (3.14) together with the stability and non-cavitation conditions (3.15). It follows from Lemma 6.3 that $\underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell \delta)^{-4} \|\boldsymbol{\phi}_\ell^{\rm K}{}'\|_{H^{m-1}}^2 \lesssim 1$ for $\ell = 1,2$. Moreover, the time derivatives $(\partial_t \zeta^{\rm K}, \partial_t \boldsymbol{\phi}_1^{\rm K}, \partial_t \boldsymbol{\phi}_2^{\rm K})$ satisfy (5.4) and $(\boldsymbol{u}_\ell^{\rm K}, w_\ell^{\rm K})$ $(\ell = 1,2)$, which are defined by (3.4) with $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ replaced by $(\boldsymbol{\phi}_1^{\rm K}, \boldsymbol{\phi}_2^{\rm K})$, satisfy (5.8). Putting

$$\zeta^{\rm res} := \zeta^{\rm K} - \zeta^{\rm IW}, \qquad \boldsymbol{\phi}_\ell^{\rm res} := \boldsymbol{\phi}_\ell^{\rm K} - \tilde{\boldsymbol{\phi}}_\ell^{\rm IW} \quad (\ell = 1,2),$$

we will show that $(\zeta^{\rm res}, \boldsymbol{\phi}_1^{\rm res}, \boldsymbol{\phi}_2^{\rm res})$ can be estimated by the errors $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_0)$. To this end, we are going to evaluate

$$E_k^{\rm res}(t) := \|\zeta^{\rm res}(t)\|_{H^k}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla \boldsymbol{\phi}_\ell^{\rm res}(t)\|_{H^k}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_\ell^{\rm res}{}'(t)\|_{H^k}^2)$$

for an appropriate integer k by making use of energy estimates similar to the ones obtained in Sections 5 and 6 for the proof of the well-posedness of the initial value problem for the Kakinuma model (2.18)–(2.20). Here, we note that $E_k^{\rm res}(0) = 0$.

As in the case of the energy estimate for the Kakinuma model, we first need to evaluate times derivatives $(\partial_t \zeta^{\rm res}, \partial_t \boldsymbol{\phi}_1^{\rm res}, \partial_t \boldsymbol{\phi}_2^{\rm res})$ in terms of $E_k^{\rm res}$. By taking difference between the first components of the first two equations in (3.5) and (7.4), $\partial_t \zeta^{\rm res}$ can be written in two ways as

$$\begin{aligned} \partial_t \zeta^{\rm res} &= -\underline{h}_1 \{ \mathcal{L}_{1,0}^{\rm K} \boldsymbol{\phi}_1^{\rm res} + (\mathcal{L}_{1,0}^{\rm K} - \mathcal{L}_{1,0}^{\rm IW}) \tilde{\boldsymbol{\phi}}_1^{\rm IW} + \mathfrak{r}_{1,0} \} \\ &= \underline{h}_2 \{ \mathcal{L}_{2,0}^{\rm K} \boldsymbol{\phi}_2^{\rm res} + (\mathcal{L}_{2,0}^{\rm K} - \mathcal{L}_{2,0}^{\rm IW}) \tilde{\boldsymbol{\phi}}_2^{\rm IW} + \mathfrak{r}_{2,0} \}, \end{aligned}$$

where $\mathcal{L}_{1,0}^{\rm K} = \mathcal{L}_{1,0}(H_1^{\rm K}, \delta, \underline{h}_1)$, $H_1^{\rm K} = 1 - \underline{h}_1^{-1} \zeta^{\rm K}$, and similar simplifications are used, and $\mathfrak{r}_{\ell,0}$ is the 0th component of the error $\mathfrak{r}_\ell$ for $\ell = 1,2$. Therefore, we have

$$\begin{aligned} \|\partial_t \zeta^{\rm res}\|_{H^{k-1}} \lesssim \underline{h}_\ell \{ & \|\nabla \boldsymbol{\phi}_\ell^{\rm res}\|_{H^k} + \|\boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^k} \\ & + \|\zeta^{\rm res}\|_{H^k} (\|\nabla \tilde{\boldsymbol{\phi}}_\ell^{\rm IW}\|_{H^m} + \|\tilde{\boldsymbol{\phi}}_\ell^{\rm IW}{}'\|_{H^m}) + \|\mathfrak{r}_{\ell,0}\|_{H^{k-1}} \} \end{aligned}$$

for $\ell=1,2$ and $|k|\leq m$. Hence, by the technique used in the proof of Lemma 5.2 we obtain

$$\begin{aligned}
\|\partial_t\zeta^{\mathrm{res}}\|_{H^{k-1}}^2 &\lesssim \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\{\|\nabla\boldsymbol{\phi}_\ell^{\mathrm{res}}\|_{H^k}^2+\|\boldsymbol{\phi}_\ell^{\mathrm{res}\prime}\|_{H^k}^2 \\
&\qquad +\|\zeta^{\mathrm{res}}\|_{H^k}^2(\|\nabla\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}}\|_{H^m}^2+\|\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}\prime}\|_{H^m}^2)+\|\mathfrak{r}_{\ell,0}\|_{H^{k-1}}^2\} \\
&\lesssim E_k^{\mathrm{res}}+\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\|\mathfrak{r}_\ell\|_{H^{k-1}}^2
\end{aligned}$$

for $|k|\leq m$. We proceed to evaluate $(\partial_t\boldsymbol{\phi}_1^{\mathrm{res}},\partial_t\boldsymbol{\phi}_2^{\mathrm{res}})$. We recall that $(\partial_t\boldsymbol{\phi}_1^{\mathrm{K}},\partial_t\boldsymbol{\phi}_2^{\mathrm{K}})$ satisfy (5.5) with $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=(\zeta^{\mathrm{K}},\boldsymbol{\phi}_1^{\mathrm{K}},\boldsymbol{\phi}_2^{\mathrm{K}})$ and note that, differentiating the first three equations of (7.2) with respect to t and using the last equation in (7.4), $(\partial_t\tilde{\boldsymbol{\phi}}_1^{\mathrm{IW}},\partial_t\tilde{\boldsymbol{\phi}}_2^{\mathrm{IW}})$ also satisfy (5.5) with $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=(\zeta^{\mathrm{IW}},\tilde{\boldsymbol{\phi}}_1^{\mathrm{IW}},\tilde{\boldsymbol{\phi}}_2^{\mathrm{IW}})$ and f_4 added with the error term $-\mathfrak{r}_0$. By taking the difference between these equations, we have therefore

$$\begin{cases}
\mathcal{L}_{1,i}^{\mathrm{IW}}\partial_t\boldsymbol{\phi}_1^{\mathrm{res}}=f_{1,i}^{\mathrm{res}} \quad \text{for} \quad i=1,2,\dots,N, \\
\mathcal{L}_{2,i}^{\mathrm{IW}}\partial_t\boldsymbol{\phi}_2^{\mathrm{res}}=f_{2,i}^{\mathrm{res}} \quad \text{for} \quad i=1,2,\dots,N^*, \\
\underline{h}_1\mathcal{L}_{1,0}^{\mathrm{IW}}\partial_t\boldsymbol{\phi}_1^{\mathrm{res}}+\underline{h}_2\mathcal{L}_{2,0}^{\mathrm{IW}}\partial_t\boldsymbol{\phi}_2^{\mathrm{res}}=\nabla\cdot\boldsymbol{f}_3^{\mathrm{res}}, \\
-\underline{\rho}_1\boldsymbol{l}_1^{\mathrm{IW}}\cdot\partial_t\boldsymbol{\phi}_1^{\mathrm{res}}+\underline{\rho}_2\boldsymbol{l}_2^{\mathrm{IW}}\cdot\partial_t\boldsymbol{\phi}_2^{\mathrm{res}}=f_4^{\mathrm{res}},
\end{cases}$$

where

$$\begin{cases}
f_{1,i}^{\mathrm{res}}=f_{1,i}^{\mathrm{K}}-\tilde{f}_{1,i}^{\mathrm{IW}}+(\mathcal{L}_{1,i}^{\mathrm{IW}}-\mathcal{L}_{1,i}^{\mathrm{K}})\partial_t\boldsymbol{\phi}_1^{\mathrm{K}} \quad \text{for} \quad i=1,2,\dots,N, \\
f_{2,i}^{\mathrm{res}}=f_{2,i}^{\mathrm{K}}-\tilde{f}_{2,i}^{\mathrm{IW}}+(\mathcal{L}_{2,i}^{\mathrm{IW}}-\mathcal{L}_{2,i}^{\mathrm{K}})\partial_t\boldsymbol{\phi}_2^{\mathrm{K}} \quad \text{for} \quad i=1,2,\dots,N^*, \\
\boldsymbol{f}_3^{\mathrm{res}}=\boldsymbol{f}_3^{\mathrm{K}}-\tilde{\boldsymbol{f}}_3^{\mathrm{IW}}+\underline{h}_1((\boldsymbol{a}_{1,0}^{\mathrm{K}}-\boldsymbol{a}_{1,0}^{\mathrm{IW}})\otimes\nabla)^{\mathrm{T}}\partial_t\boldsymbol{\phi}_1^{\mathrm{K}} \\
\qquad +\underline{h}_2\{((\boldsymbol{a}_{2,0}^{\mathrm{K}}-\boldsymbol{a}_{2,0}^{\mathrm{IW}})\otimes\nabla)^{\mathrm{T}}\partial_t\boldsymbol{\phi}_2^{\mathrm{K}}-((\boldsymbol{b}_{2,0}^{\mathrm{K}}-\boldsymbol{b}_{2,0}^{\mathrm{IW}})\cdot\partial_t\boldsymbol{\phi}_2^{\mathrm{K}})\underline{h}_2^{-1}\nabla b\}, \\
f_4^{\mathrm{res}}=f_4^{\mathrm{K}}-\tilde{f}_4^{\mathrm{IW}}+\mathfrak{r}_0-\underline{\rho}_1(\boldsymbol{l}_1^{\mathrm{IW}}-\boldsymbol{l}_1^{\mathrm{K}})\cdot\partial_t\boldsymbol{\phi}_1^{\mathrm{K}}+\underline{\rho}_2(\boldsymbol{l}_2^{\mathrm{IW}}-\boldsymbol{l}_2^{\mathrm{K}})\cdot\partial_t\boldsymbol{\phi}_2^{\mathrm{K}}.
\end{cases}$$

Here, $f_{1,i}^{\mathrm{K}}$, $f_{2,i}^{\mathrm{K}}$, $\boldsymbol{f}_3^{\mathrm{K}}$, f_4^{K} (respectively $\tilde{f}_{1,i}^{\mathrm{IW}}$, $\tilde{f}_{2,i}^{\mathrm{IW}}$, $\tilde{\boldsymbol{f}}_3^{\mathrm{IW}}$, $\tilde{f}_4^{\mathrm{IW}}$) are those in (5.6) with $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=(\zeta^{\mathrm{K}},\boldsymbol{\phi}_1^{\mathrm{K}},\boldsymbol{\phi}_2^{\mathrm{K}})$ (respectively $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=(\zeta^{\mathrm{IW}},\tilde{\boldsymbol{\phi}}_1^{\mathrm{IW}},\tilde{\boldsymbol{\phi}}_2^{\mathrm{IW}})$), $\boldsymbol{a}_{\ell,0}^{\mathrm{K}}=\boldsymbol{a}_{\ell,0}(H_\ell^{\mathrm{K}})$ and $\boldsymbol{b}_{2,0}^{\mathrm{K}}=\boldsymbol{b}_{2,0}(H_2^{\mathrm{K}})$, where $\boldsymbol{a}_{\ell,0}(H_\ell)$ and $\boldsymbol{b}_{2,0}(H_2)$ are the 0th columns of the matrixes $A_\ell(H_\ell)$ and $B_2(H_2)$ defined by (6.2) and (6.4), respectively, and so on. Note the relations $\mathcal{L}_{1,0}\boldsymbol{\phi}_1=-\nabla\cdot((\boldsymbol{a}_{1,0}\otimes\nabla)^{\mathrm{T}}\boldsymbol{\phi}_1)$ and $\mathcal{L}_{2,0}\boldsymbol{\phi}_2=-\nabla\cdot((\boldsymbol{a}_{2,0}\otimes\nabla)^{\mathrm{T}}\boldsymbol{\phi}_2-(\boldsymbol{b}_{2,0}\cdot\boldsymbol{\phi}_2)\underline{h}_2^{-1}\nabla b)$. Therefore, by Lemma 5.1 we have, for $1\leq k\leq m+1$,

$$\begin{aligned}
&\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\big(\|\nabla\partial_t\boldsymbol{\phi}_\ell^{\mathrm{res}}\|_{H^{k-1}}^2+(\underline{h}_\ell\delta)^{-2}\|\partial_t\boldsymbol{\phi}_\ell^{\mathrm{res}\prime}\|_{H^{k-1}}^2\big) \\
&\lesssim \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\underline{h}_\ell\delta)^2\|\boldsymbol{f}_\ell^{\mathrm{res}\prime}\|_{H^{k-1}}^2+\min\Big\{\frac{\underline{\rho}_1}{\underline{h}_1},\frac{\underline{\rho}_2}{\underline{h}_2}\Big\}\|\boldsymbol{f}_3^{\mathrm{res}}\|_{H^{k-1}}^2+\min\Big\{\frac{\underline{h}_1}{\underline{\rho}_1},\frac{\underline{h}_2}{\underline{\rho}_2}\Big\}\|f_4^{\mathrm{res}}\|_{H^k}^2.
\end{aligned}$$

We will evaluate each term in the right-hand side. For $1\leq k\leq m-1$, we see that

$$\begin{aligned}
\|\boldsymbol{f}_\ell^{\mathrm{res}\prime}\|_{H^{k-1}} &\lesssim \underline{h}_\ell^{-1}\|\zeta^{\mathrm{res}}\|_{H^k}(\|\nabla\boldsymbol{\phi}_\ell^{\mathrm{K}}\|_{H^m}+(\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\phi}_\ell^{\mathrm{K}\prime}\|_{H^m})\underline{h}_\ell^{-1}\|\partial_t\zeta^{\mathrm{K}}\|_{H^{m-1}} \\
&\quad +(\|\nabla\boldsymbol{\phi}_\ell^{\mathrm{res}}\|_{H^k}+(\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\phi}_\ell^{\mathrm{res}\prime}\|_{H^k})\underline{h}_\ell^{-1}\|\partial_t\zeta^{\mathrm{K}}\|_{H^{m-1}} \\
&\quad +(\|\nabla\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}}\|_{H^m}+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}\prime}\|_{H^m})\underline{h}_\ell^{-1}\|\partial_t\zeta^{\mathrm{res}}\|_{H^{k-1}} \\
&\quad +\underline{h}_\ell^{-1}\|\zeta^{\mathrm{res}}\|_{H^k}(\|\nabla\partial_t\boldsymbol{\phi}_\ell^{\mathrm{K}}\|_{H^{m-1}}+(\underline{h}_\ell\delta)^{-2}\|\partial_t\boldsymbol{\phi}_\ell^{\mathrm{K}\prime}\|_{H^{m-1}})
\end{aligned}$$

for $\ell=1,2$,

$$\begin{aligned}
\|\boldsymbol{f}_3^{\mathrm{res}}\|_{H^{k-1}} \lesssim \sum_{\ell=1,2}&\{\|\boldsymbol{u}_\ell^{\mathrm{K}}-\tilde{\boldsymbol{u}}_\ell^{\mathrm{IW}}\|_{H^k}\|\partial_t\zeta^{\mathrm{K}}\|_{H^{m-1}}+\|\tilde{\boldsymbol{u}}_\ell^{\mathrm{IW}}\|_{H^m}\|\partial_t\zeta^{\mathrm{res}}\|_{H^{k-1}} \\
&+\|\zeta^{\mathrm{res}}\|_{H^k}(\|\nabla\partial_t\boldsymbol{\phi}_\ell^{\mathrm{K}}\|_{H^{m-1}}+\|\partial_t\boldsymbol{\phi}_\ell^{\mathrm{K}\prime}\|_{H^{m-1}})\},
\end{aligned}$$

and

$$\|f_4^{\rm res}\|_{H^k} \lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \bigl\{ (\|\boldsymbol{u}_\ell^{\rm K}\|_{H^m} + \|\tilde{\boldsymbol{u}}_\ell^{\rm IW}\|_{H^m}) \|\boldsymbol{u}_\ell^{\rm K} - \tilde{\boldsymbol{u}}_\ell^{\rm IW}\|_{H^k}$$
$$+ (\underline{h}_\ell\delta)^{-2} (\|w_\ell^{\rm K}\|_{H^m} + \|\tilde{w}_\ell^{\rm IW}\|_{H^m}) \|w_\ell^{\rm K} - \tilde{w}_\ell^{\rm IW}\|_{H^k}$$
$$+ \underline{h}_\ell^{-1} \|\zeta^{\rm res}\|_{H^k} \|\partial_t \boldsymbol{\phi}_\ell^{\rm K}{}'\|_{H^{m-1}} \bigr\} + \|\zeta^{\rm res}\|_{H^k} + \|\mathfrak{r}_0\|_{H^k}.$$

Moreover, for any $0 \le k \le m$ we have also

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \bigl(\|\boldsymbol{u}_\ell^{\rm K} - \tilde{\boldsymbol{u}}_\ell^{\rm IW}\|_{H^k}^2 + (\underline{h}_\ell\delta)^{-2} \|w_\ell^{\rm K} - \tilde{w}_\ell^{\rm IW}\|_{H^k}^2 \bigr) \lesssim E_k^{\rm res}. \tag{7.6}$$

Summarizing the above estimates and using $\underline{h}_1^{-1}, \underline{h}_2^{-1} \lesssim 1$ we obtain, for $1 \le k \le m-1$,

$$\begin{aligned}\|\partial_t\zeta^{\rm res}\|_{H^{k-1}}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \bigl(\|\nabla\partial_t \boldsymbol{\phi}_\ell^{\rm res}\|_{H^{k-1}}^2 + (\underline{h}_\ell\delta)^{-2} \|\partial_t \boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^{k-1}}^2 \bigr) \\ \lesssim E_k^{\rm res} + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\mathfrak{r}_\ell\|_{H^{k-1}}^2 + \|\mathfrak{r}_0\|_{H^k}^2.\end{aligned} \tag{7.7}$$

We need also to evaluate $\underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{-4} \|\boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^{k-1}}^2$ for $\ell = 1,2$ in terms of $E_k^{\rm res}$. In view of

$$\begin{cases} \mathcal{L}_{1,i}^{\rm IW} \boldsymbol{\phi}_1^{\rm res} = \mathcal{L}_{1,i}^{\rm IW} \boldsymbol{\phi}_1^{\rm K} = (\mathcal{L}_{1,i}^{\rm IW} - \mathcal{L}_{1,i}^{\rm K}) \boldsymbol{\phi}_1^{\rm K} =: h_{1,i}^{\rm res} \quad \text{for} \quad i = 1,2,\dots,N, \\ \mathcal{L}_{2,i}^{\rm IW} \boldsymbol{\phi}_2^{\rm res} = \mathcal{L}_{2,i}^{\rm IW} \boldsymbol{\phi}_2^{\rm K} = (\mathcal{L}_{2,i}^{\rm IW} - \mathcal{L}_{2,i}^{\rm K}) \boldsymbol{\phi}_2^{\rm K} =: h_{2,i}^{\rm res} \quad \text{for} \quad i = 1,2,\dots,N^*, \end{cases}$$

Lemma 6.3 yields $(\underline{h}_\ell\delta)^{-2} \|\boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^{k-1}} \lesssim \|\nabla\boldsymbol{\phi}_\ell^{\rm res}\|_{H^k} + \|\boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^k} + \|\boldsymbol{h}_\ell^{\rm res}{}'\|_{H^{k-1}}$ and we have $\|\boldsymbol{h}_\ell^{\rm res}{}'\|_{H^{k-1}} \lesssim (\|\nabla\boldsymbol{\phi}_\ell^{\rm K}\|_{H^m} + \|\boldsymbol{\phi}_\ell^{\rm K}{}'\|_{H^m} + (\underline{h}_\ell\delta)^{-2} \|\boldsymbol{\phi}_\ell^{\rm K}{}'\|_{H^{m-1}}) \|\zeta^{\rm res}\|_{H^k}$ for $1 \le k \le m$. Therefore, for $1 \le k \le m$ we obtain

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{-4} \|\boldsymbol{\phi}_\ell^{\rm res}{}'\|_{H^{k-1}}^2 \lesssim E_k^{\rm res}. \tag{7.8}$$

Now, by deriving equations for spatial derivatives of $(\zeta^{\rm res}, \boldsymbol{\phi}_1^{\rm res}, \boldsymbol{\phi}_2^{\rm res})$ and applying the energy estimate obtained in Subsection 6.1 we will evaluate $E_k^{\rm res}$. Let β be a multi-index such that $1 \le |\beta| \le k$. Applying ∂^β to the Kakinuma model (3.5) for $(\zeta^{\rm K}, \boldsymbol{\phi}_1^{\rm K}, \boldsymbol{\phi}_2^{\rm K})$ and to (7.4) for $(\zeta^{\rm IW}, \tilde{\boldsymbol{\phi}}_1^{\rm IW}, \tilde{\boldsymbol{\phi}}_2^{\rm IW})$ and taking the difference between the resulting equations, we obtain

$$\begin{cases} \boldsymbol{l}_1^{\rm K} (\partial_t + \boldsymbol{u}_1^{\rm K}\cdot\nabla) \partial^\beta \zeta^{\rm res} + \underline{h}_1 L_1^{\rm K,pr} \partial^\beta \boldsymbol{\phi}_1^{\rm res} = \boldsymbol{f}_{1,\beta}^{\rm res}, \\ \boldsymbol{l}_2^{\rm K} (\partial_t + \boldsymbol{u}_2^{\rm K}\cdot\nabla) \partial^\beta \zeta^{\rm res} - \underline{h}_2 L_2^{\rm K,pr} \partial^\beta \boldsymbol{\phi}_2^{\rm res} = \boldsymbol{f}_{2,\beta}^{\rm res}, \\ \underline{\rho}_1 \boldsymbol{l}_1^{\rm K} \cdot (\partial_t + \boldsymbol{u}_1^{\rm K}\cdot\nabla) \partial^\beta \boldsymbol{\phi}_1^{\rm res} - \underline{\rho}_2 \boldsymbol{l}_2^{\rm K} \cdot (\partial_t + \boldsymbol{u}_1^{\rm K}\cdot\nabla) \partial^\beta \boldsymbol{\phi}_2^{\rm res} - a^{\rm K} \partial^\beta \zeta^{\rm res} = f_{0,\beta}^{\rm res}, \end{cases}$$

where

$$\begin{cases} \boldsymbol{f}_{1,\beta}^{\rm res} := \boldsymbol{f}_{1,\beta}^{\rm K} - \tilde{\boldsymbol{f}}_{1,\beta}^{\rm IW} - \underline{h}_1 \partial^\beta \mathfrak{r}_1 + \underline{h}_1 (L_1^{\rm IW,pr} - L_1^{\rm K,pr}) \partial^\beta \tilde{\boldsymbol{\phi}}_1^{\rm IW} \\ \qquad + \bigl(\boldsymbol{l}_1^{\rm IW} (\partial_t + \tilde{\boldsymbol{u}}_1^{\rm IW}\cdot\nabla) - \boldsymbol{l}_1^{\rm K} (\partial_t + \boldsymbol{u}_1^{\rm K}\cdot\nabla) \bigr) \partial^\beta \zeta^{\rm IW}, \\ \boldsymbol{f}_{2,\beta}^{\rm res} := \boldsymbol{f}_{2,\beta}^{\rm K} - \tilde{\boldsymbol{f}}_{2,\beta}^{\rm IW} - \underline{h}_2 \partial^\beta \mathfrak{r}_2 - \underline{h}_2 (L_2^{\rm IW,pr} - L_2^{\rm K,pr}) \partial^\beta \tilde{\boldsymbol{\phi}}_2^{\rm IW} \\ \qquad + \bigl(\boldsymbol{l}_2^{\rm IW} (\partial_t + \tilde{\boldsymbol{u}}_2^{\rm IW}\cdot\nabla) - \boldsymbol{l}_2^{\rm K} (\partial_t + \boldsymbol{u}_2^{\rm K}\cdot\nabla) \bigr) \partial^\beta \zeta^{\rm IW}, \\ f_{0,\beta}^{\rm res} := f_{0,\beta}^{\rm K} - \tilde{f}_{0,\beta}^{\rm IW} - \partial^\beta \mathfrak{r}_0 - (\tilde{a}^{\rm IW} - a^{\rm K}) \partial^\beta \zeta^{\rm IW} \\ \qquad + \underline{\rho}_1 \bigl(\boldsymbol{l}_1^{\rm IW} (\partial_t + \tilde{\boldsymbol{u}}_1^{\rm IW}\cdot\nabla) - \boldsymbol{l}_1^{\rm K} (\partial_t + \boldsymbol{u}_1^{\rm K}\cdot\nabla) \bigr) \cdot \partial^\beta \tilde{\boldsymbol{\phi}}_1^{\rm IW} \\ \qquad - \underline{\rho}_2 \bigl(\boldsymbol{l}_2^{\rm IW} (\partial_t + \tilde{\boldsymbol{u}}_2^{\rm IW}\cdot\nabla) - \boldsymbol{l}_2^{\rm K} (\partial_t + \boldsymbol{u}_2^{\rm K}\cdot\nabla) \bigr) \cdot \partial^\beta \tilde{\boldsymbol{\phi}}_2^{\rm IW}. \end{cases}$$

Here, $\boldsymbol{f}^{\mathrm{K}}_{1,\beta}$, $\boldsymbol{f}^{\mathrm{K}}_{2,\beta}$, and $f^{\mathrm{K}}_{0,\beta}$ are those in (6.15)–(6.17) with $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=(\zeta^{\mathrm{K}},\boldsymbol{\phi}^{\mathrm{K}}_1,\boldsymbol{\phi}^{\mathrm{K}}_2)$, and so on. As we saw, all the assumptions in Proposition 6.2 are satisfied, so that we have

$$\mathscr{E}(\partial^\beta \boldsymbol{U}^{\mathrm{res}}(t)) \lesssim \int_0^t \mathscr{F}^{\mathrm{res}}_\beta(\tau)\mathrm{d}\tau,$$

where $\boldsymbol{U}^{\mathrm{res}} := (\zeta^{\mathrm{res}},\boldsymbol{\phi}^{\mathrm{res}}_1,\boldsymbol{\phi}^{\mathrm{res}}_2)^{\mathrm{T}}$, $\mathscr{E}$ is defined in (6.11), and

$$\begin{aligned}\mathscr{F}^{\mathrm{res}}_\beta &= \|f^{\mathrm{res}}_{0,\beta}\|_{H^1}(\|\partial_t\zeta^{\mathrm{res}}\|_{H^{k-1}}+\|\zeta^{\mathrm{res}}\|_{H^k})\\ &\quad + \sum_{\ell=1,2}\underline{\rho}_\ell(\|\boldsymbol{f}^{\mathrm{res}}_{\ell,\beta}\|_{L^2}+\|\zeta^{\mathrm{res}}\|_{H^k})(\|\nabla\partial_t\boldsymbol{\phi}^{\mathrm{res}}_\ell\|_{H^{k-1}}+\|\nabla\boldsymbol{\phi}^{\mathrm{res}}_\ell\|_{H^k}).\end{aligned}$$

In view of $\|(\zeta^{\mathrm{IW}},\zeta^{\mathrm{K}})\|_{H^m}\lesssim 1$, straightforward calculations yield

$$\begin{aligned}\|\boldsymbol{f}^{\mathrm{res}}_{\ell,\beta}\|_{L^2} &\lesssim (\|\partial_t\zeta^{\mathrm{IW}}\|_{H^{m-1}}+\|\tilde{\boldsymbol{u}}^{\mathrm{IW}}_\ell\|_{H^m})\|\zeta^{\mathrm{res}}\|_{H^k}\\ &\quad + \underline{h}_\ell(\|\nabla\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_\ell\|_{H^m}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^m}+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^{m-1}})\|\zeta^{\mathrm{res}}\|_{H^k}\\ &\quad + \underline{h}_\ell(\|\nabla\tilde{\boldsymbol{\phi}}^{\mathrm{res}}_\ell\|_{H^k}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{res}\prime}_\ell\|_{H^k}+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}^{\mathrm{res}\prime}_\ell\|_{H^{k-1}})\\ &\quad + \|\partial_t\zeta^{\mathrm{res}}\|_{H^{k-1}}+\|\tilde{\boldsymbol{u}}^{\mathrm{IW}}_\ell-\boldsymbol{u}^{\mathrm{K}}_\ell\|_{H^k}+\underline{h}_\ell\|\mathfrak{r}_\ell\|_{H^k}\end{aligned}$$

for $\ell=1,2$ and $\frac{n}{2}<k\leq m-1$. As for $f^{\mathrm{res}}_{0,\beta}$, we note the relation

$$\begin{aligned}&\big\{\big([\partial^\beta,\boldsymbol{l}^{\mathrm{K}}_2]-\boldsymbol{l}'_2(H^{\mathrm{K}}_2)(\underline{h}_2^{-1}\partial^\beta\zeta^{\mathrm{K}})\big)-\big([\partial^\beta,\boldsymbol{l}^{\mathrm{IW}}_2]-\boldsymbol{l}'_2(H^{\mathrm{IW}}_2)(\underline{h}_2^{-1}\partial^\beta\zeta^{\mathrm{IW}})\big)\big\}^{\mathrm{T}}\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_2\\ &= \int_0^1\big\{[\partial^\beta,\boldsymbol{l}'_2(sH^{\mathrm{IW}}+(1-s)H^{\mathrm{K}})]\\ &\qquad - \boldsymbol{l}''_2(sH^{\mathrm{IW}}+(1-s)H^{\mathrm{K}})\underline{h}_2^{-1}\partial^\beta(s\zeta^{\mathrm{IW}}+(1-s)\zeta^{\mathrm{K}})\big\}^{\mathrm{T}}(\underline{h}_2^{-1}\zeta^{\mathrm{res}})\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_2\\ &\quad + \boldsymbol{l}'_2(sH^{\mathrm{IW}}+(1-s)H^{\mathrm{K}})\big\{[\partial^\beta,\underline{h}_2^{-1}\zeta^{\mathrm{res}}]-(\partial^\beta(\underline{h}_2^{-1}\zeta^{\mathrm{res}}))\big\}^{\mathrm{T}}\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_2\mathrm{d}s.\end{aligned}$$

Therefore, straightforward calculations yield

$$\begin{aligned}\|f^{\mathrm{res}}_{0,\beta}\|_{H^1} &\lesssim \sum_{l=1,2}\underline{\rho}_\ell\big\{(\|\nabla\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_\ell\|_{H^{m-2}}+\|\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^{m-2}})\|\zeta^{\mathrm{res}}\|_{H^k}\\ &\quad + (\|\tilde{\boldsymbol{u}}^{\mathrm{IW}}_\ell\|_{H^m}+\|\boldsymbol{u}^{\mathrm{K}}_\ell\|_{H^m})(\|\nabla\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_\ell\|_{H^m}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^m})\|\zeta^{\mathrm{res}}\|_{H^k}\\ &\quad + (\underline{h}_\ell\delta)^{-2}\|w^{\mathrm{K}}_\ell\|_{H^m}\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^m}\|\zeta^{\mathrm{res}}\|_{H^k}+\|\nabla\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{res}}_\ell\|_{H^{k-1}}+\|\partial_t\tilde{\boldsymbol{\phi}}^{\mathrm{res}\prime}_\ell\|_{H^{k-1}}\\ &\quad + \|\boldsymbol{u}^{\mathrm{K}}_\ell\|_{H^m}(\|\nabla\tilde{\boldsymbol{\phi}}^{\mathrm{res}}_\ell\|_{H^k}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{res}\prime}_\ell\|_{H^k})+(\underline{h}_\ell\delta)^{-2}\|w^{\mathrm{K}}_\ell\|_{H^m}\|\tilde{\boldsymbol{\phi}}^{\mathrm{res}\prime}_\ell\|_{H^k}\\ &\quad + (\|\boldsymbol{u}^{\mathrm{K}}_\ell\|_{H^m}+\|\tilde{\boldsymbol{u}}^{\mathrm{IW}}_\ell\|_{H^m}+\|\nabla\tilde{\boldsymbol{\phi}}^{\mathrm{IW}}_\ell\|_{H^m}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^m})\|\tilde{\boldsymbol{u}}^{\mathrm{IW}}_\ell-\boldsymbol{u}^{\mathrm{K}}_\ell\|_{H^k}\\ &\quad + (\underline{h}_\ell\delta)^{-2}(\|w^{\mathrm{K}}_\ell\|_{H^m}+\|\tilde{w}^{\mathrm{IW}}_\ell\|_{H^m}+\|\tilde{\boldsymbol{\phi}}^{\mathrm{IW}\prime}_\ell\|_{H^m})\|\tilde{w}^{\mathrm{IW}}_\ell-w^{\mathrm{K}}_\ell\|_{H^k}\big\}+\|\mathfrak{r}_0\|_{H^{k+1}}\end{aligned}$$

for $\frac{n}{2}<k\leq m-2$. In view of the above estimates and (7.6)–(7.8) we obtain $\mathscr{F}^{\mathrm{res}}_\beta \lesssim E^{\mathrm{res}}_k+\mathfrak{R}_k$ with $\mathfrak{R}_k := \|\mathfrak{r}_0\|^2_{H^{k+1}}+\sum_{l=1,2}\underline{\rho}_\ell\underline{h}_\ell\|\mathfrak{r}_\ell\|^2_{H^k}$. We note that the multi-index β is assumed to satisfy $1\leq|\beta|\leq k$. As for the case $\beta=0$, we have $\frac{\mathrm{d}}{\mathrm{d}t}E^{\mathrm{res}}_0(t)\lesssim E^{\mathrm{res}}_k(t)$, hence $E^{\mathrm{res}}_0(t)\lesssim\int_0^t E^{\mathrm{res}}_k(\tau)\mathrm{d}\tau$. Summarizing the above estimates we obtain $E^{\mathrm{res}}_k(t)\lesssim\int_0^t(E^{\mathrm{res}}_k(\tau)+\mathfrak{R}_k(\tau))\mathrm{d}\tau$ for $\frac{n}{2}<k\leq m-2$. Putting $k=m-4(N+1)$ and applying Gronwall's inequality and (7.5) in Proposition 7.6 we obtain $E^{\mathrm{res}}_{m-4(N+1)}(t)\lesssim(\underline{h}_1\delta)^{4N+2}+(\underline{h}_2\delta)^{4N+2}$ for $0\leq t\leq\min\{T,T^{\mathrm{IW}}\}$.

It remains to evaluate $\phi_\ell^{\mathrm{IW}} - \phi_\ell^{\mathrm{K}}$ for $\ell = 1,2$. Let $(\boldsymbol{\phi}_1^{\mathrm{IW}}, \boldsymbol{\phi}_2^{\mathrm{IW}})$ be the solution to (3.19) with $(\zeta, \phi_1, \phi_2) = (\zeta^{\mathrm{IW}}, \phi_1^{\mathrm{IW}}, \phi_2^{\mathrm{IW}})$. Then, we have $\phi_\ell^{\mathrm{K}} - \phi_\ell^{\mathrm{IW}} = \boldsymbol{l}_\ell^{\mathrm{K}} \cdot \boldsymbol{\phi}_\ell^{\mathrm{res}} + (\boldsymbol{l}_\ell^{\mathrm{K}} - \boldsymbol{l}_\ell^{\mathrm{IW}}) \cdot \tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}} + \boldsymbol{l}_\ell^{\mathrm{IW}} \cdot (\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}} - \boldsymbol{\phi}_\ell^{\mathrm{IW}})$, so that for any $0 \leq k \leq m-1$

$$\begin{aligned}\|\nabla \phi_\ell^{\mathrm{K}} - \nabla \phi_\ell^{\mathrm{IW}}\|_{H^k} &\lesssim \|\nabla \boldsymbol{\phi}_\ell^{\mathrm{res}}\|_{H^k} + \|\boldsymbol{\phi}_\ell^{\mathrm{res}\prime}\|_{H^k} + \underline{h}_\ell^{-1} \|\zeta^{\mathrm{res}}\|_{H^{k+1}} \|\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}\prime}\|_{H^m} \\ &\quad + \|\nabla(\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}} - \boldsymbol{\phi}_\ell^{\mathrm{IW}})\|_{H^k} + \|\tilde{\boldsymbol{\phi}}_\ell^{\mathrm{IW}\prime} - \boldsymbol{\phi}_\ell^{\mathrm{IW}\prime}\|_{H^k}.\end{aligned}$$

Therefore, the previous result together with Lemma 7.5 implies

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \|\nabla \phi_\ell^{\mathrm{K}} - \nabla \phi_\ell^{\mathrm{IW}}\|_{H^{m-(4N+5)}}^2 \lesssim (\underline{h}_1 \delta)^{4N+2} + (\underline{h}_2 \delta)^{4N+2}.$$

This completes the proof of Theorem 3.9.

8 Approximation of Hamiltonians; proof of Theorem 3.10

As was shown in the companion paper [14, Theorem 8.4], the Kakinuma model (2.18) enjoys a Hamiltonian structure analogous to the one exhibited on the full model for interfacial gravity waves by T. B. Benjamin and T. J. Bridges in [3]. In this section, we will prove Theorem 3.10, which states that the Hamiltonian $\mathscr{H}^{\mathrm{K}}(\zeta, \phi)$ of the Kakinuma model defined in (2.25) approximates the Hamiltonian $\mathscr{H}^{\mathrm{IW}}(\zeta, \phi)$ of the full model defined in (2.22) with an error of order $O((\underline{h}_1 \delta)^{4N+2} + (\underline{h}_2 \delta)^{4N+2})$.

8.1 Preliminary elliptic estimates

We consider the following transmission problem

$$(8.1) \qquad \begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \Phi_\ell = 0 & \text{in} \quad \Omega_\ell \qquad (\ell = 1,2), \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_\ell = 0 & \text{on} \quad \Sigma_\ell \qquad (\ell = 1,2), \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_2 - \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_1 = r_S & \text{on} \quad \Gamma, \\ \underline{\rho}_2 \Phi_2 - \underline{\rho}_1 \Phi_1 = \phi & \text{on} \quad \Gamma, \end{cases}$$

where the rigid-lid Σ_1 of the upper layer Ω_1, the bottom Σ_2 of the lower layer Ω_2, and the interface Γ are defined by $z = \underline{h}_1$, $z = -\underline{h}_2 + b(\boldsymbol{x})$, and $z = \zeta(\boldsymbol{x})$, respectively, $I_\delta := \mathrm{diag}(1, \ldots, 1, \delta^{-1})$, $\nabla_X := (\nabla, \partial_z)^{\mathrm{T}} = (\partial_1, \ldots, \partial_n, \partial_z)$, and $\boldsymbol{n}$ is an upward normal vector, specifically, $\boldsymbol{n} = \boldsymbol{e}_z$ on Σ_1, $\boldsymbol{n} = (-\nabla b, 1)^{\mathrm{T}}$ on Σ_2, and $\boldsymbol{n} = (-\zeta, 1)^{\mathrm{T}}$ on Γ.

Lemma 8.1. *Let c, M be positive constants. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying $\underline{h}_1 \delta, \underline{h}_2 \delta \leq 1$, if $\zeta, b \in W^{1,\infty}$, $H_1 = 1 - \underline{h}_1^{-1} \zeta$, and $H_2 = 1 + \underline{h}_2^{-1} \zeta - \underline{h}_2^{-1} b$ satisfy*

$$\begin{cases} \underline{h}_1^{-1} \|\zeta\|_{W^{1,\infty}} + \underline{h}_2^{-1} \|\zeta\|_{W^{1,\infty}} + \underline{h}_2^{-1} \|b\|_{W^{1,\infty}} \leq M, \\ H_1(\boldsymbol{x}) \geq c, \quad H_2(\boldsymbol{x}) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

then for any (r_S, ϕ) satisfying $\nabla \phi \in H^{-\frac{1}{2}}$ and $(-\Delta)^{-\frac{1}{2}} r_S \in H^{\frac{1}{2}}$ there exists a solution (Φ_1, Φ_2) to the transmission problem (8.1). The solution is unique up to an additive constant of the form $(\underline{\rho}_2 \mathcal{C}, \underline{\rho}_1 \mathcal{C})$ and satisfies

$$(8.2) \qquad \begin{aligned}&\sum_{\ell=1,2} \underline{\rho}_\ell \|I_\delta \nabla_X \Phi_\ell\|_{L^2(\Omega_\ell)}^2 \\ &\leq C\big(\|((\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-1} \Lambda_{1,0} \Lambda_{2,0})^{\frac{1}{2}} \phi\|_{L^2}^2 + \underline{\rho}_1 \underline{\rho}_2 \|(\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-\frac{1}{2}} r_S\|_{L^2}^2\big),\end{aligned}$$

where $\Lambda_{1,0} = \Lambda_1(0,\delta,\underline{h}_1)$ and $\Lambda_{2,0} = \Lambda_2(0,0,\delta,\underline{h}_2)$ are Dirichlet-to-Neumann maps in the case $\zeta(\boldsymbol{x}) \equiv b(\boldsymbol{x}) \equiv 0$. Particularly, if we further impose $\phi \in \mathring{H}^1$, $(-\Delta)^{-\frac{1}{2}} r_S \in H^1$, the natural restrictions (2.14), and $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$ with a positive constant $\underline{h}_{\min}$, then we have

$$\sum_{\ell=1,2} \underline{\rho}_\ell \|I_\delta \nabla_X \Phi_\ell\|_{L^2(\Omega_\ell)}^2 \leq C\|\nabla\phi\|_{L^2}^2 + C \min_{\ell=1,2} \left\{ \frac{\underline{\rho}_\ell}{\underline{h}_\ell} \|((-\Delta)^{-\frac{1}{2}} + \underline{h}_\ell \delta) r_S\|_{L^2}^2 \right\}, \tag{8.3}$$

where the constant C depends also on $\underline{h}_{\min}$.

Proof. The existence and the uniqueness of the solution is standard, so that we focus on deriving the uniform estimate of the solution. To this end, it is convenient to transform the water regions Ω_1 and Ω_2 into simple domains $\Omega_{1,0} = \mathbf{R}^n \times (0, \underline{h}_1)$ and $\Omega_{2,0} = \mathbf{R}^n \times (-\underline{h}_2, 0)$ by using diffeomorphisms $\Theta_\ell(\boldsymbol{x}, z) = (\boldsymbol{x}, \theta_\ell(\boldsymbol{x}, z))\colon \Omega_{\ell,0} \to \Omega_\ell$ ($\ell = 1,2$), respectively, where $\theta_1(\boldsymbol{x}, z) = (1 - \underline{h}_1^{-1}\zeta(\boldsymbol{x}))z + \zeta(\boldsymbol{x})$ and $\theta_2(\boldsymbol{x}, z) = (1 + \underline{h}_2^{-1}(\zeta(\boldsymbol{x}) - b(\boldsymbol{x})))z + \zeta(\boldsymbol{x})$. Put $\tilde{\Phi}_\ell = \Phi_\ell \circ \Theta_\ell$ ($\ell = 1,2$). Then, the transmission problem (8.1) is transformed into

$$\begin{cases} \nabla_X \cdot I_\delta \mathcal{P}_\ell I_\delta \nabla_X \tilde{\Phi}_\ell = 0 & \text{in} \quad \Omega_{\ell,0} \qquad (\ell = 1,2), \\ \boldsymbol{e}_z \cdot I_\delta \mathcal{P}_\ell I_\delta \nabla_X \tilde{\Phi}_\ell = 0 & \text{on} \quad \Sigma_{\ell,0} \qquad (\ell = 1,2), \\ \boldsymbol{e}_z \cdot I_\delta \mathcal{P}_2 I_\delta \nabla_X \tilde{\Phi}_2 - \boldsymbol{e}_z \cdot I_\delta \mathcal{P}_1 I_\delta \nabla_X \tilde{\Phi}_1 = r_S & \text{on} \quad \Gamma_0, \\ \underline{\rho}_2 \tilde{\Phi}_2 - \underline{\rho}_1 \tilde{\Phi}_1 = \phi & \text{on} \quad \Gamma_0, \end{cases}$$

where $\Sigma_{1,0}$, $\Sigma_{2,0}$, and Γ_0 are represented as $z = \underline{h}_1$, $z = -\underline{h}_2$, and $z = 0$, respectively, and

$$\mathcal{P}_\ell := \det\left(\frac{\partial\Theta_\ell}{\partial X}\right) I_\delta^{-1} \left(\frac{\partial\Theta_\ell}{\partial X}\right)^{-1} I_\delta^2 \left(\left(\frac{\partial\Theta_\ell}{\partial X}\right)^{-1}\right)^{\mathrm{T}} I_\delta^{-1} \qquad (\ell = 1,2).$$

We note that $\|I_\delta \nabla_X \Phi_\ell\|_{L^2(\Omega_\ell)} \simeq \|I_\delta \nabla_X \tilde{\Phi}_\ell\|_{L^2(\Omega_{\ell,0})}$ ($\ell = 1,2$). Let (Ψ_1, Ψ_2) be a solution to the transmission problem

$$\begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \Psi_\ell = 0 & \text{in} \quad \Omega_{\ell,0} \qquad (\ell = 1,2), \\ \boldsymbol{e}_z \cdot I_\delta^2 \nabla_X \Psi_\ell = 0 & \text{on} \quad \Sigma_{\ell,0} \qquad (\ell = 1,2), \\ \boldsymbol{e}_z \cdot I_\delta^2 \nabla_X \Psi_2 - \boldsymbol{e}_z \cdot I_\delta^2 \nabla_X \Psi_1 = r_S & \text{on} \quad \Gamma_0, \\ \underline{\rho}_2 \Psi_2 - \underline{\rho}_1 \Psi_1 = \phi & \text{on} \quad \Gamma_0, \end{cases}$$

and put $\Phi_\ell^{\mathrm{res}} = \tilde{\Phi}_\ell - \Psi_\ell$ ($\ell = 1,2$). Then, we can decompose

$$|I_\delta \nabla_X \Phi_\ell^{\mathrm{res}}|^2 - I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot (I - \mathcal{P}_\ell) I_\delta \nabla_X \tilde{\Phi}_\ell = \nabla_X \Phi_\ell^{\mathrm{res}} \cdot \{(I_\delta \mathcal{P}_\ell I_\delta \nabla_X \tilde{\Phi}_\ell - I_\delta^2 \nabla_X \Psi_\ell)\}$$

for $\ell = 1,2$ and $\underline{\rho}_1 \Phi_1^{\mathrm{res}} = \underline{\rho}_2 \Phi_2^{\mathrm{res}}$ on $z = 0$. Therefore, denoting the unit outward normal vector to $\partial\Omega_{\ell,0}$ by N_ℓ ($\ell = 1,2$) we have

$$\begin{aligned} &\sum_{\ell=1,2} \underline{\rho}_\ell \int_{\Omega_{\ell,0}} \big(|I_\delta \nabla_X \Phi_\ell^{\mathrm{res}}|^2 - I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot (I - \mathcal{P}_\ell) I_\delta \nabla_X \tilde{\Phi}_\ell\big) \mathrm{d}X \\ &\quad = \sum_{\ell=1,2} \int_{\partial\Omega_{\ell,0}} \underline{\rho}_\ell \Phi_\ell^{\mathrm{res}} (N_\ell \cdot I_\delta \mathcal{P}_\ell I_\delta \nabla_X \tilde{\Phi}_\ell - N_\ell \cdot I_\delta^2 \nabla_X \Psi_\ell) \mathrm{d}S \\ &\quad = \sum_{\ell=1,2} \int_{\mathbf{R}^n} \underline{\rho}_1 \big[\Phi_1^{\mathrm{res}} \{(\boldsymbol{e}_z \cdot I_\delta \mathcal{P}_2 I_\delta \nabla_X \tilde{\Phi}_2 - \boldsymbol{e}_z \cdot I_\delta^2 \nabla_X \Psi_2) \\ &\qquad\qquad - (\boldsymbol{e}_z \cdot I_\delta \mathcal{P}_1 I_\delta \nabla_X \tilde{\Phi}_1 - \boldsymbol{e}_z \cdot I_\delta^2 \nabla_X \Psi_1)\}\big]\big|_{z=0} \mathrm{d}\boldsymbol{x} \\ &\quad = 0, \end{aligned}$$

so that we obtain

$$\sum_{\ell=1,2} \underline{\rho}_\ell \int_{\Omega_{\ell,0}} |I_\delta \nabla_X \Phi_\ell^{\mathrm{res}}|^2 \mathrm{d}X = \sum_{\ell=1,2} \underline{\rho}_\ell \int_{\Omega_{\ell,0}} I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot (I - \mathcal{P}_\ell) I_\delta \nabla_X \tilde{\Phi}_\ell \mathrm{d}X.$$

Similarly, in view of the decomposition

$$\begin{aligned} I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot \mathcal{P}_\ell I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} - I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot (I - \mathcal{P}_\ell) I_\delta \nabla_X \Psi_\ell \\ = \nabla_X \Phi_\ell^{\mathrm{res}} \cdot \{ (I_\delta \mathcal{P}_\ell I_\delta \nabla_X \tilde{\Phi}_\ell - I_\delta^2 \nabla_X \Psi_\ell) \} \end{aligned}$$

for $\ell = 1, 2$, we obtain

$$\sum_{\ell=1,2} \underline{\rho}_\ell \int_{\Omega_{\ell,0}} I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot \mathcal{P}_\ell I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} = \sum_{\ell=1,2} \underline{\rho}_\ell \int_{\Omega_{\ell,0}} I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \cdot (I - \mathcal{P}_\ell) I_\delta \nabla_X \Psi_\ell \mathrm{d}X.$$

It follows from these two identities that

$$\sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \Phi_\ell^{\mathrm{res}} \|_{L^2(\Omega_{\ell,0})}^2 \lesssim \min \left\{ \sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \tilde{\Phi}_\ell \|_{L^2(\Omega_{\ell,0})}^2, \sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \Psi_\ell \|_{L^2(\Omega_{\ell,0})}^2 \right\},$$

which yields the equivalence

$$\sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \tilde{\Phi}_\ell \|_{L^2(\Omega_{\ell,0})}^2 \simeq \sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \Psi_\ell \|_{L^2(\Omega_{\ell,0})}^2.$$

Therefore, it is sufficient to evaluate the right-hand side of the above equation. In other words, the evaluation is reduced to the simple case $\zeta(\boldsymbol{x}) \equiv b(\boldsymbol{x}) \equiv 0$.

Putting $\psi_\ell = \Psi_\ell|_{z=0}$ ($\ell = 1, 2$), we see that

$$\sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \Psi_\ell \|_{L^2(\Omega_{\ell,0})}^2 = \sum_{\ell=1,2} \underline{\rho}_\ell (\Lambda_{\ell,0} \psi_\ell, \psi_\ell)_{L^2}$$

and that

$$\begin{cases} \Lambda_{1,0} \psi_1 + \Lambda_{2,0} \psi_2 = r_S, \\ \underline{\rho}_2 \psi_2 - \underline{\rho}_1 \psi_1 = \phi. \end{cases}$$

Particularly, we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-1} \begin{pmatrix} -\Lambda_{2,0} \phi + \underline{\rho}_2 r_S \\ \Lambda_{1,0} \phi + \underline{\rho}_1 r_S \end{pmatrix}.$$

Therefore,

$$\sum_{\ell=1,2} \underline{\rho}_\ell \| I_\delta \nabla_X \Psi_\ell \|_{L^2(\Omega_{\ell,0})}^2 = \begin{cases} \| ((\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-1} \Lambda_{1,0} \Lambda_{2,0})^{\frac{1}{2}} \phi \|_{L^2}^2 & \text{if } r_S = 0, \\ \underline{\rho}_1 \underline{\rho}_2 \| (\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-\frac{1}{2}} r_S \|_{L^2}^2 & \text{if } \phi = 0. \end{cases}$$

Hence, by the linearity of the problem we obtain (8.2).

Finally, in order to show (8.3) it is sufficient to evaluate the symbols of the Fourier multipliers $(\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-1} \Lambda_{1,0} \Lambda_{2,0}$ and $\underline{\rho}_1 \underline{\rho}_2 (\underline{\rho}_1 \Lambda_{2,0} + \underline{\rho}_2 \Lambda_{1,0})^{-1}$. We recall that the symbol of the

Dirichlet-to-Neumann map $\Lambda_{\ell,0}$ is given by $\sigma(\Lambda_{\ell,0}) = \delta^{-1}|\boldsymbol{\xi}|\tanh(\underline{h}_\ell\delta|\boldsymbol{\xi}|)$ for $\ell = 1, 2$. In view of $0 \le \tanh\xi \le \xi$ for $\xi \ge 0$, we have

$$\begin{aligned}
\sigma((\underline{\rho}_1\Lambda_{2,0} + \underline{\rho}_2\Lambda_{1,0})^{-1}\Lambda_{1,0}\Lambda_{2,0}) &\le \min\left\{\frac{\sigma(\Lambda_{1,0})}{\underline{\rho}_1}, \frac{\sigma(\Lambda_{2,0})}{\underline{\rho}_2}\right\} \\
&\le \min\left\{\frac{\underline{h}_1}{\underline{\rho}_1}, \frac{\underline{h}_2}{\underline{\rho}_2}\right\}|\boldsymbol{\xi}|^2 \\
&\le 2|\boldsymbol{\xi}|^2,
\end{aligned}$$

where we used (2.15). In view of $\tanh\xi \simeq (1+\xi)^{-1}\xi$ for $\xi \ge 0$ and the relation (2.14), we have

$$\begin{aligned}
\sigma(\underline{\rho}_1\underline{\rho}_2(\underline{\rho}_1\Lambda_{2,0} + \underline{\rho}_2\Lambda_{1,0})^{-1}) &\simeq \frac{\underline{\rho}_1\underline{\rho}_2}{\underline{h}_1\underline{h}_2}\frac{(1+\underline{h}_1\delta|\boldsymbol{\xi}|)(1+\underline{h}_2\delta|\boldsymbol{\xi}|)}{(1+\delta|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2} \\
&\lesssim \min\left\{\frac{\underline{\rho}_1}{\underline{h}_1}\underline{\rho}_2\frac{1+\underline{h}_1\delta|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|^2}, \frac{\underline{\rho}_2}{\underline{h}_2}\underline{\rho}_1\frac{1+\underline{h}_2\delta|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|^2}\right\} \\
&\lesssim \min\left\{\frac{\underline{\rho}_1}{\underline{h}_1}(|\boldsymbol{\xi}|^{-1} + \underline{h}_1\delta)^2, \frac{\underline{\rho}_2}{\underline{h}_2}(|\boldsymbol{\xi}|^{-1} + \underline{h}_2\delta)^2\right\},
\end{aligned}$$

where we used $1 \lesssim \underline{h}_1, \underline{h}_2$. These estimates imply (8.3). The proof is complete. □

8.2 Completion of the proof of Theorem 3.10

Now we are ready to prove Theorem 3.10. We recall the definitions (3.3) of $\boldsymbol{l}_1(H_1)$, $\boldsymbol{l}_2(H_2)$ and (3.6) of the operators $\mathcal{L}_{1,i}(H_1,\delta,\underline{h}_1)$ and $\mathcal{L}_{2,i}(H_2,b,\delta,\underline{h}_2)$. These depend on N, so that we denote them by $\boldsymbol{l}_1^{(N)}(H_1)$, $\boldsymbol{l}_2^{(N)}(H_2)$ and $\mathcal{L}_{1,i}^{(N)}(H_1,\delta,\underline{h}_1)$ and $\mathcal{L}_{2,i}^{(N)}(H_2,b,\delta,\underline{h}_2)$, respectively, in the following argument. For given (ζ,ϕ), let Φ be the solution to the transmission problem (8.1) with $r_S = 0$ and let $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and $(\tilde{\boldsymbol{\phi}}_1, \tilde{\boldsymbol{\phi}}_2)$ be the solutions to the problems

$$\begin{cases}
\mathcal{L}_{1,i}^{(N)}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 = 0 \quad \text{for} \quad i = 1,2,\dots,N, \\
\mathcal{L}_{2,i}^{(N)}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0 \quad \text{for} \quad i = 1,2,\dots,N^*, \\
\underline{h}_1\mathcal{L}_{1,0}^{(N)}(H_1,\delta,\underline{h}_1)\boldsymbol{\phi}_1 + \underline{h}_2\mathcal{L}_{2,0}^{(N)}(H_2,b,\delta,\underline{h}_2)\boldsymbol{\phi}_2 = 0, \\
\underline{\rho}_2\boldsymbol{l}_2^{(N)}(H_2)\cdot\boldsymbol{\phi}_2 - \underline{\rho}_1\boldsymbol{l}_1^{(N)}(H_1)\cdot\boldsymbol{\phi}_1 = \phi
\end{cases}$$

and

$$\begin{cases}
\mathcal{L}_{1,i}^{(2N+2)}(H_1,\delta,\underline{h}_1)\tilde{\boldsymbol{\phi}}_1 = 0 \quad \text{for} \quad i = 1,2,\dots,2N+2, \\
\mathcal{L}_{2,i}^{(2N+2)}(H_2,b,\delta,\underline{h}_2)\tilde{\boldsymbol{\phi}}_2 = 0 \quad \text{for} \quad i = 1,2,\dots,2N^*+2, \\
\underline{h}_1\mathcal{L}_{1,0}^{(2N+2)}(H_1,\delta,\underline{h}_1)\tilde{\boldsymbol{\phi}}_1 + \underline{h}_2\mathcal{L}_{2,0}^{(2N+2)}(H_2,b,\delta,\underline{h}_2)\tilde{\boldsymbol{\phi}}_2 = 0, \\
\underline{\rho}_2\boldsymbol{l}_2^{(2N+2)}(H_2)\cdot\tilde{\boldsymbol{\phi}}_2 - \underline{\rho}_1\boldsymbol{l}_1^{(2N+2)}(H_1)\cdot\tilde{\boldsymbol{\phi}}_1 = \phi,
\end{cases}$$

respectively, and define $(\Phi_1^{\mathrm{app}}, \Phi_2^{\mathrm{app}})$ and $(\tilde{\Phi}_1^{\mathrm{app}}, \tilde{\Phi}_2^{\mathrm{app}})$ by (2.24) and

$$\begin{cases}
\tilde{\Phi}_1^{\mathrm{app}}(\boldsymbol{x},z) = \displaystyle\sum_{i=0}^{2N+2}(1-\underline{h}_1^{-1}z)^{2i}\tilde{\phi}_{1,i}(\boldsymbol{x}), \\
\tilde{\Phi}_2^{\mathrm{app}}(\boldsymbol{x},z) = \displaystyle\sum_{i=0}^{2N^*+2}(1+\underline{h}_2^{-1}(z-b(\boldsymbol{x})))^{p_i}\tilde{\phi}_{2,i}(\boldsymbol{x}),
\end{cases}$$

respectively. Then, by the definitions of the Hamiltonian functionals $\mathscr{H}^{\rm IW}(\zeta,\phi)$ and $\mathscr{H}^{\rm K}(\zeta,\phi)$ given in Section 2.3, we have

$$
\begin{aligned}
2(\mathscr{H}^{\rm IW}(\zeta,\phi)-\mathscr{H}^{\rm K}(\zeta,\phi)) &= \sum_{\ell=1,2}\underline{\rho}_\ell\int_{\Omega_\ell}(|I_\delta\nabla_X\Phi_\ell|^2-|I_\delta\nabla_X\Phi_\ell^{\rm app}|^2){\rm d}X \\
&= \sum_{\ell=1,2}\underline{\rho}_\ell\int_{\Omega_\ell}(|I_\delta\nabla_X\Phi_\ell|^2-|I_\delta\nabla_X\tilde{\Phi}_\ell^{\rm app}|^2){\rm d}X \\
&\quad +\sum_{\ell=1,2}\underline{\rho}_\ell\int_{\Omega_\ell}(|I_\delta\nabla_X\tilde{\Phi}_\ell^{\rm app}|^2-|I_\delta\nabla_X\Phi_\ell^{\rm app}|^2){\rm d}X \\
&=: I_1+I_2.
\end{aligned}
$$

We will evaluate I_1 and I_2, separately.

In order to evaluate I_1, we put $\Phi_\ell^{\rm res}=\Phi_\ell-\tilde{\Phi}_\ell^{\rm app}$ $(\ell=1,2)$, so that

$$
\begin{aligned}
|I_1| &= \left|\sum_{\ell=1,2}\underline{\rho}_\ell\int_{\Omega_\ell}I_\delta\nabla_X\Phi_\ell^{\rm res}\cdot I_\delta\nabla_X(\Phi_\ell+\tilde{\Phi}_\ell^{\rm app}){\rm d}X\right| \\
&\leq \sum_{\ell=1,2}\underline{\rho}_\ell\|I_\delta\nabla_X\Phi_\ell^{\rm res}\|_{L^2(\Omega_\ell)}(\|I_\delta\nabla_X\Phi_\ell\|_{L^2(\Omega_\ell)}+\|I_\delta\nabla_X\tilde{\Phi}_\ell^{\rm app}\|_{L^2(\Omega_\ell)}).
\end{aligned}
$$

It follows from Lemma 8.1 that $\sum_{\ell=1,2}\underline{\rho}_\ell\|I_\delta\nabla_X\Phi_\ell\|_{L^2(\Omega_\ell)}^2\lesssim\|\nabla\phi\|_{L^2}^2$. We see also that

$$
\begin{aligned}
\sum_{\ell=1,2}\underline{\rho}_\ell\|I_\delta\nabla_X\tilde{\Phi}_\ell^{\rm app}\|_{L^2(\Omega_\ell)}^2 &= \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(L_\ell^{(2N+2)}\tilde{\boldsymbol{\phi}}_\ell,\tilde{\boldsymbol{\phi}}_\ell)_{L^2} \\
&\lesssim \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla\tilde{\boldsymbol{\phi}}_\ell\|_{L^2}^2+(\underline{h}_\ell\delta)^{-2}\|\tilde{\boldsymbol{\phi}}'\|_{L^2}^2) \\
&\lesssim \|\nabla\phi\|_{L^2}^2,
\end{aligned}
$$

where we used Lemma 5.1 and (2.15). In order to evaluate $\|I_\delta\nabla_X\Phi_\ell^{\rm res}\|_{L^2(\Omega_\ell)}$, we first notice that $(\Phi_1^{\rm res},\Phi_2^{\rm res})$ satisfy

$$
\begin{cases}
\nabla_X\cdot I_\delta^2\nabla_X\Phi_\ell^{\rm res}=R_\ell & \text{in}\quad \Omega_\ell \qquad (\ell=1,2), \\
\boldsymbol{n}\cdot I_\delta^2\nabla_X\Phi_1^{\rm res}=0 & \text{on}\quad \Sigma_1, \\
\boldsymbol{n}\cdot I_\delta^2\nabla_X\Phi_2^{\rm res}=\underline{h}_2 r_B & \text{on}\quad \Sigma_2, \\
\underline{\rho}_2\Phi_2^{\rm res}-\underline{\rho}_1\Phi_1^{\rm res}=0 & \text{on}\quad \Gamma, \\
\Lambda_1[\Phi_1^{\rm res}|_{z=\zeta}]+\Lambda_2[\Phi_2^{\rm res}|_{z=\zeta}]=r_S,
\end{cases}
$$

where

$$
\begin{cases}
R_\ell=-\nabla_X\cdot I_\delta^2\nabla_X\tilde{\Phi}_\ell^{\rm app} \qquad (\ell=1,2), \\
r_B=-\underline{h}_2^{-1}(-\nabla b,1)^{\rm T}\cdot I_\delta^2(\nabla_X\tilde{\Phi}_\ell^{\rm app})|_{z=-\underline{h}_2+b}, \\
r_S=\displaystyle\sum_{\ell=1,2}(\underline{h}_\ell\Lambda_\ell^{(2N+2)}-\Lambda_\ell)[\tilde{\Phi}_\ell^{\rm app}|_{z=\zeta}].
\end{cases}
$$

Here, we note that R_ℓ $(\ell=1,2)$ can be written the form

$$
\begin{cases}
R_1(\boldsymbol{x},z)=\displaystyle\sum_{i=0}^{2N+2}(1-\underline{h}_1^{-1}z)^{2i}r_{1,i}(\boldsymbol{x}), \\
R_2(\boldsymbol{x},z)=\displaystyle\sum_{i=0}^{2N^*+2}(1+\underline{h}_2^{-1}(z-b(\boldsymbol{x})))^{p_i}r_{2,i}(\boldsymbol{x}).
\end{cases}
$$

Estimates for the residuals $(r_{1,0}, r_{1,1}, \dots, r_{1,2N+2})$, $(r_{2,0}, r_{2,0}, \dots, r_{2,2N^*+2})$, and r_B were given in [20, Lemmas 6.4 and 6.9] and their proofs. In fact, we have

$$\begin{aligned}\|(r_{1,0}, r_{1,1}, \dots, r_{1,2N+2})\|_{L^2} &\lesssim \|\tilde{\phi}_{1,2N+2}\|_{H^2} \\ &\lesssim (\underline{h}_1\delta)^{4N+2}\|\nabla\tilde{\boldsymbol{\phi}}_1\|_{H^{4N+3}}\end{aligned}$$

and

$$\begin{aligned}\|(r_{2,0}, r_{2,1}, \dots, r_{2,2N^*+2})\|_{L^2} + \|r_B\|_{L^2} &\lesssim \|(\tilde{\phi}_{2,2N^*+1}, \tilde{\phi}_{2,2N^*+2})\|_{H^2} \\ &\lesssim (\underline{h}_2\delta)^{4N+2}(\|\nabla\tilde{\boldsymbol{\phi}}_2\|_{H^{4N+3}} + \|\tilde{\boldsymbol{\phi}}_2'\|_{H^{4N+3}}).\end{aligned}$$

We decompose $\Phi_\ell^{\mathrm{res}} = \Phi_\ell^{\mathrm{res},1} + \Phi_\ell^{\mathrm{res},2}$, where $(\Phi_1^{\mathrm{res},1}, \Phi_2^{\mathrm{res},1})$ is a unique solution to the problem

$$\begin{cases}\nabla_X \cdot I_\delta^2 \nabla_X \Phi_\ell^{\mathrm{res},1} = R_\ell & \text{in} \quad \Omega_\ell \qquad (\ell = 1,2), \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_1^{\mathrm{res},1} = 0 & \text{on} \quad \Sigma_1, \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_2^{\mathrm{res},1} = \underline{h}_2 r_B & \text{on} \quad \Sigma_2, \\ \Phi_\ell^{\mathrm{res},1} = 0 & \text{on} \quad \Gamma \qquad (\ell = 1,2),\end{cases}$$

so that $(\Phi_1^{\mathrm{res},2}, \Phi_2^{\mathrm{res},2})$ satisfy

$$\begin{cases}\nabla_X \cdot I_\delta^2 \nabla_X \Phi_\ell^{\mathrm{res},2} = 0 & \text{in} \quad \Omega_\ell \qquad (\ell = 1,2), \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_\ell^{\mathrm{res},2} = 0 & \text{on} \quad \Sigma_\ell \qquad (\ell = 1,2), \\ \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_2^{\mathrm{res},2} - \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_1^{\mathrm{res},2} = r_S & \text{on} \quad \Gamma, \\ \underline{\rho}_2 \Phi_2^{\mathrm{res},2} - \underline{\rho}_1 \Phi_1^{\mathrm{res},2} = 0 & \text{on} \quad \Gamma,\end{cases} \tag{8.4}$$

where we used the relations $\Lambda_1[\Phi_1^{\mathrm{res},2}|_{z=\zeta}] = -\boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_1^{\mathrm{res},2}$ and $\Lambda_2[\Phi_2^{\mathrm{res},2}|_{z=\zeta}] = \boldsymbol{n} \cdot I_\delta^2 \nabla_X \Phi_2^{\mathrm{res},2}$ on Γ. It is easy to see that

$$\begin{aligned}\|I_\delta \nabla_X \Phi_1^{\mathrm{res},1}\|_{L^2(\Omega_1)}^2 &\lesssim (\underline{h}_1\delta)^2 \|R_1\|_{L^2(\Omega_1)}^2 \\ &\lesssim \underline{h}_1(\underline{h}_1\delta)^2 \|(r_{1,0}, r_{1,1}, \dots, r_{1,2N+2})\|_{L^2} \\ &\lesssim \underline{h}_1(\underline{h}_1\delta)^{2(4N+3)} \|\nabla\tilde{\boldsymbol{\phi}}_1\|_{H^{4N+3}}^2\end{aligned}$$

and that

$$\begin{aligned}\|I_\delta \nabla_X \Phi_2^{\mathrm{res},1}\|_{L^2(\Omega_2)}^2 &\lesssim \underline{h}_2(\underline{h}_2\delta)^2 (\underline{h}_2^{-1}\|R_2\|_{L^2(\Omega_2)}^2 + \|r_B\|_{L^2}^2) \\ &\lesssim \underline{h}_2(\underline{h}_2\delta)^2 (\|(r_{2,0}, r_{2,1}, \dots, r_{2,2N^*+2})\|_{L^2} + \|r_B\|_{L^2}^2) \\ &\lesssim \underline{h}_2(\underline{h}_2\delta)^{2(4N+3)} (\|\nabla\tilde{\boldsymbol{\phi}}_2\|_{H^{4N+3}} + \|\tilde{\boldsymbol{\phi}}_2'\|_{H^{4N+3}}).\end{aligned}$$

Therefore, by Lemma 5.1 together with (2.15) we have

$$\sum_{\ell=1,2} \underline{\rho}_\ell \|I_\delta \nabla_X \Phi_\ell^{\mathrm{res},1}\|_{L^2(\Omega_\ell)}^2 \lesssim ((\underline{h}_1\delta)^{4N+3} + (\underline{h}_2\delta)^{4N+3})^2 \|\nabla\phi\|_{H^{4N+3}}^2.$$

On the other hand, it follows from Lemmas 8.1, 4.5, 7.2, and 5.1 that

$$\begin{aligned}
\sum_{\ell=1,2} \underline{\rho}_\ell \|I_\delta \nabla_X \Phi_\ell^{\mathrm{res},2}\|_{L^2(\Omega_\ell)}^2 &\lesssim \min_{\ell=1,2} \frac{\underline{\rho}_\ell}{\underline{h}_\ell} \|((-\Delta)^{-\frac12} + \underline{h}_\ell\delta) r_S\|_{L^2}^2 \\
&\lesssim \sum_{\ell=1,2} \frac{\underline{\rho}_\ell}{\underline{h}_\ell} \|((-\Delta)^{-\frac12} + \underline{h}_\ell\delta)(\underline{h}_\ell \Lambda_\ell^{(2N+2)} - \Lambda_\ell)[\tilde{\Phi}_\ell^{\mathrm{app}}|_{z=\zeta}]\|_{L^2}^2 \\
&\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{2(4N+2)} \|\nabla(\tilde{\Phi}_\ell^{\mathrm{app}}|_{z=\zeta})\|_{H^{4N+3}}^2 \\
&\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{2(4N+2)} (\|\nabla \tilde{\boldsymbol{\phi}}_\ell\|_{H^{4N+3}}^2 + \|\tilde{\boldsymbol{\phi}}_\ell'\|_{H^{4N+3}}^2) \\
&\lesssim ((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})^2 \|\nabla\phi\|_{H^{4N+3}}^2.
\end{aligned}$$

Summarizing the above estimates, we obtain $|I_1| \lesssim ((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2}) \|\nabla\phi\|_{H^{4N+3}} \|\nabla\phi\|_{L^2}$.

We proceed to evaluate I_2, which can be written as

$$\begin{aligned}
I_2 &= \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (L_\ell^{(2N+2)} \tilde{\boldsymbol{\phi}}_\ell, \tilde{\boldsymbol{\phi}}_\ell)_{L^2} - \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (L_\ell^{(N)} \boldsymbol{\phi}_\ell, \boldsymbol{\phi}_\ell)_{L^2} \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}$$

In view of (3.8), we see that

$$\begin{aligned}
I_{2,1} &= \underline{\rho}_1 \underline{h}_1 (\mathcal{L}_{1,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_1, \boldsymbol{l}_1^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_1)_{L^2} + \underline{\rho}_2 \underline{h}_2 (\mathcal{L}_{2,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_2, \boldsymbol{l}_2^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_2)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_2, \underline{\rho}_2 \boldsymbol{l}_2^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_2 - \underline{\rho}_1 \boldsymbol{l}_1^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_1)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_2, \phi)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_2, \underline{\rho}_2 \boldsymbol{l}_2^{(N)} \cdot \boldsymbol{\phi}_2 - \underline{\rho}_1 \boldsymbol{l}_1^{(N)} \cdot \boldsymbol{\phi}_1)_{L^2} \\
&= \underline{\rho}_1 \underline{h}_1 (\mathcal{L}_{1,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_1, \boldsymbol{l}_1^{(N)} \cdot \boldsymbol{\phi}_1)_{L^2} + \underline{\rho}_2 \underline{h}_2 (\mathcal{L}_{2,0}^{(2N+2)} \tilde{\boldsymbol{\phi}}_2, \boldsymbol{l}_2^{(N)} \cdot \boldsymbol{\phi}_2)_{L^2} \\
&= \underline{\rho}_1 \underline{h}_1 \sum_{i=0}^{N} \sum_{j=0}^{2N+2} (L_{1,ij} \tilde{\phi}_{1,j}, \phi_{1,i})_{L^2} + \underline{\rho}_2 \underline{h}_2 \sum_{i=0}^{N^*} \sum_{j=0}^{2N^*+2} (L_{2,ij} \tilde{\phi}_{2,j}, \phi_{2,i})_{L^2} \\
&= \underline{\rho}_1 \underline{h}_1 \sum_{i=0}^{N} \sum_{j=0}^{2N+2} (L_{1,ji} \phi_{1,i}, \tilde{\phi}_{1,j})_{L^2} + \underline{\rho}_2 \underline{h}_2 \sum_{i=0}^{N^*} \sum_{j=0}^{2N^*+2} (L_{2,ji} \phi_{2,i}, \tilde{\phi}_{2,j})_{L^2},
\end{aligned}$$

where we used $L_{\ell,ij}^* = L_{\ell,ji}$. Similarly, we see also that

$$\begin{aligned}
I_{2,2} &= \underline{\rho}_1 \underline{h}_1 (\mathcal{L}_{1,0}^{(N)} \boldsymbol{\phi}_1, \boldsymbol{l}_1^{(N)} \cdot \boldsymbol{\phi}_1)_{L^2} + \underline{\rho}_2 \underline{h}_2 (\mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \boldsymbol{l}_2^{(N)} \cdot \boldsymbol{\phi}_2)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \underline{\rho}_2 \boldsymbol{l}_2^{(N)} \cdot \boldsymbol{\phi}_2 - \underline{\rho}_1 \boldsymbol{l}_1^{(N)} \cdot \boldsymbol{\phi}_1)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \phi)_{L^2} \\
&= (\underline{h}_2 \mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \underline{\rho}_2 \boldsymbol{l}_2^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_2 - \underline{\rho}_1 \boldsymbol{l}_1^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_1)_{L^2} \\
&= \underline{\rho}_1 \underline{h}_1 (\mathcal{L}_{1,0}^{(N)} \boldsymbol{\phi}_1, \boldsymbol{l}_1^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_1)_{L^2} + \underline{\rho}_2 \underline{h}_2 (\mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \boldsymbol{l}_2^{(2N+2)} \cdot \tilde{\boldsymbol{\phi}}_2)_{L^2} \\
&= \underline{\rho}_1 \underline{h}_1 \sum_{j=0}^{2N+2} (H_1^{2j} \mathcal{L}_{1,0}^{(N)} \boldsymbol{\phi}_1, \tilde{\phi}_{1,j})_{L^2} + \underline{\rho}_2 \underline{h}_2 \sum_{j=0}^{2N^*+2} (H_2^{p_j} \mathcal{L}_{2,0}^{(N)} \boldsymbol{\phi}_2, \tilde{\phi}_{2,j})_{L^2}.
\end{aligned}$$

Here, it follows from (3.8) that $H_1^{2j}\mathcal{L}_{1,0}^{(N)}\boldsymbol{\phi}_1 = \sum_{i=0}^{N} L_{1,ji}\phi_{1,i}$ and $H_2^{p_j}\mathcal{L}_{2,0}^{(N)}\boldsymbol{\phi}_1 = \sum_{i=0}^{N^*} L_{2,ji}\phi_{2,i}$ hold only for $j=0,1,\ldots,N$ and for $j=0,1,\ldots,N^*$, respectively. Therefore, we have

$$
\begin{aligned}
I_{2,2} &= \underline{\rho}_1\underline{h}_1\sum_{i=0}^{N}\sum_{j=0}^{N}(L_{1,ji}\phi_{1,i},\tilde{\phi}_{1,j})_{L^2} + \underline{\rho}_2\underline{h}_2\sum_{i=0}^{N^*}\sum_{j=0}^{N^*}(L_{2,ji}\phi_{2,i},\tilde{\phi}_{2,j})_{L^2} \\
&\quad + \underline{\rho}_1\underline{h}_1\sum_{i=0}^{N}\sum_{j=N+1}^{2N+2}(H_1^{2j}L_{1,0i}\phi_{1,i},\tilde{\phi}_{1,j})_{L^2} + \underline{\rho}_2\underline{h}_2\sum_{i=0}^{N^*}\sum_{j=N^*+1}^{2N^*+2}(H_2^{p_j}L_{2,0i}\phi_{2,i},\tilde{\phi}_{2,j})_{L^2},
\end{aligned}
$$

so that

$$
\begin{aligned}
I_2 &= \underline{\rho}_1\underline{h}_1\sum_{i=0}^{N}\sum_{j=N+1}^{2N+2}((L_{1,ji}-H_1^{2j}L_{1,0i})\phi_{1,i},\tilde{\phi}_{1,j})_{L^2} \\
&\quad + \underline{\rho}_2\underline{h}_2\sum_{i=0}^{N^*}\sum_{j=N^*+1}^{2N^*+2}((L_{2,ji}-H_2^{p_j}L_{2,0i})\phi_{2,i},\tilde{\phi}_{2,j})_{L^2} \\
&= \underline{\rho}_1\underline{h}_1\sum_{i=0}^{N}\sum_{j=N+1}^{2N+2}((L_{1,ji}-H_1^{2j}L_{1,0i})(\phi_{1,i}-\tilde{\phi}_{1,i}),\tilde{\phi}_{1,j})_{L^2} \\
&\quad + \underline{\rho}_2\underline{h}_2\sum_{i=0}^{N^*}\sum_{j=N^*+1}^{2N^*+2}((L_{2,ji}-H_2^{p_j}L_{2,0i})(\phi_{2,i}-\tilde{\phi}_{2,i}),\tilde{\phi}_{2,j})_{L^2} \\
&\quad - \underline{\rho}_1\underline{h}_1\sum_{i=N+1}^{2N+2}\sum_{j=N+1}^{2N+2}((L_{1,ji}-H_1^{2j}L_{1,0i})\tilde{\phi}_{1,i},\tilde{\phi}_{1,j})_{L^2} \\
&\quad - \underline{\rho}_2\underline{h}_2\sum_{i=N^*+1}^{2N^*+2}\sum_{j=N^*+1}^{2N^*+2}((L_{2,ji}-H_2^{p_j}L_{2,0i})\tilde{\phi}_{2,i},\tilde{\phi}_{2,j})_{L^2}.
\end{aligned}
$$

Hence, denoting by $\boldsymbol{\varphi}_1 = (\varphi_{1,0},\varphi_{1,1},\ldots,\varphi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\varphi}_2 = (\varphi_{2,0},\varphi_{2,1},\ldots,\varphi_{2,N^*})^{\mathrm{T}}$ with $\varphi_{\ell,i} = \phi_{\ell,i}-\tilde{\phi}_{\ell,i}$ we obtain

$$
\begin{aligned}
|I_2| &\lesssim \sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell(\|\nabla\boldsymbol{\varphi}_\ell\|_{L^2}^2 + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\varphi}_\ell'\|_{L^2}^2) \\
&\quad + \underline{\rho}_1\underline{h}_1\|(\tilde{\phi}_{1,N+1},\tilde{\phi}_{1,N+2},\ldots,\tilde{\phi}_{1,2N+2})\|_{H^1}^2 \\
&\quad + \underline{\rho}_2\underline{h}_2\|(\tilde{\phi}_{2,N^*+1},\tilde{\phi}_{2,N^*+2},\ldots,\tilde{\phi}_{2,2N^*+2})\|_{H^1}^2 \\
&\quad + \underline{\rho}_1\underline{h}_1(\underline{h}_1\delta)^{-2}\|(\tilde{\phi}_{1,N+1},\tilde{\phi}_{1,N+2},\ldots,\tilde{\phi}_{1,2N+2})\|_{L^2}^2 \\
&\quad + \underline{\rho}_2\underline{h}_2(\underline{h}_2\delta)^{-2}\|(\tilde{\phi}_{2,N^*+1},\tilde{\phi}_{2,N^*+2},\ldots,\tilde{\phi}_{2,2N^*+2})\|_{L^2}^2.
\end{aligned}
$$

Here, we note that $(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)$ satisfy

$$
\begin{cases}
\mathcal{L}_{1,i}^{(N)}\boldsymbol{\varphi}_1 = r_{1,i} \quad \text{for} \quad i=0,1,\ldots,N, \\
\mathcal{L}_{2,i}^{(N)}\boldsymbol{\varphi}_2 = r_{2,i} \quad \text{for} \quad i=0,1,\ldots,N^*, \\
\underline{h}_1\mathcal{L}_{1,0}^{(N)}\boldsymbol{\varphi}_1 + \underline{h}_2\mathcal{L}_{2,0}^{(N)}\boldsymbol{\varphi}_2 = \nabla\cdot(\underline{h}_1\boldsymbol{r}_{3,1}+\underline{h}_2\boldsymbol{r}_{3,2}), \\
\underline{\rho}_2\boldsymbol{l}_2^{(N)}\cdot\boldsymbol{\varphi}_2 - \underline{\rho}_1\boldsymbol{l}_1^{(N)}\cdot\boldsymbol{\varphi}_1 = \underline{\rho}_1 r_{4,1} + \underline{\rho}_2 r_{4,2},
\end{cases}
$$

where

$$\begin{cases} r_{1,i} = -\sum_{j=N+1}^{2N+2}(L_{1,ij} - H^{2i}L_{1,0j})\tilde{\phi}_{1,j} \quad \text{for} \quad i = 0,1,\dots,N, \\ r_{2,i} = -\sum_{j=N^*+1}^{2N^*+2}(L_{2,ij} - H^{p_i}L_{2,0j})\tilde{\phi}_{2,j} \quad \text{for} \quad i = 0,1,\dots,N^*, \\ \nabla\cdot\boldsymbol{r}_{3,1} = \sum_{j=N+1}^{2N+2} L_{1,0j}\tilde{\phi}_{1,j}, \quad \nabla\cdot\boldsymbol{r}_{3,2} = \sum_{j=N^*+1}^{2N^*+2} L_{2,0j}\tilde{\phi}_{2,j}, \\ r_{4,1} = \sum_{j=N+1}^{2N+2} H_1^{2j}\tilde{\phi}_{1,j}, \quad r_{4,2} = -\sum_{j=N^*+1}^{2N^*+2} H_2^{p_j}\tilde{\phi}_{2,j}. \end{cases}$$

We put $\boldsymbol{r}_1' = (0, r_{1,1}, \dots, r_{1,N})^{\mathrm{T}}$ and $\boldsymbol{r}_2' = (0, r_{2,1}, \dots, r_{2,N})^{\mathrm{T}}$. Then, with a suitable decomposition $\boldsymbol{r}_\ell = \boldsymbol{r}_\ell^{\mathrm{high}} + (\underline{h}_\ell\delta)^{-2}\boldsymbol{r}_\ell^{\mathrm{low}}$ for $\ell = 1,2$, and using the linearity of (5.1), we see by Lemma 5.1 that

$$\begin{aligned} &\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\nabla\boldsymbol{\varphi}_\ell\|_{L^2}^2 + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{\varphi}_\ell'\|_{L^2}^2) \\ &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\|\boldsymbol{r}_\ell^{\mathrm{high}}\|_{H^{-1}}^2 + (\underline{h}_\ell\delta)^{-2}\|\boldsymbol{r}_\ell^{\mathrm{low}}\|_{L^2}^2 + \|\boldsymbol{r}_{3,\ell}\|_{L^2}^2 + \|r_{4,\ell}\|_{H^1}^2) \\ &\lesssim \underline{\rho}_1\underline{h}_1\|(\tilde{\phi}_{1,N+1}, \tilde{\phi}_{1,N+2}, \dots, \tilde{\phi}_{1,2N+2})\|_{H^1}^2 \\ &\quad + \underline{\rho}_2\underline{h}_2\|(\tilde{\phi}_{2,N^*+1}, \tilde{\phi}_{2,N^*+2}, \dots, \tilde{\phi}_{2,2N^*+2})\|_{H^1}^2 \\ &\quad + \underline{\rho}_1\underline{h}_1(\underline{h}_1\delta)^{-2}\|(\tilde{\phi}_{1,N+1}, \tilde{\phi}_{1,N+2}, \dots, \tilde{\phi}_{1,2N+2})\|_{L^2}^2 \\ &\quad + \underline{\rho}_2\underline{h}_2(\underline{h}_2\delta)^{-2}\|(\tilde{\phi}_{2,N^*+1}, \tilde{\phi}_{2,N^*+2}, \dots, \tilde{\phi}_{2,2N^*+2})\|_{L^2}^2. \end{aligned}$$

Moreover, it follows from [20, Lemmas 5.2 and 5.4] that

$$\begin{aligned} &\|(\tilde{\phi}_{1,N+1}, \tilde{\phi}_{1,N+2}, \dots, \tilde{\phi}_{1,2N+2})\|_{H^k} \lesssim (\underline{h}_1\delta)^{2N+2-k}\|\nabla\tilde{\boldsymbol{\phi}}_1\|_{H^{2N+1}} \\ &\|(\tilde{\phi}_{2,N^*+1}, \tilde{\phi}_{2,N^*+2}, \dots, \tilde{\phi}_{2,2N^*+2})\|_{H^k} \lesssim (\underline{h}_2\delta)^{2N+2-k}(\|\nabla\tilde{\boldsymbol{\phi}}_2\|_{H^{2N+1}} + \|\tilde{\boldsymbol{\phi}}_2'\|_{H^{2N+1}}), \end{aligned}$$

for $k = 0,2$, and hence also for $k = 1$ by interpolation, so that

$$\begin{aligned} |I_2| &\lesssim \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell (\underline{h}_\ell\delta)^{4N+2}(\|\nabla\tilde{\boldsymbol{\phi}}_\ell\|_{H^{2N+1}}^2 + \|\tilde{\boldsymbol{\phi}}_\ell'\|_{H^{2N+1}}^2) \\ &\lesssim ((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})\|\nabla\phi\|_{H^{2N+1}}^2 \\ &\lesssim ((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})\|\nabla\phi\|_{H^{4N+2}}\|\nabla\phi\|_{L^2}, \end{aligned}$$

where we used Lemma 5.1 with (2.15), and interpolation. This completes the proof of Theorem 3.10.

References

[1] D. M. Ambrose, J. L. Bona, and D. P. Nicholls. On ill-posedness of truncated series models for water waves. *Proc. R. Soc. A*, **470** (2014), 20130849.

[2] G. A. Athanassoulis and K. A. Belibassakis. A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions. *J. Fluid Mech.*, **389** (1999), 275–301.

[3] T. B. Benjamin and T. J. Bridges. Reappraisal of the Kelvin–Helmholtz problem. Part 1. Hamiltonian structure. *J. Fluid Mech.*, **333** (1997), 301–325.

[4] J. Boussinesq. Addition au mémoire sur la théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire, etc. *J. Math. Pures Appl.*, **17** (1873), 47–52.

[5] D. Bresch and M. Renardy. Well-posedness of two-layer shallow water flow between two horizontal rigid plates. *Nonlinearity*, **24** (2011), 1081–1088.

[6] W. Choi. On Rayleigh expansion for nonlinear long water waves. *J. Hydrodyn.*, **31** (2019), 1115–1126.

[7] W. Choi. High-order strongly nonlinear long wave approximation and solitary wave solution. *J. Fluid Mech.*, **945** (2022), Paper No. A15, 38.

[8] W. Choi and R. Camassa. Fully nonlinear internal waves in a two-fluid system. *J. Fluid Mech.*, **396** (1999), 1–36.

[9] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, **108** (1993), 73–83.

[10] D. G. Dommermuth and D. K. Yue. A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid Mech.*, **184** (1987), 267–288.

[11] V. Duchêne. Many Models for Water Waves. Open Math Notes, OMN:202109.111309, (2021).

[12] V. Duchêne. Boussinesq/Boussinesq systems for internal waves with a free surface, and the KdV approximation. *ESAIM Math. Model. Numer. Anal.*, **46** (2012), 145–185.

[13] V. Duchêne and T. Iguchi. A Hamiltonian structure of the Isobe–Kakinuma model for water waves. *Water Waves*, **3** (2021), 193–211.

[14] V. Duchêne and T. Iguchi. A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part I: Structures and well-posedness. To appear in *Ann. Inst. H. Poincaré Anal. Non Linéaire.*

[15] V. Duchêne and B. Melinand. Rectification of a deep water model for surface gravity waves. To appear in *Pure and Applied Analysis.*

[16] P. Guyenne. HOS simulations of nonlinear water waves in complex media. In *Nonlinear Water Waves*, Tutorials, Schools, and Workshops in the Mathematical Sciences, pages 53–69. Birkhäuser, Cham, 2019.

[17] G. Klopman, B. van Groesen, and M. W. Dingemans. A variational approach to Boussinesq modelling of fully nonlinear water waves. *J. Fluid Mech.*, **657** (2010), 36–63.

[18] T. Iguchi. A shallow water approximation for water waves. *J. Math. Kyoto Univ.*, **49** (2009), 13–55.

[19] T. Iguchi. Isobe–Kakinuma model for water waves as a higher order shallow water approximation. *J. Differential Equations*, **265** (2018), 935–962.

[20] T. Iguchi. A mathematical justification of the Isobe–Kakinuma model for water waves with and without bottom topography. *J. Math. Fluid Mech.*, **20** (2018), 1985–2018.

[21] M. Isobe. A proposal on a nonlinear gentle slope wave equation. Proceedings of Coastal Engineering, Japan Society of Civil Engineers, **41** (1994), 1–5 [Japanese].

[22] M. Isobe. Time-dependent mild-slope equations for random waves. Proceedings of 24th International Conference on Coastal Engineering, ASCE, 285–299, 1994.

[23] T. Kakinuma. [title in Japanese]. Proceedings of Coastal Engineering, Japan Society of Civil Engineers, **47** (2000), 1–5 [Japanese].

[24] T. Kakinuma. A set of fully nonlinear equations for surface and internal gravity waves. Coastal Engineering V: Computer Modelling of Seas and Coastal Regions, 225–234, WIT Press, 2001.

[25] T. Kakinuma. A nonlinear numerical model for surface and internal waves shoaling on a permeable beach. Coastal engineering VI: Computer Modelling and Experimental Measurements of Seas and Coastal Regions, 227–236, WIT Press, 2003.

[26] V. Kamotski and G. Lebeau. On 2D Rayleigh–Taylor instabilities. *Asymptotic Analysis*, **42** (2005), 1–27.

[27] D. Lannes. A stability criterion for two-fluid interfaces and applications. *Arch. Ration. Mech. Anal.*, **208** (2013), 481–567.

[28] D. Lannes. The water waves problem: mathematical analysis and asymptotics. Math. Surveys Monogr., **188**, American Mathematical Society, Providence, RI, 2013.

[29] D. Lannes and M. Ming. The Kelvin-Helmholtz instabilities in two-fluids shallow water models. In *Hamiltonian partial differential equations and applications*, volume 75 of *Fields Inst. Commun.*, pages 185–234. Fields Inst. Res. Math. Sci., Toronto, ON, 2015.

[30] J. C. Luke. A variational principle for a fluid with a free surface. *J. Fluid Mech.*, **27** (1967), 395–397.

[31] Y. Matsuno. Hamiltonian formulation of the extended Green–Naghdi equations. *Phys. D*, **301/302** (2015), 1–7.

[32] Y. Matsuno. Hamiltonian structure for two-dimensional extended Green–Naghdi equations. *Proc. R. Soc. A*, **472** (2016), 20160127.

[33] M. Miyata. An internal solitary wave of large amplitude. *La Mer.* **23** (1985), 43–48.

[34] Y. Murakami and T. Iguchi. Solvability of the initial value problem to a model system for water waves. *Kodai Math. J.*, **38** (2015), 470–491.

[35] R. Nemoto and T. Iguchi. Solvability of the initial value problem to the Isobe–Kakinuma model for water waves. *J. Math. Fluid Mech.*, **20** (2018), 631–653.

[36] D. P. Nicholls. High-order perturbation of surfaces short course: boundary value problems. In *Lectures on the theory of water waves*, volume 426 of *London Math. Soc. Lecture Note Ser.*, pages 1–18. Cambridge Univ. Press, Cambridge, 2016.

[37] T. Iguchi, N. Tanaka and A. Tani. On the two-phase free boundary problem for two-dimensional water waves. *Math. Ann.*, **309** (1997), 199–223.

[38] C. E. Papoutsellis and G. A. Athanassoulis. A new efficient hamiltonian approach to the nonlinear water-wave problem over arbitrary bathymetry. arXiv preprint:1704.03276.

[39] J. W. S. Rayleigh. On waves. *Philos. Mag.*, **1** (1876), 251–271.

[40] J. J. Stoker. The formation of breakers and bores. The theory of nonlinear wave propagation in shallow water and open channels. *Comm. Pure Appl. Math.*, **1** (1948), 1–87.

[41] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, and R. L. Milton. A new numerical method for surface hydrodynamics. *J. Geophys. Res.*, **92** (1987), 11803–11824.

[42] J. Wilkening and V. Vasan. Comparison of five methods of computing the Dirichlet-Neumann operator for the water wave problem. In *Nonlinear wave equations: analytic and computational techniques*, volume 635 of *Contemp. Math.*, pages 175–210. Amer. Math. Soc., Providence, RI, 2015.

[43] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.*, **9** (1968), 190–194.

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