Many Models for Water Waves

A unified theoretical approach

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Caveat lector This document is a slightly improved version of my "Habilitation à Diriger des Recherches" memoir, which is meant as a prerequisite before the supervision of Ph.D. students. As such the presentation of the results is very strongly biased towards my own production. That being written, I tried to offer a general picture, and hopefully the content of this memoir can be valuable for others. I plan to update this memoir from time to time when novel material fitting in the picture will arise. Please do not hesitate to contact me when you notice typos or mistakes,[1] or if you have any question, comment or query, using the email address provided on the front page.

In this document we will derive, discuss, and justify as much as possible a large class of models describing in an approximate manner the propagation of waves at the surface of water, at the interface between two homogeneous fluids, or in the bulk of a continuously density-stratified fluid. In our idealized frameworks, these waves propagate from an initial perturbation of the rest state under the influence of gravity forces. Let us unveil a little bit of the following material in order to warn the potentially disappointed reader.

- Our motivation is theoretical, in the sense that practical direct use of the results is not the main objective. The problem of the propagation of water waves is one example of partial differential equations which may be written under a compact formulation but forecasts a fascinating variety of phenomena, while enjoying a rich mathematical structure. It is hence a formidable toy on which one can apply advanced tools of modelization. Yet it is impossible not to have in mind that practical applications are just a few steps away, and many choices in the modelization procedure are grounded on applicative views, for instance robustness of the models or easy implementation.

- The "master" equations, that is the system of equations from which all subsequent simplified models are derived, already incorporates many idealizations. To name a few, we will neglect earth curvature, the Coriolis force, wind forcing, any dissipative effect and—most of the time—surface tension. In the "water waves" case we will consider homogeneous fluids and potential flows. We will also restrict our analysis to laminar (*i.e.* regular) rather than turbulent flows.

- While the equations at stake are of dispersive nature, we will use little or none of the advanced tools on dispersive equations, nor will we report the latest mathematical developments involving paradifferential calculus, normal forms, KAM theory, *etc.* Our mathematical tools are old but robust: on one hand the elliptic theory to derive models from approximate solutions to a Laplace problem; and on the other hand the energy method to justify rigorously the resulting evolution equations (being of quasilinear hyperbolic nature). The heart of the matter consists in using these tools in a refined manner so as to offer error estimates uniform with respect to the important parameters at stake.

- Given their number and diversity, it is impossible to present all relevant models based on the water waves system, even restricting to a specific asymptotic regime (the shallow water regime in our case). This document focuses on models which preserve as much as possible the structure (and in particular the Hamiltonian formulation) of the master equations, as well as mathematical properties (typically the well-posedness of the initial-value problem). That such models are often historical and among the most studied is not, to my opinion, a coincidence. Hence most of this work is dedicated to fairly standard models in oceanography. The aim of this document is to present such models together with more recent ones in a unified framework, and to address the state of their rigorous justification.

[1] This document was written over a several-years period of time, affected by sleep deprivation, and I am often negligent anyhow. I spot inconsistencies, typos or clumsiness each time I proofread some portion of this manuscript. I advise against relying on results presented therein without some proof checking.

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List of Figures

Figures listed in **bold** expose interplays between models presented in this work.

Prologue

Y'a tant de vagues, et tant d'idées qu'on n'arrive plus à décider le faux du vrai

— Michel Berger, *Le paradis blanc*

Contents

Foreword

In this monograph we aim at describing the evolution in time of a body of fluid—typically water. Of course the features of the dynamics depend greatly on the framework, and in particular on the scales involved. As a rule of thumb, we will be motivated by the description of the motion of the surface of water as seen by a human eye. These are often referred to as surface gravity waves, or simply *water waves*. As any wanderer knows, despite the restrictive framework, water waves are still remarkably diverse, and this is what makes them a fascinating subject of study.[2] In order to get a grasp at the behavior of water waves in a given situation, one typically uses simplified models. Below we give examples of a few such models[3] which appeared in the early literature,[4] with the aim at emphasizing the diversity of possible waves and the hope of giving an insight at the possible mechanisms involved in the full picture. The models described further on in this work are refinements of such models.

[2] To quote Feynman during his well-known Lectures on Physics (Vol. I, Ch. 51: Waves): "Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."

[3] We do not attempt at exhaustiveness. The relentless reader will find more in the present document and much more in the literature, using for instance [294, 268, 373, 64, 269] as starting points.

[4] The interested reader will find in [133] a detailed historical account on the early studies on water waves.

There are many ways to formally derive the models presented below. Considering the Saint-Venant system for instance, a typical way consists in integrating the horizontal velocity over the fluid layer and invoking a closure formula, based on physical principles such as energy conservation. One can also use some *ad hoc* hypotheses, such as columnar motion and hydrostatic approximation. Or a loose assumption that derivatives of a function are smaller than the function itself. Our strategy, called *asymptotic modeling*, is akin to the latter one, and provides a justification of the former ones, with quantitative estimates of the inaccuracies. We start with the so-called *full Euler system* presented in Section 1 (or more precisely, for models in this Prologue, the *water waves system* presented in Section 2) whose solutions are regarded as “exact” (although, admittedly, the derivation of the equations relies on many oversimplifications). Using the typical scales of the flow, we can extract dimensionless parameters describing the strength of the main mechanisms involved; see Section 2.4. The asymptotic models are obtained through a description of the operators involved in the water waves system using assumptions on the size of these parameters, which will be called the *asymptotic regime*.

The complete rigorous justification of models in a given asymptotic regime typically proceeds in two steps. First we prove that sufficiently regular solutions to the water waves system satisfy the equations of the model—or the other way around—up to a small remainder term, measured by the size of the dimensionless parameters and data in a prescribed metric space; this is called *consistency*. Anticipating with future notations and results, we find that the water waves system is consistent with the acoustic wave equation (i) with precision $\mathcal{O}(\mu+\varepsilon)$, with the linearized (Airy) equations (ii) with precision $\mathcal{O}(\varepsilon)$, with the Saint-Venant system (iii) with precision $\mathcal{O}(\mu)$, with all the Boussinesq systems (vii) with precision $\mathcal{O}(\mu^2+\mu\varepsilon)$, *etc.* This is however not sufficient, and there remains to prove that for a large class of sufficiently regular initial data (typically a neighborhood of the rest state in the aforementioned metric space), there exist unique solutions to both the water waves system and the asymptotic model, and that the two remain close on the relevant timescale. Following Lannes [268], we call the former property (uniform) *well-posedness*, and the latter *convergence*.

An important portion of this monograph is dedicated to the rigorous justification in the above sense—together with the study of a few basic properties—of standard and less-standard models for the propagation of surface, interfacial and internal gravity waves.

i The linear acoustic wave equation

Arguably the simplest (partial differential) equation describing the motion of water waves, already put forward by Lagrange [265], is the following:

$$\partial_t^2 \zeta = gd \, \Delta_{\boldsymbol{x}} \zeta \, . \tag{i}$$

Here ζ represents the deformation of the free surface, in the sense that the surface of the body of water at time t is parameterized as

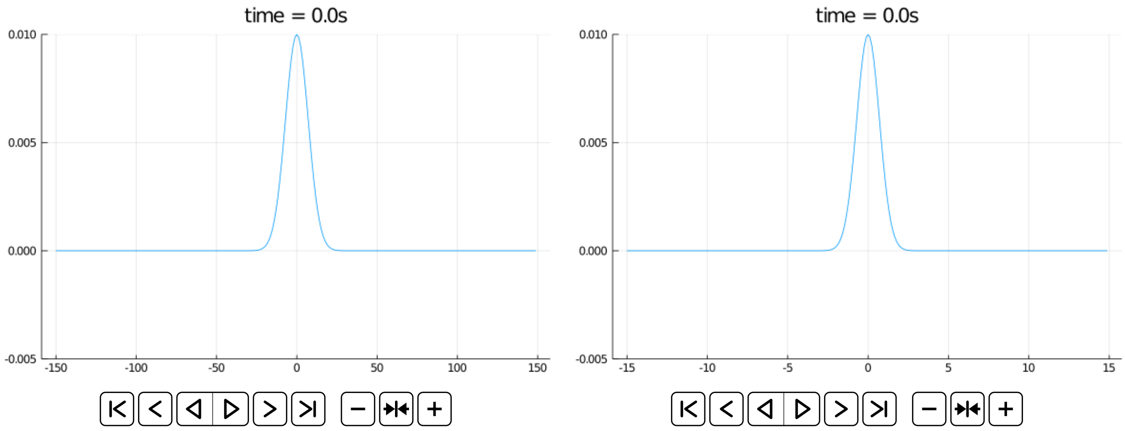
$$\Gamma_{\rm top} = \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \zeta(t, \boldsymbol{x})\}.$$

Hence the function ζ depends on time, t, and horizontal space variable, $\boldsymbol{x}$. For simplicity we assume that the horizontal variable lies in the full space $\mathbb{R}^d$. The constant g denotes the gravity acceleration and d is the depth of the layer. Equation (i) is called the ***linear acoustic wave equation*** as it governs the propagation of infinitesimally small acoustic waves through a material medium. It is only a coincidence that it also describes—very roughly, remember Feynman's quote—water waves. In fact the above equation describes infinitely small and infinitely long water waves.

In the special case of horizontal dimension $d = 1$,[5] the solution of the initial-value problem is easily found as

$$\zeta(t, x) = \frac{1}{2}\Big(\zeta(0, x + c_0 t) + \zeta(0, x - c_0 t)\Big) + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} \partial_t \zeta(0, y) \, {\rm d}y.$$

with $c_0 = \sqrt{gd}$. Hence the wave decomposes into the superposition of a right-going and a left-going components, both translating with velocity c_0. This is shown in Figure i where the evolution of the surface deformation when taken initially as Gaussians (with zero initial velocities) according to eq. (i) and eq. (ii) in dimension $d = 1$ is represented.



(a) Linear acoustic wave equation, eq. (i)

(b) Linearized water waves system, eq. (ii)

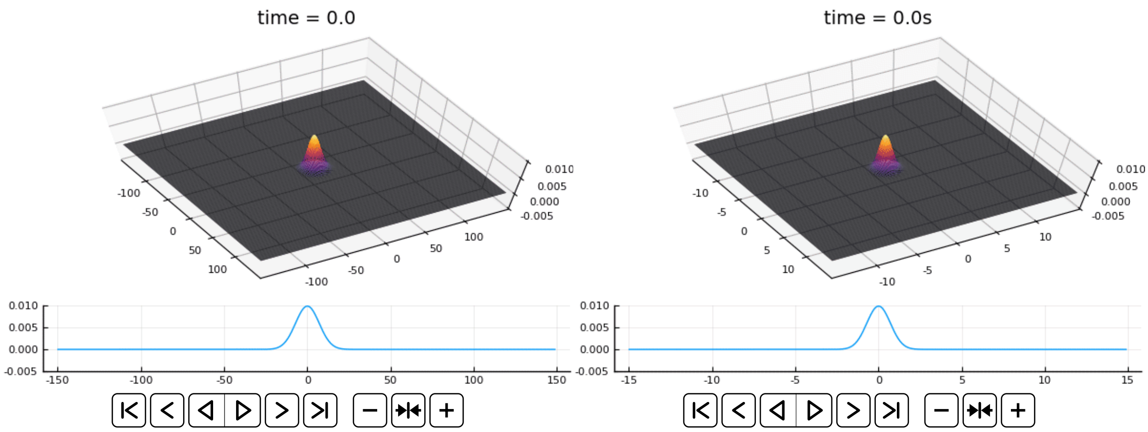
Figure i: Disintegration of Gaussian initial data, $\zeta(t = 0, x) = 0.01 \exp(-(0.1\,x)^2)$ (left) and $\zeta(t = 0, x) = 0.01 \exp(-x^2)$ (right), with zero initial velocities. $g = 9.81\,{\rm m.s}^{-2}$, $d = 1\,{\rm m}$.

In dimension $d = 2$, the solution is less explicit, but a formula can still be written—at least for sufficiently regular initial data—with the use of Green's function (we could also use Fourier representation as in the next section):

$$\zeta(t, \boldsymbol{x}) = \frac{1}{2\pi c_0}\int_{|\boldsymbol{y}-\boldsymbol{x}|\le c_0 t} \frac{\partial_t \zeta(0, \boldsymbol{y})}{\sqrt{(c_0 t)^2 - |\boldsymbol{y} - \boldsymbol{x}|^2}} \, {\rm d}\boldsymbol{y} + \frac{1}{2\pi c_0}\partial_t \int_{|\boldsymbol{y}-\boldsymbol{x}|\le c_0 t} \frac{\zeta(0, \boldsymbol{y})}{\sqrt{(c_0 t)^2 - |\boldsymbol{y} - \boldsymbol{x}|^2}} \, {\rm d}\boldsymbol{y}$$

[5] The one-dimensional framework $d = 1$ is relevant for instance for waves propagating along a narrow channel.

We can observe that the solution satisfies causality (but not Huygens' principle): waves must be given enough time to propagate between two specified points. Again, c_0 is a good measure of the (scalar) velocity of waves according to eq. (i). Less obvious is the fact that for sufficiently smooth and decaying initial data, the amplitude of the solution decays for large time as $(c_0 t)^{-1/2}$. Figure ii represents the evolution of the surface deformation when taken initially as Gaussians (with zero initial velocities) according to eq. (i) and eq. (ii), in dimension $d = 2$.



(a) Linear acoustic wave equation, eq. (i)

(b) Linearized water waves system, eq. (ii)

Figure ii: Disintegration of Gaussian initial data, with zero initial velocities. $g = 9.81\,\mathrm{m.s}^{-2}$, $d = 1\,\mathrm{m}$. The bottom plot represents the solution on $\{(x, y) \ : \ y = 0\}$.

ii The linearized (Airy) water waves equations

The following equations describe the propagation of infinitesimally small waves without the long wave assumption of the previous section: it is the linearized system about the rest-state solution to the water waves equations, whose solutions shall be considered as "exact", and which is introduced in Section 2.1. Consider the ***linearized water waves equations*** as

$$\begin{cases} \partial_t \zeta - \mathcal{G}_0 \psi = 0, \\ \partial_t \psi + g\zeta = 0 \end{cases} \tag{ii}$$

where $\mathcal{G}_0 = |D| \tanh(d|D|)$ is the Fourier multiplier operator defined on sufficiently regular solutions by

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \qquad \widehat{\mathcal{G}_0 \psi}(\boldsymbol{\xi}) = |\boldsymbol{\xi}| \tanh(d|\boldsymbol{\xi}|) \widehat{\psi}(\boldsymbol{\xi}).$$

Here, g, d and ζ are as above and ψ represents the trace of the velocity potential at the surface. Equation (ii) is a system of linear constant-coefficient equations of the form

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \mathsf{L}(D) \begin{pmatrix} \zeta \\ \psi \end{pmatrix}$$

where $\mathsf{L}(D)$ is a matrix with Fourier multiplier coefficients.

Formally taking the limit $d \to 0$, we may replace $\tanh(d|\boldsymbol{\xi}|)$ with $d|\boldsymbol{\xi}|$ in $\mathcal{G}_0$, and then we recover the acoustic wave equation, eq. (i). In fact using eq. (ii) instead of eq. (i) in the left side of Figure i and Figure ii yields a very similar outcome; such is not the case for the narrower initial data used for right sides.

Modal analysis Plane waves of the form $(\zeta, \psi) = (\zeta_0 e^{i(\boldsymbol{\xi}\cdot\boldsymbol{x}-\omega t)}, \psi_0 e^{i(\boldsymbol{\xi}\cdot\boldsymbol{x}-\omega t)})$ are solutions to eq. (ii) provided that $i\omega\psi_0 = g\zeta_0$ and the *dispersion relation* holds [250, 7]:

$$\omega(\boldsymbol{\xi})^2 = g|\boldsymbol{\xi}| \tanh(d|\boldsymbol{\xi}|).$$

In other words, we can explicitly solve the equation in the Fourier space:

$$\begin{pmatrix} \widehat{\zeta}(t,\boldsymbol{\xi}) \\ \widehat{\psi}(t,\boldsymbol{\xi}) \end{pmatrix} = \exp(\mathsf{L}(\boldsymbol{\xi})t) \begin{pmatrix} \widehat{\zeta}(0,\boldsymbol{\xi}) \\ \widehat{\psi}(0,\boldsymbol{\xi}) \end{pmatrix} = \begin{pmatrix} \cos(|\omega(\boldsymbol{\xi})|t) & \frac{|\omega(\boldsymbol{\xi})|}{g} \sin(|\omega(\boldsymbol{\xi})|t) \\ -\frac{g}{|\omega(\boldsymbol{\xi})|} \sin(|\omega(\boldsymbol{\xi})|t) & \cos(|\omega(\boldsymbol{\xi})|t) \end{pmatrix} \begin{pmatrix} \widehat{\zeta}(0,\boldsymbol{\xi}) \\ \widehat{\psi}(0,\boldsymbol{\xi}) \end{pmatrix}.$$

For such plane wave solutions, ω is called the (angular) *frequency*, $\boldsymbol{\xi}$ the (angular) *wave vector* (wavenumber if $d = 1$), and $|\boldsymbol{\xi}|$ the (angular) *wavenumber*. *Phase velocities* describe the velocity in a given direction of a plane wave with wave vector $\boldsymbol{\xi}$, and satisfy

$$\boldsymbol{c}_p \cdot \boldsymbol{\xi} = \omega(\boldsymbol{\xi}).$$

The *group velocity* represents the traveling velocity of a wave packet about wave vector $\boldsymbol{\xi}$, and is given by

$$\boldsymbol{c}_g = \nabla_{\boldsymbol{\xi}}(\omega(\boldsymbol{\xi})).$$

Misusing these definitions, we shall also refer to

$$c_p = \frac{|\omega(\boldsymbol{\xi})|}{|\boldsymbol{\xi}|} = \sqrt{gd}\left(\frac{\tanh(d|\boldsymbol{\xi}|)}{d|\boldsymbol{\xi}|}\right)^{1/2}$$

as the phase velocity, and to

$$c_g = |\boldsymbol{c}_g| = \sqrt{gd}\left(\frac{1}{2}\left(\frac{\tanh(d|\boldsymbol{\xi}|)}{d|\boldsymbol{\xi}|}\right)^{1/2} + \frac{\operatorname{sech}^2(d|\boldsymbol{\xi}|)}{2}\left(\frac{d|\boldsymbol{\xi}|}{\tanh(d|\boldsymbol{\xi}|)}\right)^{1/2}\right)$$

as the group velocity. They are represented in Figure iii. That the phase velocity is different (and greater) than the group velocity manifests the essential feature of the (linearized) water waves equations as being dispersive. Notice however that for small-normed wave vectors, $d|\boldsymbol{\xi}| \ll 1$, both velocities converge to $c_0 = \sqrt{gd}$, the velocity of (non-dispersive) infinitely long waves. In the opposite direction, for $d|\boldsymbol{\xi}| \gg 1$, we have $c_g \sim \frac{1}{2}c_p \sim \frac{\sqrt{g}}{2|\boldsymbol{\xi}|^{1/2}}$.

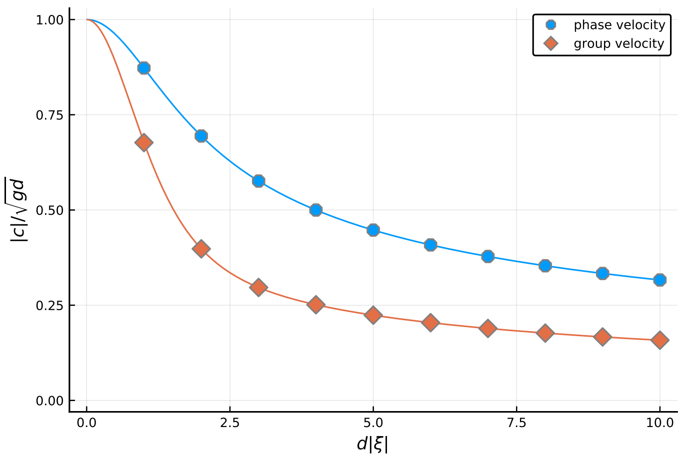


Figure iii: Phase and group velocities of the linearized water waves system.

Large-time behavior We can infer the large-time behavior of the solution, at least in dimension $d = 1$, through the stationary phase theorem on oscillatory integrals; see *e.g.* [390]. Indeed, for any $c \in \mathbb{R}$, and initial data such that $(\widehat{\zeta}(0,\cdot), |\omega|(\cdot)\widehat{\psi}(0,\cdot)) \in L^1(\mathbb{R})^2$, we have from the above

$$\zeta(t,ct) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{i(c\xi-\omega(\xi))t}\left(\widehat{\zeta}(0,\xi) + i\frac{\omega(\xi)}{g}\widehat{\psi}(0,\xi)\right) + e^{i((c\xi+\omega(\xi))t}\left(\widehat{\zeta}(0,\xi) - i\frac{\omega(\xi)}{g}\widehat{\psi}(0,\xi)\right)\mathrm{d}\xi$$

where we denote $\omega(\xi) = \mathrm{sgn}(\xi)(g|\xi|\tanh(d|\xi|))^{1/2}$, and use a standard convention for the Fourier transform. We deduce that the following holds for sufficiently decaying and regular initial data.

i. For any $c \in (-\infty, -\sqrt{gd}) \cup (\sqrt{gd}, +\infty)$, one has for any $n \in \mathbb{N}$,
$$|\zeta|(t,ct) = \mathcal{O}(t^{-n}).$$

ii. For any $c \in (-\sqrt{gd}, \sqrt{gd}) \setminus \{0\}$, one has
$$|\zeta|(t,ct) \sim_{t\to\infty} \frac{1}{4\pi}(2!)^{1/2}\Gamma(\tfrac{3}{2})|A(\xi_c)|\big(|\omega''(\xi_c)|\,t\big)^{-\frac{1}{2}}$$
where ξ_c is defined by the relation $c = \omega'(\xi_c)$ and $A(\xi_c) \stackrel{\text{def}}{=} \widehat{\zeta}(0,\xi_c) + \mathrm{sgn}(c)i\frac{\omega(\xi_c)}{g}\widehat{\psi}(0,\xi_c)$; unless $A(\xi_c) = 0$ in which case the decay is at least $\mathcal{O}(t^{-1})$.

iii. If $c \in \{-\sqrt{gd}, \sqrt{gd}\}$, one has
$$|\zeta|(t,ct) \sim_{t\to\infty} \frac{1}{4\pi}(3!)^{\frac{1}{3}}\Gamma(\tfrac{4}{3})|A(0)|\big(d^2\sqrt{gd}\,t\big)^{-\frac{1}{3}} \approx a\,((d^2/\lambda^2)\sqrt{gd}/\lambda t)^{-\frac{1}{3}},$$
with $A(0) \stackrel{\text{def}}{=} \lim_{\xi\to 0} A(\xi)$ (notice we require regularity only on $\xi\widehat{\psi}(0,\xi)$); unless $A(0) = 0$, in which case the decay is at least $\mathcal{O}(t^{-\frac{2}{3}})$. The last approximation is meant in a loose sense, where we set $A(0) \approx a\lambda$. This allows to hint at the timescale for which dispersive mechanisms have a bearing on the behavior of the flow, which is large compared with the time period of long waves, $T \stackrel{\text{def}}{=} \lambda/\sqrt{gd}$, when $d^2/\lambda^2 \ll 1$.

Above, Γ is the Euler Gamma function: $\Gamma(s) \stackrel{\text{def}}{=} \int_0^{+\infty} \tau^{s-1}e^{-\tau}\,\mathrm{d}\tau$. A loose interpretation of the above is that for large time, the dominant part of the wave which will remain visible is the large wavelength component, traveling at velocity $|c| \approx c_0 = \sqrt{gd}$.

iii The Saint-Venant system

Our first nonlinear system is the so-called shallow water, or ***Saint-Venant system*** system [370]:

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla\zeta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0}, \end{cases} \tag{iii}$$

where $h \stackrel{\text{def}}{=} d+\zeta$ represents the water depth, and $\boldsymbol{u}$ a horizontal velocity (it can be the layer-averaged horizontal velocity, velocity at a certain depth, or $\nabla\psi$). Pursuing the analogy of Section i, one can notice that the Saint-Venant system is equivalent to the isentropic, *compressible* Euler equation for ideal gases with the pressure law $p(\rho) \propto \rho^2$ (identifying ρ with h).

System (iii) is hence a prototype of quasilinear hyperbolic systems. Hyperbolicity amounts to the non-cavitation assumption, that is restricting data to $\{(\zeta,\boldsymbol{u}) \ : \ d+\zeta > 0\}$.[6] Indeed, the system in dimension $d = 2$ reads

$$\partial_t \begin{pmatrix} \zeta \\ u_x \\ u_y \end{pmatrix} + \begin{pmatrix} u_x & h & 0 \\ g & u_x & 0 \\ 0 & 0 & u_x \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ u_x \\ u_y \end{pmatrix} + \begin{pmatrix} u_y & 0 & h \\ 0 & u_y & 0 \\ g & 0 & u_y \end{pmatrix} \partial_y \begin{pmatrix} \zeta \\ u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and the eigenvalues of the associated symbol (see *e.g.* [310]) are

$$\boldsymbol{u}\cdot\boldsymbol{\xi} \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{\xi} \pm \sqrt{gh}|\boldsymbol{\xi}|\,.$$

Notice here again the "sound speed" of long surface gravity waves as being $c_0 = \sqrt{gd}$.

In dimension $d = 1$, as any quasilinear system of two scalar balance laws, eq. (iii) enjoys a basis of Riemann invariants. The Riemann invariants are explicit in this case: setting $r_\pm = u \pm 2\sqrt{gh}$, the system (iii) is equivalent to

$$\begin{cases} \partial_t r_+ + \frac{3r_+ + r_-}{4}\partial_x r_+ = 0, \\ \partial_t r_- + \frac{3r_- + r_+}{4}\partial_x r_- = 0. \end{cases} \tag{iv}$$

Notice that $\frac{3r_+ + r_-}{4} = u+\sqrt{gh}$ and $\frac{3r_- + r_+}{4} = u-\sqrt{gh}$, consistently with the hyperbolicity discussion. The diagonal formulation, eq. (iv), allows to construct *simple waves*, *i.e.* solutions of the form

$$(r_+, r_-) = \boldsymbol{R}(\theta(t,x))$$

where θ is a scalar function. For instance, any sufficiently regular solution to eq. (iv) with initial data satisfying $u\,|_{t=0} = 2\sqrt{gh\,|_{t=0}} - 2\sqrt{gd}$, the second equation yields $r_- \equiv -2\sqrt{gd}$ for all times, from which we deduce $r_+ = 2\sqrt{gd} + 2u$, where $u(t,x)$ satisfies the ***inviscid Burgers*** (or Hopf) equation

$$\partial_t u + \big(\sqrt{gd} + \tfrac{3}{2}u\big)\partial_x u = 0. \tag{v}$$

Conversely, any solution to the above equation provides a particular solution to eq. (iv) by setting $(r_-, r_+) = (2\sqrt{gd} + 2u, -2\sqrt{gd})$, or equivalently a solution to eq. (iii) with $\zeta = g^{-1}\big(\sqrt{gd}u + \frac{1}{4}u^2\big)$.

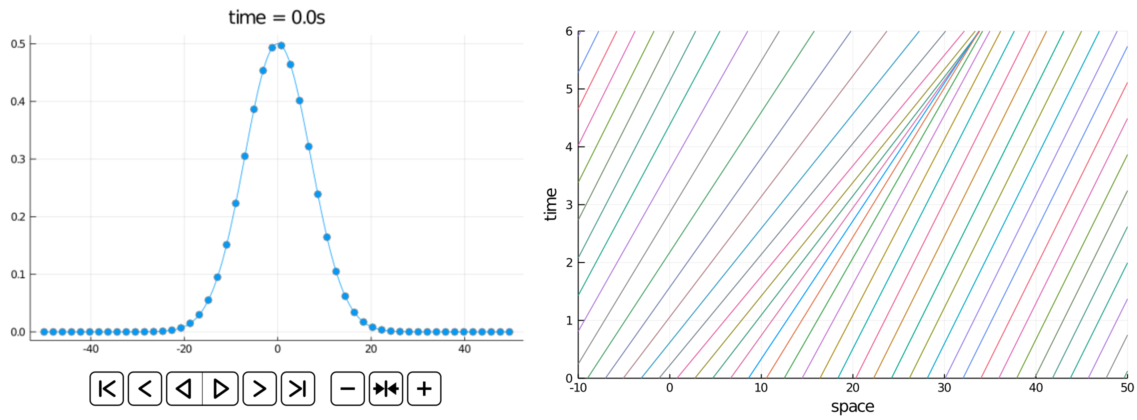
Equation (v) may be solved by the hodograph transform, or the characteristics method, and exhibits a new phenomenon with respect to the linear equations discussed in previous sections: finite-time singularity formation. Assume u is a Lipschitz solution to eq. (v) and define, for any $x_0 \in \mathbb{R}$, $v_{x_0}(t) \stackrel{\text{def}}{=} u(t, x_{x_0}(t))$ where $x_{x_0}(t)$ is defined by the initial condition $x_{x_0}(t) = x_0$ and the ordinary differential equation $x'_{x_0}(t) = \sqrt{gd} + \frac{3}{2}u(t, x_{x_0}(t))$. Chain rule and eq. (v) yields $v'_{x_0}(t) = 0$,

[6] Sufficiently regular solutions with initial data in the hyperbolicity domain cannot leave the hyperbolicity domain due to first equation (mass conservation) in eq. (iii). Indeed, denoting $h_{\boldsymbol{x}_\star}(t) = h(t, \boldsymbol{x}_{\boldsymbol{x}_\star}(t))$ where $\boldsymbol{x}_{\boldsymbol{x}_\star}(t)$ is defined by the final condition $\boldsymbol{x}_{\boldsymbol{x}_\star}(t_\star) = \boldsymbol{x}_\star$ and the ordinary differential equation $\boldsymbol{x}'_{\boldsymbol{x}_\star}(t) = \boldsymbol{u}(t, \boldsymbol{x}_{\boldsymbol{x}_\star}(t))$ for $t \in [0, t_\star]$, we find $h(t_\star, \boldsymbol{x}_\star) = h_{\boldsymbol{x}_\star}(t_\star) = h(0, \boldsymbol{x}_{\boldsymbol{x}_\star}(0)) \exp(-\int_0^{t_\star}(\nabla\cdot\boldsymbol{u})(t, \boldsymbol{x}_{\boldsymbol{x}_\star}(t))\,\mathrm{d}t) > 0$.

and hence $v_{x_0}(t) = u(0, x_0)$ and finally $x_{x_0}(t) = x_0 + (\sqrt{gd} + \frac{3}{2}u(0, x_0))t$. In other words, the solution is constant along the characteristics defined by $x_{x_0}(t)$, for any $x_0 \in \mathbb{R}$, and the characteristics are straight lines. This allows to define and describe solutions as long as two characteristics do not cross, *i.e.* as long as for any $t \in (0, T)$, there does not exists $x_0 \neq x_1 \in \mathbb{R}$ with

$$x_0 + \big(\sqrt{gd} + \tfrac{3}{2}u(0, x_0)\big)t = x_1 + \big(\sqrt{gd} + \tfrac{3}{2}u(0, x_1)\big)t \quad \Longleftrightarrow \quad \tfrac{u(0,x_1) - u(0,x_0)}{x_1 - x_0} = -\tfrac{2}{3t}.$$

Hence we see that for any Lipschitz initial data $u(t = 0, \cdot) = u_0 \in W^{1,\infty}(\mathbb{R})$, the solution described above (which is unique) exists on the time domain $[0, T^\star)$ where $T^\star = -\frac{2}{3}(\inf_{\mathbb{R}} u_0')^{-1}$ with the convention $T^\star = \infty$ if $\inf_{\mathbb{R}} u_0' \geq 0$. In the situation where $\inf_{\mathbb{R}} u_0' < 0$ (in particular for any non-trivial u_0 such that $u_0 \to 0$ as $|x| \to \infty$), there exists indeed a singularity formation as $t \to T^\star$: since the solution remains bounded but $\inf_{\mathbb{R}} \partial_x u(t, \cdot) \to -\infty$ as $t \nearrow T^\star$, we say that a shock, or a *wavebreaking*, occurs. We represent this situation in Figure iv.



(a) Evolution in time

(b) Characteristics

Figure iv: Wavebreaking of a simple wave according to eq. (v). The initial data for ζ is the Gaussian $\zeta(t = 0, x) = 0.5\exp(-(0.1\,x)^2)$ and corresponding velocity. $g = 9.81\,\mathrm{m.s}^{-2}$, $d = 1\,\mathrm{m}$.

Going back to the system case, eq. (iv), each of the Riemann invariants, $r_\pm$, is constant along characteristics curves defined by

$$x_{\pm,x_0}(0) = x_0, \quad x_{\pm,x_0}'(t) = \frac{1}{4}(3r_\pm + r_\mp)(t, x_{\pm,x_0}(t)).$$

However the characteristics curves are no longer straight lines in general. Still we can infer the behavior of solutions for instance if we assume that initial data $(\zeta(t = 0, \cdot), u(t = 0, \cdot)) \stackrel{\text{def}}{=} (\zeta_0, u_0)$ have compact support, say in $(-\lambda, \lambda)$, and are are sufficiently small so that there exists $c \in (0, c_0)$ with

$$r_{+,0} \stackrel{\text{def}}{=} u_0 + 2\sqrt{d + \zeta_0} \in (2c_0 - c, 2c_0 + c) \quad \text{and} \quad r_{-,0} \stackrel{\text{def}}{=} u_0 - 2\sqrt{d + \zeta_0} \in (-2c_0 - c, -2c_0 + c).$$

Because the Riemann invariants are constant along characteristics, we have, as long as the solution remains regular, $\frac{3r_+ + r_-}{4} \in (c_0 - c, c_0 + c)$ and $\frac{3r_- + r_+}{4} \in (-c_0 - c, -c_0 + c)$, and as a consequence

$$r_+(t, x) \equiv 2c_0 \text{ if } x \leq -\lambda + (c_0 - c)t \quad \text{and} \quad r_-(t, x) \equiv -2 \text{ if } x \geq \lambda - (c_0 - c)t.$$

If the initial data is sufficiently small in order to ensure that no shock formation occurs before $T_\star = \frac{\lambda}{c - c_0}$, we can afterwards decompose the flow as the superposition of two simple waves described by Hopf equations, and in particular a shock inevitably occurs after sufficiently large time.

iv Boussinesq systems

In his celebrated manuscript [59], Boussinesq introduced the first models for the propagation of surface gravity waves taking into account both (first order) nonlinear and dispersive effects. While restricted in the original work to unidirectional waves, models with similar flavor were later on obtained for general waves. Eventually, one may obtain a full family of systems [295, 51], often called $(abcd)$ ***Boussinesq systems***, of the form[7]

$$\begin{cases} \partial_t(\zeta - bd^2\Delta\zeta) + \nabla\cdot(h\boldsymbol{u} + ad^3\nabla\nabla\cdot\boldsymbol{u}) = 0, \\ \partial_t(\boldsymbol{u} - dd^2\nabla\nabla\cdot\boldsymbol{u}) + g\nabla(\zeta + cd^2\Delta\zeta) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \mathbf{0}, \end{cases} \tag{vii}$$

where $\mathfrak{p} = (a,b,c,d) \in \mathbb{R}^4$ is such that (when neglecting surface tension) $a+b+c+d = \frac{1}{3}$. In eq. (vii) the precise meaning of the velocity variable depends on the choice of the parameters. The freedom in the choice of $(a,b,c,d) \in \mathfrak{p}$ is at the same time a blessing—for instance one may tune parameters so as to enhance the accuracy of the dispersion relation—and a curse, since important properties of the system will typically depend on the choice of $(a,b,c,d) \in \mathfrak{p}$. In particular, the initial-value problem of a subfamily of eq. (vii) is strongly ill-posed, as can be seen from modal instabilities of the linearized equations about the rest state: the dispersion relation being

$$\omega(\boldsymbol{\xi})^2 = gd|\boldsymbol{\xi}|^2\frac{(1-a|d\boldsymbol{\xi}|^2)(1-c|d\boldsymbol{\xi}|^2)}{(1+b|d\boldsymbol{\xi}|^2)(1+d|d\boldsymbol{\xi}|^2)}$$

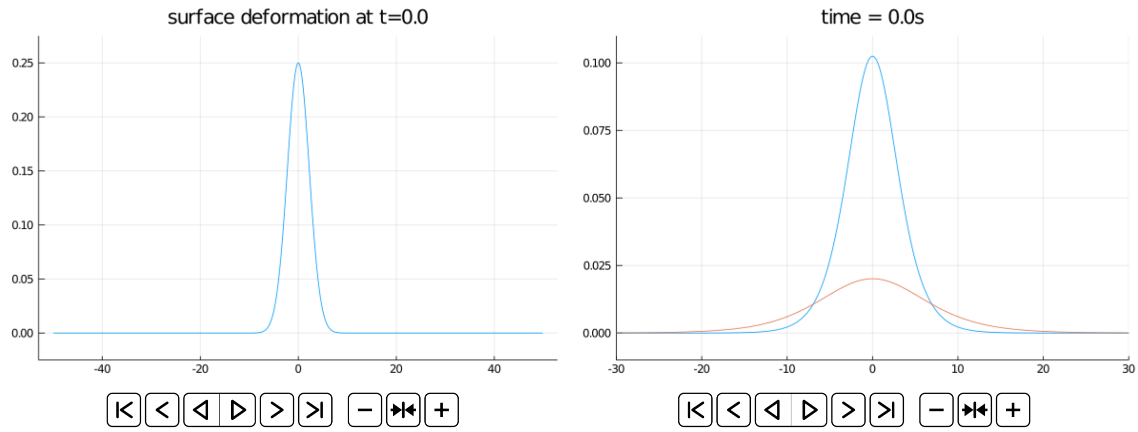
with right-hand side taking arbitrarily large negative values at large wavenumbers, $|\boldsymbol{\xi}|$, for ill-chosen $(a,b,c,d) \in \mathfrak{p}$. Incidentally, this is also the case for the original "bad" Boussinesq equation, eq. (vi). This is a useful reminder that consistency is not the only property to look for in a model.

In the other way, it is expected that for "good" choices of $(a,b,c,d) \in \mathfrak{p}$, dispersive properties of the Boussinesq systems prevent the wavebreaking scenario in the Saint-Venant model, eq. (iii). As a matter of fact, for several families of parameters, $(a,b,c,d) \in \mathfrak{p}$, global-in-time existence and uniqueness of solutions have been proved (see [375, Remark 1.1]) and—to the author's knowledge—the emergence of finite-time singularity has not been proved or numerically witnessed on any of the models, at least for solutions maintaining positive layer depth. In the situation of long waves and relatively large amplitude, the solution typically generates a zone of rapid oscillations (or modulations) often called *dispersive shock wave*, in place of the shock predicted by the Saint-Venant system. Properties of these dispersive shock waves will typically depend on the choice of $(a,b,c,d) \in \mathfrak{p}$, and is not expected to accurately describe the real-life phenomenon.

An important property of nonlinear *and* dispersive equations such as eq. (vii) is that they allow the existence of *traveling waves*, that is solutions that maintain their shape while propagating at a constant velocity, including *solitary waves* which in addition bear finite energy. Once again the reader will find in [133] the fascinating and tumultuous story of the discovery and progressive acceptance of these waves. Existence and properties of traveling waves again typically depend on the choice of $(a,b,c,d) \in \mathfrak{p}$. We however expect that they exist at least for small supercritical velocities, $0 < c - c_0 \ll 1$, and grow in amplitude with the velocity parameter; see *e.g.* [142]. We show examples in Figure v.

[7] The transport term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is often replaced with $\frac{1}{2}\nabla(|\boldsymbol{u}|^2)$, trading the direct comparison with the Saint-Venant system, eq. (iii), with conservative form. The change is immaterial in dimension $d=1$, or when $\operatorname{rot}\boldsymbol{u} = 0$. Similar systems can be derived using momentum-type variables instead of velocity variables, thus slightly altering the nonlinear/dispersive interplay; see [185]. These systems, sometimes called Abbott systems [2, 3], have conservative form. Other *ad hoc* transformations can be performed, for instance to improve the mathematical properties of the system; see [53]. Finally, the models can also be written as second order scalar equations similar to eq. (i), as in the original work of Boussinesq [59, (26), p. 75]:

$$\partial_t^2\zeta = gd\,\Delta\big(\zeta + \tfrac{3}{2d}\zeta^2 + \tfrac{d^2}{3}\Delta\zeta\big)\,. \tag{vi}$$



(a) Disintegration of Gaussian initial data

(b) Traveling waves

Figure v: Left: Disintegration of the Gaussian $\zeta(t=0,x) = 0.25\exp(-0.1x^2)$, with zero initial velocity. Right: Solitary wave solutions with velocities $c = 1.05c_0$ and $c = 1.01c_0$. Both according to system (vii) with $-a = b = d = \frac{1}{3}$, $c = 0$, $g = 9.81\,\mathrm{m.s}^{-2}$, $d = 1\,\mathrm{m}$.

It would be impossible to review all known results on Boussinesq systems and closely related (symmetric, Abbott, *etc.*) variants. Let me lazily refer to [150, 269, 375]—in addition to previous references—and references therein, and conclude with a last warning. The Boussinesq systems typically lose important properties of the original water waves equations and in particular its Hamiltonian structure (see Section 2.2). Hence unless the parameters $(a,b,c,d) \in \mathfrak{p}$ are well-chosen,we do not expect energy conservation, or Galilean invariance, *etc.*

v The Korteweg–de Vries and Whitham equations

It was mentioned in the previous section that Boussinesq's original motivation was the study of unidirectional waves, and in particular solitary waves. Using such assumption one may derive[8] (as did Boussinesq) simplified scalar equations, of which the most famous is the ***Korteweg–de Vries equation*** [61, 262] for right-going waves in dimension $d = 1$:

$$\partial_t\zeta + \sqrt{gd}\,\partial_x(\zeta + \tfrac{3}{4d}\zeta^2 + \tfrac{1}{6}d^2\partial_x^2\zeta) = 0\,. \tag{viii}$$

One of the many reasons for the importance of the Korteweg–de Vries equation is the family of explicit solitary wave solutions[9]

$$\zeta(t,x) = \zeta_c(x-ct), \qquad \zeta_c(x) \stackrel{\text{def}}{=} 2d(\tfrac{c}{c_0}-1)\,\mathrm{sech}^2\Big(\sqrt{\tfrac{3}{2d^2}(\tfrac{c}{c_0}-1)}\,x\Big)$$

where the velocity variable, c, may take any value $c > c_0 = \sqrt{gd}$.

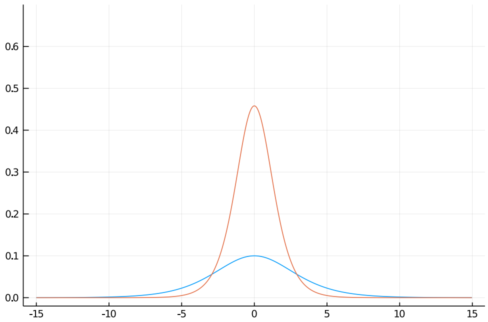
[8] We will not discuss in this document the interesting question of justifying such scalar equations from aforementioned systems of equations. Let me just mention that this justification is relatively straightforward for well-prepared initial data accounting for the assumption of unidirectional propagation, and much more involved for general initial data where we want to express that the flow can be decomposed at first order as the superposition of two counter-propagating unidirectional waves. Let me also refer—once again—to [268] and references therein (see also [36] for a recent development) for all details concerning the Korteweg–de Vries equation and to [178] for the Whitham equation.

[9] Far from being simply some entertaining special solutions, solitary waves play a very important role as they allow to describe the large-time dynamics of generic solutions; a phenomenon designated as *soliton resolution*. We will not discuss further on this feature as it relies on the integrability of the Korteweg–de Vries equation, a property which is not shared by other models in this document.

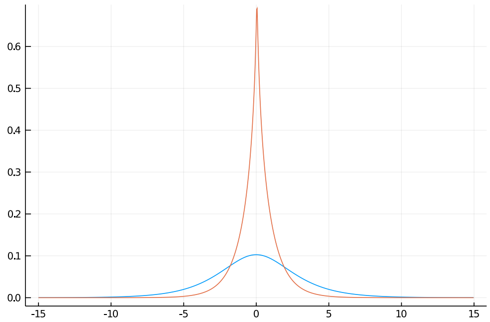
The existence of traveling waves with arbitrarily large amplitude and arbitrarily large velocity may found undesirable as nonphysical [393]. Such is the case also for the global-in-time well-posedness properties, preventing the aforedescribed wavebreaking scenario. With this in mind, Whitham [412] introduced the following equation[10] which is now called ***Whitham equation***:

$$\partial_t\zeta + \sqrt{gd}\partial_x\Big(\sqrt{\tfrac{\tanh(d|D|)}{d|D|}}\zeta + \tfrac{3}{4d}\zeta^2\Big) = 0\,. \tag{x}$$

arguing that the fact that its linear dispersion relation reproduces exactly one branch of the dispersion relation of eq. (ii) would authorize wavebreaking and peaked traveling waves of extreme height. This prediction turned out to be valid, as recently shown in [219, 173, 402, 374]. A numerical comparison of solitary wave solutions to the Korteweg–de Vries and Whitam equations is shown in Figure vi.



(a) KdV equation, eq. (viii)



(b) Whitham equation, eq. (x)

Figure vi: Solitary waves of unidirectional models with velocity $c = 1.05c_0$ (blue, smaller) and $c = 1.2290408c_0$ (red, larger). $g = 9.81\,\mathrm{m.s}^{-2}$, $d = 1\,\mathrm{m}$, $c_0 = \sqrt{gd}$.

[10]He also proposed [413, §13.14]

$$\partial_t\zeta + \sqrt{g\tfrac{\tanh(d|D|)}{|D|}}\partial_x\zeta + \big(3\sqrt{g(d+\zeta)} - 3\sqrt{gd}\big)\partial_x\zeta = 0\,, \tag{ix}$$

where the advection term fits the decomposition in Riemann invariants of the Saint-Venant system.

CHAPTER A

The "master" equations

Le problème de l'établissement [...] des équations différentielles du mouvement, et ensuite de leur intégration approchée, aura encore sa difficulté souvent grande. Mais il ne présentera plus, envisagé ainsi, cette désespérante énigme contre laquelle des esprits distingués se sont heurtés en vain.

— Adhémar Barré de Saint-Venant, *Comptes rendus des séances de l'Académie des sciences, séance du 18 mars 1872*

Contents

Foreword

In this chapter, we introduce and provide a preliminary study of the systems of equations from which asymptotic models are derived in subsequent chapters. The presentation, as well as most of the notations, are borrowed from Lannes' book [268]. However concision has been pursued and I cannot encourage enough a thorough reading of the book for a detailed account.

We shall first write down in Section 1 the most general system of equations which is considered in this work, that is Euler equations for a layer of (non-necessarily homogeneous) incompressible ideal fluid, coupled with boundary conditions accounting for the impermeable bottom and the free surface. The only external force acting on the system will be the gravity force, assumed constant and vertical. We refer to the system we obtain, namely eq. (1.1), as the ***full Euler system***. Then we focus on particular settings.

The homogeneous and irrotational framework is particularly rewarding, as it allows to rewrite the whole system as two evolution equations for unknown functions of time and horizontal space variables only. This is system (2.2), which we refer to as the ***water waves equations***. Some information concerning the water waves equations are given in Section 2.

Prominently important in the water waves equations is the *Dirichlet-to-Neumann operator*, which is defined after solving a Laplace problem on the fluid domain with Dirichlet and Neumann boundary conditions. Its study, and in particular the asymptotic expansions which allow to derive asymptotic models, is postponed to Section 4, culminating with Proposition 4.10 and Proposition 4.15.

Meanwhile we make a small step outside the world of homogeneous and potential flows to consider ***interfacial waves*** between two layers of homogeneous fluids with irrotational velocities, in Section 3. Additional Dirichlet/Neumann operators appearing in this framework are tackled in Section 4.5.

1 The full Euler system

Let us introduce the equations which will serve as a reference for any other models in this manuscript. These equations are meant to predict the evolution of an infinite layer of a fluid (typically water) delimited above by a free surface and below by a rigid bottom under the effect of gravity. We will always assume that the upper surface and lower bottom of the fluid can be described through the graph of regular functions—so that no surging waves are allowed—and as such we can denote the domain of the fluid (see Figure 1.1) as

$$\Omega^t \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ -d + b(t, \boldsymbol{x}) < z < \zeta(t, \boldsymbol{x})\}.$$

As apparent in this definition, $\boldsymbol{x} \in \mathbb{R}^d$ with $d \in \{1, 2\}$ (when the dimension is prescribed we shall denote $\boldsymbol{x} = x$ when $d = 1$ and $\boldsymbol{x} = (x, y)$ when $d = 2$) is the horizontal variable and z the vertical variable. We represent the depth at rest by $d > 0$. We also denote the bottom topography and the surface deformation

$$\Gamma_{\text{bot}} \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = -d + b(t, \boldsymbol{x})\},$$
$$\Gamma_{\text{top}} \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \zeta(t, \boldsymbol{x})\},$$

although we occasionally misname the surface deformation as the function ζ instead of Γ_{top} and the bottom topography as b instead of Γ_{bot}.

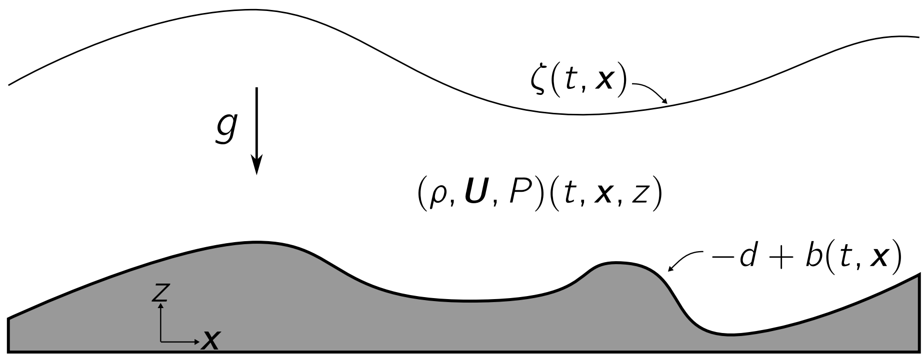


Figure 1.1: Sketch of the domain and notations.

The main goal is to predict the evolution of the surface deformation, Γ_{top}, together with the velocity field inside the layer. To this aim, we introduce the following ***full Euler equations***:

$$\partial_t \rho + \nabla_{x,z} \cdot (\rho \boldsymbol{U}) = 0 \quad \text{in } \Omega^t, \tag{1.1a}$$
$$\rho \partial_t \boldsymbol{U} + \rho(\boldsymbol{U} \cdot \nabla_{x,z})\boldsymbol{U} = -\nabla_{x,z} P - \rho g \boldsymbol{e}_z \quad \text{in } \Omega^t, \tag{1.1b}$$
$$\operatorname{div} \boldsymbol{U} = 0 \quad \text{in } \Omega^t, \tag{1.1c}$$
$$\partial_t \zeta = w - \nabla \zeta \cdot U \quad \text{on } \Gamma_{\text{top}}, \tag{1.1d}$$
$$\partial_t b = w - \nabla b \cdot U \quad \text{on } \Gamma_{\text{bot}}, \tag{1.1e}$$
$$P - p_{\text{atm}} = -\sigma \nabla \cdot \left(\frac{\nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}} \right) \quad \text{on } \Gamma_{\text{top}}, . \tag{1.1f}$$

The first three equations are the Euler equations for incompressible fluids, and represent the *conservation of mass, of momentum and the incompressibility constraint.* The fourth one is called the *kinematic boundary condition* and corresponds to the assumption that no fluid particle shall cross the surface (in fact fluid particles at the surface are forever "trapped" at the surface). Similarly, the subsequent one is the *impermeability condition* ensuring that no fluid particle shall cross the bottom. We assume that the pressure jump at the surface is proportional to the mean curvature of the surface, with the constant σ denoting the *surface tension* coefficient. Finally, we impose that the density does not vanish on the fluid domain or on the boundaries and enforce the *non-cavitation assumption*, *i.e.* the depth of the layer nowhere vanishes.

Here, $\rho(t, \boldsymbol{x}, z)$ is the fluid mass density at time t and position $(\boldsymbol{x}, z)$. We denote by $\boldsymbol{U}(t, \boldsymbol{x}, z)$ the velocity of the fluid particle at time t and position $(\boldsymbol{x}, z)$. We also denote $\boldsymbol{U} = (U, w)$ where U is the horizontal component and w the vertical one. We denote $\nabla_{\boldsymbol{x},z}$ the $(d+1)$-dimensional gradient operator while ∇ is the d-dimensional horizontal gradient operator. P denotes the pressure inside the fluid; it is not an unknown but rather the Lagrange multiplier associated with the incompressibility constraint, eq. (1.1c), and can be deduced from other unknowns at any time instant by solving the equation obtained when taking the divergence of eq. (1.1b). Finally, $p_{\rm atm}$ is the (prescribed) atmospheric pressure at the surface, g is the (constant) acceleration of gravity, and $\boldsymbol{e}_z$ is the vertical upward unit vector.

Additional assumptions Many assumptions were made so as to write eq. (1.1), and we shall add other important ones even before we move towards the derivation of asymptotic models. For instance we neglected the effects of compressibility, viscosity, and friction at the bottom. This is motivated by the fact that when considering a large body of water with relatively mild behavior, these effects are expected to have almost no contribution on the evolution of the flow. We have also neglected the Coriolis effect, as well as the curvature of earth. This assumption is valid provided we consider a body of water which is not too large. Hence our framework is restricted between two extremes, the rule of thumb being that we describe waves that a human eye can see (see [268] for a more detailed and quantitative discussion). As our aim is to highlight only the relevant mechanism in the propagation of surface gravity waves, it makes sense to discard as early as possible any unnecessary complexities. In the same spirit, we shall discard the surface tension effects:

$$\sigma = 0, \ \textit{i.e.}\ P = p_{\rm atm} \qquad \text{on } \Gamma_{\rm top}. \tag{1.2a}$$

However it turns out that the surface tension component, although *a priori* negligible, has very important theoretical consequences for some problems because it strongly modifies the high frequency behavior of the equations (in particular the linear group and phase velocity are no longer bounded and decreasing with the size of the wavenumber). This has strong consequences for instance when looking for traveling waves solutions, or for the well-posedness theory in the bilayer setting. This being said, we will use eq. (1.2a) for the sake of concision when deriving models; a version of the models with surface tension effects are always easy to deduce.

We will also assume thereafter that the bottom is time-independent:

$$\partial_t b = 0 \qquad \text{on } \Gamma_{\rm bot}, \tag{1.2b}$$

and that the atmospheric pressure at the surface is uniform in space,

$$\nabla p_{\rm atm} = 0 \qquad \text{on } \Gamma_{\rm top}. \tag{1.2c}$$

The above assumptions are made for concision and clarity, but again it is not difficult to add—at least formally—the effects of atmospheric or topographic changes in the models, which can be then used for studying the generation of waves in addition to their propagation. We refer for instance to [189, 305] for such study.

Such is not the case concerning the following assumptions which will be very important for the mathematical analysis: we assume that there exist $\rho_\star$ and $d_\star$ positive constants such that

$$\rho \geq \rho_\star > 0 \qquad \text{in } \Omega^t, \tag{1.2d}$$

$$d + \zeta - b \geq d_\star > 0 \qquad \text{in } \mathbb{R}^d. \tag{1.2e}$$

The former assumption is quite natural in the oceanographic context, but not if eq. (1.1) is applied to the atmospheric motion. The non-cavitation assumption, eq. (1.2e), is much more consequential, and prevents any study near the shore, and in particular shoaling effects. We refer to [361, 319, 320] (and [272, 387, 72] concerning important models in this manuscript) for some works dealing with this situation. We mention here that our unknowns, and in particular the surface deformation, ζ, will be assumed to vanish at infinity through finite energy assumptions. In particular we have the far field conditions

$$|\zeta|, |\boldsymbol{U}| \to 0, \quad \rho(\cdot, z) \to \underline{\rho}(z) \qquad \text{as } |\boldsymbol{x}| \to \infty. \tag{1.2f}$$

Most of the results extend to the periodic setting with almost no modifications (d is then the layer-depth average ensuring that ζ, b are mean-zero), but we expect they may also be extended to the more relevant Kato's uniformly local Sobolev spaces; see [12]. We will also assume sufficient regularity on all variables so that the identities above hold on the classical, pointwise sense. Let us now introduce the two crucial (and arguable) additional assumptions from which the so-called water waves equations is derived. We shall, unless otherwise specified, assume that the fluid is homogeneous, *i.e.* there exists a constant $\rho_0 > 0$ such that

$$\rho \equiv \rho_0 \qquad \text{in } \Omega^t. \tag{1.3a}$$

This assumption needs only to be made initially in time, as it is automatically propagated for positive times thanks to the mass conservation, eq. (1.1a) and incompressibility constraint, eq. (1.1c).

Finally there is one last very important assumption: we shall most of the time restrict ourselves to irrotational (or potential) flows, namely

$$\operatorname{rot} \boldsymbol{U} = \boldsymbol{0} \qquad \text{in } \Omega^t. \tag{1.3b}$$

In the homogeneous setting, because all the forces in the right-hand side of eq. (1.1b) are potential, the irrotational assumption, eq. (1.3b), needs only to be set initially, and it is automatically propagated by the equations for positive times. Restricting motions to homogeneous potential flows turns out to be an extremely rewarding assumption, as it allows to rewrite the entire set of equations as only two scalar evolution equations for unknowns depending only the time and horizontal space variables. This striking reduction, which is described in the following Section, should be seen as a warning that much of the diversity of the waves of the original system, eq. (1.1), has been discarded through the assumptions of eq. (1.3a) and eq. (1.3b).

2 The water waves system

2.1 Derivation

We shall rewrite in this section the ***full Euler system***, eq. (1.1)-(1.2), in the homogeneous—eq. (1.3a)—and irrotational—eq. (1.3b)—framework as a simple-looking system of two scalar evolution equations. This is the so-called Zakharov-Craig-Sulem formulation which we will refer to simply as *the water waves equations*. The irrotationality assumption induces

$$\boldsymbol{U} = \nabla_{\boldsymbol{x},z}\Phi \qquad \text{in } \Omega^t, \tag{1.3b$'$}$$

where $\Phi(t, \boldsymbol{x}, z) \in \mathbb{R}$ is the velocity potential, defined by $\boldsymbol{U}$ up to a time-dependent additive constant. We can then rewrite the momentum equation and incompressibility constraint in terms of the velocity potential:

$$\partial_t\Phi + \frac{1}{2}|\nabla_{\boldsymbol{x},z}\Phi|^2 = -\frac{1}{\rho_0}(P - P_{\rm atm}) - gz \qquad \text{in } \Omega^t, \tag{1.1b$'$}$$

$$\Delta_{\boldsymbol{x},z}\Phi = 0 \qquad \text{in } \Omega^t, \tag{1.1c$'$}$$

The former is called the Bernoulli equation. As it has been obtained from an integration in space, it should include a time-dependent source term, which has been set to zero by choosing suitably the time-dependent additive constant in Φ. The latter equation is of course Laplace's equation, hence the potential is harmonic. With this in mind, we introduce its trace at the surface,

$$\psi(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \Phi(t,\boldsymbol{x},\zeta(t,\boldsymbol{x})),$$

and notice that the velocity potential, Φ, is uniquely determined (under reasonable hypotheses, see below) by the knowledge of (ζ, b, ψ) after solving

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi = 0 & \text{in } \Omega^t, \\ \Phi = \psi & \text{on } \Gamma_{\text{top}}, \\ \partial_z\Phi - (\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi = 0 & \text{on } \Gamma_{\text{bot}}, \end{cases} \tag{2.1}$$

the last equation being provided by eq. (1.1e), $\nabla_{\boldsymbol{x}}$ being the d-dimensional horizontal gradient operator. The following result is standard in the theory of elliptic operators (see Appendix III for the definition of functional spaces).

Proposition 2.1. *Let $(\zeta, b) \in W^{2,\infty}(\mathbb{R}^d)$ such that eq. (1.2e) holds. Then for any $\psi \in \mathring{H}^2(\mathbb{R}^d)$, there exists a unique $\Phi \in \mathring{H}^2(\Omega^t)$ strong solution to eq. (2.1).*

Following Craig, Sulem and Sulem [130, 129], it is then convenient to introduce the Dirichlet-to-Neumann operator returning the rescaled normal component of the velocity at the surface:

Definition 2.2 (Dirichlet-to-Neumann operator)**.** *Under the assumptions of Proposition 2.1, the Dirichlet-to-Neumann operator*

$$\mathcal{G}[\zeta,b] : \begin{array}{ccc} \mathring{H}^2(\mathbb{R}^d) & \to & H^{1/2}(\mathbb{R}^d) \\ \psi & \mapsto & (\partial_z\Phi - (\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\zeta} \end{array}$$

where $\Phi \in \mathring{H}^2(\Omega^t)$ is the solution to eq. (2.1) provided by Proposition 2.1, is well-defined and continuous. If, moreover, $\zeta, b, \psi \in \mathring{H}^{2+s_\star}(\mathbb{R}^d)$ with $s_\star > d/2$, then $\mathcal{G}[\zeta,b]\psi \in H^{s_\star}(\mathbb{R}^d) \subset \mathcal{C}^0(\mathbb{R}^d)$.

We provide a proof of these Propositions in Section 4. Sharper results are provided in [268] together with a thorough description of many properties of the Dirichlet-to-Neumann operator. Let us collect some of them for future reference.

Proposition 2.3. *Under the assumptions of Proposition 2.1 the Dirichlet-to-Neumann operator satisfies the following*

- *Identity of mass conservation:*

  $$\mathcal{G}[\zeta,b]\psi = -\nabla\cdot(h\overline{\boldsymbol{u}})$$

  where $h = d + \zeta - b$ and $\overline{\boldsymbol{u}} \stackrel{\text{def}}{=} \frac{1}{h}\int_{-d+b}^{\zeta} \nabla_{\boldsymbol{x}}\Phi \,\mathrm{d}z$. In particular, $\mathcal{G}[\zeta,b]\psi \in (\mathring{H}^2(\mathbb{R}^d))'$.

- *Symmetry:*

  $$\big\langle \psi_1, \mathcal{G}[\zeta,b]\psi_2 \big\rangle_{\mathring{H}^2-(\mathring{H}^2)'} = \big\langle \psi_2, \mathcal{G}[\zeta,b]\psi_1 \big\rangle_{\mathring{H}^2-(\mathring{H}^2)'}.$$

- *Positivity:*

  $$\big\langle \psi, \mathcal{G}[\zeta,b]\psi \big\rangle_{\mathring{H}^2-(\mathring{H}^2)'} \approx \big|\Lambda^{-1/2}\nabla\psi\big|_{L^2(\mathbb{R}^d)}^2.$$

- *Shape derivative:*

  $$\mathrm{d}_\zeta\,\mathcal{G}[\zeta,b](\delta\zeta)\psi = -\mathcal{G}[\zeta,b]((\delta\zeta)\underline{w}) - \nabla\cdot((\delta\zeta)\underline{\boldsymbol{u}})$$

  where $\mathrm{d}_\zeta\,\mathcal{G}[\zeta,b](\delta\zeta)\psi$ is the derivative of the mapping $\zeta \mapsto \mathcal{G}[\zeta,b]\psi$ in the direction $\delta\zeta$, and we denote $\underline{w} = \frac{\mathcal{G}[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi}{1+|\nabla\zeta|^2}$ and $\underline{\boldsymbol{u}} = \nabla\psi - \underline{w}\nabla\zeta$. One easily checks, by the above identity and chain rules, that $\boldsymbol{U}\big|_{z=\zeta} = (\nabla_{\boldsymbol{x},z}\Phi)\big|_{z=\zeta} = (\underline{\boldsymbol{u}}, \underline{w})$.

By the use of the chain rule, we can now rewrite the (trace at the surface of the) Bernoulli equation, eq. (1.1b′), as well as the kinematic boundary condition at the surface, eq. (1.1d):

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta, b]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \dfrac{(\mathcal{G}[\zeta, b]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0. \end{cases} \tag{2.2}$$

We call the closed set of equations (2.2) the ***water waves system*** in order to distinguish it from the *full Euler equations*, eq. (1.1). It is easy to see, following the lines above, that any sufficiently regular solution to the full Euler equations, (1.1) with (1.2)-(1.3), satisfies the water waves system, eq. (2.2). The converse can also be verified. The analysis has been detailed even for mildly regular data—and in particular a very rough topography—in [10].

We conclude this section by noticing that eq. (2.2) may be equivalently written as

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{\boldsymbol{u}}) = 0, \\ \partial_t \psi + g\zeta + \underline{\boldsymbol{u}} \cdot \nabla\psi - \frac{1}{2}\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}} - \underline{w}^2 = 0, \end{cases} \tag{2.2'}$$

using the notations and identities of Proposition 2.3.

2.2 Variational structure

A remarkable property of the water waves equations, eq. (2.2), as put forward by Zakharov [424] is its canonical Hamiltonian structure. Indeed, define the Hamiltonian as the total energy, summing up the potential and kinetic energies (up to multiplying with ρ the mass density):

$$\begin{aligned} \mathscr{H}(\zeta, \psi) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \psi\mathcal{G}[\zeta, b]\psi \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d}\int_{-d+b}^{\zeta} gz + \frac{1}{2}|\nabla_{\boldsymbol{x},z}\Phi|^2 \,\mathrm{d}z - \frac{1}{2}(d - b)^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Then one can show (using Proposition 2.3) that eq. (2.2) reads

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathscr{H} \\ \delta_\psi \mathscr{H} \end{pmatrix}$$

where $\delta_\zeta\mathscr{H}$ and $\delta_\psi\mathscr{H}$ denote the functional derivatives: for instance

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \lim_{\epsilon \to 0} \frac{\mathscr{H}(\zeta, \psi + \epsilon\varphi) - \mathscr{H}(\zeta, \psi)}{\epsilon} = \int_{\mathbb{R}^d} (\delta_\psi \mathscr{H})\varphi \,\mathrm{d}\boldsymbol{x}.$$

We may associate a Lagrangian to the Hamiltonian structure: define

$$\mathscr{L}_{\mathrm{Z}} = \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \psi\partial_t\zeta \,\mathrm{d}\boldsymbol{x} - \mathscr{H}(\zeta, \psi) \,\mathrm{d}t.$$

Then we see that the water waves equations, eq. (2.2) follows from Hamilton's principle

$$\delta\mathscr{L}_{\mathrm{Z}} = 0.$$

One may notice that, using the conservation of mass as a constraint, one can rewrite the Lagrangian as the *difference* between the kinetic and potential energies.

As noticed in [316], the Hamiltonian structure is closely related to Luke's variational formulation [289]. Indeed we can recover the water waves equations from Hamilton's principle using the Lagrangian action

$$\begin{aligned}\mathscr{L}_{\rm L} &= \int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{-d+b}^{\zeta}\frac{P-p_{\rm atm}}{\rho_0}\,\mathrm{d}z\,+\frac12(d-b)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= -\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{-d+b}^{\zeta}\partial_t\Phi+\frac12|\nabla_{\boldsymbol{x},z}\Phi|^2+gz\,\mathrm{d}z\,+\frac12(d-b)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\end{aligned}$$

where the second term is added for the sake of the finiteness of the integrals. As proof, let us observe that

$$\mathscr{L}_{\rm L}+\mathscr{L}_{\rm Z}=\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\int_{-d+b}^{\zeta}\Phi\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

and the right-hand side does not contribute to Hamilton's principle. The Lagrangian action $\mathscr{L}_{\rm Z}$ is somewhat more favorable than Luke's counterpart, $\mathscr{L}_{\rm L}$, as the former is well-defined for variables having finite energy, $(\zeta,\psi)\in L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d)$, while the latter demands additional decay assumptions at infinity.

Alternatively, we may specifically treat the conservation of mass as a constraint and write down the Lagrangian action as

$$\mathscr{L}\overset{\rm def}{=}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\frac12\psi\mathcal{G}[\zeta,b]\psi-\frac{g}{2}\zeta^2+\varphi(\partial_t\zeta-\mathcal{G}[\zeta,b]\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where φ is a Lagrange multiplier associated with the conservation of mass constraint. Again the water waves equations are obtained from Hamilton's principle since

$$\delta_\varphi\mathscr{L}=\partial_t\zeta-\mathcal{G}[\zeta,b]\psi,\quad \delta_\zeta\mathscr{L}=(\frac12\psi-\varphi)\delta_\zeta\mathcal{G}[\zeta,b]\psi-g\zeta-\partial_t\varphi,\quad \delta_\psi\mathscr{L}=\mathcal{G}[\zeta,b]\psi-\mathcal{G}[\zeta,b]\varphi.$$

Such formulation is quite handy to quickly (but formally) derive asymptotic models, which then enjoy by construction a variational structure as well; see for instance [107], and Section 8.1.2.

One of the nice outcomes of the Hamiltonian structure is that it relates, through Noether's theorem, symmetry groups and conserved quantities (invariants) of the system.

2.2.1 Group symmetries

Some relevant group symmetries are as follows. If (ζ,ψ) is a solution to eq. (2.2), then for any $\theta\in\mathbb{R}$, $(\zeta^\theta,\psi^\theta)$ also satisfies eq. (2.2), where

- Variation of base level for the velocity potential
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x})=\big(\zeta,\psi+\theta\big)(t,\boldsymbol{x}).$$
- Horizontal translation along the direction $\boldsymbol{e}\in\mathbb{R}^d$ (in the flat bottom case)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x})=\big(\zeta,\psi\big)(t,\boldsymbol{x}-\theta\boldsymbol{e}).$$
- Time translation
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x})=\big(\zeta,\psi\big)(t-\theta,\boldsymbol{x}).$$
- Galilean boost along the direction $\boldsymbol{e}\in\mathbb{R}^d$ (in the flat bottom case)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x})=\big(\zeta,\psi+\theta\boldsymbol{e}\cdot\boldsymbol{x}\big)(t,\boldsymbol{x}-\theta\boldsymbol{e}t).$$
- Horizontal rotation (in dimension $d=2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp\cdot\nabla b=0$)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x})=\big(\zeta,\psi\big)(t,R_\theta\boldsymbol{x})$$
where R_θ is the rotation matrix of angle θ.

2.2.2 Preserved quantities

We have the following related preserved quantities (often one should multiply with ρ to reconcile with the physical meaning).

- Excess of mass
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = 0, \qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \,\mathrm{d}\boldsymbol{x}.$$
- Horizontal impulse (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \nabla\psi \,\mathrm{d}\boldsymbol{x} \qquad (\text{if } b \equiv 0).$$
- Total energy
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = 0 \qquad \mathscr{H} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \psi\mathcal{G}[\zeta, b]\psi \,\mathrm{d}\boldsymbol{x}.$$
- Horizontal coordinate of mass centroid times mass (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{C} = \int_{\mathbb{R}^d} \zeta \nabla\psi \,\mathrm{d}\boldsymbol{x}, \qquad \mathscr{C} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \qquad (\text{if } b \equiv 0).$$
- Angular impulse (in dimension $d = 2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp \cdot \nabla b = 0$)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{A} = 0, \qquad \mathscr{A} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x}^\perp \cdot \nabla\psi \,\mathrm{d}\boldsymbol{x}.$$
  where $(x, y)^\perp \stackrel{\text{def}}{=} (-y, x)$.

The horizontal impulse and horizontal momentum are directly related after integration by parts: for instance in dimension $d = 1$ and in the flat bottom case
$$\mathscr{M} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_{-d}^{\zeta} \partial_x \Phi \,\mathrm{d}z \,\mathrm{d}x = \mathscr{I} + \lim_{x\to+\infty}\int_{-d}^{\zeta} \Phi \,\mathrm{d}z - \lim_{x\to-\infty}\int_{-d}^{\zeta} \Phi \,\mathrm{d}z,$$
and the latter terms are time-independent (but do not necessarily vanish) as a consequence of the Bernoulli equation and our boundary conditions.

The quantities are preserved in a stronger sense: their integrand satisfies a *conservation law*, which we do not write out explicitly. The interested reader will find in [46] a full account on symmetry groups and conserved quantities of the full Euler system.

2.3 The linearized system

The system (2.2) linearized about the trivial solution, $(\zeta = 0, \psi = 0)$—and hence also around $(\zeta = 0, \psi = \boldsymbol{u} \cdot \boldsymbol{x})$, where $\boldsymbol{u} \in \mathbb{R}^d$ is constant, by Galilean invariance—is explicitly solvable in the flat bottom case. Indeed, setting $\zeta = \epsilon\zeta^0, \psi = \epsilon\psi^0$ and $b = 0$, keeping only first-order terms with respect to small ϵ, one is left with the system
$$\begin{cases} \partial_t \zeta^0 - \mathcal{G}_0 \psi^0 = 0, \\ \partial_t \psi^0 + g\zeta^0 = 0 \end{cases} \tag{2.3}$$
where $\mathcal{G}_0 \stackrel{\text{def}}{=} \mathcal{G}[0, 0]\psi^0 = (\partial_z \Phi^0)\,|_{z=0}$ and Φ^0 is the unique solution to
$$\begin{cases} \Delta_{x,z}\Phi^0 = 0 & \text{in } \mathbb{R}^d \times (-d, 0), \\ \Phi^0 = \psi^0 & \text{on } \mathbb{R}^d \times \{0\}, \\ \partial_z \Phi^0 = 0 & \text{on } \mathbb{R}^d \times \{-d\}. \end{cases} \tag{2.4}$$

Remark 2.4. *It is possible to clarify the above vague statement and rigorously justify eq. (2.3) as an asymptotic model in a small amplitude regime, in the same way we justify shallow water asymptotic models in this document.*

One can "explicitly" solve the Laplace problem, eq. (2.4), using the Fourier transform:

$$\Phi^0 = \frac{\cosh((z+d)|D|)}{\cosh(d|D|)}\psi^0 \tag{2.5}$$

and hence

$$\mathcal{G}_0\psi^0 = |D|\tanh(d|D|)\psi^0.$$

Here we use the convention for the Fourier multiplier operator (see Definition III.1) defined by

$$\widehat{F(D)\varphi}(\boldsymbol{\xi}) = F(\boldsymbol{\xi})\widehat{\varphi}(\boldsymbol{\xi}),$$

where the Fourier transform is applied only on the horizontal variable. Plugging this formula into eq. (2.3), we recognize the linearized "Airy" equations introduced in Section ii.

It is very natural to look for dispersive estimates on the flow map of eq. (2.3). For simplicity, we will restrict our discussion here to dimension $d=1$; see [146, 137] for extensions to more general frameworks. Diagonalizing

$$\mathsf{L}(D) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \mathcal{G}_0 \\ -g & 0 \end{pmatrix} = \mathsf{P}(D)\begin{pmatrix} i(g\mathcal{G}_0)^{1/2} & 0 \\ 0 & -i(g\mathcal{G}_0)^{1/2} \end{pmatrix}\mathsf{P}(D)^{-1}, \qquad \mathsf{P}(D) = \begin{pmatrix} \mathcal{G}_0^{1/2} & \mathcal{G}_0^{1/2} \\ ig^{1/2} & -ig^{1/2} \end{pmatrix},$$

we find it is sufficient to study the following semigroup

$$\mathsf{S}_t = \exp\left(it(g\mathcal{G}_0)^{1/2}\right) = \exp\left(it(g|D|\tanh(d|D|))^{1/2}\right).$$

To this aim, we shall use the standard Littlewood–Paley dyadic decomposition. Let $\chi \in C_0^\infty((-2,2))$ be such that $\chi(\xi)=1$ for $|\xi|\leq 1$, χ is even and non-increasing on $\mathbb{R}^+$, and let $\tilde\chi(\xi) \stackrel{\text{def}}{=} \chi(\xi)-\chi(2\xi)$, and Δ_λ the frequency cut-off operators defined for $\lambda \in 2^{\mathbb{N}}$ by

$$\Delta_{2^N} = \tilde\chi(|D|/(2^N)) \quad (N\in\mathbb{N}^\star).$$

Hence $\sum_{\lambda\in 2^{\mathbb{Z}}}\Delta_\lambda = \mathrm{Id}$. We have the following dispersive estimate.

Lemma 2.5. *Let $d=1$. For any $g,d>0$ and $2\leq r\leq\infty$, there exists $C>0$ such that for any $t\in\mathbb{R}^\star$ and $\lambda=2^N$ with $N\in\mathbb{N}^\star$ and $f\in L^{r'}(\mathbb{R})$ with $\frac{1}{r}+\frac{1}{r'}=1$,*

$$\left|\mathsf{S}_t\Delta_\lambda f\right|_{L^r} \leq C\left(\lambda^{3/4}|t|^{-1/2}\right)^{1-2/r}\left|\Delta_\lambda f\right|_{L^{r'}}.$$

Proof. It suffices to prove the estimate for $r=\infty$ (and hence $r'=1$),[11] the general case is deduced by interpolating with the $r=r'=2$ estimate (using Plancherel equality)

$$\left|\mathsf{S}_t\Delta_\lambda f\right|_{L^2} \leq \left|\Delta_\lambda f\right|_{L^2}.$$

We have

$$(\mathsf{S}_t\Delta_\lambda f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\xi+m(\xi)t)}\tilde\chi(\xi/\lambda)\widehat{f}(\xi)\,\mathrm{d}\xi = \frac{\lambda}{2\pi}\int_{\mathbb{R}} e^{i(\lambda x\xi+m(\lambda\xi)t)}\tilde\chi(\xi)\widehat{f}(\lambda\xi)\,\mathrm{d}\xi,$$

where we denote $m(\xi)=(g|\xi|\tanh(d|\xi|))^{1/2}$. Hence

$$\left|\mathsf{S}_t\Delta_\lambda f\right|_{L^\infty} \lesssim \left|I_{\lambda,t}\right|_{L^\infty}\left|\widehat{f}\right|_{L^\infty} \lesssim \left|I_{\lambda,t}\right|_{L^\infty}\left|f\right|_{L^1}$$

[11] The L^1-L^∞ estimate is of course consistent with the large-time behavior described in Section ii. In particular, the power of λ is sharp, and the estimate does not hold uniformly for any $N\in\mathbb{Z}$.

where

$$I_{\lambda,t}(x) = \lambda \int_{\mathbb{R}} e^{it\phi_\lambda(\xi)} \tilde{\chi}(\xi)\,\mathrm{d}\xi$$

where $\phi_\lambda(\xi) = \lambda\xi x/t + m(\lambda\xi)$. The above is an oscillatory integral which can be estimated thanks to [390, Corollary, p. 334], and using that

$$-m''(\xi) \gtrsim |\xi|(1+|\xi|^2)^{-5/4}.$$

It follows, for any $\lambda \in 2^N$ with $N \in \mathbb{N} \cup \{-1\}$,

$$\big|\mathsf{S}_t\Delta_\lambda f\big|_{L^\infty} \lesssim \big|I_{\lambda,t}\big|_{L^\infty}\big|f\big|_{L^1} \lesssim \lambda|\lambda^{1/2}t|^{-1/2}\big|f\big|_{L^1}.$$

The result follows using the identity $\Delta_\lambda = (\Delta_{\lambda/2} + \Delta_\lambda + \Delta_{2\lambda})\Delta_\lambda$. □

As usual, Strichartz estimates follow from the dispersive estimates in Lemma 2.5.

Proposition 2.6. *Let $d=1$. For any $g,d>0$ and $2 \le r \le \infty$, as well as $2 \le q < \infty$ such that $\frac{2}{q} = \frac{d}{2}(1-\frac{2}{r})$, there exists $C>0$ such that for any $t \in \mathbb{R}^\star$ and $\lambda = 2^N$ with $N \in \mathbb{N}^\star$, any $f \in L^2(\mathbb{R})$,*

$$\big|\mathsf{S}_t\Delta_\lambda f\big|_{L^q_tL^r_x} \le C\lambda^{3d/8(1-2/r)}\big|f\big|_{L^2}.$$

Proof. The proof is standard. Let q' and r' such that $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$. By Minkowski inequality, Lemma 2.5 (replacing Δ_λ with Δ_λ^2, which is obviously valid) and the Hardy–Littlewood–Sobolev inequality with $\frac{1}{q'}+\frac{d}{2}(1-\frac{2}{r}) = 1+\frac{1}{q}$, we have for any F such that $\Delta_\lambda F \in L^{q'}_tL^{r'}_x$

$$\begin{aligned}
\Big\|\int_{\mathbb{R}} \mathsf{S}_{t-s}\Delta_\lambda^2 F(s,\cdot)\,\mathrm{d}s\Big\|_{L^q_tL^r_x} &\le \Big|\int_{\mathbb{R}} \big|\mathsf{S}_{t-s}\Delta_\lambda^2 F(s,\cdot)\big|_{L^r_x}\,\mathrm{d}s\Big|_{L^q_t} \\
&\lesssim \Big|\int_{\mathbb{R}} \big(\lambda^{3d/4}|t-s|^{-1/2}\big)^{1-2/r}\big|\Delta_\lambda F\big|_{L^{r'}_x}\,\mathrm{d}s\Big|_{L^q_t} \\
&\lesssim \lambda^{3d/4(1-2/r)}\Big|\big|\Delta_\lambda F\big|_{L^{r'}_x}\Big|_{L^{q'}_t} = \lambda^{3d/4(1-2/r)}\big\|\Delta_\lambda F\big\|_{L^{q'}_tL^{r'}_x}.
\end{aligned}$$

We conclude with the infamous $TT^\star$ argument; see *e.g.* [397]. □

Remark 2.7. *Proposition 2.6 is still valid when $d \ge 2$; see [146, 137]. The Strichartz estimate exhibits a maximal regularizing effect of $d/8$ derivatives, with respect to the naive estimate following from Bernstein inequality and Parseval equality:*

$$\big|\mathsf{S}_t\Delta_\lambda f\big|_{L^r_x} \lesssim \lambda^{d/2(1-2/r)}\big|\mathsf{S}_t\Delta_\lambda f\big|_{L^2_x} \le \lambda^{d/2(1-2/r)}\big|\Delta_\lambda f\big|_{L^2} \lesssim \big|f\big|_{H^{d/2(1-2/r)}}.$$

We did not include the low-frequency component, $\lambda = 2^N$ with $N \in \mathbb{Z}$ since then the time decay of the dispersive estimate is weaker; see again [137]. Loosely speaking, the conclusion is that dispersive effects are quite weak.

2.4 Non-dimensionalization

We discussed previously the relevance of neglecting some effects (viscosity, friction, *etc.*) based on vague comments on the typical scales of the setting. These comments can be made quantitative after scaling the variables so as to extract the relevant dimensionless parameters which allow to measure the respective strength of various mechanisms. This step is also of tremendous importance to our goal since we shall motivate asymptotic models based on a smallness assumption of such a parameter. The following scaling appears naturally after solving explicitly the linearized system around the trivial solution, eq. (2.3).

We set

$$\boldsymbol{x} = \frac{\mathsf{x}}{\lambda} \quad ; \quad z = \frac{\mathsf{z}}{d} \quad ; \quad t = \mathsf{t}\frac{\sqrt{gd}}{\lambda}$$

and

$$\zeta = \frac{\zeta}{a_{\rm top}} \quad ; \quad b = \frac{b}{a_{\rm bot}} \quad ; \quad \Phi = \varPhi\frac{d}{a_{\rm top}\lambda\sqrt{gd}}.$$

In these formulae, we introduced a typical horizontal wavelength denoted λ as well as $a_{\rm top}$ (resp. $a_{\rm bot}$) denoting the typical amplitude of the surface deformation (resp. bottom topography). We also recognize $c_0 \stackrel{\rm def}{=} \sqrt{gd}$ which is the celerity of infinitesimally long and small waves, and $T = \lambda/c_0$ their time period. With this scaling and introducing the dimensionless parameters

$$\varepsilon = \frac{a_{\rm top}}{d} \quad ; \quad \beta = \frac{a_{\rm bot}}{d} \quad ; \quad \mu = \frac{d^2}{\lambda^2}, \tag{2.6}$$

the dimensional water waves equations, eq. (2.2), becomes

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = 0, \end{cases} \tag{2.7}$$

where we define the (dimensionless) Dirichlet-to-Neumann operator as

$$\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = \left(\partial_z\Phi - \mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi\right)\big|_{z=\varepsilon\zeta}$$

where Φ is the unique solution to

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ -1+\beta b(\boldsymbol{x}) < z < \varepsilon\zeta(\boldsymbol{x})\}, \\ \Phi = \psi & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta(\boldsymbol{x})\}, \\ \partial_z\Phi - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi = 0 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z = -1+\beta b(\boldsymbol{x})\}. \end{cases} \tag{2.8}$$

Now all the variables except for the dimensionless parameters are typically of size $\mathcal{O}(1)$.[12] It is clear from the above that ε measures the strength of the nonlinear effects in the systems, while β measures the magnitude of topography effects. Finally the parameter μ is the so-called shallowness parameter. A small value of the shallowness parameter amounts to assuming that most of the energy of the wave is located at low (spatial) frequencies, and that in some sense "derivatives of the unknowns are smaller than the unknowns". We have also found in the previous section that smallness of the shallowness parameter is related to the weakness of dispersive effects.

As a rule of thumb, typical values of these dimensionless parameters in the context of coastal oceanography range as

$$\varepsilon \in [0, 0.1] \quad ; \quad \beta \in [0, 0.5] \quad ; \quad \mu \in (0, 0.01].$$

Our models will be derived from the assumption that

$$\mu \ll 1 \quad ; \quad \varepsilon, \beta = \mathcal{O}(1) \quad ,$$

although (most of) our results will hold for any triple of parameters $(\mu, \varepsilon, \beta)$ in the shallow-water regime defined in Definition III.2.

[12] More precisely, our results will be valid uniformly for data in a given ball around the origin of a Banach space (typically Sobolev-based spaces with a given index of regularity) and the dependency with respect to the scales of the setting will be measured only through the two dimensionless parameters, ε and μ. Of course, describing a whole set of functions using only a handful of parameters is quite restrictive, and as a consequence our results offer only a rough description of the solutions.

Remark 2.8. *Keeping track of the surface tension effects would yield*

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = \frac{1}{\mathrm{Bo}}\nabla\cdot\left(\frac{\nabla\zeta}{\sqrt{1+\mu\varepsilon^2|\nabla\zeta|^2}}\right), \end{cases} \tag{2.9}$$

where the Bond number, $\mathrm{Bo} = \frac{\rho_0 g\lambda^2}{\sigma}$, *measures the ratio of gravity forces over capillary forces.*

Our choice of scaling has been done having in mind the applications to coastal oceanography, and in particular the range $\mu \in (0,\mu^\star]$. *Another natural framework is that of deep water* $\mu \in [\mu_\star,\infty)$, *for which the usual scaling is*

$$\boldsymbol{x} = \frac{\mathsf{X}}{\lambda} \quad ; \quad z = \frac{\mathsf{z}}{\lambda} \quad ; \quad t = \mathsf{t}\frac{\sqrt{g\lambda}}{\lambda}$$

(the scaling in the variable z *is somehow immaterial and appears only in the Laplace problem) and*

$$\zeta = \frac{\zeta}{a_{\mathrm{top}}} \quad ; \quad b = \frac{b}{a_{\mathrm{bot}}} \quad ; \quad \Phi = \Phi\frac{1}{a_{\mathrm{top}}\sqrt{g\lambda}}.$$

With this scaling the dimensionless problem becomes

$$\begin{cases} \partial_t\zeta - \frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \epsilon\dfrac{(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta,\beta b]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\epsilon^2|\nabla\zeta|^2)} = \frac{1}{\mathrm{Bo}}\nabla\cdot\left(\frac{\nabla\zeta}{\sqrt{1+\epsilon^2|\nabla\zeta|^2}}\right), \end{cases} \tag{2.10}$$

where we introduce a convenient new dimensionless parameter,

$$\epsilon = \varepsilon\sqrt{\mu} = \frac{a_{\mathrm{top}}}{\lambda},$$

representing the typical steepness *of the wave. Notice that in the limit* $\mu \to \infty$, *one has*

$$\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta,\beta b]\psi = \mathcal{G}^\infty[\epsilon\zeta,\beta b]\psi = (\partial_z\Phi^\infty - (\epsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi^\infty)\,\big|_{z=\epsilon\zeta}$$

where Φ^∞ *is the unique solution to*

$$\begin{cases} \Delta_{\boldsymbol{x}}\Phi^\infty + \partial_z^2\Phi^\infty = 0 & \text{in } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \;:\; -\infty < z < \epsilon\zeta(\boldsymbol{x})\}, \\ \Phi^\infty = \psi & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \;:\; z = \epsilon\zeta(\boldsymbol{x})\}, \\ \partial_z\Phi^\infty \to 0 & \text{as } z \to -\infty. \end{cases}$$

Hence eq. (2.10) *also makes sense in the infinite-layer framework* ($\mu = \infty$), *and in this case any reference to the depth* d *has disappeared, as it should.*

If one wants to cover the full range of values $\mu \in (0,\infty)$, *then may as in [268] use the scaling*

$$\boldsymbol{x} = \frac{\mathsf{X}}{\lambda} \quad ; \quad z = \frac{\mathsf{z}}{\lambda} \quad ; \quad t = \mathsf{t}\frac{\sqrt{gd\nu}}{\lambda}$$

and

$$\zeta = \frac{\zeta}{a_{\mathrm{top}}} \quad ; \quad b = \frac{b}{a_{\mathrm{bot}}} \quad ; \quad \Phi = \Phi\frac{1}{a_{\mathrm{top}}\sqrt{gd/\nu}}.$$

with $\nu = \min(1, 1/\sqrt{\mu})$ *(or, say,* $\tanh(\sqrt{\mu})/\sqrt{\mu}$*) which yields*

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla\psi|^2 - \frac{\varepsilon}{\nu}\dfrac{(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \sqrt{\mu}\varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} \\ \qquad\qquad = \frac{1}{\mathrm{Bo}}\nabla\cdot\left(\frac{\nabla\zeta}{\sqrt{1+\mu\varepsilon^2|\nabla\zeta|^2}}\right). \end{cases} \tag{2.11}$$

We conclude this section by noticing that system (2.7) may be equivalently written as

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \varepsilon\underline{\boldsymbol{u}}\cdot\nabla\psi - \frac{\varepsilon}{2}\underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{u}} - \frac{\mu\varepsilon}{2}\underline{w}^2 = 0, \end{cases} \tag{2.7'}$$

where we denote $(\underline{\boldsymbol{u}},\underline{w}) \stackrel{\text{def}}{=} (\nabla_{\boldsymbol{x}}\Phi, \frac{1}{\mu}\partial_z\Phi)\big|_{z=\varepsilon\zeta}$ and used the identities

$$\underline{\boldsymbol{u}} = \nabla\psi - \mu\varepsilon\underline{w}\nabla\zeta \quad \text{and} \quad \underline{w} = \varepsilon\underline{\boldsymbol{u}}\cdot\nabla\zeta + \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = \frac{\varepsilon\nabla\psi\cdot\nabla\zeta + \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi}{1+\mu\varepsilon^2|\nabla\zeta|^2}.$$

2.5 Well-posedness

The full justification of asymptotic models in the shallow water regime is made possible by a well-posedness result in Sobolev spaces proved by Iguchi [224] and Lannes [268] (improving the earlier work in collaboration with Alvarez-Samaniego [14]), and of which we provide below a rough statement. The main point in this result is that it allows to control the size of solutions on a relevant time interval, *uniformly with respect parameters in the shallow water regime*, defined as follows.

Definition (Shallow water asymptotic regime)**.** *Given $\mu^\star > 0$, we let*

$$\mathfrak{p}_{\rm SW} = \big\{(\mu,\varepsilon,\beta) \ : \ \mu\in(0,\mu^\star], \ \varepsilon\in[0,1], \ \beta\in[0,1]\big\}.$$

Theorem 2.9. *Let $d\in\{1,2\}$, $\mu^\star>0$, $h_\star>0$, $a_\star>0$, $M^\star\geq 0$, and $N\geq 5$. There exists $C,T>0$ and an operator $\mathfrak{a} : H^{N+1}(\mathbb{R}^d)\times \mathring{H}^{N+1}(\mathbb{R}^d)\times W^{N+1,\infty}(\mathbb{R}^d)\to \mathcal{C}(\mathbb{R}^d)$ such that for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$, for any $(\zeta_0,\psi_0)\in H^{N+1}(\mathbb{R}^d)\times\mathring{H}^{N+1}(\mathbb{R}^d)$ and $b\in W^{N+1,\infty}(\mathbb{R}^d)$ satisfying*

$$M_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_0\big|_{H^{N+1}} + \big|\varepsilon\nabla\psi_0\big|_{H^N} + \big|\beta b\big|_{W^{N+1,\infty}} \leq M^\star$$

and

$$\inf_{\mathbb{R}^d}\big(1+\varepsilon\zeta_0-\beta b\big)\geq h_\star>0 \quad \textit{and} \quad \inf_{\mathbb{R}^d}\mathfrak{a}[\varepsilon\zeta_0,\varepsilon\psi_0,\beta b]\geq a_\star>0,$$

there exists a unique $(\zeta,\psi)\in\mathcal{C}^0([0,T/M_0);H^N(\mathbb{R}^d)\times\mathring{H}^{N-1/2}(\mathbb{R}^d))$ classical solution to eq. (2.7) with initial data $(\zeta,\psi)\big|_{t=0}=(\zeta_0,\psi_0)$. Moreover, one has for any $t\in(0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^N} + \big|\nabla\psi(t,\cdot)\big|_{H^{N-3/2}} \leq C\big(\big|\zeta_0\big|_{H^{N+1}} + \big|\nabla\psi_0\big|_{H^N}\big)$$

and $\inf_{\mathbb{R}^d}\big(1+\varepsilon\zeta(t,\cdot)-\beta b\big)\geq h_\star/2$, $\inf_{\mathbb{R}^d}\mathfrak{a}[\varepsilon\zeta,\varepsilon\psi,\beta b]\geq a_\star/2$.

Remark 2.10. *The loss of derivatives in the statement is only apparent: we can define an energy functional and a corresponding functional space which is propagated by the flow,*[13] *and we can prove in fact the well-posedness in the sense of Hadamard (i.e. with the continuity of the flow with respect to the initial data). Oversimplifying, Theorem 2.9 relies in one part on estimates on the Dirichlet-to-Neumann operator refining the elliptic theory covered in Section 4, and on the other part on energy estimates, viewing the water waves equations as a quasilinear system—when written using the appropriate (Alinhac's) unknowns—in the same way we will consider the Saint-Venant system (in Section 5.3), the Green-Naghdi system (in Section 8.5 and Section 8.6) and fully dispersive counterparts (in Section 10.5) and the Isobe–Kakinuma systems (in Section 13.6) later on.*

[13] Specifically, the proof relies on the control of the energy functional

$$\mathcal{E}_s(t) \stackrel{\text{def}}{=} \big|\zeta(t,\cdot)\big|_{H^N}^2 + \big|(\mathcal{G}_0^\mu)^{1/2}\psi\big|_{H^{N-1}}^2 + \sum_{|\boldsymbol{k}|=0}^{N}\big|\big(\partial^{\boldsymbol{k}}(\mathcal{G}_0^\mu)^{1/2}\psi - \varepsilon\underline{w}\partial^{\boldsymbol{k}}(\mathcal{G}_0^\mu)^{1/2}\zeta\big)(t,\cdot)\big|_{L^2}^2,$$

with $\boldsymbol{k}\in\mathbb{N}^d$ multi-indices and $\mathcal{G}_0^\mu = \frac{1}{\sqrt{\mu}}|D|\tanh(\sqrt{\mu}|D|)\approx\frac{|D|^2}{1+\sqrt{\mu}|D|}$; see [268, §4].

Incidentally, a nice proof of the quasilinear nature of the water waves system was recently provided in [369]. The operator $\mathfrak{a}$ *naturally arises as a hyperbolicity condition on the system. Specifically, it is defined as*[14]

$$\mathfrak{a}[\varepsilon\zeta, \varepsilon\psi, \beta b] \stackrel{\text{def}}{=} 1 + \varepsilon(\partial_t + \varepsilon\underline{\boldsymbol{u}} \cdot \nabla)\underline{w},$$

where we recall $(\underline{\boldsymbol{u}}, \underline{w}) \stackrel{\text{def}}{=} (\nabla_{\boldsymbol{x}}\Phi, \frac{1}{\mu}\partial_z\Phi)\big|_{z=\varepsilon\zeta}$. *It is also physically motivated since it is equivalent to the Rayleigh–Taylor criterion, namely* $\inf_{\mathbb{R}^d}(-\partial_z P\big|_{z=\varepsilon\zeta}) > 0$*; see [268, Proposition 4.29]. The water waves equations is ill-posed if this criterion is violated [170]. The Rayleigh–Taylor criterion is automatically satisfied as soon as we restrict the set of parameters* $\mathfrak{p}_{\rm SW}$ *to* $\varepsilon\mu$ *or* $\varepsilon^2\beta\mu$ *being sufficiently small; see [268, §4.3.5].*

Theorem 2.9 is only one of the well-posedness results on the water waves equations, and is neither the oldest nor the sharpest one. The reader can refer to [268, footnote 4, p. 102] for earlier important references. Among them one can point out as specially relevant to our future discussion the works by Ovsjannikov [350, 351] and Kano and Nishida [244, 245, 243], which set the foundation—after the formal expansion procedure described by Friedrichs in [392, Appendix A]—for the rigorous justification of shallow water models for data with analytic regularity, and [339, 422, 117] for pioneering explorations in finite-regularity (Sobolev) spaces.

As for posterior results, Alazard, Burq and Zuily (see [11] among other works) have extracted the paradifferential structure of the water waves equations, which allowed to considerably lower the regularity threshold for which well-posedness holds. More recently, an impressive body of literature has been dedicated to the delicate study of the large-time behavior of solutions—such as global or almost-global existence results, scattering or modified scattering for small initial data—depending on the dimension d, the domain $\mathbb{R}^d$ or $\mathbb{T}^d$, the presence of surface tension, *etc.*. A comprehensive account with extensive references can be found for instance in [136, 6, 332]. These latter results typically rely on the dispersive nature of the system and hence do not hold uniformly with respect to the parameter $\mu \ll 1$. A detailed study of the interplay between these results and smallness assumptions of the shallowness parameter is yet to be accomplished.

In the opposite direction, the existence time is finite in general at least in the infinite-depth situation and dimension $d = 1$: (i) regular initial data exist whose interface is a graph and such that the corresponding solution to the water waves system generates a finite time singularity, as the interface ceases to be a graph; and (ii) regular initial data exist such that the solution to the water waves system written in coordinates that allow overturning waves produces a self-intersecting interface in finite time; see [82, 81, 80] and the survey [115].

Another interesting question arises when comparing Theorem 2.9 and the result in [11] as in the latter the bottom topography can be very wild, due to smoothing effects on the contribution of the bottom topography. Yet this smoothing effect is not uniform with respect to the parameter $\mu \ll 1$ (see discussion in [268, §2.5.3, 3.7.2 and A.4]), and the important issue of the behavior of shallow water waves over "rough", random or non-smooth topographies is still not fully understood from a mathematical viewpoint, despite important advances in [210, 366, 337, 186, 367, 122, 123, 127, 86]; see also [85] for an extended discussion and other relevant references. Concerning the influence of large but smooth bottom topography one can also ask whether the existence time in Theorem 2.9 can be proved to be uniform with respect to the parameter β. A partial result, using surface tension has been obtained in [308]; see also the discussion in Section 8.7.

2.6 Traveling waves

It is not the place—nor the author—to review the rich theory of traveling waves for the water waves system, let alone the much richer one concerning the full Euler equation. Let me simply mention, for comparison with corresponding results on subsequent models, that in the unidimensional ($d = 1$)

[14] The right-hand side may be defined pointwise in time (that is, without involving time derivatives) from $(\varepsilon\zeta_0, \varepsilon\psi_0, \beta b)$ after solving the Laplace problem and using the water waves equations; see [268, §4.3.1].

and flat bottom ($b \equiv 0$) framework, the existence of smooth solitary waves[15] has been obtained by Lavrent′ev [274], Friedrichs and Hyers [187], Beale [43] and Mielke [313]. These waves are of small amplitude and supercritical ($c > 1$). The limit of large (but finite) amplitude solutions has been investigated by Amick and Toland [400, 23], leading to the proof of Stokes' conjecture that (solitary) waves of greatest height are characterized by a sharp crest with angle $2\pi/3$; see [22, 359].

These results only scratch the surface of the theory—spanning over more than a century—concerning special solutions to eq. (2.7) or the infinite-depth counterpart, with or without surface tension (see Remark 2.8), which include periodic traveling waves [279, 394], three-dimensional ($d = 2$) waves [128, 203, 67], standing waves [360] and quasi-periodic waves; see [183] and references in the Literature paragraph therein, and the surveys in [112, 64, 211] for more details.

[15] that is $(\zeta, \psi) \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R})^2$ solutions to eq. (2.7) of the form

$$(\zeta, \psi)(t, x) = (\zeta_c, \psi_c)(x - ct), \qquad \lim_{|x| \to \infty} |(\zeta_c, \psi_c')|(x) = 0.$$

3 Interfacial waves system

While the two assumptions of homogeneity, eq. (1.3a), and irrotationality, eq. (1.3b), are extremely beneficial, as we have seen in the preceding section, they can be seen as too restrictive and indeed miss important physical phenomena; see *e.g.* [235, 213] and references therein. We will not consider the case of homogeneous water waves with vorticity in this work,[16] but will slightly unveil some properties in the wild world of inhomogeneous water waves.

The simplest setting one can imagine is that of interfacial waves, where the fluid consists in two layers separated by a free interface. This is physically sound in locations where the water contains fresh/warm water above denser salted/cold water, and if the separation between the two (the *pycnocline*) is sharp. If the fluids inside each layer are assumed homogeneous and flows irrotational, then we can build an extension to the water waves equations in this bilayer framework. While having a similar structure as the water wave equations, eq. (2.2), and in particular consisting in a handful of scalar equations for unknown functions of time and horizontal space variables only, the resulting equations are more intricate and possess interesting new features. Among them we shall emphasize the role of the density contrast, and the emergence of Kelvin–Helmholtz instabilities.

3.1 Derivation

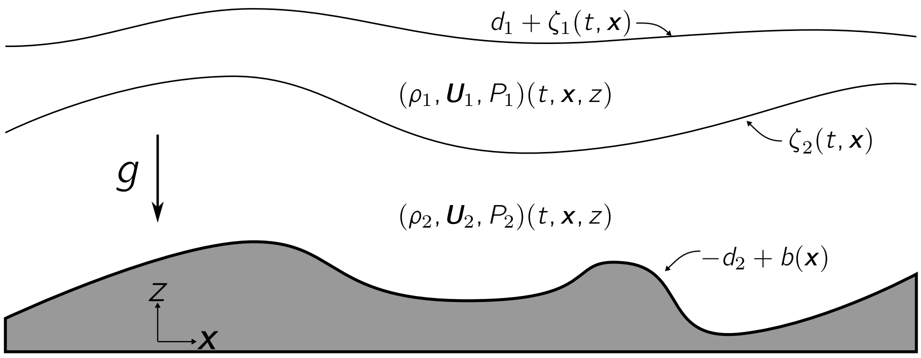


Figure 3.1: Sketch of the domain and notations.

In this section we consider special solutions to the full Euler system, eq. (1.1)-(1.2), satisfying that the fluid can be split into two layers with homogeneous densities and irrotational velocities; see Figure 3.1. Denote the two fluid domains as

$$\Omega_1^t \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ \zeta_2(t, \boldsymbol{x}) < z < d_1 + \zeta_1(t, \boldsymbol{x})\}$$

and

$$\Omega_2^t \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ -d_2 + b(\boldsymbol{x}) < z < \zeta_2(t, \boldsymbol{x})\}$$

delimited by the free surface, the interface and the bottom topography:

$$\begin{aligned}
\Gamma_{\text{top}} &\stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = d_1 + \zeta_1(t, \boldsymbol{x})\}, \\
\Gamma_{\text{int}} &\stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \zeta_2(t, \boldsymbol{x})\}, \\
\Gamma_{\text{bot}} &\stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = -d_2 + b(\boldsymbol{x})\}.
\end{aligned}$$

[16] Let me simply mention [84] for an extension of Theorem 2.9 to the rotational framework and [113] for a seminal work on traveling waves with vorticity.

Assuming[17] that the fluid is homogeneous in each layer:

$$\rho \equiv \rho_\ell > 0 \qquad \text{in } \Omega_\ell^t \qquad (\ell \in \{1,2\}),$$

and that the flows are potential in each layer:

$$\boldsymbol{U} = \nabla_{x,z}\Phi_\ell \qquad \text{in } \Omega_\ell^t \qquad (\ell \in \{1,2\}),$$

and imposing the kinematic boundary condition at the interface:[18]

$$\partial_t\zeta_2 = \partial_z\Phi_1 - (\nabla\zeta_2)\cdot\nabla_x\Phi_1 = \partial_z\Phi_2 - (\nabla\zeta_2)\cdot\nabla_x\Phi_2 \qquad \text{in } \Gamma_{\rm int},$$

and the continuity of the pressure at the interface (in addition to the surface), we can follow the strategy in Section 2.1 and (see *e.g.* [152]) rewrite the full Euler equations (1.1)-(1.2) equivalently as the following system, which we refer to as the ***interfacial waves system***.

$$\begin{cases} \partial_t\zeta_1 - \mathcal{G}_1[\zeta_1,\zeta_2,b](\varphi_1,\psi_2) = 0, \\ \partial_t\zeta_2 - \mathcal{G}_2[\zeta_2,b]\psi_2 = 0, \\ \partial_t\varphi_1 + g\zeta_1 + \frac{1}{2}|\nabla\varphi_1|^2 - \dfrac{(\mathcal{G}_1[\zeta_1,\zeta_2,b]\varphi_1 + \nabla\zeta_1\cdot\nabla\varphi_1)^2}{2(1+|\nabla\zeta_1|^2)} = 0, \\ \partial_t\big(\rho_2\psi_2 - \rho_1\psi_1\big) + g(\rho_2-\rho_1)\zeta_2 + \frac{1}{2}\big(\rho_2|\nabla\psi_2|^2 - \rho_1|\nabla\psi_1|^2\big) \\ \qquad - \dfrac{\rho_2(\mathcal{G}_2[\zeta_2,b]\psi_2 + \nabla\zeta_2\cdot\nabla\psi_2)^2 - \rho_1(\mathcal{G}_2[\zeta_2,b]\psi_2 + \nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta_2|^2)} = 0, \end{cases} \tag{3.1}$$

where $\mathcal{G}_1[\zeta_1,\zeta_2,b](\varphi_1,\psi_2)$, $\mathcal{G}_2[\zeta_2,b]\psi_2$ and $\psi_1 = \mathcal{H}[\zeta_1,\zeta_2,b](\varphi_1,\psi_2)$ are defined by

$$\begin{aligned} \mathcal{G}_2[\zeta_2,b]\psi_2 &= \big(\partial_z\Phi_2 - (\nabla\zeta_2)\cdot\nabla_x\Phi_2\big)\big|_{z=\zeta_2}, \\ \mathcal{G}_1[\zeta_1,\zeta_2,b](\varphi_1,\psi_2) &= \big(\partial_z\Phi_1 - (\nabla\zeta_1)\cdot\nabla_x\Phi_1\big)\big|_{z=d_1+\zeta_1}, \\ \psi_1 = \mathcal{H}[\zeta_1,\zeta_2,b](\varphi_1,\psi_2) &= \Phi_1\big|_{z=\zeta_2}, \end{aligned}$$

where Φ_2 is uniquely determined by

$$\begin{cases} \Delta_{x,z}\Phi_2 = 0 & \text{in } \Omega_2^t, \\ \Phi_2 = \psi_2 & \text{on } \Gamma_{\rm int}, \\ \partial_z\Phi_2 - (\nabla b)\cdot\nabla_x\Phi_2 = 0 & \text{on } \Gamma_{\rm bot}, \end{cases} \tag{3.2}$$

and then Φ_1 is uniquely determined by

$$\begin{cases} \Delta_{x,z}\Phi_1 = 0 & \text{in } \Omega_1^t, \\ \Phi_1 = \varphi_1 & \text{on } \Gamma_{\rm top}, \\ \partial_z\Phi_1 - (\nabla\zeta_2)\cdot\nabla_x\Phi_1 = \mathcal{G}_2[\zeta_2,b]\psi_2 & \text{on } \Gamma_{\rm int}. \end{cases} \tag{3.3}$$

By the superposition principle, we have the decomposition

$$\Phi_1 = \Phi_{1,\rm D} + \Phi_{1,\rm N}$$

where

$$\begin{cases} \Delta_{x,z}\Phi_{1,\rm D} = 0 & \text{in } \Omega_1^t, \\ \Phi_{1,\rm D} = \varphi_1 & \text{on } \Gamma_{\rm top}, \\ \partial_z\Phi_{1,\rm D} - (\nabla\zeta_2)\cdot\nabla_x\Phi_{1,\rm D} = 0 & \text{on } \Gamma_{\rm int}, \end{cases} \qquad \begin{cases} \Delta_{x,z}\Phi_{1,\rm N} = 0 & \text{in } \Omega_1^t, \\ \Phi_{1,\rm N} = 0 & \text{on } \Gamma_{\rm top}, \\ \partial_z\Phi_{1,\rm N} - (\nabla\zeta_2)\cdot\nabla_x\Phi_{1,\rm N} = \mathcal{G}_2[\zeta_2,b]\psi_2 & \text{on } \Gamma_{\rm int}, \end{cases}$$

[17] As in the one-layer framework, homogeneity and irrotationality assumptions are needed only at initial time, and then propagate for positive times.

[18] Hence we assume that particles of fluid initially at the interface remain trapped at the interface, or in other words that no mixing occurs between the two fluids. Notice in particular that while the normal component of the velocity is continuous at the interface, the horizontal—or tangential—components cannot remain continuous even if this holds at initial time.

and we consistently decompose, with hopefully obvious definitions,

$$\mathcal{G}_1[\zeta_1,\zeta_2,b](\varphi_1,\psi_2) = \mathcal{G}_{1,\mathrm{D}}[\zeta_1,\zeta_2]\varphi_1 + \mathcal{G}_{1,\mathrm{N}}[\zeta_1,\zeta_2]\mathcal{G}_2[\zeta_2,b]\psi_2$$
$$\mathcal{H}[\zeta_1,\zeta_2,b](\varphi_1,\psi_2) = \mathcal{H}_{\mathrm{D}}[\zeta_1,\zeta_2]\varphi_1 + \mathcal{H}_{\mathrm{N}}[\zeta_1,\zeta_2]\mathcal{G}_2[\zeta_2,b]\psi_2$$

In particular the operators $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$ are well-defined (and continuous) in suitable functional Sobolev or Beppo Levi spaces by the following result, which follows from the analysis in Section 4, and in particular Proposition 4.5 and Proposition 4.20.

Proposition 3.1. *Let $(\zeta_1,\zeta_2,b)\in W^{2,\infty}(\mathbb{R}^d)$ such that*

$$\begin{cases} d_2+\zeta_2-b\geq d_\star>0 \\ d_1+\zeta_1-\zeta_2\geq d_\star>0 \end{cases} \qquad \text{in } \mathbb{R}^d. \tag{3.4}$$

Then for any $\varphi_1,\psi_2\in \mathring{H}^2(\mathbb{R}^d)$, there exist unique $\Phi_\ell\in\mathring{H}^2(\Omega_\ell^t)$ ($\ell\in\{1,2\}$) strong solution to eq. (3.2)-(3.3).

3.1.1 The rigid-lid assumption

In the study of interfacial waves in the oceanographic context, one often uses the so-called *rigid-lid* assumption, based on the observation that the amplitude of interfacial waves are of several order of magnitudes greater than surface waves. Hence we assume altogether that the surface is flat (this is only an approximation of course, whose validity is discussed below, and here we depart from exact solutions to the full Euler equations), that is

$$\Omega_1^t \stackrel{\text{def}}{=} \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ \zeta_2(t,\boldsymbol{x})<z<d_1\} \quad ; \quad \Gamma_{\mathrm{top}} \stackrel{\text{def}}{=} \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z=d_1+\zeta_1(t,\boldsymbol{x})\},$$

and eq. (3.1) becomes

$$\begin{cases} \partial_t\zeta_2-\mathcal{G}_2[\zeta_2,b]\psi_2=0, \\ \partial_t\big(\rho_2\psi_2-\rho_1\psi_1\big)+g(\rho_2-\rho_1)\zeta_2+\frac12\big(\rho_2|\nabla\psi_2|^2-\rho_1|\nabla\psi_1|^2\big) \\ \qquad\qquad -\dfrac{\rho_2(\mathcal{G}_2[\zeta_2,b]\psi_2+\nabla\zeta_2\cdot\nabla\psi_2)^2-\rho_1(\mathcal{G}_2[\zeta_2,b]\psi_2+\nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta_2|^2)}=0, \end{cases} \tag{3.5}$$

where $\mathcal{G}_2[\zeta_2,b]\psi_2$ and $\psi_1=\mathcal{H}[\zeta_2,b]\psi_2$ are defined (mind the abuse of notation) by

$$\mathcal{G}_2[\zeta_2,b]\psi_2 = (\partial_z\Phi_2-(\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_2)\big|_{z=\zeta_2},$$
$$\psi_1=\mathcal{H}[\zeta_2,b]\psi_2=\Phi_1\big|_{z=\zeta_2},$$

where Φ_2 is uniquely determined as above by

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi_2=0 & \text{in } \Omega_2^t, \\ \Phi_2=\psi_2 & \text{on } \Gamma_{\mathrm{int}}, \\ \partial_z\Phi_2-(\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi_2=0 & \text{on } \Gamma_{\mathrm{bot}}, \end{cases} \tag{3.6}$$

and then Φ_1 is determined up to a harmless additive constant by the Laplace problem with rigid-lid

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi_1=0 & \text{in } \Omega_1^t, \\ \partial_z\Phi_1=0 & \text{on } \Gamma_{\mathrm{top}}, \\ \partial_z\Phi_1-(\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_1=\mathcal{G}_2[\zeta_2,b]\psi_2 & \text{on } \Gamma_{\mathrm{int}}. \end{cases} \tag{3.7}$$

Again, the operators $\mathcal{G}_2$ and $\mathcal{H}$ are well-defined (and continuous) in suitable functional Sobolev or Beppo Levi spaces; see [54, Appendix A] or [267, Proposition 1]. System (3.5) is seemingly simpler that system (3.1) since it contains only two scalar equations. It allows to concentrate on the propagation of interfacial waves while the bilayer system with free surface describes the coupled propagation of surface and interfacial waves. In the limit of small density contrast, $1 - \frac{\rho_1}{\rho_2} \ll 1$, one can observe a decoupling of the two modes of propagation, the surface waves propagating much faster than interfacial waves. The validity of the rigid-lid assumption lies in that regime, which is somewhat similar to the weakly incompressible limit. This analogy is quite apparent (and rigorously justified) for the hydrostatic equations, studied in Section 6.2.5.

Another formulation equivalent to eq. (3.5) is the following:

$$\begin{cases} \partial_t \zeta_2 = \mathcal{G}_1[\zeta_2]\psi_1 = \mathcal{G}_2[\zeta_2, b]\psi_2, \\ \partial_t \psi_1 + g\zeta_2 + \frac{1}{2}|\nabla\psi_1|^2 - \dfrac{(\mathcal{G}_1[\zeta_2]\psi_1 + \nabla\zeta_2 \cdot \nabla\psi_1)^2}{2(1 + |\nabla\zeta_2|^2)} = -\frac{1}{\rho_1} p_{\rm int} \\ \partial_t \psi_2 + g\zeta_2 + \frac{1}{2}|\nabla\psi_2|^2 - \dfrac{(\mathcal{G}_2[\zeta_2, b]\psi_2 + \nabla\zeta_2 \cdot \nabla\psi_2)^2}{2(1 + |\nabla\zeta_2|^2)} = -\frac{1}{\rho_2} p_{\rm int}, \end{cases} \tag{3.8}$$

where $\mathcal{G}_2$ is as above and (mind the abuse of notation)

$$\mathcal{G}_1[\zeta_2]\psi_1 = (\partial_z \Phi_1 - (\nabla\zeta_2) \cdot \nabla_x \Phi_1)\,\big|_{z=\zeta_2}$$

where Φ_1 is uniquely determined by the Laplace problem

$$\begin{cases} \Delta_{x,z}\Phi_1 = 0 & \text{in } \Omega_1^t, \\ \partial_z \Phi_1 = 0 & \text{on } \Gamma_{\rm top}, \\ \Phi_1 = \psi_1 & \text{on } \Gamma_{\rm int}. \end{cases} \tag{3.9}$$

The variable $p_{\rm int}$, representing the pressure at the interface, can be interpreted as a Lagrange multiplier associated with the compatibility condition $\mathcal{G}_1[\zeta_2]\psi_1 = \mathcal{G}_2[\zeta_2, b]\psi_2$. While eq. (3.8) seems more complicated than the previous formulation, it is beneficial that both $\mathcal{G}_1$ and $\mathcal{G}_2$ are Dirichlet-to-Neumann operators, which are studied in details in Section 4, while $\mathcal{H}[\zeta_2, b] = \mathcal{G}_1[\zeta_2]^{-1} \circ \mathcal{G}_2[\zeta_2, b]$ requires the analysis of its inverse; see again [267].

3.1.2 The Boussinesq approximation

Another standard approximation for systems with weak density contrast is the so-called *Boussinesq approximation*. Here we neglect the density difference in all but buoyancy terms (that is the ones with a g prefactor) in the Bernoulli equation in eq. (3.5):

$$\begin{cases} \partial_t \zeta_2 - \mathcal{G}_2[\zeta_2, b]\psi_2 = 0, \\ \partial_t\big(\psi_2 - \psi_1\big) + g(\rho_2 - \rho_1)\zeta_2 + \frac{1}{2}\big(|\nabla\psi_2|^2 - |\nabla\psi_1|^2\big) \\ \qquad - \dfrac{(\mathcal{G}_2[\zeta_2, b]\psi_2 + \nabla\zeta_2 \cdot \nabla\psi_2)^2 - (\mathcal{G}_2[\zeta_2, b]\psi_2 + \nabla\zeta_2 \cdot \nabla\psi_1)^2}{2(1 + |\nabla\zeta_2|^2)} = 0. \end{cases} \tag{3.5'}$$

(the definitions of $\psi_1 = \mathcal{H}[\zeta_2, b]\psi_2$ and the operators $\mathcal{G}_1$ and $\mathcal{H}$ are unchanged). This yields some striking simplifications in the analysis, in particular on the hydrostatic equations; see Section 6.2.1.

We could do the same approximation in the free-surface equations, eq. (3.1), but this would be somehow inconsistent as the rigid-lid assumption and Boussinesq approximation stem from the same hypothesis.

3.1.3 Interfacial tension

As we shall discuss below, the Cauchy problem for eq. (3.5)—or eq. (3.5')—is ill-posed in Sobolev spaces, due to the so-called Kelvin–Helmholtz instabilities. However, these instabilities disappear (at least from the modal study as we shall see below; see [267] for the rigorous nonlinear study) when interfacial tension effects are added. [19] Taking into account a density jump proportional to the mean curvature at the interface yields the following generalization of eq. (3.5):

$$\begin{cases} \partial_t\zeta_2 - \mathcal{G}_2[\zeta_2,b]\psi_2 = 0, \\ \partial_t\big(\rho_2\psi_2 - \rho_1\psi_1\big) + g(\rho_2-\rho_1)\zeta_2 + \frac{1}{2}\big(\rho_2|\nabla\psi_2|^2 - \rho_1|\nabla\psi_1|^2\big) \\ \qquad -\dfrac{\rho_2(\mathcal{G}_2[\zeta_2,b]\psi_2 + \nabla\zeta_2\cdot\nabla\psi_2)^2 - \rho_1(\mathcal{G}_2[\zeta_2,b]\psi_2 + \nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta_2|^2)} = \sigma\nabla\cdot\Big(\frac{\nabla\zeta_2}{\sqrt{1+|\nabla\zeta_2|^2}}\Big), \end{cases} \tag{3.10}$$

where σ is the interfacial tension coefficient. Of course we could do the same for the free-surface equations, eq. (3.1), with or without surface tension.

3.2 Variational structure

Zakharov's canonical Hamiltonian formulation to the water waves equations (recall Section 2.2) can be generalized to the bilayer framework, and the Hamiltonian functional is still the total (perturbation of the) energy of the system, written with suitable variables. This was put forward in [45, 120] in the rigid-lid (or rather infinite-depth) case, and in [20, 121] for the free-surface case.

The free-surface case Consider

$$\begin{aligned}\mathscr{H} &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} g(\rho_2-\rho_1)\zeta_2^2 + g\rho_1\zeta_1^2 + (\rho_2\psi_2-\rho_1\psi_1)\mathcal{G}_2[\zeta_2,b]\psi_2 + \rho_1\varphi_1\mathcal{G}_1[\zeta_1,\zeta_2,b](\varphi_1,\psi_2)\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} C + \int_{-d_2+b}^{\zeta_2} \rho_2 g z + \frac{\rho_2}{2}|\nabla_{\boldsymbol{x},z}\Phi_2|^2\,\mathrm{d}z + \int_{\zeta_2}^{d_1+\zeta_1} \rho_1 g z + \frac{\rho_1}{2}|\nabla_{\boldsymbol{x},z}\Phi_1|^2\,\mathrm{d}z.\end{aligned}$$

where C is a constant tailored so that the second integral is finite. Viewing $\mathscr{H}$ as a functional for $(\zeta_1,\zeta_2,\xi_1 \stackrel{\text{def}}{=} \rho_1\varphi_1, \xi_2 \stackrel{\text{def}}{=} \rho_2\psi_2-\rho_1\psi_1)$, one can check that—at least formally—eq. (3.1) reads

$$\partial_t\begin{pmatrix}\zeta_1\\ \xi_1\\ \zeta_2\\ \xi_2\end{pmatrix} = \begin{pmatrix}\delta_{\xi_1}\mathscr{H}\\ -\delta_{\zeta_1}\mathscr{H}\\ \delta_{\xi_2}\mathscr{H}\\ -\delta_{\zeta_2}\mathscr{H}\end{pmatrix}.$$

Alternatively, we can define the Lagrangian of the system with the difference between potential and kinetic energy; see [121, (2.23)].

Associated with the Hamiltonian formulation and natural symmetry groups of the system are preserved quantities (invariants). Related to the variation of base level for the velocity potentials are the obvious conservation of the excess of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z}_\ell = 0, \qquad \mathscr{Z}_\ell \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\zeta_\ell\,\mathrm{d}\boldsymbol{x} \qquad (\ell\in\{1,2\}).$$

From horizontal translation invariance (in the flat bottom case) we obtain the conservation of the horizontal impulse

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\zeta_1\nabla\xi_1 + \zeta_2\nabla\xi_2\,\mathrm{d}\boldsymbol{x} \qquad (\text{if } b\equiv 0).$$

From time translation invariance we obtain the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = 0.$$

[19]It should be warned however that there is no interfacial tension effects in the oceanographic context where the top fluid is warm and fresh water and the bottom fluid is cold and salted water. Interfacial tension should be considered as a mathematical artifact, in view of explaining the existence and robustness of interfacial waves despite Kelvin–Helmholtz instabilities. See the discussion in the prologue of Chapter E.

The rigid-lid case Consider now

$$\begin{aligned}\mathscr{H} &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} g(\rho_2-\rho_1)\zeta_2^2 + (\rho_2\psi_2-\rho_1\psi_1)G_2[\zeta_2,b]\psi_2\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^d} C + \int_{-d_2+b}^{\zeta_2} \rho_2 g z + \frac{\rho_2}{2}|\nabla_{\boldsymbol{x},z}\Phi_2|^2\,\mathrm{d}z + \int_{\zeta_2}^{d_1} \rho_1 g z + \frac{\rho_1}{2}|\nabla_{\boldsymbol{x},z}\Phi_1|^2\,\mathrm{d}z.\end{aligned}$$

where C is a constant tailored so that the second integral is finite. Viewing $\mathscr{H}$ as a functional for $(\zeta_2, \xi_2 \stackrel{\text{def}}{=} \rho_2\psi_2 - \rho_1\psi_1)$, one obtains the canonical formulation for eq. (3.5):

$$\partial_t \begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix} = \begin{pmatrix}\delta_{\xi_2}\mathscr{H}\\ -\delta_{\zeta_2}\mathscr{H}\end{pmatrix}.$$

Alternatively, we can define the Lagrangian of the system with the difference between potential and kinetic energy; see [121, above (2.15)].

Associated with the Hamiltonian formulation and natural symmetry groups of the system are preserved quantities (invariants). Related to the variation of base level for the velocity potentials are the obvious conservation of the excess of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = 0, \qquad\qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta_2\,\mathrm{d}\boldsymbol{x}.$$

From horizontal translation invariance (in the flat bottom case) we obtain the conservation of the horizontal impulse

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad\qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta_2 \nabla\xi_2\,\mathrm{d}\boldsymbol{x} \qquad\qquad (\text{if } b\equiv 0).$$

From time translation invariance we obtain the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = 0.$$

See [45] for more details.

3.3 Modal analysis

The free-surface case We linearize (3.1) about the trivial solution ($\zeta_1 = \zeta_2 = 0, \varphi_1 = \psi_2 = 0$). While this can be straightforwardly generalized to $\varphi_1 = \psi_2 = \boldsymbol{u}\cdot\boldsymbol{x}$ by Galilean invariance, it should be noticed that the assumption $\varphi_1 = \psi_2$ is strong and consequential, and the case of different background velocities is tackled in the rigid-lid framework below. Setting $\zeta_\ell = \epsilon\zeta_\ell^0, \psi_\ell = \epsilon\psi_\ell^0$ ($\ell \in \{1,2\}$) and $b = 0$, keeping only first-order terms with respect to small ϵ, one obtains

$$\begin{cases} \partial_t\zeta_1^0 - G_1[0,0,0](\varphi_1^0,\psi_2^0) = 0,\\ \partial_t\zeta_2^0 - G_2[0,0]\psi_2^0 = 0,\\ \partial_t\varphi_1^0 + g\zeta_1^0 = 0,\\ \partial_t\big(\rho_2\psi_2^0 - \rho_1\mathcal{H}[0,0,0](\varphi_1^0,\psi_2^0)\big) + g(\rho_2-\rho_1)\zeta_2^0 = 0,\end{cases} \tag{3.11}$$

where $G_1[0,0,0]$, $G_2[0,0]$ and $\mathcal{H}[0,0,0]$ are explicitly found by solving in (horizontal) Fourier space

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi_2^0 = 0 & \text{in } \mathbb{R}^d\times(-d_2,0),\\ \Phi_2^0 = \psi_2^0 & \text{on } \mathbb{R}^d\times\{0\},\\ \partial_z\Phi_2^0 = 0 & \text{on } \mathbb{R}^d\times\{-d_2\},\end{cases} \quad \text{and} \quad \begin{cases} \Delta_{\boldsymbol{x},z}\Phi_1^0 = 0 & \text{in } \mathbb{R}^d\times(0,d_1),\\ \Phi_1^0 = \varphi_1^0 & \text{on } \mathbb{R}^d\times\{d_1\},\\ \partial_z\Phi_1^0 = G_2[0,0]\psi_2^0 & \text{on } \mathbb{R}^d\times\{0\}.\end{cases}$$

We find

$$\Phi_2^0 = \frac{\cosh((z+d_2)|D|)}{\cosh(d_2|D|)}\psi_2^0 \quad ; \quad \Phi_1^0 = \frac{\cosh(z|D|)}{\cosh(d_1|D|)}\varphi_1^0 + \frac{\sinh((z-d_1)|D|)}{\cosh(d_1|D|)}\tanh(d_2|D|)\psi_2^0$$

and hence

$$\begin{aligned} \mathcal{G}_2[0,0]\psi_2^0 &= |D|\tanh(d_2|D|)\psi_2^0, \\ \mathcal{G}_1[0,0,0](\varphi_1^0,\psi_2^0) &= |D|\tanh(d_1|D|)\psi_2^0 + \frac{|D|\tanh(d_2|D|)}{\cosh(d_1|D|)}\psi_2^0, \\ \mathcal{H}[0,0,0](\varphi_1^0,\psi_2^0) &= \frac{1}{\cosh(d_1|D|)}\varphi_1^0 - \tanh(d_1|D|)\tanh(d_2|D|)\psi_2^0. \end{aligned}$$

Plugging these expressions into eq. (3.11) yields the dispersion relation

$$\omega(\boldsymbol{\xi})^4 - g|\boldsymbol{\xi}|\rho_2\frac{\tanh(d_1|\boldsymbol{\xi}|)+\tanh(d_2|\boldsymbol{\xi}|)}{\rho_2+\rho_1\tanh(d_1|\boldsymbol{\xi}|)\tanh(d_2|\boldsymbol{\xi}|)}\omega(\boldsymbol{\xi})^2 + g^2|\boldsymbol{\xi}|^2\frac{(\rho_2-\rho_1)\tanh(d_1|\boldsymbol{\xi}|)\tanh(d_2|\boldsymbol{\xi}|)}{\rho_2+\rho_1\tanh(d_1|\boldsymbol{\xi}|)\tanh(d_2|\boldsymbol{\xi}|)} = 0.$$

This equation has two non-negative solutions (and their opposite) if $0 \le \rho_1 \le \rho_2$ (we implicitly assume $g>0$), corresponding to the "stable" case wherein the lower fluid is heavier than the upper one. Taking the limit $\rho_1 \to 0$, we observe

$$\omega_\ell(\boldsymbol{\xi})^2 \to g|\boldsymbol{\xi}|\tanh(d_\ell|\boldsymbol{\xi}|) \quad \text{as } \rho_1 \to 0 \qquad (\ell \in \{1,2\}),$$

and we recognize the dispersion relation of the (one-layer) linearized water waves equations, with reference depth d_1 and d_2. Taking the limit $\rho_1 \to \rho_2$, we observe a strong separation between the two modes of propagation:

$$\begin{cases} \omega_+(\boldsymbol{\xi})^2 \to g|\boldsymbol{\xi}|\tanh((d_1+d_2)|\boldsymbol{\xi}|) \\ \omega_-(\boldsymbol{\xi})^2 \sim g|\boldsymbol{\xi}|\frac{(\rho_2-\rho_1)\tanh(d_1|D|)\tanh(d_2|D|)}{\rho_2\tanh(d_1|D|)+\rho_1\tanh(d_2|D|)} \end{cases} \qquad \text{as } \rho_1 \nearrow \rho_2.$$

The first solution corresponds to the barotropic (or surface) mode, while the second one corresponds to the baroclinic mode, and is of interest for the propagation of interfacial waves. While for fixed wave number, the baroclinic mode has considerably lower group or phase velocity than the corresponding barotropic one, it should be noticed that there always exist a larger wavenumber for which the barotropic velocity will coincide. See [124, 125, 126] for a study of the effect of long baroclinic (interfacial) waves to short barotropic (surface) waves, and Figure 3.2 for an illustration.

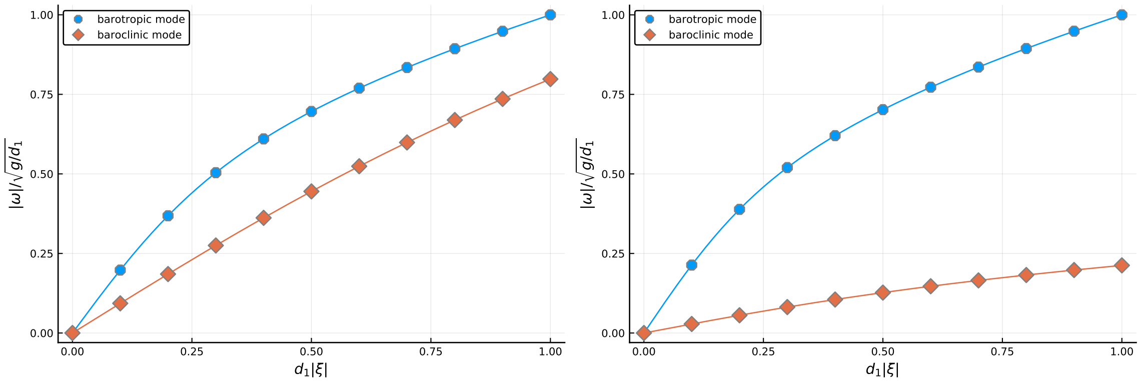
The rigid-lid case Now we linearize eq. (3.8), about the constant shear solution:

$$(\zeta_2 = \epsilon\zeta^0, \psi_1 = \boldsymbol{u}_1\cdot\boldsymbol{x} + \epsilon\psi_1^0, \psi_2 = \boldsymbol{u}_2\cdot\boldsymbol{x} + \epsilon\psi_2^0)$$

in the flat bottom case, $b \equiv 0$. Notice that contrary to above, we use here different (but constant) background velocities in the two layers. The shear velocity $\boldsymbol{v} \stackrel{\text{def}}{=} \boldsymbol{u}_2 - \boldsymbol{u}_1$ is invariant by Galilean transformation. In this framework we mimic the behavior of small perturbations to *any* solutions.[20] Keeping only first-order terms in ϵ yields the following closed system for the perturbations ζ^0 and $\psi^0 = \frac{\rho_2}{\rho_1+\rho_2}\psi_2^0 - \frac{\rho_1}{\rho_1+\rho_2}\psi_1^0$ (see *e.g.* [266, 87, 45, 273])

$$\begin{cases} \partial_t\zeta^0 + \boldsymbol{c}(D)\cdot\nabla\zeta^0 - b(D)\psi^0 = 0, \\ \partial_t\psi^0 + a(D)\zeta^0 + \boldsymbol{c}(D)\cdot\nabla\psi^0 = 0, \end{cases}$$

[20] The reader can refer to Section 15.2.2 for a display of such approach, and in particular to Remark 15.6 for a comparison with the linearization about the constant shear solutions. For our purpose the outcome—that is the presence of high frequency instabilities due to shear velocities—is captured equally by both approaches.



(a) Small upper density, $\rho_1/\rho_2 = 0.1$.   (b) Small density contrast, $\rho_1/\rho_2 = 0.9$.

Figure 3.2: Wave frequencies, $|\omega|(d_1|\boldsymbol{\xi}|)$, according to eq. (3.11). $d_2 = 4d_1$.

where

$$\begin{aligned}
a(D) &\stackrel{\text{def}}{=} g\frac{\rho_2-\rho_1}{\rho_1+\rho_2} - \frac{\rho_1\rho_2/(\rho_1+\rho_2)}{\rho_2\tanh(d_1|D|)+\rho_1\tanh(d_2|D|)}\frac{(\boldsymbol{v}\cdot D)^2}{|D|},\\
b(D) &\stackrel{\text{def}}{=} \frac{(\rho_1+\rho_2)\tanh(d_1|D|)\tanh(d_2|D|)}{\rho_2\tanh(d_1|D|)+\rho_1\tanh(d_2|D|)}|D|,\\
\boldsymbol{c}(D) &\stackrel{\text{def}}{=} \frac{\rho_2\tanh(d_1|D|)\boldsymbol{u}_2+\rho_1\tanh(d_2|D|)\boldsymbol{u}_1}{\rho_2\tanh(d_1|D|)+\rho_1\tanh(d_2|D|)},
\end{aligned}$$

and hence we have the dispersion relation

$$(\omega(\boldsymbol{\xi}) - \boldsymbol{c}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2 = a(\boldsymbol{\xi})b(\boldsymbol{\xi}). \tag{3.12}$$

Notice incidentally that for $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \mathbf{0}$, we obtain the dispersion relation of the baroclinic mode in the small density contrast limit.

Since $b(\boldsymbol{\xi}) > 0$ for $\boldsymbol{\xi} \neq \mathbf{0}$, we find that the Fourier mode with wave vector $\boldsymbol{\xi}$ is exponentially amplified (*i.e.* unstable) if $a(\boldsymbol{\xi}) < 0$. This occurs for $|\boldsymbol{\xi}|$ sufficiently large as soon as $\boldsymbol{v} \neq \mathbf{0}$, which brings to light the role of shear velocities. This phenomenon is known as *Kelvin–Helmholtz instabilities*. Moreover, the exponential rate grows proportionally to $|\boldsymbol{v}\cdot\boldsymbol{\xi}|$ as $|\boldsymbol{\xi}| \to \infty$ and in particular takes arbitrarily large values.

In the opposite direction, as $|\boldsymbol{\xi}| \to 0$, we have $a(\boldsymbol{\xi}) \stackrel{\text{def}}{=} g\frac{\rho_2-\rho_1}{\rho_1+\rho_2} - \frac{\rho_1\rho_2/(\rho_1+\rho_2)}{\rho_2 d_1+\rho_1 d_2}(\frac{\boldsymbol{v}\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|})^2 + \mathcal{O}(|\boldsymbol{v}||\boldsymbol{\xi}|^2)$, so that Kelvin–Helmholtz modal instabilities do not appear for small wavenumbers provided that $\rho_2 > \rho_1$ and the shear velocity is sufficiently small. This is consistent with the well-posedness results obtained on the (nonlinear) Saint-Venant systems presented in Section 6.

In order to tame Kelvin–Helmholtz instabilities, it has been suggested to include surface tension effects, as in Section 3.1.3. Indeed for eq. (3.10) the dispersion relation becomes

$$(\omega(\boldsymbol{\xi}) - \boldsymbol{c}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2 = a_\sigma(\boldsymbol{\xi})b(\boldsymbol{\xi}), \qquad a_\sigma(\boldsymbol{\xi}) = a(\boldsymbol{\xi}) + \frac{\sigma}{\rho_1+\rho_2}|\boldsymbol{\xi}|^2$$

and hence, for any $\sigma > 0$, $a_\sigma(\boldsymbol{\xi}) > 0$ for $|\boldsymbol{\xi}|$ sufficiently large. Moreover, for $\boldsymbol{v}$ sufficiently small, Fourier modes are stable for all wave vectors. We refer again to [273] for an extended modal analysis, and to [267] (and references therein) for the rigorous nonlinear approach. See Figure 3.3 for a numerical illustration.

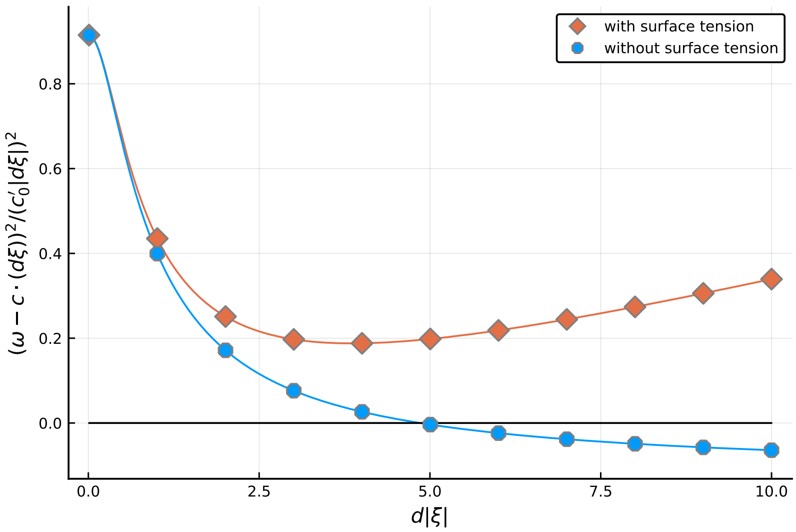


Figure 3.3: Dispersion relation. We plot $(\omega(d\boldsymbol{\xi}) - \boldsymbol{c}(d\boldsymbol{\xi})\cdot d\boldsymbol{\xi})^2/|c_0' d\boldsymbol{\xi}|^2 = a_\sigma(d\boldsymbol{\xi})b(d\boldsymbol{\xi})/|c_0' d\boldsymbol{\xi}|^2$; negative values indicate unstable modes.
We set $\rho_1/\rho_2 = 0.9$, $d_1 = d_2/4$, $2\boldsymbol{v} = c_0' = \sqrt{g\frac{(\rho_2-\rho_1)d_1 d_2}{\rho_2 d_1 + \rho_1 d_2}}$.
In the presence of surface tension, $\mathrm{Bo} = \frac{g(\rho_2-\rho_1)d_1^2}{\sigma} = 15$.

3.4 Non-dimensionalization

As for the water waves equations (see Section 2.4), we non-dimensionalize the equations as a first step before introducing asymptotic models.

The free-surface case We set

$$\boldsymbol{x} = \frac{\mathsf{x}}{\lambda} \quad ; \quad z = \frac{\mathsf{z}}{d_1} \quad ; \quad t = \mathsf{t}\frac{c_0'}{\lambda}$$

and

$$\zeta_1 = \frac{\zeta}{a_{\rm top}} \quad ; \quad \zeta_2 = \frac{\zeta}{a_{\rm int}} \quad ; \quad b = \frac{b}{a_{\rm bot}} \quad ; \quad \Phi_\ell = \Phi_\ell \frac{d_1}{a_{\rm int}\lambda c_0'}.$$

In these formulae, we introduced a typical horizontal wavelength denoted λ as well as $a_{\rm top}$ (resp. $a_{\rm int}$, $a_{\rm bot}$) denoting the typical amplitude of the surface deformation (resp. interface deformation, bottom topography). We also set $c_0' \stackrel{\rm def}{=} \sqrt{g\frac{(\rho_2-\rho_1)d_1 d_2}{\rho_2 d_1+\rho_1 d_2}}$ which is the celerity of infinitesimally long and small internal waves—based on eq. (3.12)—and $T = \lambda/c_0'$ their time period. We also introduce the dimensionless parameters

$$\varepsilon = \frac{a_{\rm int}}{d_1} \quad ; \quad \beta = \frac{a_{\rm bot}}{d_1} \quad ; \quad \mu = \frac{d^2}{\lambda^2} \quad ; \quad \alpha = \frac{a_{\rm top}}{a_{\rm int}} \quad ; \quad \delta = \frac{d_1}{d_2} \quad ; \quad \gamma = \frac{\rho_1}{\rho_2}. \tag{3.13}$$

In addition to ε measuring the strength of the nonlinear effects in the systems, β measuring the magnitude of topography effects and μ the shallowness parameter, we added three dimensionless parameters; namely α the amplitude ratio of surface deformations to interface deformations (being somewhat artificial, it is straightforwardly removed by redefining ζ_1), δ the ratio of the upper-layer to the lower-layer depth, and γ the ratio of the mass density between the two fluids. For stability reasons explained in Section 3.3, we shall always assume $\gamma \in [0, 1)$. However the interval is not closed, and the interesting limit (both from a physical and mathematical perspectives) of weak density contrast, $\gamma \nearrow 1$, will be studied in the hyperbolic framework in Section 6.2.5. Setting $\gamma = 0$ and $\delta = 1$, we recover the one-layer water waves equations. We have used d_1 as the reference depth; a choice which is harmless as we shall give some upper and lower bounds on the depth ratio, δ; see for instance [373] and references therein for some studies in the physically sound regime where the lower layer is much deeper than the upper layer. To summarize, the results in this manuscript concerning interfacial waves will be restricted to parameters in the following set.

Definition (Shallow water/Shallow water asymptotic regime)**.** *Given* $\mu^\star, \delta_\star, \delta^\star > 0$*, we let*

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} = \big\{(\mu, \varepsilon, \beta, \alpha, \delta, \gamma) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1], \ \alpha \in [0,1], \ \delta \in [\delta_\star, \delta^\star], \gamma \in [0,1)\big\}.$$

Using the above scalings the dimensional free-surface interfacial waves equations, eq. (3.1), becomes

$$\begin{cases} \alpha\partial_t\zeta_1 - \frac{1}{\mu}\mathcal{G}_1^{\mu,\delta}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2, \beta b](\varphi_1, \psi_2) = 0, \\ \partial_t\zeta_2 - \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 = 0, \\ \partial_t\varphi_1 + \frac{\delta+\gamma}{1-\gamma}\alpha\zeta_1 + \frac{\varepsilon}{2}|\nabla\varphi_1|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_1^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2, \beta b](\varphi_1, \psi_2) + \alpha\varepsilon\nabla\zeta_1\cdot\nabla\varphi_1)^2}{2(1+\mu|\alpha\varepsilon\nabla\zeta_1|^2)} = 0, \\ \partial_t\big(\psi_2 - \gamma\psi_1\big) + (\delta+\gamma)\zeta_2 + \frac{\varepsilon}{2}\big(|\nabla\psi_2|^2 - \gamma|\nabla\psi_1|^2\big) \\ \qquad -\mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_2)^2 - \gamma(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)} = 0, \end{cases} \tag{3.14}$$

where $\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2$, $\mathcal{G}_1^{\mu,\delta}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2, \beta b](\varphi_1, \psi_2) = \mathcal{G}_{1,\rm D}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\varphi_1 + \mathcal{G}_{1,\rm N}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2$, and $\psi_1 = \mathcal{H}_{\rm D}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\varphi_1 + \mathcal{H}_{\rm N}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2$ are defined by

$$\begin{aligned} \mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 &= (\partial_z\Phi_2 - \mu(\varepsilon\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_2)\big|_{z=\varepsilon\zeta_2}, \\ \mathcal{G}_{1,\rm D}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\varphi_1 &= (\partial_z\Phi_{1,\rm D} - \mu(\alpha\varepsilon\nabla\zeta_1)\cdot\nabla_{\boldsymbol{x}}\Phi_{1,\rm D})\big|_{z=1+\alpha\varepsilon\zeta_1}, \\ \mathcal{G}_{1,\rm N}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 &= (\partial_z\Phi_{1,\rm N} - \mu(\alpha\varepsilon\nabla\zeta_1)\cdot\nabla_{\boldsymbol{x}}\Phi_{1,\rm N})\big|_{z=1+\alpha\varepsilon\zeta_1}, \\ \mathcal{H}_{\rm D}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2, b]\varphi_1 &= \Phi_{1,\rm D}\big|_{z=\varepsilon\zeta_2}, \\ \mathcal{H}_{\rm N}^{\mu}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2, b]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 &= \Phi_{1,\rm N}\big|_{z=\varepsilon\zeta_2}, \end{aligned}$$

where Φ_2 is uniquely determined by

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ -1/\delta + \beta b < z < \varepsilon\zeta_2\}, \\ \Phi_2 = \psi_2 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta_2\}, \\ \partial_z\Phi_2 - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi_2 = 0 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = -1/\delta + \beta b\}, \end{cases}$$

and then $\Phi_{1,\rm D}$ and $\Phi_{1,\rm N}$ are uniquely determined by

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_{1,\rm D} + \partial_z^2\Phi_{1,\rm D} = 0 & \text{in } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ \varepsilon\zeta_2 < z < 1 + \alpha\varepsilon\zeta_1\}, \\ \Phi_{1,\rm D} = \varphi_1 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = 1 + \alpha\varepsilon\zeta_1\}, \\ \partial_z\Phi_{1,\rm D} - \mu(\varepsilon\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_{1,\rm D} = 0 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta_2\}. \end{cases}$$

and

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_{1,\rm N} + \partial_z^2\Phi_{1,\rm N} = 0 & \text{in } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ \varepsilon\zeta_2 < z < 1 + \alpha\varepsilon\zeta_1\}, \\ \Phi_{1,\rm N} = 0 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = 1 + \alpha\varepsilon\zeta_1\}, \\ \partial_z\Phi_{1,\rm N} - \mu(\varepsilon\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_{1,\rm N} = \mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 & \text{on } \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta_2\}. \end{cases}$$

The rigid-lid case We proceed as above, although $\alpha\zeta_1$ is now irrelevant, so that the shallow water regime in the rigid-lid situation is defined as follows.

Definition (Shallow water/Shallow water asymptotic regime). *Given $\mu^\star, \delta_\star, \delta^\star > 0$, we let*

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} = \big\{(\mu,\varepsilon,\beta,\delta,\gamma)\ :\ \mu\in(0,\mu^\star],\ \varepsilon\in[0,1],\ \beta\in[0,1],\ \delta\in[\delta_\star,\delta^\star],\gamma\in[0,1)\big\}.$$

Then we can check that eq. (3.8) becomes

$$\begin{cases} \partial_t\zeta_2 = \frac{1}{\mu}\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1 = \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2, \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_1|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)} = -\gamma^{-1}p_{\rm int} \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_2|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_2)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)} = -p_{\rm int}, \end{cases} \tag{3.15}$$

where $\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2$ and $\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1$ are defined by

$$\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 = \big(\partial_z\Phi_2 - \mu(\varepsilon\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_2\big)\big|_{z=\varepsilon\zeta_2},$$
$$\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1 = \big(\partial_z\Phi_1 - \mu(\varepsilon\nabla\zeta_2)\cdot\nabla_{\boldsymbol{x}}\Phi_1\big)\big|_{z=\varepsilon\zeta_2},$$

where Φ_1 and Φ_2 are uniquely determined by the Laplace problems

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ -1/\delta+\beta b < z < \varepsilon\zeta_2\}, \\ \Phi_2 = \psi_2 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=\varepsilon\zeta_2\}, \\ \partial_z\Phi_2 - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi_2 = 0 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z = -1/\delta+\beta b\}, \end{cases}$$

and

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_1 + \partial_z^2\Phi_1 = 0 & \text{in } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ \varepsilon\zeta_2 < z < 1\}, \\ \partial_z\Phi_1 = 0 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=1\}, \\ \Phi_1 = \psi_1 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=\varepsilon\zeta_2\}. \end{cases}$$

Again, the variable $p_{\rm int}$ can be interpreted as a Lagrange multiplier associated with the compatibility condition $\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1 = \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2$, and physically represents the dimensionless pressure at the interface, scaled as

$$p_{\rm int} \stackrel{\rm def}{=} \frac{d_1}{\rho_2 a_{\rm int}(c_0')^2}\mathsf{p}_{\rm int}.$$

Using the formulation with only two equations, namely eq. (3.5), and adding surface tension effects for the sake of completeness (and hence using rather eq. (3.10)), we obtain

$$\begin{cases} \partial_t\zeta_2 = \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2, \\ \partial_t(\psi_2-\gamma\psi_1) + (\delta+\gamma)\zeta_2 + \frac{\varepsilon}{2}\big(|\nabla\psi_2|^2 - \gamma|\nabla\psi_1|^2\big) \\ \qquad -\dfrac{(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_2)^2 - \gamma(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_1 + \varepsilon\nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)} \\ \qquad\qquad = \frac{\delta+\gamma}{\rm Bo}\nabla\cdot\Big(\frac{\nabla\zeta_2}{\sqrt{1+\mu|\varepsilon\nabla\zeta_2|^2}}\Big), \end{cases} \tag{3.16}$$

where ${\rm Bo} = \frac{g(\rho_2-\rho_1)\lambda^2}{\sigma}$ is the Bond dimensionless number measuring the ratio of gravity forces over capillary forces (obviously, neglecting surface tension effects consists in setting ${\rm Bo}=\infty$), and $\psi_1 = \mathcal{G}_1^{\mu}[\varepsilon\zeta_2]^{-1}\circ\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2$. Once again, we immediately recover the one-layer case, eq. (2.9), by setting $\gamma=0$ and $\delta=1$. The Boussinesq approximation consists in setting $\gamma=1$ in eq. (3.16);[21]

[21] although not (necessarily) in the prefactors $(\delta+\gamma)$ and $\frac{\delta+\gamma}{\rm Bo}$. Of course, γ therein can indeed be set to 1 by a slightly different choice of scalings.

notice the contribution of the buoyancy term is not neglected, whereas it would have been if we had naively set $\rho_1 = \rho_2$ in eq. (3.10). Alternatively, we can use the formulation (3.15) and replace $\gamma^{-1}p_{\rm int}$ with $p_{\rm int}$ in the second equation (or use a reference density in both pressure contributions; see Remark 6.18).

3.5 Well-posedness

The well-posedness result for the water waves system presented in Section 3.5 does not extend to interfacial gravity waves systems, as soon as $\rho_1 \neq 0$. Indeed, recall that the modal analysis in Section 3.3 shows that in the absence of surface tension, the interfacial waves system—at least in the rigid-lid framework—exhibits strong (Kelvin–Helmholtz) instabilities. Consistently, ill-posedness results in finite-regularity spaces have obtained in [170, 227, 277, 242, 417]. However, the (local-in-time) well-posedness of the initial-value problem may be restored by including the effects of surface tension [16, 19, 92, 384, 385]. Moreover, Lannes exhibits in [267] a stability criterion for the existence of strong regular solutions to the (rigid-lid) system with surface tension, that is eq. (3.16), which is obviously not uniform with respect to the Bond dimensionless number measuring the ratio of gravity forces over capillary forces, $\mathrm{Bo} = \frac{g(\rho_2-\rho_1)\lambda^2}{\sigma}$, but weakens as the shallow water parameter decreases. More precisely the nonlinear criterion therein is satisfied as soon as the dimensionless number $\Upsilon \stackrel{\rm def}{=} \gamma^2\mu\varepsilon^4\,\mathrm{Bo}$ is sufficiently small. Again this is in full accordance with the modal analysis; see [273].

3.6 Traveling waves

In Section 2.6 I refused to survey the vast literature on special solutions to the water waves system. The literature in the non-homogeneous case—even restricting the study of two layers with irrotational homogeneous flows—is even more intricate, with specific new phenomena including the role of the rigid-lid versus free-surface assumption, the role of (weak or strong) interfacial tension, and the existence of bores and generalized solitary waves. A comprehensive review of known results is yet to be accomplished as far as I am aware. Let me lazily refer to [90, 211] for some relevant references, in particular concerning the existence of solitary waves, periodic waves and fronts.

4 The Laplace problem and Dirichlet-to-Neumann operator

We provide here a brief account on some essential results concerning the Laplace problem underlying the Dirichlet-to-Neumann operator. Indeed, crucial estimates on and approximations of the Dirichlet-to-Neumann operators follow from related estimates on the Laplace problem. The latter are obtained following standard tools of elliptic problems, with special attention to the dependence with respect to the boundary of the domain—since it stands for a variable of the time-evolution problem—and to dimensionless parameters. Most of the material of this section is given with more details and sharper estimates in [268, Ch. 2&3]. However, significant modifications have been made so as to provide as simple proofs as possible.

Recall the (scaled) Dirichlet-to-Neumann operator is defined for sufficiently smooth data as

$$\mathcal{G}^{\mu}[\varepsilon\zeta, \beta b]\psi = \left(\partial_z \Phi - \mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi\right)\big|_{z=\varepsilon\zeta}$$

where Φ is the unique solution (see below) to

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega = \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ -1+\beta b < z < \varepsilon\zeta\}, \\ \Phi = \psi & \text{on } \Gamma_{\rm top} = \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta\}, \\ \partial_z\Phi - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi = 0 & \text{on } \Gamma_{\rm bot} = \{(\boldsymbol{x}, z)\in\mathbb{R}^{d+1} \ : \ z = -1+\beta b\}. \end{cases} \tag{4.1}$$

In this section, we drop any reference to the time variable, which acts as a parameter. We always assume thereafter the non-cavitation assumption:

Assumption 4.1. *We have $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ and satisfy*

$$\forall \boldsymbol{x}\in\mathbb{R}^d, \qquad h(\boldsymbol{x}) = 1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0.$$

4.1 Flattening the domain

It is convenient to change variables so as to rewrite the constant-coefficient Laplace equations in a variable domain as variable-coefficient equations in a fixed domain; here the strip $\mathcal{S} \stackrel{\rm def}{=} \mathbb{R}^d \times (-1, 0)$. We choose here the most obvious diffeomorphism for simplicity, since we are not too concerned by regularity issues. [22] Let us define

$$\Sigma : \begin{array}{ccc} \mathcal{S} & \to & \Omega \\ (\boldsymbol{x}, z) & \mapsto & \big(\ \boldsymbol{x}\, , \, (1+\varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}))z + \varepsilon\zeta(\boldsymbol{x})\ \big) \end{array}.$$

Of course this defines a diffeomorphism from the strip, $\mathcal{S}$, to the fluid domain, Ω, by Assumption 4.1. For sufficiently regular $\Phi, \psi, \mathrm{R}, r_{\rm bot}$ satisfying

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi + \partial_z^2\Phi = \mathrm{R} & \text{in } \Omega, \\ \Phi = \psi & \text{on } \Gamma_{\rm top}, \\ \partial_z\Phi - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi = r_{\rm bot} & \text{on } \Gamma_{\rm bot}, \end{cases} \tag{4.2}$$

we have that $\mathit{\Phi} \stackrel{\rm def}{=} \Phi\circ\Sigma$, and $R \stackrel{\rm def}{=} \mathrm{R}\circ\Sigma$ satisfies

$$\begin{cases} \frac{1}{\partial_z\sigma}\nabla^{\mu}_{\boldsymbol{x},z}\cdot P(\Sigma)\nabla^{\mu}_{\boldsymbol{x},z}\mathit{\Phi} = R & \text{in } \mathbb{R}^d\times(-1,0), \\ \mathit{\Phi} = \psi & \text{on } \mathbb{R}^d\times\{0\}, \\ \boldsymbol{e}_{d+1}\cdot P(\Sigma)\nabla^{\mu}_{\boldsymbol{x},z}\mathit{\Phi} = r_{\rm bot} & \text{on } \mathbb{R}^d\times\{-1\}. \end{cases} \tag{4.3}$$

[22] see [268] or [224] for more involved—regularizing—diffeomorphisms which are useful for obtaining optimal regularity estimates. The latter ones are crucial when studying the well-posedness of the water waves problem, but not so much for deriving asymptotic models since we allow losses of derivatives.

where we denote $\nabla^\mu_{\boldsymbol{x},z} = (\sqrt{\mu}\nabla, \partial_z)^\top$, $\boldsymbol{e}_{d+1}$ is the unit (upward) vector in the vertical direction, $\sigma(\boldsymbol{x}, z) \stackrel{\text{def}}{=} (1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}))z + \varepsilon\zeta(\boldsymbol{x})$ and

$$P(\Sigma) \stackrel{\text{def}}{=} \begin{pmatrix} (\partial_z\sigma)\,\mathrm{Id}_d & -\sqrt{\mu}\nabla_{\boldsymbol{x}}\sigma \\ -\sqrt{\mu}\nabla_{\boldsymbol{x}}^\top\sigma & \frac{1+\mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma} \end{pmatrix}.$$

That the two problems are equivalent for sufficiently regular solutions can be straightforwardly checked by chain rules. It also holds true for less regular, *variational solutions*, from which the elliptic theory can be built on. To this aim, it is convenient to subtract the trace of the velocity potential, ψ, to the solutions, so as to work with functions with value zero at the surface. Specifically we introduce the following functional spaces: $H^1_{0,\mathrm{top}}(\Omega)$ the completion of $\mathcal{D}(\Omega \cup \Gamma_{\mathrm{bot}})$ in $H^1(\Omega)$, and $H^1_{0,\mathrm{top}}(\mathcal{S})$ the completion of $\mathcal{D}(\mathbb{R}^d \times [-1, 0))$ in $H^1(\mathcal{S})$. Notice that the above closures could equivalently use the Beppo-Levi (semi) norm $\|\bullet\|_{\mathring{H}^1} \stackrel{\text{def}}{=} \|\nabla_{\boldsymbol{x},z}\bullet\|_{L^2}$ thanks to Poincaré's inequality: for any $\phi \in \mathcal{D}(\Omega \cup \Gamma_{\mathrm{bot}})$,

$$\begin{aligned}\|\phi\|^2_{L^2(\Omega)} = \iint_\Omega |\phi(\boldsymbol{x}, z)|^2 \,\mathrm{d}z\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d}\int_{-1+\beta b(\boldsymbol{x})}^{\epsilon\zeta(\boldsymbol{x})} \Big| \int_z^{\varepsilon\zeta(\boldsymbol{x})} \partial_z\phi(\boldsymbol{x}, z')\,\mathrm{d}z'\Big|^2 \,\mathrm{d}z\,\mathrm{d}\boldsymbol{x} \\ \le \Big(\sup_{\mathbb{R}^d}(1 + \varepsilon\zeta - \beta b)\Big)^2 \|\partial_z\phi\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.4}$$

We shall also make use of the following trace formula:

$$\big|\phi\,\big|_{z=-1+\beta b}\big|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\Big(\int_{-1+\beta b(\boldsymbol{x})}^{\epsilon\zeta(\boldsymbol{x})} \partial_z\phi(\boldsymbol{x}, z)\,\mathrm{d}z\Big)^2 \mathrm{d}\boldsymbol{x} \le \Big(\sup_{\mathbb{R}^d}(1 + \varepsilon\zeta - \beta b)\Big) \|\partial_z\phi\|^2_{L^2(\Omega)}. \tag{4.5}$$

By a density argument, eq. (4.4) and eq. (4.5) hold for any $\phi \in H^1_{0,\mathrm{top}}(\Omega)$, and obviously replacing the domain Ω with the strip $\mathcal{S}$. The latter is easily extended to $\phi \in H^1(\Omega)$ using a smooth truncation function.

Definition 4.2 (Variational solutions). *Let $\psi \in \mathring{H}^1(\mathbb{R}^d)$ and ζ, b satisfying Assumption 4.1. We say that Φ is a variational solution to eq. (4.1) if there exists $\widetilde{\Phi} \in H^1_{0,\mathrm{top}}(\Omega)$ such that $\Phi = \psi + \widetilde{\Phi}$ and for any $\varphi \in H^1_{0,\mathrm{top}}(\Omega)$,*

$$\iint_\Omega \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi} \cdot \nabla^\mu_{\boldsymbol{x},z}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = -\mu \iint_\Omega \nabla\psi \cdot \nabla_{\boldsymbol{x}}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z.$$

Let additionally $\boldsymbol{R} \in L^2(\mathcal{S})$. We say that $\mathit{\Phi}$ is a variational solution to eq. (4.3) with remainder terms $(\partial_z\sigma)R = \nabla^\mu_{\boldsymbol{x},z} \cdot \boldsymbol{R}$ and $r_{\mathrm{bot}} = \boldsymbol{e}_{d+1} \cdot \boldsymbol{R}\big|_{z=-1}$ if there exists $\widetilde{\mathit{\Phi}} \in H^1_{0,\mathrm{top}}(\mathcal{S})$ such that $\mathit{\Phi} = \psi + \widetilde{\mathit{\Phi}}$ and for any $\varphi \in H^1_{0,\mathrm{top}}(\mathcal{S})$,

$$\iint_{\mathcal{S}} \nabla^\mu_{\boldsymbol{x},z}\widetilde{\mathit{\Phi}} \cdot P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \iint_{\mathcal{S}} \big(\boldsymbol{R} - P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\psi\big) \cdot \nabla^\mu_{\boldsymbol{x},z}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z.$$

In the formula above we identified $\boldsymbol{x} \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\mathbb{R}^d)$ and $(\boldsymbol{x}, z) \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\Omega)$ or $\mathring{H}^1(\mathcal{S})$.

Remark 4.3. *We have specified a particular form of remainder terms because these are the ones appearing in the proof of Proposition 4.5, below. We could treat in a similar way more general remainder terms, which would then be useful to justify asymptotic models, as in [268]. We however use a slightly different path, and Proposition 4.5 will be used* in fine *only with $\boldsymbol{R} = \boldsymbol{0}$.*

Remark 4.4 (Assumption on the bottom topography). *In the following, we choose to work with regular but not asymptotically flat topography,* i.e. *$b \in W^{n,\infty}(\mathbb{R}^d)$. All the results below are valid (with the same proof) for less regular but square-integrable $b \in H^n(\mathbb{R}^d)$, and may be refined to $b \in L^\infty(\mathbb{R}^d) \cap \mathring{H}^n(\mathbb{R}^d)$.*

4.2 The Laplace problem

The following result guarantees the existence and uniqueness of the above variational solutions, consistency with respect to strong and classical solutions for sufficiently regular data, as well as very useful regularity estimates. These results are not sharp; see [268, Sect. 2] for a much more thorough analysis.

Proposition 4.5. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. We have the following.*

i. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$ and $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and any $\psi \in \mathring{H}^1(\mathbb{R}^d)$, there exists a unique variational solution, Φ, to eq. (4.1). Setting additionally $\boldsymbol{R} \in L^2(\mathcal{S})^{d+1}$, there exists a unique variational solution, $\mathit{\Phi}$, to eq. (4.3). If $\boldsymbol{R} = \boldsymbol{0}$, one has $\mathit{\Phi} = \Phi \circ \Sigma$.

If, moreover, $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$, $\boldsymbol{R} \in H^1(\mathcal{S})^{d+1}$ and $\psi \in \mathring{H}^2(\mathbb{R}^d)$, then $\nabla_{\boldsymbol{x},z}\mathit{\Phi} \in H^1(\mathcal{S})$ and $\mathit{\Phi}$ is a strong solution to eq. (4.3), i.e. *the identities hold in $L^2(\mathcal{S}) \times \mathring{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.*

ii. Let $M \geq 0$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$ and any $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\big|\varepsilon\zeta\big|_{W^{1,\infty}} + \big|\beta b\big|_{W^{1,\infty}} \leq M$, for any $\boldsymbol{R} \in L^2(\mathcal{S})^{d+1}$, the variational solution to eq. (4.3) satisfies

$$\big\|\nabla^\mu_{\boldsymbol{x},z}\mathit{\Phi}\big\|_{L^2(\mathcal{S})} \leq C\big(\big\|\boldsymbol{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{L^2(\mathbb{R}^d)}\big).$$

iii. Let $k \in \mathbb{N}^\star$, $M \geq 0$. There exists C_k such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, $\zeta \in H^{\max\{1+k, 2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{1+k,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

for any $\psi \in \mathring{H}^{1+k}(\mathbb{R}^d)$ and $\boldsymbol{R} \in L^2(\mathcal{S})^{d+1}$ such that $\Lambda^k\boldsymbol{R} \in L^2(\mathcal{S})^{d+1}$, the strong solution to eq. (4.3) satisfies $\Lambda^k\nabla^\mu_{\boldsymbol{x},z}\mathit{\Phi} \in L^2(\mathcal{S})$ and

$$\begin{aligned}\big\|\Lambda^k\nabla^\mu_{\boldsymbol{x},z}\mathit{\Phi}\big\|_{L^2(\mathcal{S})} &\leq C_k\big(\big\|\Lambda^k\boldsymbol{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^k}\big)\\ &\quad + C_k\Big\langle\big(\big|\varepsilon\nabla\zeta\big|_{H^k} + \big|\beta\nabla b\big|_{W^{k,\infty}}\big)\big(\big\|\Lambda^{1+s_\star}\boldsymbol{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^{1+s_\star}}\big)\Big\rangle_{k>1+s_\star}.\end{aligned}$$

Moreover, for $k \geq 1 + s_\star$ and if $\boldsymbol{R} = \boldsymbol{0}$, it holds that $\mathit{\Phi} \in \mathcal{C}^2(\mathcal{S})$ and is a classical solution to eq. (4.3), i.e. *the identities hold pointwise everywhere.*

Proof. The bilinear form

$$(\varphi_1, \varphi_2) \mapsto \iint_\Omega \mu(\nabla_{\boldsymbol{x}}\varphi_1)\cdot(\nabla_{\boldsymbol{x}}\varphi_2) + (\partial_z\varphi_1)(\partial_z\varphi_2)\,{\rm d}\boldsymbol{x}\,{\rm d}z$$

is continuous and coercive on the Hilbert space $(H^1_{0,\rm top}(\Omega), \big\|\bullet\big\|_{H^1})$ by Poincaré's inequality (4.4). Using that $(\boldsymbol{x}, z) \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\Omega)$, the linear form

$$\varphi \mapsto -\iint_\Omega \mu\nabla\psi\cdot\nabla_{\boldsymbol{x}}\varphi\,{\rm d}\boldsymbol{x}\,{\rm d}z,$$

is well-defined and continuous. Hence the existence and uniqueness of a variational solution to eq. (4.2) follows by Lax–Milgram theorem (or Riesz representation Lemma).

The existence and uniqueness of a variational solution to eq. (4.3) is obtained in the same way, using that $P(\Sigma)$ is coercive thanks to Assumption 4.1.

In order to check that the variational solutions correspond, namely that $\mathit{\Phi} = \Phi\circ\Sigma$ when $\boldsymbol{R} = \boldsymbol{0}$, it suffices to change variables in the integrals, since $(\nabla^\mu_{\boldsymbol{x},z}\Phi)\circ\Sigma = (J^{-1}_{\Sigma,\mu})^\top\nabla^\mu_{\boldsymbol{x},z}\mathit{\Phi}$, and

$$P(\Sigma) = \det(J_{\Sigma,\mu})(J^{-1}_{\Sigma,\mu})(J^{-1}_{\Sigma,\mu})^\top \quad \text{with} \quad (J^{-1}_{\Sigma,\mu})^\top = \begin{pmatrix} {\rm Id}_d & \frac{-\sqrt{\mu}\nabla_{\boldsymbol{x}}\sigma}{\partial_z\sigma}\\ \boldsymbol{0}^\top & \frac{1}{\partial_z\sigma}\end{pmatrix}.$$

The estimate of item ii is a obtained by using the test function $\varphi = \widetilde{\Phi}$ in the variational identity, and the uniform coercivity of $P(\Sigma)$.

Now we prove that $\widetilde{\Phi} \in H^2(\mathcal{S})$ if $\psi \in \mathring{H}^2(\mathbb{R}^d)$ and $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$. For $h > 0$ and $\boldsymbol{e} \in \mathbb{R}^d$, let

$$\widetilde{\Phi}_{h\boldsymbol{e}} \stackrel{\text{def}}{=} (D_{h\boldsymbol{e}}\widetilde{\Phi}) : (\boldsymbol{x}, z) \mapsto \frac{\widetilde{\Phi}(\boldsymbol{x} + h\boldsymbol{e}, z) - \widetilde{\Phi}(\boldsymbol{x}, z)}{h}.$$

We have, using the test function $-D_{-h\boldsymbol{e}}D_{h\boldsymbol{e}}\widetilde{\Phi} \in H^1_{0,\mathrm{top}}(\mathcal{S})$,

$$\iint_{\mathcal{S}} \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}_{h\boldsymbol{e}} \cdot P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}_{h\boldsymbol{e}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = + \iint_{\mathcal{S}} D_{h\boldsymbol{e}}\big(\boldsymbol{R} - P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\psi\big) \cdot \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}_{h\boldsymbol{e}} \,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}$$
$$- \iint_{\mathcal{S}} \big[D_{h\boldsymbol{e}}, P(\Sigma)\big]\nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi} \cdot \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}_{h\boldsymbol{e}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z.$$

Using the previous estimate yields

$$\big\|\nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}_{h\boldsymbol{e}}\big\|_{L^2(\mathcal{S})} \le C(h_\star^{-1}, \big|\varepsilon\zeta\big|_{W^{2,\infty}}, \big|\beta b\big|_{W^{2,\infty}})\big(\big|\Lambda^1\boldsymbol{R}\big|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^1(\mathbb{R}^d)}\big),$$

were we used that for any $v \in H^1(\mathbb{R}^d)$, $\big|v_{h\boldsymbol{e}}\big|_{L^2(\mathbb{R}^d)} = \big|\int_0^1 \boldsymbol{e}\cdot\nabla v(\boldsymbol{x} + hr\boldsymbol{e})\,\mathrm{d}r\big|_{L^2(\mathbb{R}^d)} \le \big|\boldsymbol{e}\cdot\nabla v\big|_{L^2(\mathbb{R}^d)}$, by Minkowski's inequality. By Poincaré's inequality, we obtain a bound on $\big\|\widetilde{\Phi}_{h\boldsymbol{e}}\big\|_{H^1(\mathcal{S})}$ which is independent of h. Hence since $H^1(\mathcal{S})$ is a reflexive Banach space, there exists $\Psi \in H^1(\mathcal{S})$ and a subsequence $(\widetilde{\Phi}_{h_n\boldsymbol{e}})_n$ with $h_n \searrow 0$ such that $\Phi_{h_n\boldsymbol{e}} \rightharpoonup \Psi$. By uniqueness of the limit in $L^2(\mathcal{S})$, we deduce that $\Psi = \mathbf{e}\cdot\nabla_{\boldsymbol{x}}\widetilde{\Phi} \in H^1(\mathcal{S})$. The limit satisfies the inequality above, and the estimate of item iii holds for $k = 1$ and $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$. In order to control the derivative in the vertical variable, we decompose for any $\varphi \in \mathcal{D}(\mathcal{S})$,

$$\iint_{\mathcal{S}} \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi} \cdot P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \iint_{\mathcal{S}} \frac{1 + \mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}(\partial_z\widetilde{\Phi})(\partial_z\varphi) + \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi} \cdot P_0(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z.$$

Thanks to the estimate above, we may integrate by parts in the horizontal variable and deduce that

$$\iint_{\mathcal{S}} \frac{1 + \mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}(\partial_z\widetilde{\Phi})(\partial_z\varphi) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \iint_{\mathcal{S}} F\varphi \,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}$$

with $F = \nabla^\mu_{\boldsymbol{x},z} \cdot \Big(P_0(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi} - \boldsymbol{R} + P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\psi\Big) \in L^2(\mathcal{S})$. Hence $\frac{1+\mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}\partial_z\widetilde{\Phi}$ is weakly differentiable in the vertical variable and $\partial_z\Big(\frac{1+\mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}\partial_z\widetilde{\Phi}\Big) \in L^2(\mathcal{S})$. By the positivity and regularity of $\frac{1+\mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}$, we deduce that $\partial_z\widetilde{\Phi}$ is weakly differentiable in the vertical variable and $\partial_z^2\widetilde{\Phi} \in L^2(\mathcal{S})$. In particular, the Laplace equation holds in $L^2(\mathcal{S})$. By the trace formula (4.5), the boundary condition in eq. (4.3) is well-defined (and satisfied) in $L^2(\mathbb{R}^d)$.

Let us now estimate higher order derivatives, by induction on $k \in \mathbb{N}$. For $k \ge 2$, let us define $\boldsymbol{k} \in (\mathbb{N}^\star)^d$ a multi-index such that $|\boldsymbol{k}| = k$, and $\partial^{\boldsymbol{k}} = \partial_{x_1}^{\boldsymbol{k}_1}\partial_{x_2}^{\boldsymbol{k}_2}$. Using that Φ is a strong solution to eq. (4.3), we may differentiate the identities and deduce that $\partial^{\boldsymbol{k}}\Phi$ is a distributional solution to

$$\begin{cases} \nabla_{\boldsymbol{x},z} \cdot P^\mu(\Sigma)\nabla_{\boldsymbol{x},z}\Phi_{\boldsymbol{k}} = \nabla^\mu_{\boldsymbol{x},z} \cdot \big(\partial^{\boldsymbol{k}}\boldsymbol{R} - [\partial^{\boldsymbol{k}}, P(\Sigma)]\nabla^\mu_{\boldsymbol{x},z}\Phi\big) & \text{in } \mathbb{R}^d \times (-1, 0), \\ \Phi_{\boldsymbol{k}} = \partial^{\boldsymbol{k}}\psi & \text{on } \mathbb{R}^d \times \{0\}, \\ \boldsymbol{e}_{d+1} \cdot P^\mu(\Sigma)\nabla_{\boldsymbol{x},z}\Phi_{\boldsymbol{k}} = \boldsymbol{e}_{d+1} \cdot \big(\partial^{\boldsymbol{k}}\boldsymbol{R} - [\partial^{\boldsymbol{k}}, P(\Sigma)]\nabla^\mu_{\boldsymbol{x},z}\Phi\big) & \text{on } \mathbb{R}^d \times \{-1\}. \end{cases} \tag{4.6}$$

Using the previously obtained estimate of item ii (*i.e.* with $k = 0$) and using the product and commutator estimates in Appendix II, we obtain that the variational solution satisfies

$$\big\|\nabla^\mu_{\boldsymbol{x},z}\Phi_{\boldsymbol{k}}\big\|_{L^2} \le C\Big(\big\|\Lambda^{k-1}\nabla^\mu_{\boldsymbol{x},z}\Phi\big\|_{L^2(\mathcal{S})} + \big\|\Lambda^k\boldsymbol{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^k}\Big)$$
$$+ C\Big\langle\big(\big|\varepsilon\nabla\zeta\big|_{H^k} + \big|\beta\nabla b\big|_{W^{k,\infty}}\big)\big\|\Lambda^{s_\star}\nabla^\mu_{\boldsymbol{x},z}\Phi\big\|_{L^2(\mathcal{S})}\Big\rangle_{k>1+s_\star}$$

with $C = C(h_\star^{-1}, \left|\varepsilon\zeta\right|_{H^{2+s_\star}}, \left|\beta b\right|_{W^{2+s_\star,\infty}})$. The desired estimates follow by induction, and one readily observes that the distributional and variational solutions must coincide, *i.e.* $\mathit{\Phi}_{\boldsymbol{k}} = \partial^{\boldsymbol{k}}\mathit{\Phi}$.

To conclude, we notice that for $k \geq 1 + s_\star$ and if $\boldsymbol{R} = \boldsymbol{0}$, using that $\mathit{\Phi}$ is a strong solution and the aforementioned decomposition and the trace formula (4.5), one obtains easily $\partial_z^2\mathit{\Phi} \in \mathcal{C}^0(\mathcal{S})$, so that $\mathit{\Phi} \in \mathcal{C}^2(\mathcal{S})$, and the Laplace equation holds in a classical sense. □

4.3 The Dirichlet-to-Neumann operator

We now apply what we have learned on the Laplace problem, to the Dirichlet-to-Neumann operator. We start with two handy identities.

Lemma 4.6. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\psi \in \mathring{H}^2(\mathbb{R}^d)$. Then we have*

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = -\mu\nabla \cdot (h\overline{\boldsymbol{u}})$$

where $h = 1 + \varepsilon\zeta - \beta b$ and

$$\overline{\boldsymbol{u}} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi \,\mathrm{d}z = \frac{1}{h}\int_{-1}^{0} (\partial_z\sigma)\nabla_{\boldsymbol{x}}\mathit{\Phi} - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\mathit{\Phi} \,\mathrm{d}z,$$

recalling that Φ is the (strong) solution to eq. (4.1), $\mathit{\Phi} = \Phi \circ \Sigma$ where $\Sigma(\boldsymbol{x}, z) = (\boldsymbol{x}, \sigma(\boldsymbol{x}, z))$ and $\sigma(\boldsymbol{x}, z) \stackrel{\rm def}{=} (1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}))z + \varepsilon\zeta(\boldsymbol{x})$.

Proof. Using that Φ is a strong solution to eq. (4.1), we test the equation against $\varphi(\boldsymbol{x}, z) = \varphi(\boldsymbol{x})$ with $\varphi \in \mathcal{D}(\mathbb{R}^d)$, and infer $\int_\Omega (\mu\Delta_{\boldsymbol{x}}\Phi + \partial_z^2\Phi)\varphi = 0$. It follows, using Green's identity,

$$\begin{aligned} 0 &= -\int_\Omega \mu(\nabla_{\boldsymbol{x}}\Phi) \cdot (\nabla_{\boldsymbol{x}}\varphi) + (\partial_z\Phi)(\partial_z\varphi)\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \left((\partial_z\Phi - \mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi)\,\varphi\right)\big|_{z=\varepsilon\zeta}\,\mathrm{d}\boldsymbol{x} \\ &= -\mu\int_{\mathbb{R}^d}\left(\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi\,\mathrm{d}z\right)\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} (\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi)\varphi\,\mathrm{d}\boldsymbol{x}, \end{aligned}$$

and the result follows from integration by parts. The identity with $\mathit{\Phi}$ follows by changing variables on the integral above. □

Lemma 4.7. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\psi \in \mathring{H}^2(\mathbb{R}^d)$. Then $\mathit{\Phi}$ the (strong) solution to eq. (4.3) with $\boldsymbol{R} = \boldsymbol{0}$ satisfies*

$$\mathit{\Phi} + \mu\ell[\varepsilon\zeta, \beta b]\mathit{\Phi} = \psi$$

where, denoting $h = 1 + \varepsilon\zeta - \beta b$ and $\sigma = hz + \varepsilon\zeta$,

$$\begin{aligned} \ell[\varepsilon\zeta, \beta b]\mathit{\Phi}(\cdot, z) \stackrel{\rm def}{=} &\int_z^0 \left(h(\nabla_{\boldsymbol{x}}\sigma)\cdot(\nabla_{\boldsymbol{x}}\mathit{\Phi}) - |\nabla_{\boldsymbol{x}}\sigma|^2(\partial_z\mathit{\Phi})\right)(\cdot, z')\,\mathrm{d}z' \\ &- h\int_z^0\int_{-1}^{z'} \nabla_{\boldsymbol{x}}\cdot\left((\partial_z\sigma)(\nabla_{\boldsymbol{x}}\mathit{\Phi}) - (\nabla_{\boldsymbol{x}}\sigma)(\partial_z\mathit{\Phi})\right)(\cdot, z'')\,\mathrm{d}z''\,\mathrm{d}z'. \end{aligned}$$

Proof. Denoting $\mathit{\Psi} = \mathit{\Phi} + \mu\ell[\varepsilon\zeta, \beta b]\mathit{\Phi}$, we have that $\mathit{\Psi}, \partial_z\mathit{\Psi}, \partial_z^2\mathit{\Psi} \in L^2(\mathcal{S})$. By direct algebraic computations, one checks that

$$\ell[\varepsilon\zeta, \beta b]\mathit{\Phi} = -z\,(\beta\nabla b)\cdot(h\nabla_{\boldsymbol{x}}\mathit{\Phi} - (\beta\nabla b)\partial_z\mathit{\Phi})\,\big|_{z=-1} - h\int_z^0\int_{-1}^{z'} \nabla_{\boldsymbol{x},z}\cdot P_1(\Sigma)\nabla_{\boldsymbol{x},z}\mathit{\Phi}(\cdot, z'')\,\mathrm{d}z''\,\mathrm{d}z'$$

with $P_1(\Sigma) = \begin{pmatrix} (\partial_z\sigma)\,\mathrm{Id}_d & -\nabla_{\boldsymbol{x}}\sigma \\ -\nabla_{\boldsymbol{x}}^\top\sigma & \frac{|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma} \end{pmatrix}$. Using $P(\Sigma) = \frac{1}{h}\begin{pmatrix} \mathrm{O}_d & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{pmatrix} + \mu P_1(\Sigma)$, and that $\mathit{\Phi}$ is a strong solution to eq. (4.3), one readily checks that $\partial_z^2\mathit{\Psi} = 0$ on $\mathcal{S}$, $\partial_z\mathit{\Psi}\,\big|_{z=-1} = 0$, and $\mathit{\Psi}\,\big|_{z=0} = \psi$. □

Proposition 4.8. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$ and $k \in \mathbb{N}^\star$. Then there exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta \in H^{\max\{k+3, 1+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+3, 1+s_\star\}}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\big|\varepsilon\zeta\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{1+s_\star,\infty}} \leq M,$$

any $\varPsi \in \mathring{H}^1(\mathcal{S})$ such that $\Lambda^{k+2}\nabla_{\boldsymbol{x},z}\varPsi \in L^2(\mathcal{S})$, one has $\Lambda^k \ell[\varepsilon\zeta, \beta b]\varPsi \in H^1(\mathcal{S})$ and

$$\begin{aligned}\big\|\Lambda^k \nabla_{\boldsymbol{x},z}\big(\ell[\varepsilon\zeta, \beta b]\varPsi\big)\big\|_{L^2(\mathcal{S})} &\leq C\big\|\Lambda^{k+2}\nabla_{\boldsymbol{x},z}\varPsi\big\|_{L^2(\mathcal{S})} \\ &\quad + C\Big\langle \big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big\|\Lambda^{s_\star}\nabla_{\boldsymbol{x},z}\varPsi\big\|_{L^2(\mathcal{S})}\Big\rangle_{k+2>s_\star}.\end{aligned}$$

Proof. The result follows directly from the definition and product estimates in Appendix II. □

It is now simple to deduce approximate expressions of $\overline{\boldsymbol{u}}$ with arbitrary precision in terms of powers of μ.

Proposition 4.9. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0,1,2\}$. There exists C_n such that the following holds. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta \in H^{\max\{k+2n+1, 2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+2n+1, 2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\begin{array}{ccc}\mathring{H}^{k+1}(\mathbb{R}^d) & \to & H^k(\mathbb{R}^d) \\ \psi & \mapsto & \overline{\boldsymbol{u}} = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi(\cdot, z)\,{\rm d}z\end{array}$$

where Φ is the unique solution to eq. (4.1), is well-defined and continuous, and the following holds.

- *If $n = 0$, then for any $\psi \in \mathring{H}^{k+1}(\mathbb{R}^d)$ and denoting $h = 1 + \varepsilon\zeta - \beta b$ one has*

$$\big|h\overline{\boldsymbol{u}}\big|_{H^k} \leq C_0\left(\big|\nabla\psi\big|_{H^k} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+1}} + \big|\beta b\big|_{W^{k+1,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k>1+s_\star}\right).$$

- *If $n = 1$, then for any $\psi \in \mathring{H}^{k+3}(\mathbb{R}^d)$*

$$\big|h\overline{\boldsymbol{u}} - h\nabla\psi\big|_{H^k} \leq C_1\,\mu\left(\big|\nabla\psi\big|_{H^{k+2}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\right).$$

- *If $n = 2$, then for any $\psi \in \mathring{H}^{k+5}(\mathbb{R}^d)$*

$$\begin{aligned}&\big|h\overline{\boldsymbol{u}} - h\nabla\psi + \mu h\mathcal{T}[h, \beta\nabla b]\nabla\psi\big|_{H^k} \\ &\qquad \leq C_2\,\mu^2\left(\big|\nabla\psi\big|_{H^{k+4}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\right),\end{aligned}$$

  where we define

$$\mathcal{T}[h, \beta\nabla b]\boldsymbol{u} \stackrel{\rm def}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u}) + \frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big) - h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}\Big) + (\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b). \quad (4.7)$$

Proof. By Lemma 4.6, we have the identity

$$h\overline{\boldsymbol{u}} = \int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi\,{\rm d}z = \int_{-1}^{0} (\partial_z\sigma)\nabla_{\boldsymbol{x}}\varPhi - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\varPhi\,{\rm d}z,$$

where $\varPhi$ is the solution to eq. (4.3) with $\boldsymbol{R} = \mathbf{0}$. The result for $n = 0$ follows from product estimates in Appendix II and Proposition 4.5. Plugging above the identity in Lemma 4.7, we obtain

$$h\overline{\boldsymbol{u}} = h\nabla\psi - \mu \int_{-1}^{0} (\partial_z\sigma)\nabla_{\boldsymbol{x}}(\ell[\varepsilon\zeta, \beta b]\varPhi) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\ell[\varepsilon\zeta, \beta b]\varPhi)\,\mathrm{d}z.$$

The result for $n = 1$ is deduced, using additionally Proposition 4.8. Plugging again the identity in Lemma 4.7 in the identity above yields the result for $n = 2$, using that

$$\begin{aligned}\ell[\varepsilon\zeta, \beta b]\psi &= \int_{z}^{0} h((1+z')\nabla h + \beta\nabla b)\cdot(\nabla\psi)\,\mathrm{d}z' - h\int_{z}^{0}\int_{-1}^{z'} \nabla\cdot(h\nabla\psi)\,\mathrm{d}z''\,\mathrm{d}z' \\ &= (z + \frac{z^2}{2})h^2\nabla\cdot\nabla\psi - zh(\beta\nabla b)\cdot(\nabla\psi)\end{aligned}$$

and hence, after tedious computations,

$$\int_{-1}^{0} h\nabla_{\boldsymbol{x}}(\ell[\varepsilon\zeta, \beta b]\psi) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\ell[\varepsilon\zeta, \beta b]\psi)\,\mathrm{d}z = h\mathcal{T}[h, \beta\nabla b]\nabla\psi.$$

This concludes the proof. $\square$

The following result is an obvious consequence of Proposition 4.9 and the identity of Lemma 4.6.

Proposition 4.10. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0, 1, 2\}$. There exists C_n such that the following holds. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta \in H^{\max\{k+2n+1, 2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+2n+1,2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b] : \begin{array}{ccc} \mathring{H}^{k+1}(\mathbb{R}^d) & \to & H^{k-1}(\mathbb{R}^d) \\ \psi & \mapsto & (\partial_z\Phi - \mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi)\,\big|_{z=\varepsilon\zeta} \end{array}$$

where Φ is the unique solution to eq. (4.1), is well-defined and continuous, and the following holds.

- *If $n = 0$, then for any $\psi \in \mathring{H}^{k+1}(\mathbb{R}^d)$*
$$\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi\big|_{H^{k-1}} \leq C_0\left(\big|\nabla\psi\big|_{H^k} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+1}} + \big|\beta b\big|_{W^{k+1,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k>1+s_\star}\right).$$
- *If $n = 1$, then for any $\psi \in \mathring{H}^{k+3}(\mathbb{R}^d)$*
$$\begin{aligned}&\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi + \nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi)\big|_{H^{k-1}} \\ &\qquad\leq C_1\,\mu\left(\big|\nabla\psi\big|_{H^{k+2}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\right).\end{aligned}$$
- *If $n = 2$, then for any $\psi \in \mathring{H}^{k+5}(\mathbb{R}^d)$, denoting $h = 1 + \varepsilon\zeta - \beta b$ and $\mathcal{T}$ as in eq. (4.7)*
$$\begin{aligned}&\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi + \nabla\cdot(h\nabla\psi) - \mu\nabla\cdot\big(h\mathcal{T}[h, \beta\nabla b]\nabla\psi\big)\big|_{H^{k-1}} \\ &\qquad\leq C_2\,\mu^2\left(\big|\nabla\psi\big|_{H^{k+4}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\right).\end{aligned}$$

The strategy for deriving asymptotic models for the water waves equations, eq. (2.7), is now fairly obvious: we simply plug the truncated expansion at the desired order (in terms of powers of μ) in the system, and withdraw any negligible contribution. The result above allows to rigorously justifies such an approximation in the sense of consistency. The first-order system we obtain this way is the so-called Saint-Venant system, studied in Section 5. One can obtain at next order the Green-Naghdi system, studied in Section 8.

4.4 Improved "fully dispersive" estimates

In this section we refine the results obtained in Section 4.3 so that all the expansions are exact at the linear level (*i.e.* if $\beta = \varepsilon = 0$). Our first task is to study the inverse of the operator $\mathrm{Id} + \mu\ell_0$ where

$$\ell_0\Phi(\cdot, z) \stackrel{\text{def}}{=} -\int_z^0 \int_{-1}^{z'} \Delta_{\boldsymbol{x}}\Phi(\cdot, z'')\,\mathrm{d}z''\,\mathrm{d}z'.$$

Lemma 4.11. *For any $d \in \mathbb{N}^\star$, $\mu > 0$, the following holds. For any $F \in H^1_{0,\mathrm{top}}(\mathcal{S})$, there exists a unique $\Phi_0 \in H^1_{0,\mathrm{top}}(\mathcal{S})$ solution to*

$$\Phi_0 + \mu\ell_0\Phi_0 = F,$$

which we denote $\Phi_0 \stackrel{\text{def}}{=} (\mathrm{Id} + \mu\ell_0)^{-1}F$; and one has

$$\left\|\nabla^\mu_{\boldsymbol{x},z}\Phi_0\right\|_{L^2(\mathcal{S})} \leq \left\|\partial_z F\right\|_{L^2(\mathcal{S})}.$$

Moreover, for any $k \in \mathbb{N}$, if $\Lambda^k F \in H^1_{0,\mathrm{top}}(\mathcal{S})$, then $\Lambda^k(\mathrm{Id} + \mu\ell_0)^{-1}F = (\mathrm{Id} + \mu\ell_0)^{-1}\Lambda^k F \in H^1_{0,\mathrm{top}}(\mathcal{S})$.

For any $\psi \in \mathring{H}^1(\mathbb{R}^d)$,

$$\Phi_0(\cdot, z) = \frac{\cosh(\sqrt{\mu}|D|(z+1))}{\cosh(\sqrt{\mu}|D|)}\psi$$

is the unique $\Phi_0 \in \mathring{H}^1(\mathcal{S})$ solution to $\Phi_0 + \mu\ell_0\Phi_0 = \psi$, identifying $\boldsymbol{x} \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\mathbb{R}^d)$ and $(\boldsymbol{x}, z) \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\mathcal{S})$.

Proof. It is straightforward to check that Φ_0 is a (variational, that is in the sense of Definition 4.2) solution to

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_0 + \partial_z^2\Phi_0 = \partial_z^2 F & \text{in } \mathcal{S}, \\ \Phi_0 = 0 & \text{on } \mathbb{R}^d \times \{0\}, \\ \partial_z\Phi_0 = \partial_z F & \text{on } \mathbb{R}^d \times \{-1\}. \end{cases}$$

While the above could be explicitly solved using Fourier multipliers, it is simpler to refer to Proposition 4.5 with $\varepsilon\zeta = \beta b = 0$ and $\boldsymbol{R} = (0, \partial_z F)$, $\psi = 0$. It is straightforward to check that the multiplicative constant therein can be set to $C = 1$.

For the second part of the statement, existence and uniqueness follows as above, and we do solve in Fourier space the corresponding Laplace problem (as in Section 2.3), *i.e.*

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi_0 + \partial_z^2\Phi_0 = 0F & \text{in } \mathcal{S}, \\ \Phi_0 = \psi & \text{on } \mathbb{R}^d \times \{0\}, \\ \partial_z\Phi_0 = 0 & \text{on } \mathbb{R}^d \times \{-1\}. \end{cases}$$

to obtain the desired expression. ☐

Our results will be deduced from the following key identity.

Lemma 4.12. *Under the assumptions of Lemma 4.7, Φ the (strong) solution to eq. (4.3) with $\boldsymbol{R} = \boldsymbol{0}$ satisfies*

$$(\mathrm{Id} + \mu\ell_0)(\Phi - \Phi_0) = -\mu(\ell[\varepsilon\zeta, \beta b] - \ell_0)\Phi$$

where $\Phi_0(\cdot, z) = \frac{\cosh(\sqrt{\mu}|D|(z+1))}{\cosh(\sqrt{\mu}|D|)}\psi$ and, denoting $h = 1 + \varepsilon\zeta - \beta b$ and $\sigma = hz + \varepsilon\zeta$,

$$\ell[\varepsilon\zeta, \beta b]\Phi(\cdot, z) \stackrel{\text{def}}{=} \int_z^0 \big(h(\nabla_{\boldsymbol{x}}\sigma)\cdot(\nabla_{\boldsymbol{x}}\Phi) - |\nabla_{\boldsymbol{x}}\sigma|^2(\partial_z\Phi)\big)(\cdot, z')\,\mathrm{d}z'$$
$$- h\int_z^0 \int_{-1}^{z'} \nabla_{\boldsymbol{x}}\cdot\big((\partial_z\sigma)(\nabla_{\boldsymbol{x}}\Phi) - (\nabla_{\boldsymbol{x}}\sigma)(\partial_z\Phi)\big)(\cdot, z'')\,\mathrm{d}z''\,\mathrm{d}z'.$$

Proof. The result is an obvious consequence of the identities

$$\Phi + \mu\ell[\varepsilon\zeta, \beta b]\Phi = \psi = \Phi_0 + \mu\ell_0\Phi_0.$$

proved respectively in Lemma 4.7 and Lemma 4.11. $\square$

We now extend Proposition 4.8.

Proposition 4.13. *Under the assumptions and using the notations of Proposition 4.8, one has*

$$\big\|\Lambda^k\nabla_{\boldsymbol{x},z}\big(\ell[\varepsilon\zeta,\beta b]\Psi - \ell_0\Psi\big)\big\|_{L^2(\mathcal{S})} \leq C\big(\big|\varepsilon\zeta\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{1+s_\star,\infty}}\big)\big\|\Lambda^{k+2}\nabla_{\boldsymbol{x},z}\Psi\big\|_{L^2(\mathcal{S})} + C\Big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big\|\Lambda^{s_\star}\nabla_{\boldsymbol{x},z}\Psi\big\|_{L^2(\mathcal{S})}\Big\rangle_{k+2>s_\star}.$$

Proof. The result follows as for Proposition 4.8 from product estimates in Appendix II, and using the cancellation stemming from the fact that $\ell_0 = \ell[0,0]$. $\square$

It is now simple to deduce the desired approximate expressions of $\overline{\boldsymbol{u}}$.

Proposition 4.14. *Under the assumptions and using the notations of Proposition 4.9, one has*

- *if* $n = 1$,

$$\big|h\overline{\boldsymbol{u}} - h\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi\big|_{H^k} \leq C_1\,\mu\Big(\big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}}\big)\big|\nabla\psi\big|_{H^{k+2}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\Big);$$

- *if* $n = 2$,

$$\big|h\overline{\boldsymbol{u}} - h\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi + \mu h\mathcal{T}[h,\beta\nabla b]\nabla\psi + \frac{\mu}{3}h\nabla\Delta\psi\big|_{H^k} \leq C_2\,\mu^2\Big(\big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}}\big)\big|\nabla\psi\big|_{H^{k+4}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\Big).$$

Proof. Recall that by Lemma 4.6, we have the identity

$$\begin{aligned} h\overline{\boldsymbol{u}} &= \int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Phi\,\mathrm{d}z = \int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi\,\mathrm{d}z, \\ &= \int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}(\Phi-\Phi_0) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\Phi-\Phi_0)\,\mathrm{d}z + \int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi_0 - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi_0\,\mathrm{d}z \end{aligned}$$

where Φ is the solution to eq. (4.3) with $\boldsymbol{R} = \boldsymbol{0}$, and $\Phi_0 = \frac{\cosh(\sqrt{\mu}|D|(z+1))}{\cosh(\sqrt{\mu}|D|)}\psi$. Plugging above the identity in Lemma 4.12, we obtain

$$h\overline{\boldsymbol{u}} = h\overline{\boldsymbol{u}}_0 - \mu\int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi_{\geq 1} - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi_{\geq 1}\,\mathrm{d}z.$$

with $h\overline{\boldsymbol{u}}_0 \stackrel{\text{def}}{=} \int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi_0 - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi_0\,\mathrm{d}z$ and $\Phi_{\geq 1} \stackrel{\text{def}}{=} (\mathrm{Id} + \mu\ell_0)^{-1}(\ell[\varepsilon\zeta,\beta b] - \ell_0)\Phi$. By direct algebra, we find

$$h\overline{\boldsymbol{u}}_0 = h\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi + (\varepsilon\nabla\zeta)(\mathrm{Id} - \tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|})\psi - (\beta\nabla b)(\tfrac{1}{\cosh(\sqrt{\mu}|D|)} - \tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|})\psi.$$

The result for $n=1$ is deduced, using product estimates in Appendix II, Proposition 4.5 to control Φ, and Lemma 4.11 and Proposition 4.13 to control Φ_1. The result for $n=0$ follows from product estimates in Appendix II and Proposition 4.5. Plugging once more the identity in Lemma 4.12, we find

$$h\overline{\boldsymbol{u}} = h\overline{\boldsymbol{u}}_0 + \mu h\overline{\boldsymbol{u}}_1 - \mu\int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi_2 - (\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi_2\,\mathrm{d}z.$$

with $h\overline{\boldsymbol{u}}_1 \stackrel{\text{def}}{=} \int_{-1}^{0}(\partial_z\sigma)\nabla_{\boldsymbol{x}}\Phi_1-(\nabla_{\boldsymbol{x}}\sigma)\partial_z\Phi_1\,\mathrm{d}z$ where $\Phi_1 \stackrel{\text{def}}{=} (\mathrm{Id}+\mu\ell_0)^{-1}(\ell[\varepsilon\zeta,\beta b]-\ell_0)\Phi_0$, and denoting $\Phi_{\geq 2} \stackrel{\text{def}}{=} \big((\mathrm{Id}+\mu\ell_0)^{-1}(\ell[\varepsilon\zeta,\beta b]-\ell_0)\big)\big((\mathrm{Id}+\mu\ell_0)^{-1}(\ell[\varepsilon\zeta,\beta b]-\ell_0)\big)\Phi$. The contribution of the latter is estimated by using once gain Lemma 4.11 and Proposition 4.13. The contribution from $h\overline{\boldsymbol{u}}_1$ is deduced from $\Phi_1 = (\ell[\varepsilon\zeta,\beta b]-\ell_0)\psi + \mathcal{O}(\mu)$, using as in Proposition 4.9

$$\int_{-1}^{0} h\nabla_{\boldsymbol{x}}(\ell[\varepsilon\zeta,\beta b]\psi) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\ell[\varepsilon\zeta,\beta b]\psi)\,\mathrm{d}z = h\mathcal{T}[h,\beta\nabla b]\nabla\psi.$$

and computing

$$\int_{-1}^{0} h\nabla_{\boldsymbol{x}}(\ell_0\psi) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\ell_0\psi)\,\mathrm{d}z = \frac{-1}{3}\nabla(\nabla\cdot\nabla\psi) + (\Delta\psi)\big(\frac{1}{6}\nabla h - \frac{\varepsilon}{2}\nabla\zeta\big),$$

where the last term compensates the corresponding contribution in $h\overline{\boldsymbol{u}}_0$. The proof is complete. $\square$

The following result is an obvious consequence of Proposition 4.14 and the identity of Lemma 4.6.

Proposition 4.15. *Under the assumptions and using the notations of Proposition 4.9, one has*

- *if $n=1$, then*

$$\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot\Big((1+\varepsilon\zeta-\beta b)\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi\Big)\Big|_{H^{k-1}} \\ \leq C_1\,\mu\,\Big(\big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}}\big)\big|\nabla\psi\big|_{H^{k+2}} \\ + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\Big);$$

- *if $n=2$, then*

$$\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot\Big(h\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi + \mu h\mathcal{T}[h,\beta\nabla b]\nabla\psi + \frac{\mu}{3}h\nabla\Delta\psi\Big)\Big|_{H^{k-1}} \\ \leq C_2\,\mu^2\,\Big(\big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}}\big)\big|\nabla\psi\big|_{H^{k+4}} \\ + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\Big).$$

Remark 4.16.

- *It is remarkable that all the results we obtain in this section improve the ones in Section 4.3 (in terms of asymptotic precision) without asking extra regularity on the data. In fact, using the regularizing effects of the operator $(\mathrm{Id}+\mu\ell_0)^{-1}$, it is conceivable that the same results hold with less regular data.*
- *While we do not display explicitly a formula for the velocity potential inside the fluid domain, the proof offers such (approximate) expression, for variables in the flat strip. A nice compact formula is exhibited in [177], namely*

$$\Phi(\cdot,z) = \psi + (1+\varepsilon\zeta-\beta b)^2\big(\tfrac{\cosh(\sqrt{\mu}|D|(z+1))}{\cosh(\sqrt{\mu}|D|)} - \mathrm{Id}\big)\psi + \mathcal{O}(\mu^2(\varepsilon+\beta))$$

which directly echoes (and refines) the widely used weakly dispersive approximation,

$$\Phi(\cdot,z) = \psi - \mu(1+\varepsilon\zeta-\beta b)^2\frac{(z+1)^2}{2}\Delta\psi + \mathcal{O}(\mu^2),$$

as well as the linear approximation, eq. (2.5).

4.5 Dirichlet-to-Dirichlet, Neumann-to-Dirichlet, Neumann-to-Neumann

Section 4 has been so far dedicated to the Dirichlet-to-Neumann operator, since it appears in the water waves problem for homogeneous fluids, eq. (2.2). When considering interfaces between fluids (typically with different densities), as in eq. (3.1), then other operators are involved, which are the subject of this section. As always, the results are not sharp in terms of regularity. We have aimed at conciseness, borrowing as much as possible from the study of the Dirichlet-to-Neumann operator.

4.5.1 The Dirichlet-to-Dirichlet operator

We refer as the Dirichlet-to-Dirichlet operator the following:

$$\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi = \Phi\,\big|_{z=-1+\beta b}$$

where Φ is the unique solution to eq. (4.1), which we recall for convenience:

$$\begin{cases} \mu\Delta_{\boldsymbol{x}}\Phi+\partial_z^2\Phi=0 & \text{in } \Omega=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ -1+\beta b<z<\varepsilon\zeta\},\\ \Phi=\psi & \text{on } \Gamma_{\rm top}=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=\varepsilon\zeta\},\\ \partial_z\Phi-\mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi=0 & \text{on } \Gamma_{\rm bot}=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=-1+\beta b\}.\end{cases} \tag{4.8}$$

We follow the strategy in Section 4.3 for the Dirichlet-to-Neumann operator, with the help of the following obvious identity:

Lemma 4.17. *Let $d,s_\star\in\mathbb{N}^\star$, $s_\star>d/2$, $h_\star>0$, $\mu^\star>0$, and $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$. Let $\zeta,b\in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\psi\in\mathring{H}^2(\mathbb{R}^d)$. Then we have*

$$\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi=\psi-\int_{-1+\beta b}^{\varepsilon\zeta}\partial_z\Phi\,{\rm d}z=\psi-\int_{-1}^{0}\partial_z\mathit{\Phi}\,{\rm d}z,$$

where Φ is the (strong) solution to eq. (4.8) and $\mathit{\Phi}=\Phi\circ\Sigma$ where $\Sigma(\boldsymbol{x},z)=(\boldsymbol{x},\sigma(\boldsymbol{x},z))$ and $\sigma(\boldsymbol{x},z)\stackrel{\rm def}{=}(1+\varepsilon\zeta(\boldsymbol{x})-\beta b(\boldsymbol{x}))z+\varepsilon\zeta(\boldsymbol{x})$.

Proposition 4.18. *Let $d,s_\star\in\mathbb{N}^\star$, $s_\star>d/2$, $h_\star>0$, $\mu^\star>0$, $M\geq 0$, $k\in\mathbb{N}^\star$, $n\in\{0,1,2\}$, and put $m=\max(\{k+2n+1,k+1,2+s_\star\})$. There exists $C_n>0$ such that the following holds. For any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$, any $\zeta\in H^m(\mathbb{R}^d)$ and $b\in W^{m,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}}+\big|\beta b\big|_{W^{2+s_\star,\infty}}\leq M,$$

the operator

$$\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]:\begin{array}{ccc}\mathring{H}^{k+1}(\mathbb{R}^d) & \to & \mathring{H}^{k}(\mathbb{R}^d)\\ \psi & \mapsto & \Phi\,\big|_{z=-1+\beta b}\end{array}$$

where Φ is the unique solution to eq. (4.8), is well-defined and continuous, and the following holds.

- *If $n=0$, then for any $\psi\in\mathring{H}^{k+1}(\mathbb{R}^d)$*

$$\big|\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\psi\big|_{H^k}\leq C_0\sqrt{\mu}\,\Big(\big|\nabla\psi\big|_{H^k}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+1}}+\big|\beta b\big|_{W^{k+1,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k>1+s_\star}\Big).$$

- *If $n=1$, then for any $\psi\in\mathring{H}^{k+2}(\mathbb{R}^d)$*

$$\big|\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\psi\big|_{H^k}\leq C_1\,\mu\,\Big(\big|\nabla\psi\big|_{H^{k+1}}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+2}}+\big|\beta b\big|_{W^{k+2,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+1>1+s_\star}\Big).$$

- *If $n=2$, then for any $\psi\in\mathring{H}^{k+4}(\mathbb{R}^d)$*

$$\begin{aligned}&\big|\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\big(\psi+\mu\tfrac12 h^2\nabla\cdot\nabla\psi-\mu h(\beta\nabla b)\cdot(\nabla\psi)\big)\big|_{H^k}\\ &\qquad\leq C_2\,\mu^2\,\Big(\big|\nabla\psi\big|_{H^{k+3}}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+4}}+\big|\beta b\big|_{W^{k+4,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+3>1+s_\star}\Big).\end{aligned}$$

Proof. The result for $n=0$ follows immediately from Proposition 4.5 (with $\boldsymbol{R}=\mathbf{0}$) and Cauchy-Schwarz inequality in Lemma 4.17. Now, using the identity in Lemma 4.7 yields

$$\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\psi=\mu\int_{-1}^{0}\partial_z(\ell[\varepsilon\zeta,\beta b]\varPhi)\,\mathrm{d}z$$

and hence

$$\big|\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\psi\big|_{H^k}\leq\mu\big\|\Lambda^k\partial_z\ell[\varepsilon\zeta,\beta b]\varPhi\big\|_{L^2(\mathcal{S})},$$

where we used Cauchy-Schwarz inequality in the vertical variable. The result for $n=1$ is deduced from Proposition 4.8 (or rather a slightly improved version, taking into account that only the vertical derivative is involved).

Plugging again the identity in Lemma 4.7 in the identity above yields

$$\mathcal{G}^{\mu}_{\rm D2D}[\varepsilon\zeta,\beta b]\psi-\psi=\mu\int_{-1}^{0}\partial_z\big(\ell[\varepsilon\zeta,\beta b](\psi-\mu\ell[\varepsilon\zeta,\beta b]\varPhi)\big)\,\mathrm{d}z$$

and the result for $n=2$ follows, using Proposition 4.8 and the identity

$$\partial_z\big(\ell[\varepsilon\zeta,\beta b]\psi\big)=(1+z)h^2\nabla\cdot\nabla\psi-h(\beta\nabla b)\cdot(\nabla\psi)\,.$$

This concludes the proof. □

4.5.2 The Neumann-to-Neumann operator

We refer as the Neumann-to-Neumann operator the following:

$$\mathcal{G}^{\mu}_{\rm N2N}[\varepsilon\zeta,\beta b]\varpi=(\partial_z\Phi-\mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi)\,\big|_{z=\varepsilon\zeta}$$

where Φ is the unique solution to

$$\begin{cases}\mu\Delta_{\boldsymbol{x}}\Phi+\partial_z^2\Phi=0 & \text{in } \Omega=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ -1+\beta b<z<\varepsilon\zeta\},\\ \Phi=0 & \text{on } \Gamma_{\rm top}=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=\varepsilon\zeta\},\\ \partial_z\Phi-\mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi=\varpi & \text{on } \Gamma_{\rm bot}=\{(\boldsymbol{x},z)\in\mathbb{R}^{d+1}\ :\ z=-1+\beta b\}.\end{cases}\tag{4.9}$$

We can solve eq. (4.9) in the same way we solved eq. (4.8), and in particular introduce the Laplace problem on the flattened domain:

$$\begin{cases}\frac{1}{\partial_z\sigma}\nabla^{\mu}_{\boldsymbol{x},z}\cdot P(\Sigma)\nabla^{\mu}_{\boldsymbol{x},z}\varPhi=0 & \text{in } \mathbb{R}^d\times(-1,0),\\ \varPhi=0 & \text{on } \mathbb{R}^d\times\{0\},\\ \boldsymbol{e}_{d+1}\cdot P(\Sigma)\nabla^{\mu}_{\boldsymbol{x},z}\varPhi=\varpi & \text{on } \mathbb{R}^d\times\{-1\}.\end{cases}\tag{4.10}$$

where we denote as above $\sigma(\boldsymbol{x},z)\stackrel{\rm def}{=}(1+\varepsilon\zeta(\boldsymbol{x})-\beta b(\boldsymbol{x}))z+\varepsilon\zeta(\boldsymbol{x})$ and

$$P(\Sigma)\stackrel{\rm def}{=}\begin{pmatrix}(\partial_z\sigma)\,\mathrm{Id}_d & -\sqrt{\mu}\nabla_{\boldsymbol{x}}\sigma\\ -\sqrt{\mu}\nabla_{\boldsymbol{x}}^{\top}\sigma & \frac{1+\mu|\nabla_{\boldsymbol{x}}\sigma|^2}{\partial_z\sigma}\end{pmatrix}.$$

Definition 4.19 (Variational solutions). *Let $\varpi\in L^2(\mathbb{R}^d)$ and ζ,b satisfying Assumption 4.1. We say that $\Phi\in H^1_{0,\rm top}(\Omega)$ is a variational solution to eq. (4.9) if for any $\varphi\in H^1_{0,\rm top}(\Omega)$,*

$$\iint_{\Omega}\nabla^{\mu}_{\boldsymbol{x},z}\Phi\cdot\nabla^{\mu}_{\boldsymbol{x},z}\varphi\,\big|_{z=-1+\beta b}\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}=-\int_{\mathbb{R}^d}\varpi\varphi\,\mathrm{d}\boldsymbol{x}=\iint_{\Omega}\varpi\partial_z\varphi\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}.$$

We say that $\varPhi\in H^1_{0,\rm top}(\mathcal{S})$ is a variational solution to eq. (4.10) for any $\varphi\in H^1_{0,\rm top}(\mathcal{S})$,

$$\iint_{\mathcal{S}}\nabla^{\mu}_{\boldsymbol{x},z}\varPhi\cdot P(\Sigma)\nabla^{\mu}_{\boldsymbol{x},z}\varphi\,\big|_{z=-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z=-\int_{\mathbb{R}^d}\varpi\varphi\,\mathrm{d}\boldsymbol{x}=\iint_{\mathcal{S}}\varpi\partial_z\varphi\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}.$$

We can then follow the proof of Proposition 4.5 with straightforward adjustments to obtain the following counterpart.

Proposition 4.20. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. We have the following.*

i. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$ and $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and any $\varpi \in L^2(\mathbb{R}^d)$, there exists a unique variational solution, Φ, to eq. (4.9), a unique variational solution, $\mathit{\Phi}$, to eq. (4.10); and $\mathit{\Phi} = \Phi \circ \Sigma$.

If, moreover, $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$, $\varpi \in H^1(\mathbb{R}^d)$, then one has $\mathit{\Phi} \in H^2(\mathcal{S})$ and $\mathit{\Phi}$ is a strong solution to eq. (4.3), i.e. *the identities hold in $L^2(\mathcal{S}) \times H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$.*

ii. Let $M \geq 0$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$ and any $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $|\varepsilon\zeta|_{W^{1,\infty}} + |\beta b|_{W^{1,\infty}} \leq M$, the variational solution to eq. (4.10) satisfies

$$\|\nabla^\mu_{\boldsymbol{x},z} \mathit{\Phi}\|_{L^2(\mathcal{S})} \leq C |\varpi|_{L^2(\mathbb{R}^d)}.$$

iii. Let $k \in \mathbb{N}^\star$, $M \geq 0$. There exists C_k such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, $\zeta \in H^{\max\{1+k, 2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{1+k,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and

$$|\varepsilon\zeta|_{H^{2+s_\star}} + |\beta b|_{W^{2+s_\star,\infty}} \leq M,$$

for any $\varpi \in H^k(\mathbb{R}^d)$, the strong solution to eq. (4.10) satisfies $\Lambda^k \mathit{\Phi} \in H^1(\mathcal{S})$ and

$$\|\Lambda^k \nabla^\mu_{\boldsymbol{x},z} \mathit{\Phi}\|_{L^2(\mathcal{S})} \leq C_k \Big(|\varpi|_{H^k} + \big\langle \big(|\varepsilon \nabla \zeta|_{H^k} + |\beta \nabla b|_{W^{k,\infty}} \big) |\varpi|_{H^{1+s_\star}} \big\rangle_{k > 1+s_\star} \Big).$$

Moreover, for $k \geq 1 + s_\star$, it holds that $\mathit{\Phi} \in \mathcal{C}^2(\mathcal{S})$ and is a classical solution to eq. (4.10), i.e. *the identities hold pointwise everywhere.*

We may then proceed with the counterpart to Lemma 4.6 and Lemma 4.7.

Lemma 4.21. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\varpi \in H^1(\mathbb{R}^d)$. Then we have*

$$\mathcal{G}^\mu_{\rm N2N}[\varepsilon\zeta, \beta b]\varpi = \varpi - \mu \nabla \cdot (h \overline{\boldsymbol{u}})$$

where $h = 1 + \varepsilon\zeta - \beta b$ and

$$\overline{\boldsymbol{u}} = \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}} \Phi \, \mathrm{d}z = \frac{1}{h} \int_{-1}^{0} (\partial_z \sigma) \nabla_{\boldsymbol{x}} \mathit{\Phi} - (\nabla_{\boldsymbol{x}} \sigma) \partial_z \mathit{\Phi} \, \mathrm{d}z,$$

where Φ is the (strong) solution to eq. (4.9), and $\mathit{\Phi} = \Phi \circ \Sigma$.

Lemma 4.22. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\varpi \in H^1(\mathbb{R}^d)$. Then $\mathit{\Phi}$ the (strong) solution to eq. (4.10) satisfies*

$$\mathit{\Phi} + \mu \ell[\varepsilon\zeta, \beta b]\mathit{\Phi} = h \varpi z$$

where $h = 1 + \varepsilon\zeta - \beta b$ and ℓ is defined as in Lemma 4.7:

$$\begin{aligned}\ell[\varepsilon\zeta, \beta b]\mathit{\Phi}(\cdot, z) \stackrel{\text{def}}{=} \int_z^0 \big(h(\nabla_{\boldsymbol{x}} \sigma) \cdot (\nabla_{\boldsymbol{x}} \mathit{\Phi}) - |\nabla_{\boldsymbol{x}} \sigma|^2 (\partial_z \mathit{\Phi}) \big)(\cdot, z') \, \mathrm{d}z' \\ - h \int_z^0 \int_{-1}^{z'} \nabla_{\boldsymbol{x}} \cdot \big((\partial_z \sigma)(\nabla_{\boldsymbol{x}} \mathit{\Phi}) - (\nabla_{\boldsymbol{x}} \sigma)(\partial_z \mathit{\Phi}) \big)(\cdot, z'') \, \mathrm{d}z'' \, \mathrm{d}z'.\end{aligned}$$

Proposition 4.23. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0,1,2,3\}$. There exists C_n such that the following holds. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{SW}}$, any $\zeta \in H^{\max\{k+n+1,2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+n+1,2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\left|\varepsilon\zeta\right|_{H^{2+s_\star}} + \left|\beta b\right|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\mathcal{G}^\mu_{\mathrm{N2N}}[\varepsilon\zeta, \beta b] : \begin{array}{ccc} H^k(\mathbb{R}^d) & \to & H^{k-1}(\mathbb{R}^d) \\ \varpi & \mapsto & (\partial_z \Phi - \mu(\varepsilon\nabla\zeta)\cdot\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\varepsilon\zeta} \end{array}$$

where Φ is the unique solution to eq. (4.9), is well-defined and continuous, and the following holds.

- *If $n = 0$, then for any $\varpi \in H^k(\mathbb{R}^d)$*

$$\left|\mathcal{G}^\mu_{\mathrm{N2N}}[\varepsilon\zeta, \beta b]\varpi - \varpi\right|_{H^{k-1}} \leq C_0 \sqrt{\mu}\left(\left|\varpi\right|_{H^k} + \left\langle\left(\left|\varepsilon\zeta\right|_{H^{k+1}} + \left|\beta b\right|_{W^{k+1,\infty}}\right)\left|\varpi\right|_{H^{1+s_\star}}\right\rangle_{k>1+s_\star}\right).$$

- *If $n = 1$, then for any $\varpi \in H^{k+1}(\mathbb{R}^d)$*

$$\left|\mathcal{G}^\mu_{\mathrm{N2N}}[\varepsilon\zeta, \beta b]\varpi - \varpi\right|_{H^{k-1}} \leq C_1 \mu\left(\left|\varpi\right|_{H^{k+1}} + \left\langle\left(\left|\varepsilon\zeta\right|_{H^{k+2}} + \left|\beta b\right|_{W^{k+2,\infty}}\right)\left|\varpi\right|_{H^{1+s_\star}}\right\rangle_{k+1>1+s_\star}\right).$$

- *If $n = 2$, then for any $\varpi \in H^{k+2}(\mathbb{R}^d)$ and denoting $h = 1 + \varepsilon\zeta - \beta b$*

$$\begin{aligned}&\left|\mathcal{G}^\mu_{\mathrm{N2N}}[\varepsilon\zeta, \beta b]\varpi - \varpi + \mu\nabla\cdot\left(h(\varpi\varepsilon\nabla\zeta - \tfrac{1}{2}h\nabla\varpi)\right)\right|_{H^{k-1}} \\ &\qquad\leq C_2\,\mu^{3/2}\left(\left|\varpi\right|_{H^{k+2}} + \left\langle\left(\left|\varepsilon\zeta\right|_{H^{k+3}} + \left|\beta b\right|_{W^{k+3,\infty}}\right)\left|\varpi\right|_{H^{1+s_\star}}\right\rangle_{k+2>1+s_\star}\right).\end{aligned}$$

- *If $n = 3$, then for any $\varpi \in H^{k+3}(\mathbb{R}^d)$*

$$\begin{aligned}&\left|\mathcal{G}^\mu_{\mathrm{N2N}}[\varepsilon\zeta, \beta b]\varpi - \varpi + \mu\nabla\cdot\left(h(\varpi\varepsilon\nabla\zeta - \tfrac{1}{2}h\nabla\varpi)\right)\right|_{H^{k-1}} \\ &\qquad\leq C_3\,\mu^{2}\left(\left|\varpi\right|_{H^{k+3}} + \left\langle\left(\left|\varepsilon\zeta\right|_{H^{k+4}} + \left|\beta b\right|_{W^{k+4,\infty}}\right)\left|\varpi\right|_{H^{1+s_\star}}\right\rangle_{k+3>1+s_\star}\right).\end{aligned}$$

Proof. The proof follows exactly as (the cases $n = 0$ and $n = 1$) in the proof of Proposition 4.9 and Proposition 4.10, using Proposition 4.20 in place of Proposition 4.5, Lemma 4.21 in place of Lemma 4.6 and Lemma 4.22 in place of Lemma 4.7. The cost of $\mu^{1/2}$ prefactor stems from the anisotropic gradient $\nabla^\mu_{\boldsymbol{x},z}$ in Proposition 4.5, and can be removed by incrementing the regularity index and using Poincaré's inequality, eq. (4.4). □

4.5.3 The Neumann-to-Dirichlet operator

We refer as the Neumann-to-Dirichlet operator the following:

$$\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi = \Phi\big|_{z=-1+\beta b}$$

where Φ is the unique solution to eq. (4.9). Here we can borrow the idea of Section 4.5.1 and results of Section 4.5.2. We have the following identity.

Lemma 4.24. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{SW}}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and $\varpi \in H^1(\mathbb{R}^d)$. Then we have*

$$\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi = -\int_{-1+\beta b}^{\varepsilon\zeta} \partial_z\Phi\,\mathrm{d}z = -\int_{-1}^{0}\partial_z\mathit{\Phi}\,\mathrm{d}z,$$

where Φ is the (strong) solution to eq. (4.9) and $\mathit{\Phi} = \Phi\circ\Sigma$ where $\Sigma(\boldsymbol{x}, z) = (\boldsymbol{x}, \sigma(\boldsymbol{x}, z))$ and $\sigma(\boldsymbol{x}, z) \stackrel{\text{def}}{=} (1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}))z + \varepsilon\zeta(\boldsymbol{x})$.

We deduce the last result of this section.

Proposition 4.25. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0,1,2\}$. There exists C_n such that the following holds. For any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{SW}}$, any $\zeta \in H^{\max\{k+n+1,2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+n+1,2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\left|\varepsilon\zeta\right|_{H^{2+s_\star}} + \left|\beta b\right|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b] : \begin{array}{ccc} H^k(\mathbb{R}^d) & \to & H^k(\mathbb{R}^d) \\ \varpi & \mapsto & \Phi\,\big|_{z=-1+\beta b} \end{array}$$

where Φ is the unique solution to eq. (4.9), is well-defined and continuous, and the following holds.

- *If $n = 0$, then for any $\varpi \in H^k(\mathbb{R}^d)$*

$$\left|\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi\right|_{H^k} \leq C_0 \left(\left|\varpi\right|_{H^k} + \left\langle \left(\left|\varepsilon\zeta\right|_{H^{k+1}} + \left|\beta b\right|_{W^{k+1,\infty}} \right) \left|\varpi\right|_{H^{1+s_\star}} \right\rangle_{k>1+s_\star} \right).$$

- *If $n = 1$, then for any $\varpi \in H^{k+1}(\mathbb{R}^d)$ and denoting $h = 1 + \varepsilon\zeta - \beta b$*

$$\left|\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi + h\varpi\right|_{H^k} \leq C_1 \sqrt{\mu} \left(\left|\varpi\right|_{H^{k+1}} + \left\langle \left(\left|\varepsilon\zeta\right|_{H^{k+2}} + \left|\beta b\right|_{W^{k+2,\infty}} \right) \left|\varpi\right|_{H^{1+s_\star}} \right\rangle_{k+1>1+s_\star} \right).$$

- *If $n = 2$, then for any $\varpi \in H^{k+2}(\mathbb{R}^d)$*

$$\left|\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi + h\varpi\right|_{H^k} \leq C_2\, \mu \left(\left|\varpi\right|_{H^{k+2}} + \left\langle \left(\left|\varepsilon\zeta\right|_{H^{k+3}} + \left|\beta b\right|_{W^{k+3,\infty}} \right) \left|\varpi\right|_{H^{1+s_\star}} \right\rangle_{k+2>1+s_\star} \right).$$

Proof. The result for $n = 0$ follows from Lemma 4.24 and Proposition 4.20. Now, using the identity in Lemma 4.22 yields

$$\mathcal{G}^\mu_{\mathrm{N2D}}[\varepsilon\zeta, \beta b]\varpi + h\varpi = \mu \int_{-1}^{0} \partial_z(\ell[\varepsilon\zeta, \beta b]\varPhi)\, \mathrm{d}z$$

and the result for $n \in \{1, 2\}$ is deduced from Proposition 4.8 (or rather a slightly improved version, taking into account that only the vertical derivative is involved). As aforementioned, the $\mu^{1/2}$ prefactor when $n = 1$ stems from the anisotropic gradient $\nabla^\mu_{\boldsymbol{x},z}$ in Proposition 4.5, and can be removed by incrementing the regularity index and using Poincaré's inequality, eq. (4.4). $\square$

CHAPTER B

Hydrostatic models

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."

— Lewis Carroll, *Alice in Wonderland*

Contents

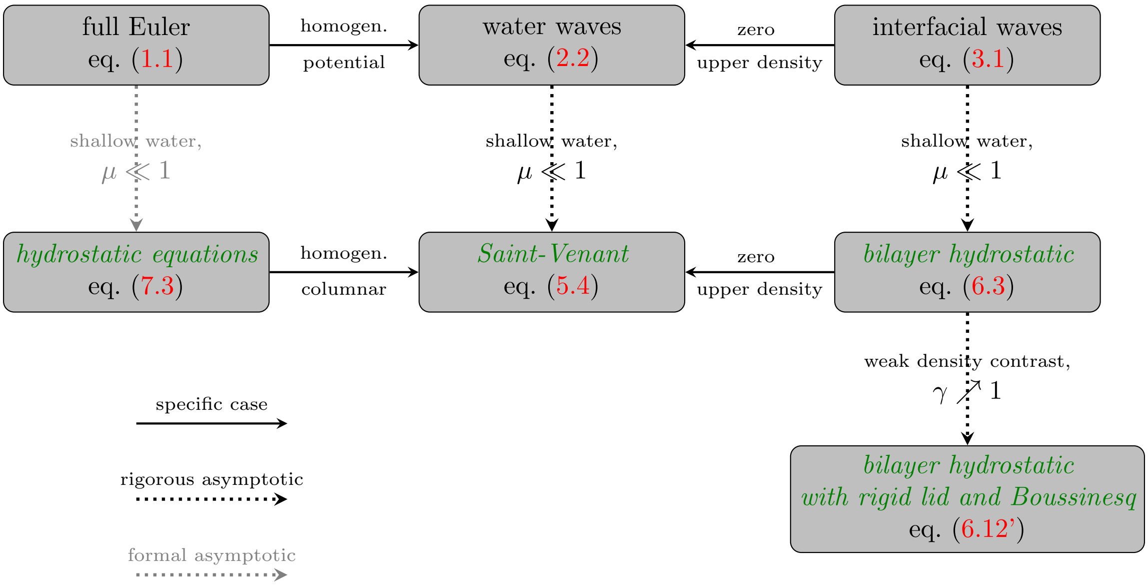


Figure B: Models in Chapter B (in *green*) and some filiations.

Foreword

We start our journey towards asymptotic models with ones among the oldest and simplest-looking. The so-called *hydrostatic models* can be formally derived from the "master" full Euler equations by using the *hydrostatic assumption*, that is approximating the pressure terms using an explicit formula stemming from neglecting the velocity advection terms in the horizontal momentum conservation equation, eq. (1.1b), specifically

$$-\partial_z P = \rho g,$$

which we can integrate using the known pressure at the free surface, by eq. (1.1f). Additionally, one often adds the assumption of *columnar motion*, stating that the horizontal velocity (approximately) does not depend on the vertical variable. When both assumptions are made, then we quickly obtain models with the rewarding properties that the vertical space variable has disappeared from the equations and only (first order) differential operators are involved.

Yet we shall not assume *a priori*—but indirectly prove—the hydrostatic assumption nor the columnar motion and will rather justify models asymptotically—with quantitative error estimates—in the shallow water regime, as $\mu \ll 1$; and using the irrotationality assumption in lieu of columnar motion.

Our first model is derived from the water waves system, that is assuming that the density is homogeneous and the flow potential (recall this allows to discard the vertical variable except in the Dirichlet-to-Neumann operator). We then obtain the well-known and much-studied ***Saint-Venant system***, eq. (5.4), already introduced in Section iii. Its derivation and rigorous justification, together with a very short description of some of its properties, is the subject of Section 5.

Then we move in Section 6 to the ***bilayer*** framework, with two layers of homogeneous potential flows. The situation is slightly messier as models differ whether we use the *free-surface* framework (in which case we obtain eq. (6.3)) or the *rigid-lid* framework (in which case we obtain eq. (6.12)), and in the latter one often uses the so-called *Boussinesq approximation* (which yields eq. (6.12')). It turns out the rigid-lid assumption and Boussinesq approximation both follow from the same assumption of weak density contrast, as shown in Section 6.2.5.

In Section 6.3 we quickly extend the analysis to $N \geq 2$ layers as above. While this ***multilayer*** framework may appear artificial, it is expected to approximate (as $N \gg 1$) the setting of continuously stratified flows, in view of withdrawing the assumptions of homogeneous density and potential flows while keeping the hydrostatic approximation in the shallow water regime.

Hydrostatic equations for ***continuously stratified*** flows are discussed in more details in Section 7. As we mention in Section 7.3, very little is known on these equations, despite the fact that they are at the core of the *primitive equations* which are widely used in studies and numerical simulations of geophysical flows. This offers stimulating mathematical challenges.

5 The Saint-Venant system

We now introduce the simplest fully nonlinear shallow water model for the water waves system. It is obtained by plugging the approximation

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi)+\mathcal{O}(\mu). \tag{5.1}$$

stemming from Proposition 4.10 into the water waves equations, eq. (2.7), and withdrawing all terms of size $\mathcal{O}(\mu)$. One obtains the system

$$\begin{cases} \partial_t\zeta+\nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi)=0, \\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\nabla\psi|^2=0. \end{cases} \tag{5.2}$$

One usually rewrites eq. (5.2) using a velocity variable

$$\begin{cases} \partial_t\zeta+\nabla\cdot\big((1+\varepsilon\zeta-\beta b)\boldsymbol{u}\big)=0, \\ \partial_t\boldsymbol{u}+\nabla\zeta+\varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\boldsymbol{0}. \end{cases} \tag{5.3}$$

System (5.3) is obtained immediately from eq. (5.2), taking the gradient of the second equation and setting $\boldsymbol{u}\stackrel{\text{def}}{=}\nabla\psi$. It is also valid if we set $\boldsymbol{u}\stackrel{\text{def}}{=}\overline{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}}$ is the layer-averaged horizontal velocity,

$$\overline{\boldsymbol{u}}\stackrel{\text{def}}{=}\frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Phi\,\mathrm{d}z,$$

in which case the first equation, representing the conservation of mass, is exactly satisfied by solutions of the water waves equations eq. (2.7) (by Lemma 4.6), and only the last (d-dimensional) equation is a $\mathcal{O}(\mu)$ approximation (and is a valid approximation even out of the irrotational framework; see [83]).

Using physical variables, (5.2) yields the ***Saint-Venant system***

$$\begin{cases} \partial_t h+\nabla\cdot(h\boldsymbol{u})=0, \\ \partial_t\boldsymbol{u}+g\nabla(h+b)+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\boldsymbol{0}, \end{cases} \tag{5.4}$$

with $h=d+\zeta-b$. By analogy with the compressible Euler equation, one recognizes here that the "sound speed" of long surface gravity waves in a layer of depth d is $c=\sqrt{gd}$.

System (5.4) is the prototype of hyperbolic quasilinear systems, the strong hyperbolicity being guaranteed by the non-cavitation assumption, $h>0$; see below. In fact, in the flat bottom case, $b\equiv 0$, the Saint-Venant system corresponds to the isentropic, *compressible* Euler equation for ideal gases with the pressure law $p(\rho)\propto\rho^2$ (identifying ρ with h). As such the literature on the Saint-Venant system is extremely vast, and we will cover here only very partial results which are directly useful for our purposes, and in particular can be extended to other more sophisticated models in subsequent sections.

5.1 Hamiltonian structure

System (5.2) inherits a canonical Hamiltonian structure from the water waves equations (see Section 2.2):

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\delta_\zeta\mathscr{H}_{\rm SV}\\ \delta_\psi\mathscr{H}_{\rm SV}\end{pmatrix}$$

with

$$\mathscr{H}_{\rm SV}(\zeta,\psi)\stackrel{\text{def}}{=}\frac12\int_{\mathbb{R}^d}\zeta^2+(1+\varepsilon\zeta-\beta b)|\nabla\psi|^2\ \mathrm{d}\boldsymbol{x}.$$

In fact, one could have (formally) derived the Saint-Venant system by plugging the approximation in eq. (5.1) directly into the water waves Hamiltonian functional, and derive the Saint-Venant system from Hamilton's principle on

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \zeta\partial_t\psi\,\mathrm{d}\boldsymbol{x} + \mathscr{H}_{\rm SV}\,\mathrm{d}t.$$

The Saint-Venant system enjoys the same symmetry groups as the water waves equations (again, see Section 2.2), and consistent preserved quantities, in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\rm SV} = 0, \quad \text{where} \quad \mathscr{Z} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \zeta\,\mathrm{d}\boldsymbol{x}\,,\ \mathscr{I} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \zeta\nabla\psi\,\mathrm{d}\boldsymbol{x}.$$

Written with the velocity variable $\boldsymbol{u}$, the system still enjoys a (non-canonical) symplectic form (see *e.g.* [386]). In dimension $d = 2$, one has

$$\partial_t\begin{pmatrix}\zeta\\ u_x\\ u_y\end{pmatrix} = -\begin{pmatrix}0 & \partial_x & \partial_y\\ \partial_x & 0 & -q\\ \partial_y & q & 0\end{pmatrix}\begin{pmatrix}\delta_\zeta\mathscr{H}_{\rm SV}\\ \delta_{u_x}\mathscr{H}_{\rm SV}\\ \delta_{u_y}\mathscr{H}_{\rm SV}\end{pmatrix}.$$

where $q = \varepsilon\frac{\operatorname{curl}\boldsymbol{u}}{h} = \varepsilon\frac{\partial_x u_y - \partial_y u_x}{1+\varepsilon\zeta-\beta b}$ and (misusing notations)

$$\mathscr{H}_{\rm SV}(\zeta,\boldsymbol{u}) \stackrel{\rm def}{=} \frac12\int_{\mathbb{R}^d} \zeta^2 + (1+\varepsilon\zeta-\beta b)|\boldsymbol{u}|^2\ \mathrm{d}\boldsymbol{x}.$$

Of course, in our situation, $q \equiv 0$ if $\boldsymbol{u} = \nabla\psi$, but it turns out the Saint-Venant system is also relevant for non-potential flows; see [83, 84] for a rigorous justification. Within this formalism, one can check that the time and space invariance of the Hamiltonian yield the conservation of total energy and momentum,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\rm SV} = 0 \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(1+\varepsilon\zeta-\beta b)\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \boldsymbol{0},$$

while Casimir invariants are, for any function C,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} hC(q)\,\mathrm{d}\boldsymbol{x} = 0,$$

which yields the conservation of mass—with $C(q) = 1$—and circulation—with $C(q) = q$—as special cases. It is of course straightforward to derive conservation laws associated with any of these preserved quantities.

5.2 Hyperbolicity

As mentioned previously, eq. (5.3) is a quasilinear system of first-order evolution equations, *i.e.* can be written under the form

$$\partial_t\boldsymbol{U} + \sum_{i=1}^d \mathcal{A}_i(\boldsymbol{U})\partial_{x_i}\boldsymbol{U} = \boldsymbol{F}(t,x,\boldsymbol{U}), \tag{5.5}$$

where $\boldsymbol{U} : \mathbb{R}^+\times\mathbb{R}^d \to \mathbb{R}^n$ (here, $n = 1+d$) represents the unknowns, and $\mathcal{A}_i : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ (where $\mathcal{M}_n(\mathbb{R})$ denotes $n\times n$ square matrices with real coefficients) and $\boldsymbol{F} : \mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^n \to \mathbb{R}^n$ are given and smooth.

We shall not recall the rich theory of such systems (see for instance [49, 310]), but recall a few facts adapted to our particular system, and which in particular allow to obtain the well-posedness and stability results stated in Theorem 5.3 and Theorem 5.7.

Recall the *principal symbol* of eq. (5.5) is

$$\mathcal{A}(\boldsymbol{U},\boldsymbol{\xi}) \stackrel{\text{def}}{=} \sum_{i=1}^{d} \boldsymbol{\xi}\mathcal{A}_i(\boldsymbol{U})$$

where $\boldsymbol{\xi} = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d$. One can check that for eq. (5.3), the characteristic equation

$$\det(\mathcal{A}(\boldsymbol{U},\boldsymbol{\xi}) - \lambda\,\mathrm{Id}_{1+d}) = 0$$

admits $d+1$ real solutions for any $\boldsymbol{\xi} \neq \mathbf{0}$ as soon as $h = 1+\varepsilon\zeta > 0$:

$$\lambda_\delta = \varepsilon(\boldsymbol{u}\cdot\boldsymbol{\xi}) + \delta\sqrt{h|\boldsymbol{\xi}|^2}$$

where $\delta \in \{-1,1\}$ if $d=1$ and $\delta \in \{-1,0,1\}$ if $d=2$. Because all the eigenvalues of $\mathcal{A}(\boldsymbol{U},\boldsymbol{\xi})$ are distinct for any for $\boldsymbol{\xi} \neq \mathbf{0}$ and any $\boldsymbol{U} \in \mathbb{R}^{1+d}_{h>0} \stackrel{\text{def}}{=} \{(\zeta,\boldsymbol{u}) \in \mathbb{R}^{1+d} \ : \ 1+\varepsilon\zeta-\beta b > 0\}$, the Saint-Venant system is *strictly hyperbolic*.

As a consequence, the symbol is smoothly diagonalizable with real eigenvalues, which in turn allows to construct a symbolic symmetrizer, namely $\mathcal{S} : (\boldsymbol{U},\boldsymbol{\xi}) \in \mathbb{R}^{1+d}_{h>0} \times \mathbb{R}^d \setminus \{\mathbf{0}\} \to \mathcal{M}_{1+d}(\mathbb{R})$, smooth and homogeneous of degree 0 in the second variable such that for all $(\boldsymbol{U},\boldsymbol{\xi}) \in \mathbb{R}^{1+d}_{h>0}\times\mathbb{R}^d\setminus\{\mathbf{0}\}$, $S(\boldsymbol{U},\boldsymbol{\xi})$ is symmetric and definite positive, and $\mathcal{S}(\boldsymbol{U},\boldsymbol{\xi})\mathcal{A}(\boldsymbol{U},\boldsymbol{\xi})$ is symmetric.

In our case, it is in fact easy to see that the system is *symmetric-hyperbolic* in the sense of Friedrichs, namely we can exhibit a (non-symbolic) explicit symmetrizer $\mathcal{S} \in \mathcal{C}^\infty(\mathbb{R}^{1+d}_{h>0}, \mathcal{M}_{1+d}(\mathbb{R}))$ such that for all $\boldsymbol{U} \in \mathbb{R}^{1+d}_{h>0}$, $\mathcal{S}(\boldsymbol{U})$ is symmetric and definite positive, and for all $i \in \{1,\dots,d\}$, $\mathcal{S}(\boldsymbol{U})\mathcal{A}_i(\boldsymbol{U})$ is symmetric. An example of such symmetrizer is

$$\mathcal{S}(\boldsymbol{U}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^\top & h\,\mathrm{Id}_d \end{pmatrix}, \qquad \boldsymbol{U} = (\zeta,\boldsymbol{u}),\ h = 1+\varepsilon\zeta-\beta b.$$

In other words, the Saint-Venant system is symmetric if one multiplies the second equation with the depth.

5.3 Rigorous justification

In this section we provide the full justification of the Saint-Venant system as an asymptotic model for water waves in the shallow water regime (Definition III.2) that is for parameters in the set

$$\mathfrak{p}_{\mathrm{SW}} = \big\{(\mu,\varepsilon,\beta) \ : \ \mu \in (0,\mu^\star],\ \varepsilon \in [0,1],\ \beta \in [0,1]\big\}.$$

Thanks to the results of Section 4, and in particular Proposition 4.10, it is now straightforward to justify eq. (5.2) in the sense of *consistency*.

Theorem 5.1 (Consistency)**.** *Let $d, s_\star \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$. Let $s \in \mathbb{N}$ and $M^\star \geq 0$. There exists $C > 0$ such that for any $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\mathrm{SW}}$, any $b \in W^{\max\{s+4,2+s_\star\},\infty}(\mathbb{R}^d)$, any $T > 0$ and any $(\zeta,\psi) \in L^\infty(0,T; H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water waves equations, eq. (2.7), satisfying*

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad h(t,\boldsymbol{x}) \stackrel{\text{def}}{=} 1+\varepsilon\zeta(t,\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0. \tag{5.6}$$

uniformly for $t \in (0,T)$ and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\big) + \big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\nabla\psi) = r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 = r_2, \end{cases}$$

and one has for almost every $t \in (0,T)$

$$\big|r_1(t,\cdot)\big|_{H^s} \leq C\,\mu\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big),$$
$$\big|r_2(t,\cdot)\big|_{H^{s+2}} \leq C\,\mu\varepsilon\,\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big).$$

Proof. The Proposition is an immediate consequence of Proposition 4.10 with $n=1$ and $k=s+1$ for the first equation, and with $n=1$ and $k=s+3$ and $k=1+s_\star$ (with the product and composition estimates of Appendix II) for the second equation. □

Remark 5.2. *One could also prove that, provided that $\partial_t\zeta, \partial_t\nabla\psi$ are sufficiently regular—or inferring their regularity from the water waves equations, eq. (2.7)—then $\overline{\boldsymbol{u}}$ defined as in Proposition 4.9, satisfies*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{\boldsymbol{u}}) = 0, \\ \partial_t\overline{\boldsymbol{u}} + \nabla\zeta + \varepsilon(\overline{\boldsymbol{u}}\cdot\nabla)\overline{\boldsymbol{u}} = \boldsymbol{r}, \end{cases}$$

with an explicit estimate for $\big\|\boldsymbol{r}\big\|_{L^\infty(0,T;H^s)} = \mathcal{O}(\mu)$. The first equation, representing the conservation of mass, is an exact identity by Lemma 4.6. The second equation follows from Proposition 4.9 as well as an estimate on

$$\big\|\partial_t\nabla\psi - \partial_t\overline{\boldsymbol{u}}\big\|_{L^\infty(0,T;H^s)} = \mathcal{O}(\mu),$$

which would be obtained by applying Proposition 4.5 to $\partial_t\Phi$ the strong solution to

$$\begin{cases} \frac{1}{\partial_z\sigma}\nabla^\mu_{\boldsymbol{x},z}\cdot P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\partial_t\Phi = -\frac{1}{\partial_z\sigma}\nabla^\mu_{\boldsymbol{x},z}\cdot\big[\partial_t, P(\Sigma)\big]\nabla^\mu_{\boldsymbol{x},z}\Phi & \text{in } \mathbb{R}^d\times(-1,0), \\ \partial_t\Phi = \partial_t\psi & \text{on } \mathbb{R}^d\times\{0\}, \\ \boldsymbol{e}_{d+1}\cdot P(\Sigma)\nabla^\mu_{\boldsymbol{x},z}\partial_t\Phi = -\boldsymbol{e}_{d+1}\cdot\big[\partial_t, P(\Sigma)\big]\nabla^\mu_{\boldsymbol{x},z}\Phi & \text{on } \mathbb{R}^d\times\{-1\}, \end{cases}$$

and differentiating (with respect to time) the identity provided by Lemma 4.6 and Lemma 4.7:

$$\overline{\boldsymbol{u}} = \nabla\psi - \frac{\mu}{h}\int_{-1}^0 (\partial_z\sigma)\nabla_{\boldsymbol{x}}(\ell[\varepsilon\zeta,\beta b]\Phi) - (\nabla_{\boldsymbol{x}}\sigma)\partial_z(\ell[\varepsilon\zeta,\beta b]\Phi)\,\mathrm{d}z.$$

The consistency alone is not sufficient to provide a full justification of the Saint-Venant system eq. (5.3). Fortunately, classical results on hyperbolic systems (see *e.g.* [49, 310], and Section 8.6) provide the well-posedness theory and stability estimates which allow a stronger notion of justification.

Theorem 5.3 (Local well-posedness)**.** *Let $d\in\mathbb{N}^\star$, $h_\star>0$, $s_\star>d/2$, $s\geq 1+s_\star$ and $M^\star>0$. There exists $T>0$ and $C>0$ such that for any $\varepsilon,\beta\in[0,1]$, any $b\in W^{s+1,\infty}(\mathbb{R}^d)$, and any $(\zeta_0,\boldsymbol{u}_0)\in H^s(\mathbb{R}^d)^{1+d}$ satisfying eq. (5.6) and*

$$M_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\boldsymbol{u}_0\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

there exists a unique $(\zeta,\boldsymbol{u})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)^{1+d})\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)^{1+d})$ classical solution to the Saint-Venant system, eq. (5.3), with initial data $(\zeta,\boldsymbol{u})\,|_{t=0}=(\zeta_0,\boldsymbol{u}_0)$; and one has for any $t\in[0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}(t,\cdot)\big|_{H^s} \leq C\times\Big(\big|\zeta_0\big|_{H^s} + \big|\boldsymbol{u}_0\big|_{H^s}\Big)$$

and eq. (5.6) holds with constant $h_\star/2$.

Remark 5.4. *Theorem 5.3 may be improved in order to obtain existence and control of the solution on the time interval $t\in[0,T/\widetilde{M}_0]$ with $\widetilde{M}_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\boldsymbol{u}_0\big|_{H^{1+s_\star}}$; see [62].*

Remark 5.5. *Uniqueness in Theorem 5.3 allows to define $T_{\max}$ the supremum of $T>0$ such that the Cauchy problem has a solution $(\zeta,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^s(\mathbb{R}^d)^{1+d})\cap\mathcal{C}^1([0,T];H^{s-1}(\mathbb{R}^d)^{1+d})$ which remains in the hyperbolic domain, $\inf_{(t,\boldsymbol{x})\in[0,T]\times\mathbb{R}^d}\big(1+\varepsilon\zeta(t,\boldsymbol{x})\big)>0$. The use of tame estimates as in Remark II.12 yields the following blowup criterion:*

$$T_{\max}<\infty \quad\Longrightarrow\quad \lim_{t\nearrow T_{\max}}\Big(\big\|\zeta\big\|_{L^\infty(0,t;W^{1,\infty})}+\big\|\boldsymbol{u}\big\|_{L^\infty(0,t;W^{1,\infty})}\Big)\to\infty,$$

since the hyperbolicity criterion remains satisfied as a consequence of the conservation of mass; see footnote 6 page vii. In particular, for given initial data, $T_{\max}$ and the maximal solution do not depend on the choice of the regularity index, $s>1+d/2$.

Remark 5.6. *In order to prove the well-posedness of the Cauchy problem associated with eq. (5.7) in the sense of Hadamard, one should state the continuity of the flow map*

$$\varphi^t:(\zeta_0,\boldsymbol{u}_0)\in H^s(\mathbb{R}^d)^{1+d}\mapsto(\zeta(t,\cdot),\boldsymbol{u}(t,\cdot))\in H^s(\mathbb{R}^d)^{1+d}.$$

While this property holds true (see the aforementioned references), it is not significant for our purposes, where we are happy to ask an extra derivative on the initial data to ensure that the flow map is Lipschitz. This result is a particular case of the stability property, Theorem 5.7, below.

Theorem 5.7 (Stability)**.** *Let $d\in\mathbb{N}^\star$, $s_\star>d/2$, $h_\star>0$, $s\in\mathbb{N}$, $M^\star\geq 0$, $n_0\overset{\text{def}}{=}\max\{s,1+s_\star\}$, $n\overset{\text{def}}{=}\max\{s+1,1+s_\star\}$. There exists $C>0$ such that for any $\varepsilon,\beta\in[0,1]$, for any $b\in W^{n,\infty}(\mathbb{R}^d)$, for any $T^\star>0$ and $(\zeta^0,\boldsymbol{u}^0)\in\mathcal{C}^0([0,T^\star];H^{n_0}(\mathbb{R}^d)^{1+d})$ satisfying the Saint-Venant system, eq. (5.3), as well as any $(\zeta,\boldsymbol{u})\in L^\infty(0,T^\star;H^n(\mathbb{R}^d)^{1+d})$ satisfying*

$$\begin{cases}\partial_t\zeta+\nabla\cdot\big((1+\varepsilon\zeta-\beta b)\boldsymbol{u}\big)=r,\\ \partial_t\boldsymbol{u}+\nabla\zeta+\varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\boldsymbol{r},\end{cases}\tag{5.7}$$

with $(r,\boldsymbol{r})\in L^1(0,T^\star;H^s(\mathbb{R}^d)^{1+d})$, and assuming that $h=1+\varepsilon\zeta-\beta b$ and $h^0=1+\varepsilon\zeta^0-\beta b$ satisfy eq. (5.6) uniformly for $t\in[0,T^\star]$ and

$$M\overset{\text{def}}{=}\operatorname*{ess\,sup}_{t\in[0,T^\star]}\Big(\big|(\varepsilon\zeta(t,\cdot),\varepsilon\boldsymbol{u}(t,\cdot))\big|_{H^n\times H^n}+\big|(\varepsilon\zeta^0(t,\cdot),\varepsilon\boldsymbol{u}^0(t,\cdot))\big|_{H^{n_0}\times H^{n_0}}\Big)+\big|\beta b\big|_{W^{n,\infty}}\leq M_\star,$$

one has, for any $t\in[0,T^\star]$,

$$\begin{aligned}\big|(\zeta-\zeta^0)(t,\cdot)\big|_{H^s}+\big|(\boldsymbol{u}-\boldsymbol{u}^0)(t,\cdot)\big|_{H^s}\leq Ce^{CMt}\Big(\big|(\zeta-\zeta^0)(t=0,\cdot)\big|_{H^s}+\big|(\boldsymbol{u}-\boldsymbol{u}^0)(t=0,\cdot)\big|_{H^s}\Big)\\ +C\int_0^t e^{CM(t-\tau)}\left(\big|r(\tau,\cdot)\big|_{H^s}+\big|\boldsymbol{r}(\tau,\cdot)\big|_{H^s}\right)\mathrm{d}\tau\,.\end{aligned}$$

The following result is a direct consequence of Theorem 5.1, Theorem 5.3 and Theorem 5.7.

Theorem 5.8 (Convergence)**.** *Let $d\in\mathbb{N}^\star$, $s\in\mathbb{N}$, $s_\star>d/2$, $h_\star>0$, $\mu^\star>0$, and $M^\star\geq 0$. There exists $T>0$ and $C>0$ such that for any $(\mu,\varepsilon,\beta)\in(0,\mu^\star]\times[0,1]^2$, any $b\in W^{\max\{s+4,2+s_\star\},\infty}(\mathbb{R}^d)$, any $T^\star>0$ and any $(\zeta,\psi)\in\mathcal{C}^0([0,T^\star];H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)\times\mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d))$ classical solution to the water waves equations, eq. (2.7), satisfying eq. (5.6) uniformly on $[0,T^\star]$ and*

$$M\overset{\text{def}}{=}\sup_{t\in[0,T^\star]}\Big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}}+\big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{\max\{s+1,1+s_\star\}}}\Big)+\big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}}\leq M^\star,$$

there exists a unique $(\zeta_{\rm SV},\boldsymbol{u}_{\rm SV})\in\mathcal{C}^0([0,T/M];H^{\max\{s,1+s_\star\}}(\mathbb{R}^d)^{1+d})$ solution to the Saint-Venant system (5.3) with initial data $(\zeta_{\rm SV},\boldsymbol{u}_{\rm SV})\,|_{t=0}=(\zeta,\nabla\psi)\,|_{t=0}$ and for any $t\in(0,\min\{T^\star,T/M\}]$,

$$\big|(\zeta-\zeta_{\rm SV})(t,\cdot)\big|_{H^s}+\big|(\nabla\psi-\boldsymbol{u}_{\rm SV})(t,\cdot)\big|_{H^s}\leq C\,\mu\,t\left(\big\|\zeta\big\|_{L^\infty(0,t;H^{s+4})}+\big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+3})}\right).$$

Remark 5.9. *That for regular initial data (satisfying the non-cavitation and Rayleigh–Taylor criteria), uniquely defined regular solutions to the water waves equations exist on the time scale of Theorem 5.8 is provided by Theorem 2.9.*

6 The bilayer hydrostatic systems

We introduce the extensions of the Saint-Venant system displayed in Section 5 which describe infinitely long interfacial waves, following the framework presented in Section 3. The case of two homogeneous layers with a free interface and a free surface is studied in Section 6.1. The rigid-lid setting is studied in Section 6.2. Finally, the extension to multiple layers is tackled in Section 6.3. Incidentally, while we could use the terminology of "multilayer Saint-Venant system", these should not be confused with the multilayer systems "with mass exchange" discussed for instance in [31, 30] as in the latter the layers are not prescribed by the density stratification.

6.1 The free-surface case

We recall that the full Euler equations describing the motion of two layers of incompressible, homogeneous, inviscid fluids under the assumption of potential flows being subject to vertical gravity forces can be written as a system of four equations on the deformation of the surface and interface (denoted respectively ζ_1 and ζ_2), and the trace of the velocity potential of the upper fluid at the surface and the lower fluid at the interface (denoted respectively φ_1 and ψ_2). These equations are given in eq. (3.1)—and eq. (3.14) for the dimensionless version—and will be referred in this section as the *interfacial waves system*.

Following the strategy from Proposition 4.10 and Proposition 4.23, we find that the operators defined therein satisfy

$$\begin{aligned}
\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 &= -\nabla\cdot\big(h_2\nabla\psi_2\big)+\mathcal{O}(\mu),\\
\frac{1}{\mu}\mathcal{G}_1^{\mu,\delta}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2,\beta b](\varphi_1,\psi_2) &= -\nabla\cdot\big(h_1\nabla\varphi_1\big)-\nabla\cdot\big(h_2\nabla\psi_2\big)+\mathcal{O}(\mu),\\
\psi_1 = \mathcal{H}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2,\beta b](\varphi_1,\psi_2) &= \varphi_1+\mathcal{O}(\mu),
\end{aligned}$$

where $h_1 = 1+\varepsilon\alpha\zeta_1-\varepsilon\zeta_2$ and $h_2=\delta^{-1}+\varepsilon\zeta_2-\beta b$. Plugging these approximations into eq. (3.14) and withdrawing all terms of size $\mathcal{O}(\mu)$ yields

$$\begin{cases}
\alpha\partial_t\zeta_1+\nabla\cdot\big(h_1\nabla\psi_1\big)+\nabla\cdot\big(h_2\nabla\psi_2\big)=0,\\
\partial_t\zeta_2+\nabla\cdot\big(h_2\nabla\psi_2\big)=0,\\
\partial_t\psi_1+\frac{\delta+\gamma}{1-\gamma}\alpha\zeta_1+\frac{\varepsilon}{2}|\nabla\psi_1|^2=0,\\
\partial_t\big(\psi_2-\gamma\psi_1\big)+(\delta+\gamma)\zeta_2+\frac{\varepsilon}{2}\big(|\nabla\psi_2|^2-\gamma|\nabla\psi_1|^2\big)=0.
\end{cases}\tag{6.1}$$

As for the (one-layer) Saint-Venant system, it is customary to rewrite eq. (6.1) using velocity variables, that is

$$\begin{cases}
\alpha\partial_t\zeta_1+\nabla\cdot\big(h_1\boldsymbol{u}_1\big)+\nabla\cdot\big(h_2\boldsymbol{u}_2\big)=0,\\
\partial_t\zeta_2+\nabla\cdot\big(h_2\boldsymbol{u}_2\big)=0,\\
\partial_t\boldsymbol{u}_1+\frac{\delta+\gamma}{1-\gamma}\alpha\nabla\zeta_1+\varepsilon(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1=0,\\
\partial_t\boldsymbol{u}_2+\gamma\frac{\delta+\gamma}{1-\gamma}\alpha\nabla\zeta_1+(\delta+\gamma)\nabla\zeta_2+\varepsilon(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2=0.
\end{cases}\tag{6.2}$$

System (6.2) is obtained immediately from eq. (6.1), defining $\boldsymbol{u}_\ell \stackrel{\text{def}}{=} \nabla\psi_\ell$ ($\ell\in\{1,2\}$) and after a little algebra. It is also valid if we set $\boldsymbol{u}_\ell \stackrel{\text{def}}{=} \overline{\boldsymbol{u}}_\ell$ ($\ell\in\{1,2\}$) where $\overline{\boldsymbol{u}}_\ell$ are the layer-averaged horizontal velocities,

$$\overline{\boldsymbol{u}}_1=\frac{1}{h_1}\int_{\varepsilon\zeta_2}^{1+\varepsilon\alpha\zeta_1}\nabla_{\boldsymbol{x}}\Phi_1\,\mathrm{d}z,\qquad \overline{\boldsymbol{u}}_2=\frac{1}{h_2}\int_{-\delta^{-1}+\beta b}^{\varepsilon\zeta_2}\nabla_{\boldsymbol{x}}\Phi_2\,\mathrm{d}z,$$

in which case the first two equations, representing the conservation of mass, are exactly satisfied by solutions of the interfacial waves equations eq. (3.14), and only the last two (d-dimensional) equations are $\mathcal{O}(\mu)$ approximations (and are expected to be valid even out of the irrotational framework).

Physical variables Using physical variables (recall Section 3.4), system (6.2) reads

$$\begin{cases} \partial_t h_1 + \nabla \cdot \big(h_1 \boldsymbol{u}_1\big) = 0, \\ \partial_t h_2 + \nabla \cdot \big(h_2 \boldsymbol{u}_2\big) = 0, \\ \rho_1 \partial_t \boldsymbol{u}_1 + g\rho_1 \nabla(h_1 + h_2 + b) + \rho_1(\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1 = \mathbf{0}, \\ \rho_2 \partial_t \boldsymbol{u}_2 + g\rho_1 \nabla h_1 + g\rho_2 \nabla(h_2 + b) + \rho_2(\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u}_2 = \mathbf{0}, \end{cases} \tag{6.3}$$

with $h_1(t,\boldsymbol{x}) \stackrel{\text{def}}{=} d_1 + \zeta_1(t,\boldsymbol{x}) - \zeta_2(t,\boldsymbol{x})$ and $h_2(t,\boldsymbol{x}) \stackrel{\text{def}}{=} d_2 + \zeta_2(t,\boldsymbol{x}) - b(\boldsymbol{x})$. We refer to this system as the (free-surface) ***bilayer hydrostatic system*** because it can be formally derived from the hydrostatic assumption (and columnar motion), that is setting $\partial_z P = -\rho g$ in the full Euler equations (1.1)-(1.2); see for instance [201, § 6.2].

6.1.1 Hamiltonian structure

The bilayer hydrostatic system, eq. (6.1), enjoys a canonical formulation which is directly inherited from the corresponding one of the interfacial waves system described in Section 3.2. Introducing

$$\mathscr{H}_{\text{hydro}} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \gamma\tfrac{\delta+\gamma}{1-\gamma}(\alpha\zeta_1)^2 + (\delta+\gamma)\zeta_2^2 + \gamma h_1|\nabla\psi_1|^2 + h_2|\nabla\psi_2|^2 \,\mathrm{d}\boldsymbol{x}$$

where $h_1 = 1 + \varepsilon\alpha\zeta_1 - \varepsilon\zeta_2$ and $h_2 = \delta^{-1} + \varepsilon\zeta_2 - \beta b$ and viewing $\mathscr{H}_{\text{hydro}}$ as a functional for $(\alpha\zeta_1, \zeta_2, \xi_1 \stackrel{\text{def}}{=} \gamma\psi_1, \xi_2 \stackrel{\text{def}}{=} \psi_2 - \gamma\psi_1)$, one can check that eq. (6.1) reads

$$\partial_t \begin{pmatrix} \alpha\zeta_1 \\ \xi_1 \\ \zeta_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_{\xi_1}\mathscr{H}_{\text{hydro}} \\ -\delta_{\alpha\zeta_1}\mathscr{H}_{\text{hydro}} \\ \delta_{\xi_2}\mathscr{H}_{\text{hydro}} \\ -\delta_{\zeta_2}\mathscr{H}_{\text{hydro}} \end{pmatrix}.$$

Alternatively, we can define the Lagrangian of the system with the difference between potential and kinetic energy; see [121, (2.23)]. Associated with the Hamiltonian formulation and natural symmetry groups of the system are preserved quantities (invariants). Related to the variation of base level for the velocity potentials are the obvious conservation of the excess of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z}_\ell = 0, \qquad \mathscr{Z}_\ell \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta_\ell \,\mathrm{d}\boldsymbol{x} \qquad (\ell \in \{1,2\}).$$

From horizontal translation invariance (in the flat bottom case) we obtain the conservation of the horizontal impulse

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \alpha\zeta_1\nabla\xi_1 + \zeta_2\nabla\xi_2 \,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0)$$

and hence the horizontal momentum which under the hydrostatic assumption simply reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}_{\text{hydro}} = 0, \qquad \mathscr{M}_{\text{hydro}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \gamma h_1\nabla\psi_1 + h_2\nabla\psi_2 \,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$

From time translation invariance we obtain the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\text{hydro}} = 0.$$

As in Section 5.1, we may also devise a (non-canonical) symplectic form for the equations with the velocity variables, eq. (6.2). We omit to write it down.

6.1.2 Hyperbolicity

Conservative form Equations (6.2) (or (6.3)) is—as in the one-layer case—a first-order quasilinear system. Some crucial differences arise however in the bilayer framework. Firstly, the system can (apparently) no longer be written under conservative form, that is

$$\partial_t \boldsymbol{U} + \operatorname{div} \boldsymbol{F} = \boldsymbol{G}$$

with well-chosen functions $\boldsymbol{F}(\boldsymbol{U})$ and $\boldsymbol{G}(\boldsymbol{U}, b)$. Indeed, the standard way of writing eq. (6.2) in a conservative way is to used momentum variables, that is $\boldsymbol{m}_\ell \stackrel{\text{def}}{=} h_\ell \boldsymbol{u}_\ell$ ($\ell \in \{1,2\}$) so that we have

$$\begin{cases} \alpha\partial_t\zeta_1 + \nabla\cdot\big(h_1\boldsymbol{u}_1\big) + \nabla\cdot\big(h_2\boldsymbol{u}_2\big) = 0, \\ \partial_t\zeta_2 + \nabla\cdot\big(h_2\boldsymbol{u}_2\big) = 0, \\ \partial_t(h_1\boldsymbol{u}_1) + \frac{\delta+\gamma}{1-\gamma}\alpha h_1\nabla\zeta_1 + \varepsilon\nabla\cdot(h_1\boldsymbol{u}_1\otimes\boldsymbol{u}_1) = \boldsymbol{0}, \\ \partial_t(h_2\boldsymbol{u}_2) + \gamma\frac{\delta+\gamma}{1-\gamma}\alpha h_2\nabla\zeta_1 + (\delta+\gamma)h_2\nabla\zeta_2 + \varepsilon\nabla\cdot(h_2\boldsymbol{u}_2\otimes\boldsymbol{u}_2) = \boldsymbol{0}, \end{cases} \tag{6.4}$$

and we recognize non-conservative contributions of the form $\zeta_2\nabla\zeta_1$ in the last equation. Let us remark however that this fact is of utmost importance in the theory of discontinuous solutions but not so much for regular solutions, and that in our irrotational framework we can use the identities

$$\big(\boldsymbol{u}_\ell\cdot\nabla\big)\boldsymbol{u}_\ell = \big((\nabla\psi_\ell)\cdot\nabla\big)\nabla\psi_\ell = \frac{1}{2}\nabla\big(|\nabla\psi_\ell|^2\big)$$

to rewrite equivalently (6.2) (for irrotational data) under conservative form.

Remark 6.1. *Setting $\gamma = 0$ and $\delta = 1$, we recover the one-layer (with free surface) situation, that is the Saint-Venant system.*

Hyperbolicity Another difference with respect to the (one-layer) Saint-Venant equations, and which is much more important for our purposes, is that the hyperbolicity domain (see Section 5.2) can no longer be described by simple explicit formulae, and in particular the non-cavitation assumptions

$$\inf_{\mathbb{R}^d} h_1 > 0, \quad \inf_{\mathbb{R}^d} h_2 > 0$$

(which we always assume thereafter) are not sufficient to guarantee hyperbolicity. However the hyperbolicity can be clarified through a nice "geometrical approach", dating back to Ovsjannikov [352] (see also [39] and [408]). We quickly recall this approach in the situation of flat bottom (since the the bottom contributions are order-zero terms) and horizontal dimension $d = 1$ (since the general case can be recovered thanks to the rotational invariance property—see [329, 157]—and symbolic analysis).Then eq. (6.2) can be written as

$$\partial_t\boldsymbol{U} + \mathcal{A}(\boldsymbol{U})\partial_x\boldsymbol{U} = 0, \qquad \boldsymbol{U} \stackrel{\text{def}}{=} \begin{pmatrix} h_1 \\ h_2 \\ \varepsilon u_1 \\ \varepsilon u_2 \end{pmatrix}, \quad \mathcal{A}(\boldsymbol{U}) = \begin{pmatrix} \varepsilon u_1 & 0 & h_1 & 0 \\ 0 & \varepsilon u_2 & 0 & h_2 \\ \varrho^{-1} & \varrho^{-1} & \varepsilon u_1 & 0 \\ \gamma\varrho^{-1} & \varrho^{-1} & 0 & \varepsilon u_2 \end{pmatrix}$$

where $\varrho \stackrel{\text{def}}{=} \frac{1-\gamma}{\delta+\gamma}$ measures the density contrast. The key idea consists in writing the characteristic polynomial of $\mathcal{A}$ under the form

$$P(\lambda) = \big((\varepsilon u_2 - \lambda)^2 - \varrho^{-1}h_2\big)\big((\varepsilon u_1 - \lambda)^2 - \varrho^{-1}h_1\big) - \gamma\varrho^{-2}h_1h_2$$

and to define

$$p_\ell = \frac{\lambda - \varepsilon u_\ell}{\sqrt{h_\ell}}, \quad \ell \in \{1,2\}.$$

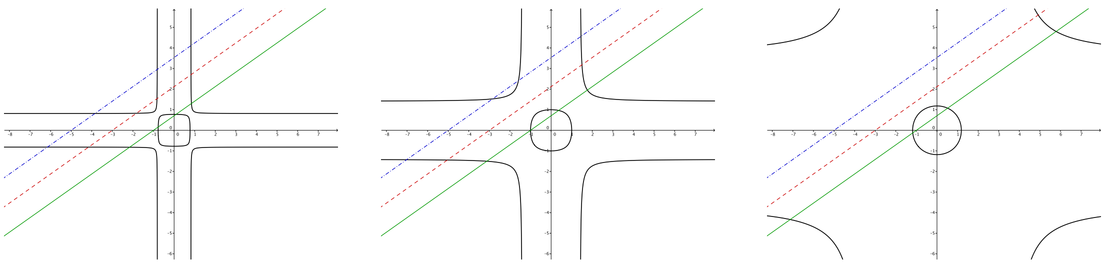
Hence real solutions of the characteristic polynomials correspond in a one-to-one manner to solutions $p_1, p_2 \in \mathbb{R}$ to

$$(p_2^2 - \varrho^{-1})(p_1^2 - \varrho^{-1}) = \gamma\varrho^{-2}, \quad p_1\sqrt{h_1} + \varepsilon u_1 = p_2\sqrt{h_2} + \varepsilon u_2. \tag{6.5}$$

Geometrically, the solutions of the above are the intersection of a fourth-order curve which is the union of a closed curve (resembling a circle) and fourfold unbounded curve (resembling four hyperbolas), and the straight line with slope $\sqrt{h_1/h_2}$ passing through $(p_1, p_2) = (0, \varepsilon\frac{u_1-u_2}{\sqrt{h_2}})$. From this interpretation and straightforward scaling arguments, we see that when h_1/h_2 and γ are fixed, there exists two critical values V_-, V_+ such that

- If $(\varepsilon|u_1 - u_2|)^2 < \frac{h_2}{\varrho}V_-(\frac{h_1}{h_2}, \gamma)$ or $(\varepsilon|u_1 - u_2|)^2 > \frac{h_2}{\varrho}V_+(\frac{h_1}{h_2}, \gamma)$, there exists four distinct real solutions to the characteristic polynomial, and hence the system is strictly hyperbolic.
- If $\frac{h_2}{\varrho}V_-(\frac{h_1}{h_2}, \gamma) < (\varepsilon|u_1 - u_2|)^2 < \frac{h_2}{\varrho}V_+(\frac{h_1}{h_2}, \gamma)$, there exists only two (simple) real solutions to the characteristic polynomial, and hence the system is of mixed elliptic-hyperbolic type.

Shortly put, the linearized bilayer hydrostatic system is stable if and only if the shear velocity is sufficiently small or sufficiently large. Moreover we see that when shear velocities are sufficiently small, we can distinguish two pairs of solution, depending on whether they cross the "inner" closed curve or "outer" unbounded curve. The former correspond to the so-called *baroclinic mode*, and the latter to the *barotropic mode*; see *e.g.* [201, § 6.2].



(a) $\gamma = 0.1$ (b) $\gamma = 0.5$ (c) $\gamma = 0.9$

Figure 6.1: Solutions to eq. (6.5). Solutions to the quartic equation are in black (plain). Solutions to the linear equation with $\varepsilon(u_1 - u_2) = 1$ (green, plain), $\varepsilon(u_1 - u_2) = 3$ (red, dashed) and $\varepsilon(u_1 - u_2) = 5$ (blue, dot-dashed). Parameters are $h_1 = 1$, $h_2 = \frac{1}{\delta} = 2$.

Special limits It is interesting to study the limit of vanishing density contrast, that is $\gamma \nearrow 1$. In this case, and recalling the definition $\varrho = \frac{1-\gamma}{\delta+\gamma}$, one may check that the aforementioned closed curve converges towards the circle around the origin with squared radius $(1-\gamma)\varrho^{-1} = \delta+\gamma \to 1+\delta$ $(\gamma \nearrow 1)$. The unbounded curves are in the outer region delimited by the circle around the origin with squared radius ϱ^{-1} and hence go to infinity (due to our choice of scaling when non-dimensionalizing which allows to concentrate on the baroclinic mode), and in the limit we have the simple criterion that the system is of strict hyperbolic type if and only if

$$\big(\varepsilon|u_1 - u_2|\big)^2 < (1+\delta)(h_1 + h_2).$$

This will correspond to the corresponding criterion for the rigid-lid system in the same limit; see Section 6.2.3 and discussion in [26].

In the opposite direction, if we consider the limit $\gamma \searrow 0$, then the quadratic curve degenerates in the set $\{-\varrho^{-1}, \varrho^{-1}\}^2$. Then there are always four real eigenvalues (counting multiplicities), that is $\lambda = \varepsilon u_\ell \pm \sqrt{\varrho^{-1}h_\ell}$ $(\ell \in \{1, 2\})$, and the system is strongly hyperbolic as soon as the non-cavitation assumption is satisfied. This corresponds to the one-layer situation, with the addition of a spurious propagation mode for the free surface which does not interact with the propagation mode of the interface.

6.1.3 Rigorous justification

Let us now rigorously justify the bilayer, hydrostatic equations, eq. (6.3), as an asymptotic model for the interfacial waves system, eq. (3.1). It is informative to compare the results below with the corresponding ones on the Saint-Venant system in Section 5.3.

In this section we denote, given $\mu^\star > 0$, $\delta_\star > 0$, and $\delta^\star > 0$:

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} \stackrel{\rm def}{=} \big\{(\mu,\varepsilon,\beta,\alpha,\delta,\gamma) \ : \ \mu\in(0,\mu^\star],\ \varepsilon\in[0,1],\ \beta\in[0,1],\ \alpha\in(0,1],\ \delta\in[\delta_\star,\delta^\star],\gamma\in[0,1)\big\}.$$

We state and prove below two consistency results, the first one being adapted to the formulation of eq. (6.1)—or more precisely an artificially extended system involving a variable ψ_1 (representing the trace of the upper velocity potential at the interface) coupled with the constraint $\psi_1=\varphi_1$—and the second one being adapted to the formulation of eq. (6.2), and more useful in practice.

Theorem 6.2 (Consistency). *Let $d\in\mathbb{N}^\star$, $s\in\mathbb{N}$, $s_\star>d/2$, $\mu^\star>0$, $\delta_\star>0$, $\delta^\star>0$, $h_\star>0$ and $M^\star\geq 0$. There exist $C>0$ such that for any $(\mu,\varepsilon,\beta,\alpha,\delta,\gamma)\in\mathfrak{p}_{\frac{\rm SW}{\rm SW}}$, any $b\in W^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)$, any $T>0$ and any $(\zeta_1,\zeta_2,\varphi_1,\psi_2)\in L^\infty(0,T;H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2\times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2)$ solution to the interfacial waves equations, eq. (3.14) with ψ_1 satisfying the constraint equation*

$$\psi_1=\mathcal{H}^\mu_{\rm D}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1+\mathcal{H}^\mu_{\rm N}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2,$$

and such that

$$\forall t\in[0,T],\quad \forall \boldsymbol{x}\in\mathbb{R}^d,\qquad \begin{cases} h_1(t,\boldsymbol{x})\stackrel{\rm def}{=}1+\alpha\varepsilon\zeta_1(t,\boldsymbol{x})-\varepsilon\zeta_2(t,\boldsymbol{x})\geq h_\star>0,\\ h_2(t,\boldsymbol{x})\stackrel{\rm def}{=}\delta^{-1}+\varepsilon\zeta_2(t,\boldsymbol{x})-\beta b(\boldsymbol{x})\geq h_\star>0,\end{cases}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)}\Big(\big|(\alpha\varepsilon\zeta_1(t,\cdot),\varepsilon\zeta_2(t,\cdot))\big|_{H^{2+s_\star}}+\big|(\varepsilon\nabla\varphi_1(t,\cdot),\varepsilon\nabla\psi_2(t,\cdot))\big|_{H^{1+s_\star}}\Big)+\big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}}\leq M^\star,$$

one has

$$\begin{cases} \alpha\partial_t\zeta_1+\nabla\cdot\big(h_1\nabla\varphi_1\big)+\nabla\cdot\big(h_2\nabla\psi_2\big)=r_1,\\ \partial_t\zeta_2+\nabla\cdot\big(h_2\nabla\psi_2\big)=r_2,\\ \partial_t\varphi_1+\frac{\delta+\gamma}{1-\gamma}\alpha\zeta_1+\frac{\varepsilon}{2}|\nabla\varphi_1|^2=r_3,\\ \partial_t\big(\psi_2-\gamma\psi_1\big)+(\delta+\gamma)\zeta_2+\frac{\varepsilon}{2}\big(|\nabla\psi_2|^2-\gamma|\nabla\psi_1|^2\big)=r_4,\\ \psi_1-\varphi_1=r_5,\end{cases}$$

and one has for almost every $t\in(0,T)$

$$\big|(r_1(t,\cdot),r_2(t,\cdot))\big|_{(H^s)^2}\leq C\,\mu\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}}+\big|\zeta_2(t,\cdot)\big|_{H^{s+4}}+\big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}}+\big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$
$$\big|(r_3(t,\cdot),r_4(t,\cdot))\big|_{(H^{s+1})^2}\leq C\,\mu\varepsilon M\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}}+\big|\zeta_2(t,\cdot)\big|_{H^{s+4}}+\big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}}+\big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$
$$\big|(r_5(t,\cdot))\big|_{H^{s+2}}\leq C\,\mu\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}}+\big|\zeta_2(t,\cdot)\big|_{H^{s+4}}+\big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}}+\big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$

with $M\stackrel{\rm def}{=}\big|\nabla\varphi_1(t,\cdot)\big|_{H^{1+s_\star}}+\big|\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}$.

Moreover, denoting $\boldsymbol{u}_1 = \nabla\varphi_1$ *and* $\boldsymbol{u}_2 = \nabla(\psi_2 - \gamma\psi_1 + \gamma\varphi_1)$*, one has*

$$\begin{cases} \alpha\partial_t\zeta_1 + \nabla\cdot\big(h_1\boldsymbol{u}_1\big) + \nabla\cdot\big(h_2\boldsymbol{u}_2\big) = \tilde{r}_1, \\ \partial_t\zeta_2 + \nabla\cdot\big(h_2\boldsymbol{u}_2\big) = \tilde{r}_2, \\ \partial_t\boldsymbol{u}_1 + \frac{\delta+\gamma}{1-\gamma}\alpha\nabla\zeta_1 + \varepsilon(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1) = \tilde{\boldsymbol{r}}_3, \\ \partial_t\boldsymbol{u}_2 + \gamma\frac{\delta+\gamma}{1-\gamma}\alpha\nabla\zeta_1 + (\delta+\gamma)\nabla\zeta_2 + \varepsilon(\boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2) = \tilde{\boldsymbol{r}}_4 \end{cases}$$

with, for almost every $t\in(0,T)$

$$\big|(\tilde{r}_1(t,\cdot),\tilde{r}_2(t,\cdot))\big|_{(H^s)^2} \leq C\,\mu\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}} + \big|\zeta_2(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$
$$\big|(\tilde{\boldsymbol{r}}_3(t,\cdot),\tilde{\boldsymbol{r}}_4(t,\cdot))\big|_{(H^s)^{2d}} \leq C\,\mu\varepsilon\tilde{M}\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}} + \big|\zeta_2(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big).$$

with $\tilde{M} \stackrel{\rm def}{=} \big|\nabla\varphi_1(t,\cdot)\big|_{H^{2+s_\star}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{2+s_\star}}$.

Proof. The estimate for r_2 is obtained as the corresponding one in Theorem 5.1—that is by Proposition 4.10 with $n=1$ and $k=s+1$—using the identity

$$\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 = \frac{\delta}{\mu}\mathcal{G}^{\mu/\delta^2}[\varepsilon\delta\zeta,\beta\delta b]\psi_2.$$

Concerning r_1, we use $\mathcal{G}_1^{\mu,\delta}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2,\beta b](\varphi_1,\psi_2) = \mathcal{G}_{1,\mathrm{D}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1 + \mathcal{G}_{1,\mathrm{N}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2$ and remark

$$\frac{1}{\mu}\mathcal{G}_{1,\mathrm{D}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1 = \frac{1}{\mu}\mathcal{G}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1$$

so that we have as above[23]

$$\big|\frac{1}{\mu}\mathcal{G}_{1,\mathrm{D}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1 + \nabla\cdot(h_1\nabla\varphi_1)\big|_{H^s} \leq C\,\mu\,\big(\big|\alpha\zeta_1\big|_{H^{s+4}} + \big|\zeta_2\big|_{H^{s+4}} + \big|\nabla\varphi_1\big|_{H^{s+3}}\big)$$

with C as in the statement (we will always use this convention). Now, by using Proposition 4.23 with $n=1$ and $k=s+1$ in conjunction with Proposition 4.10 with $n=0$ and $k\in\{s+3,2+s_\star\}$ (and using the standard trick stemming from interpolation Lemma II.3 and Lemma II.13 and Young's inequality as in Proposition II.7) we deduce

$$\frac{1}{\mu}\big|\mathcal{G}_{1,\mathrm{N}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 - \mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2\big|_{H^s} \leq C\,\mu\,\big(\big|\alpha\zeta_1\big|_{H^{s+4}} + \big|\zeta_2\big|_{H^{s+4}} + \big|\nabla\psi_2\big|_{H^{s+3}}\big).$$

and by Proposition 4.10 with $n=1$ and $k=s+1$

$$\big|\frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta_2,\beta b]\psi_2 + \nabla\cdot\big((\delta^{-1}+\varepsilon\zeta_2-\beta b)\nabla\psi_2\big)\big|_{H^s} \leq C\,\mu\,\big(\big|\zeta_2\big|_{H^{s+4}} + \big|\nabla\psi_2\big|_{H^{s+3}}\big),$$

and the estimate on r_1 follows from the triangular inequality.

The estimate for r_3 follows as above, but using (in addition to product and composition estimates, Proposition II.7 and Proposition II.11) Proposition 4.23 with $n=0$ and $k=s+2$, while the estimate for r_4 uses Proposition 4.23 with $n=0$ and $k=s+3$.

We conclude with the estimate for r_5. Recall

$$\psi_1 = \mathcal{H}_{\mathrm{D}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\varphi_1 + \mathcal{H}_{\mathrm{N}}^{\mu}[\alpha\varepsilon\zeta_1,\varepsilon\zeta_2]\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2$$

[23]We need to take into account that the role of bottom topography is played by the interface deformation, and we have $\zeta_2\in H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)$ while $b\in W^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)$. We could ask more regularity on the data and use the continuous Sobolev embedding $H^{\sigma+s_\star}(\mathbb{R}^d)\subset W^{\sigma}(\mathbb{R}^d)$ for $\sigma\geq 0$, but in fact we claim that the announced result holds without asking more regularity on the data, adapting the proof of Proposition 4.10 by using exclusively product and commutator estimates in Appendix II.2 in place of the ones in Appendix II.3.

By Proposition 4.18 with $n = 1$ and $k = s + 2$ (and the same caveat as in the above footnote) we have

$$\left|\mathcal{H}^{\mu}_{\mathrm{D}}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\varphi_1 - \varphi_1\right|_{H^{s+2}} \le C\,\mu\,\left(\left|\alpha\zeta_1\right|_{H^{s+4}} + \left|\zeta_2\right|_{H^{s+4}} + \left|\nabla\psi\right|_{H^{s+3}}\right).$$

By Proposition 4.25 with $n = 0$ and $k = s+2$ in conjunction with Proposition 4.10 with $n = 0$ and $k \in \{s+3, 2+s_\star\}$, we obtain as above

$$\left|\mathcal{H}^{\mu}_{\mathrm{N}}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\mathcal{G}^{\mu,\delta}_2[\varepsilon\zeta_2, \beta b]\psi_2\right|_{H^{s+2}} \le C\,\mu\,\left(\left|\alpha\zeta_1\right|_{H^{s+4}} + \left|\zeta_2\right|_{H^{s+4}} + \left|\nabla\psi_2\right|_{H^{s+3}}\right).$$

and the first statement follows.

The second statement then follows from the identities

$$\psi_2 = (\psi_2 - \gamma\psi_1) + \gamma\varphi_1 + \gamma(\psi_1 - \varphi_1) \quad \text{ and } \quad \psi_1 = \varphi_1 + (\psi_1 - \varphi_1)$$

from which we infer

$$\tilde r_1 = r_1 + \gamma\nabla\cdot\big(h_2 r_5\big), \qquad \tilde r_2 = r_2 + \gamma\nabla\cdot\big(h_2 r_5\big), \qquad \tilde{\boldsymbol{r}}_3 = \nabla r_3,$$

$$\tilde{\boldsymbol{r}}_4 = \nabla(r_4 + \gamma r_3) + \frac{\varepsilon}{2}\nabla\left((\gamma\nabla r_5)\cdot((\gamma-1)\nabla r_5 + 2\nabla(\psi_2 - \gamma\psi_1 + (\gamma-1)\varphi_1))\right).$$

We can then apply previously obtained estimates and product estimates to control each component as required. □

Remark 6.3. *Notice the size of ψ_1 does not enter the estimates; it can be devised from its definition and Propositions used in the proof. The second statement allows to obtain—withdrawing remainder terms—a closed set of evolution equations for the unknowns $\zeta_1, \zeta_2, \varphi_1, \psi_2 - \gamma\psi_1$. It is possible to simply replace ψ_1 with φ_1 in the fourth equation of the first statement, yet in this case the residual r_4 must be modified to take into account the control of $\partial_t(\varphi_1 - \psi_1)$, which can be devised provided that we have an* a priori *control on the time derivative of the data; see Remark 5.2 and [152] for the rigorous analysis.*

In the same spirit, we could justify the equations eq. (6.2), with $\boldsymbol{u}_\ell = \overline{\boldsymbol{u}}_\ell$ ($\ell \in \{1, 2\}$) the layer-averaged velocities, in which case the first two (mass conservation) equations hold exactly; see again Remark 5.2 and [152].

Finally, if we were to study the limit $\gamma \searrow 0$, it would be necessary to refine the estimate for r_3 and $\tilde{\boldsymbol{r}}_3$ as depending on $(\gamma^{1/2}\varphi_1, \gamma^{1/2}\psi_1)$, which arise naturally in energy functionals of the system.

Let us now turn to the justification of the bilayer hydrostatic equations, eq. (6.3), in the stronger sense including the well-posedness of the Cauchy problem, and the convergence analysis. A difference with respect to the study in Section 5.3 is that the hyperbolic domain of the equations are not simply the non-cavitation assumptions, but rather require the shear velocity are either sufficiently small or sufficiently large (see Section 6.1.2). Since our nonlinear framework will require finite-energy solutions, shear necessarily decay to 0 as $|x| \to \infty$, and hence we use a smallness assumption on the shear velocity as a *sufficient*—since explicit formula for the hyperbolic domain are not available—criteria for hyperbolicity, and eventually well-posedness in Sobolev spaces.

Another important difference is that the limit $\gamma \nearrow 1$ is *singular*, as coefficients of the equations diverge in this limit. In order to secure uniform energy estimates—and eventually time of existence—we must employ techniques akin to singular limits such as the incompressible limit of the Euler equations. The mathematical analysis of such limits have a rich history that we shall not detail; the interested reader can refer to [379, 190, 299, 9] for a comprehensive introduction to the theory. However our system does not readily fits—as far as I know—into a general setting on which ready-to-use theorems can be applied. The result stated below—as well as the ones in Section 6.2.5—was proved in [156, 157].

Theorem 6.4 (Local well-posedness). *Let $d \in \{1,2\}$, $s_\star > d/2$, $s \geq 1+s_\star$, $\delta_\star > 0$, $\delta^\star > 0$, $h_\star > 0$, $M^\star > 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu,\varepsilon,\beta,\alpha,\delta,\gamma) \in \mathfrak{p}_{\frac{\rm SW}{\rm SW}}$, any $b \in W^{s+1,\infty}(\mathbb{R}^d)$, and any $(\zeta_{0,1},\zeta_{0,2},\boldsymbol{u}_{0,1},\boldsymbol{u}_{0,2}) \in H^s(\mathbb{R}^d)^{2+2d}$ satisfying*[24]

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad \begin{cases} h_{1,0}(\boldsymbol{x}) \stackrel{\rm def}{=} 1 + \alpha\varepsilon\zeta_{1,0}(\boldsymbol{x}) - \varepsilon\zeta_{2,0}(\boldsymbol{x}) \geq h_\star > 0, \\ h_{2,0}(\boldsymbol{x}) \stackrel{\rm def}{=} \delta^{-1} + \varepsilon\zeta_{2,0}(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star + \frac{|\boldsymbol{u}_{2,0}-\boldsymbol{u}_{1,0}|^2}{\delta+\gamma}, \end{cases} \tag{6.6}$$

$$M_0 \stackrel{\rm def}{=} \tfrac{1}{\sqrt{1-\gamma}}\big|\alpha\varepsilon\zeta_{0,1}\big|_{H^{1+s_\star}} + \big|\varepsilon\zeta_{0,2}\big|_{H^{1+s_\star}} + \big|\varepsilon\boldsymbol{u}_{0,1}\big|_{H^{1+s_\star}} + \big|\varepsilon\boldsymbol{u}_{0,2}\big|_{H^{1+s_\star}} + \tfrac{1}{\sqrt{1-\gamma}}\big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

there exists a unique $(\zeta_1,\zeta_2,\boldsymbol{u}_1,\boldsymbol{u}_2) \in \mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)^{2+2d}) \cap \mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)^{2+2d})$ classical solution to eq. (6.2), with initial data $(\zeta_1,\zeta_2,\boldsymbol{u}_1,\boldsymbol{u}_2)\,|_{t=0} = (\zeta_{1,0},\zeta_{2,0},\boldsymbol{u}_{1,0},\boldsymbol{u}_{2,0})$; and one has, for any $t \in [0,T/M_0]$, eq. (6.6) holds with constant $h_\star/2$ and

$$\begin{aligned} &\big|\tfrac{\alpha}{\sqrt{1-\gamma}}\zeta_1(t,\cdot)\big|_{H^s} + \big|\zeta_2(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}_1(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}_2(t,\cdot)\big|_{H^s} \\ &\qquad\qquad \leq C \times \Big(\big|\tfrac{\alpha}{\sqrt{1-\gamma}}\zeta_{0,1}\big|_{H^s} + \big|\zeta_{0,2}\big|_{H^s} + \big|\boldsymbol{u}_{0,1}\big|_{H^s} + \big|\boldsymbol{u}_{0,2}\big|_{H^s}\Big). \end{aligned}$$

Proof. A first result follows, as for Theorem 5.3, from the standard theory on symmetrizable quasi-linear systems. Indeed, if we denote $\boldsymbol{U} = (\frac{\alpha}{\sqrt{\varrho}}\zeta_1,\zeta_2,\boldsymbol{u}_1,\boldsymbol{u}_2)^\top = (\frac{\alpha}{\sqrt{\varrho}}\zeta_1,\zeta_2,u_{1,x_1},u_{2,x_2})^\top$ with

$$\varrho \stackrel{\rm def}{=} \frac{1-\gamma}{\delta+\gamma}$$

then eq. (6.2) reads in dimension $d=2$ (modifications are straightforward when $d=1$)

$$\partial_t\boldsymbol{U} + \mathcal{A}_1(\varepsilon\boldsymbol{U},\beta b)\partial_{x_1}\boldsymbol{U} + \mathcal{A}_2(\varepsilon\boldsymbol{U},\beta b)\partial_{x_2}\boldsymbol{U} = \boldsymbol{F}(\varepsilon\boldsymbol{U},\beta b)$$

with $\boldsymbol{F}(\boldsymbol{U},\beta b) \stackrel{\rm def}{=} \big(\frac{\beta\nabla b\cdot\boldsymbol{u}_2}{\sqrt{\varrho}},\beta\nabla b\cdot\boldsymbol{u}_2,0,0,0,0\big)^\top$ and

$$\mathcal{A}_1(\varepsilon\boldsymbol{U},\beta b) \stackrel{\rm def}{=} \begin{pmatrix} \varepsilon u_{1,x_1} & \frac{\varepsilon}{\sqrt{\varrho}}(u_{2,x_1}-u_{1,x_1}) & \frac{1}{\sqrt{\varrho}}h_1 & 0 & \frac{1}{\sqrt{\varrho}}h_2 & 0 \\ 0 & \varepsilon u_{2,x_1} & 0 & 0 & h_2 & 0 \\ \frac{1}{\sqrt{\varrho}} & 0 & \varepsilon u_{1,x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon u_{1,x_1} & 0 & 0 \\ \frac{\gamma}{\sqrt{\varrho}} & \delta+\gamma & 0 & 0 & \varepsilon u_{2,x_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon u_{2,x_2} \end{pmatrix}$$

with $h_1 \stackrel{\rm def}{=} 1+\varepsilon\alpha\zeta_1-\varepsilon\zeta_2$, $h_2 \stackrel{\rm def}{=} \delta^{-1}+\varepsilon\zeta_2-\beta b$ and where (as noticed in [329]) by the rotational invariance property of the system, the symbol of the system reads

$$\mathcal{A}(\varepsilon\boldsymbol{U},\beta b,\boldsymbol{\xi}) \stackrel{\rm def}{=} \mathcal{A}_1(\varepsilon\boldsymbol{U},\beta b)\xi_1 + \mathcal{A}_2(\varepsilon\boldsymbol{U},\beta b)\xi_2 = \mathcal{Q}(\boldsymbol{\xi})^\top\mathcal{A}_1(\varepsilon\mathcal{Q}(\boldsymbol{\xi})\boldsymbol{U},\beta b)\mathcal{Q}(\boldsymbol{\xi})|\boldsymbol{\xi}|$$

with

$$\mathcal{Q}(\xi) \stackrel{\rm def}{=} \frac{1}{|\boldsymbol{\xi}|}\begin{pmatrix} |\boldsymbol{\xi}| & & & & & \\ & |\boldsymbol{\xi}| & & & & \\ & & \xi_1 & \xi_2 & & \\ & & -\xi_2 & \xi_1 & & \\ & & & & \xi_1 & \xi_2 \\ & & & & -\xi_2 & \xi_1 \end{pmatrix}.$$

[24] This criterion is not sharp. In particular it does not correspond to the explicit conditions for hyperbolicity given in Section 6.1.2 as $\gamma \nearrow 1$ or $\gamma \searrow 0$.

This allows to infer (either by direct inspection, or using the fact that the Hessian of the total energy provides a symmetrizer of the unidimensional system) a *symbolic* symmetrizer of the system of the form

$$\mathcal{S}(\varepsilon \boldsymbol{U}, \beta b, \boldsymbol{\xi}) \stackrel{\text{def}}{=} \mathcal{Q}(\boldsymbol{\xi})^\top \mathcal{S}_1(\varepsilon \mathcal{Q}(\boldsymbol{\xi})\boldsymbol{U}, \beta b)\mathcal{Q}(\boldsymbol{\xi})|\boldsymbol{\xi}|$$

with

$$\mathcal{S}_1(\varepsilon \boldsymbol{U}, \beta b) \stackrel{\text{def}}{=} \begin{pmatrix} \gamma & & & & & \\ & \delta+\gamma & & & \varepsilon(u_{2,x_1}-u_{1,x_1}) & \\ & & \gamma h_1 & & & \\ & & & \gamma h_1 & & \\ & \varepsilon(u_{2,x_1}-u_{1,x_1}) & & & h_2 & \\ & & & & & h_2 \end{pmatrix}.$$

Yet, due to the $\varrho^{-1/2}$ prefactor in coefficients of the system, directly applying the standard theory on symmetrizable quasilinear systems (see [310] for instance) yields a disappointing time of existence of the form $\tilde{T} \gtrsim \varrho^{1/2}T/M_0$. Let us explain now explain why the uniform time of existence holds.

The first remark is that when using $\boldsymbol{V} = (\frac{\alpha}{\sqrt{\varrho}}\zeta_1, \zeta_2, \boldsymbol{u}_2 - \gamma\boldsymbol{u}_1, h_1\boldsymbol{u}_1 + h_2\boldsymbol{u}_2)^\top$ as unknowns, then the system has the form

$$\partial_t \boldsymbol{V} + \varrho^{-1/2}\mathsf{L}\boldsymbol{V} + \mathcal{B}_1(\varepsilon\boldsymbol{V}, \beta b)\partial_{x_1}\boldsymbol{V} + \mathcal{B}_2(\varepsilon\boldsymbol{V}, \beta b)\partial_{x_2}\boldsymbol{V} = \boldsymbol{G}(\varepsilon\boldsymbol{V}, \beta b)$$

where L is constant-coefficients and coefficients in $\mathcal{B}_1$ and $\mathcal{B}_2$ are smooth with respect to their variables and *uniformly bounded with respect to* $\varrho \in (0,1]$. The second remark is that the symmetrizer of the system inferred by the one exhibited above and the change of variables, specifically $\mathcal{T}(\varepsilon\boldsymbol{V}, \beta b, \xi) \stackrel{\text{def}}{=} \mathcal{F}[\boldsymbol{V}]^\top \mathcal{S}(F(\varepsilon\boldsymbol{V}), \beta b, \boldsymbol{\xi})\mathcal{F}[\boldsymbol{V}]$ where $\mathcal{F}[\boldsymbol{V}]$ is the Jacobian of the change of variables $\boldsymbol{U} = F(\boldsymbol{V})$, satisfies

$$\mathcal{T}(\varepsilon\boldsymbol{V}, \beta b, \xi) = \mathsf{T} + \mathcal{T}(\varepsilon\Pi\boldsymbol{V}, \beta b, \xi)\Pi + \varrho^{1/2}\tilde{\mathcal{T}}(\varepsilon\boldsymbol{V}, \beta b, \xi)$$

where T is constant-coefficients, coefficients in T, $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are uniformly bounded with respect to $\varrho \in (0,1]$, and Π denotes the orthogonal projection onto $\ker \mathsf{L}$. A close look at the energy estimates in the *a priori* control of the Sobolev norm of solutions shows that it involves only terms which are uniformly bounded with respect to $\varrho \in (0,1]$ since $\varrho^{-1/2}\mathsf{T}(D)\mathsf{L}(D)$ is self-adjoint and commutes with space derivatives and Fourier multipliers, and $\big\|\partial_t\big(\mathcal{T}(\varepsilon\boldsymbol{V}, \beta b, \xi)\big)\big\| \lesssim \big\|\Pi\partial_t\boldsymbol{V}\big\| + \sqrt{\varrho}\big\|\partial_t\boldsymbol{V}\big\| = \mathcal{O}(1)$. From this one deduces the existence and control of the solution as stated in Theorem 6.4. □

Remark 6.5. *As discussed in the proof, the limit of* weak density contrast, *that is $\gamma \nearrow 1$, is a singular limit of the system. Such limit yields formally the Boussinesq approximation; see Section 6.2.1. In Section 6.2.5 we justify rigorously the Boussinesq approximation in conjunction of the rigid-lid assumption, in the limit of weak density contrast.*

Remark 6.6. *The restriction $\big|\alpha\varepsilon\zeta_{0,1}\big|_{H^{1+s_\star}} \leq \sqrt{1-\gamma}M_\star$ appearing in Theorem 6.4 is natural as it balances the contributions of the free-surface deformation and interface deformations in the potential energy of the system; see Section 6.1.1. If we remove the $\sqrt{1-\gamma}$ prefactor, then the time of existence and control of the solution is bounded from below by $\tilde{T} \gtrsim T/\sqrt{1-\gamma}$, which is the timescale of surface gravity waves; compare c_0' defined in Section 3.4 with c_0 used in Section 2.4.*

The restriction $\big|\beta b\big|_{W^{s+1,\infty}} \leq \sqrt{1-\gamma}M^\star$ is less physically motivated, and possibly reveals a shortcoming of the result.

Remark 6.7. *Remark 5.5 (on the maximal time of existence and blowup criterion) and Remark 5.6 (on the continuity of the flow map) apply* mutatis mutandis.

Associated with the proof of the well-posedness by the energy method is a stability estimate, as in Theorem 5.7. We do not write it down and jump to the convergence result which directly follows from the stability, well-posedness and consistency of the bilayer hydrostatic equations, eq. (6.2).

Theorem 6.8 (Convergence)**.** *Let $d \in \{1,2\}$, $s \in \mathbb{N}$, $s_\star > d/2$, $\delta_\star > 0$, $\delta^\star > 0$, $h_\star > 0$, and $M^\star > 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta, \alpha, \delta, \gamma) \in \mathfrak{p}_{\underline{\mathrm{SW}}}$ and any $b \in W^{\max\{s+4,2+s_\star\},\infty}(\mathbb{R}^d)$, $(\zeta_1, \zeta_2, \varphi_1, \psi_2) \in L^\infty(0,T; H^{\max\{s+4,3+s_\star\}}(\mathbb{R}^d)^2 \times \mathring{H}^{\max\{s+4,3+s_\star\}}(\mathbb{R}^d)^2)$ solution to the interfacial waves equations, eq. (3.14) with ψ_1 satisfying the constraint equation $\psi_1 = \mathcal{H}^\mu_{\mathrm{D}}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\varphi_1 + \mathcal{H}^\mu_{\mathrm{N}}[\alpha\varepsilon\zeta_1, \varepsilon\zeta_2]\mathcal{G}^{\mu,\delta}_2[\varepsilon\zeta_2, \beta b]\psi_2$, and such that*

$$\begin{aligned} M \stackrel{\rm def}{=} \sup_{t\in[0,T^\star]} \Big(&\tfrac{1}{\sqrt{1-\gamma}}\big|\alpha\varepsilon\zeta_1(t,\cdot)\big|_{H^{\max\{s+1,3+s_\star\}}} + \big|\varepsilon\zeta_2(t,\cdot)\big|_{H^{\max\{s+1,3+s_\star\}}} \\ &+ \big|\varepsilon\nabla\varphi_1(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}} + \big|\varepsilon\nabla\varphi_1(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}}\Big) + \tfrac{1}{\sqrt{1-\gamma}}\big|\beta b\big|_{W^{\max\{s+4,3+s_\star\},\infty}} \leq M^\star, \end{aligned}$$

and $(\alpha\zeta_1, \zeta_2, \boldsymbol{u}_1, \boldsymbol{u}_2)(t,\cdot)$ satisfies eq. (6.6) uniformly on $[0,T^\star]$, where we denote $\boldsymbol{u}_1 = \nabla\varphi_1$ and $\boldsymbol{u}_2 = \nabla(\psi_2 - \gamma\psi_1 + \gamma\varphi_1)$, there exists a unique $(\zeta_{1,\mathrm{SV}}, \zeta_{2,\mathrm{SV}}, \boldsymbol{u}_{1,\mathrm{SV}}, \boldsymbol{u}_{2,\mathrm{SV}}) \in \mathcal{C}^0([0,T/M]; H^s(\mathbb{R}^d)^{2+2d})$ classical solution to eq. (6.2), with initial data $(\zeta_{1,\mathrm{SV}}, \zeta_{2,\mathrm{SV}}, \boldsymbol{u}_{1,\mathrm{SV}}, \boldsymbol{u}_{2,\mathrm{SV}})\,|_{t=0} = (\zeta_1, \zeta_2, \boldsymbol{u}_1, \boldsymbol{u}_2)\,|_{t=0}$; and one has for any $t \in (0, \min\{T^\star, T/M\}]$,[25]

$$\begin{aligned} &\big|\alpha(\zeta_1 - \zeta_{1,\mathrm{SV}})(t,\cdot)\big|_{H^s} + \big|(\zeta_2 - \zeta_{2,\mathrm{SV}})(t,\cdot)\big|_{H^s} + \big|(\boldsymbol{u}_1 - \boldsymbol{u}_{1,\mathrm{SV}})(t,\cdot)\big|_{H^s} + \big|(\boldsymbol{u}_2 - \boldsymbol{u}_{2,\mathrm{SV}})(t,\cdot)\big|_{H^s} \\ &\quad\leq C\,\mu\, t\,\big(\big\|\alpha\zeta_1\big\|_{L^\infty(0,t;H^{s+4})} + \big\|\zeta_2\big\|_{L^\infty(0,t;H^{s+4})} + \big\|\nabla\varphi_1\big\|_{L^\infty(0,t;H^{s+3})} + \big\|\nabla\psi_2\big\|_{L^\infty(0,t;H^{s+3})}\big)\,. \end{aligned}$$

Remark 6.9. *Contrarily to the homogeneous situation with free surface (see Remark 5.9), the existence and uniqueness of solutions to the interfacial waves equations for a large class of initial data (with finite regularity) does not hold, due to the so-called Kelvin–Helmholtz instabilities. This issue is discussed in more details in Chapter E.*

6.2 The rigid-lid approximation

We now consider the bilayer system under the rigid-lid assumption. Hence the upper boundary of the system is no longer free, but fixed and flat. The full Euler equations are given in eq. (3.5)—or equivalently eq. (3.8)—using physical variables and eq. (3.15)—or equivalently eq. (3.16)—using dimensionless variables. By using Proposition 4.10 (and straightforward rescaling) we infer the approximations

$$\frac{1}{\mu}\mathcal{G}^{\mu,\delta}_2[\varepsilon\zeta_2, \beta b]\psi_2 = -\nabla\cdot(h_2\nabla\psi_2) + \mathcal{O}(\mu), \qquad h_2 \stackrel{\rm def}{=} \delta^{-1} + \varepsilon\zeta_2 - \beta b,$$

$$\frac{1}{\mu}\mathcal{G}^{\mu}_1[\varepsilon\zeta_2]\psi_1 = \nabla\cdot(h_1\nabla\psi_1) + \mathcal{O}(\mu), \qquad h_1 \stackrel{\rm def}{=} 1 - \varepsilon\zeta_2$$

which, when plugged into eq. (3.15) and withdrawing all terms of size $\mathcal{O}(\mu)$, yield the system

$$\begin{cases} \partial_t\zeta_2 = \nabla\cdot(h_1\nabla\psi_1) = -\nabla\cdot(h_2\nabla\psi_2), \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_1|^2 = -\gamma^{-1}p_{\rm int}, \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_2|^2 = -p_{\rm int}. \end{cases} \tag{6.7}$$

We recall that $p_{\rm int}$ (physically representing the pressure at the interface) is not an unknown but a Lagrange multiplier associated with the compatibility condition $\nabla\cdot(h_1\nabla\psi_1) = -\nabla\cdot(h_2\nabla\psi_2)$. We can rewrite the above equivalently as two evolution equations, corresponding to $\mathcal{O}(\mu)$ approximations of the formulation eq. (3.16), with

$$\begin{cases} \partial_t\zeta_2 + \nabla\cdot(h_2\nabla\psi_2) = 0, \\ \partial_t(\psi_2 - \gamma\psi_1) + (\delta+\gamma)\zeta_2 + \frac{\varepsilon}{2}(|\nabla\psi_2|^2 - \gamma|\nabla\psi_1|^2) = 0, \end{cases} \tag{6.8}$$

[25] Recall $\boldsymbol{u}_1 = \nabla\varphi_1$ and $\boldsymbol{u}_2 = \nabla\psi_2 + \gamma\nabla(\varphi_1 - \psi_1)$ and hence, by Theorem 6.2,
$$\big|(\nabla\psi_2 - \boldsymbol{u}_2)(t,\cdot)\big|_{H^{s+1}} \leq C\,\mu\,\big(\big|\alpha\zeta_1(t,\cdot)\big|_{H^{s+4}} + \big|\zeta_2(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\varphi_1(t,\cdot)\big|_{H^{s+3}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big).$$

where ψ_1 is determined (up to a harmless constant) from (ψ_2, ζ_2, b) after solving the elliptic equation

$$\nabla \cdot (h_1 \nabla \psi_1) = -\nabla \cdot (h_2 \nabla \psi_2).$$

Using velocity variables $\boldsymbol{u}_\ell = \nabla \psi_\ell$ or $\boldsymbol{u}_\ell = \overline{\boldsymbol{u}}_\ell$ ($\ell \in \{1,2\}$) as in Section 6.1, we find respectively

$$\begin{cases} \partial_t \zeta_2 = \nabla \cdot (h_1 \boldsymbol{u}_1) = -\nabla \cdot (h_2 \boldsymbol{u}_2), \\ \partial_t \boldsymbol{u}_1 + \frac{\delta+\gamma}{1-\gamma} \nabla \zeta_2 + \varepsilon (\boldsymbol{u}_1 \cdot \nabla) \boldsymbol{u}_1 = -\gamma^{-1} \nabla p_{\rm int}, \\ \partial_t \boldsymbol{u}_2 + \frac{\delta+\gamma}{1-\gamma} \nabla \zeta_2 + \varepsilon (\boldsymbol{u}_2 \cdot \nabla) \boldsymbol{u}_2 = -\nabla p_{\rm int}, \end{cases} \tag{6.9}$$

and, denoting $\boldsymbol{v} \stackrel{\rm def}{=} \boldsymbol{u}_2 - \gamma \boldsymbol{u}_1$,

$$\begin{cases} \partial_t \zeta_2 + \nabla \cdot (h_2 \boldsymbol{u}_2) = 0, \\ \partial_t \boldsymbol{v} + (\delta + \gamma) \nabla \zeta_2 + \frac{\varepsilon}{2} \nabla (|\boldsymbol{u}_2|^2 - \gamma |\boldsymbol{u}_1|^2) = \boldsymbol{0}, \end{cases} \tag{6.10}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are uniquely determined from $\boldsymbol{v}$ by solving

$$\nabla \cdot \big((h_1 + \gamma h_2) \boldsymbol{u}_1 \big) = -\nabla \cdot (h_2 \boldsymbol{v}), \quad \nabla \cdot \big((h_1 + \gamma h_2) \boldsymbol{u}_2 \big) = \nabla \cdot (h_1 \boldsymbol{v})$$

in the space of gradient vector fields. Hence a nonlocal operator is involved, which was put forward in [54] (see also [209, 163]). Only when $d = 1$ can we simply put $u_1 = \frac{-h_2 v}{h_1 + \gamma h_2}$ and $u_2 = \frac{h_1 v}{h_1 + \gamma h_2}$ and infer

$$\begin{cases} \partial_t \zeta_2 + \partial_x \big(\frac{h_1 h_2}{h_1 + \gamma h_2} v \big) = 0, \\ \partial_t v + (\delta + \gamma) \partial_x \zeta_2 + \frac{\varepsilon}{2} \partial_x \big(\frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} v^2 \big) = 0. \end{cases} \tag{6.11}$$

Physical variables Using physical variables (recall Section 3.4) yields

$$\begin{cases} \partial_t h_1 + \nabla \cdot (h_1 \boldsymbol{u}_1) = 0 \\ \partial_t h_2 + \nabla \cdot (h_2 \boldsymbol{u}_2) = 0, \\ \rho_1 \partial_t \boldsymbol{u}_1 + g \rho_1 \nabla (h_2 + b) + \rho_1 (\boldsymbol{u}_1 \cdot \nabla) \boldsymbol{u}_1 = -\nabla p_{\rm int}, \\ \rho_2 \partial_t \boldsymbol{u}_2 + g \rho_2 \nabla (h_2 + b) + \rho_2 (\boldsymbol{u}_2 \cdot \nabla) \boldsymbol{u}_2 = -\nabla p_{\rm int}, \end{cases} \tag{6.12}$$

with $h_1(t, \boldsymbol{x}) \stackrel{\rm def}{=} d_1 - \zeta_2(t, \boldsymbol{x})$ and $h_2(t, \boldsymbol{x}) \stackrel{\rm def}{=} d_2 + \zeta_2(t, \boldsymbol{x}) - b(\boldsymbol{x})$ and

$$\begin{cases} \partial_t h_2 + \nabla \cdot (h_2 \boldsymbol{u}_2) = 0, \\ \rho_2 \partial_t \boldsymbol{v} + g(\rho_2 - \rho_1) \nabla (h_2 + b) + \rho_2 (\boldsymbol{u}_2 \cdot \nabla) \boldsymbol{u}_2 - \rho_1 (\boldsymbol{u}_1 \cdot \nabla) \boldsymbol{u}_1 = \boldsymbol{0}, \end{cases} \tag{6.13}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are uniquely determined from $\boldsymbol{v}$ by solving

$$\nabla \cdot \big((\rho_2 h_1 + \rho_1 h_2) \boldsymbol{u}_1 \big) = -\rho_2 \nabla \cdot (h_2 \boldsymbol{v}), \quad \nabla \cdot \big((\rho_2 h_1 + \rho_1 h_2) \boldsymbol{u}_2 \big) = \rho_2 \nabla \cdot (h_1 \boldsymbol{v})$$

in the space of gradient vector fields.

6.2.1 The Boussinesq approximation

As mentioned in Section 3.1.2, a standard approximation in oceanography is the so-called Boussinesq approximation, which consists in neglecting the density difference in all but buoyancy terms. This yields the following systems corresponding to eq. (6.7)–(6.13).

$$\begin{cases} \partial_t \zeta_2 = \nabla \cdot (h_1 \nabla \psi_1) = -\nabla \cdot (h_2 \nabla \psi_2), \\ \partial_t \psi_1 + \gamma \frac{\delta+\gamma}{1-\gamma} \zeta_2 + \frac{\varepsilon}{2} |\nabla \psi_1|^2 = -p_{\rm int}, \\ \partial_t \psi_2 + \frac{\delta+\gamma}{1-\gamma} \zeta_2 + \frac{\varepsilon}{2} |\nabla \psi_2|^2 = -p_{\rm int} \end{cases} \tag{6.7'}$$

and, equivalently,

$$\begin{cases} \partial_t\zeta_2 + \nabla\cdot(h_2\nabla\psi_2) = 0, \\ \partial_t(\psi_2 - \psi_1) + (\delta+\gamma)\zeta_2 + \frac{\varepsilon}{2}(|\nabla\psi_2|^2 - |\nabla\psi_1|^2) = 0. \end{cases} \tag{6.8'}$$

where ψ_1 is determined (up to a harmless constant) from (ψ_2, ζ_2, b) after solving the elliptic equation

$$\nabla\cdot(h_1\nabla\psi_1) = -\nabla\cdot(h_2\nabla\psi_2).$$

Using velocity variables,

$$\begin{cases} \partial_t\zeta_2 = \nabla\cdot(h_1\boldsymbol{u}_1) = -\nabla\cdot(h_2\boldsymbol{u}_2), \\ \partial_t\boldsymbol{u}_1 + \gamma\frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \varepsilon(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 = -\nabla p_{\rm int}, \\ \partial_t\boldsymbol{u}_2 + \frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \varepsilon(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 = -\nabla p_{\rm int}, \end{cases} \tag{6.9'}$$

and, denoting $\boldsymbol{v} \stackrel{\rm def}{=} \boldsymbol{u}_2 - \boldsymbol{u}_1$,

$$\begin{cases} \partial_t\zeta_2 + \nabla\cdot(h_2\boldsymbol{u}_2) = 0, \\ \partial_t\boldsymbol{v} + (\delta+\gamma)\nabla\zeta_2 + \frac{\varepsilon}{2}\nabla(|\boldsymbol{u}_2|^2 - |\boldsymbol{u}_1|^2) = 0, \end{cases} \tag{6.10'}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are uniquely determined from $\boldsymbol{v}$ by solving

$$\nabla\cdot\big((h_1+h_2)\boldsymbol{u}_1\big) = -\nabla\cdot(h_2\boldsymbol{v}), \quad \nabla\cdot\big((h_1+h_2)\boldsymbol{u}_2\big) = \nabla\cdot(h_1\boldsymbol{v})$$

in the space of gradient vector fields. When $d = 1$, the above reduces to

$$\begin{cases} \partial_t\zeta_2 + \partial_x\big(\frac{h_1h_2}{h_1+h_2}v\big) = 0, \\ \partial_t v + (\delta+\gamma)\partial_x\zeta_2 + \frac{\varepsilon}{2}\partial_x\big(\frac{h_1^2-h_2^2}{(h_1+h_2)^2}v^2\big) = 0. \end{cases} \tag{6.11'}$$

Physical variables Using physical variables, we obtain

$$\begin{cases} \partial_t h_2 = \nabla\cdot(h_1\boldsymbol{u}_1) = -\nabla\cdot(h_2\boldsymbol{u}_2), \\ \partial_t\boldsymbol{u}_1 + g\frac{\rho_1}{\rho_0}\nabla(h_2+b) + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 = -\frac{1}{\rho_0}\nabla p_{\rm int}, \\ \partial_t\boldsymbol{u}_2 + g\big(\frac{\rho_2-\rho_1}{\rho_2} + \frac{\rho_1}{\rho_0}\big)\nabla(h_2+b) + (\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 = -\frac{1}{\rho_0}\nabla p_{\rm int}, \end{cases} \tag{6.12'}$$

where ρ_0 is a reference density (which plays a role uniquely for the definition of $p_{\rm int}$) and

$$\begin{cases} \partial_t h_2 + \nabla\cdot(h_2\boldsymbol{u}_2) = 0, \\ \rho_2\partial_t\boldsymbol{v} + g(\rho_2-\rho_1)\nabla(h_2+b) + \rho_2(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 - \rho_2(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 = \boldsymbol{0}, \end{cases} \tag{6.13'}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are uniquely determined from $\boldsymbol{v}$ by solving

$$\nabla\cdot\big((h_1+h_2)\boldsymbol{u}_1\big) = -\nabla\cdot(h_2\boldsymbol{v}), \quad \nabla\cdot\big((h_1+h_2)\boldsymbol{u}_2\big) = \nabla\cdot(h_1\boldsymbol{v})$$

in the space of gradient vector fields.

Riemann invariants A nice outcome of the Boussinesq approximation for the bilayer hydrostatic system with rigid lid in dimension $d = 1$, eq. (6.11'), is that one can obtain explicit formula for Riemann invariants, similar to the ones presented for the Saint-Venant system in Section iii, as functions of $h = h_2 - h_1$ and $v = u_2 - u_1$. Even more striking is the fact that, after appropriate scaling, the corresponding equations for the Riemann invariants is exactly eq. (iv). As a consequence, there is a one-to-one correspondence between solutions to the Saint-Venant system (in dimension $d = 1$) satisfying the non-cavitation assumption and solutions to eq. (6.11') satisfying

i. the non-cavitation assumptions $h_1 > 0$ and $h_2 > 0$;

ii. the hyperbolicity condition $\varepsilon^2|v|^2 < (h_1 + h_2)(\delta + \gamma)$.

In particular, one deduces that solutions to eq. (6.11') cannot leave the hyperbolicity domain defined by i. and ii. as long as they remain regular. Coupled with the standard blowup criterion for quasilinear systems [132, Theorem 7.8.1], the finite-time breakdown of classical solutions may only occur as a gradient catastrophe, *i.e.* $\big|\partial_x\zeta(t,\cdot)\big|_{L^\infty} + \big|\partial_x v(t,\cdot)\big|_{L^\infty} \to \infty$ as $t \nearrow T_\star < \infty$.

Without the Boussinesq approximation, the system still possesses Riemann invariants (as any hyperbolic system of two conservation laws). Yet explicit formulae appear out of reach, and the hyperbolicity domain is no longer flow-invariant, although it is possible to exhibit a subdomain which enjoys this property; see [317, 103, 57] for more details.

6.2.2 Hamiltonian structure

The bilayer hydrostatic system, eq. (6.8), enjoys a canonical formulation which is directly inherited from the corresponding one of the interfacial waves system described in Section 3.2. Introducing

$$\mathscr{H}_{\rm hydro} \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} (\delta+\gamma)\zeta_2^2 + \gamma h_1|\nabla\psi_1|^2 + h_2|\nabla\psi_2|^2 \,{\rm d}\boldsymbol{x}$$

with $h_1 = 1 - \varepsilon\zeta_2$ and $h_2 = \delta^{-1} + \varepsilon\zeta_2$ and viewing $\mathscr{H}_{\rm hydro}$ as a functional for $(\zeta_2, \xi_2 \stackrel{\rm def}{=} \psi_2 - \gamma\psi_1)$ and using the constraint

$$\nabla\cdot(h_1\nabla\psi_1) = -\nabla\cdot(h_2\nabla\psi_2),$$

one can check that eq. (6.8) reads

$$\partial_t \begin{pmatrix} \zeta_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_{\xi_2}\mathscr{H}_{\rm hydro} \\ -\delta_{\zeta_2}\mathscr{H}_{\rm hydro} \end{pmatrix}.$$

Associated with the Hamiltonian formulation and natural symmetry groups of the system are preserved quantities (invariants). Related to the variation of base level for the velocity potentials are the obvious conservation of the excess of mass:

$$\frac{\rm d}{{\rm d}t}\mathscr{Z} = 0, \qquad\qquad \mathscr{Z} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \zeta_2 \,{\rm d}\boldsymbol{x}.$$

From horizontal translation invariance (in the flat bottom case) we obtain the conservation of the horizontal impulse

$$\frac{\rm d}{{\rm d}t}\mathscr{I} = 0, \qquad\qquad \mathscr{I} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \zeta_2\nabla(\psi_2 - \gamma\psi_1)\,{\rm d}\boldsymbol{x} \qquad\qquad (\text{if } \beta b \equiv 0)$$

and hence the horizontal momentum (using $h_1 + h_2 \equiv 1 + \delta^{-1}$) which under the hydrostatic assumption reads

$$\frac{\rm d}{{\rm d}t}\mathscr{M}_{\rm hydro} = 0, \qquad\qquad \mathscr{M}_{\rm hydro} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \gamma h_1\nabla\psi_1 + h_2\nabla\psi_2 \,{\rm d}\boldsymbol{x} \qquad\qquad (\text{if } \beta b \equiv 0).$$

From time translation invariance we obtain the conservation of the energy

$$\frac{\rm d}{{\rm d}t}\mathscr{H}_{\rm hydro} = 0.$$

6.2.3 Hyperbolicity

In contrast with the free-surface setting, the bilayer hydrostatic system in the rigid-lid case cannot be written as a quasilinear system of first-order evolution equations, as nonlocal effects arise from either the presence of a Lagrange multiplier or the projection onto gradient vector fields, except in the case of horizontal dimension $d = 1$ or very specific cases (for instance, if $\gamma = 0$, we recover the original Saint-Venant system, that is the one-layer free-surface case). Consistently, the hyperbolicity analysis of this section is restricted to eq. (6.11) which can be written (in the flat bottom situation) as

$$\partial_t \boldsymbol{U} + \mathcal{A}(\boldsymbol{U})\partial_x \boldsymbol{U} = 0, \qquad \boldsymbol{U} \stackrel{\text{def}}{=} \begin{pmatrix} \zeta_2 \\ v \end{pmatrix}, \quad \mathcal{A}(\boldsymbol{U}) = \begin{pmatrix} \varepsilon\frac{h_1^2-\gamma h_2^2}{(h_1+\gamma h_2)^2}v & \frac{h_1h_2}{h_1+\gamma h_2} \\ (\delta+\gamma)-\gamma\varepsilon^2\frac{(h_1+h_2)^2}{(h_1+\gamma h_2)^3}v^2 & \varepsilon\frac{h_1^2-\gamma h_2^2}{(h_1+\gamma h_2)^2}v \end{pmatrix}.$$

One sees immediately that $\mathcal{A}(\boldsymbol{U})$ has two distinct real eigenvalues,

$$\lambda_\pm \stackrel{\text{def}}{=} \varepsilon\frac{h_1^2-\gamma h_2^2}{(h_1+\gamma h_2)^2}v \pm \sqrt{\frac{h_1h_2}{h_1+\gamma h_2}\Big((\delta+\gamma)-\gamma\varepsilon^2\frac{(h_1+h_2)^2}{(h_1+\gamma h_2)^3}v^2\Big)}$$

if and only if, in addition to the non-cavitation assumption $h_1 > 0$, $h_2 > 0$, the following hyperbolicity condition holds:

$$(\delta+\gamma) \ - \ \gamma\varepsilon^2\frac{(h_1+h_2)^2}{(h_1+\gamma h_2)^3}v^2 \ > \ 0,$$

which in view of the identities $u_1 = \frac{-h_2 v}{h_1+\gamma h_2}$ and $u_2 = \frac{h_1 v}{h_1+\gamma h_2}$, reads simply

$$(\delta+\gamma) \ - \ \varepsilon^2\frac{\gamma}{h_1+\gamma h_2}(u_2-u_1)^2 \ > \ 0.$$

This should be compared with the analysis of Section 6.1.2. In the rigid-lid framework we obtain that hyperbolicity holds only for sufficiently small shear velocities, with an explicit criterion. Notice everything agrees in the limit of weak density contrast: setting $\gamma = 1$ above—which corresponds to using the Boussinesq approximation—we recover the free-surface criterion in the limit $\gamma \nearrow 1$. This is an agreement to the analysis in Section 6.2.5 below which suggests that the Boussinesq approximation and the rigid-lid assumption are two sides of the same coin.

Incidentally, setting $\gamma = 0$, we obtain as in the one-layer Saint-Venant system that strict hyperbolicity holds as soon as the non-cavitation assumption holds.

6.2.4 Rigorous justification

In this section we rigorously justify the bilayer hydrostatic equations, eq. (6.12), as an asymptotic model for the interfacial waves system with the rigid-lid assumption, eq. (3.8), in direct analogy with the results obtained in the free-surface framework in Section 6.1.3.

In this section we denote, given $\mu^\star > 0$, $\delta_\star > 0$ and $\delta^\star > 0$:

$$\mathfrak{p}_{\underset{\rm SW}{\rm SW}} \stackrel{\text{def}}{=} \big\{(\mu,\varepsilon,\beta,\delta,\gamma) \ : \ \mu\in(0,\mu^\star], \ \varepsilon\in[0,1], \ \beta\in[0,1], \ \delta\in[\delta_\star,\delta^\star], \gamma\in[0,1)\big\}.$$

Theorem 6.10 (Consistency)**.** *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, $s_\star > d/2$, $\mu^\star > 0$, $\delta_\star > 0$, $\delta^\star > 0$, $h_\star > 0$ and $M^\star \geq 0$. There exists $C > 0$ such that for any $(\mu,\varepsilon,\beta,\delta,\gamma) \in \mathfrak{p}_{\underset{\rm SW}{\rm SW}}$, any $b \in W^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)$, any $T > 0$ and any $(\zeta,\psi_1,\psi_2) \in L^\infty(0,T;H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)\times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2)$ solution to the interfacial waves equations, eq. (3.15), satisfying*

$$\forall t\in[0,T], \quad \forall \boldsymbol{x}\in\mathbb{R}^d, \qquad \begin{cases} h_1(t,\boldsymbol{x}) \stackrel{\text{def}}{=} 1-\varepsilon\zeta(t,\boldsymbol{x}) \geq h_\star > 0, \\ h_2(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \delta^{-1}+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x}) \geq h_\star > 0, \end{cases}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi_1(t,\cdot)\big|_{H^{1+s_\star}} + \big|\varepsilon\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}\big) + \big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta_2 - \nabla\cdot(h_1\nabla\psi_1) = r_1, \\ \partial_t\zeta_2 + \nabla\cdot(h_2\nabla\psi_2) = r_2, \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_1|^2 = -\gamma^{-1}p_{\rm int} + r_3, \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\nabla\psi_2|^2 = -p_{\rm int} + r_4, \end{cases}$$

and one has for almost every $t\in(0,T)$

$$\big|(r_1(t,\cdot),r_2(t,\cdot))\big|_{(H^s)^2} \leq C\,\mu\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+3}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$
$$\big|(r_3(t,\cdot),r_4(t,\cdot))\big|_{(H^{s+1})^2} \leq C\,\mu\varepsilon M\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+3}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+3}}\big),$$

with $M \stackrel{\rm def}{=} \big|\nabla\psi_1(t,\cdot)\big|_{H^{1+s_\star}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}$.

Proof. The proof is exactly the same as the one of Theorem 5.1—that is a direct consequence of Proposition 4.10 and estimates in Sobolev spaces—once we remark the identities

$$\frac{1}{\mu}\mathcal{G}_1^\mu[\varepsilon\zeta_2]\psi_1 = -\frac{1}{\mu}\mathcal{G}^\mu[-\varepsilon\zeta_2,0]\psi_1, \qquad \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 = \frac{\delta}{\mu}\mathcal{G}^{\mu/\delta^2}[\varepsilon\delta\zeta,\beta\delta b]\psi_2.$$

Notice that the functional spaces for $p_{\rm int}$ and time derivatives have not been described but are immaterial at this stage. □

Remark 6.11. *Our consistency result is lazy as we allow to satisfy the mass conservation equations up to a small remainder term. It is possible but more complicated to justify the equations eq. (6.9) with* $\boldsymbol{u}_\ell = \overline{\boldsymbol{u}}_\ell$ *(*$\ell\in\{1,2\}$*) the layer-averaged velocities, in which case the mass conservation identities hold exactly; see Remark 5.2 and [163] for the rigorous analysis.*

The well-posedness of the initial-value problem for the bilayer hydrostatic system with the rigid-lid assumption, eq. (6.12), has been studied[26] in [209] and then improved in [63] (it is also a particular case of the result of [159], stated in Theorem 15.9). The work [209] is restricted to irrotational velocity fields—which is not problematic in our context since the equations do stem from potential flows—and the criterion on the initial data to secure the well-posedness of the initial-value problem is sharper in [63]. Both works are restricted to the flat bottom situation, although variable bottoms could be handled by their method (and [159] does allow variable bottoms). We reproduce below their result.

Theorem 6.12 (Local well-posedness)**.** *Let* $d\in\{1,2\}$*,* $s>1+d/2$*,* $\delta_\star>0$*,* $\delta^\star>0$*,* $h_\star>0$ *,* $a_\star>0$ *and* $M^\star>0$*. There exist* $T>0$ *and* $C>0$ *such that for any* $(\mu,\varepsilon,\beta,\alpha,\delta,\gamma)\in\mathfrak{p}_{\frac{\rm SW}{\rm SW}}$ *and any* $(\zeta_{0,2},\boldsymbol{u}_{0,1},\boldsymbol{u}_{0,2})\in H^s(\mathbb{R}^d)^{1+2d}$ *satisfying the non-cavitation assumptions*

$$\forall\boldsymbol{x}\in\mathbb{R}^d,\qquad \begin{cases} h_{1,0}(\boldsymbol{x}) \stackrel{\rm def}{=} 1-\varepsilon\zeta_{2,0}(\boldsymbol{x}) \geq h_\star>0, \\ h_{2,0}(\boldsymbol{x}) \stackrel{\rm def}{=} \delta^{-1}+\varepsilon\zeta_{2,0}(\boldsymbol{x}) \geq h_\star>0, \end{cases} \tag{6.14}$$

the additional hyperbolicity assumption

$$\forall\boldsymbol{x}\in\mathbb{R}^d,\qquad (\delta+\gamma)-\gamma\varepsilon^2\frac{1}{h_{1,0}(\boldsymbol{x})+\gamma h_{2,0}(\boldsymbol{x})}|\boldsymbol{u}_{2,0}(\boldsymbol{x})-\boldsymbol{u}_{1,0}(\boldsymbol{x})|^2 \geq a_\star>0 \tag{6.15}$$

[26] In the dimension $d=2$. When $d=1$ the system reduces to a (symmetrizable) quasilinear hyperbolic system of order one which can be treated by standard methods, as for the Saint–Venant system in Section 5.3.

the rigid-lid constraint

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad \nabla \cdot \big(h_{1,0}\boldsymbol{u}_{1,0} + h_{2,0}\boldsymbol{u}_{2,0}\big)(\boldsymbol{x}) = 0, \tag{6.16}$$

and

$$M_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_{0,2}\big|_{H^s} + \big|\varepsilon\boldsymbol{u}_{0,1}\big|_{H^s} + \big|\varepsilon\boldsymbol{u}_{0,2}\big|_{H^s} \leq M^\star,$$

there exists a unique $(\zeta_2, \boldsymbol{u}_1, \boldsymbol{u}_2) \in \mathcal{C}^0([0, T/M_0]; H^s(\mathbb{R}^d)^{1+2d})$ *(and a corresponding pressure,* $p_{\rm int}$*) solution to eq. (6.9) with* $\beta b \equiv 0$ *and initial data* $(\zeta_2, \boldsymbol{u}_1, \boldsymbol{u}_2)\,|_{t=0} = (\zeta_{2,0}, \boldsymbol{u}_{1,0}, \boldsymbol{u}_{2,0})$*; and one has for any* $t \in [0, T/M_0]$

$$\big|\zeta_2(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}_1(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}_2(t,\cdot)\big|_{H^s} \leq C \times \Big(\big|\zeta_{0,2}\big|_{H^s} + \big|\boldsymbol{u}_{0,1}\big|_{H^s} + \big|\boldsymbol{u}_{0,2}\big|_{H^s}\Big)$$

and eq. (6.14) (resp. (6.15)) holds with constant $h_\star/2$ *(resp.* $a_\star/2$*).*

Remark 6.13. *The result holds—setting* $\gamma = 1$ *in the left-hand side of eq. (6.15)—for the system with Boussinesq approximation, eq. (6.9'), with straightforward adjustments of the proof.*

Neither in [209]—whose proof is based on the energy method—or in [63]—whose proof is based on an abstract result of [217] relying on the semigroup theory of evolution equations—can be found a stability result from which a convergence result analogous to Theorem 6.8 would be deduced. While there is no doubt that such result indeed holds, we conclude here this section, and provide in the next one a justification of eq. (6.12) (or rather eq. (6.12') since the Boussinesq approximation arise) from the free-surface hydrostatic equations, eq. (6.3), in the limit of weak density contrast.

6.2.5 The weak density contrast limit

In this section we motivate the use of the rigid-lid assumption—at least in the hydrostatic setting—by showing that solutions to the system in the physically relevant free-surface framework can indeed be approximated by solutions in rigid-lid framework (and using the Boussinesq approximation), in the weak density contrast situation where $1 - \gamma \ll 1$. Shortly put, we draw the lower arrow in Figure 6.2. The results in this section are taken from [156, 157]. They are analogous to standard results in the theory of singular limits, and in particular weakly incompressible flows. The reader can refer to [379, 190, 299, 9] for an introduction to the standard and not-so-standard results in this field.

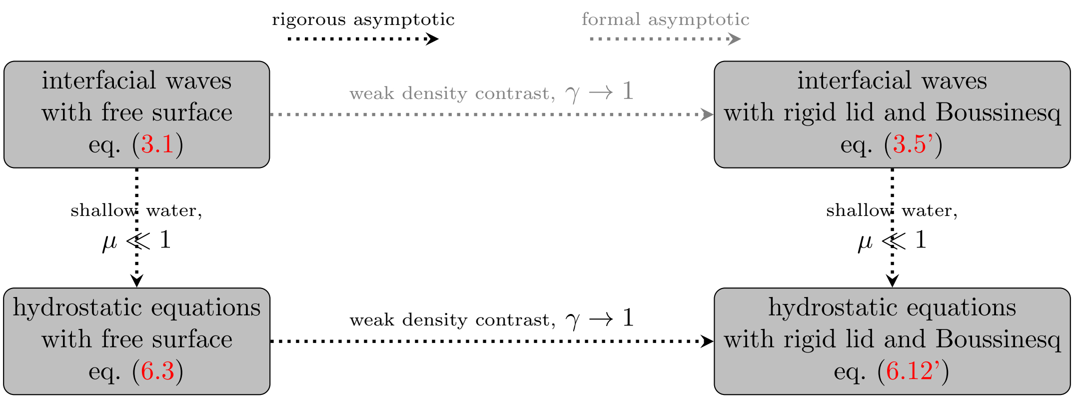


Figure 6.2: The hydrostatic and weak density contrast limits.

Theorem 6.14 (Weak convergence). *Under the assumptions of Theorem 6.4 with* $\alpha=\beta=\sqrt{1-\gamma}$, *denote* $\boldsymbol{U}^\gamma=(\zeta_1^\gamma,\zeta_2^\gamma,\boldsymbol{u}_1^\gamma,\boldsymbol{u}_2^\gamma)\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)^{2+2d})\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)^{2+2d})$ *the classical solutions to eq. (6.2) indexed by* $\gamma\in(0,1)$. *Then for any sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *with* $\gamma_n\nearrow 1$, *the sequence* $\boldsymbol{U}^{\gamma_n}$ *converges weakly (in the sense of distributions and up to a subsequence) towards* $\boldsymbol{U}^{\rm RL}=(\zeta_1^{\rm RL},\zeta_2^{\rm RL},\boldsymbol{u}_1^{\rm RL},\boldsymbol{u}_2^{\rm RL})\in L^\infty(0,T/M_0;H^s(\mathbb{R}^d))^{2+2d}$ *satisfying*

$$\begin{cases}\nabla\cdot(h_1^{\rm RL}\boldsymbol{u}_1^{\rm RL})+\nabla\cdot(h_2^{\rm RL}\boldsymbol{u}_2^{\rm RL})=0,\\ \partial_t\zeta_2^{\rm RL}+\nabla\cdot(h_2^{\rm RL}\boldsymbol{u}_2^{\rm RL})=0,\\ \partial_t\boldsymbol{u}_1^{\rm RL}+\varepsilon(\boldsymbol{u}_1^{\rm RL}\cdot\nabla)\boldsymbol{u}_1^{\rm RL}=-\nabla p^{\rm RL},\\ \partial_t\boldsymbol{u}_2^{\rm RL}+(\delta+1)\nabla\zeta_2^{\rm RL}+\varepsilon(\boldsymbol{u}_2^{\rm RL}\cdot\nabla)\boldsymbol{u}_2^{\rm RL}=-\nabla p^{\rm RL},\end{cases}\tag{6.9”}$$

where $h_1^{\rm RL}=1-\varepsilon\zeta_2^{\rm RL}$ *and* $h_2^{\rm RL}=\delta^{-1}+\varepsilon\zeta_2^{\rm RL}$.

Proof. The strategy is fairly standard; see *e.g.* [373, §2.2]. The existence of a weakly converging subsequence (still denoted $\boldsymbol{U}^{\gamma_n}$), $\boldsymbol{U}^{\gamma_n}\rightharpoonup\boldsymbol{U}^{\rm RL}$, with $\boldsymbol{U}^{\rm RL}\in L^\infty([0,T/M_0];H^s(\mathbb{R}^d))^{2+2d}$ is a consequence of the uniform control on the solutions provided by Theorem 6.4, and Banach—Alaoglu theorem.

Then notice that, by eq. (6.2), $(\partial_t\zeta_2^\gamma,\partial_t(\boldsymbol{u}_2^\gamma-\boldsymbol{u}_1^\gamma))\in L^\infty(0,T/M_0;H^{s-1}(\mathbb{R}^d))^{1+d}$ is uniformly bounded with respect to $\gamma\in(0,1)$. This, using the Aubin–Lions Lemma and the fact that the embedding $H^s(\mathbb{R}^d)\subset H^{s'}(\mathbb{R}^d)$ is locally compact, implies strong convergence for a subsequence $(\zeta_2^{\gamma_n},\boldsymbol{u}_2^{\gamma_n}-\boldsymbol{u}_1^{\gamma_n})\to(\zeta_2^{\rm RL},\boldsymbol{u}_2^{RL}-\boldsymbol{u}_1^{RL})$ in $\mathcal{C}^0([0,T/M_0];H^{s'}_{\rm loc}(\mathbb{R}^d)^{1+d})$ for any $s-1<s'<s$. Furthermore we have, for $\ell\in\{1,2\}$, the same result on $({\rm Id}-\nabla\Delta^{-1}\nabla\cdot)(\boldsymbol{u}_\ell^\gamma)$ and, after some manipulations, we may infer that a subsequence $\boldsymbol{u}_\ell^{\gamma_n}-\nabla\Delta^{-1}\nabla\cdot\frac{h_1^{\gamma_n}\boldsymbol{u}_1^{\gamma_n}+h_2^{\gamma_n}\boldsymbol{u}_2^{\gamma_n}}{1+\delta^{-1}}\to\boldsymbol{u}_\ell^{\rm RL}$ in $\mathcal{C}^0([0,T/M_0];H^{s'}_{\rm loc}(\mathbb{R}^d)^d)$.

By the standard Sobolev embedding, choosing $1+\frac{d}{2}<s'<s$ and using a Cantor diagonal process, convergence in $\mathcal{C}^0([0,T/M_0];H^{s'}_{\rm loc}(\mathbb{R}^d))$ yields pointwise convergence (of a subsequence) for the functions and their derivatives in $[0,T/M_0]\times\mathbb{R}^d$, which allows to pass to the limit on every nonlinear contributions to eq. (6.2) and eventually infer that the limit $\boldsymbol{U}$ does satisfy eq. (6.9”), where $-\nabla p^{\rm RL}$ therein is defined by the left-hand side, which is proved to be potential and equal for both equations. □

Remark 6.15. *Notice that eq. (6.9”) coincides with the rigid-lid system with Boussinesq approximation, (6.9’), up to setting* $\nabla p_{\rm int}+\gamma\frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2^{\rm RL}=\nabla p^{\rm RL}$ *and harmlessly scaling the time variable and velocities to replace* $\delta+1$ *with* $\delta+\gamma$. *In particular, we can use Theorem 6.12 (see also Remark 6.13) and time reversibility of the equations and infer that* $(\zeta_2^{\rm RL},\boldsymbol{u}_1^{\rm RL},\boldsymbol{u}_2^{\rm RL})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d))^{1+2d}$. *In particular, the trace at time* $t=0$ *is well-defined and satisfies* $\nabla\cdot(h_1^{\rm RL}\boldsymbol{u}_1^{\rm RL}+h_2^{\rm RL}\boldsymbol{u}_2^{\rm RL})\,|_{t=0}=0$.

It is important to stress out that the convergence in the above theorem is weak. Such results are not fully satisfactory as they allow undetectable wild behavior, such as asymptotically rapid oscillations (in time) as $\gamma\nearrow 1$. In particular, the initial data for the limit does not, in general, coincide with the original initial data. This phenomenon of boundary layer in time is already apparent in the proof and in the above remark, and made even clearer in the following theorem.

Theorem 6.16 (Strong convergence). *Under the assumptions of Theorem 6.4 with* $\alpha=\beta=\sqrt{1-\gamma}$, *denote* $\boldsymbol{U}^\gamma\stackrel{\rm def}{=}(\zeta_1^\gamma,\zeta_2^\gamma,\boldsymbol{u}_1^\gamma,\boldsymbol{u}_2^\gamma)\stackrel{\rm def}{=}(\zeta_1^\gamma,\widetilde{\boldsymbol{U}}^\gamma)\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)^{2+2d})$ *the solution to eq. (6.2), indexed by* $\gamma\in(0,1)$, *and* $h_1^\gamma\stackrel{\rm def}{=}1-\varepsilon\zeta_2^\gamma$, $h_2^\gamma\stackrel{\rm def}{=}\delta^{-1}+\varepsilon\zeta_2^\gamma$.

i. If one has

$$\zeta_1^\gamma\,|_{t=0}\qquad\textit{and}\qquad\nabla\cdot(h_1^\gamma\boldsymbol{u}_1^\gamma+h_2^\gamma\boldsymbol{u}_2^\gamma)\,|_{t=0}\tag{6.17}$$

then, denoting $\boldsymbol{U}^{\rm RL}\stackrel{\rm def}{=}(\zeta_2^{\rm RL},\boldsymbol{u}_1^{\rm RL},\boldsymbol{u}_2^{\rm RL})\in\mathcal{C}^0([0,T/M_0];H^{s-1}(\mathbb{R}^d)^{1+2d})$ *the solution to eq. (6.9”) with corresponding initial data* $\boldsymbol{U}^{\rm RL}\,|_{t=0}=\widetilde{\boldsymbol{U}}^\gamma\,|_{t=0}$ *and* $p^{\rm RL}\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d))$ *the cor-*

responding pressure, we have for any $t \in [0, T/M_0]$,

$$\frac{1}{\sqrt{1-\gamma}}\big|\alpha\zeta_1^\gamma - \tfrac{1-\gamma}{\delta+\gamma}p^{\rm RL}\big|_{H^{s-1}} + \big|(\widetilde{\boldsymbol U}^\gamma - \boldsymbol U^{\rm RL})(t,\cdot)\big|_{H^{s-1}} \leq C(1+t)\sqrt{1-\gamma}\Big(\big|\zeta_{0,2}\big|_{H^s} + \big|\boldsymbol u_{0,1}\big|_{H^s} + \big|\boldsymbol u_{0,2}\big|_{H^s}\Big),$$

where C *depends uniquely on* $s_\star > d/2$, $s \geq 1 + s_\star$, $\delta_\star > 0$, $\delta^\star > 0$, $h_\star > 0$, $M^\star > 0$.

ii. If $d = 2$ *then, denoting* $\boldsymbol U^{\rm RL} \stackrel{\rm def}{=} (\zeta_2^{\rm RL}, \boldsymbol u_1^{\rm RL}, \boldsymbol u_2^{\rm RL}) \in \mathcal C^0([0,T/M_0]; H^s(\mathbb R^d)^{1+2d})$ *the classical solution to eq. (6.9") with initial data*

$$\zeta_2^{\rm RL}\,|_{t=0} = \zeta_2^\gamma\,|_{t=0}\,, \qquad \boldsymbol u_\ell^{\rm RL}\,|_{t=0} = \boldsymbol u_\ell^\gamma\,|_{t=0} - \nabla\Delta^{-1}\nabla\cdot\boldsymbol w_0, \qquad (\ell \in \{1,2\})$$

where $\boldsymbol w_0 = \frac{(h_1^\gamma \boldsymbol u_1^\gamma + h_2^\gamma \boldsymbol u_2^\gamma)|_{t=0}}{1+\delta^{-1}}$ *and* $p^{\rm RL} \in \mathcal C^0([0,T/M_0]; H^s(\mathbb R^d))$ *the corresponding pressure; and* $(\zeta_1^{\rm ac}, \boldsymbol w^{\rm ac}) \in \mathcal C^0(\mathbb R; H^s(\mathbb R^d))^{1+d}$ *the solution to the acoustic wave equations*

$$\partial_t\zeta_1^{\rm ac} + \nabla\cdot\boldsymbol w^{\rm ac} = 0, \qquad \partial_t\boldsymbol w^{\rm ac} + \frac{1}{1-\gamma}\nabla\zeta^{\rm ac} = 0, \tag{6.18}$$

with initial data $(\zeta_1^{\rm ac}, \boldsymbol w^{\rm ac})\,|_{t=0} = (\zeta_1^\gamma\,|_{t=0}\,, \nabla\Delta^{-1}\nabla\cdot\boldsymbol w_0)$. *Then for any* $0 \leq s' < s$, *one has*

$$\sup_{t\in[0,T/M_0]}\Big(\frac{1}{\sqrt{1-\gamma}}\big|(\zeta_1^\gamma - \tfrac{1-\gamma}{\delta+\gamma}p^{\rm RL} - \zeta_1^{\rm ac})(t,\cdot)\big|_{H^{s'}} + \big|(\zeta_2^\gamma - \zeta_2^{\rm RL})(t,\cdot)\big|_{H^{s'}} + \big|(\boldsymbol u_1^\gamma - \boldsymbol u_1^{\rm RL} - \boldsymbol w^{\rm ac})(t,\cdot)\big|_{H^{s'}} + \big|(\boldsymbol u_2^\gamma - \boldsymbol u_2^{\rm RL} - \boldsymbol w^{\rm ac})(t,\cdot)\big|_{H^{s'}}\Big) \to 0 \quad (as\ \gamma \nearrow 1).$$

Proof. First we remark that for T' sufficiently small, $\boldsymbol U^{\rm RL} \in \mathcal C^1([0,T']; H^{s-1}(\mathbb R^d)^{1+2d})$ is well-defined by Theorem 6.12 (see also Remark 6.13), and the fact that $p^{\rm RL} \in \mathcal C^1([0,T']; H^{s-1}(\mathbb R^d))$ is defined from $(\zeta_2^{\rm RL}, \boldsymbol u_1^{\rm RL}, \boldsymbol u_2^{\rm RL})$ through the Poisson equation stemming from the rigid-lid constraint, $\nabla\cdot(h_1^{\rm RL}\boldsymbol u_1^{\rm RL} + h_2^{\rm RL}\boldsymbol u_2^{\rm RL}) = 0$, that is[27]

$$\nabla\cdot\big((h_1^{\rm RL} + h_2^{\rm RL})\nabla p^{\rm RL}\big) = -\nabla\cdot\big((\delta+1)h_2^{\rm RL}\nabla\zeta_2^{\rm RL} + \varepsilon\nabla\cdot(h_1^{\rm RL}\boldsymbol u_1^{\rm RL}\otimes\boldsymbol u_1^{\rm RL} + h_2^{\rm RL}\boldsymbol u_2^{\rm RL}\otimes\boldsymbol u_2^{\rm RL}\big). \tag{6.19}$$

Because—see the proof of Theorem 6.14—there exists a sequence $\{\boldsymbol U^{\gamma_n}\}_{n\in\mathbb N}$ such that $\widetilde{\boldsymbol U}^{\gamma_n}$ converges (strongly) in $\mathcal C^0([0,T/M_0]; H^{s'}_{\rm loc}(\mathbb R^d)^{1+d})$ to $\boldsymbol U^{\rm RL}$ a solution to eq. (6.9") satisfying $\widetilde{\boldsymbol U}^\gamma = \boldsymbol U^{\rm RL}\,|_{t=0}$, we find that $(\zeta_2^{\rm RL}, \boldsymbol u_1^{\rm RL}, \boldsymbol u_2^{\rm RL})$ can be extended at least to the time $T' = T/M_0$. It is then an easy task to show that $\widetilde{\boldsymbol U}^{\rm RL} \stackrel{\rm def}{=} (\frac{1-\gamma}{\delta+\gamma}p^{\rm RL}, \boldsymbol U^{\rm RL})$ satisfies eq. (6.2) up to an additive remainder term scaling as $\sqrt{1-\gamma}$ as $\gamma \nearrow 1$ (in the anisotropic energy functional, that is applying a $\frac{1}{\sqrt{1-\gamma}}$ prefactor to the first component). Then applying the uniform energy estimates from the proof of Theorem 6.4 to the equation satisfied by the difference $\boldsymbol U^\gamma - \widetilde{\boldsymbol U}^{\rm RL}$, the claimed estimate follows.

In the second part of the statement, the construction and estimates on the rigid-lid contribution, still denoted $\boldsymbol U^{\rm RL}$, is obtained exactly as above since $\boldsymbol U^{\rm RL}$ satisfies the desired initial conditions. The solutions to the acoustic wave equations are well-known; see [34] for instance. We have in particular $(\zeta_1^{\rm ac}, \boldsymbol w^{\rm ac}) \in \mathcal C^0(\mathbb R; H^s(\mathbb R^d))^{1+d}$, $({\rm Id} - \nabla\Delta^{-1}\nabla\cdot)\boldsymbol w^{\rm ac} = ({\rm Id} - \nabla\Delta^{-1}\nabla\cdot)\boldsymbol w_0 = \boldsymbol 0$,

$$\frac{1}{1-\gamma}\big|\zeta_1^{\rm ac}(t,\cdot)\big|_{H^s}^2 + (1+\delta^{-1})\big|\boldsymbol w(t,\cdot)\big|_{H^s}^2 = \frac{1}{1-\gamma}\big|\zeta_1^\gamma\,|_{t=0}\big|_{H^s}^2 + (1+\delta^{-1})\big|\boldsymbol w_0\big|_{H^s}^2$$

[27] Notice that the fact that we can set $p^{\rm RL}$ with finite energy from $\nabla p^{\rm RL} \in \mathcal C^0([0,T']; H^{s-1}(\mathbb R^d))$ stems from the Boussinesq approximation; since in eq. (6.19) $h_1^{\rm RL} + h_2^{\rm RL}$ is constant and the right-hand side is the action of differential operators with symbol homogeneous of degree 2 acting on functions in $L^2(\mathbb R^d)$. It is shown in [70, 71] that, without the Boussinesq approximation, initial conditions do generate horizontal pressure imbalances which in turn yield an apparently paradoxical evolution in time of the total horizontal momentum.

for any $t \in \mathbb{R}$; and, since $d = 2$ and using time rescaling, the Strichartz estimates

$$\big\|\tfrac{1}{\sqrt{1-\gamma}}\zeta_1^{\rm ac}\big\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} + \big\|\sqrt{1+\delta^{-1}}\boldsymbol{w}^{\rm ac}\big\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \Big(\sqrt{\tfrac{1-\gamma}{1+\delta^{-1}}}\Big)^{1/p}\Big(\big|\tfrac{1}{\sqrt{1-\gamma}}\zeta_1^\gamma\,|_{t=0}\big|_{H^\sigma}^2 + \big|\sqrt{1+\delta^{-1}}\boldsymbol{w}_0\big|_{H^\sigma}\Big)$$

where p, q satisfy $2 < p, q < \infty$ and $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \sigma$ and $\frac{2}{p} + \frac{d-1}{q} = \frac{d-1}{2}$; say $p = q = 6$ and $\sigma = \frac{1}{2}$. These estimates allow to control quadratic contributions as follows: let $f \in L^\infty(0, T; H^s(\mathbb{R}^d))$ with $s > \frac{1}{6}$ and $g \in L^6(0, T; L^6(\mathbb{R}^d))$. Then by Hölder inequality (Lemma II.1) and continuous Sobolev embedding (Lemma II.4), we have

$$\big\|fg\big\|_{L^1(0,T;L^2(\mathbb{R}^d))} \lesssim \big\|f\big\|_{L^6(0,T;L^6(\mathbb{R}^d))}\big\|g\big\|_{L^6(0,T;L^6(\mathbb{R}^d))} \lesssim T^{1/6}\big\|f\big\|_{L^\infty(0,T;H^s(\mathbb{R}^d))}\big\|g\big\|_{L^6(0,T;L^6(\mathbb{R}^d))}.$$

Such estimates allow to infer that

$$\boldsymbol{U}^{\rm app} \stackrel{\rm def}{=} \big(\tfrac{1-\gamma}{\delta+\gamma}p^{\rm RL} + \zeta_1^{\rm ac}, \zeta_2^{\rm RL}, \boldsymbol{u}_1^{\rm RL} + \boldsymbol{w}^{\rm ac}, \boldsymbol{u}_2^{\rm RL} + \boldsymbol{w}^{\rm ac}\big)$$

satisfies the free-surface equations, eq. (6.2), up to an additional term $\boldsymbol{R}$ such that

$$\big\|\boldsymbol{R}\big\|_{L^1(0,T;L^2(\mathbb{R}^d))} \to 0 \qquad (\text{as } \gamma \nearrow 1).$$

(in the anisotropic energy functional, that is applying a $\frac{1}{\sqrt{1-\gamma}}$ prefactor to the first component) with plenty of room to spare. Using the uniform energy estimates from the proof of Theorem 6.4 to the equation satisfied by the difference $\boldsymbol{U}^\gamma - \boldsymbol{U}^{\rm app}$, we obtain the claimed limit when $s' = 0$. The case $0 \le s' < s$ follows from the logarithmic convexity of Sobolev norms; Lemma II.3. □

Remark 6.17. *In the first scenario of Theorem 6.16, eq. (6.17) is an assumption of* well-prepared initial data. *It allows to suppress (at first order) the contribution of the "fast mode"*[28] *appearing in the second scenario, and which is responsible for the aforementioned boundary layer in time. It could easily be relaxed to* $\big|\alpha\zeta_1^\gamma\,|_{t=0}\big|_{H^s} = \mathcal{O}(1-\gamma)$ *and* $\big|\nabla\cdot(h_1^\gamma\boldsymbol{u}_1^\gamma + h_2^\gamma\boldsymbol{u}_2^\gamma)\,|_{t=0}\big|_{H^s} = \mathcal{O}(\sqrt{1-\gamma})$.

In the second scenario we construct an approximate solution as the superposition of the "fast mode" and the "slow mode".[29] *The condition $d = 2$ is essential to ensure, thanks to the (asymptotically rapid) decreasing of the amplitude of the fast mode which follows from dispersive estimates, that the interaction between the fast and slow modes, as well as the nonlinear self-interactions of the fast mode, vanish in the limit $\gamma \nearrow 1$. One could provide an explicit rate of convergence, yet much weaker than the one for well-prepared initial data.*

When $d = 2$, an assumption of spatial localization can replace the dispersive estimates to prove that the fast and slow modes do not interact; see [156, Proposition 4.4] and [157, Remark 1.6].

6.3 The multilayer case

The bilayer hydrostatic systems presented in Section 6.1 and Section 6.2 can naturally be extended to a framework with an arbitrary number (denoted N) of homogeneous layers; see Figure 6.3. Let us write the ***multilayer hydrostatic equations*** with a free surface, using physical variables (see, for instance, [35]).

$$\begin{cases} \partial_t h_\ell + \nabla\cdot\big(h_\ell\boldsymbol{u}_\ell\big) = 0 & (\ell \in \{1,\dots,N\}), \\ \rho_\ell\partial_t\boldsymbol{u}_\ell + g\displaystyle\sum_{1\le\ell'<\ell}\rho_{\ell'}\nabla h_{\ell'} + g\rho_\ell\nabla\zeta_\ell + \rho_\ell(\boldsymbol{u}_\ell\cdot\nabla)\boldsymbol{u}_\ell = \boldsymbol{0} & (\ell \in \{1,\dots,N\}), \end{cases} \tag{6.20}$$

[28] the "acoustic component" in the context of weakly compressible flows, the "barotropic mode" in our context.
[29] the "incompressible component" in the context of weakly compressible flows, the "baroclinic mode" in our context.

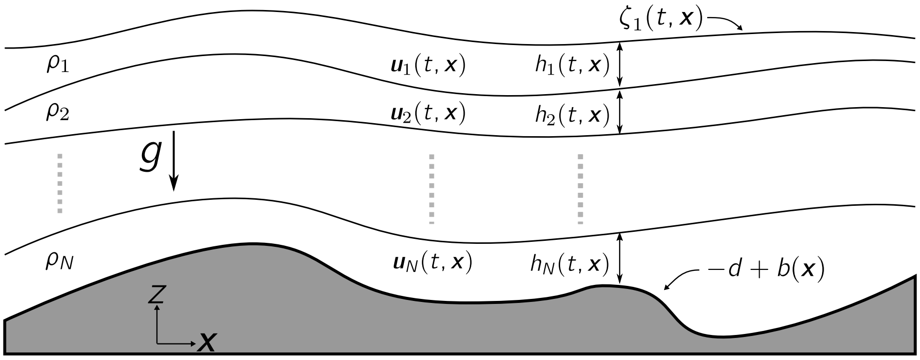


Figure 6.3: Sketch of the domain and notations in the multilayer framework.

where ζ_ℓ denotes the deformation of the ℓ^{th} interface (with ζ_1 the free surface and, by convention, $\zeta_{N+1} = b$ the fixed bottom), $h \stackrel{\text{def}}{=} d_\ell + \zeta_\ell - \zeta_{\ell+1}$ the depth of the ℓ^{th} layer and $\boldsymbol{u}_\ell$ a horizontal velocity (say, layer-averaged) associated with the layer ℓ^{th} layer. Notice $\nabla\zeta_\ell = \sum_{\ell\leq\ell'\leq N} \nabla h_\ell + \nabla b$ so the above system is a closed set of equations. Of course, ρ_ℓ is the constant density of the ℓ^{th} layer and d_ℓ its depth at rest, while as always g represents the constant vertical gravity acceleration.

Rigid lid In the rigid-lid framework these equations reduce to

$$\begin{cases} \partial_t h_\ell + \nabla\cdot\big(h_\ell \boldsymbol{u}_\ell\big) = 0 & (\ell \in \{1,\dots,N\}),\\ \rho_\ell \partial_t \boldsymbol{u}_\ell + g \displaystyle\sum_{1\leq\ell'<\ell} \rho_{\ell'}\nabla h_{\ell'} + g\rho_\ell\nabla\zeta_\ell + \rho_\ell(\boldsymbol{u}_\ell\cdot\nabla)\boldsymbol{u}_\ell = -\nabla p_{\text{lid}} & (\ell \in \{1,\dots,N\}),\end{cases} \tag{6.21}$$

where we set $\zeta_1 \stackrel{\text{def}}{=} \sum_{\ell=1}^N (h_\ell - d_\ell) + b = 0$, and p_{lid} represents the pressure at the rigid lid and is the Lagrange multiplier associated with the constraint

$$\sum_{\ell=1}^N \nabla\cdot\big(h_\ell \boldsymbol{u}_\ell\big) = 0.$$

Applying the Boussinesq approximation yields

$$\begin{cases} \partial_t h_\ell + \nabla\cdot\big(h_\ell \boldsymbol{u}_\ell\big) = 0 & (\ell \in \{1,\dots,N\}),\\ \rho_0 \partial_t \boldsymbol{u}_\ell + g \displaystyle\sum_{1\leq\ell'<\ell} \rho_{\ell'}\nabla h_{\ell'} + g\rho_\ell\nabla\zeta_\ell + \rho_0(\boldsymbol{u}_\ell\cdot\nabla)\boldsymbol{u}_\ell = -\nabla p_{\text{lid}} & (\ell \in \{1,\dots,N\}),\end{cases} \tag{6.21'}$$

where ρ_0 is a reference density, say $\rho_0 = \rho_1$.

Remark 6.18. *The convention for Boussinesq approximation here is different from the one used in Section 6.2.1, and in particular eq.* (6.12'), *unless we choose above $\rho_0 = \rho_2$ as a reference density (and, of course, $N = 2$). The convention in the bilayer setting is convenient because the resulting system does not depend on the choice of the reference density, except for the harmless physical interpretation of the Lagrange multiplier. However, when $N \geq 3$ layers are involved, our choice of Boussinesq approximation provides stronger simplifications (see* e.g. *[102]). Moreover, in the limit of continuous stratification presented in Section 6.3.3, we recover the hydrostatic equations with rigid-lid and Boussinesq approximation in Eulerian coordinates, that is eq.* (7.4)-(7.4b') *(following the computations in Section 7.2).*

6.3.1 Hyperbolicity

Systems (6.20) as well as eq. (6.21)—and eq. (6.21') if $d = 1$ are first-order quasilinear system. Yet starting with three layers, it becomes impossible to gather precise information on the domain of hyperbolicity of these systems, except in very specific situations (see [103, 188, 391]). Thanks to a perturbative analysis, we are however able to state the important fact that, for any number of layer, N, eq. (6.20) is hyperbolic provided that

- the non-cavitation assumption holds: $\inf_{h_\ell} > 0$ for any $\ell \in \{1, \dots, N\}$,
- the density stratification is stable: $\rho_1 < \rho_2 < \dots < \rho_N$,
- the shear velocities, $|\boldsymbol{u}_{\ell+1} - \boldsymbol{u}_\ell|$ for $\ell \in \{1, \dots, N-1\}$, are sufficiently small.

This follows from a perturbative analysis. Indeed, if we set $\boldsymbol{u}_1 = \dots = \boldsymbol{u}_N \stackrel{\text{def}}{=} \boldsymbol{u}$ and restrict to the dimension $d = 1$ and the flat bottom situation, then writing eq. (6.20) in compact form as

$$\partial_t \boldsymbol{U} + \mathcal{A}(\boldsymbol{U})\partial_x \boldsymbol{U} = 0, \qquad \boldsymbol{U} \stackrel{\text{def}}{=} (h_1, h_2 \dots, h_N, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_N)^\top$$

then the $(2N) \times (2N)$ square matrix $\mathcal{A}(\boldsymbol{U})$ has $2N$ distinct real eigenvalues:

$$\lambda_{\pm\ell} = \boldsymbol{u} \pm \mu_\ell \quad (\ell \in \{1, \dots, N\}), \qquad 0 < \mu_N < \dots < \mu_2 < \mu_1.$$

This comes from the fact that the eigenvalue problem for $\mathcal{A}$ can be rewritten as the eigenvalue problem for a real, symmetric tridiagonal matrix with non-zero (positive) off-diagonals entries, that is a *discrete Sturm–Liouville problem*; see *e.g.* [48, 35, 154]. The hyperbolicity of the full problem, that is with bottom topography, dimension $d = 2$ and including (sufficiently small) shear velocities follows from the fact that bottom contributions are order-zero, the rotational invariance property described in the proof of Theorem 6.4, and a perturbative analysis, respectively; see [157, Appendix B].

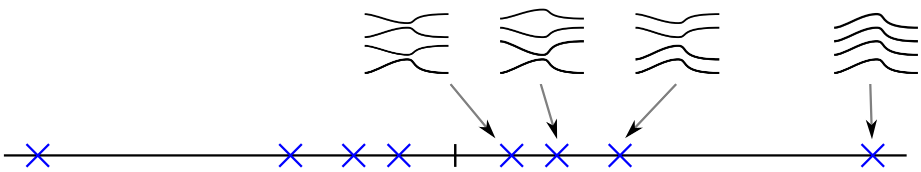


Figure 6.4: Sketch of eigenvalues and eigenvectors with $N = 4$.

A nice outcome of the Sturm–Liouville theory [388] is the fact that the eigenvalues are ordered following the number of sign changes of the eigenvectors; see Figure 6.4. This allows to characterize the modes of propagation in a multilayer system. The fastest mode is the *barotropic mode*, and surface and interface deformations are either all elevation, or all depression. Then polarity changes increase by one in *baroclinic* modes with decreasing velocities. See for instance [420] for observations in the South China Sea, and references therein for theoretical investigations, laboratory experiments, numerical analyses and field observations.

6.3.2 The weak density contrast limit

As in the bilayer case, we observe a strong separation between the barotropic mode and the baroclinic modes in the limit of weak density contrast, that is

$$\frac{\rho_{\ell+1} - \rho_\ell}{\rho_N} \ll 1, \qquad (\ell \in \{1, 2, \dots, N-1\}).$$

In fact the analysis presented in Section 6.2.5 can be extended *mutatis mutandis*, and in [157] it is shown that we can rigorously approximate solutions to the free-surface system, eq. (6.20) with solutions to the rigid-lid system with Boussinesq approximation, eq. (6.21'), and superposing the solution of an acoustic wave equation, eq. (6.18), in the foregoing limit and for sufficiently small initial surface perturbations. The main—and important—difference with the bilayer framework is that the condition of "sufficiently small shear velocities", that is eq. (6.6) when $N = 2$, is no longer explicit. In particular, there is no reason to believe that it should be uniform with respect to the number of layers, N; see discussions in [157] and in [328]. Indeed, as N grows, the eigenvalues of the problem without shear velocities described above accumulate around the given value of the horizontal velocity, and the aforementioned perturbative analysis depends extensively on (the inverse of) the shrinking spectral gaps.

6.3.3 The continuous stratification limit

The main interest in considering multilayer systems is that we hope it describes accurately, if the number of layers N is large, the situation of continuous stratification.[30] Simply put, we approximate the continuous density by a step function (in addition to the hydrostatic assumption, here). Despite the fact that we have no explicit or uniform-in-N criterion for hyperbolicity (or "nonlinear stability", or well-posedness) for the multilayer hydrostatic systems (see [1, 215, 365, 102] and the discussion above), we shall formally investigate this *continuous stratification limit* in this section. To this aim we consider eq. (6.20) with N arbitrary and $d_\ell^{(N)}$, $\rho_\ell^{(N)}$ ($\ell \in \{1,\dots,N\}$) satisfying

$$d_\ell^{(N)}/N = \underline{h}(\ell/N), \qquad \rho_\ell^{(N)} = \underline{\rho}(\ell/N), \qquad \ell \in \{1,\dots,N\},$$

with $\underline{h}, \underline{\rho} : [0,1] \to \mathbb{R}$ two continuous functions. Denoting (assuming the limits exist)

$$\eta(t,\boldsymbol{x},r) = \lim_{\ell=\lfloor rN \rfloor} \frac{\zeta_{\ell+1}^{(N)}(t,\boldsymbol{x}) - \zeta_\ell^{(N)}(t,\boldsymbol{x})}{N}, \qquad \boldsymbol{u}(t,\boldsymbol{x},r) = \lim_{\ell=\lfloor rN \rfloor} \boldsymbol{u}_\ell^{(N)}(t,\boldsymbol{x}), \qquad r \in (0,1)$$

we find, formally passing to the limit in eq. (6.20),

$$\begin{cases} \partial_t \eta + \nabla_{\boldsymbol{x}} \cdot \big((\underline{h} + \eta) \boldsymbol{u} \big) = 0, \\ \underline{\rho} \partial_t \boldsymbol{u} + g \displaystyle\int_0^r \underline{\rho} \nabla_{\boldsymbol{x}} \eta \, \mathrm{d}r' + g\underline{\rho} \Big(\nabla_{\boldsymbol{x}} b + \int_r^1 \nabla_{\boldsymbol{x}} \eta \, \mathrm{d}r' \Big) + \underline{\rho} (\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}}) \boldsymbol{u} = \boldsymbol{0}. \end{cases} \tag{6.22}$$

Here the variable $r \in (0,1)$ is the isopycnal coordinate. At any time t, we can identify a location in physical space $(\boldsymbol{x}, z) \in \Omega^t$ to $(\boldsymbol{x}, r) \in \mathbb{R}^d \times (0,1)$ through the identity

$$\rho(t,\boldsymbol{x},z) = \underline{\rho}(r),$$

provided that $\underline{\rho}$ (which we can choose without loss of generality) is strictly monotonic and continuous, and $\rho(t,\boldsymbol{x},\cdot) : (-d + b(\boldsymbol{x}), \zeta(t,\boldsymbol{x})) \to (\underline{\rho}(0), \underline{\rho}(1))$ is one-to-one and onto. In the following, we always consider $\underline{\rho}$ increasing and $\rho(t,\boldsymbol{x},\cdot)$ decreasing. Then $\boldsymbol{u}(t,\boldsymbol{x},r)$ represents the horizontal velocity at $(t,\boldsymbol{x},z)$ and $\int_0^r \underline{h}(r') + \eta(t,\boldsymbol{x},r') \, \mathrm{d}r' = \zeta(t,\boldsymbol{x}) - z$ (from which we infer the necessary conditions $\int_0^r \underline{h}(r') \, \mathrm{d}r' = d$ and $\underline{h} + \eta > 0$). We give more precisions on the relationship between Eulerian and isopycnal coordinates, starting directly from the continuous framework, in Section 7.2.

[30] The only rigorous result on this limit to my knowledge is [89] for traveling waves in the non-hydrostatic framework.

Rigid lid The rigid-lid counterpart can of eq. (6.22) can be derived in the same way from eq. (6.21), and we infer

$$\begin{cases} \partial_t \eta + \nabla_x \cdot \big((\underline{h} + \eta) \boldsymbol{u} \big) = 0, \\ b + \int_0^1 \eta \, \mathrm{d}r = d, \\ \underline{\rho} \partial_t \boldsymbol{u} + g \displaystyle\int_0^r \underline{\rho} \nabla_x \eta \, \mathrm{d}r' + g \underline{\rho} \Big(\nabla_x b + \int_r^1 \nabla_x \eta \, \mathrm{d}r' \Big) + \underline{\rho} (\boldsymbol{u} \cdot \nabla_x) \boldsymbol{u} = - \nabla_x p_{\mathrm{lid}}, \end{cases} \tag{6.23}$$

where p_{lid} represents the pressure at the rigid lid and is the Lagrange multiplier associated with the constraint of the second equation. Using the Boussinesq approximation yields

$$\begin{cases} \partial_t \eta + \nabla_x \cdot \big((\underline{h} + \eta) \boldsymbol{u} \big) = 0, \\ b + \int_0^1 \eta \, \mathrm{d}r = d, \\ \underline{\rho} \partial_t \boldsymbol{u} + g \displaystyle\int_0^r \underline{\rho} \nabla_x \eta \, \mathrm{d}r' + g \underline{\rho} \Big(\nabla_x b + \int_r^1 \nabla_x \eta \, \mathrm{d}r' \Big) + \underline{\rho} (\boldsymbol{u} \cdot \nabla_x) \boldsymbol{u} = - \frac{\underline{\rho}}{\rho_0} \nabla_x p_{\mathrm{lid}}, \end{cases} \tag{6.23'}$$

where ρ_0 is a reference density, say $\rho_0 = \rho_1$.

Hyperbolicity It is informative to try to extend the discussion on the hyperbolicity and mode decomposition of the bilayer system (Section 6.3.1) to the continuous density framework. For simplicity, let us restrict to the case of dimension $d = 1$ and flat bottom. Then eq. (6.22) can be written as

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + \mathcal{A}(\eta, u) \partial_x \begin{pmatrix} \eta \\ u \end{pmatrix} = \mathbf{0}, \qquad \mathcal{A}(\eta, u) \stackrel{\mathrm{def}}{=} \begin{pmatrix} u & \underline{h} + \eta \\ g\mathcal{M} & u \end{pmatrix}$$

with $\mathcal{M} : L^1(0,1) \to \mathcal{C}^1(0,1)$ is the operator defined by

$$\begin{aligned} (\mathcal{M}\zeta)(r) &\stackrel{\mathrm{def}}{=} \int_0^r \underline{\rho}(r') \zeta(r') \, \mathrm{d}r' + \underline{\rho}(r) \int_r^1 \zeta(r') \, \mathrm{d}r' \\ &= \int_0^r \underline{\rho}'(r') \int_r^1 \zeta(r'') \, \mathrm{d}r'' \, \mathrm{d}r' + \underline{\rho}(0) \int_0^1 \zeta(r') \, \mathrm{d}r', \end{aligned}$$

and is related to the *Montgomery potential*; see Section 7.2. If we neglect shear velocities and consider $u \equiv u_\star$, then

$$\mathcal{A}(\eta, u_\star) \begin{pmatrix} s \\ f \end{pmatrix} = \lambda \begin{pmatrix} s \\ f \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} (\lambda - u_\star)^2 s = g(\underline{h} + \eta) \mathcal{M} s, \\ (\underline{h} + \eta) f = (\lambda - u_\star) s. \end{cases}$$

The first line of the right-hand side is a Sturm–Liouville problem [395]. Indeed, defining

$$S(r) \stackrel{\mathrm{def}}{=} \int_r^1 s(r') \, \mathrm{d}r',$$

we obtain

$$(\lambda - u_\star)^2 s = g(\underline{h} + \eta) \mathcal{M} s \quad \Longleftrightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}r} \Big(\frac{1}{g(\underline{h} + \eta)} \frac{\mathrm{d}}{\mathrm{d}r} S \Big) = \frac{\underline{\rho}'}{(\lambda - u_\star)^2} S.$$

From this we infer that provided that $\inf_r \underline{h} + \eta > 0$ and $\inf_r \underline{\rho}' > 0$ (we recognize here the non-cavitation and stable density stratification assumptions introduced in the multilayer system), there exists an countable infinite number of real eigenvalues

$$\lambda_{\pm \ell} = u_\star \pm \mu_\ell \quad (\ell \in \mathbb{N}), \qquad 0 < \cdots < \mu_\ell < \cdots < \mu_2 < \mu_1,$$

and the aforementioned oscillation (or polarity change) property holds for S which represents the deformation from rest of the isopycnals, *i.e.* sheets of fluid particles with given density.

Hence the discussion of hyperbolicity and mode separation of Section 6.3.1 does extend to the setting of continuous density stratification *in the absence of shear velocities*. However one cannot use perturbative arguments (see [248]) to extend the analysis to non-zero shear velocities, due to the fact that eigenvalues accumulate at $u_\star$. This certainly explains why even second baroclinic modes (that is with two polarity changes) are rarely observed [420]. Yet the celebrated result of Miles [315] and Howard [216] applies in the hydrostatic framework in particular and shows that, in dimension $d = 1$ (see [191, Remark 1.3] when $d = 2$), no eigenvalue with non-trivial imaginary part may arise as soon as shear velocities are sufficiently small, more precisely

$$\forall r \in (0,1), \qquad \frac{1}{4}\,|\underline{\boldsymbol{u}}'|^2 \leq g(\underline{h}+\eta)\frac{-\underline{\rho}'}{\underline{\rho}}.$$

This shows the spectral stability but this does not imply—by far!—other stability results, such as a continuous semigroup property on the linearized operator, let alone well-posedness of the nonlinear initial-value problem. Let me lazily refer to [192] for a detailed discussion on a related problem, and more generally to [191] for a very nice exposition to hydrodynamic stability theory.

6.4 Discussion and open questions

We have already expressed that Theorem 6.4—and therefore results relying on it—suffers from two serious shortcomings. The first one is that the conditions on the initial data therein does not meet the conditions for hyperbolicity described in Section 6.1.2. The second one is that it involves a stringent smallness condition on the bottom topography as $\gamma \nearrow 1$. These shortcomings may be only technical, and require only a closer look at the structure of the system. Recall that in the one-layer (Saint-Venant) situation, the related questions of large-time existence in weakly nonlinear situations with strong bottom variations was answered in [62]; see Remark 5.4.

For quasilinear systems such as the bilayer or multilayer hydrostatic systems presented in this section, one of the main focus is the domain of hyperbolicity. It is quite frustrating that so little is known despite quite a lot of works in this direction. Summarizing what has been discussed above (see also [408] for a review and more references), and restricting the discussion to dimension $d = 1$:

- we know explicitly the hyperbolicity domain and Riemann invariants for the system with $N = 2$ layers, rigid lid and Boussinesq approximation;
- we know explicitly the hyperbolicity domain for the system with $N = 2$ layers and rigid lid;
- we can characterize the hyperbolicity domain for the system with $N = 2$ layers and free surface;
- we know that the system with $N \geq 2$ layers is strictly hyperbolic for sufficiently small shear velocities.

Hence outside of the situation of $N = 2$ (or very specific situations if $N = 3$, see *e.g.* [407]), our knowledge is very limited. In particular the domain of hyperbolicity is expected to be larger than the domain of strict hyperbolicity and possible scenarios for eigenvalue crossings [104] remain to be understood.

As a consequence, we have—again, by far—not enough material to study the limit of continuous stratification described above, despite the fact that it is one of the main motivation behind multilayer models. It should be noted however that the strategy of crudely approximating a continuous density by a step function may be too rough, and some works have been dedicated to loosen this premise; see *e.g.* [188] and references therein. Also, we assume potential flows in each layer, so that the vorticity in the continuous model is approximated by “Dirac” vortex sheets located at interfaces. This again may be too rough, and the authors in [93] have exhibited the regularizing effect of incorporating vorticity in multilayer models. Shortly put, the multilayer hydrostatic systems discussed in this manuscript might not be the best ones to consider for practical purposes. However, for any model of multilayer type, in order to hope that good stability results hold uniformly with respect to the number of layers, these should also hold for the (limit) continuously stratified model introduced in Section 6.3.3 and discussed in more details in Section 7. For this continuous model again results are sparse despite important efforts and in particular deciding the well-posedness of the initial-value problem for the latter (in spaces of finite regularity) is, to the author’s opinion, an important open mathematical challenge.

7 The continuously stratified hydrostatic systems

Here we introduce hydrostatic equations for continuously stratified incompressible flows. These equations are often called *primitive equations* in studies of geophysical flows, although the equations typically include Coriolis force, more complicated equations of state (including the role of "tracers"such as temperature and salinity) as well as diffusion and/or viscosity effects (and some authors use the term *primitive equations* for non-hydrostatic equations). We will therefore refrain from using this otherwise engaging terminology. The interested reader may refer to [357] among *many* inspiring textbooks on geophysical flows for a mathematically oriented introduction.

7.1 Formal derivation

In this section we formally derive the hydrostatic counterparts of the full Euler equations, eq. (1.1), with fixed bottom , constant atmospheric pressure, and neglecting surface tension. We first introduce a scaling based on[31] Section 2.4. We set

$$\boldsymbol{x} = \frac{\mathsf{x}}{\lambda} \quad ; \quad z = \frac{\mathsf{z}}{d} \quad ; \quad t = \frac{\sqrt{gd}}{\lambda}\mathsf{t}$$

and (notice the different scaling for the horizontal and vertical velocities)

$$\zeta = \frac{\zeta}{a_{\rm top}} \quad ; \quad b = \frac{b}{a_{\rm bit}} \quad ; \quad \rho = \frac{\rho}{\rho_0} \quad ; \quad U = \frac{1}{\sqrt{gd}}\mathsf{U} \quad ; \quad w = \frac{\lambda}{d\sqrt{gd}}\mathsf{w} \quad ; \quad P = \frac{1}{\rho_0 gd}\mathsf{P}.$$

In addition to the typical horizontal wavelength denoted λ, the depth of the layer at rest d, the typical amplitude of surface waves $a_{\rm top}$, the typical amplitude of bottom deformations $a_{\rm bot}$, we introduced ρ_0 the typical mass density. With this scaling and recalling the dimensionless parameters

$$\varepsilon = \frac{a_{\rm top}}{d} \quad ; \quad \beta = \frac{a_{\rm bot}}{d} \quad ; \quad \mu = \frac{d^2}{\lambda^2}$$

we have

$$\begin{aligned}
&\partial_t \rho + \nabla_{\boldsymbol{x}} \cdot (\rho U) + \partial_z(\rho w) = 0 && \text{in } \Omega^t, && (7.1a)\\
&\rho\partial_t U + \rho(U\cdot\nabla_{\boldsymbol{x}} + w\partial_z)U = -\nabla_{\boldsymbol{x}} P && \text{in } \Omega^t, && (7.1b)\\
&\mu\big(\rho\partial_t w + \rho(U\cdot\nabla_{\boldsymbol{x}} + w\partial_z)w\big) = -\partial_z P - \rho && \text{in } \Omega^t, && (7.1c)\\
&\nabla_{\boldsymbol{x}} \cdot U + \partial_z w = 0 && \text{in } \Omega^t, && (7.1d)\\
&\varepsilon\partial_t\zeta = w - (\varepsilon\nabla\zeta)\cdot U && \text{on } \Gamma_{\rm top}, && (7.1e)\\
&w = (\beta\nabla b)\cdot U && \text{on } \Gamma_{\rm bot}, && (7.1f)\\
&P = p_{\rm atm} && \text{on } \Gamma_{\rm top}. && (7.1g)
\end{aligned}$$

where $\Omega^t \stackrel{\rm def}{=} \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ -1+\beta b(\boldsymbol{x}) < z < \varepsilon\zeta(t,\boldsymbol{x})\}$, $\Gamma_{\rm top} \stackrel{\rm def}{=} \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta(t,\boldsymbol{x})\}$, and $\Gamma_{\rm bot} \stackrel{\rm def}{=} \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z = -1+\beta b(\boldsymbol{x})\}$. The hydrostatic equations now stem from setting $\mu = 0$ in eq. (7.1c), so that we infer

$$P(t,\boldsymbol{x},z) \approx P(t,\boldsymbol{x},\varepsilon\zeta(t,\boldsymbol{x})) + \int_z^{\varepsilon\zeta} \rho(t,\boldsymbol{x},z')\,{\rm d}z',$$

[31] yet slightly different, as we wish the nonlinearity parameter ε to measure the deviation to non-trivial *shear flows*:
$$\rho(t,\boldsymbol{x},z) = \underline{\rho}(z) + \varepsilon\widetilde{\rho}(t,\boldsymbol{x},z), \quad U(t,\boldsymbol{x},z) = \underline{U}(z) + \varepsilon\widetilde{U}(t,\boldsymbol{x},z), \quad w(t,\boldsymbol{x},z) = \varepsilon\widetilde{w}(t,\boldsymbol{x},z).$$

and plugging this approximation in eq. (7.1b). While we lost the evolution equation for w,[32] we can use the incompressibility condition, eq. (7.1e), and the boundary condition, eq. (7.1f), to infer

$$w(t,\boldsymbol{x},z) = (\beta\nabla b(\boldsymbol{x}))\cdot U(t,\boldsymbol{x},-1+\beta b(\boldsymbol{x})) - \int_{-1+\beta b(\boldsymbol{x})}^{z} (\nabla_{\boldsymbol{x}}\cdot U)(t,\boldsymbol{x},z')\,\mathrm{d}z'.$$

After an obvious algebra on eq. (7.1f), the system of equations eq. (7.1) then becomes

$$\partial_t\rho + U\cdot\nabla_{\boldsymbol{x}}\rho + w\partial_z\rho = 0 \qquad \text{in } \Omega^t, \tag{7.2a}$$

$$\rho\partial_t U + \rho(U\cdot\nabla_{\boldsymbol{x}} + w\partial_z)U = -\nabla_{\boldsymbol{x}}P \qquad \text{in } \Omega^t, \tag{7.2b}$$

$$\varepsilon\partial_t\zeta + \nabla\cdot\Big(\int_{-1+\beta b}^{\varepsilon\zeta} U(\cdot,z)\,\mathrm{d}z\Big) = 0 \qquad \text{on } \mathbb{R}^d, \tag{7.2c}$$

$$P(\cdot,\boldsymbol{x},z) = p_{\mathrm{atm}} + \int_z^{\varepsilon\zeta}\rho(\cdot,\boldsymbol{x},z')\,\mathrm{d}z' \qquad \text{in } \Omega^t, \tag{7.2d}$$

$$w(\cdot,\boldsymbol{x},z) = (\beta\nabla b(\boldsymbol{x}))\cdot U(\cdot,\boldsymbol{x},-1+\beta b(\boldsymbol{x})) - \int_{-1+\beta b(\boldsymbol{x})}^{z}(\nabla_{\boldsymbol{x}}\cdot U)(\cdot,\boldsymbol{x},z')\,\mathrm{d}z' \qquad \text{in } \Omega^t. \tag{7.2e}$$

Physical variables Scaling back to physical variables we get the ***hydrostatic Euler equations***

$$\partial_t\rho + U\cdot\nabla_{\mathsf{x}}\rho + w\partial_z\rho = 0 \qquad \text{in } \mathit{\Omega}^t, \tag{7.3a}$$

$$\rho\partial_t U + \rho(U\cdot\nabla_{\mathsf{x}} + w\partial_z)U = -\nabla_{\mathsf{x}}P \qquad \text{in } \mathit{\Omega}^t, \tag{7.3b}$$

$$\partial_t\zeta + \nabla\cdot\Big(\int_{-d+b}^{\zeta} U(\cdot,z)\,\mathrm{d}z\Big) = 0 \qquad \text{on } \mathbb{R}^d, \tag{7.3c}$$

$$P(\cdot,\mathsf{x},z) = p_{\mathrm{atm}} + g\int_z^{\zeta}\rho(\cdot,\mathsf{x},z')\,\mathrm{d}z' \qquad \text{in } \mathit{\Omega}^t, \tag{7.3d}$$

$$w(\cdot,\mathsf{x},z) = \nabla b(\mathsf{x})\cdot U(\cdot,\mathsf{x},-d+b(\mathsf{x})) - \int_{-d+b(\mathsf{x})}^{z}(\nabla_{\mathsf{x}}\cdot U)(\cdot,\mathsf{x},z')\,\mathrm{d}z' \qquad \text{in } \mathit{\Omega}^t. \tag{7.3e}$$

Remark 7.1 (Homogeneous setting)**.** *As in the full Euler equations, if we assume that the density is constant at a given time, then this property propagates in time, by the conservation of mass, eq. (7.3a). In this case, we obtain the so-called* ***Benney system*** *[47],*[33] *which consists in replacing* $-\frac{\nabla_{\mathsf{x}}P}{\rho}$ *with* $-g\nabla\zeta$ *in eq. (7.3b), coupled with eq. (7.3c) and eq. (7.3e). Notice this is* not *the Saint-Venant system introduced in Section 5. However, if we assume additionally that* $\partial_z U \equiv 0$ *(*i.e. columnar motion*) at a given time, this property propagates in time and the system reduces to the Saint-Venant system, eq. (5.4). Hence we have the diagram in Figure 7.1.*

[32] It is said the variable w is no longer *prognostic* but *diagnostic*.

[33] Incidentally, the Benney system has a quite striking connection with kinetic equations. By using a Lagrangian formulation akin to the isopycnal coordinates described later on [425], the equations turn out to be a representation of the Vlasov—Poisson equation, where the "Coulomb potential" is replaced by the Dirac mass; see [37, 38]. The system is not only Hamiltonian, but also "quasi-integrable"; see aforementioned references.

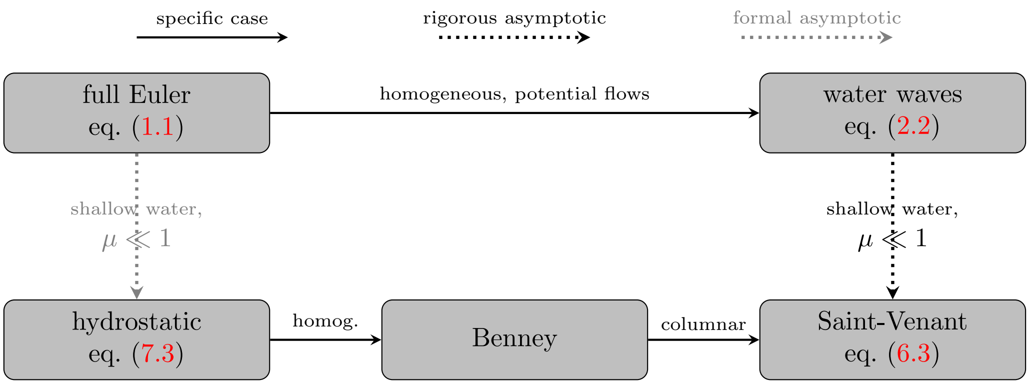


Figure 7.1: Some hydrostatic models.

Rigid-lid As we did in Section 6, we can introduce a version of the above systems in the rigid-lid framework. The rigid-lid counterpart to eq. (7.2) is

$$\begin{aligned}
&\partial_t \rho + U \cdot \nabla_{\boldsymbol{x}} \rho + w \partial_z \rho = 0 && \text{in } \mathbb{R}^d \times (-1,0), && \text{(7.4a)}\\
&\rho \partial_t U + \rho (U \cdot \nabla_{\boldsymbol{x}} + w \partial_z) U = -\nabla_{\boldsymbol{x}} P && \text{in } \mathbb{R}^d \times (-1,0), && \text{(7.4b)}\\
&w\,|_{z=0} = 0 && \text{on } \mathbb{R}^d, && \text{(7.4c)}\\
&P(\cdot, \boldsymbol{x}, z) = p_{\rm lid}(\cdot, \boldsymbol{x}) + \int_z^0 \rho(\cdot, \boldsymbol{x}, z')\,\mathrm{d}z' && \text{in } \mathbb{R}^d \times (-1,0), && \text{(7.4d)}\\
&w(\cdot, \boldsymbol{x}, z) = \beta \nabla b(\boldsymbol{x}) \cdot U(\cdot, \boldsymbol{x}, -1 + \beta b(\boldsymbol{x})) - \int_{-1+\beta b(\boldsymbol{x})}^{z} (\nabla_{\boldsymbol{x}} \cdot U)(\cdot, \boldsymbol{x}, z')\,\mathrm{d}z' && \text{in } \mathbb{R}^d \times (-1,0). && \text{(7.4e)}
\end{aligned}$$

where $p_{\rm lid}$ represents the pressure at the rigid lid and is the Lagrange multiplier associated with the constraint $w\,|_{z=0} = 0$. Scaling back to physical variables, we get

$$\begin{aligned}
&\partial_t \rho + U \cdot \nabla_{\mathsf{x}} \rho + w \partial_z \rho = 0 && \text{in } \mathbb{R}^d \times (-d,0), && \text{(7.5a)}\\
&\rho \partial_t U + \rho (U \cdot \nabla_{\mathsf{x}} + w \partial_z) U = -\nabla_{\mathsf{x}} P && \text{in } \mathbb{R}^d \times (-d,0), && \text{(7.5b)}\\
&w\,|_{z=0} = 0 && \text{on } \mathbb{R}^d, && \text{(7.5c)}\\
&P(\cdot, \mathsf{x}, z) = g p_{\rm lid}(\cdot, \mathsf{x}) + g \int_z^0 \rho(\cdot, \mathsf{x}, z')\,\mathrm{d}z' && \text{in } \mathbb{R}^d \times (-d,0), && \text{(7.5d)}\\
&w(\cdot, \mathsf{x}, z) = (\nabla b(\mathsf{x})) \cdot U(\cdot, \mathsf{x}, -d + b(\mathsf{x})) - \int_{-d+b(\mathsf{x})}^{z} (\nabla_{\mathsf{x}} \cdot U)(\cdot, \mathsf{x}, z')\,\mathrm{d}z' && \text{in } \mathbb{R}^d \times (-d,0). && \text{(7.5e)}
\end{aligned}$$

Applying the Boussinesq approximation consists in replacing eq. (7.4b) with

$$\partial_t U + (U \cdot \nabla_{\boldsymbol{x}} + w \partial_z) U = -\nabla_{\boldsymbol{x}} P \qquad \text{in } \mathbb{R}^d \times (-1,0), \qquad \text{(7.4b')}$$

or eq. (7.5b) with

$$\partial_t U + (U \cdot \nabla_{\mathsf{x}} + w \partial_z) U = -\frac{1}{\rho_0} \nabla_{\mathsf{x}} P \qquad \text{in } \mathbb{R}^d \times (-d,0), \qquad \text{(7.5b')}$$

where ρ_0 is a constant reference density.

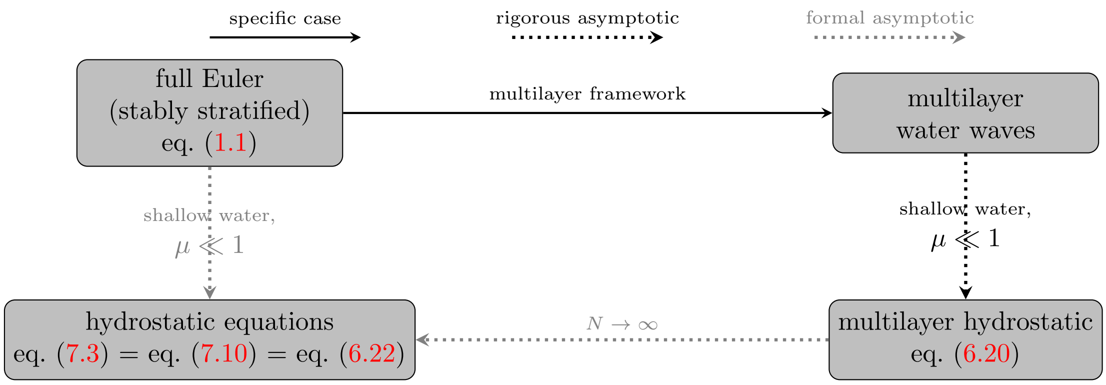


Figure 7.2: Hydrostatic limits and stratification.

7.2 Isopycnal coordinates

In this section we (formally) show that eq. (6.22), obtained as the limit of multilayer hydrostatic systems, is nothing but our hydrostatic system (7.3) written in isopycnal coordinates. This completes the diagram in Figure 7.2.

We follow closely the exposition in [215]. First we assume that at any $(t, \boldsymbol{x})$, the density distribution $\rho(t, \boldsymbol{x}, \cdot)$ is invertible, and that the density varies (continuously) between prescribed values at the bottom and at the surface. We denote $H(t, \boldsymbol{x}, \cdot)$ its inverse, representing the vertical height of the particle densities with prescribed density (given by the last variable) at time t and horizontal position $\boldsymbol{x}$. By chain rule and (7.3d), one has

$$\partial_\varrho\big(P(t, \boldsymbol{x}, H(t, \boldsymbol{x}, \varrho))\big) = (\partial_z P)(t, \boldsymbol{x}, H(t, \boldsymbol{x}, \varrho)) \times \partial_\varrho H(t, \boldsymbol{x}, \varrho) = -g\varrho\partial_\varrho H(t, \boldsymbol{x}, \varrho).$$

Thus defining the *Montgomery potential*

$$\Psi(t, \boldsymbol{x}, \varrho) \stackrel{\text{def}}{=} P(t, \boldsymbol{x}, H(t, \boldsymbol{x}, \varrho)) + g\varrho H(t, \boldsymbol{x}, \varrho),$$

one has

$$\partial_\varrho \Psi(t, \boldsymbol{x}, \varrho) = gH(t, \boldsymbol{x}, \varrho). \tag{7.6}$$

In the same way, we infer from chain rule

$$\begin{aligned}\nabla_{\boldsymbol{x}} P(t, \boldsymbol{x}, z) + gz\nabla_{\boldsymbol{x}}\rho(t, \boldsymbol{x}, z) &= \nabla_{\boldsymbol{x}}\big(\Psi(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z))\big) \\ &= (\nabla_{\boldsymbol{x}}\Psi)(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) + gH(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z))\nabla_{\boldsymbol{x}}\rho(t, \boldsymbol{x}, z),\end{aligned}$$

and hence

$$\nabla_{\boldsymbol{x}} P(t, \boldsymbol{x}, z) = (\nabla_{\boldsymbol{x}}\Psi)(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)).$$

Next, we define

$$\boldsymbol{u}(t, \boldsymbol{x}, \varrho) \stackrel{\text{def}}{=} U(t, \boldsymbol{x}, H(t, \boldsymbol{x}, \varrho))$$

and we infer from the identity $U(t, \boldsymbol{x}, z) = \boldsymbol{u}(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z))$ and chain rule

$$\begin{aligned}\partial_t U(t, \boldsymbol{x}, z) &= (\partial_t \boldsymbol{u})(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) + (\partial_t \rho(t, \boldsymbol{x}, z))(\partial_\varrho \boldsymbol{u})(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) \\ \big((U \cdot \nabla_{\boldsymbol{x}})U\big)(t, \boldsymbol{x}, z) &= \big((\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{u}\big)(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) + \big((\partial_\varrho \boldsymbol{u})\boldsymbol{u}\big)(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) \cdot \nabla_{\boldsymbol{x}}\rho(t, \boldsymbol{x}, z) \\ \big(w\partial_z U\big)(t, \boldsymbol{x}, z) &= \big(w\partial_z \rho\big)(t, \boldsymbol{x}, z)\big(\partial_\varrho \boldsymbol{u}\big)(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z))\end{aligned}$$

Plugging these expression in eq. (7.3b) and using eq. (7.3a) yields

$$\varrho\big(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big) + \nabla_{\boldsymbol{x}}\Psi = 0. \tag{7.7}$$

Next we concentrate on the mass conservation equation. From the identity $\rho(t,\boldsymbol{x},H(t,\boldsymbol{x},\varrho)) = \varrho$ we gather

$$\begin{aligned}
\partial_t\rho(t,\boldsymbol{x},z) + \partial_z\rho(t,\boldsymbol{x},z)\big(\partial_t H\big)(t,\boldsymbol{x},\rho(t,\boldsymbol{x},z)) &= 0,\\
\nabla_{\boldsymbol{x}}\rho(t,\boldsymbol{x},z) + \partial_z\rho(t,\boldsymbol{x},z)\big(\nabla_{\boldsymbol{x}}H\big)(t,\boldsymbol{x},\rho(t,\boldsymbol{x},z)) &= 0,\\
\partial_z\rho(t,\boldsymbol{x},z)\big(\partial_\varrho H\big)(t,\boldsymbol{x},\rho(t,\boldsymbol{x},z)) &= 1.
\end{aligned}$$

Plugging into eq. (7.3a) yields

$$\partial_t H(\cdot,\varrho) + \big(\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}H\big)(\cdot,\varrho) - w(\cdot,H(\cdot,\varrho)) = 0.$$

Differentiating the above with respect to ϱ and denoting

$$h(t,\boldsymbol{x},\varrho) \stackrel{\text{def}}{=} -\partial_\varrho H(t,\boldsymbol{x},\varrho) \tag{7.8}$$

yields

$$\partial_t h(\cdot,\varrho) + \big(\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}h\big)(\cdot,\varrho) - \big((\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}H\big)(\cdot,\varrho) + \big(\partial_\varrho H\big)(\cdot,\varrho)\big(\partial_z w\big)(\cdot,H(\cdot,\varrho)) = 0.$$

Finally, using the incompressibility constraint, $\partial_z w = -\nabla\cdot U$ and chain rule, we infer

$$\partial_t h + \nabla_{\boldsymbol{x}}\cdot(h\boldsymbol{u}) = 0. \tag{7.9}$$

Collecting eq. (7.6),(7.7),(7.8),(7.9), we find

$$\begin{cases}
\partial_t h + \nabla_{\boldsymbol{x}}\cdot(h\boldsymbol{u}) = 0,\\
\varrho\big(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big) + \nabla_{\boldsymbol{x}}\Psi = 0,\\
\partial_\varrho\Psi = gH(t,\boldsymbol{x},\varrho), \qquad h = -\partial_\varrho H.
\end{cases} \tag{7.10}$$

In the situation where the density is equal to ρ_0 at the surface and ρ_1 at the bottom, we have by definition $H(t,\boldsymbol{x},\rho_1) = -d + b(\boldsymbol{x})$ and $\Psi(t,\boldsymbol{x},\rho_0) = p_{\rm atm} + g\rho_0 H(t,\boldsymbol{x},\rho_0)$ so that for any $\varrho\in(\rho_0,\rho_1)$,

$$\begin{aligned}
\nabla_{\boldsymbol{x}}\Psi(t,\boldsymbol{x},\varrho) &= g\rho_0\nabla_{\boldsymbol{x}}H(t,\boldsymbol{x},\rho_0) + g\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}H(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'\\
&= g\varrho\nabla b(\boldsymbol{x}) + g\rho_0\int_{\rho_0}^{\rho_1}\nabla_{\boldsymbol{x}}h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + g\int_{\rho_0}^{\varrho}\int_{\varrho'}^{\rho_1}\nabla_{\boldsymbol{x}}h(t,\boldsymbol{x},\varrho'')\,\mathrm{d}\varrho''\,\mathrm{d}\varrho'\\
&= g\varrho\nabla b(\boldsymbol{x}) + g\int_{\rho_0}^{\varrho}\varrho'\nabla_{\boldsymbol{x}}h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + g\varrho\int_{\varrho}^{\rho_1}\nabla_{\boldsymbol{x}}h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'.
\end{aligned}$$

Plugging this formula in eq. (7.10), the first two equations provide a closed system of evolution equations for the variables $(h,\boldsymbol{u})$, which are the ***continuously stratified hydrostatic Euler equations*** in isopycnal coordinates.

Remark 7.2. *Applying the change of coordinate* $\varrho = \rho_0 + r(\rho_1-\rho_0)$ *in eq.* (7.10)*, we recognize eq.* (6.22) *with* $\underline{\rho}(r) = \rho_0 + r(\rho_1-\rho_0)$ *and* $(\underline{h}+\eta)(\cdot,r) = (\rho_1-\rho_0)h(\cdot,\rho_0+r(\rho_1-\rho_0))$.

In the rigid-lid framework, we have the same equations, eq. (7.10), yet with boundary conditions $H(t,\boldsymbol{x},\rho_1) = -d + b(\boldsymbol{x})$, $H(t,\boldsymbol{x},\rho_0) = 0$ and $\Psi(t,\boldsymbol{x},\rho_0) = p_{\rm lid}$, where $p_{\rm lid}$ can be seen as a Lagrange multiplier associated with the constraint stemming from the rigid-lid assumption:

$$\int_{\rho_0}^{\rho_1}\big(\nabla\cdot(h\boldsymbol{u})\big)(\cdot,\varrho)\,\mathrm{d}\varrho = -\partial_t\int_{\rho_0}^{\rho_1}h(\cdot,\varrho)\,\mathrm{d}\varrho = \partial_t\big(H(t,\boldsymbol{x},\rho_1) - H(t,\boldsymbol{x},\rho_0)\big) = 0, \tag{7.11}$$

and thus we have for any $\varrho \in (\rho_0, \rho_1)$,

$$\begin{aligned}\nabla_{\boldsymbol{x}}\Psi(t,\boldsymbol{x},\varrho) &= \nabla p_{\rm lid} + g\int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} H(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' \\ &= \nabla p_{\rm lid} + g\varrho\nabla b(\boldsymbol{x}) + g\rho_0\int_{\rho_0}^{\rho_1}\nabla_{\boldsymbol{x}} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + g\int_{\rho_0}^{\varrho}\int_{\varrho'}^{\rho_1}\nabla_{\boldsymbol{x}} h(t,\boldsymbol{x},\varrho'')\,\mathrm{d}\varrho''\,\mathrm{d}\varrho' \\ &= \nabla p_{\rm lid} + g\varrho\nabla b(\boldsymbol{x}) + g\int_{\rho_0}^{\varrho}\varrho'\nabla_{\boldsymbol{x}} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + g\varrho\int_{\varrho}^{\rho_1}\nabla_{\boldsymbol{x}} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'.\end{aligned}$$

Plugging this formula in eq. (7.10), and adding the constraint eq. (7.11), provides a closed system of evolution equations for the variables $(h, \boldsymbol{u})$.

7.3 Discussion and open questions

As we mentioned in Section 6.3.3 and Section 6.4, our knowledge on the hydrostatic equations, eq. (7.3) or eq. (7.10), is very limited. In particular, the well-posedness of the initial-value problem in functional spaces of finite regularity (see [264] in for analytic data) and in the inhomogeneous case (say with stable stratification, $\partial_z\rho < 0$) is completely open. In order to understand the structure of the equations it is interesting to try out the standard energy method on the equations and see where it fails. As a second step this will motivate us to consider physically-grounded additional terms with regularizing effects.

Demise of the energy method in Eulerian coordinates Let us first consider the equations in Eulerian coordinates, using the rigid-lid assumption and flat bottom case for simplicity (we could also consider the free-surface case and general topography with simple modifications of the argument), that is eq. (7.5). A natural functional space for this system is

$$\rho(t,\boldsymbol{x},z) \stackrel{\rm def}{=} \underline{\rho}(z) + \widetilde{\rho}(t,\boldsymbol{x},z), \qquad U(t,\boldsymbol{x},z) \stackrel{\rm def}{=} \underline{U}(z) + \widetilde{U}(t,\boldsymbol{x},z)$$

with $(\underline{\rho},\underline{U}) \in W^{s,\infty}((-d,0))^{1+d}$ given and unknowns $(\widetilde{\rho}(t,\cdot),\widetilde{U}(t,\cdot)) \in H^s(\mathbb{R}^d\times(-d,0))^{1+d}$ with $s\in\mathbb{N}$ sufficiently large. If we consider the equations for $\widetilde{\rho}(t,\cdot),\widetilde{U}(t,\cdot)$ and apply the differential operator $\partial^{\boldsymbol{k}}$ for a multi-index $\boldsymbol{k} = (\boldsymbol{k}_{\boldsymbol{x}}, \boldsymbol{k}_z) \in \mathbb{N}^{d+1}$ with $0 \le |\boldsymbol{k}| \le s$ we find that smooth solutions to eq. (7.5) must satisfy

$$\begin{cases} \partial_t\partial^{\boldsymbol{k}}\widetilde{\rho} + \big((\underline{U}+\widetilde{U})\cdot\nabla_{\boldsymbol{x}} + \widetilde{w}\partial_z\big)\partial^{\boldsymbol{k}}\rho - (\underline{\rho}' + \partial_z\widetilde{\rho})\int_{-d}^{z}(\nabla_{\boldsymbol{x}}\cdot\partial^{\boldsymbol{k}}\widetilde{U})(\cdot,z')\,\mathrm{d}z' = r_{\boldsymbol{k}}, \\ \partial_t\partial^{\boldsymbol{k}}\widetilde{U} + \big((\underline{U}+\widetilde{U})\cdot\nabla_{\boldsymbol{x}} + \widetilde{w}\partial_z\big)\partial^{\boldsymbol{k}}\widetilde{U} - (\underline{U}' + \partial_z\widetilde{U})\int_{-d}^{z}(\nabla_{\boldsymbol{x}}\cdot\partial^{\boldsymbol{k}}\widetilde{U})(\cdot,z')\,\mathrm{d}z' \\ \qquad + \frac{1}{\underline{\rho}+\widetilde{\rho}}\nabla\partial^{\boldsymbol{k}}p_{\rm lid} + \frac{g}{\underline{\rho}+\widetilde{\rho}}\int_z^0\partial^{\boldsymbol{k}}\nabla_{\boldsymbol{x}}\widetilde{\rho}(\cdot,z')\,\mathrm{d}z' = R_{\boldsymbol{k}}, \\ \int_{-d}^0(\nabla_{\boldsymbol{x}}\cdot\partial^{\boldsymbol{k}}\widetilde{U})(\cdot,z)\,\mathrm{d}z = 0, \end{cases}$$

where $\widetilde{w} = -\int_{-d}^{z}(\nabla_{\boldsymbol{x}}\cdot\widetilde{U})(\cdot,z)'\,\mathrm{d}z'$ and $(r_{\boldsymbol{k}},R_{\boldsymbol{k}}) \in L^2((\mathbb{R}^d\times(-d,0))^{1+d}$. In the first two terms of the first two equations we recognize the advection along the flow, and these will typically be harmless contributions. Then we observe the unbounded contributions from the pressure terms, as well as the key contributions stemming from the fact that $\widetilde{w}$ is a diagnostic variable, so that $\widetilde{w} \notin H^s(\mathbb{R}^d\times(-d,0))^{1+d}$. It turns out that when testing the first equation with respect to $\frac{-g}{\underline{\rho}'+\partial_z\widetilde{\rho}}\partial^{\boldsymbol{k}}\widetilde{\rho}$ and the second one with respect to $(\underline{\rho}+\widetilde{\rho})\partial^{\boldsymbol{k}}\widetilde{U}$, we observe a compensation (after integrating by parts) between the contributions from two among the three terms. Since the contribution of the rigid-lid pressure vanishes identically thanks to the rigid-lid constraint[34] the remaining term on the second

[34] It should be pointed out that the remainder term $R_{\boldsymbol{k}}$ contains—outside of the Boussinesq approximation framework—contributions from $p_{\rm lid} \in \mathring{H}^s(\mathbb{R}^d)$ which are bounded through elliptic estimates on the Poisson equation

$$\nabla_{\boldsymbol{x}}\cdot\left(\Big(\int_{-d}^0\frac{1}{\underline{\rho}+\widetilde{\rho}}\,\mathrm{d}z\Big)\nabla p_{\rm lid}\right) = -\int_{-d}^0\nabla_{\boldsymbol{x}}\cdot\left(\big((\underline{U}+\widetilde{U})\cdot\nabla_{\boldsymbol{x}}+\widetilde{w}\partial_z\big)(\underline{U}+\widetilde{U}) + \frac{g}{\underline{\rho}+\widetilde{\rho}}\int_z^0\nabla_{\boldsymbol{x}}\widetilde{\rho}(\cdot,z')\,\mathrm{d}z'\right)\mathrm{d}z.$$

equation can be singled out as sole responsible for the failure of energy estimates. More precisely we have, after integration by parts and using the boundary conditions $\widetilde{w}\,|_{z=-d} = \widetilde{w}\,|_{z=0} = 0$,

$$\mathcal{E}'(t) \leq C\,\mathcal{E}(t) + \iint_{\mathbb{R}^d\times(-d,0)} (\underline{\rho}+\widetilde{\rho})(\underline{U}'+\partial_z\widetilde{U})\cdot(\partial^{\boldsymbol{k}}\widetilde{U})\left(\int_{-d}^{z}(\nabla_{\boldsymbol{x}}\cdot\partial^{\boldsymbol{k}}\widetilde{U})(\cdot,z')\,\mathrm{d}z'\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}z,$$

where C depends on $d, s, d, g, \underline{M}, \widetilde{M}$ and $m_\star$ with

$$\left|\underline{\rho}\right|_{W^{s,\infty}((-d,0))} + \left|\underline{U}\right|_{W^{s,\infty}((-d,0))} \leq \underline{M}, \qquad \left|\widetilde{\rho}\right|_{H^s(\mathbb{R}^d\times(-d,0))} + \left|\widetilde{U}\right|_{H^s(\mathbb{R}^d\times(-d,0))} \leq \widetilde{M},$$

$$\inf_{(0,t)\times\mathbb{R}^d\times(-d,0)}\left(\left\{\underline{\rho}+\widetilde{\rho}, -\underline{\rho}'-\partial_z\widetilde{\rho}\right\}\right) \geq m_\star > 0,$$

and we define

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{|\boldsymbol{k}|=0}^{s}\iint_{\mathbb{R}^d\times(-d,0)} \frac{-g\big(\partial^{\boldsymbol{k}}\widetilde{\rho}\big)^2}{\underline{\rho}'+\partial_z\widetilde{\rho}} + (\underline{\rho}+\widetilde{\rho})\left|\partial^{\boldsymbol{k}}\widetilde{U}\right|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \approx \left\|\widetilde{\rho}(t,\cdot)\right\|_{H^s}^2 + \left\|\widetilde{U}(t,\cdot)\right\|_{H^s}^2.$$

Because the operator (which incidentally also plays an essential role in the inviscid Prandtl equations as the equations are the same but only boundary conditions differ; see *e.g.* [299])

$$L : U \mapsto \int_{-d}^{z} \nabla_{\boldsymbol{x}}\cdot U$$

is neither bounded nor skew-symmetric in $L^2(\mathbb{R}^d\times(-d,0)))$, we cannot infer a control of the last term in the above differential inequality, and the energy methods fails at this point.

A few comments are in order. Obviously the fact that we did not succeed to make the energy method work does not mean that it *cannot* work (modifying the functional space, choice of unknowns, symmetrizer, or looking for clever nonlinear cancellations). In particular, Masmoudi and Wong exhibit in [300] a key cancellation for the *homogeneous* hydrostatic equations (in dimension $d=1$) from which they deduce a local well-posedness result for the initial-value problem in Sobolev space $H^s(\mathbb{R}^d\times(-d,0))$. Their result holds under the local Rayleigh condition

$$\partial_z^2(\underline{U}+\widetilde{U}) \geq m_\star > 0,$$

which in turns coincide (with non-strict inequality) with the stability criteria for the linearized equations about shear flows; see in particular [363] for an ill-posedness result when this criteria is violated. For comparison, let us recall that the celebrated result of Miles [315] and Howard [216] shows that, in dimension $d=1$ (see [191, Remark 1.3] when $d=2$), no unstable mode of the linearized system about a shear velocity may arise as soon as the local Richardson number is greater than $1/4$ everywhere, that is

$$\forall z\in(-d,0), \qquad \frac{1}{4}\left|\underline{U}'(z)\right|^2 \leq g\frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}.$$

This gives hope for a well-posedness of the initial value problem of the (linearized or nonlinear) equations under such assumption, but again no result is known in that respect. Finally, as mentioned above, the failure of the energy estimate can be tracked back to the fact that $\widetilde{w}$ is a diagnostic variable rather than a prognostic variable. In the non-hydrostatic equations, one obtains by standard methods a control of the unknowns $(\widetilde{\rho},\widetilde{U},\widetilde{w})\in H^s(\mathbb{R}^d\times(-1,0))^{1+d+1}$, using integration by parts to deal with advection terms and elliptic estimates on the Poisson equation satisfied by the pressure; see [138]. Of course, the latter estimates are not uniform with respect to the shallow water parameter, μ, and hence solutions to the (non-hydrostatic) inhomogeneous Euler equations provide no result on the hydrostatic problem.

Demise of the energy method in isopycnal coordinates It is interesting to compare the result of the energy method on the formulation using Eulerian coordinates, and the one using isopycnal coordinates. Let us consider eq. (7.10) in the flat bottom situation, and set

$$H(t,\boldsymbol{x},\varrho) = \int_\varrho^{\rho_1} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' \stackrel{\text{def}}{=} \int_\varrho^{\rho_1} \underline{h}(\varrho')\,\mathrm{d}\varrho' + \underline{H}(t,\boldsymbol{x},\varrho), \qquad \boldsymbol{u}(t,\boldsymbol{x},\varrho) \stackrel{\text{def}}{=} \underline{\boldsymbol{u}}(\varrho) + \widetilde{\boldsymbol{u}}(t,\boldsymbol{x},\varrho),$$

with given $(\underline{h},\underline{\boldsymbol{u}}) \in W^{s,\infty}((\rho_0,\rho_1))^{1+d}$ and unknowns $(\widetilde{H}(t,\cdot),\widetilde{\boldsymbol{u}}(t,\cdot)) \in H^s(\mathbb{R}^d\times(\rho_0,\rho_1))^{1+d}$ with $s\in\mathbb{N}$ sufficiently large. If we consider the equations for $\widetilde{H}(t,\cdot),\widetilde{U}(t,\cdot)$ (hence integrating the first equation) and apply the operator $\partial^{\boldsymbol{k}}$ for a multi-index $\boldsymbol{k}=(\boldsymbol{k}_{\boldsymbol{x}},\boldsymbol{k}_z)\in\mathbb{N}^{d+1}$ with $0\le|\boldsymbol{k}|\le s$ we find that smooth solutions to eq. (7.10) must satisfy

$$\begin{cases} \partial_t\partial^{\boldsymbol{k}}\widetilde{H} + \int_\varrho^{\rho_1}\big((\underline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}h\big)(\cdot,\varrho')\,\mathrm{d}\varrho' + \int_\varrho^{\rho_1}\big(h(\nabla_{\boldsymbol{x}}\cdot\partial^{\boldsymbol{k}}\widetilde{\boldsymbol{u}})\big)(\cdot,\varrho')\,\mathrm{d}\varrho' = r_{\boldsymbol{k}},\\ \partial_t\partial^{\boldsymbol{k}}\widetilde{\boldsymbol{u}} + \big((\underline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\big)\partial^{\boldsymbol{k}}\widetilde{\boldsymbol{u}} + \frac{1}{\varrho}\nabla\partial^{\boldsymbol{k}}p_{\rm lid} + \frac{g}{\varrho}\int_{\rho_0}^{\varrho}\partial^{\boldsymbol{k}}\nabla_{\boldsymbol{x}}\widetilde{H}(\cdot,\varrho')\,\mathrm{d}\varrho' = R_{\boldsymbol{k}}, \end{cases}$$

where $h=-\partial_\varrho H=\underline{h}-\partial_\varrho\widetilde{H}$ and $(r_{\boldsymbol{k}},R_{\boldsymbol{k}})\in L^2(\mathbb{R}^d\times(\rho_0,\rho_1))^{1+d}$. Once again we recognize a compensation between the last two terms of the right-hand side provided that we test the first equation with respect to $g\partial^{\boldsymbol{k}}\widetilde{H}$ and the second one with $\varrho h\partial^{\boldsymbol{k}}\widetilde{\boldsymbol{u}}$. Using integration by parts to deal with the aforementioned and advection terms and the identity

$$\int_\varrho^{\rho_1}\big((\underline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}h\big)(\cdot,\varrho')\,\mathrm{d}\varrho' = (\underline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}H + \int_\varrho^{\rho_1}\big((\underline{\boldsymbol{u}}'+\partial_\varrho\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}H\big)(\cdot,\varrho')\,\mathrm{d}\varrho',$$

as well as the control of $p_{\rm lid}\in\mathring{H}^s(\mathbb{R}^d)$[35] we infer

$$\begin{aligned}\mathcal{E}'(t) \le C\,\mathcal{E}(t) &- \iint_{\mathbb{R}^d\times(\rho_0,\rho_1)} g\big(\partial^{\boldsymbol{k}}\widetilde{H}\big)\left(\int_\varrho^{\rho_1}\big((\underline{\boldsymbol{u}}'+\partial_\varrho\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}H\big)(\cdot,\varrho')\,\mathrm{d}\varrho'\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho\\ &- \int_{\mathbb{R}^d} g\big(\partial^{\boldsymbol{k}}p_{\rm lid}\big)\left(\int_{\rho_0}^{\rho_1}\big((\underline{\boldsymbol{u}}'+\partial_\varrho\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\partial^{\boldsymbol{k}}H\big)(\cdot,\varrho)\,\mathrm{d}\varrho\right)\mathrm{d}\boldsymbol{x}\end{aligned}$$

where C depends on $d,s,\rho_0,\rho_1,g,\underline{M},\widetilde{M}$ and $m_\star$ with

$$\big|\underline{h}\big|_{W^{s,\infty}((\rho_0,\rho_1))} + \big|\underline{\boldsymbol{u}}\big|_{W^{s,\infty}((\rho_0,\rho_1))} \le \underline{M}, \qquad \big|\widetilde{H}\big|_{H^s(\mathbb{R}^d\times(\rho_0,\rho_1))} + \big|\widetilde{\boldsymbol{u}}\big|_{H^s(\mathbb{R}^d\times(\rho_0,\rho_1))} \le \widetilde{M},$$
$$\inf_{(0,t)\times\mathbb{R}^d\times(\rho_0,\rho_1)}\big(\underline{h}-\partial_\varrho\underline{H}\big) \ge m_\star > 0,$$

(the latter "non-cavitation" assumption accounting for the stable stratification, $-\partial_z\varrho>0$) and

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \frac12\sum_{|\boldsymbol{k}|=0}^{s}\iint_{\mathbb{R}^d\times(\rho_0,\rho_1)} g\big(\partial^{\boldsymbol{k}}\widetilde{H}\big)^2 + (\underline{h}-\partial_\varrho\widetilde{H})\big|\partial^{\boldsymbol{k}}\widetilde{\boldsymbol{u}}\big|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho \approx \big\|\widetilde{H}(t,\cdot)\big\|_{H^s}^2 + \big\|\widetilde{\boldsymbol{u}}(t,\cdot)\big\|_{H^s}^2.$$

Again we see that some contributions cannot be bounded by $\mathcal{E}(t)$, and the energy method fails. It is however interesting to notice that, as with Eulerian coordinates, the additional terms stem from the contribution of shear velocities[36], but in contrast with Eulerian coordinates the loss of derivative applies to H (and hence ϱ) rather than U. This motivates the following discussion.

[35] by elliptic estimates on the Poisson equation stemming from eq. (7.11):

$$\nabla\cdot\left(\int_{\rho_0}^{\rho_1}\frac{h(\cdot,\varrho)}{\varrho}\,\mathrm{d}\varrho\,\nabla p_{\rm lid}\right) = -\int_{\rho_0}^{\rho_1}\nabla_{\boldsymbol{x}}\cdot\left(\big(h(\underline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}})\cdot\nabla_{\boldsymbol{x}}\big)\widetilde{\boldsymbol{u}} + \frac{gh}{\varrho}\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\widetilde{H}(\cdot,\varrho')\,\mathrm{d}\varrho'\right)\mathrm{d}\varrho.$$

[36] in both cases one can solve the linearized equations about no-shear flows: $U'(z)\equiv0$ or $\boldsymbol{u}'(\varrho)\equiv0$; see the discussion in Section 6.3.3. In particular, recall the above Miles–Howard criterion reads in isopycnal coordinates

$$\forall\varrho\in(\rho_0,\rho_1),\qquad \frac14|\underline{\boldsymbol{u}}'(\varrho)|^2 \le \frac{g\underline{h}(\varrho)}{\varrho}.$$

Some natural regularizations Arguably the most natural way[37] to regularize the hydrostatic system is the parabolic approach which we use for instance in the proof of Theorem 8.3 in Section 8.6. By looking at the above analysis on the system in Eulerian coordinates, one realizes that it is sufficient to add the parabolic regularization on the momentum conservation equation of eq. (7.3), and that this regularization may occur only on the horizontal space variable. Specifically, one may replace eq. (7.3b) with

$$\rho\partial_t U + \rho(U\cdot\nabla_x + w\partial_z)U + \nabla_x P = \nu\rho\Delta_x U$$

with $\nu > 0$ a positive constant. For this system with partial viscosity (in the rigid-lid framework with Boussinesq approximation), Cao, Li and Titi [75, 76] proved local-in-time existence and uniqueness of strong solutions for initial data in $H^1(\mathbb{T}^d\times(-d,0))$ (with extra boundary conditions) and global existence assuming slightly more regularity. This is part of a line of research of the authors where many other situations were studied; see the review [280] for discussion and many important references.

It should be noticed that considering partial (horizontal) viscosity bares physical significance. Obviously we cannot relate the viscosity to molecular viscosity, considering Navier–Stokes equations rather than Euler equations as master equations: firstly because the parameter measuring the effect of molecular viscosity (basically the inverse of the Reynolds number) is ridiculously small in oceanographic scales, and secondly because in the hydrostatic limit ($\mu \searrow 0$) originally isotropic viscosity become predominant in the vertical rather than horizontal direction. Yet anisotropic horizontal viscosity is often considered as a good design for *eddy viscosity* modeling the large-scale dissipative effects from small-scale turbulence. This is due to the observation that (quoting Gent and McWilliams [198] when describing the conclusions of Iselin [231] and Montgomery [331]) “mixing of material properties by eddies in the stably stratified parts of the oceans occurs mostly along surfaces of constant density or isopycnal coordinates”, and hence horizontal viscosity approximates isopycnal dissipation effects.

In their seminal work, Gent and McWilliams [198] (see also [199, 254]) proposed a parameterization of eddies through isopycnal diffusivity. In the Eulerian coordinates, this consists in replacing eq. (7.3a)–(7.3b)

$$\begin{aligned} &\partial_t\rho + (U+U^\star)\cdot\nabla_x\rho + (w+w^\star)\partial_z\rho = 0 && \text{in } \Omega^t,\\ &\rho\partial_t U + \rho((U+U^\star)\cdot\nabla_x + (w+w^\star)\partial_z)U = -\nabla_x P && \text{in } \Omega^t,\end{aligned}$$

where (in the simplest situation where the diffusivity coefficient, κ, is assumed constant)

$$U^\star \overset{\text{def}}{=} \kappa\partial_z\left(\frac{\nabla_x\rho}{\partial_z\rho}\right) \qquad \text{and} \qquad w^\star \overset{\text{def}}{=} -\kappa\nabla_x\cdot\left(\frac{\nabla_x\rho}{\partial_z\rho}\right).$$

Yet the effect of such diffusivity is—unsurprisingly—clearer in isopycnal coordinates: one replaces eq. (7.10) with

$$\begin{cases} \partial_t h + \nabla_x\cdot(h(\boldsymbol{u}+\boldsymbol{u}^\star)) = 0,\\ \varrho\big(\partial_t\boldsymbol{u} + ((\boldsymbol{u}+\boldsymbol{u}^\star)\cdot\nabla_x)\boldsymbol{u}\big) + \nabla_x\Psi = 0,\\ \partial_\varrho\Psi = gH(t,\boldsymbol{x},\varrho), \quad h = -\partial_\varrho H,\end{cases}$$

where

$$\boldsymbol{u}^\star \overset{\text{def}}{=} -\kappa\frac{\nabla_x h}{h}.$$

Hence we see that diffusivity appears as a parabolic contribution on the first equation, whose regularizing effects act on the variable h (and hence H), as demanded by the above analysis of the energy method on the formulation with isopycnal coordinates.

[37] Other possibilities include the Leray-α and LANS-α models; see [298, 200, 212] and references therein.

CHAPTER C

Weakly dispersive models

> parce que, [les Anciens] s'étant élevés jusqu'à un certain degré où ils nous ont portés, le moindre effort nous fait monter plus haut, et avec moins de peine et moins de gloire nous nous trouvons au-dessus d'eux. C'est de là que nous pouvons découvrir des choses qu'il leur était impossible d'apercevoir.
>
> — Blaise Pascal, *traité du vide*

Contents

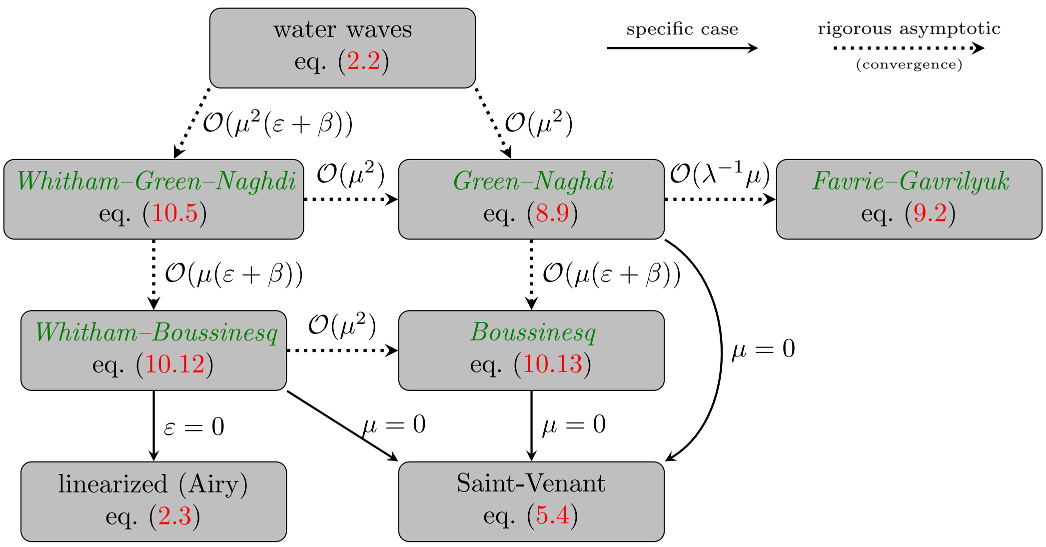


Figure C: Models in Chapter C (in *green*) and some filiations.

Foreword

This chapter is devoted to the derivation and analysis of *weakly dispersive models*. These models refine the hydrostatic equations studied in Chapter B (in fact, more precisely, the Saint-Venant equations since here we restrict the analysis to the water waves framework; see Chapter E for an extension to the bilayer framework) by introducing dispersive effects at first order. More refined models are presented in Chapter D.

Section 8 has been meant as a showcase for a thorough study of water waves models. Here we introduce and analyze in much details the so-called ***(Serre–)Green–Naghdi model***. Firstly the model is quickly derived as an asymptotic model from the expansion of the Dirichlet-to-Neumann operator obtained in Section 4—and more precisely Proposition 4.10. Yet the result which follows (namely the *consistency* of the model) is far from being sufficient to validate the Green–Naghdi equations as a good model for water waves. Firstly, its rigorous justification must be completed by *well-posedness*, *stability* and *convergence* results, which are expressed in Section 8.5. They follow from careful energy estimates in suitable functional spaces carried on in Section 8.6. In a looser way, we also expect "good" models to retain important properties of the master equations (here the water waves system). Here we focus mostly on the variational structure of the equations: we observe in Section 8.1 that the Green–Naghdi equations not only preserve Zakharov's canonical Hamiltonian structure of the water waves system, but it also enjoys a deeper Lagrangian formalism which embeds the system inside a natural family of conservation laws, which can be interpreted as equations for compressible fluids with inertia effects. Hence the structure of the equations becomes richer as we simplify the equations from the water waves system to the Green–Naghdi equations (and then from the Green–Naghdi equations to the Saint-Venant system). This explains in my opinion why the Green–Naghdi equations, among many other loosely equivalent models, has attracted so much attention from diverse communities. We review some basic properties of the Green–Naghdi equations: preserved quantities (Section 8.2), modal analysis and dispersion relation (Section 8.3), solitary wave solutions (Section 8.4). Finally, some open questions are discussed in Section 8.7.

Of course I do not claim that the Green–Naghdi model is perfect! One of its main drawback is certainly that numerically approximating the equations turns out to be quite costly. In Section 9 we explore some equations which have been proposed by Favrie and Gavrilyuk [181] to circumvent this issue. The equations are constructed using the aforementioned Lagrangian formalism, using a strategy akin to relaxation limits. Hence the system contains additional unknowns as well as a (large) parameter which is expected to measure the precision of solutions to the augmented equations with respect to solutions to the original Green–Naghdi equations, at least when initial data are well-prepared. The rigorous study of this *singular limit* is described in Section 9.5 and Section 9.6, based on [158]. Again, the Section is concluded by perspectives and open questions.

In Section 10 we introduce a *fully dispersive* analogue of the Green–Naghdi system, which we name ***Whitham–Green–Naghdi***. When linearized about trivial equilibrium solutions, fully dispersive models coincide with the corresponding (Airy) linearized water waves equations, as introduced in Section ii and Section 2.3. Interest in such fully dispersive models in the context of long water waves started with the work of Whitham, which proposed eq. (x) and eq. (ix) as suitable modifications of the standard Korteweg-de Vries equation, eq. (viii), with the view of reproducing at least qualitatively important features of water waves such as wavebreaking and peaked traveling waves. Much more recently, the interest was renewed as Whitham's predictions were proved to be valid [219, 173, 402, 374]. Yet the question of validating fully dispersive models as asymptotic models with improved accuracy with respect to their standard counterparts was mostly left aside. The precision of the Whitham–Green–Naghdi model (respectively ***Whitham–Boussinesq***) we introduce in Section 10 (respectively Section 10.6) significantly improves the precision of the Green–Naghdi (respectively ***Boussinesq***) model for weak nonlinearities and mild bottom variations (see Appendix I.5 for numerical illustrations) with the important price to pay that nonlocal operators (Fourier multipliers) are involved. These models also allow to rigorously justify the Whitham equations and observe a similar improvement with respect to the Korteweg–de Vries equation [178].

8 The Green–Naghdi system

We introduce a weakly dispersive fully nonlinear shallow water model, known in the literature as the (Serre–)Green–Naghdi system. To this aim we use the second-order approximation stemming from Proposition 4.10:[38]

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot\big(h(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b])^{-1}\nabla\psi\big)+\mathcal{O}(\mu^2), \tag{8.1}$$

where we recall the notations $h=1+\varepsilon\zeta-\beta b$ and

$$\mathcal{T}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u})+\frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big)-h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}\Big)+(\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b).$$

Plugging this expansion into eq. (2.7) and withdrawing $\mathcal{O}(\mu^2)$ terms yields

$$\begin{cases} \partial_t\zeta+\nabla\cdot(h\boldsymbol{u})=0,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\boldsymbol{u}|^2=\mu\varepsilon\mathcal{R}[h,\beta\nabla b,\boldsymbol{u}],\end{cases} \tag{8.2}$$

where

$$\mathcal{R}[h,\beta\nabla b,\boldsymbol{u}] \stackrel{\text{def}}{=} \frac{\boldsymbol{u}}{3h}\cdot\nabla(h^3\nabla\cdot\boldsymbol{u})+\frac{1}{2}h^2(\nabla\cdot\boldsymbol{u})^2 \\ -\frac{1}{2}\Big(\frac{\boldsymbol{u}}{h}\cdot\nabla\big(h^2(\beta\nabla b\cdot\boldsymbol{u})\big)+h(\beta\nabla b\cdot\boldsymbol{u})\nabla\cdot\boldsymbol{u}+(\beta\nabla b\cdot\boldsymbol{u})^2\Big)$$

and $\boldsymbol{u}$ is deduced from (ζ,ψ) after solving the equation [39]

$$h\nabla\psi \;=\; h\boldsymbol{u}+\mu h\mathcal{T}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\text{def}}{=} \mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u}. \tag{8.5}$$

[38]It would be tempting to rather use directly the approximation

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot(h\nabla\psi)+\mu\nabla\cdot\big(h\mathcal{T}[h,\beta\nabla b]\nabla\psi\big)+\mathcal{O}(\mu^2).$$

However, due to the fact that the operator $\boldsymbol{u}\mapsto h\boldsymbol{u}-\mu h\mathcal{T}[h,\beta\nabla b]\boldsymbol{u}$ is not positive definite, the resulting system suffers from strong instabilities at high frequencies in the sense that the linearized system about the trivial solution $(\zeta=0,\psi=0)$, explicitly solvable in Fourier space in the flat-bottom setting, exhibits unstable modes whose amplitude grows exponentially and arbitrarily rapidly for large wavenumbers. We shall not write down this system, but the interested reader may find it in [412, (10)-(11)] and [119, (14)-(15)] (in the one-dimensional and flat bottom situation). As pointed out in [314, (1.8a),(1.8b)], this system reduces to the original (also ill-posed) Boussinesq (or Kaup) system when the amplitude is small, that is withdrawing $\mathcal{O}(\mu(\varepsilon+\beta))$ terms.

[39]Notice that by Proposition 4.9, we have

$$\boldsymbol{u}=\overline{\boldsymbol{u}}+\mathcal{O}(\mu^2),\qquad \overline{\boldsymbol{u}}=\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Phi(\cdot,z)\,\mathrm{d}z, \tag{8.3}$$

where Φ is the unique solution to eq. (4.1). This allows to recognize the first equation as the conservation of mass. As for the second equation, we notice that denoting

$$w \stackrel{\text{def}}{=} (\beta\nabla b)\cdot\boldsymbol{u}-h\nabla\cdot\boldsymbol{u},$$

eq. (8.2) can be written as

$$\begin{cases} \partial_t\zeta+\nabla\cdot(h\boldsymbol{u})=0,\\ \partial_t\psi+\zeta+\varepsilon\boldsymbol{u}\cdot\nabla\psi-\frac{\varepsilon}{2}\boldsymbol{u}\cdot\boldsymbol{u}-\frac{\varepsilon\mu}{2}w^2=0.\end{cases} \tag{8.4}$$

This formulation echoes the formulation of the water waves equations displayed in eq. (2.7'). Indeed, using Lemma 4.6 and chain rule, one has

$$(\nabla_{\boldsymbol{x}}\Phi)\Big|_{z=1+\varepsilon\zeta}=\nabla\psi-(\varepsilon\nabla\zeta)(\partial_z\Phi)\Big|_{z=1+\varepsilon\zeta}\quad\text{and}\quad \mu^{-1}(\partial_z\Phi)\Big|_{z=1+\varepsilon\zeta}=(\varepsilon\nabla\zeta)\cdot(\nabla_{\boldsymbol{x}}\Phi)\Big|_{z=1+\varepsilon\zeta}-\nabla\cdot(h\overline{\boldsymbol{u}}),$$

and we deduce from Proposition 4.9

$$\boldsymbol{u}=\nabla\psi+\mathcal{O}(\mu)=(\nabla_{\boldsymbol{x}}\Phi)\Big|_{z=1+\varepsilon\zeta}+\mathcal{O}(\mu)\quad\text{and}\quad w=\mu^{-1}(\partial_z\Phi)\Big|_{z=1+\varepsilon\zeta}+\mathcal{O}(\mu),$$

from which the consistency of eq. (8.4) with eq. (2.7') (after non-dimensionalizing) is easily checked.

Taking the gradient of the second equation, one can check (see [162] for details) that the system can be written with only differential operators in terms of the unknowns ζ and $\boldsymbol{u}$, namely

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \big(\mathrm{Id} + \mu\mathcal{T}[h,\beta\nabla b]\big)\partial_t\boldsymbol{u} + \nabla\zeta + \varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\varepsilon\mathcal{Q}[h,\beta\nabla b,\boldsymbol{u}] = \boldsymbol{0}, \end{cases} \tag{8.6}$$

where

$$\begin{aligned}\mathcal{Q}[h,\beta\nabla b,\boldsymbol{u}] &\stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}) - (\nabla\cdot\boldsymbol{u})^2\big)\Big) \\ &\quad + \frac{\beta}{2h}\Big(\nabla\big(h^2(\boldsymbol{u}\cdot\nabla)^2 b\big) - h^2\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}) - (\nabla\cdot\boldsymbol{u})^2\big)\nabla b\Big) + \beta\big((\boldsymbol{u}\cdot\nabla)^2 b\big)(\beta\nabla b).\end{aligned}$$

An even more compact formulation, if one removes the constraint of considering first order (in time) evolution equations, is the following:

$$\begin{cases} \partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u}) = 0, \\ \varepsilon\partial_t\boldsymbol{u} + \nabla(h+\beta b) + \varepsilon^2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\mathcal{P}[h,\beta b,\varepsilon\boldsymbol{u}] = \boldsymbol{0}, \end{cases} \tag{8.7}$$

where

$$\mathcal{P}[h,\beta b,\varepsilon\boldsymbol{u}] \stackrel{\text{def}}{=} \frac{1}{h}\nabla\Big(h^2\big(\frac{\ddot{h}}{3} + \frac{\beta\ddot{b}}{2}\big)\Big) + \Big(\frac{\ddot{h}}{2} + \beta\ddot{b}\Big)(\beta\nabla b)$$

and where we denote $\dot{h} = \partial_t h + \varepsilon\boldsymbol{u}\cdot\nabla h$, $\ddot{h} = \partial_t\dot{h} + \varepsilon\boldsymbol{u}\cdot\nabla\dot{h}$, and similarly $\dot{b}, \ddot{b}$. The above formulation remains valid when the bottom has a prescribed but non-trivial time-dependent evolution; see [182, 189]. It may equivalently be written equivalently as follows [65, 184, 8, 180] (see also [206, Lemma 3.1] for a similar-looking yet different reformulation):

$$\begin{cases} \partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u}) = 0, \\ \varepsilon\partial_t\boldsymbol{u} + \nabla(h+\beta b) + \varepsilon^2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{\mu}{h}\nabla(hq) + \frac{\mu}{h}(\beta\nabla b)q_{\mathrm{b}} = \boldsymbol{0}, \\ \partial_t v + \varepsilon\boldsymbol{u}\cdot\nabla v = \frac{q}{h}, \quad \partial_t v_{\mathrm{b}} + \varepsilon\boldsymbol{u}\cdot\nabla v_{\mathrm{b}} = \frac{q_{\mathrm{b}}}{h}, \\ v = \dfrac{\dot{h}}{3} + \dfrac{\beta\dot{b}}{2}, \quad v_{\mathrm{b}} = \dfrac{\dot{h}}{2} + \beta\dot{b}. \end{cases} \tag{8.8}$$

Here, q and q_{b} are not unknowns but may be thought as the Lagrange multipliers associated with the constraints $v = \frac{\dot{h}}{3} + \frac{\beta\dot{b}}{2}$ and $v_{\mathrm{b}} = \frac{\dot{h}}{2} + \beta\dot{b}$. Physically speaking, they respectively represent the first order non-hydrostatic correction to layer-averaged pressure and the pressure at the bottom; see *e.g.* [184]. See also eq. (8.13), below, for a generalization of eq. (8.2)—or rather eq. (8.10)—to time-dependent topographies.

Using physical variables (recall Section 2.4), (8.7) yields the ***(Serre–)Green–Naghdi system***

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathcal{P}[h,b,\boldsymbol{u}] = \boldsymbol{0}. \end{cases} \tag{8.9}$$

with $h = d + \zeta - b$.

The Green–Naghdi system with its many formulations has been derived and studied numerous times, including in [383, 396, 202, 314, 42, 382, 371, 410, 73, 255, 40, 270, 107, 230, 182, 195]. Its rigorous justification as an asymptotic model in the shallow water limit in the sense provided in Section 8.5 has been obtained in [296, 284, 234, 15, 189, 162]. We provide a self-contained proof in Section 8.5 and Section 8.6, based on [189].

8.1 Variational structure

8.1.1 Hamiltonian structure

As the Saint-Venant system, eq. (8.2) inherits a canonical Hamiltonian structure from the water waves equations: Hamilton's principle on

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \zeta \partial_t \psi \, \mathrm{d}\boldsymbol{x} + \mathscr{H}_{\mathrm{GN}} \, \mathrm{d}t.$$

where $\mathscr{H}_{\mathrm{GN}}$ is the approximate Hamiltonian obtained when plugging the approximation eq. (8.1) into the Hamiltonian functional of the water waves equations (see Section 2.2), that is

$$\mathscr{H}_{\mathrm{GN}}(\zeta, \psi) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + (h\nabla\psi) \cdot \mathfrak{T}^\mu[h, \beta\nabla b]^{-1}(h\nabla\psi) \, \mathrm{d}\boldsymbol{x}$$

yields

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathscr{H}_{\mathrm{GN}} \\ \delta_\psi \mathscr{H}_{\mathrm{GN}} \end{pmatrix},$$

which corresponds to eq. (8.2). We may hence follow the discussion of Section 2.2.

We can check that, written with the velocity variable $\boldsymbol{v} = \nabla\psi$, eq. (8.2) still enjoys a (non-canonical) symplectic form. Indeed, the system is easily seen to be equivalent to

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\boldsymbol{u}) = 0, \\ (\partial_t + \varepsilon \boldsymbol{u}^\perp \operatorname{curl}) \boldsymbol{v} + \nabla\zeta + \frac{\varepsilon}{2}\nabla(|\boldsymbol{u}|^2) = \mu\varepsilon\nabla\mathcal{R}[h, \beta\nabla b, \boldsymbol{u}], \end{cases} \tag{8.10}$$

where we recall $h\boldsymbol{v} = \mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}$, $\operatorname{curl} \boldsymbol{v} \stackrel{\text{def}}{=} \partial_x v_y - \partial_y v_x$ and $\boldsymbol{u}^\perp \stackrel{\text{def}}{=} (-u_y, u_x)$. Notice that the term $\varepsilon \boldsymbol{u}^\perp \operatorname{curl} \boldsymbol{v}$ is artificial (in dimension $d = 1$, this term should be dropped) since $\boldsymbol{v} = \nabla\psi$, and contrarily to the Saint-Venant case, we do not expect that the system is still relevant outside of the irrotational setting. Yet it allows to obtain the exact same symplectic form as the Saint-Venant system, and the conclusions still apply. In dimension $d = 2$, one has

$$\partial_t \begin{pmatrix} \zeta \\ v_x \\ v_y \end{pmatrix} = - \begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & -q \\ \partial_y & q & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathscr{H} \\ \delta_{v_x} \mathscr{H} \\ \delta_{v_y} \mathscr{H} \end{pmatrix}.$$

where $q = \varepsilon \frac{\operatorname{curl} \boldsymbol{v}}{h} = \varepsilon \frac{\partial_x v_y - \partial_y v_x}{1 + \varepsilon\zeta - \beta b}$ and (misusing notations)

$$\mathscr{H}_{\mathrm{GN}}(\zeta, \boldsymbol{v}) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + (h\boldsymbol{v}) \cdot \mathfrak{T}^\mu[h, \beta\nabla b]^{-1}(h\boldsymbol{v}) \ \mathrm{d}\boldsymbol{x}$$

Within this formalism, one can check that the space and time invariance of the Hamiltonian yield the conservation of horizontal impulse (in the flat bottom case) and of total energy,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \zeta \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \boldsymbol{0} \quad \text{if } \beta b \equiv 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_{\mathrm{GN}} = 0.$$

while Casimir invariants are, for any function C,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} hC(q) \, \mathrm{d}\boldsymbol{x},$$

which yields the conservation of mass—with $C(q) = 1$—and circulation—with $C(q) = q$—as special cases.

This variational structure is equivalent to the one pointed out in [214, 283] expressed with different variables.

8.1.2 A Lagrangian structure

In [109] (see also [107]), Clamond, Dutykh and Mitsotakis derive the Green–Naghdi in a formal but very efficient way, based on the Lagrangian structure of the water waves equations. This derivation is very close in spirit to the one of the preceding section, and we reproduce it here for the sake of completeness. Recall the Lagrangian of the water waves equations (see Section 2.2) is

$$\mathscr{L} = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \frac{1}{2\mu}\psi\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi - \frac{1}{2}\zeta^2 + \varphi(\partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Using above the eq. (8.1), eq. (8.5) and withdrawing $\mathcal{O}(\mu^2)$ terms yields

$$\begin{aligned}\mathscr{L}_{\rm GN} &= \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \frac{1}{2}\boldsymbol{u}\cdot\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u} - \frac{1}{2}\zeta^2 + \varphi(\partial_t\zeta + \nabla\cdot(h\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \frac{1}{2}\int_{t_0}^{t_1} \int_{\mathbb{R}^d} h|\boldsymbol{u}|^2 + \frac{\mu}{3}h(h\nabla\cdot\boldsymbol{u})^2 - \mu h^2(\beta\nabla b\cdot\boldsymbol{u})\nabla\cdot\boldsymbol{u} + \mu h(\beta\nabla b\cdot\boldsymbol{u})^2 - \zeta^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\quad + \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \varphi(\partial_t\zeta + \nabla\cdot(h\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

Following Hamilton's principle, we obtain

$$0 = \delta_\varphi\mathscr{L}_{\rm GN} = \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}),$$

which is the equation of conservation of mass;

$$0 = \delta_{\boldsymbol{u}}\mathscr{L}_{\rm GN} = \mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u} - h\nabla\varphi,$$

from which we deduce $\nabla\varphi = \boldsymbol{u} - \frac{1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u})$; and

$$0 = \delta_\zeta\mathscr{L}_{\rm GN} = \frac{\varepsilon}{2}|\boldsymbol{u}|^2 + \frac{\mu\varepsilon}{2}h^2(\nabla\cdot\boldsymbol{u})^2 - \mu\varepsilon h(\beta\nabla b\cdot\boldsymbol{u})\nabla\cdot\boldsymbol{u} + \frac{\mu\varepsilon}{2}(\beta\nabla b\cdot\boldsymbol{u})^2 - \zeta - \partial_t\varphi - \varepsilon\boldsymbol{u}\cdot\nabla\varphi.$$

We indeed recognize eq. (8.2), or more precisely eq. (8.4).

8.1.3 Another Lagrangian structure

Interestingly, the Green–Naghdi system falls into another Lagrangian formalism which is not directly related to a corresponding one on the water waves equations, but rather includes the Green–Naghdi system inside a family of equations for compressible fluids with inertia effects. Indeed, define a Lagrangian action

$$\mathscr{L} = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \mathcal{L}(h, \boldsymbol{u}, \dot{h}, \dot{b}, b)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

where $h, \boldsymbol{u}$ are unknowns b is given (and $\dot{h} \stackrel{\rm def}{=} \partial_t h + \boldsymbol{u}\cdot\nabla h$, $\dot{b} \stackrel{\rm def}{=} \partial_t b + \boldsymbol{u}\cdot\nabla b$).

Then one can infer (by Hamilton's principle along virtual displacements, see [194, 197, 193]) that the critical points of the above Lagrangian submitted to the constraint of mass conservation

$$\partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0$$

satisfies the following equation:

$$\partial_t(\delta_{\boldsymbol{u}}\mathscr{L}) + \nabla\cdot(\boldsymbol{u}\otimes\delta_{\boldsymbol{u}}\mathscr{L}) + \big(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\big)^\top\delta_{\boldsymbol{u}}\mathscr{L} - h\nabla\delta_h\mathscr{L} = 0.$$

Denoting $\delta_{\boldsymbol{u}}\mathscr{L} \stackrel{\rm def}{=} h\boldsymbol{K}$, using the conservation of mass and some algebra, we infer in dimension $d = 2$

$$\partial_t\boldsymbol{K} + \boldsymbol{u}^\perp\,\mathrm{curl}\,\boldsymbol{K} + \nabla\big(\boldsymbol{K}\cdot\boldsymbol{u} - \delta_h\mathscr{L}\big) = 0 \qquad (8.11)$$

(in dimension $d = 1$, we simply have $\partial_t \boldsymbol{K} + \partial_x\big(\boldsymbol{K}\boldsymbol{u} - \delta_h\mathscr{L}\big) = 0$).

Using the specific choice

$$\mathcal{L}(h, \boldsymbol{u}) = h\big(\frac{1}{2}|\boldsymbol{u}|^2 - e(h)\big)$$

in eq. (8.11) yields the isentropic compressible Euler equations, where h denotes the density and $e(h)$ is the internal energy, and the pressure $p(h) = h^2 e'(h)$. In particular, if $e(h) = \frac{1}{2}h$, we recover the Saint-Venant system with flat bottom; see eq. (5.4).

The Green–Naghdi system is obtained using the specific choice (using dimensional variables, the corresponding dimensionless Lagrangian is easily deduced)

$$\mathcal{L}(h, \boldsymbol{u}, \dot{h}, \dot{b}, b) = \frac{h}{2}\big(|\boldsymbol{u}|^2 + \frac{1}{3}(\dot{h} + \frac{3}{2}\dot{b})^2 + \frac{1}{4}\dot{b}^2\big) - \frac{g}{2}(h+b)^2. \tag{8.12}$$

Indeed, we have

$$\begin{aligned}\delta_h\mathscr{L} &= \partial_h\mathcal{L} - \partial_t(\partial_{\dot{h}}\mathcal{L}) - \nabla\cdot((\partial_{\dot{h}}\mathcal{L})\boldsymbol{u})\\ &= \frac{1}{2}\big(|\boldsymbol{u}|^2 + \frac{1}{3}(\dot{h} + \frac{3}{2}\dot{b})^2 + \frac{1}{4}\dot{b}^2\big) - g(h+b) - \partial_t\big(h(\frac{\dot{h}}{3} + \frac{\dot{b}}{2})\big) - \nabla\cdot\big(h(\frac{\dot{h}}{3} + \frac{\dot{b}}{2})\boldsymbol{u}\big),\end{aligned}$$

and

$$\begin{aligned}h\boldsymbol{K} = \delta_{\boldsymbol{u}}\mathscr{L} &= \partial_{\boldsymbol{u}}\mathcal{L} + (\partial_{\dot{h}}\mathcal{L})\nabla h + (\partial_{\dot{b}}\mathcal{L})\nabla b\\ &= h\boldsymbol{u} + h(\frac{\dot{h}}{3} + \frac{\dot{b}}{2})\nabla h + h(\frac{\dot{h}}{2} + \dot{b})\nabla b.\end{aligned}$$

Introducing

$$\boldsymbol{v} = \boldsymbol{K} + \nabla\left(h(\frac{\dot{h}}{3} + \frac{\dot{b}}{2})\right) = \boldsymbol{u} + \frac{1}{h}\nabla\Big(h^2(\frac{1}{3}\dot{h} + \frac{1}{2}\dot{b})\Big) + \Big(\frac{1}{2}\dot{h} + \dot{b}\Big)\nabla b$$

and using the above identities, eq. (8.11) reads

$$\partial_t\boldsymbol{v} + \boldsymbol{u}^{\perp}\operatorname{curl}\boldsymbol{v} + \nabla\left(\boldsymbol{v}\cdot\boldsymbol{u} - \frac{1}{2}|\boldsymbol{u}|^2 - \frac{1}{2}(\dot{h} + \dot{b})^2 + g(h+b)\right) = 0. \tag{8.13}$$

When $\partial_t b = 0$, and using that $\dot{h} = -h\nabla\cdot\boldsymbol{u}$ by the mass conservation, we recognize immediately eq. (8.10) (up to the non-dimensionalization scaling), with $\boldsymbol{v} = \nabla\psi$. In the general setting, one can recover eq. (8.9) after tedious algebra.

8.2 Group symmetries and preserved quantities

Thanks to the Hamiltonian structure of the Green–Naghdi system, and following the study concerning the water waves system in Section 2.2, Noether's theorem relate group symmetries and conserved quantities of the system. The physically relevant ones are listed below (see for instance [148, 246] for more detailed accounts).

Group symmetries If (ζ, ψ) is a solution to eq. (8.2), then for any $\theta \in \mathbb{R}$, $(\zeta^\theta, \psi^\theta)$ also satisfies eq. (8.2), where

- Variation of base level for the velocity potential:

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi + \theta\big)(t, \boldsymbol{x}).$$

- Horizontal translation along the direction $\boldsymbol{e} \in \mathbb{R}^d$ (in the flat bottom case)
$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t, \boldsymbol{x} - \theta \boldsymbol{e}).$$
- Time translation
$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t - \theta, \boldsymbol{x}).$$
- Galilean boost along the direction $\boldsymbol{e} \in \mathbb{R}^d$ (in the flat bottom case)
$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi + \theta \boldsymbol{e} \cdot \boldsymbol{x}\big)(t, \boldsymbol{x} - \theta \boldsymbol{e} t).$$
- Horizontal rotation (in dimension $d = 2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp \cdot \nabla b = 0$)
$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t, R_\theta \boldsymbol{x})$$
where R_θ is the rotation matrix of angle θ.

Preserved quantities We have the following corresponding preserved quantities.

- Excess of mass
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = 0, \qquad\qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \, \mathrm{d}\boldsymbol{x}.$$
- Horizontal impulse (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad\qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \nabla \psi \, \mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$
- Total energy
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\rm GN} = 0, \qquad\qquad \mathscr{H}_{\rm GN} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + (h \nabla \psi) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$
- Horizontal coordinate of mass centroid times mass (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{C} = \int_{\mathbb{R}^d} \zeta \nabla \psi \, \mathrm{d}\boldsymbol{x}, \qquad\qquad \mathscr{C} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$
- Angular impulse (in dimension $d = 2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp \cdot \nabla b = 0$)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{A} = 0, \qquad\qquad \mathscr{A} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x}^\perp \cdot \nabla \psi \, \mathrm{d}\boldsymbol{x}.$$
where $(x, y)^\perp \stackrel{\text{def}}{=} (-y, x)$.

Notice also the following conserved quantity which is seemingly trivial in the formulation (8.2) but not in the formulation (8.6):
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V} = 0, \qquad\qquad \mathscr{V} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \nabla \psi \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \boldsymbol{u} + \mu \mathcal{T}[h, \beta \nabla b] \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$
See [195] for a detailed discussion. Moreover, in the flat bottom case, we deduce from the above the conservation of a quantity corresponding to the horizontal momentum (recall eq. (8.3))
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}_{\rm GN} = 0, \qquad \mathscr{M}_{\rm GN} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} h \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} h \nabla \psi \, \mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \nabla (h^3 \nabla \cdot \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} = \mathscr{V} + \mathscr{I} + 0.$$
The quantities presented here are preserved in a stronger sense: their integrand satisfies a *conservation law*, which we do not write out explicitly. We let the reader refer to [195, 148] for a more thorough account.

8.3 Modal analysis

The dispersion relation associated with eq. (8.9) (in the flat bottom case, $b \equiv 0$) when linearized about the trivial rest solution is

$$\omega(\boldsymbol{\xi})\Big(\omega(\boldsymbol{\xi})^2 - \frac{gd|\boldsymbol{\xi}|^2}{1+\frac{1}{3}d^2|\boldsymbol{\xi}|^2}\Big) = 0.$$

The solution $\omega(\boldsymbol{\xi}) = 0$ corresponds to the propagation of the "vorticity", $\mathrm{curl}(\boldsymbol{u} + \mathcal{T}[h, \nabla b]\boldsymbol{u})$, and is irrelevant to potential flows as the vorticity vanishes. The remaining modes approximate the ones of the water waves equations when $d|\boldsymbol{\xi}| \ll 1$ (see Figure 8.1), although the small-wavelength behavior ($d|\boldsymbol{\xi}| \gg 1$) is different. Notice the large-time behavior discussion of Section ii and dispersive estimates of Section 2.3 apply, *mutatis mutandis.*

Using dimensionless variables such as in eq. (8.6), the above becomes

$$\omega_{\mathrm{GN}}(\boldsymbol{\xi})\Big(\omega_{\mathrm{GN}}(\boldsymbol{\xi})^2 - \frac{|\boldsymbol{\xi}|^2}{1+\frac{\mu}{3}|\boldsymbol{\xi}|^2}\Big) = 0,$$

which should be compared with the corresponding dispersion relation of the dimensionless water waves system, eq. (2.7), linearized about the rest solution, that is

$$\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2 = \frac{1}{\sqrt{\mu}}|\boldsymbol{\xi}|\tanh(\sqrt{\mu}|\boldsymbol{\xi}|).$$

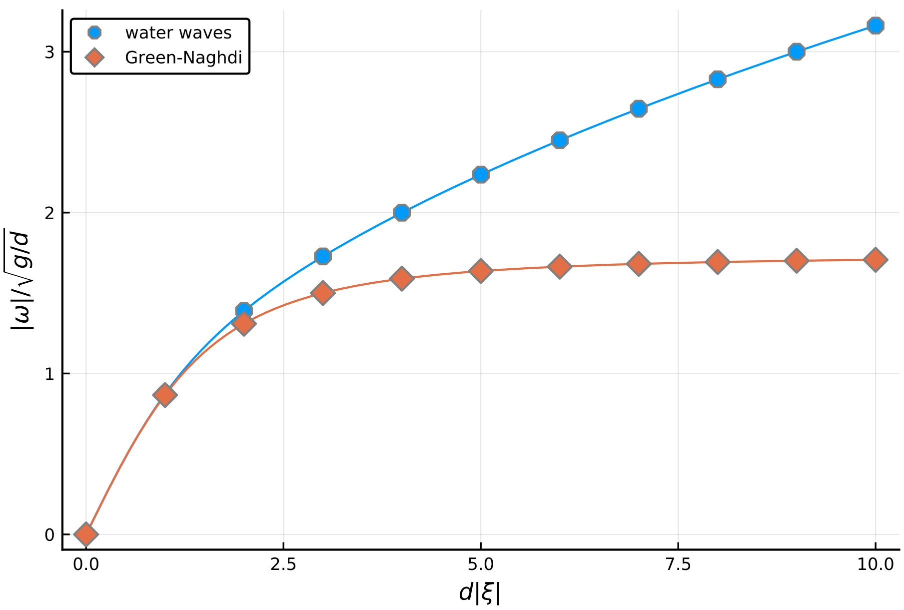


Figure 8.1: Non-trivial wave frequencies, $|\omega|(|\boldsymbol{\xi}|)$, solutions to the dispersion relation corresponding to the (linearized about rest) water waves and Green–Naghdi systems.

8.4 Solitary waves

In the unidimensional ($d = 1$) and flat bottom ($b \equiv 0$) framework, the Green–Naghdi system, eq. (8.2), enjoys an explicit family of solitary wave solutions,[40] that is satisfying

$$(\zeta, \psi)(t, x) = (\zeta_c, \psi_c)(x - ct), \qquad \lim_{|x|\to\infty} |(\zeta_c, \psi_c')|(x) = 0.$$

Denoting $h_c = 1 + \varepsilon\zeta_c$ and $\psi_c' = u_c - \frac{\mu}{3h_c}\big(h_c^3 u_c'\big)'$ and plugging the above Ansatz into eq. (8.2) yields

$$\begin{cases} -c\zeta_c' + (h_c u_c)' = 0, \\ -c\Big(u_c - \frac{\mu}{3h_c}\big(h_c^3 u_c'\big)'\Big)' + \zeta_c' + \frac{\varepsilon}{2}(u_c^2)' = \mu\varepsilon\Big(\frac{u_c}{3h_c}(h_c^3 u_c')' + \frac{1}{2}(h_c^2 u_c')^2\Big)'. \end{cases}$$

We may now integrate and, using the vanishing condition at infinity to set the integration constant, we deduce from the first equation

$$-c\zeta_c + h_c u_c = 0$$

and using this identity into the second equation yields

$$h_c - 1 - \frac{c^2}{2}\frac{h_c^2 - 1}{h_c^2} = \mu c^2\big(\frac{-1}{3h_c^2}(h_c h_c')' + \frac{1}{2h_c^2}(h_c')^2\big).$$

Multiplying with h_c' and once again integrating yields

$$(h_c - 1)^2(c^2 - h_c) = \frac{\mu c^2}{3}(h_c')^2.$$

Since $h_c \to 1$ as $|x| \to \infty$, the differential equation has a real solution only if $c > 1$, in which case there exists a unique (up to translations) solution given by

$$h_c(x) = 1 + (c^2 - 1)\,\mathrm{sech}^2\left(\sqrt{\frac{3(c^2-1)}{4c^2}}\,\frac{x - x_\star}{\sqrt{\mu}}\right), \qquad \varepsilon\zeta_c = h_c - 1, \; \varepsilon u_c = \frac{c(h_c - 1)}{h} \tag{8.14}$$

This explicit solution was provided as early as in [383]. It should be compared with the infamous solitary wave solutions to the (right-going) Korteweg-de Vries equation

$$\partial_t\zeta_{\rm KdV} + \partial_x\zeta_{\rm KdV} + \frac{3\varepsilon}{4}\partial_x\big(\zeta_{\rm KdV}^2\big) + \frac{\mu}{6}\partial_x^3\zeta_{\rm KdV},$$

namely $\zeta_{\rm KdV}(t, x) = \zeta_{c,{\rm KdV}}(x - ct)$ with

$$\varepsilon\,\zeta_{c,{\rm KdV}}(x) = 2(c - 1)\,\mathrm{sech}^2\left(\sqrt{\frac{6(c-1)}{4}}\,\frac{x - x_\star}{\sqrt{\mu}}\right).$$

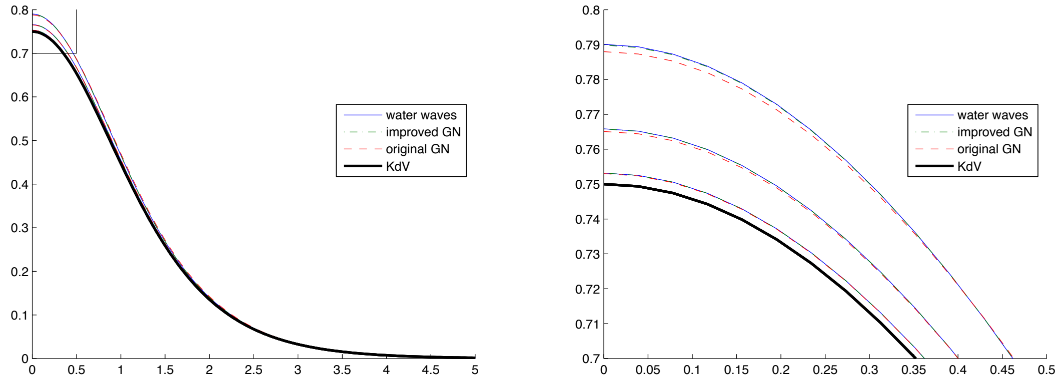
By Theorem 8.7 the above solutions provide good approximations of the traveling waves of the exact water waves equations, eq. (2.7), when $c - 1 \approx \varepsilon \approx \mu \ll 1$, that is in the long wave regime (see Definition III.3). See Figure 8.2 for a comparison of the solitary waves at a given velocity.

8.5 Rigorous justification

In this section we provide a complete rigorous justification of the Green–Naghdi system, eq. (8.2) and hence equivalently eq. (8.6), as an asymptotic model for the water waves system, eq. (2.7), in the shallow water regime (Definition III.2) that is for parameters in the set

$$\mathfrak{p}_{\rm SW} = \big\{(\mu, \varepsilon, \beta) \;:\; \mu \in (0, \mu^\star],\; \varepsilon \in [0, 1],\; \beta \in [0, 1]\big\}.$$

[40] The following discussion can be extended to construct cnoidal (*i.e.* periodic) traveling waves; see *e.g.* [79, 196].



(a) Rescaled solitary waves for $c = 1.025$, 1.01, 1.002.

(b) Close-up.

Figure 8.2: Comparison of the solutions of the KdV and Green–Naghdi models and the water waves system (the latter is numerically computed), taken from [165]. The waves are rescaled so that the Korteweg-de Vries solution does not depend on c. Consistently, we set $\varepsilon = \mu = 1$. The "improved" Green–Naghdi system is the Whitham–Green–Naghdi system presented in Section 10 and cannot be distinguished from the water waves solution.

The complete justification follows from several results: (i) a *consistency* result stating that exact solutions to the water waves system satisfy approximately the Green–Naghdi equations; (ii) a (local) *well-posedness* result on the initial-value problem for the Green–Naghdi equations which should be uniform in the shallow water regime; and (iii) a *stability* result controlling the difference between an approximate and an exact solution to the Green–Naghdi equations. Altogether, this yields the target *convergence* result which estimates the difference between solutions to the water waves system—which exist on the relevant timescale and satisfy the required bounds by Theorem 2.9—and the corresponding solutions to the Green–Naghdi model.

In order for eq. (8.2) to make sense as evolution equations, one needs first to ensure that $\mathfrak{T}^\mu$ is invertible. As a matter of fact, robust and quantitative information on the operator, $\mathfrak{T}^\mu$, and its inverse, will be crucial in our proofs. To this aim, we first introduce some relevant functional spaces. For $s \in \mathbb{N}$ we denote

$$X^s_\mu \stackrel{\text{def}}{=} \{\boldsymbol{u} \in L^2(\mathbb{R}^d)^d \ : \ \big|\boldsymbol{u}\big|^2_{X^s_\mu} \stackrel{\text{def}}{=} \sum_{|\boldsymbol{k}|=0}^{s} \big|\partial^{\boldsymbol{k}}\boldsymbol{u}\big|^2_{L^2} + \mu\big|\partial^{\boldsymbol{k}}\nabla\cdot\boldsymbol{u}\big|^2_{L^2} < \infty\},$$

$$Y^s_\mu \stackrel{\text{def}}{=} \{\boldsymbol{v} \in (X^0_\mu)' \ : \ \big|\boldsymbol{v}\big|^2_{Y^s_\mu} \stackrel{\text{def}}{=} \sum_{|\boldsymbol{k}|=0}^{s} \big|\partial^{\boldsymbol{k}}\boldsymbol{v}\big|^2_{(X^0_\mu)'} < \infty\}.$$

It turns out—see Lemma 8.9 and Lemma 8.10—that the operator $\mathfrak{T}^\mu[h, \beta\nabla b] : X^s_\mu \to Y^s_\mu$ is well-defined, one-to-one and onto provided that ζ, b are sufficiently regular and the non-cavitation assumption holds:

Assumption 8.1. *We have $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ and satisfy*

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad h(\boldsymbol{x}) \stackrel{\text{def}}{=} 1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0.$$

Theorem 8.2 (Consistency)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. Let $s \in \mathbb{N}$ and $M^\star \geq 0$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{SW}}$, any $b \in W^{\max\{s+6, 2+s_\star\}}(\mathbb{R}^d)$, any $T > 0$ and any*

$(\zeta,\psi)\in L^\infty(0,T;H^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)\times \mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d))$ *classical solution to the water waves equations, eq.* (2.7)*, satisfying Assumption* 8.1 *uniformly for* $t\in(0,T)$ *and*

$$\operatorname*{ess\,sup}_{t\in(0,T)}\big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{2+s_\star}}+\big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\big)+\big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}\leq M^\star,$$

one has

$$\begin{cases}\partial_t\zeta+\nabla\cdot(h\boldsymbol{u})=r_1,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\boldsymbol{u}|^2-\mu\varepsilon\mathcal{R}[h,\beta\nabla b,\boldsymbol{u}]=r_2,\end{cases}$$

where we denote $h=1+\varepsilon\zeta-\beta b$*,* $\boldsymbol{u}=\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}(h\nabla\psi)$*, and one has for almost every* $t\in(0,T)$

$$\big|r_1(t,\cdot)\big|_{H^s}\leq C\,\mu^2\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}}+\big|\nabla\psi(t,\cdot)\big|_{H^{s+5}}\big),$$
$$\big|r_2(t,\cdot)\big|_{H^{s+1}}\leq C\,\mu^2\varepsilon\,\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}}+\big|\nabla\psi(t,\cdot)\big|_{H^{s+5}}\big).$$

Proof. The control of r_1 is a obtained noticing that

$$\begin{aligned}\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi+\nabla\cdot\big(h\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\Big|_{H^s}&\\ \leq\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi+\nabla\cdot\big(h(\operatorname{Id}-\mu\mathcal{T}[h,\beta\nabla b])\nabla\psi\big)\Big|_{H^s}&\\ +\mu^2\big|\nabla\cdot\big(h\mathcal{T}[h,\beta\nabla b]\mathcal{T}[h,\beta\nabla b]\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\big|_{H^s}.&\end{aligned}$$

The first term is estimated by Proposition 4.10 (with $n=2$ and $k=s+1$), and we can use Lemma 8.10 to estimate the second term. The control of r_2 is obtained in the same way, using Proposition 4.10 with $n=1$ and $k=s+2$ as well as $k=1+s_\star$, together with Lemma 8.10 and the product and composition estimates in Appendix II. ☐

The local well-posedness of system (8.2) (or, equivalently, system (8.6)) has been proved in [284, 228] (dimension $d=1$, flat bottom), [234, 229] (dimension $d=1$), [15] (existence and uniqueness of a solution "with loss of derivatives" through a Nash–Moser scheme), and [189, 162] (general case). We provide a detailed proof based on [189] in Section 8.6.

Theorem 8.3 (Local well-posedness)**.** *Let* $d\in\mathbb{N}^\star$*,* $s_\star>d/2$ *and* $s\in\mathbb{N}$*,* $s\geq 1+s_\star$*,* $h_\star>0$*,* $\mu^\star>0$*, and* $M^\star\geq 0$*. There exist* $T>0$ *and* $C>0$ *such that the for any* $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$*, any* $b\in W^{s+1,\infty}(\mathbb{R}^d)$*, and any* $(\zeta_0,\boldsymbol{u}_0)\in H^s(\mathbb{R}^d)\times X^s_\mu$ *satisfying Assumption* 8.1 *and*

$$M_0\stackrel{\rm def}{=}\big|\varepsilon\zeta_0\big|_{H^{1+s_\star}}+\big|\varepsilon\boldsymbol{u}_0\big|_{X^{1+s_\star}_\mu}+\big|\beta b\big|_{W^{s+1,\infty}}\leq M^\star,$$

there exists a unique $(\zeta,\boldsymbol{u})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times X^s_\mu)\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times X^{s-1}_\mu)$ *classical solution to the Green–Naghdi system, eq.* (8.6)*, with initial data* $(\zeta,\boldsymbol{u})\mid_{t=0}=(\zeta_0,\boldsymbol{u}_0)$*; and we have for any* $t\in[0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^s}+\big|\boldsymbol{u}(t,\cdot)\big|_{X^s_\mu}\leq C\times\Big(\big|\zeta_0\big|_{H^s}+\big|\boldsymbol{u}_0\big|_{X^s_\mu}\Big)$$

and $\inf_{\boldsymbol{x}\in\mathbb{R}^d}1+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x})\geq h_\star/2$*.*

Moreover, denoting $\boldsymbol{v}=h^{-1}\mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u}$ *with* $h=1+\varepsilon\zeta-\beta b$ *and* $\mathfrak{T}^\mu$ *defined in eq.* (8.5)*, we have that* $(\zeta,\boldsymbol{v})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times Y^s_\mu)\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times Y^{s-1}_\mu)$ *is a classical solution to eq.* (8.2) *(applying the gradient to the second equation).*

Remark 8.4. *Uniqueness in Theorem* 8.3 *allows to define* $T_{\max}$ *the supremum of* $T>0$ *such that the Cauchy problem has a solution* $(\zeta,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^s(\mathbb{R}^d)\times X^s_\mu)\cap\mathcal{C}^1([0,T];H^{s-1}(\mathbb{R}^d)\times X^{s-1}_\mu)$*. We also have the blowup criterion*

$$T_{\max}<\infty\quad\Longrightarrow\quad\lim_{t\nearrow T_{\max}}\Big(\|\zeta\|_{L^\infty(0,t;H^{1+s_\star})}+\|\boldsymbol{u}\|_{L^\infty(0,t;X^{1+s_\star}_\mu)}\Big)\to\infty,$$

since the hyperbolicity criterion remains satisfied as a consequence of the conservation of mass; see footnote 6 page vii. In particular, for given initial data, $T_{\max}$ and the maximal solution do not depend on the choice of the regularity index, $s > 1 + d/2$.

Remark 8.5. *The continuity of the flow map (and hence well-posedness in the sense of Hadamard)*

$$\varphi^t : (\zeta_0, \boldsymbol{u}_0) \in H^s(\mathbb{R}^d) \times X^s_\mu \mapsto (\zeta(t,\cdot), \boldsymbol{u}(t,\cdot)) \in H^s(\mathbb{R}^d) \times X^s_\mu$$

does hold, and can be obtained using the so-called Bona–Smith technique [55]. As stated earlier, it is not significant for our purposes, where we are happy to ask an extra derivative on the initial data to ensure that the flow map is Lipschitz. This result is a particular case of the stability property, Theorem 8.6, below.

Theorem 8.6 (Stability)**.** *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $M^\star \geq 0$, and denote $n_0 \stackrel{\rm def}{=} \max\{s, 1+s_\star\}$, $n \stackrel{\rm def}{=} \max\{s+1, 1+s_\star\}$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $b \in W^{n,\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and $(\zeta^0, \boldsymbol{u}^0) \in \mathcal{C}^0([0,T^\star]; H^{n_0}(\mathbb{R}^d) \times X^{n_0}_\mu)$ satisfying the Green–Naghdi system, eq. (8.6), and any $(\zeta, \boldsymbol{u}) \in L^\infty(0, T^\star; H^n(\mathbb{R}^d) \times X^n_\mu)$ satisfying*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\boldsymbol{u}) = r, \\ \big(\operatorname{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\partial_t \boldsymbol{u} + \nabla\zeta + \varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\varepsilon\mathcal{Q}[h, \beta\nabla b, \boldsymbol{u}] = \boldsymbol{r}, \end{cases} \tag{8.15}$$

with $(r, \boldsymbol{r}) \in L^1(0, T^\star; H^s(\mathbb{R}^d) \times Y^s_\mu)$, and assuming that $h = 1 + \varepsilon\zeta - \beta b$ and $h^0 = 1 + \varepsilon\zeta^0 - \beta b$ satisfy Assumption 8.1 uniformly for $t \in [0, T^\star]$ and

$$M \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{t\in[0,T^\star]} \Big(\big|(\varepsilon\zeta, \varepsilon\boldsymbol{u})(t,\cdot)\big|_{H^n \times X^n_\mu} + \big|(\varepsilon\zeta^0, \varepsilon\boldsymbol{u}^0)(t,\cdot)\big|_{H^{n_0}\times X^{n_0}_\mu}\Big) + \big|\beta b\big|_{W^{n,\infty}} \leq M^\star,$$

then one has for any $t \in [0, T^\star]$,

$$\begin{aligned}\big|(\zeta - \zeta^0)(t,\cdot)\big|_{H^s} + \big|(\boldsymbol{u} - \boldsymbol{u}^0)(t,\cdot)\big|_{X^s_\mu} \leq C e^{CMt} \Big(\big|(\zeta - \zeta^0)(t=0,\cdot)\big|_{H^s} + \big|(\boldsymbol{u} - \boldsymbol{u}^0)(t=0,\cdot)\big|_{X^s_\mu}\Big) \\ + C\int_0^t e^{CM(t-\tau)} \big(\big|r(\tau,\cdot)\big|_{H^s} + \big|\boldsymbol{r}(\tau,\cdot)\big|_{Y^s_\mu}\big)\,{\rm d}\tau\,.\end{aligned}$$

The following result is a direct consequence of Theorem 8.2, Theorem 8.3 and Theorem 8.6.

Theorem 8.7 (Convergence)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $s \in \mathbb{N}$ and $M^\star \geq 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $b \in W^{\max\{s+6, 2+s_\star\},\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and any $(\zeta, \psi) \in \mathcal{C}^0([0,T^\star]; H^{\max\{s+6,2+s_\star\}} \times \mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d))$ solution to the water waves equations (2.7) and such that $h = 1 + \varepsilon\zeta - \beta b$ satisfies Assumption 8.1 uniformly for $t \in [0, T^\star]$ and*

$$M \stackrel{\rm def}{=} \sup_{t\in[0,T^\star]} \big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}} + \big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{\max\{s+1,1+s_\star\}}}\big) + \big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}} \leq M^\star,$$

there exists a unique $(\zeta_{\rm GN}, \boldsymbol{u}_{\rm GN}) \in \mathcal{C}^0([0, T/M]; H^{\max\{s,1+s_\star\}}(\mathbb{R}^d) \times X^{\max\{s,1+s_\star\}}_\mu)$ strong solution to the Green–Naghdi system (8.6) with initial data $\big(\zeta_{\rm GN}, \boldsymbol{u}_{\rm GN}\big)\big|_{t=0} = \big(\zeta, \mathfrak{T}^\mu[h, \beta\nabla b]^{-1}(h\nabla\psi)\big)\big|_{t=0}$ (see eq. (8.5)); and one has for any $t \in (0, \min\{T^\star, T/M\}]$,

$$\big|(\zeta - \zeta_{\rm GN})(t,\cdot)\big|_{H^s} + \big|(\nabla\psi - \boldsymbol{v}_{\rm GN})(t,\cdot)\big|_{Y^s_\mu} \leq C\,\mu^2\,t\,\big(\big\|\zeta\big\|_{L^\infty(0,t;H^{s+6})} + \big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+5})}\big)\,,$$

where we denote $\boldsymbol{v}_{\rm GN} = h_{\rm GN}^{-1}\mathfrak{T}^\mu[h_{\rm GN}, \beta\nabla b]\boldsymbol{u}_{\rm GN}$ and $h_{\rm GN} = 1 + \varepsilon\zeta_{\rm GN} - \beta b$.

8.6 Well-posedness

In this section we provide a proof of Theorem 8.3, and hence complete the analysis of Section 8.5. The strategy mimics the standard "energy method" for hyperbolic first-order quasilinear systems [13, 49, 310], with specific adjustments due to the presence of high order differential operators. In particular, we shall not use dispersive techniques such as Strichartz estimates (see for instance [397]), because we aim at results uniform with respect to $\mu \in (0, \mu^\star)$. This strategy is typical in the study of shallow water or long wave models for the water waves system, and even the study of the water waves system in the shallow water regime, Definition III.2; see [224, 268].

Let us very roughly sketch a typical strategy concerning hyperbolic symmetrizable first-order quasilinear systems (such as the Saint-Venant system; see Section 5). Consider a system of the form

$$\mathcal{S}(\boldsymbol{U})\partial_t \boldsymbol{U} + \mathcal{S}_x(\boldsymbol{U})\partial_x \boldsymbol{U} + \mathcal{S}_y(\boldsymbol{U})\partial_y \boldsymbol{U} = 0$$

where $\boldsymbol{U}(t,x,y) \in \mathbb{R}^n$ and $\mathcal{S}, \mathcal{S}_x, \mathcal{S}_y$ are smooth functions with values into $n \times n$ symmetric matrices. The Picard iteration scheme consists in proving that we can define a sequence $\boldsymbol{U}_n$ by solving inductively the linearized system:

$$\mathcal{S}(\boldsymbol{U}_n)\partial_t \boldsymbol{U}_{n+1} + \mathcal{S}_x(\boldsymbol{U}_n)\partial_x \boldsymbol{U}_{n+1} + \mathcal{S}_y(\boldsymbol{U}_n)\partial_y \boldsymbol{U}_{n+1} = 0,$$

and that the sequence converges (up to taking a subsequence) towards the desired solution of the nonlinear equation. Both the well-posedness of the Cauchy problem associated with the linearized system and the convergence result rely on robust *a priori* estimates, which can be derived as follows. Consider sufficiently smooth and decaying solutions of the system

$$\mathcal{S}(\underline{\boldsymbol{U}})\partial_t \boldsymbol{U} + \mathcal{S}_x(\underline{\boldsymbol{U}})\partial_x \boldsymbol{U} + \mathcal{S}_y(\underline{\boldsymbol{U}})\partial_y \boldsymbol{U} = 0.$$

Testing the equation against $\boldsymbol{U}$, integrating by parts and using the symmetry properties, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}(\underline{\boldsymbol{U}})\,\boldsymbol{U}\,,\,\boldsymbol{U}\big)_{L^2} \leq C(\big|\underline{\boldsymbol{U}}\big|_{W^{1,\infty}}, \big|\partial_t \underline{\boldsymbol{U}}\big|_{L^\infty})\big|\boldsymbol{U}\big|_{L^2}^2.$$

In order to control higher order derivatives, we may differentiate the system $\boldsymbol{k}$ times (where $\boldsymbol{k} \in \mathbb{N}^d$ is a multiindex) and test against $\partial^{\boldsymbol{k}}\boldsymbol{U}$. Using the gain of one derivative from commutator estimates (see Proposition II.9), we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}(\underline{\boldsymbol{U}})\,\partial^{\boldsymbol{k}}\boldsymbol{U}\,,\,\partial^{\boldsymbol{k}}\boldsymbol{U}\big)_{L^2} \leq C(\big|\underline{\boldsymbol{U}}\big|_{H^{\max\{|\boldsymbol{k}|,1+s_\star\}}}, \big|\partial_t \underline{\boldsymbol{U}}\big|_{H^{\max\{|\boldsymbol{k}|-1,s_\star\}}})\big|\boldsymbol{U}\big|_{H^{|\boldsymbol{k}|}}^2$$

where $s_\star > d/2$. By considering the above with any $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| \in \{0, \dots, s\}$ where $s \in \mathbb{N}$, $s \geq 1 + s_\star > 1 + d/2$, and provided that $\mathcal{S}(\underline{\boldsymbol{U}})$ is uniformly positive definite, we deduce a control of $\boldsymbol{U} \in L^\infty(0,T;H^s)$ (by Gronwall's estimate) and hence $\partial_t \boldsymbol{U} \in L^\infty(0,T;H^{s-1})$ (using the system of equations). This is our desired *a priori* energy estimate which, thanks to a regularization and a limiting procedure, eventually yield the well-posedness of the Cauchy problem for the linearized system. Because the control asked on the reference state $\underline{\boldsymbol{U}}$ is the same as the control we provide on the solution, $\boldsymbol{U}$, we may expect that the Picard iteration scheme converges, towards a solution of the nonlinear system.

When trying to adapt the strategy to the Green–Naghdi system, eq. (8.6), the main objection to robust energy estimates stem from the presence of nonlinear high order differential operators, as the gain of one derivative due to commutator estimates is in principle insufficient to treat commutators as order-zero remainder terms. This problem is however only apparent, and a careful study of these high order operators reveal that they are in fact of order one when considering the correct functional spaces, X^n_μ and Y^n_μ.

In Proposition 8.11, we extract the quasilinear structure of the Green–Naghdi system, which in this case is nothing but the linearized system about *constant* states, just as with hyperbolic systems. We then provide in Proposition 8.12 the key *a priori* energy estimates of the linearized system. Finally, we detail in Section 8.6.4 the proof, via a parabolic regularization of the equations, of the (local-in-time) existence and uniqueness of a solution to the Cauchy problem.

8.6.1 Some technical tools

Let us first introduce some technical tools on the functional spaces X^n_μ, Y^n_μ, and the operator $\mathfrak{T}^\mu$.

Lemma 8.8. *Let $s \in \mathbb{N}$. We have the continuous embeddings $H^{s+1}(\mathbb{R}^d)^d \subset X^s_\mu \subset H^s(\mathbb{R}^d)^d$ and $H^s(\mathbb{R}^d)^d \subset Y^s_\mu \subset H^{s-1}(\mathbb{R}^d)^d$. There exists C_s, independent of $\mu > 0$, such that the following inequalities hold as soon as the right-hand side is finite:*

$$\big|\boldsymbol{u}\big|_{H^s} \leq \big|\boldsymbol{u}\big|_{X^s_\mu}, \qquad\qquad \big|\boldsymbol{u}\big|_{X^s_\mu} \leq C_s\big|\boldsymbol{u}\big|_{H^{s+1}}, \tag{8.16}$$

$$\big|\boldsymbol{v}\big|_{H^{s-1}} \leq C_s\big|\boldsymbol{v}\big|_{Y^s_\mu}, \qquad\qquad \big|\boldsymbol{v}\big|_{Y^s_\mu} \leq \big|\boldsymbol{v}\big|_{H^s}. \tag{8.17}$$

We also have the non-uniform continuous embedding

$$\big|\nabla f\big|_{Y^s_\mu} \leq C_s\frac{1}{\sqrt{\mu}}\big|f\big|_{H^s}, \qquad\qquad \big|\nabla\cdot\boldsymbol{u}\big|_{H^s} \leq C_s\frac{1}{\sqrt{\mu}}\big|\boldsymbol{u}\big|_{X^s_\mu}. \tag{8.18}$$

Proof. The continuous embeddings $H^1(\mathbb{R}^d)^d \subset X^0_\mu \subset L^2(\mathbb{R}^d)^d$ are straightforward, and the corresponding $L^2(\mathbb{R}^d)^d \subset Y^0_\mu \subset H^{-1}(\mathbb{R}^d)^d$ follow by duality. The estimate (8.18) with $n = 0$ is easily checked, as for any $\boldsymbol{u} \in X^0_\mu$,

$$\big|\langle\nabla f, \boldsymbol{u}\rangle_{(X^0_\mu)'-X^0_\mu}\big| = \big|\big(f, \nabla\cdot\boldsymbol{u}\big)_{L^2}\big| \leq \frac{1}{\sqrt{\mu}}\big|f\big|_{L^2}\big|\boldsymbol{u}\big|_{X^0_\mu}.$$

The case $s \in \mathbb{N}^\star$ is reduced to the case $s = 0$ by considering $\partial^{\boldsymbol{k}}\boldsymbol{u}, \partial^{\boldsymbol{k}}\boldsymbol{v}, \partial^{\boldsymbol{k}} f$ with $0 \leq |\boldsymbol{k}| \leq s$. □

Lemma 8.9. *Let $h_\star > 0$, $\mu^\star > 0$ and $M > 0$. Then there exists $C > 0$ such that for any $(\varepsilon, \beta, \mu) \in \mathcal{P}_{\rm SW}$, any $b \in W^{1,\infty}$ and $h \in L^\infty$ satisfying Assumption 4.1 and*

$$\big|h\big|_{L^\infty} + \big|\beta\nabla b\big|_{L^\infty} \leq M,$$

$\mathfrak{T}^\mu[h, \beta\nabla b] : X^0_\mu \to (X^0_\mu)'$ is a well-defined topological isomorphism, and one has

$$\begin{aligned}
&\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in X^0_\mu, && \langle\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}_1, \boldsymbol{u}_2\rangle_{(X^0_\mu)'-X^0_\mu} = \langle\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}_2, \boldsymbol{u}_1\rangle_{(X^0_\mu)'},\\
&\forall \boldsymbol{u} \in X^0_\mu, && \big|\mathfrak{T}^\mu[h, \beta\nabla b]\big|_{(X^0_\mu)'} \leq C\,\big|\boldsymbol{u}\big|_{X^0_\mu},\\
&\forall \boldsymbol{v} \in (X^0_\mu)', && \big|\mathfrak{T}^\mu[h, \beta\nabla b]^{-1}\boldsymbol{v}\big|_{X^0_\mu} \leq C\,\big|\boldsymbol{v}\big|_{(X^0_\mu)'}.
\end{aligned}$$

Proof. We establish the estimates for $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}, \boldsymbol{v} \in \mathcal{S}(\mathbb{R}^d)^d$ so that all the terms are well-defined, and the $((X^0_\mu)' - X^0_\mu)$ duality product coincides with the L^2 inner product. The result for less regular functions is obtained by density of $\mathcal{S}(\mathbb{R}^d)^d$ in X^0_μ and continuous linear extension.

By definition of $\mathfrak{T}^\mu$ in (8.5) and after integration by parts, one has

$$\begin{aligned}
\big(\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}_1, \boldsymbol{u}_2\big)_{L^2} = \int_{\mathbb{R}^d} & h\boldsymbol{u}_1\cdot\boldsymbol{u}_2 + \frac{\mu}{3}h^3(\nabla\cdot\boldsymbol{u}_1)(\nabla\cdot\boldsymbol{u}_2)\\
& - \frac{\mu}{2}h^2\big((\nabla\cdot\boldsymbol{u}_2)(\beta\nabla b\cdot\boldsymbol{u}_1) + (\beta\nabla b\cdot\boldsymbol{u}_2)(\nabla\cdot\boldsymbol{u}_1)\big) + \mu h(\beta\nabla b\cdot\boldsymbol{u}_1)(\beta\nabla b\cdot\boldsymbol{u}_2),
\end{aligned}$$

from which the symmetry is evident. Applying Cauchy–Schwarz inequality, we have

$$\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in X^0_\mu, \quad \big|\langle\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}_1, \boldsymbol{u}_2\rangle_{(X^0_\mu)'-X^0_\mu}\big| \leq C(\big|h\big|_{L^\infty}, \big|\beta\nabla b\big|_{L^\infty})\big|\boldsymbol{u}_1\big|_{X^0_\mu}\big|\boldsymbol{u}_2\big|_{X^0_\mu},$$

and the first estimate follows by duality. The second one is obtained when rewriting

$$\big(\mathfrak{T}^\mu[h, \beta\nabla b]\boldsymbol{u}, \boldsymbol{u}\big)_{L^2} = \int_{\mathbb{R}^d} h|\boldsymbol{u}|^2 + \frac{\mu}{12}h^3|\nabla\cdot\boldsymbol{u}|^2 + \frac{\mu}{4}h|h\nabla\cdot\boldsymbol{u} - 2\beta\nabla b\cdot\boldsymbol{u}|^2.$$

This shows that $\mathfrak{T}^\mu[h,\beta\nabla b] : X^0_\mu \to (X^0_\mu)'$ is continuous and coercive, so that the operator version of Lax-Milgram theorem ensures that $\mathfrak{T}^\mu[h,\beta\nabla b]$ is an isomorphism. The continuity of the inverse is a consequence of the coercivity of $\mathfrak{T}^\mu[h,\beta\nabla b]$:

$$\big|\boldsymbol{u}\big|^2_{X^0_\mu} \le C(h_\star^{-1})|\langle \mathfrak{T}^\mu[h,\beta b]\boldsymbol{u},\boldsymbol{u}\rangle_{(X^0_\mu)'}| \le \big|\mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u}\big|_{(X^0_\mu)'}\big|\boldsymbol{u}\big|_{X^0_\mu},$$

and setting $\boldsymbol{u} = \mathfrak{T}^\mu[h,\beta\nabla b]^{-1}\boldsymbol{v}$ above. □

Lemma 8.10. *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}^\star$, $s_\star > d/2$, and $h_\star > 0$, $M > 0$, $\mu^\star > 0$. There exists $C > 0$ such that for any $(\varepsilon,\beta,\mu) \in \mathcal{P}_{\rm SW}$, for any $b \in W^{\max\{s+1,2+s_\star\},\infty}(\mathbb{R}^d)$* [41] *and $\zeta \in H^{\max\{s,1+s_\star\}}(\mathbb{R}^d)$ satisfying Assumption 4.1 and*

$$\big|\varepsilon\zeta\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \le M,$$

the following holds.

- *For any $\boldsymbol{u} \in X^s_\mu$, $\mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u} \in Y^s_\mu$ and*

$$\big|\mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u}\big|_{Y^s_\mu} \le C \times \Big(\big|\boldsymbol{u}\big|_{X^s_\mu} + \Big\langle (\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{u}\big|_{X^{s_\star}_\mu}\Big\rangle_{s>s_\star}\Big).$$

- *For any $\boldsymbol{v} \in Y^s_\mu$. Then $\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}\boldsymbol{v} \in X^s_\mu$ and*

$$\big|\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}\boldsymbol{v}\big|_{X^s_\mu} \le C \times \Big(\big|\boldsymbol{v}\big|_{Y^s_\mu} + \Big\langle (\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{v}\big|_{Y^{s_\star}_\mu}\Big\rangle_{s>1+s_\star}\Big).$$

Proof. Let us denote for simplicity $\mathfrak{T}^\mu \stackrel{\rm def}{=} \mathfrak{T}^\mu[h,\beta\nabla b]$. We also introduce $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| = s$. We have for any $\boldsymbol{u},\boldsymbol{w} \in \mathcal{S}(\mathbb{R}^d)^d$,

$$\begin{aligned}|\big(\partial^{\boldsymbol{k}}(\mathfrak{T}^\mu\boldsymbol{u}),\boldsymbol{w}\big)_{L^2}| = \big(\partial^{\boldsymbol{k}}(h\boldsymbol{u}),\boldsymbol{w}\big)_{L^2} &+ \frac{\mu}{3}\big(\partial^{\boldsymbol{k}}(h^3\nabla\cdot\boldsymbol{u}),\nabla\cdot\boldsymbol{w}\big)_{L^2}\\ &- \frac{\mu}{2}\big(\partial^{\boldsymbol{k}}(h^2(\beta\nabla b)\cdot\boldsymbol{u}),\nabla\cdot\boldsymbol{w}\big)_{L^2} - \frac{\mu}{2}\big(\partial^{\boldsymbol{k}}(h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}),\boldsymbol{w}\big)_{L^2}\\ &+ \mu\big(\partial^{\boldsymbol{k}}(h(\beta\nabla b)(\beta\nabla b)\cdot\boldsymbol{u}),\boldsymbol{w}\big)_{L^2}.\end{aligned}$$

Hence, by product estimates, *i.e.* Proposition II.7 and Proposition II.14, we have

$$|\big(\partial^{\boldsymbol{k}}(\mathfrak{T}^\mu\boldsymbol{u}),\boldsymbol{w}\big)_{L^2}| \le C(M)\Big(\big|\boldsymbol{u}\big|_{X^s_\mu} + \Big\langle (\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{u}\big|_{X^{s_\star}_\mu}\Big\rangle_{s>s_\star}\Big)\big|\boldsymbol{w}\big|_{X^0_\mu}.$$

The first result is deduced by density and continuity arguments. Now, notice

$$\begin{aligned}|\big([\partial^{\boldsymbol{k}},\mathfrak{T}^\mu]\boldsymbol{u},\boldsymbol{w}\big)_{L^2}| = \big([\partial^{\boldsymbol{k}},h]\boldsymbol{u},\boldsymbol{w}\big)_{L^2} &+ \frac{\mu}{3}\big([\partial^{\boldsymbol{k}},h^3]\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{w}\big)_{L^2}\\ &- \frac{\mu}{2}\big([\partial^{\boldsymbol{k}},h^2(\beta\nabla b)\cdot]\boldsymbol{u},\nabla\cdot\boldsymbol{w}\big)_{L^2} - \frac{\mu}{2}\big([\partial^{\boldsymbol{k}},h^2(\beta\nabla b)]\nabla\cdot\boldsymbol{u},\boldsymbol{w}\big)_{L^2}\\ &+ \mu\big([\partial^{\boldsymbol{k}},h(\beta\nabla b)(\beta\nabla b)\cdot]\boldsymbol{u},\boldsymbol{w}\big)_{L^2}.\end{aligned}$$

Using commutator estimates, Proposition II.9 and Proposition II.15, we deduce

$$|\big([\partial^{\boldsymbol{k}},\mathfrak{T}^\mu]\boldsymbol{u},\boldsymbol{w}\big)_{L^2}| \le C(M)\Big(\big|\boldsymbol{u}\big|_{X^{s-1}_\mu} + \Big\langle (\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{u}\big|_{X^{s_\star}_\mu}\Big\rangle_{s>1+s_\star}\Big)\big|\boldsymbol{w}\big|_{X^0_\mu}.$$

[41] the result holds as well assuming instead that $b \in L^\infty(\mathbb{R}^d) \cap H^{\max\{s+1,2+s_\star\},\infty}(\mathbb{R}^d)$, replacing $\big|\beta b\big|_{W^{2+s_\star,\infty}}$ with $\big|\beta\nabla b\big|_{H^{1+s_\star}} + \big|\beta b\big|_{L^\infty}$, and $\big|\beta\nabla b\big|_{W^{s,\infty}}$ with $\big|\beta\nabla b\big|_{H^s}$.

By density and continuity arguments, we infer that for any $\boldsymbol{u} \in X_\mu^{s-1}$, $\big[\partial^{\boldsymbol{k}}, \mathfrak{T}^\mu\big]\boldsymbol{u} \in (X_\mu^0)'$ and

$$\big|\big[\partial^{\boldsymbol{k}}, \mathfrak{T}^\mu\big]\boldsymbol{u}\big|_{(X_\mu^0)'} \leq C(M)\Big(\big|\boldsymbol{u}\big|_{X_\mu^{s-1}} + \Big\langle(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{u}\big|_{X_\mu^{s_\star}}\Big\rangle_{s>1+s_\star}\Big).$$

Now, we make use of the identity

$$\big[\partial^{\boldsymbol{k}}, (\mathfrak{T}^\mu)^{-1}\big]\boldsymbol{v} = -(\mathfrak{T}^\mu)^{-1}\big[\partial^{\boldsymbol{k}}, \mathfrak{T}^\mu\big](\mathfrak{T}^\mu)^{-1}\boldsymbol{v}.$$

Combining the above and by Lemma 8.9, we find

$$\begin{aligned}
\big|\partial^{\boldsymbol{k}}((\mathfrak{T}^\mu)^{-1}\boldsymbol{v})\big|_{X_\mu^0} &= \big|(\mathfrak{T}^\mu)^{-1}\partial^{\boldsymbol{k}}\boldsymbol{v} - (\mathfrak{T}^\mu)^{-1}\big[\partial^{\boldsymbol{k}}, \mathfrak{T}^\mu\big](\mathfrak{T}^\mu)^{-1}\boldsymbol{v}\big|_{X_\mu^0} \\
&\leq C_0\big|\partial^{\boldsymbol{k}}\boldsymbol{v} - \big[\partial^{\boldsymbol{k}}, \mathfrak{T}^\mu\big](\mathfrak{T}^\mu)^{-1}\boldsymbol{v}\big|_{(X_\mu^0)'} \\
&\leq C(M)\Big(\big|\boldsymbol{v}\big|_{Y_\mu^s} + \big|(\mathfrak{T}^\mu)^{-1}\boldsymbol{v}\big|_{X_\mu^{s-1}} + \Big\langle(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|(\mathfrak{T}^\mu)^{-1}\boldsymbol{v}\big|_{X_\mu^{s_\star}}\Big\rangle_{s>1+s_\star}\Big).
\end{aligned}$$

The result follows by induction on s, and by density of $\mathcal{S}(\mathbb{R}^d)^d$ in Y_μ^s. □

8.6.2 The quasilinear structure

Proposition 8.11. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $\boldsymbol{k} \in \mathbb{N}^d \setminus \{\boldsymbol{0}\}$ multi-index and denote $n = \max\{|\boldsymbol{k}|, 1 + s_\star\}$. Let $\mu^\star > 0$, $T > 0$ and $M^\star \geq 0$. Then there exists $C, \tilde{C} > 0$ such that the following holds for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta \in \mathcal{C}^0([0,T]; H^n(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T]; H^{n-1}(\mathbb{R}^d))$, $b \in W^{n+1,\infty}(\mathbb{R}^d)$ such that Assumption 8.1 holds uniformly on $[0,T]$ and any $\boldsymbol{u} \in \mathcal{C}^0([0,T]; X_\mu^n) \cap \mathcal{C}^1([0,T]; X_\mu^{n-1})$ such that system eq. (8.6) holds and*

$$M \stackrel{\rm def}{=} \big\|\varepsilon\zeta\big\|_{L^\infty(0,T;H^{1+s_\star})} + \big\|\varepsilon\boldsymbol{u}\big\|_{L^\infty(0,T;X_\mu^{1+s_\star})} + \big|\beta b\big|_{W^{n+1,\infty}} \leq M^\star.$$

Then $\zeta^{(\boldsymbol{k})} \stackrel{\rm def}{=} \partial^{\boldsymbol{k}}\zeta$ and $\boldsymbol{u}^{(\boldsymbol{k})} \stackrel{\rm def}{=} \partial^{\boldsymbol{k}}\boldsymbol{u}$ satisfy

$$\begin{cases} \partial_t\zeta^{(\boldsymbol{k})} + \varepsilon\boldsymbol{u}\cdot\nabla\zeta^{(\boldsymbol{k})} + h\nabla\cdot\boldsymbol{u}^{(\boldsymbol{k})} = r_{(\boldsymbol{k})}, \\ \big(\operatorname{Id} + \mu\mathcal{T}[h, \beta\nabla b]\big)\partial_t\boldsymbol{u}^{(\boldsymbol{k})} + \nabla\zeta^{(\boldsymbol{k})} + \varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}^{(\boldsymbol{k})} + \mu\varepsilon\mathcal{Q}[h, \beta\nabla b, \boldsymbol{u}]\boldsymbol{u}^{(\boldsymbol{k})} = \boldsymbol{r}_{(\boldsymbol{k})}, \end{cases} \tag{8.19}$$

with $h = 1 + \varepsilon\zeta - \beta b$ and (abusing notations)

$$\begin{aligned}
\mathcal{Q}[h, \beta\nabla b, \boldsymbol{u}]\boldsymbol{u}^{(\boldsymbol{k})} &\stackrel{\rm def}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}^{(\boldsymbol{k})})\big)\Big) \\
&\quad + \frac{\beta}{2h}\Big(\nabla\big(h^2(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}^{(\boldsymbol{k})}\cdot\nabla b)\big) - h^2\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}^{(\boldsymbol{k})})\big)\nabla b\Big) + \beta\big((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}^{(\boldsymbol{k})}\cdot\nabla b)\big)(\beta\nabla b),
\end{aligned}$$

and where $r_{(\boldsymbol{k})}, \boldsymbol{r}_{(\boldsymbol{k})}$ enjoy the estimate (discarding the reference to $t \in [0,T]$)

$$\big|r_{(\boldsymbol{k})}\big|_{L^2} + \big|\boldsymbol{r}_{(\boldsymbol{k})}\big|_{Y_\mu^0} \leq C\ M\ \Big(\big|\zeta\big|_{H^{|\boldsymbol{k}|}} + \big|\boldsymbol{u}\big|_{X_\mu^{|\boldsymbol{k}|}}\Big). \tag{8.20}$$

Moreover, for any $\tilde{\zeta}, \tilde{\boldsymbol{u}}$ satisfying the same assumptions and denoting $\tilde{r}_{(\boldsymbol{k})}, \tilde{\boldsymbol{r}}_{(\boldsymbol{k})}$ the corresponding residuals, one has

$$\begin{aligned}
\big|r_{(\boldsymbol{k})} - \tilde{r}_{(\boldsymbol{k})}\big|_{L^2} + \big|\boldsymbol{r}_{(\boldsymbol{k})} - \tilde{\boldsymbol{r}}_{(\boldsymbol{k})}\big|_{Y_\mu^0} &\leq \tilde{C}\ M\ \Big(\big|\zeta - \tilde{\zeta}\big|_{H^{|\boldsymbol{k}|}} + \big|\boldsymbol{u} - \tilde{\boldsymbol{u}}\big|_{X_\mu^{|\boldsymbol{k}|}}\Big) \\
&\quad + \Big\langle\tilde{C}\ M_{\boldsymbol{k}}\ \Big(\big|\zeta - \tilde{\zeta}\big|_{H^{s_\star}} + \big|\boldsymbol{u} - \tilde{\boldsymbol{u}}\big|_{X_\mu^{s_\star}}\Big)\Big\rangle_{|\boldsymbol{k}|>1+s_\star}.
\end{aligned} \tag{8.21}$$

with $M_{\boldsymbol{k}} \stackrel{\rm def}{=} \big|\varepsilon\zeta\big|_{H^{|\boldsymbol{k}|}} + \big|\varepsilon\tilde{\zeta}\big|_{H^{|\boldsymbol{k}|}} + \big|\varepsilon\boldsymbol{u}\big|_{X_\mu^{|\boldsymbol{k}|}} + \big|\varepsilon\tilde{\boldsymbol{u}}\big|_{X_\mu^{|\boldsymbol{k}|}} + \beta\big|\nabla b\big|_{W^{n,\infty}}$.

Proof. Within this proof, we use the following convenient notation. We denote

$$a \sim_{L^2} b \quad \Longleftrightarrow \quad a - b = r,$$
$$\boldsymbol{a} \sim_{Y^0_\mu} \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{a} - \boldsymbol{b} = \boldsymbol{r}$$

with $\big|r\big|_{L^2}$ and $\big|\boldsymbol{r}\big|_{Y^0_\mu}$ satisfying (8.20) with $C = C(|\boldsymbol{k}|, \mu^\star, h_\star^{-1}, M^\star)$.

First equation. We start by applying the operator $\partial^{\boldsymbol{k}}$ to the first equation of eq. (8.6):

$$\partial_t \zeta^{(\boldsymbol{k})} + \partial^{\boldsymbol{k}}(h\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla h) = 0.$$

By Proposition II.9, Proposition II.15 and using the obvious continuous embedding $X^0_\mu \subset L^2(\mathbb{R}^d)^d$, one finds

$$\partial^{\boldsymbol{k}} \nabla \cdot (h\boldsymbol{u}) \sim_{L^2} h\nabla \cdot \partial^{\boldsymbol{k}} \boldsymbol{u} + \varepsilon \boldsymbol{u} \cdot \nabla \partial^{\boldsymbol{k}} \zeta$$

as desired.

Second equation. Here we multiply eq. $(8.6)_2$ with h before applying the operator $\partial^{\boldsymbol{k}}$. Proceeding a above and using the dual continuous embedding $L^2(\mathbb{R}^d)^d \subset Y^0_\mu$, we have

$$\partial^{\boldsymbol{k}} \big(\varepsilon h(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big) \sim_{Y^0_\mu} \varepsilon h(\boldsymbol{u} \cdot \nabla)\partial^{\boldsymbol{k}} \boldsymbol{u}.$$

Now we consider the contribution of

$$\mu\varepsilon h \mathcal{Q}[h, \mathbf{0}, \boldsymbol{u}] = \frac{-\mu\varepsilon}{3} \nabla \Big(h^3 \big((\boldsymbol{u} \cdot \nabla)(\nabla \cdot \boldsymbol{u}) - (\nabla \cdot \boldsymbol{u})^2 \big) \Big),$$

Thanks to the μ prefactor and using the non-uniform embeddings of Lemma 8.8, the second term gives no contribution, and the first one satisfies

$$\frac{-\mu\varepsilon}{3} \partial^{\boldsymbol{k}} \nabla \Big(h^3 \big((\boldsymbol{u} \cdot \nabla)(\nabla \cdot \boldsymbol{u}) \Big) \sim_{Y^0_\mu} \frac{-\mu\varepsilon}{3} \nabla \Big(h^3 \big((\boldsymbol{u} \cdot \nabla)(\partial^{\boldsymbol{k}} \nabla \cdot \boldsymbol{u}) \big) \Big).$$

We proceed similarly with the contribution of

$$\mu\varepsilon h \mathcal{Q}_{\rm b}[h, \beta\nabla b, \boldsymbol{u}] \stackrel{\rm def}{=} \frac{\mu\varepsilon\beta}{2} \Big(\nabla \big(h^2 (\boldsymbol{u} \cdot \nabla)^2 b \big) - h^2 \big((\boldsymbol{u} \cdot \nabla)(\nabla \cdot \boldsymbol{u}) - (\nabla \cdot \boldsymbol{u})^2 \big) \nabla b \Big) + \mu\varepsilon\beta h \big((\boldsymbol{u} \cdot \nabla)^2 b \big)(\beta \nabla b)$$

and deduce

$$\mu\varepsilon \partial^{\boldsymbol{k}} \Big(h \mathcal{Q}_{\rm b}[h, \beta\nabla b, \boldsymbol{u}] \Big) \sim_{Y^0_\mu} \frac{\mu\varepsilon\beta}{2} \Big(\nabla \big(h^2 (\boldsymbol{u} \cdot \nabla)(\partial^{\boldsymbol{k}} u \cdot \nabla b) \big) - h^2 \big((\boldsymbol{u} \cdot \nabla)(\partial^{\boldsymbol{k}} \nabla \cdot \boldsymbol{u}) \big) \nabla b \Big)$$
$$+ \mu\varepsilon\beta h \big((\boldsymbol{u} \cdot \nabla)(\partial^{\boldsymbol{k}} \boldsymbol{u} \cdot \nabla b) \big)(\beta \nabla b).$$

There remains the contribution of

$$h\big(\operatorname{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\partial_t \boldsymbol{u} = h\partial_t \boldsymbol{u} - \frac{\mu}{3} \nabla (h^3 \nabla \cdot \partial_t \boldsymbol{u}) + \frac{\mu}{2} \Big(\nabla \big(h^2 (\beta \nabla b) \cdot \partial_t \boldsymbol{u} \big) - h^2 (\beta \nabla b) \nabla \cdot \partial_t \boldsymbol{u} \Big)$$
$$+ \mu h \beta (\nabla b \cdot \partial_t \boldsymbol{u})(\beta \nabla b).$$

Let us first notice that we have $\partial_t \boldsymbol{u} \in X^j_\mu$ for any $j \in \{0, \ldots, \max\{s_\star, k-1\}\}$ by using eq. $(8.6)_2$. Indeed, applying the operator $\mathfrak{T}^\mu[h, \beta\nabla b]^{-1} = \big(h\big(\operatorname{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\big)^{-1}$, we have, by Lemma 8.10,

$$\big|\partial_t \boldsymbol{u}\big|_{X^j_\mu} \leq C(|\boldsymbol{k}|, \mu^\star, h_\star^{-1}, M^\star)\big(\big|\zeta\big|_{H^{j+1}} + \big|\boldsymbol{u}\big|_{X^{j+1}_\mu}\big).$$

We may then proceed as above to prove that

$$\partial^{\boldsymbol{k}} \Big(h\big(\operatorname{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\partial_t \boldsymbol{u} \Big) \sim_{Y^0_\mu} h\big(\operatorname{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\partial^{\boldsymbol{k}} \partial_t \boldsymbol{u}.$$

This concludes the proof of eq. (8.20). The proof of eq. (8.21) is obtained in the exact same way. $\square$

8.6.3 A priori energy estimates

Proposition 8.12. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$, $T > 0$ and $M > 0$. Then there exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $(\underline{\zeta}, \underline{\boldsymbol{u}}) \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^d)^{1+d}) \cap W^{1,\infty}(0, T; L^\infty(\mathbb{R}^d)^{1+d})$ and $b \in W^{1,\infty}(\mathbb{R}^d)$ such that $\underline{h} = 1 + \varepsilon\underline{\zeta} - \beta b$, satisfies Assumption 8.1 uniformly for $t \in (0, T)$ and*

$$\big\|\varepsilon\underline{\zeta}\big\|_{L^\infty(0,T;W^{1,\infty})} + \big\|\varepsilon\partial_t\underline{\zeta}\big\|_{L^\infty(0,T;L^\infty)} + \big\|\varepsilon\underline{\boldsymbol{u}}\big\|_{L^\infty(0,T;W^{1,\infty})} + \big|\beta b\big|_{W^{1,\infty}} \le M$$

as well as any $(\zeta, \boldsymbol{u}) \in L^\infty(0, T; H^1(\mathbb{R}^d) \times X^1_\mu) \cap W^{1,\infty}(0, T; L^2(\mathbb{R}^d) \times X^0_\mu)$ satisfying

$$\begin{cases} \partial_t\zeta + \varepsilon\underline{\boldsymbol{u}} \cdot \nabla\zeta + \underline{h}\nabla \cdot \boldsymbol{u} = r, \\ \big(\operatorname{Id} + \mu\mathcal{T}[\underline{h}, \beta\nabla b]\big)\partial_t\boldsymbol{u} + \nabla\zeta + \varepsilon(\underline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u} + \mu\varepsilon\mathcal{Q}[\underline{h}, \beta\nabla b, \underline{\boldsymbol{u}}]\boldsymbol{u} = \boldsymbol{r}, \end{cases} \tag{8.22}$$

with

$$\mathcal{Q}[\underline{h}, \beta\nabla b, \underline{\boldsymbol{u}}]\boldsymbol{u} \stackrel{\rm def}{=} \frac{-1}{3\underline{h}}\nabla\Big(h^3\big((\underline{\boldsymbol{u}} \cdot \nabla)(\nabla \cdot \boldsymbol{u})\big)\Big) + \frac{\beta}{2\underline{h}}\Big(\nabla\big(\underline{h}^2(\underline{\boldsymbol{u}} \cdot \nabla)(\boldsymbol{u} \cdot \nabla b)\big) - \underline{h}^2\big((\underline{\boldsymbol{u}} \cdot \nabla)(\nabla \cdot \boldsymbol{u})\big)\nabla b\Big) + \beta\big((\underline{\boldsymbol{u}} \cdot \nabla)(\boldsymbol{u} \cdot \nabla b)\big)(\beta\nabla b)$$

and $(r, \boldsymbol{r}) \in L^\infty(0, T; L^2(\mathbb{R}^d) \times Y^0_\mu)$, the following holds. Denoting

$$\mathcal{E}(t) \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} |\zeta|^2 + \underline{h}|\boldsymbol{u}|^2 + \ \mu\,\underline{h}\mathcal{T}[\underline{h}, \beta\nabla b]\boldsymbol{u} \cdot \boldsymbol{u}\,{\rm d}\boldsymbol{x},$$

we have

$$\frac{\rm d}{{\rm d}t}\mathcal{E} \le C\ M\ \mathcal{E} + C\ \big(\big|r\big|_{L^2} + \big|\boldsymbol{r}\big|_{Y^0_\mu}\big)\ \mathcal{E}^{1/2},$$

and as a consequence

$$\mathcal{E}^{1/2}(t) = \mathcal{E}^{1/2}(0)e^{CMt/2} + C\int_0^t e^{CM(t-\tau)/2}\big(\big|r(\tau, \cdot)\big|_{L^2} + \big|\boldsymbol{r}(\tau, \cdot)\big|_{Y^0_\mu}\big)\,{\rm d}\tau.$$

Proof. We test the first equation of eq. (8.22) against ζ and the second against $\underline{h}\boldsymbol{u}$. It follows, after some algebra,

$$\frac{\rm d}{{\rm d}t}\mathcal{E} = \mathcal{I}_1 + \mu\mathcal{I}_2 + \mu\mathcal{I}_{\rm b} + \mathcal{I}_{\rm r}$$

with

$$\begin{aligned} \mathcal{I}_1 &= \frac{1}{2}\int_{\mathbb{R}^d} \varepsilon(\nabla \cdot \underline{\boldsymbol{u}})|\zeta|^2 + 2(\varepsilon\nabla\underline{\zeta} - \beta\nabla b) \cdot \boldsymbol{u}\zeta + \varepsilon\big(\partial_t\underline{\zeta} + \nabla \cdot (\underline{h}\,\underline{\boldsymbol{u}})\big)|\boldsymbol{u}|^2\,{\rm d}\boldsymbol{x}, \\ \mathcal{I}_2 &= \frac{\varepsilon}{6}\int_{\mathbb{R}^d} \big(3\underline{h}^2\partial_t\underline{\zeta} + \nabla \cdot (\underline{h}^3\underline{\boldsymbol{u}})\big)(\nabla \cdot \boldsymbol{u})^2\,{\rm d}\boldsymbol{x}, \\ \mathcal{I}_{\rm b} &= \frac{\varepsilon}{2}\int_{\mathbb{R}^d} -\big(2\underline{h}\partial_t\underline{\zeta} + \nabla \cdot (\underline{h}^2\underline{\boldsymbol{u}})\big)(\beta\nabla b \cdot \boldsymbol{u})\nabla \cdot \boldsymbol{u} + \big(\partial_t\underline{\zeta} + \nabla \cdot (\underline{h}\,\underline{\boldsymbol{u}})\big)(\beta\nabla b \cdot \underline{\boldsymbol{u}})^2\,{\rm d}\boldsymbol{x}, \\ \mathcal{I}_{\rm r} &= \int_{\mathbb{R}^d} r\zeta + \underline{h}\boldsymbol{r} \cdot \boldsymbol{u}\,{\rm d}\boldsymbol{x}. \end{aligned}$$

We deduce immediately, by Cauchy–Schwarz inequality,

$$\varepsilon|\mathcal{I}_1| + \varepsilon|\mathcal{I}_2| + \varepsilon|\mathcal{I}_{\rm b}|\ \le\ C\ M\ \big(\big|\zeta\big|_{L^2}^2 + \big|\boldsymbol{u}\big|_{X^0_\mu}^2\big)$$

where $C = C(\mu^\star, \big|\varepsilon\partial_t\underline{\zeta}\big|, \big|\underline{h}\big|_{W^{1,\infty}}, \big|\varepsilon\underline{\boldsymbol{u}}\big|_{W^{1,\infty}}, \big|\beta\nabla b\big|_{L^\infty})$. In the same way, one has immediately

$$|\mathcal{I}_{\rm r}| \le \big|r\big|_{L^2}\big|\zeta\big|_{L^2} + \big|\boldsymbol{r}\big|_{Y^0_\mu}\big|\underline{h}\boldsymbol{u}\big|_{X^0_\mu} \le \big|r\big|_{L^2}\big|\zeta\big|_{L^2} + C(\mu^\star, \big|\underline{h}\big|_{W^{1,\infty}})\big|\boldsymbol{r}\big|_{Y^0_\mu}\big|\boldsymbol{u}\big|_{X^0_\mu}.$$

There only remains to use Lemma 8.9 to deduce

$$\big|\zeta\big|_{L^2}^2 + \big|\boldsymbol{u}\big|_{X^0_\mu}^2 \le C(h_\star^{-1})\mathcal{E},$$

and the result follows. □

8.6.4 Completion of the proof

The well-posedness result of Theorem 8.3 may be deduced, exploiting energy estimates similar to the ones derived above. Several strategies are known, including the use of regularizing operators as in [311] or a duality method as in [13, 49]. However both these methods rely on pseudo-differential tools which we have not introduced in these notes.[42] Hence we will follow here the a parabolic regularization approach as advocated in [189]. The strategy is standard, and similar to that applied for instance to the Navier–Stokes and Euler equations in [247], as described *e.g.* in [398].

8.6.4.1 Step 1: local existence for the regularized system. We introduce the regularized system

$$\begin{cases} \partial_t\zeta_\nu - \nu\Delta\zeta_\nu + \nabla\cdot(h_\nu \boldsymbol{u}_\nu) = 0 \\ h_\nu\big(\operatorname{Id} + \mu\mathcal{T}[h_\nu, \beta\nabla b]\big)(\partial_t \boldsymbol{u}_\nu - \nu\Delta\boldsymbol{u}_\nu) + h_\nu\nabla\zeta_\nu + h_\nu\varepsilon(\boldsymbol{u}_\nu\cdot\nabla)\boldsymbol{u}_\nu + \mu\varepsilon\underline{h}_\nu\mathcal{Q}[h_\nu, \beta\nabla b, \boldsymbol{u}_\nu] = \boldsymbol{0}, \end{cases}$$

where $h_\nu \stackrel{\text{def}}{=} 1 + \varepsilon\zeta_\nu - \beta b$ and $\nu > 0$ is a parameter which will eventually go to zero. By Lemma 8.9, we may invert the operator $\mathfrak{T}^\mu[h, \beta\nabla b] = h_\nu\big(\operatorname{Id} + \mu\mathcal{T}[h_\nu, \beta\nabla b]\big)$ (for sufficiently regular data) and write the system under the abstract form

$$\partial_t \begin{pmatrix} \zeta_\nu \\ \boldsymbol{u}_\nu \end{pmatrix} - \nu\Delta \begin{pmatrix} \zeta_\nu \\ \boldsymbol{u}_\nu \end{pmatrix} = \boldsymbol{F}(\zeta_\nu, \boldsymbol{u}_\nu). \tag{8.23}$$

By Duhamel's formula, solutions $(\zeta_\nu, \boldsymbol{u}_\nu) \in \mathcal{C}^0([0, T_\nu]; H^s(\mathbb{R}^d) \times X^s_\mu) \cap \mathcal{C}^1([0, T_\nu]; H^{s-1}(\mathbb{R}^d) \times X^{s-1}_\mu)$ to eq. (8.23) satisfy

$$\begin{pmatrix} \zeta_\nu \\ \boldsymbol{u}_\nu \end{pmatrix}(t) = e^{\nu t\Delta} \begin{pmatrix} \zeta_\nu \\ \boldsymbol{u}_\nu \end{pmatrix}(0) + \int_0^t e^{\nu(t-\tau)\Delta}\boldsymbol{F}(\zeta_\nu, \boldsymbol{u}_\nu)(\tau)\,\mathrm{d}\tau. \tag{8.24}$$

Here, $e^{\nu t\Delta}$ is the heat flow, defined as the Fourier multiplier (see Definition III.1)

$$\widehat{e^{\nu t\Delta} f}(\boldsymbol{\xi}) = e^{-\nu t|\boldsymbol{\xi}|^2}\widehat{f}(\boldsymbol{\xi}),$$

(applied to all components). We have, by Plancherel's formula for any $s \in \mathbb{R}$ and $t \geq 0$

$$\big\|e^{\nu t\Delta}\big\|_{H^s \to H^s} \leq 1$$

and for any $s' \geq s$ there exists $C_{s'-s}$ such that for any $t > 0$

$$\big\|e^{\nu t\Delta}\big\|_{H^s \to H^{s'}} \leq C_{s'-s}\big(1 + (\nu t)^{-\frac{s'-s}{2}}\big).$$

Here we exhibited the regularizing effect of the heat operator. The estimate above indicates that we can gain regularity in space by using integrability in time: by Hölder's inequality, we have

$$\big\|e^{\nu t\Delta}\big\|_{L^p(0,T;H^s) \to L^{p'}(0,T;H^{s'})} \leq C_{s,s',p,q}$$

for any $1 \leq p' < p \leq \infty$ if $0 \leq \frac{s'-s}{2} < \frac{1}{p'} - \frac{1}{p}$.

Proposition 8.13. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$ and $\nu > 0$. Let $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{SW}}$, $s \in \mathbb{N}$, $s > d/2 + 1$ and $\zeta_0 \in H^s(\mathbb{R}^d)$ and $b \in W^{s+1,\infty}(\mathbb{R}^d)$ be such that Assumption 8.1 holds, and $\boldsymbol{u}_0 \in X^s_\mu$. Then there exists $T_\nu > 0$ and $(\zeta_\nu, \boldsymbol{u}_\nu) \in \mathcal{C}^0([0, T_\nu]; H^s(\mathbb{R}^d) \times X^s_\mu) \cap L^2(0, T_\nu; H^{s+1}(\mathbb{R}^d) \times X^{s+1}_\mu)$ solution to (8.24). Moreover, this solution is unique.*

[42] in fact in our framework we only need a generalization of the product and commutator estimates given in Appendix II for Fourier multipliers such as $(\operatorname{Id} - \nu\Delta)^{-1/2}$, which follow for instance using the Littlewood-Paley theory, *i.e.* dyadic decomposition of the frequency space.

Proof. Here we only need to use the standard Banach fixed point argument on

$$\boldsymbol{\Phi}:\begin{pmatrix}\zeta_0\\ \boldsymbol{u}_\nu\end{pmatrix}\mapsto e^{\nu t\Delta}\begin{pmatrix}\zeta_0\\ \boldsymbol{u}_0\end{pmatrix}+\int_0^t e^{\nu(t-\tau)\Delta}\boldsymbol{F}(\zeta_\nu,\boldsymbol{u}_\nu)(\tau)\,\mathrm{d}\tau.$$

Consider $Z^s_{T_\nu}=\mathcal{C}^0([0,T_\nu];H^s(\mathbb{R}^d)\times X^s_\mu)\cap L^2(0,T_\nu;H^{s+1}(\mathbb{R}^d)\times X^{s+1}_\mu)$, endowed with the norm

$$\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^s_{T_\nu}}\stackrel{\rm def}{=}\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{L^\infty(0,T_\nu;H^s\times X^s_\mu)}+\nu^{1/2}\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{L^2(0,T_\nu;H^{s+1}\times X^{s+1}_\mu)}.$$

Given $R>0$ and $h_\star>0$, we denote

$$B_{R,h_\star}=\Big\{(\zeta_\nu,\boldsymbol{u}_\nu)\in Z^s_{T_\nu}\ :\ \big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^s_{T_\nu}}\le R,\ \inf_{t\in(0,T_\nu)}1+\varepsilon\zeta_\nu-\beta b\ge h_\star\Big\}.$$

By product estimates, Proposition II.7, and Lemma 8.10, we have for any $(\zeta_\nu,\boldsymbol{u}_\nu)\in B_{R,h_\star}$,

$$\big\|\boldsymbol{F}(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^{s-1}_{T_\nu}}\le C(R,h_\star^{-1}).$$

Thanks to the regularizing properties of the heat operator (using the energy method or by Plancherel's formula), there exists $C(T_\nu)$ such that

$$\big\|(e^{\nu t\Delta}\zeta_0,e^{\nu t\Delta}\boldsymbol{u}_0)\big\|_{Z^s_{T_\nu}}\le C(T_\nu)\big(\big|\zeta_\nu(0)\big|_{H^s}+\big|\boldsymbol{u}_\nu(0)\big|_{X^s_\mu}\big)$$

and $c_\nu=C(\nu,T_\nu)$ with $c_\nu\to0$ as $T_\nu\to0$ such that

$$\Big\|\int_0^t e^{\nu(t-\tau)\Delta}\boldsymbol{F}(\zeta_\nu,\boldsymbol{u}_\nu)(\tau)\,\mathrm{d}\tau\Big\|_{Z^s_{T_\nu}}\le c_\nu\big\|\boldsymbol{F}(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^{s-1}_{T_\nu}}.$$

Moreover, we have

$$\big\|\partial_t e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;L^\infty)}=\big\|\Delta e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;L^\infty)}\lesssim\big\|e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;H^{s+1})}.$$

Hence we can choose R sufficiently large and then T_ν sufficiently small so that $\boldsymbol{\Phi}$ maps $B_{R,h_\star/2}$ into itself. Using again product estimates, Proposition II.7, and Lemma 8.10, we infer that for any $(\zeta^\ell_\nu,\boldsymbol{u}^\ell_\nu)\in B_{R,h_\star/2}$ (with $\ell\in\{1,2\}$),

$$\big\|\boldsymbol{F}(\zeta^1_\nu,\boldsymbol{u}^1_\nu)-\boldsymbol{F}(\zeta^2_\nu,\boldsymbol{u}^2_\nu)\big\|_{Z^{s-1}_{T_\nu}}\le C(R,h_\star^{-1})\big\|(\zeta^1_\nu-\zeta^2_\nu,\boldsymbol{u}^1_\nu-\boldsymbol{u}^2_\nu)\big\|_{Z^s_{T_\nu}}$$

and in turn that, lowering T_ν if necessary, $\boldsymbol{\Phi}$ is a contraction mapping. The result follows from Banach fixed-point theorem. □

Proposition 8.14. *Let $d\in\mathbb{N}^\star$, $h_\star>0$, $\mu^\star>0$ and $\nu>0$, $s_\star>d/2$. Let $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$. Let $\zeta_0\in H^\infty(\mathbb{R}^d)$ and $b\in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 8.1 holds, and $\boldsymbol{u}_0\in H^\infty(\mathbb{R}^d)^d$. Then there exists a unique $T_{\max}>0$ and a unique $(\zeta_\nu,\boldsymbol{u}_\nu)\in\mathcal{C}^\infty([0,T_{\max});H^\infty(\mathbb{R}^d)^{1+d})$ maximal solution to (8.23). Moreover, if $T_{\max}<\infty$, then*

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^{1+s_\star})}+\big\|\boldsymbol{u}_\nu\big\|_{L^\infty(0,T^\star;X^{1+s_\star}_\mu)}\to\infty\qquad\textit{or}\qquad\inf_{\mathbb{R}^d}1+\varepsilon\zeta_\nu-\beta b\to0.$$

Proof. By iterating Proposition 8.13, we have for any given $s\ge s_\star$, the existence and uniqueness of a maximal Cauchy development, *i.e.* $T^\star>0$ and a mild solution $(\zeta_\nu,\boldsymbol{u}_\nu)\in\mathcal{C}^0([0,T^\star);H^s(\mathbb{R}^d)\times X^s_\mu)$ such that if $T_\star<\infty$,

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^s)}+\big\|\boldsymbol{u}_\nu\big\|_{L^\infty(0,T^\star;X^s_\mu)}\to\infty\qquad\text{or}\qquad\inf_{\mathbb{R}^d}1+\varepsilon\zeta_\nu-\beta b\to0.$$

By the uniqueness of the mild solution, the solutions do not depend on the regularity index, s, as long as their domain of existence coincide. In principle, the maximal time of existence, $T_\star$, may depend on the regularity index, s, we consider. Such is not the case thanks to the independent blowup criterion: if $T_\star < \infty$, then

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^{1+s_\star})} + \big\|\boldsymbol{u}_\nu\big\|_{L^\infty(0,T^\star;X_\mu^{1+s_\star})} \to \infty \quad \text{or} \quad \inf_{\mathbb{R}^d} 1+\varepsilon\zeta_\nu - \beta b \to 0.$$

This blowup criterion is obtained by contradiction and using tame product estimates of Proposition II.7, and Lemma 8.10. Indeed, assuming that the above quantities are bounded (respectively from above and below), we find, following the steps of Proposition 8.13 that

$$\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}} \le C_1\big|(\zeta_\nu,\boldsymbol{u}_\nu)(\tau_1)\big|_{H^s\times X^s_\mu} + C_2\nu^{-1/2}(\tau_2-\tau_1)^{1/2}\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{L^\infty(\tau_1,\tau_2;H^s\times X^s_\mu)},$$

for any $0<\tau_1<\tau_2<T^\star$ and denoting

$$\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}} = \big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{L^\infty(\tau_1,\tau_2;H^s\times X^s_\mu)} + \nu^{1/2}\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{L^2(\tau_1,\tau_2;H^{s+1}\times X^{s+1}_\mu)}.$$

We can then choose τ_1 sufficiently large to absorb the second term of the right-hand side, and deduce an estimate on $\big\|(\zeta_\nu,\boldsymbol{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}}$, uniform with respect to $\tau_2\in(\tau_1,T^\star)$, from which the contradiction follows.

There only remains to prove that the solution has the desired regularity (in time). Differentiating the Duhamel formula with respect to time, we have that $(\zeta_\nu,\boldsymbol{u}_\nu)$ satisfies eq. (8.23) in the sense of spacetime distributions, and hence $(\zeta_\nu,\boldsymbol{u}_\nu)\in\mathcal{C}^1([0,T^\star);H^\infty(\mathbb{R}^d)^{1+d})$. We may then iterate eq. (8.23) to obtain the desired regularity. □

8.6.4.2 Step 2: local existence for smooth solutions. In order to be able to construct (smooth) solutions to eq. (8.6) from (smooth) solutions to the parabolic regularization, eq. (8.23), we need to obtain uniform energy estimates.

Proposition 8.15. *Let $d\in\mathbb{N}^\star$, $h_\star>0$, $\mu^\star>0$, $s_\star>d/2$, $s\in\mathbb{N}$ and $M^\star\ge 0$. Then there exists $T>0$ and $C>0$ such that for any $\nu\in(0,1]$, any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$, any $\zeta_0\in H^\infty(\mathbb{R}^d)$ and $b\in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 8.1 holds, and $\boldsymbol{u}_0\in H^\infty(\mathbb{R}^d)^d$ such that*

$$M \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\boldsymbol{u}_0\big|_{X_\mu^{1+s_\star}} + \big|\beta b\big|_{W^{\max\{s+1,2+s_\star+1\},\infty}} \le M^\star,$$

the unique maximal solution to eq. (8.6) provided by Proposition 8.14 satisfies $T^\star\ge T/M$ and for any $t\in[0,T/M]$, $\inf_{\boldsymbol{x}\in\mathbb{R}^d}(1+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x}))\ge h_\star/2$ and

$$\big|\zeta_\nu(t,\cdot)\big|_{H^s} + \big|\boldsymbol{u}_\nu(t,\cdot)\big|_{X^s_\mu} \le C\times\big(\big|\zeta_0\big|_{H^s} + \big|\boldsymbol{u}_0\big|_{X^s_\mu}\big).$$

Proof. The estimate (and hence the lower bound on the maximal time of existence by the blowup criterion) follow from a priori energy estimates similar to Proposition 8.11 and Proposition 8.12. There are however additional terms to be taken care of. Let us just consider a handful of them. After testing the contribution

$$h_\nu\big(\operatorname{Id}+\mu\mathcal{T}[h_\nu,\beta\nabla b]\big)(\partial_t\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu-\nu\Delta\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu)$$

against $\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu$, we have to estimate the additional contributions

$$\big(h_\nu\big(\operatorname{Id}+\mu\mathcal{T}[h_\nu,\beta\nabla b]\big)(\partial_t\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu-\nu\Delta\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu),\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu\big)_{L^2}$$

Let us discard the terms stemming from the variable topography and concentrate on

$$-\nu\big(h_\nu\Delta\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu,\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu\big)_{L^2} - \frac{\nu\mu}{3}\big(h_\nu^3\nabla\cdot\Delta\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu,\nabla\cdot\partial^{\boldsymbol{k}}\boldsymbol{u}_\nu\big)_{L^2}.$$

Now we have, assuming $d = 2$ and denoting $\boldsymbol{u}_\nu = (\boldsymbol{u}_{\nu,x}, \boldsymbol{u}_{\nu,y})$,

$$-\nu\big(h_\nu \Delta \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu, \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu\big)_{L^2} = \nu \int_{\mathbb{R}^d} h_\nu \big(|\nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,x}|^2 + |\nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,y}|^2\big)\, \mathrm{d}\boldsymbol{x} \\ + \nu(\nabla h_\nu \cdot \nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,x}, \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,x})_{L^2} + \nu(\nabla h_\nu \cdot \nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,y}, \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,y})_{L^2},$$

and hence

$$-\nu\big(h_\nu \Delta \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu, \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu\big)_{L^2} - \frac{\nu}{2} \int_{\mathbb{R}^d} h \big(|\nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,x}|^2 + |\nabla \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,y}|^2\big)\, \mathrm{d}\boldsymbol{x} \\ \leq \frac{\nu}{2} \int_{\mathbb{R}^d} h^{-1} \big(|\nabla h \cdot \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,x}|^2 + |\nabla h \cdot \partial^{\boldsymbol{k}} \boldsymbol{u}_{\nu,y}|^2\big)\, \mathrm{d}\boldsymbol{x}.$$

The second term is estimated similarly, using that

$$\nabla \cdot \Delta \operatorname{Id} \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu = \Delta \nabla \cdot \partial^{\boldsymbol{k}} \boldsymbol{u}_\nu.$$

Using the boundedness of ν, we may then conclude as in Proposition 8.12. We obtain in fact the improved differential inequality

$$\mathcal{E}_s'(t) + \nu \mathcal{E}_{s+1}(t) \leq C\ M\ \mathcal{E}_s(t)$$

where

$$\mathcal{E}_s(t) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{|\boldsymbol{k}| \leq s} \int_{\mathbb{R}^d} |\partial^{\boldsymbol{k}} \zeta|^2 + h |\partial^{\boldsymbol{k}} \boldsymbol{u}|^2 + \ \mu\, h \mathcal{T}[h, \beta \nabla b] \partial^{\boldsymbol{k}} \boldsymbol{u} \cdot \partial^{\boldsymbol{k}} \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} \approx \big|\zeta_\nu\big|_{H^s}^2 + \big|\boldsymbol{u}_\nu\big|_{X^s_\mu}^2.$$

However we need to ensure that $\inf_{\boldsymbol{x} \in \mathbb{R}^d} (1 + \varepsilon \zeta(t, \boldsymbol{x}) - \beta b(\boldsymbol{x})) \geq h_\star/2$ for any $t \in [0, T/M]$. This is obtained using the positivity of the heat kernel, so that

$$|\zeta(t, \boldsymbol{x}) - \zeta(0, \boldsymbol{x})| \leq \big|\partial_t \zeta - \nu \Delta \zeta\big|_{L^1(0,T; L^\infty(\mathbb{R}^d))} \lesssim \big|h \boldsymbol{u}\big|_{L^1(0,T; H^{1+s_\star}(\mathbb{R}^d))}.$$

This concludes the proof. □

Proposition 8.16. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$, $s_\star \in \mathbb{N}$, $s_\star > d/2$, $s \in \mathbb{N}$ and $M^\star \geq 0$. Then there exists $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta_0 \in H^\infty(\mathbb{R}^d)$ and $b \in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 8.1 holds, and $\boldsymbol{u}_0 \in H^\infty(\mathbb{R}^d)$ such that*

$$\big|\varepsilon \zeta_0\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{\max\{1+s, 2+s_\star\},\infty}} + \big|\beta b\big|_{L^\infty} + \big|\varepsilon \boldsymbol{u}_0\big|_{X^{1+s_\star}_\mu} \leq M,$$

there exists $(\zeta, \boldsymbol{u}) \in \mathcal{C}^\infty([0, T/M]; H^\infty(\mathbb{R}^d)^{1+d})$ solution to (8.6) and satisfying for any $t \in [0, T/M]$,

$$\big|\zeta(t, \cdot)\big|_{H^s} + \big|\boldsymbol{u}(t, \cdot)\big|_{X^s_\mu} \leq C \times \big(\big|\zeta_0\big|_{H^s} + \big|\boldsymbol{u}_0\big|_{X^s_\mu}\big).$$

and $\inf_{\boldsymbol{x} \in \mathbb{R}^d} (1 + \varepsilon \zeta(t, \boldsymbol{x}) - \beta b(\boldsymbol{x})) \geq h_\star/2$.

Sketch of the proof. We introduce (ν_n) a sequence such that $\nu_n \searrow 0$. By Proposition 8.15, there exists C, T, *independent of* n and a sequence $(\zeta_{\nu_n}, \boldsymbol{u}_{\nu_n}) \in \mathcal{C}^\infty([0, T/M]; H^\infty(\mathbb{R}^d)^{1+d})$, uniformly bounded and equicontinuous (by Sobolev embedding) and satisfying (as well as an arbitrary number of derivatives) the desired estimate. By weak compactness, there exists a converging subsequence. From Arzelá–Ascoli theorem, the convergence holds locally uniformly. Hence we can take limits and deduce that the limit satisfies (8.6). The desired bound is a direct consequence of the identical (uniform in n) estimate on $(\zeta_{\nu_n}, \boldsymbol{u}_{\nu_n})$. □

8.6.4.3 Step 3: Existence and uniqueness of classical solutions We are now in position to prove Theorem 8.3.

Proof of Theorem 8.3. We start with the uniqueness. Let us consider two classical solution to eq. (8.15), $(\zeta_1, \boldsymbol{u}_1)$ and $(\tilde{\zeta}_2, \tilde{\boldsymbol{u}}_2) \in \mathcal{C}^0([0,T/M_0]; H^s(\mathbb{R}^d) \times X^s_\mu) \cap \mathcal{C}^1([0,T/M_0]; H^{s-1}(\mathbb{R}^d) \times X^{s-1}_\mu)$, with same initial data $(\zeta_i, \boldsymbol{u}_i)\,|_{t=0} = (\zeta_0, \boldsymbol{u}_0)$. By Proposition 8.11 (with $\boldsymbol{k} = \boldsymbol{0}$), we have that the difference $(\zeta, \boldsymbol{u}) \stackrel{\text{def}}{=} (\zeta_2 - \zeta_1, \boldsymbol{u}_2 - \boldsymbol{u}_1)$ satisfies

$$\begin{cases} \partial_t \zeta + \varepsilon \boldsymbol{u}_2 \cdot \nabla \zeta + h_2 \nabla \cdot \boldsymbol{u} = r, \\ \big(\mathrm{Id} + \mu \mathcal{T}[h_2, \beta \nabla b]\big)\partial_t \boldsymbol{u} + \nabla \zeta + \varepsilon(\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u} + \mu\varepsilon \mathcal{Q}[h_2, \beta\nabla b, \boldsymbol{u}_2]\boldsymbol{u} = \boldsymbol{r}, \end{cases}$$

where the right-hand side satisfies, using in particular eq. (8.21),

$$\forall t \in (0, T/M_0], \qquad \big|r(t,\cdot)\big|_{L^2} + \big|\boldsymbol{r}(t,\cdot)\big|_{Y^0_\mu} \leq C(M)\, M\, \Big(\big|\zeta(t,\cdot)\big|_{L^2} + \big|\boldsymbol{u}(t,\cdot)\big|_{X^0_\mu}\Big),$$

where

$$M = \big\|\varepsilon\zeta_1\big\|_{L^\infty(0,t;H^{s_\star})} + \big\|\varepsilon\boldsymbol{u}_1\big\|_{L^\infty(0,t;X^{s_\star}_\mu)} + \big\|\varepsilon\zeta_2\big\|_{L^\infty(0,t;H^{s_\star})} + \big\|\varepsilon\boldsymbol{u}_2\big\|_{L^\infty(0,t;X^{s_\star}_\mu)} + \big|\beta\nabla b\big|_{H^{1+s_\star}}.$$

We can now use Proposition 8.12 to deduce that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} |\zeta|^2 + h_2|\boldsymbol{u}|^2 + \ \mu\, h_2 \mathcal{T}[h_2, \beta\nabla b]\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x},$$

where $h_2 = 1 + \varepsilon\zeta_2 - \beta b$, satisfies[43]

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq C(M)\ M\ \mathcal{E},$$

and hence, since $\mathcal{E}(0) = 0$, $\mathcal{E} \equiv 0$. By Lemma 8.9, we deduce the uniqueness.

Now for the existence, we shall construct a solution as the limit of a Cauchy sequence of smooth solutions. The argument is classical and often referred to as the Bona-Smith technique [55] although it appeared already in the work of Kato [247]. We introduce the one-parameter family of mollifiers: for any $\iota > 0$,

$$\mathsf{J}^\iota \stackrel{\text{def}}{=} \chi(\nu|D|), \qquad \chi(\xi) = \mathbf{1}_{|\xi|\leq 1}.$$

We shall use the following limits which follow by Plancherel's theorem and dominated convergence: for any $\zeta \in H^s(\mathbb{R}^d)$,

$$\big|\zeta - \mathsf{J}^\iota\zeta\big|_{H^s} + \iota^{-1}\big|\zeta - \mathsf{J}^\iota\zeta\big|_{H^{s-1}} + \iota\big|\mathsf{J}^\iota\zeta\big|_{H^{s+1}} \to 0, \tag{8.25a}$$

and for any $\boldsymbol{u} \in X^s_\mu$,

$$\big|\boldsymbol{u} - \mathsf{J}^\iota\boldsymbol{u}\big|_{X^s_\mu} + \iota^{-1}\big|\boldsymbol{u} - \mathsf{J}^\iota\boldsymbol{u}\big|_{X^{s-1}_\mu} + \iota\big|\mathsf{J}^\iota\boldsymbol{u}\big|_{X^{s+1}_\mu} \to 0. \tag{8.25b}$$

By Proposition 8.16 with initial data $(\zeta^\iota_0, \boldsymbol{u}^\iota_0) \stackrel{\text{def}}{=} (\mathsf{J}^\iota\zeta_0, \mathsf{J}^\iota\boldsymbol{u}_0) \in H^\infty(\mathbb{R}^d)^{1+d}$ (and assuming at first that the bottom topography is smooth [44]), for any $s' \in \mathbb{N}$ there exists $C_0, T > 0$, *independent of* ι, and $(\zeta_\iota, \boldsymbol{u}_\iota) \in \mathcal{C}^\infty([0,T/M_0]; H^\infty(\mathbb{R}^d)^{1+d})$ solution to (8.6) and satisfying for any $t \in [0, T/M_0]$

$$\big|\zeta_\iota(t,\cdot)\big|_{H^{s'}} + \big|\boldsymbol{u}_\iota(t,\cdot)\big|_{X^{s'}_\mu} \leq C_0\big(\big|\mathsf{J}^\iota\zeta_0\big|_{H^{s'}} + \big|\mathsf{J}^\iota\boldsymbol{u}_0\big|_{X^{s'}_\mu}\big) \tag{8.26}$$

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[43] Here we use that the hyperbolicity criterion $\inf_{\boldsymbol{x}\in\mathbb{R}^d} h_2 > 0$ remains satisfied as a consequence of the conservation of mass; see footnote 6 page vii.

[44] In order to deal with non-smooth topographies, we may consider the sequence of solutions corresponding to the mollified topographies

$$b^\iota \stackrel{\text{def}}{=} \rho_\iota \star b = \int_{\mathbb{R}^d} \frac{1}{\iota^d}\rho(\frac{\cdot - \boldsymbol{y}}{\iota})b(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

where ρ is smooth, non-negative with compact support, and $\big|\rho\big|_{L^1} = 1$.

and $\inf_{\boldsymbol{x}\in\mathbb{R}^d}(1+\varepsilon\zeta_\iota(t,\boldsymbol{x})-\beta b(\boldsymbol{x}))\geq h_\star/2$.

We wish to prove that, given a decreasing sequence $\iota_n\to 0$, the constructed $(\zeta_{\iota_n},\boldsymbol{u}_{\iota_n})$ is a Cauchy sequence. Proceeding as above, we may estimate the difference between two solution

$$(\zeta_{m,n},\boldsymbol{u}_{m,n})\stackrel{\text{def}}{=}(\zeta_{\iota_n}-\zeta_{\iota_m},\boldsymbol{u}_{\iota_n}-\boldsymbol{u}_{\iota_m})$$

by Proposition 8.11 and Proposition 8.12. One gets, for any $\boldsymbol{k}\in\mathbb{N}^d$, $0\leq|\boldsymbol{k}|\leq s$, and any $n>m$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\boldsymbol{k}}\leq C_{\boldsymbol{k}}(M)M\ \mathcal{E}_{\boldsymbol{k}}+C_{\boldsymbol{k}}(M)\mathcal{E}_{\boldsymbol{k}}^{1/2}\big(\big|\zeta_{\iota_m}\big|_{H^{|\boldsymbol{k}|+1}}+\big|\boldsymbol{u}_{\iota_m}\big|_{X_\mu^{|\boldsymbol{k}|+1}}\big)\big(\big|\zeta_{m,n}\big|_{H^{s_\star}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s_\star}}\big),$$

where we denote

$$M\stackrel{\text{def}}{=}\big\|\varepsilon\zeta_{\iota_n}\big\|_{L^\infty(0,t;H^{s_\star})}+\big\|\varepsilon\boldsymbol{u}_{\iota_n}\big\|_{L^\infty(0,t;X_\mu^{s_\star})}+\big\|\varepsilon\zeta_{\iota_m}\big\|_{L^\infty(0,t;H^{s_\star})}+\big\|\varepsilon\boldsymbol{u}_{\iota_m}\big\|_{L^\infty(0,t;X_\mu^{s_\star})}+\big|\beta\nabla b\big|_{W^{s,\infty}}$$

and

$$\mathcal{E}_{\boldsymbol{k}}(t)\stackrel{\text{def}}{=}\frac{1}{2}\int_{\mathbb{R}^d}|\partial^{\boldsymbol{k}}\zeta_{m,n}|^2+h_n|\partial^{\boldsymbol{k}}\boldsymbol{u}_{m,n}|^2+\ \mu h_n(\mathcal{T}[h_m,\beta\nabla b]\partial^{\boldsymbol{k}}\boldsymbol{u}_{m,n})\cdot(\partial^{\boldsymbol{k}}\boldsymbol{u}_{m,n})\,\mathrm{d}\boldsymbol{x},$$

where $h_n\stackrel{\text{def}}{=}1+\varepsilon\zeta_{\iota_n}-\beta b_{\iota_n}\geq h_\star/2$. As a consequence, using eq. (8.26) with $s'=s$ and eq. (8.25), there exists $C'>0$ such that for any m,n,

$$\big|\zeta_{m,n}\big|_{H^{s-1}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s-1}}=C'\big(\big|\zeta_{m,n}\big|_{H^{s-1}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s-1}}\big)(t=0)$$

and in turn

$$\begin{aligned}\big|\zeta_{m,n}\big|_{H^{s}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s}}&=C'\big(\big|\zeta_{m,n}\big|_{H^{s}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s}}\big)(t=0)\\&\quad+C't\big(\big|\zeta_{\iota_m}\big|_{H^{s+1}}+\big|\boldsymbol{u}_{\iota_m}\big|_{X_\mu^{s+1}}\big)\big(\big|\zeta_{m,n}\big|_{H^{s-1}}+\big|\boldsymbol{u}_{m,n}\big|_{X_\mu^{s-1}}\big)(t=0).\end{aligned}$$

Using eq. (8.26) with $s'=s+1$ and applying eq. (8.25), we deduce that

$$\limsup_{m,n\to\infty}\Big(\big\|\zeta_{m,n}\big\|_{L^\infty(0,T/M;H^s)}+\big\|\boldsymbol{u}_{m,n}\big\|_{L^\infty(0,T/M;X_\mu^s)}\Big)=0,$$

and hence the sequence strongly converges in $\mathcal{C}^0([0,T/M];H^s\times X_\mu^s)$ towards $(\zeta,\boldsymbol{u})$, satisfying the desired initial condition. We have that $(\partial_t\zeta_{\iota_n},\partial_t\boldsymbol{u}_{\iota_n})\rightharpoonup(\partial_t\zeta,\partial_t\boldsymbol{u})$ in the sense of distributions, and hence $(\zeta,\boldsymbol{u})$ is a solution to eq. (8.6) in the sense of distributions. It follows from Lemma 8.10 that $(\partial_t\zeta,\partial_t\boldsymbol{u})\in\mathcal{C}^0([0,T/M];H^{s-1}\times X_\mu^{s-1})$ and hence we have constructed a classical solution. It satisfies the desired estimate by eq. (8.26).

In order to be fully complete, we need to check the second assertion, that is that $(\zeta,\boldsymbol{v})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times Y_\mu^s)\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times Y_\mu^{s-1})$ is a classical solution to eq. (8.2) (applying the gradient to the second equation), where $\boldsymbol{v}=h^{-1}\mathfrak{T}^\mu[h,\beta\nabla b]\boldsymbol{u}$ with $h=1+\varepsilon\zeta-\beta b$. The desired regularity is a direct consequence of Lemma 8.10, and the fact that $(\zeta,\boldsymbol{v})$ satisfies the desired equations follows from tedious algebra, which are detailed in [162, §6]. □

8.7 Discussion and open questions

It should acknowledged that, despite numerous works on the Green–Naghdi system, very little is known concerning the behavior of solutions, except in very particular cases (such as traveling waves). I give below a list of natural questions which in my opinion would deserve additional investigations.

Singularity formation, shock-like solutions

The first very natural follow-up interrogation after Theorem 8.3 is whether the time of existence of solutions stated therein is optimal. In particular, we do not know whether the Green–Naghdi equations are globally well-posed, even for small initial data. We however have no result on finite-time (or infinite-time) singularity formations. Numerical investigations have been for now fairly inconclusive:

- In [151] the authors' efforts to numerically exhibit singularity formations have been vain;
- In [196] the exhibited "shock-like" structures appear to be smooth.

This question is particularly interesting because the natural Boussinesq (that is, weakly-nonlinear) system correspond to the Green–Naghdi system (8.6), that is in dimension $d = 1$ and with flat bottom

$$\begin{cases} \partial_t\zeta + \partial_x(hu) = 0, \\ \big(1 - \frac{\mu}{3}\partial_x^2\big)\partial_t u + \partial_x\zeta + \varepsilon u\partial_x u = 0, \end{cases} \tag{8.27}$$

is one of the few Boussinesq systems which are known to be globally well-posed, provided that the non-cavitation assumption $\inf_{\mathbb{R}}(1 + \varepsilon\zeta) > 0$ is initially satisfied, by [380, 21]; see also [327]. This does not prevent the existence of shock-like structures as exhibited for eq. (8.27) in the work of El, Hoefer and Shearer [175] but only their formation from smooth initial data. Moreover, the non-cavitation assumption is essential: Bae and Granero-Belinchón showed in [33] that if the non-cavitation assumption initially fails to hold at one single point and some symmetry assumptions are enforced, then solutions to eq. (8.6) (or rather an equivalent reformulation when the non-cavitation assumption holds) preserve these assumptions for positive time and cannot remain smooth globally in time. Finally, let me point out that in the presence of surface tension, in dimension $d = 1$ with flat bottom, the existence of global weak solutions (for small data) and solutions exhibiting finite-time singularity (with non-empty intersection) have been proved by Guelmame in [204, 205].

Stability of traveling waves

In this section we restrict the discussion to the flat bottom situation, and dimension $d = 1$. We have seen in Section 8.4 the existence—and in fact explicit formula—of solitary wave solutions to eq. (8.6), and *cnoidal* (*i.e.* periodic) traveling wave solutions could be derived in a similar way. For these structures to be observable in practice (disregarding all the approximations that have been made to derive the equations), one should ensure that they satisfy a notion of *stability*. Roughly speaking, we ask that for an initial data in the vicinity of traveling wave profile, the solution remains in the vicinity of the corresponding traveling wave solution. As usual, since the system enjoys a continuous family of solutions with any supercritical velocity, the notion of stability can only be *orbital*, that is up to horizontal translation. In other words we are looking at the stability of profiles.

Based on the Hamiltonian structure of the Green–Naghdi system, we may interpret traveling waves solutions to (8.6) as critical points to the functional $\mathscr{H}_{\mathrm{GN}}(\zeta, v) - c\mathscr{I}_{\mathrm{GN}}(\zeta, v)$, *i.e.*

$$c\zeta = \delta_v\mathscr{H}_{\mathrm{GN}}(\zeta, v) \quad \text{ and } \quad cv = \delta_\zeta\mathscr{H}_{\mathrm{GN}}(\zeta, v),$$

where $\mathscr{I}_{\mathrm{GN}}(\zeta, v) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d}\zeta v$ and $v = h^{-1}\mathfrak{T}^\mu[h, 0]u$. However these critical points are neither minimizers nor maximizers of the functional, and hence give no direct information to the stability of the solutions. In [165], a constrained minimization problem is introduced which allows to prove the existence—together with a weak notion of stability—of solitary waves for a larger class of equations, including (8.6), yet the functionals at stake are not preserved quantities of the system, and hence even the standard weak "conditional" notion of stability fails to hold by this method.

In [282, 283], Li shows that solitary wave solutions of the eq. (8.6) equations with sufficiently small supercritical velocity $0 < c - 1 \ll 1$ are (orbitally) *linearly stable* (see details therein) for infinitely small and exponentially decaying perturbations. Later on Carter and Cienfuegos numerically studied in [79] the linear stability of *cnoidal waves* and found that sufficiently large or steep cnoidal waves exhibit linear instability, with relatively small growth rate. The results do not pass to the limit of infinitely long waves, that is solitary waves; see the discussion in [79]. Finally, the *modulational stability* of small-amplitude *bore* solutions is proved in [174]. To my knowledge, no nonlinear stability result is yet available. The numerical investigation in [151] pleads for the stability of solitary waves, even with large velocities, although the latter exhibit a strong sensitivity to perturbations.

Small-amplitude, large-time dynamics

A very natural question in the oceanographic context concerns the large time asymptotic behavior of solutions to the Green–Naghdi system for small data. After a straightforward rescaling of (8.6), the problem is naturally formulated in terms of solutions to the system

$$\begin{cases} \partial_t\zeta + \frac{1}{\varepsilon}\nabla\cdot(h\boldsymbol{u}) = 0, \qquad h \stackrel{\text{def}}{=} 1+\varepsilon\zeta-\beta b, \\ \big(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b]\big)\partial_t\boldsymbol{u} + \frac{1}{\varepsilon}\nabla\zeta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\mathcal{Q}[h,\beta\nabla b,\boldsymbol{u}] = 0. \end{cases} \tag{8.28}$$

Is the Cauchy problem for (8.28) locally well-posed, uniformly with respect to $\varepsilon\in(0,1]$? Is it globally well-posed for ε small enough? Can we exhibit "averaged" equations asymptotically describing a slow coherent evolution of the solution?

This type of singular limit has been widely studied in particular in the context of the low Mach number limit; see *e.g.* [190, 9, 379, 299] and references therein (it appears also in the study of the weak density contrast limit in Section 6.2.5 and in the study of the Favrie–Gavrilyuk approximation in the subsequent Section 9). As a matter of fact, when $\beta=\mu=0$ and horizontal dimension $d=2$, one recognizes the incompressible limit for the isentropic two-dimensional Euler equations, and it is tempting to elaborate on the analogy. One would then expect the solutions to (8.28) to be asymptotically described (as $\varepsilon\searrow 0$) as the superposition of two components, described thereafter.

i. The "incompressible" component, being defined as the solution to

$$\begin{cases} \nabla\cdot((1-\beta b)\boldsymbol{u}) = 0, \\ \big(\mathrm{Id}+\mu\mathcal{T}[1-\beta b,\beta\nabla b]\big)\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\ \qquad +\mu\mathcal{Q}[1-\beta b,\beta\nabla b,\boldsymbol{u}] = -\nabla p, \end{cases} \tag{8.29}$$

where the "pressure" p in $(8.29)_2$ is the Lagrange multiplier associated to the "incompressibility" constraint $(8.29)_1$.

ii. The "acoustic" component, being defined as the solution to

$$\begin{cases} \partial_t\zeta + \frac{1}{\varepsilon}\nabla\cdot((1-\beta b)\boldsymbol{u}) = 0, \\ \big(\mathrm{Id}+\mu\mathcal{T}[1-\beta b,\beta\nabla b]\big)\partial_t\boldsymbol{u} + \frac{1}{\varepsilon}\nabla\zeta = 0, \end{cases} \tag{8.30}$$

with initial data satisfying $\mathrm{curl}(\boldsymbol{u}+\mu\mathcal{T}[1-\beta b,\beta\nabla b]\boldsymbol{u})\,|_{t=0} = 0$.

System (8.29) was derived in [73, 74], and is usually referred to as the *great-lake equations*. Its well-posedness, extending the theory concerning the two-dimensional incompressible Euler equations, was subsequently provided in [278, 346]. However, we should emphasize that due to the irrotationality assumption that was used when deriving the Green–Naghdi system, it is natural to consider initial data satisfying

$$(\boldsymbol{u}+\mu\mathcal{T}[1-\beta b,\beta\nabla b]\boldsymbol{u})\,|_{t=0} = \nabla\psi,$$

and hence $\mathrm{curl}(\boldsymbol{u} + \mu\mathcal{T}[1-\beta b, \beta\nabla b]\boldsymbol{u})\,|_{t=0} = 0$. However, adding this constraint to the additional constraint $(8.29)_1$ leaves only the trivial solution. In other words, in the irrotational framework that is the only one—at least so far—for which the Green–Naghdi system is rigorously justified, the incompressible (or rigid-lid) component vanishes; see also [306] for a similar discussion on the water waves system. One thus expects that the flow is asymptotically described by (8.30) only, in the limit $\varepsilon \to 0$ and in the irrotational setting.

However, when trying to adapt the usual strategy for rigorously proving such behavior, one immediately encounters a serious difficulty in the physically relevant situation of non-trivial topography, which transpires in the the fact that our lower bound for the existence time in Theorem 8.3 depends on the size of the bottom variations in addition to the size of the initial data. When transcribed to system (8.28), this means that we are not able to obtain a lower bound on the existence time of its solutions which is uniform with respect to ε, unless $\beta = \mathcal{O}(\varepsilon)$.

For the Saint-Venant system, that is setting $\mu = 0$, Bresch and Métivier [62] have obtained such a uniform lower bound without any restriction on the amplitude bathymetry. The strategy consists in estimating first the time derivatives of the solution, and then using the system to deduce estimates on space derivatives. A related strategy (in the sense that we look for operators commuting with the singular component of the system) amounts to remark that for any $n \in \mathbb{N}$, one can control the L^2-norm of

$$\zeta_n \stackrel{\text{def}}{=} (\nabla\cdot(1-b)\nabla)^n\zeta, \qquad \boldsymbol{u}_n \stackrel{\text{def}}{=} (\nabla(1-b)\nabla\cdot)^n\boldsymbol{u}$$

by exhibiting the quasilinear system satisfied by $(\zeta_n, \boldsymbol{u}_n)$ and applying simple energy estimates. This allows to control the H^{2n}-norm of $\zeta, \boldsymbol{u}$, provided that the initial data and bottom topography are sufficiently regular. One expects a similar strategy to work for the water waves system (partial results have been obtained by the method of time derivatives in [309]), that is to control

$$\zeta_n \stackrel{\text{def}}{=} (\frac{1}{\mu}\mathcal{G}^\mu[0,\beta b])^n\zeta, \qquad \psi_n \stackrel{\text{def}}{=} (\frac{1}{\mu}\mathcal{G}^\mu[0,\beta b])^n\psi$$

Since $\mathcal{G}^\mu$ is an order-one operator, controlling ζ_n, ψ_n indeed allows to control higher regularities on ζ, ψ. The strategy however fails for the Green–Naghdi system, as the corresponding operator, namely (see eq. (8.1))

$$\frac{1}{\mu}G^\mu[0,\beta b]\bullet \approx -\nabla\cdot\Big((1-\beta b)\mathfrak{T}^\mu[1-\beta b,\beta\nabla b]^{-1}\big\{(1-\beta b)\nabla\bullet\big\}\Big)$$

is of order zero. One could easily propose different systems that do not suffer from such a shortcoming, by adding the effect of surface tension as in [309]—but then the result would depend on the size of the surface tension parameter—or modifying the system without hurting its consistency as in [307]—but then the model would presumably lose the variational structure.

In the flat bottom case ($\beta \equiv 0$) and in dimension $d = 2$, the rigorous justification of eq. (8.30) as a valid approximation (that is the fact that we can neglect quadratic nonlinearities) should follow from Strichartz estimates (see [34, § 8.3]) as in the work of Ukai [403] and Asano [27] (see also Theorem 6.16) for the weakly compressible Euler equation (that is when $\mu = 0$), although our setting is simpler since we do not have to consider the interaction with the "incompressible" component, which is trivial by the foregoing discussion. Notice finally that a by-product of this analysis (see the discussion in [190]) would prevent the possibility of a singularity formation in a time interval uniform with respect to ε for solutions to eq. (8.28) and hence in a time interval of size $\mathcal{O}(1/\varepsilon)$ in the original equations, eq. (8.6).

9 The Favrie–Gavrilyuk approximation

A difficulty arises when one tries to solve numerically the initial-value problem associated with the Green–Naghdi system, say eq. (8.6), as it is found necessary to invert the elliptic operator

$$\mathfrak{T}[h,\beta\nabla b]:\boldsymbol{u}\mapsto h\boldsymbol{u}-\frac{1}{3}\nabla(h^3\nabla\cdot\boldsymbol{u})+\frac{1}{2}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big)-h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}\Big)+h(\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b).$$

This is only a technical difficulty in the proof of the local well-posedness of the Cauchy problem (see Section 8.6, and in particular Lemma 8.9), but remains a severe issue for practical numerical simulations, as the cost of inverting this operator at each time step can be prohibitive, especially in dimension $d=2$. We refer to [105, 275, 168, 281, 323, 321, 358, 196, 8, 148, 151, 25] for several numerical schemes adapted to the Green–Naghdi system. The aforementioned issue is addressed in particular in [139, 271, 166], where the authors introduce a new class of models which enjoy the same precision as the original Green–Naghdi system—as an asymptotic model for the water waves equations—but for which the elliptic operator playing the role of $\mathfrak{T}[h]$ is independent of time. However this new model does not enjoy the nice properties of the Green–Naghdi system, and in particular the conservation of energy. A different direction of investigation is proposed by Favrie and Gavrilyuk in [181]. They relax the Lagrangian associated with the variational formulation of the Green–Naghdi system (see Section 8.1.3) by introducing new variables: instead of the one displayed in eq. (8.12), they consider (here we extend their framework to non-trivial topographies) the Lagrangian density

$$\mathcal{L}^\lambda(h,\boldsymbol{u},\dot h,\dot b,b)=\frac{h}{2}\big(|\boldsymbol{u}|^2+\frac{1}{3}(\dot\eta+\frac{3}{2}\dot b)^2+\frac{1}{4}\dot b^2\big)-\frac{g}{2}(h+b)^2-\frac{\lambda}{6h}(\eta-h)^2. \tag{9.1}$$

From this they derive the system which we refer to as the ***Favrie–Gavrilyuk system***:

$$\begin{cases}\partial_t h+\nabla\cdot(h\boldsymbol{u})=0,\\ \partial_t\boldsymbol{u}+g\nabla(h+b)+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\mathcal{P}^\lambda[\eta,h,b,\boldsymbol{u}]=\boldsymbol{0},\\ \partial_t\eta+\boldsymbol{u}\cdot\nabla\eta=w-\frac{3}{2}\dot b,\\ \partial_t w+\boldsymbol{u}\cdot\nabla w=-\frac{\lambda}{h^2}(\eta-h),\end{cases} \tag{9.2}$$

where

$$\mathcal{P}^\lambda[\eta,h,b,\boldsymbol{u}]\stackrel{\text{def}}{=}-\frac{\lambda}{3h}\nabla\Big(\frac{\eta}{h}(\eta-h)\Big)-\frac{\lambda}{2h^2}(\eta-h)\nabla b+\frac{1}{4}\ddot b\nabla b. \tag{9.3}$$

System (9.2) is a quasilinear system of balance laws with two additional unknowns and a free parameter, λ, which should be chosen large in order to hope for valuable approximations. Hence the system has some similarity with the widely studied—both from a theoretical (see references in Section 9.6) and numerical (see *e.g.* [259, 377, 304, 333, 334, 135, 236] and references therein) point of view—Euler system in the low Mach number or weakly compressible limit, and its is hoped that this purely quasilinear structure can be useful to devise efficient numerical schemes with good properties; see in particular [249] where Perfectly Matched Layer (PML) boundary conditions are proposed.

The hope is that in the limit $\lambda\to\infty$, solutions to eq. (9.2) approach solutions to eq. (8.9). Indeed we expect, using the fourth and third equations of eq. (9.2):

$$\eta=h+\mathcal{O}(\lambda^{-1})\quad\text{ and }\quad\lambda\big(\eta-h\big)=-h^2\ddot\eta=-h^2\ddot h+\mathcal{O}(\lambda^{-1}),$$

and we recover eq. (8.9) when plugging the truncated approximations in the second equation. A rigorous proof to the above formal reasoning is given (in the flat bottom situation) in [158], however with the important assumption that the initial data should be sufficiently well-prepared. This work also provides some insights to how large λ should be chosen so that solutions to eq. (9.2) are approximate solutions to the water waves system which are "as good"—or at least "not much worse"—than the ones supplied by the Green–Naghdi system. We give precise statements in Section 9.5 and sketch the proofs in Section 9.6.

In order to provide valuable descriptions of the precision of the solutions to the Favrie-Gavrilyuk equation as approximate solutions to the Green–Naghdi system, we first need to non-dimensionalize eq. (9.2). Considering w as a vertical velocity, η as a vertical length and λ as a vertical acceleration times a vertical length (or the square of a vertical velocity), we obtain

$$\begin{cases} \partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u}) = 0, \\ \varepsilon\partial_t\boldsymbol{u} + \nabla(h+\beta b) + \varepsilon^2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\mathcal{P}^\lambda[\eta, h, \beta b, \varepsilon\boldsymbol{u}] = \boldsymbol{0}, \\ \partial_t\eta + \varepsilon\boldsymbol{u}\cdot\nabla\eta = \varepsilon w - \frac{3}{2}\beta\dot{b}, \\ \varepsilon\partial_t w + \varepsilon^2\boldsymbol{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta - h\big), \end{cases} \tag{9.4}$$

where $\mathcal{P}^\lambda$ is defined in eq. (9.3). We shall consider the interplay of λ being large and μ being small.

Remark 9.1. *A somewhat more standard approach of artificial relaxation*[45] *based on the formulation eq. (8.8) consists in relaxing the constraint*[46]

$$v = \frac{\dot{h}}{3} + \frac{\dot{b}}{2}$$

with the following

$$\partial_t(hq) + \nabla\cdot(hq\boldsymbol{u}) = -\lambda\Big(v - \frac{\dot{h}}{3} - \frac{\dot{b}}{2}\Big)$$

This yields the system

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{h}\nabla(hq) + (\nabla b)(\frac{3}{2h}q + \frac{1}{4}\ddot{b}) = \boldsymbol{0}, \\ \partial_t v + \boldsymbol{u}\cdot\nabla v = \frac{q}{h}, \\ \partial_t(hq) + \nabla\cdot(hq\boldsymbol{u}) = -\lambda\Big(v - \frac{\dot{h}}{3} - \frac{\dot{b}}{2}\Big). \end{cases}$$

Denoting $\eta = h - \frac{3}{\lambda}hq$ and $w = 3v$, we obtain

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{\lambda}{3h}\nabla(\eta - h) + (\nabla b)(-\frac{\lambda}{2h^2}(\eta-h) + \frac{1}{4}\ddot{b}) = \boldsymbol{0}, \\ \partial_t\eta + \nabla\cdot(\eta\boldsymbol{u}) = w - \dot{h} - \frac{3}{2}\dot{b}, \\ \partial_t w + \boldsymbol{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta - h\big). \end{cases} \tag{9.5}$$

It turns out that eq. (9.5) is not exactly the Favrie–Gavrilyuk system, eq. (9.2) (despite having the same conserved energy), but only up to negligible terms in the limit $\lambda \to \infty$, since we expect (and prove in Section 9.5) $\eta - h = \mathcal{O}(\lambda^{-1})$. All the results in Section 9.5 are easily adapted to eq. (9.5).

[45] see *e.g.* [236] for a discussion and several examples of hyperbolic systems with stiff relaxations.

[46] We do not relax the constraint on $v_{\rm b}$ but simply use $v_{\rm b} = \frac{3}{2}v + \frac{1}{4}\dot{b}$, and hence $q_{\rm b} = \frac{3}{2}q + \frac{1}{4}h\ddot{b}$ in eq. (8.8). The authors in [41] relax both constraints simultaneously—henceforth avoiding second-order differential operators acting on bottom topography contributions—and arrive at a first-order hyperbolic systems of balance laws involving four additional unknowns instead of two; see also [206] for another relaxation system with three additional unknowns, and [56, (104)–(106)] for yet another system with two additional unknowns. See also [179, 180, 207] for previous approaches (with two additional unknowns) where the effect of bottom topography was not fully taken into account. Interestingly, Richard [364] arrives at a very similar model in the framework of (weakly) compressible flows; there the velocity of acoustic waves (to the square) plays role of the parameter λ.

For the sake of completeness, let me introduce yet another relaxation approach, which was brought up to my attention by D. Bresch. Relaxing differently the constraints in eq. (8.8), we arrive at

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t \boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{\lambda}{h}\nabla\big(h(v + \frac{h\nabla\cdot\boldsymbol{u}}{3} - \frac{\dot b}{2})\big) - \frac{\lambda}{h}(\nabla b)\big(v_{\rm b} + \frac{h\nabla\cdot\boldsymbol{u}}{2} - \dot b\big) = \boldsymbol{0}, \\ \partial_t(hv) + \nabla\cdot(hv\boldsymbol{u}) = -\lambda\big(v + \frac{h\nabla\cdot\boldsymbol{u}}{3} - \frac{\dot b}{2}\big), \\ \partial_t(hv_{\rm b}) + \nabla\cdot(hv_{\rm b}\boldsymbol{u}) = -\lambda\big(v_{\rm b} + \frac{h\nabla\cdot\boldsymbol{u}}{2} - \dot b\big). \end{cases} \tag{9.6}$$

This approach is quite different from the aforementioned ones: the resulting system is of parabolic type as second-order operators acting on $\boldsymbol{u}$ *appear in the second equation. Depending on the point of view, this feature can be seen as problematic as the approach introduces spurious energy dissipation, or desirable as the resulting system may damp undesirable high frequency oscillations.*

9.1 Variational structure

By construction, the Favrie–Gavrilyuk system, eq. (9.4), enjoys a Lagrangian structure related to the one of the Green–Naghdi system discussed in Section 8.1.3. One can wonder whether the system also enjoys variational structures related to the other ones discussed in Section 8.1 and which are more directly comparable to the ones of the water waves equations; see Section 2.2. The positive answer is provided below.

9.1.1 Another Lagrangian structure

In the spirit of Section 8.1.2, we can obtain the Favrie–Gavrilyuk system, eq. (9.4), through Hamilton's principle on the following Lagrangian action function:

$$\begin{aligned}\mathscr{L}_{\rm FG} = \int_{t_0}^{t_1}\int_{\mathbb{R}^d} &\frac12 h|\varepsilon\boldsymbol{u}|^2 + \frac{\mu}{6}h(\varepsilon w)^2 + \frac{\mu}{8}h(\beta\dot b)^2 - \frac12\big((h+\beta b)^2 - 1\big) - \frac{\lambda\mu}{6h}(\eta - h)^2 \\ &+ \varepsilon\varphi\big(\partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u})\big) + \varepsilon\vartheta\big(\partial_t\eta + \varepsilon\boldsymbol{u}\cdot\nabla\eta - \varepsilon w + \frac32\beta\dot b\big)\,{\rm d}\boldsymbol{x}\,{\rm d}t.\end{aligned}$$

Indeed, one recovers obviously the first and third equation from

$$0 = \delta_\varphi\mathscr{L}_{\rm FG} = \partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u}) \quad ; \quad 0 = \delta_\vartheta\mathscr{L}_{\rm FG} = \partial_t\eta + \varepsilon\boldsymbol{u}\cdot\nabla\eta - \varepsilon w + \frac32\beta\dot b.$$

Now, we have

$$\delta_w\mathscr{L}_{\rm FG} = \frac{\mu}{3}h\varepsilon^2 w - \varepsilon^2\vartheta \quad ; \quad \delta_\eta\mathscr{L}_{\rm FG} = -\frac{\lambda\mu}{3h}(\eta - h) - \varepsilon\partial_t\vartheta - \varepsilon^2\nabla\cdot(\boldsymbol{u}\vartheta),$$

and hence $\delta_w\mathscr{L}_{\rm FG} = \delta_\eta\mathscr{L}_{\rm FG} = 0$ yields the fourth equation. Finally,

$$\begin{aligned}\delta_{\boldsymbol{u}}\mathscr{L}_{\rm FG} &= h\varepsilon^2\boldsymbol{u} + \frac{\mu}{4}\varepsilon\beta h\nabla b(\beta\dot b) - \varepsilon^2 h\nabla\varphi + \varepsilon\vartheta(\varepsilon\nabla\eta + \frac32\beta\nabla b) \\ \delta_h\mathscr{L}_{\rm FG} &= \frac12|\varepsilon\boldsymbol{u}|^2 + \frac{\mu}{6}(\varepsilon w)^2 + \frac{\mu}{8}(\beta\dot b)^2 - (h+\beta b) + \frac{\lambda\mu}{6}(\frac{\eta^2}{h^2} - 1) - \varepsilon\partial_t\varphi - \varepsilon^2\boldsymbol{u}\cdot\nabla\varphi.\end{aligned}$$

Hence we define

$$\vartheta = \frac{\mu}{3}hw \quad ; \quad \varepsilon\nabla\varphi = \varepsilon\boldsymbol{u} + \frac{\mu}{4}\beta\nabla b(\beta\dot b) + \frac{\mu}{3}\varepsilon w(\nabla\eta + \frac32\beta\nabla b) \tag{9.7}$$

and $\delta_{\boldsymbol{u}}\mathscr{L}_{\rm FG} = \boldsymbol{0}$, $\delta_h\mathscr{L}_{\rm FG} = 0$ yields

$$(\partial_t + \varepsilon\boldsymbol{u}^\perp\,{\rm curl})(\varepsilon\nabla\varphi) + \varepsilon^2\nabla(\boldsymbol{u}\cdot\nabla\varphi) = \nabla\left(\frac12|\varepsilon\boldsymbol{u}|^2 + \frac{\mu}{6}(\varepsilon w)^2 + \frac{\mu}{8}(\beta\dot b)^2 - (h+\beta b) + \frac{\lambda\mu}{6}(\frac{\eta^2}{h^2} - 1)\right).$$

This equation can be found to be equivalent to the second equation in eq. (9.4). Notice an irrotationality condition needs to be satisfied in order to define $\nabla\varphi$, which echoes the physical irrotationality condition $\boldsymbol{v} = \nabla\psi$ that already appeared on the Green–Naghdi system. The above equations provide a conservative form of the Favrie–Gavrilyuk system, provided this irrotationality condition is (initially) satisfied.

9.1.2 Hamiltonian structure

Let us now restrict to the time-independent framework, $\partial_t b = 0$. Notice that plugging the identities (9.7) into the Lagrangian $\mathscr{L}_{\rm FG}$, and after integrating by parts, we find

$$\mathscr{L}_{\rm FG} = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \varepsilon\varphi\partial_t h + \varepsilon\vartheta\partial_t\eta \,\mathrm{d}\mathbf{x} - \mathscr{H}_{\rm FG}\,\mathrm{d}t$$

where we define

$$\mathscr{H}_{\rm FG}(h,\eta,\varepsilon\varphi,\varepsilon\vartheta) \stackrel{\rm def}{=} \int_{\mathbb{R}} \frac{1}{2}h\varepsilon^2\boldsymbol{u}\cdot\big(\boldsymbol{u}+\frac{\mu\beta}{4}(\boldsymbol{u}\cdot\nabla b)(\beta\nabla b)\big)+\frac{3\mu^{-1}}{2h}(\varepsilon\vartheta)^2+\frac{1}{2}\big((h+\beta b)^2-1\big)+\frac{\lambda\mu}{6h}(\eta-h)^2,$$

and, therein, we denote $\boldsymbol{u}[h,\eta,\varepsilon\varphi,\varepsilon\vartheta,b]$ the solution to

$$\boldsymbol{u} + \frac{\mu\beta}{4}(\boldsymbol{u}\cdot\nabla b)(\beta\nabla b) = \nabla\varphi - \frac{\vartheta}{h}(\nabla\eta + \frac{3}{2}\beta\nabla b).$$

Hence from Hamilton's principle we infer the Hamiltonian structure of eq. (9.4):

$$\partial_t \begin{pmatrix} h \\ \varepsilon\varphi \\ \eta \\ \varepsilon\vartheta \end{pmatrix} + \begin{pmatrix} -\delta_{\varepsilon\varphi}\mathscr{H}_{\rm FG} \\ +\delta_h\mathscr{H}_{\rm FG} \\ -\delta_{\varepsilon\vartheta}\mathscr{H}_{\rm FG} \\ +\delta_\eta\mathscr{H}_{\rm FG} \end{pmatrix} = \mathbf{0}. \tag{9.8}$$

This formulation is in some sense the counterpart of the formulation of the Green–Naghdi system using Zakharov's canonical variables, that is eq. (8.2).

9.2 Group symmetries and preserved quantities

Recall that thanks to its Hamiltonian structure, by Noether's theorem, symmetry groups of the Favrie–Gavrilyuk system relate to conserved quantities of the system. We have in particular the following group symmetries: using the above notations, if $(h,\varphi,\eta,\vartheta)$ is a solution to eq. (9.8), then for any $\theta\in\mathbb{R}$, $(h^\theta,\varphi^\theta,\eta^\theta,\vartheta^\theta)$ also satisfies eq. (9.8), where

- Variation of base level for the velocity potential

$$\big(h^\theta,\varphi^\theta,\eta^\theta,\vartheta^\theta\big)(t,\boldsymbol{x}) \stackrel{\rm def}{=} \big(h,\varphi+\theta,\eta,\vartheta\big)(t,\boldsymbol{x}).$$

- Horizontal translation along the direction $\boldsymbol{e}\in\mathbb{R}^d$ (in the flat bottom case)

$$\big(h^\theta,\varphi^\theta,\eta^\theta,\vartheta^\theta\big)(t,\boldsymbol{x}) \stackrel{\rm def}{=} \big(h,\varphi,\eta,\vartheta\big)(t,\boldsymbol{x}-\theta\boldsymbol{e}).$$

- Time translation (in the fixed-bottom case)

$$\big(h^\theta,\varphi^\theta,\eta^\theta,\vartheta^\theta\big)(t,\boldsymbol{x}) \stackrel{\rm def}{=} \big(h,\varphi,\eta,\vartheta\big)(t-\theta,\boldsymbol{x}).$$

We have the following corresponding preserved quantities of eq. (9.8) (or, equivalently, eq. (9.4))

- Excess of mass

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = 0, \qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \,\mathrm{d}\boldsymbol{x} \qquad (\text{where } h \stackrel{\text{def}}{=} 1 + \varepsilon\zeta - \beta b).$$

- Horizontal impulse (in the flat bottom case)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}_{\mathrm{FG}} = 0, \qquad \mathscr{I}_{\mathrm{FG}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} h\nabla\varphi + \eta\nabla\vartheta \,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0),$$

and we note that using eq. (9.7) we have (in the flat bottom case) the following identity:

$$\mathscr{I}_{\mathrm{FG}} = \varepsilon h\boldsymbol{u} + \frac{\mu\varepsilon}{3}\nabla(h\eta w),$$

hence we see a direct correspondence with respect to the conservation of the horizontal momentum (in the flat bottom case)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M} = 0, \qquad \mathscr{M} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} h\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$

- Total energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathrm{FG}} = 0,$$

where we recall that

$$\mathscr{H}_{\mathrm{FG}} = \int_{\mathbb{R}} \frac{1}{2}h\varepsilon^2\boldsymbol{u}\cdot\big(\boldsymbol{u} + \frac{\mu\beta}{4}(\boldsymbol{u}\cdot\nabla b)(\beta\nabla b)\big) + \frac{\mu}{6h}(\varepsilon h w)^2 + \frac{1}{2}\big((h+\beta b)^2 - 1\big) + \frac{\lambda\mu}{6h}(\eta - h)^2.$$

The quantities are preserved in a stronger sense: their integrand satisfies a conservation law, which are written out explicitly in [68, Appendix A] up to $\mathcal{O}(\beta^2)$ terms since the contributions of $\frac{1}{4}\dot{b}^2$ in the definition of Lagrangian, eq. (9.1), are neglected therein.

9.3 Modal analysis

Linearizing eq. (9.2) around the rest state (in the flat bottom situation) yields

$$\begin{cases} \partial_t h + d\nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\boldsymbol{u} + g\nabla h - \frac{\lambda}{3d}\nabla(\eta - h) = \mathbf{0}, \\ \partial_t\eta = w, \\ \partial_t w = -\frac{\lambda}{d^2}(\eta - h), \end{cases}$$

where d is the layer depth at infinity. From this we infer the dispersion relation

$$(\omega(|\boldsymbol{\xi}|)^2 - gd|\boldsymbol{\xi}|^2)(1 - \frac{\omega(|\boldsymbol{\xi}|)^2 d^2}{\lambda}) + \frac{1}{3}d^2|\boldsymbol{\xi}|^2\omega(|\boldsymbol{\xi}|)^2 = 0$$

provided $\lambda \neq \omega(|\boldsymbol{\xi}|)^2 d^2$. From this we infer, denoting $\tilde{\lambda} = \frac{\lambda}{gd}$,

$$\frac{\omega_\pm(|\boldsymbol{\xi}|)^2}{g/d} = \frac{1}{2}(d|\boldsymbol{\xi}|)^2 + \frac{\tilde{\lambda}}{2}\big(1 + \frac{1}{3}(d|\boldsymbol{\xi}|)^2\big) \pm \frac{1}{6}\sqrt{(3+\tilde{\lambda})^2(d|\boldsymbol{\xi}|)^4 + (6\tilde{\lambda}^2 - 18\tilde{\lambda})(d|\boldsymbol{\xi}|)^2 + 9\tilde{\lambda}^2}.$$

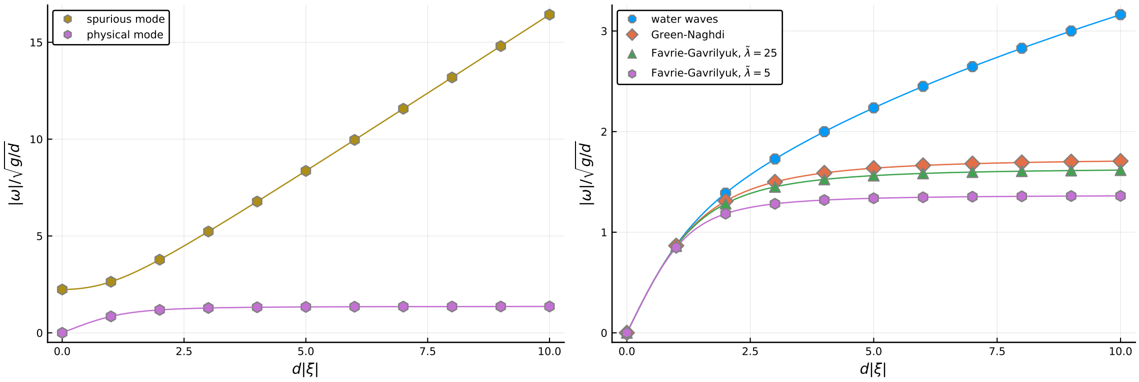
Hence there are two modes of propagation, corresponding to either ω_+ and ω_-. Denoting $\omega_{\mathrm{GN}}(\boldsymbol{\xi})$ the wave angular frequency at wave vector $\boldsymbol{\xi}$ associated with the linearized Green–Naghdi system, that is (see Section 8.3):

$$\omega_{\mathrm{GN}}(\boldsymbol{\xi})^2 = \frac{gd|\boldsymbol{\xi}|^2}{1 + \frac{1}{3}d^2|\boldsymbol{\xi}|^2},$$

we have for any $\boldsymbol{\xi} \neq \mathbf{0}$ and any $\tilde{\lambda} > 0$ (see Figure 9.1)

$$|\omega_-|(\boldsymbol{\xi}) < |\omega_{\rm GN}|(\boldsymbol{\xi}) < |\omega_+|(\boldsymbol{\xi}) \quad ; \quad |\omega_-|(\boldsymbol{\xi}) \to |\omega_{\rm GN}|(\boldsymbol{\xi})(\lambda \to \infty) \quad ; \quad \omega_+^2(\boldsymbol{\xi}) > \omega_+^2(\mathbf{0}) = \lambda/d^2.$$

Hence the mode of propagation associated with ω_- represents the "physical" wave which approach the solution to the Green–Naghdi system. The mode of propagation associated with ω_+ represents spurious waves. Notice that while they oscillate rapidly, their group velocity vanishes at small wavenumbers, and hence we may expect a resonant interaction between the two modes at the nonlinear level. This is an indication that we need to ensure that the spurious mode is small through appropriate initial conditions so as to ensure good properties at the nonlinear level.



(a) Spurious and physical modes.

(b) Comparison.

Figure 9.1: Wave frequencies, $|\omega_\pm|(|\boldsymbol{\xi}|)$, corresponding to the Favrie–Gavrilyuk system, and comparison with the water waves and the Green–Naghdi systems.

9.4 Traveling waves

The analysis of the existence of solitary waves to the Favrie–Gavrilyuk system (in the flat bottom situation) was provided in [249, Annex A], and a numerical comparison with respect to the explicit ones of the Green–Naghdi system (see Section 8.4) is carried out in [181, 68]. Yet a quantitative comparison of their profiles and the thorough analysis of the interplay between the parameter λ and the Froude number (that is the normalized velocity c of the traveling wave; after rescaling, we can relate the limit $c \searrow 1$ with the long wave limit, $\varepsilon \approx \mu \searrow 0$) remains to be accomplished.

9.5 Rigorous justification

In this section we provide a rigorous justification of the Favrie–Gavrilyuk system eq. (9.4), as an approximate system for the Green–Naghdi model (and hence the water waves equations). The detailed proof of the results stated below can be found in [158] in the flat bottom framework, the extension to small but non-flat bottom being straightforward. A sketch of the proofs is presented in Section 9.6.

Because system (9.4) is a symmetrizable hyperbolic quasilinear system, the well-posedness of the corresponding initial-value problem is provided by standard theory; see *e.g.* [49].

Theorem 9.2 (Small-time existence)**.** *Let $s \in \mathbb{R}$ with $s > 1 + d/2$. Then for any $\lambda, \mu, \varepsilon, \beta \in (0, \infty)$, for any $b \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^d))$ and any $\boldsymbol{U}_0 = (\zeta_0, \boldsymbol{u}_0, \varsigma_0, w_0) \in H^s(\mathbb{R}^d)^{d+3}$ satisfying the hyperbolicity condition*

$$h_0 \stackrel{\text{def}}{=} 1 + \varepsilon\zeta_0 - \beta b\,|_{t=0} \geq h_\star > 0, \tag{9.9}$$

there exists a unique maximal strong solution $\boldsymbol{U} = (\zeta, \boldsymbol{u}, \varsigma, w) \in \mathcal{C}^0([0,T^\star); H^s(\mathbb{R}^d))^{d+3}$ *to* (9.4) *(denoting* $h = 1 + \varepsilon\zeta - \beta b$ *and* $\eta = 1 + \varepsilon\varsigma - \beta b$*) with* $\boldsymbol{U}\,|_{t=0} = \boldsymbol{U}_0$*, where* $T^\star > 0$ *is the maximal time of existence.*

Solutions to the Favrie–Gavrilyuk system are valuable approximations to the Green–Naghdi system (in the sense of consistency) as long as several space and time derivatives of the solutions are uniformly bounded, as stated below.

Theorem 9.3 (Consistency)**.** *Let* $s \geq 2$ *and* $h_\star, M^\star > 0$*. There exists* $C > 0$ *such that for any* $\lambda, \mu, \varepsilon, \beta \in (0,\infty)$*,* $T > 0$ *and* $\boldsymbol{U} = (\zeta, \boldsymbol{u}, \varsigma, w) \in \bigcap_{j=0}^{2} \mathcal{C}^j([0,T]; H^{s-j}(\mathbb{R}^d))^{d+3}$ *strong solution to* (9.4) *satisfying* $h \stackrel{\rm def}{=} 1 + \varepsilon\zeta - \beta b \geq h_\star > 0$ *and*

$$\sup_{t\in[0,T]} \left(\big|\varepsilon \boldsymbol{U}\big|_{H^s} + \big|\varepsilon\partial_t \boldsymbol{U}\big|_{H^{s-1}} + \big|\varepsilon\partial_t^2 \boldsymbol{U}\big|_{H^{s-2}} + \big|\beta b\big|_{H^s} + \big|\beta\partial_t b\big|_{H^{s-1}} + \big|\beta\partial_t^2 b\big|_{H^{s-2}}\right) \leq M^\star,$$

the component $(\zeta, \boldsymbol{u})$ *satisfies the Green–Naghdi system, eq.* (8.7)*, up to a small remainder,* i.e.

$$\begin{cases} \partial_t h + \varepsilon\nabla\cdot(h\boldsymbol{u}) = 0, \\ \varepsilon\partial_t \boldsymbol{u} + \nabla(h+\beta b) + \varepsilon^2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu\mathcal{P}[h,\beta b,\varepsilon\boldsymbol{u}] = \varepsilon r, \end{cases} \tag{9.10}$$

with

$$\varepsilon\lambda^{-1} r = \frac{\mu}{3h}\nabla\Big(h(\ddot{\eta} + \frac{3}{2}\beta\ddot{b})(h-\eta) - h^2(\ddot{h}-\ddot{\eta})\Big) + \frac{\mu}{2}(\beta\nabla b)(\ddot{h}-\ddot{\eta}) \in \mathcal{C}^0([0,T]; H^{s-3}(\mathbb{R}^d))$$

and if one has additionally $\partial_t^j w \in L^1(0,T; H^{s+1-j}(\mathbb{R}^d))$ *for any* $j \in \{0,\dots,3\}$*, then*

$$\big|r\big|_{H^{s-3}} \leq C\mu\lambda^{-1}\sum_{j=0}^{3}\big|\partial_t^j w\big|_{H^{s+1-j}}.$$

Proof. The formula for r comes from straightforward manipulations on eq. (9.2). By Proposition II.7 and Proposition II.11 we infer $r \in \mathcal{C}^0([0,T]; H^{s-3}(\mathbb{R}^d))$ and

$$\forall t \in [0,T], \qquad \big|\varepsilon r(t,\cdot)\big|_{H^{s-3}} \leq C(M^\star)\mu\sum_{j=0}^{2}\big|\partial_t^j(\eta-h)(t,\cdot)\big|_{H^{s-j}}.$$

The desired estimate is deduced, applying Proposition II.7 to the last equation of eq. (9.2). □

With this result, we may apply Theorem 8.6 (see [189] for time-dependent topographies) to prove that the solutions to the Favrie–Gavrilyuk system produce approximate solutions (*i.e.* at a distance $\mathcal{O}(\mu\lambda^{-1})$ to the exact solution of the Green–Naghdi system with corresponding initial data) as long as the assumptions of Theorem 9.3 remain valid. Of course, there is no reason to hope *a priori* that the maximal solutions to eq. (9.2) with initial data in a given ball of $H^s(\mathbb{R}^d)^{d+3}$—or continuously embedded normed spaces—satisfy the assumption of Theorem 9.3 on a relevant time interval uniformly with respect to the parameters λ (large) and μ (small). Our main result is to prove that, under a very strong restriction on the bottom topography, it is possible to *prepare the initial data* (for η and w) so that such property holds.

All our results from now on are restricted to the following set of parameters

Definition (Favrie–Gavrilyuk asymptotic regime)**.** *Given* $\nu^\star > 0$*, we let*

$$\mathfrak{p}_{\rm FG} = \big\{(\lambda,\mu,\varepsilon,\beta) \in [0,\infty)^2 \times [0,1]^2 \;:\; \mu + \lambda^{-1} + (\lambda\mu)^{-1} \leq \nu^\star\big\}.$$

Above $\nu^\star$ should be thought as a prescribed constant of order of magnitude one. The results are valid for any choice of $\nu^\star > 0$, but of course not uniformly as $\nu^\star \to \infty$. The first two restrictions in $\mathfrak{p}_{\rm FG}$ on μ and λ^{-1} are harmless in the shallow water regime (Definition III.2) and since we can—and want to—choose λ large. The assumption on $\lambda\mu$ hints at a possibly non-uniform behavior with respect to small values of μ. Our statements include however another very restrictive assumption which is essentially an upper bound on $\lambda^{1/2}\beta$, and basically amounts in restricting our framework to the flat bottom situation for λ large. For simplicity and consistently with our assumptions so far, we also restrict ourselves later on to the time-independent situation.

In order to state the main result, we further prepare the Favrie–Gavrilyuk system through a change of variables which allows to balance singular terms. Recalling $h = 1 + \varepsilon\zeta - \beta b$ and $\eta = 1 + \varepsilon\varsigma - \beta b$ and denoting

$$\iota \stackrel{\rm def}{=} \varepsilon^{-1}(\mu\lambda)^{1/2}(\eta - h) \quad ; \quad \kappa \stackrel{\rm def}{=} \mu^{1/2}h^{-1}w, \tag{9.11}$$

we see that (9.4) is equivalent to

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = -\varepsilon^{-1}\beta\partial_t b, \\ \partial_t\boldsymbol{u} + \varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\zeta - \frac{1}{3h}\nabla\Big((\mu\lambda)^{1/2}\iota + \varepsilon\frac{\iota^2}{h}\Big) = \Big(\frac{1}{2h^2}(\mu\lambda)^{1/2}\iota - \frac{\beta\mu}{4\varepsilon}\ddot{b}\Big)(\beta\nabla b), \\ \partial_t\iota + \varepsilon\boldsymbol{u}\cdot\nabla\iota - \lambda^{1/2}(h\kappa + \mu^{1/2}h\nabla\cdot\boldsymbol{u}) = -\frac{3}{2}(\lambda\mu)^{1/2}\varepsilon^{-1}\beta\dot{b}, \\ \partial_t(h\kappa) + \varepsilon\boldsymbol{u}\cdot\nabla(h\kappa) + \lambda^{1/2}h^{-2}\iota = 0. \end{cases} \tag{9.12}$$

The following result shows that one can control solutions to eq. (9.12) on a time interval uniform with respect to λ (sufficiently large) and μ provided that the initial data is well-prepared, and the bottom topography sufficiently small.

Theorem 9.4 (Large-time existence)**.** *Let $m, s \in \mathbb{N}$ with $s > 1+d/2$, $1 \leq m \leq s$, and $h_\star, M_0^\star, \nu^\star > 0$. Set also $\delta_\star \in (0,1)$ if $m = s$. There exist $\nu_\star, T, C_0 > 0$ such that for any $(\lambda,\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm FG}$ satisfying $\lambda\mu \geq \nu_\star$, any $\tilde\lambda \in [1,\lambda\mu]$, for any (time independent) $b \in H^{s+1}(\mathbb{R}^d)$, then any maximal strong solution (provided by Theorem 9.2) $\boldsymbol{V} = (\zeta,\boldsymbol{u},\iota,\kappa) \in \mathcal{C}^0([0,T^\star); H^s(\mathbb{R}^d))^{d+3}$ to eq. (9.12) with initial data satisfying $h\,|_{t=0} = 1 + \varepsilon\zeta - \beta b\,|_{t=0} \geq h_\star$ and, additionally,*

$$M_0 \stackrel{\rm def}{=} \varepsilon\sum_{j=0}^{m}\big|\partial_t^j\boldsymbol{V}(0,\cdot)\big|_{H^{s-j}} + \varepsilon\sum_{j=m+1}^{s}\tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j\boldsymbol{V}(0,\cdot)\big|_{H^{s-j}} + \lambda^{1/2}\beta\big|b\big|_{H^s} + (\lambda\mu)^{1/2}\beta\big|b\big|_{H^{s+1}} \leq M_0^\star, \tag{9.13}$$

one has $T^\star > (M_0T)^{-1}$ and for any $t \in [0,(M_0T)^{-1}]$,

$$\varepsilon\sum_{j=0}^{m}\big|\partial_t^j\boldsymbol{V}(t,\cdot)\big|_{H^{s-j}} + \varepsilon\sum_{j=m+1}^{s}\tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j\boldsymbol{V}(t,\cdot)\big|_{H^{s-j}} \leq C_0M_0.$$

If $m = s$, we can withdraw the condition $\lambda\mu \geq \nu_\star$ and replace it with the sharper

$$(1-\delta_\star)(\lambda\mu)^{1/2} \geq \max\Big\{\big|(\varepsilon\kappa h)\,|_{t=0}\,\big|, \frac{1}{2}\big|(\varepsilon\iota h^{-1})\,|_{t=0}\,\big|\Big\}.$$

It is important to notice that the above result holds with any $\tilde\lambda \in [1,\lambda\mu]$ but not with $\tilde\lambda = \lambda$ uniformly with respect to μ small. If it were the case, then the initial assumption on the high order time derivatives of the unknown ($j \geq m+1$ in eq. (9.13)) would be irrelevant as, using the system of equations eq. (9.12) and product estimates (see Proposition II.7), we can estimate high order time derivatives of $\boldsymbol{V}$ from lower-order time derivatives, with a cost of powers of $\lambda^{1/2}$:

$$\sum_{j=m+1}^{s}\lambda^{\frac{m-j}{2}}\big|\partial_t^j\boldsymbol{V}\big|_{H^{s-j}} \leq C\Big(h_\star^{-1},\nu^\star,\varepsilon\sum_{j=0}^{m}\big|\partial_t^j\boldsymbol{V}\big|_{H^{s-j}},\lambda^{1/2}\beta\big|b\big|_{H^s},(\lambda\mu)^{1/2}\beta\big|b\big|_{H^{s+1}}\Big)\times\sum_{j=0}^{m}\big|\partial_t^j\boldsymbol{V}\big|_{H^{s-j}}.$$

In particular, in the strong dispersion regime ($\mu \approx 1$), the explicit condition

$$\varepsilon\big|\boldsymbol{V}\,|_{t=0}\,\big|_{H^s}+\varepsilon\lambda^{1/2}\big|\iota\,|_{t=0}\,\big|_{H^{s-1}}+\varepsilon\lambda^{1/2}\big|\kappa\,|_{t=0}+\mu^{1/2}\nabla\cdot\boldsymbol{u}\,|_{t=0}\,\big|_{H^{s-1}}+\lambda^{1/2}\beta\big|b\big|_{H^s}+(\lambda\mu)^{1/2}\beta\big|b\big|_{H^{s+1}}\leq M_0^\star$$

is sufficient (applying Theorem 9.4 with $\tilde\lambda = \lambda\mu \approx \lambda$ and $m = 1$) to guarantee the existence and uniform control of the corresponding solution—but not its time derivatives—on a time interval uniform with respect to λ sufficiently large. In the weak dispersion regime ($\mu \ll 1$), the assumption eq. (9.13) is a strong constraint on the initial behavior of the solution, and it is natural to ask whether it is possible, for a given initial physical state defined by $\zeta\,|_{t=0}\,, \boldsymbol{u}\,|_{t=0}$ and $(\varepsilon, \beta, \mu) \in \mathfrak{p}_{\rm SW}$ (see Definition III.2), to provide initial data for the additional components $\eta\,|_{t=0}$ and $w\,|_{t=0}$ such that the corresponding solution to eq. (9.4) satisfies eq. (9.13) uniformly with respect to large λ and small μ. We answer positively below.

Theorem 9.5 (Preparation of the initial data)**.** *Let $s, m \in \mathbb{N}$, $s > d/2+1$, $s \geq m+1$ and $h_\star, M_0^\star, \nu^\star > 0$. There exists $C_m, C_m' > 0$ such that for any $(\lambda, \mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm FG}$, for any $b \in H^{s+1}(\mathbb{R}^d)$ (time independent), and for any $(\zeta_0, \boldsymbol{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ such that $h_0 = 1 + \varepsilon\zeta_0 - \beta b \geq h_\star > 0$ and*

$$M_0 \stackrel{\rm def}{=} \varepsilon\big|\zeta_0\big|_{H^s} + \varepsilon\big|\boldsymbol{u}_0\big|_{H^s} + \lambda^{1/2}\beta\big|b\big|_{H^s} + (\lambda\mu)^{1/2}\beta\big|b\big|_{H^{s+1}} \leq M_0^\star,$$

the following holds. There exists $c^{(j)} \in H^s(\mathbb{R}^d)$ for $j \in \{1, \cdots, m\}$ such that the strong solution to eq. (9.4) with initial data $\boldsymbol{U}^{(m)}\,|_{t=0} = (\zeta_0, \boldsymbol{u}_0, \eta_0^{(m)}, w_0^{(m)})$ where

$$w_0^{(m)} = \sum_{\substack{j \text{ odd}\\ 1\leq j\leq m}} \lambda^{-(j-1)/2}c^{(j)} \qquad \text{and} \qquad \eta_0^{(m)} = h_0 + \varepsilon \sum_{\substack{j \text{ even}\\ 2\leq j\leq m}} \lambda^{-j/2}c^{(j)} \tag{9.14}$$

satisfies

$$\sum_{j=0}^{m+1} \varepsilon\big|\partial_t^j \boldsymbol{U}^{(m)}(0,\cdot)\big|_{H^{s-j}} + \lambda\varepsilon\sum_{j=0}^{m}\big|\partial_t^j(\eta^{(m)} - h^{(m)})(0,\cdot)\big|_{H^{s-j}} \leq C_m\, M_0. \tag{9.15}$$

Moreover, we have for any $j \in \{1, \cdots, m\}$

$$\begin{cases} \varepsilon\big|c^{(j)}\big|_{H^{s-j}} + \varepsilon\mu^{j/2}\big|c^{(j)}\big|_{H^s} \leq C_m'\, M_0, & \text{if } j \text{ is even,}\\ \varepsilon\big|c^{(j)}\big|_{H^{s-j}} + \varepsilon\mu^{(j-1)/2}\big|c^{(j)}\big|_{H^{s-1}} \leq C_m'\, M_0, & \text{if } j \text{ is odd.}\end{cases} \tag{9.16}$$

We can choose $c^{(1)} = -h_0\nabla\cdot\boldsymbol{u}_0 + \frac{3}{2}\beta\boldsymbol{u}_0\cdot\nabla b$ and $c^{(2)}$ the unique solution to

$$\begin{aligned}\varepsilon\mathfrak{t}[h_0, \beta\nabla b]c^{(2)} = -\nabla\cdot\big(\mathcal{I}(\beta\nabla b)\boldsymbol{a}\big) + \varepsilon^2\boldsymbol{u}_0\cdot\nabla(\nabla\cdot\boldsymbol{u}_0) - \varepsilon^2(\nabla\cdot\boldsymbol{u}_0)^2&\\ + \frac{3}{2h_0}\beta\nabla b\cdot\mathcal{I}(\beta\nabla b)\boldsymbol{a} - \frac{3}{2h_0}(\varepsilon\boldsymbol{u}_0\cdot\nabla)^2(\beta b)&\end{aligned} \tag{9.17}$$

where $\mathcal{I}(\beta\nabla b) \stackrel{\rm def}{=} \big(\mathrm{Id} + \frac{\mu}{4}(\beta\nabla b)(\beta\nabla b)^\top\big)^{-1}$, $\boldsymbol{a} \stackrel{\rm def}{=} \nabla(h+\beta b) + (\varepsilon\boldsymbol{u}_0\cdot\nabla)(\varepsilon\boldsymbol{u}_0) + \frac14(\beta(\varepsilon\boldsymbol{u}\cdot\nabla)^2 b)(\beta\nabla b)$ and

$$\begin{aligned}\mathfrak{t}[h, \beta\nabla b]\psi \stackrel{\rm def}{=} h^{-3}\psi - \frac{\mu}{3}\nabla\cdot\big(h^{-1}\mathcal{I}(\beta\nabla b)\nabla\psi\big) - \frac{\mu}{2}\nabla\cdot\big(h^{-2}(\mathcal{I}(\beta\nabla b)(\beta\nabla b))\psi\big)&\\ + \frac{\mu}{2}h^{-2}(\beta\nabla b)\cdot(\mathcal{I}(\beta\nabla b)\nabla\psi) + \frac{3\mu}{4}h^{-3}(\beta\nabla b)\cdot(\mathcal{I}(\beta\nabla b)(\beta\nabla b))\psi.&\end{aligned} \tag{9.18}$$

Remark 9.6. *The expression for $c^{(2)}$ emerges when solving*

$$(h^2\ddot h - h^2\ddot\eta)\,|_{t=0} = \big(\lambda(\eta - h) + h^2(\ddot h + \frac{3}{2}\beta\ddot b)\big)\,|_{t=0} = \mathcal{O}(\lambda^{-1}).$$

The operator $\mathfrak{t}[h, \beta\nabla b]$ is one-to-one and onto if $\inf h > 0$ and is in some sense conjugate to $\mathfrak{T}^\mu$ defined in eq. (8.5), as at least in the flat bottom case, we have for any sufficiently regular $(h, \psi, \boldsymbol{u})$

$$\mathfrak{T}^\mu[h, \mathbf{0}](h^{-1}\nabla\psi) = \nabla(h^3\mathfrak{t}[h, \mathbf{0}]\psi) \qquad \text{and} \qquad \nabla\cdot(h^{-1}\mathfrak{T}^\mu[h, \mathbf{0}]\boldsymbol{u}) = \mathfrak{t}[h, \mathbf{0}](h^3\nabla\cdot\boldsymbol{u}).$$

Remark 9.7. *A direct application of Theorem 9.4 and Theorem 9.5 shows that—at least in the flat bottom framework—for any regular initial data $(\zeta_0, \boldsymbol{u}_0)$ satisfying the non-cavitation assumption, eq. (9.9), one may associate a solution to eq. (9.4) satisfying the estimates of Theorem 9.3 uniformly with respect to μ possibly small and λ sufficiently large, on the quadratic time scale (*i.e. *inversely proportional to the size of the initial data). Henceforth we produce $(\zeta, \boldsymbol{u})$ satisfying the Green–Naghdi system up to a residual of size $\mathcal{O}(\lambda^{-1}\mu)$,* i.e. *approximate solutions in the sense of consistency. By Theorem 8.6, we deduce that the difference between such solution and the exact solution to the Green–Naghdi system, eq. (8.6) with the same initial data is of size $\mathcal{O}(\lambda^{-1}\mu t)$ on the quadratic time scale. This should be compared with Theorem 8.7 stating that the solution to the Green–Naghdi system is at a distance $\mathcal{O}(\mu^2 t)$ to the solution of the full water waves equations with corresponding initial data on the same time scale. Hence the Favrie–Gavrilyuk system produces as precise approximate solutions for long gravity waves as the Green–Naghdi system itself as soon as $\lambda \gtrsim \mu^{-1}$ and the initial data for (η, w) is suitably chosen.*

9.6 Sketch of the proof

9.6.1 A three-scale singular limit

The main tool for proving Theorem 9.4 the above results are a priori estimates, which should hold uniformly with respect to the parameters $(\lambda, \mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{FG}}$. In order to obtain these estimates, we make use of a symmetric structure which is fairly easily deduced from the formulation (9.12): the system—when $\partial_t b = 0$ and $d = 2$—can be written as

$$\mathcal{S}_t(\boldsymbol{V})\partial_t \boldsymbol{V} + \mathcal{S}_x(\boldsymbol{V})\partial_x \boldsymbol{V} + \mathcal{S}_y(\boldsymbol{V})\partial_y \boldsymbol{V} = \lambda^{1/2}\mathsf{J}^\mu \boldsymbol{V} + \boldsymbol{G}(\boldsymbol{V}),$$

where $\mathcal{S}_t, \mathcal{S}_x, \mathcal{S}_y$ are smooth functions of $\boldsymbol{V}$ with values into symmetric matrices (and uniform with respect to λ, μ), J^μ is a skew-symmetric constant-coefficient differential operator, and $\boldsymbol{G}$ is a smooth function (uniform with respect to λ, μ). Moreover $\mathcal{S}_t$ is positive definite in a hyperbolicity domain containing a neighborhood of the origin.

We are looking at a *singular limit* problem. Such problems, and in particular incompressible or low Mach number limits in the context of fluid mechanics, have a very rich history, which we shall not recall but we let the interested reader refer to, *e.g.* , [379, 190, 299, 9] for comprehensive reviews. Due to the non-trivial symmetrizer in front of the time derivative, the linearized system does not appear to be uniformly well-posed in Sobolev spaces as $\lambda \to \infty$ since small perturbations of the initial data might cause large changes in solutions. This is a noteworthy feature of the incompressible limit of the non-isentropic Euler equations, as studied in particular in [312]. However, our problem is different in nature as we do not wish to deal with large oscillations in time but rather aim at discarding them as spurious products of the approximation procedure. Hence we willingly restrict our study to *well-prepared initial data*, and as such our work is more directly related to pioneering works of Browning and Kreiss [66], Klainerman and Majda [257], and Schochet [378]. In fact our proof of Theorem 9.4 closely follows the one of [378]; while the proof of Theorem 9.5 is strongly inspired by [66]. However in both cases the proof requires significant adaptations in order to take into account the fact that the singular operator, J^μ*, is not homogeneous of order one.* More precisely, we have

$$\mathsf{J}^\mu = \begin{pmatrix} 0 & \mathbf{0}^\top & 0 & 0 \\ \mathbf{0} & \mathrm{O}_d & \mu^{1/2}\nabla & \mathbf{0} \\ 0 & \mu^{1/2}\nabla^\top & 0 & 1 \\ 0 & \mathbf{0}^\top & -1 & 0 \end{pmatrix}. \tag{9.19}$$

The most serious novel difficulty stems from the fact that the contribution from order-zero terms in J^μ are less well-behaved than order-one contributions, and that the latter are multiplied by a vanishing prefactor as $\mu \to 0$. Hence the problem—as in [91] for instance— involves three different time scales, which is the origin of the shortcoming described below Theorem 9.4. I would like to explain now this discrepancy with the more standard setting—studied in the previously mentioned references—where J is homogeneous of order one. A toy model for the latter situation could be the following:

$$\partial_t u = \frac{1}{\epsilon} h \partial_x u \quad ; \quad \partial_t h = 0. \tag{9.20}$$

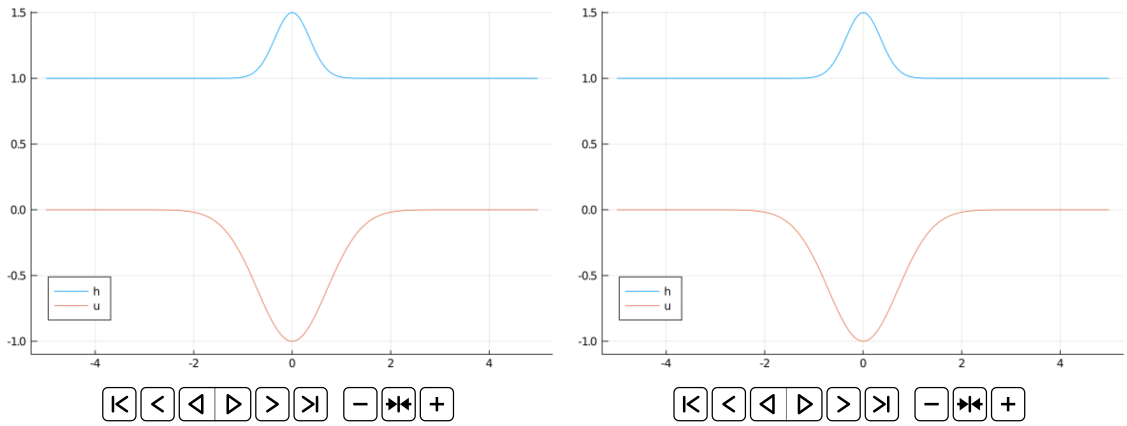
Here u is the singular variable while h is a regular variable, given and independent of time. The problem is reduced to a linear problem with variable coefficients, which is readily solvable by the methods of characteristics if we assume for instance that h, u are initially regular and for any $x \in \mathbb{R}$, $0 < h_\star \leq h(x) \leq h^\star < \infty$. We see that variations of size δ in h produce variations of size 1 on u at time $t = \epsilon/\delta$. However, the solution and its space-derivatives remain controlled for all times, uniformly with respect to ϵ small. This behavior is not shared for the toy model corresponding to J homogeneous of order zero, namely

$$\partial_t u = i \frac{1}{\epsilon} h u \quad ; \quad \partial_t h = 0. \tag{9.21}$$

The problem is now an ordinary differential equation in time where the space variable is a parameter. The solution $u(t, x) = u_0(x) \exp(ith(x)/\epsilon)$ strongly oscillates with a different rate as $h(x)$ takes different values. Hence for positive times, the solution exhibits small scale oscillations, and space-derivatives are not uniformly controlled with respect to the parameter ϵ small. If variations of h are of size δ, it is necessary to prepare the initial data $u\,|_{t=0} = \mathcal{O}(\epsilon^m)$ in order to control m space derivatives of the solution at time $t = 1/\delta$. Our situation is roughly speaking a combination of the above where the size of μ measures the relative strength of the two influences. Based on the properties satisfied by J^μ a toy model could be

$$\partial_t u = i \frac{1}{\epsilon} h \sqrt{1 - \mu \partial_x^2}\, u \quad ; \quad \partial_t h = 0.$$

Consistently with Theorem 9.4—and following the lines of its proof—preparing the initial data as $u\,|_{t=0} = \mathcal{O}(\mu^{m/2})$ is sufficient to control m space derivatives of the solution at time $t = 1/\delta$, uniformly with respect to ϵ small. Figure 9.2 illustrates of the above discussion.



(a) Homogeneous of order one, eq. (9.20).

(b) Homogeneous of order zero, eq. (9.21).

Figure 9.2: Behavior of solutions to "toy" singular problems.
Here, $\epsilon = 0.1$, $u\,|_{t=0}(x) = \exp(-x^2)$, $h(x) = 1 + \frac{1}{2}\exp(-4x^2)$, and final time is $t = 5$.

9.6.2 A priori energy estimates

Let us explain how we can provide uniform estimates satisfied by well-prepared strong solutions of the Favrie–Gavrilyuk system (9.12), which yield the large time well-posedness result of Theorem 9.4. In the spirit of [378], we define for $m, s \in \mathbb{N}$, $1 \leq m \leq s$, $\tilde\lambda \in (0, +\infty)$ and sufficiently regular functions $\boldsymbol{V}$:[47]

$$\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda}^2 \stackrel{\text{def}}{=} \sum_{j=0}^{m} \big|\partial_t^j \boldsymbol{V}\big|_{H^{s-j}}^2 + \sum_{j=m+1}^{s} \tilde\lambda^{m-j}\big|\partial_t^j \boldsymbol{V}\big|_{H^{s-j}}^2, \tag{9.22}$$

$$\begin{aligned}\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(1)}^2 \stackrel{\text{def}}{=} & \sum_{j=0}^{m-1}\sum_{|\boldsymbol{k}|=0}^{s-j} \big(\mathcal{S}_t(\boldsymbol{V})\partial_t^j\partial^{\boldsymbol{k}}\boldsymbol{V}, \partial_t^j\partial^{\boldsymbol{k}}\boldsymbol{V}\big)_{L^2} \\ & + \sum_{j=m}^{s} \tilde\lambda^{m-j}\big(\mathcal{S}_t(\boldsymbol{V})\partial_t^j\boldsymbol{V}, \partial_t^j\boldsymbol{V}\big)_{L^2},\end{aligned} \tag{9.23}$$

$$\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(2)}^2 \stackrel{\text{def}}{=} \sum_{j=m}^{s-1}\sum_{|\boldsymbol{k}|=1}^{s-j} \tilde\lambda^{m-j}\big|\partial_t^j\partial^{\boldsymbol{k}}\boldsymbol{V}\big|_{L^2}^2. \tag{9.24}$$

By convention $\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(2)} = 0$ if $m = s$. By Theorem 9.2 we define $\boldsymbol{V} \in \mathcal{C}^0([0,T]; H^s(\mathbb{R}^d)^{d+3})$ strong solutions to eq. (9.12); and iterating the equation, we have $\boldsymbol{V} \in \bigcap_{j=0}^{s} \mathcal{C}^j([0,T]; H^{s-j}(\mathbb{R}^d)^{d+3})$, hence the above is well-defined and finite.

We want to prove that any sufficiently regular solutions to eq. (9.12) satisfying reasonable hyperbolicity-type conditions enjoy, uniformly with respect to $(\lambda, \mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{FG}}$, the following estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(1)}^2 \leq C(\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda})\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda}^3 \tag{9.25}$$

with

$$\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda} \approx \big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(1)} + \big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(2)}, \tag{9.26}$$

and

$$\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(2)} \leq C(\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(1)})\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda,(1)}. \tag{9.27}$$

By Gronwall's Lemma, one immediately deduces an a priori control on $\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda}$ on the quadratic time scale provided that such control is satisfied at initial time. Unfortunately, eq. (9.27) holds only for $\tilde\lambda \leq \lambda\mu$ (we also need $\tilde\lambda \geq \nu_\star$ with $\nu_\star = C(\big\|\boldsymbol{V}\big\|_{s,m,\tilde\lambda})$), which is the reason for the corresponding assumption on Theorem 9.4 and as a consequence the need for finely prepared initial data.

The equivalence estimate, eq. (9.26) is obvious to check; let us explain how the other estimates can be derived. Recall that our system takes the form

$$\mathcal{S}_t(\boldsymbol{V})\partial_t\boldsymbol{V} + \mathcal{S}_x(\boldsymbol{V})\partial_x\boldsymbol{V} + \mathcal{S}_y(\boldsymbol{V})\partial_y\boldsymbol{V} = \lambda^{1/2}\mathsf{J}^\mu\boldsymbol{V} + \boldsymbol{G}(\boldsymbol{V}), \tag{9.28}$$

where $\mathcal{S}_t, \mathcal{S}_x, \mathcal{S}_y$ are smooth functions of $\boldsymbol{V}$ with values into symmetric matrices, J^μ is constant-coefficient and skew-symmetric, and $\boldsymbol{G}$ is a smooth function. The main difference with the framework of [378] is that in our case, J^μ is not homogeneous of order one, but contains an order-zero additional component, and depends on the second parameter μ.

Following the standard strategy for hyperbolic quasilinear systems (which eventually yields Theorem 9.2), we first seek a differential inequality for the "energy" of the system, which after integration in time yields a control of the energy for positive times. Thanks to the symmetric structure of the equation, it is immediate to obtain such an estimate, *uniformly with respect to the parameters* $(\lambda, \mu, \varepsilon, \beta) \in \mathfrak{p}_{\mathrm{FG}}$, by testing the system against $\boldsymbol{V}$. However this estimate relies on the

[47] Of course the notation in (9.23) and (9.24) is abusive as the right-hand side does not define a norm.

a priori control of the solution itself in L^∞ norm, as well as one derivative, with respect to space or time. In other words we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}_t(\boldsymbol{V})\,\boldsymbol{V},\boldsymbol{V}\big)_{L^2} \leq C\big(\big|\boldsymbol{V}\big|_{L^\infty}\big)\big(\big|\partial_t\boldsymbol{V}\big|_{L^\infty}+\big|\partial_x\boldsymbol{V}\big|_{L^\infty}+\big|\partial_y\boldsymbol{V}\big|_{L^\infty}\big)\big|\boldsymbol{V}\big|_{L^2}^2.$$

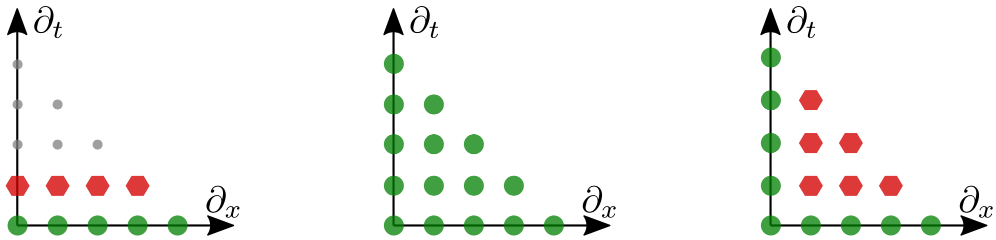
In view of obtaining a self-contained energy estimate, the standard strategy consists in differentiating the system with respect to space, and testing against derivatives of the unknown. Thanks to the regularizing properties of commutators, and using the fact that J^μ commutes with space derivatives, one deduces for $s > 1 + d/2$ a uniform differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{|\boldsymbol{k}|=0}^{s}\big(\mathcal{S}_t(\boldsymbol{V})\partial^{\boldsymbol{k}}\boldsymbol{V},\partial^{\boldsymbol{k}}\boldsymbol{V}\big)_{L^2}\right) \leq C\big(\big|\boldsymbol{V}\big|_{H^s}\big)\big(\big|\partial_t\boldsymbol{V}\big|_{H^{s-1}}+\big|\boldsymbol{V}\big|_{H^s}\big)\big|\boldsymbol{V}\big|_{H^s}^2.$$

For standard (non-singular) quasilinear systems, that is setting $\mathsf{J}^\mu \equiv 0$ in eq. (9.28), the above estimate is sufficient as we have the control

$$\big|\partial_t\boldsymbol{V}\big|_{H^{s-1}} \leq C\big(\big|\boldsymbol{V}\big|_{H^s}\big)\big|\boldsymbol{V}\big|_{H^s} \tag{9.29}$$

stemming from the fact that $\boldsymbol{V}$ satisfies eq. (9.28), and hence the differential inequality, by Gronwall's Lemma and provided that $\mathsf{S}_t(\boldsymbol{V})$ is positive definite, provides an a priori control on $\big|V\big|_{H^s}$. We express the above argument through the cartoon in Figure 9.3a. However, the argument is not useful in our framework as eq. (9.29) is not uniform with respect to $\lambda \gg 1$ due to the contribution from J^μ.



(a) Standard. (b) Browning and Kreiss [66]. (c) Schochet [378].
Green dots represent space and time derivatives of solutions controlled through energy estimates. Red hexagons represent additional terms whose control is inferred by the system of equations.

Figure 9.3: Sketch of the different strategies for a priori estimates.

The first strategy that one may have (which is the one developed in [66]) would consist in controlling time derivatives of the unknown through energy estimates as above: differentiating the system with respect to time as well as with space and using that J^μ commutes with space and time derivatives, we obtain (notice we set $m = s$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\boldsymbol{V}\big\|_{s,s,\tilde{\lambda},(1)}^2 \leq C\big(\big\|\boldsymbol{V}\big\|_{s,s,\tilde{\lambda}}^2\big)\big\|\boldsymbol{V}\big\|_{s,s,\tilde{\lambda},(1)}^2. \tag{9.30}$$

Using that $\big\|\boldsymbol{V}\big\|_{s,s,\tilde{\lambda},(1)} \approx \big\|\boldsymbol{V}\big\|_{s,s,\tilde{\lambda}}$ if $\mathcal{S}_t(\boldsymbol{V})$ is positive definite, we have indeed a self-contained energy inequality, which can be integrated in time to offer a valuable uniform a priori estimate for the solution and derivatives. This would correspond to Figure 9.3b. Notice however that the estimate which is propagated for positive times must of course be satisfied initially: the a priori control of $\big\|\boldsymbol{V}\,|_{t=0}\big\|_{s,s,\tilde{\lambda}}$ is a very strong constraint on the initial data since $m = s$.

The strategy in [378] is more subtle. The first step consists in remarking that, by using energy estimates, we may obtain a uniform energy inequality for time derivatives of the unknowns, of the form (notice we have now $m = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\boldsymbol{V}\big\|^2_{s,1,\tilde{\lambda},(1)} \leq C(\big\|\boldsymbol{V}\big\|^2_{s,1,\tilde{\lambda}})\big\|\boldsymbol{V}\big\|^2_{s,1,\tilde{\lambda}}. \tag{9.31}$$

Here, it suffices to ensure that only one time-derivative of the initial data is uniformly controlled, so that we can take advantage of a gain of a factor $\tilde{\lambda}^{-1/2}$ as soon as time derivatives are distributed. However, $\big\|\boldsymbol{V}\big\|_{s,1,\tilde{\lambda},(1)} \approx \big\|\boldsymbol{V}\big\|_{s,1,\tilde{\lambda}}$ does not hold, that is we still need to control the contribution of terms involving time *and* space derivatives of the unknowns. To this aim we do not use energy estimates (they fail due to the lack of uniform estimate for $\big|[\partial_t^j\partial^{\boldsymbol{k}}, \mathcal{S}_t]\partial_t\boldsymbol{V}\big|_{L^2}$ when $j \neq 0$ and $\boldsymbol{k} \neq \boldsymbol{0}$) but rather directly control the remaining components with respect to the former, that is eq. (9.26) and eq. (9.27). This is represented in Figure 9.3c. We cannot infer this control from a simple interpolation uniformly with respect to $\tilde{\lambda} \gg 1$, but rather will make use of eq. (9.28). This is where the precise properties of J^μ come into play, and this is where the results differ from the ones in [378]. Indeed, when J^μ is a skew-symmetric differential operator, constant-coefficient and homogeneous of order one, we can decompose the (frequency) space as the direct sum of the kernel and the characteristic space associated with non-trivial eigenvalues of its symbol. Controlling the projection of $\boldsymbol{V}$ onto the kernel (the “regular component”) is obtained as in eq. (9.29), applying first the projection to the system onto the kernel, and hence withdrawing the non-uniformly bounded contributions. One controls the other component of $\boldsymbol{V}$ (the “singular component”) in the opposite direction, projecting the system onto the singular subspace and using that the restriction of J^μ to the singular subspace is invertible, and that the inverse is a (regularizing) operator of order -1 in Sobolev spaces. While the above properties are still true in our setting where J^μ is a non-homogeneous Fourier multiplier, the inverse on the singular subspace (which is $\frac{-\mathsf{J}^\mu}{1-\mu\Delta}$; see eq. (9.19)) is not uniformly bounded with respect to the parameter $\mu \ll 1$. This is easily understood by setting $\mu = 0$, in which case J^μ is an order-zero operator, and hence the inverse cannot be regularizing; and this is the reason why we need to restrict to $\tilde{\lambda} \in [\nu_\star, \lambda\mu]$ in order to ensure that eq. (9.27) holds.

The parameter $m \in \{1, \dots, s\}$ allows to somehow “interpolate” between the two strategies of [66] and [378], and allows some flexibility on the assumption on the initial data.

9.6.3 Preparing the initial data

As already mentioned, Theorem 9.4 applies to solutions satisfying a strong initial constraint, as several time derivatives must be initially controlled. Theorem 9.5 allows to show that for any initial data defined by $(\zeta_0, \boldsymbol{u}_0, \varepsilon, \mu)$ in a given ball of $H^s \times H^s \times \mathbb{R}^+ \times \mathbb{R}^+$, and for arbitrarily large λ, we may set the initial data for the additional unknowns η, w so as to ensure that the desired initial control holds. The strategy consists in iterating the system of eq. (9.4) in order to extract explicit expressions for time derivatives in terms of space derivatives, and to iteratively set corrector terms $c^{(j)}$ so as to cancel out singular (*i.e.* non-uniformly bounded) terms in these expressions. While it is not difficult to convince oneself, after manipulating the equations and deducing expressions for the first corrector terms, that the induction process may indeed be successfully set up, one quickly realizes that the expressions are very cumbersome. Let us detail the calculations, and how the first order terms, $c^{(1)}$ and $c^{(2)}$, may be obtained.

We first notice that after differentiating eq. (9.4) with respect to time and using Proposition II.7, one has that any solution $\boldsymbol{U} = (\zeta, \boldsymbol{u}, \eta, w)$ satisfies

$$\big|\partial_t^{j+1}\boldsymbol{U}\big|_{H^{s-(j+1)}} \leq C(h_\star^{-1}, \big\|\boldsymbol{U}\big\|_{s,j})\big(\big\|\boldsymbol{U}\big\|_{s,j} + \lambda\big\|\eta - h\big\|_{s,j}\big), \tag{9.32}$$

where we denote

$$\big\|\boldsymbol{U}\big\|^2_{s,m} \stackrel{\text{def}}{=} \sum_{j=0}^{m}\big|\partial_t^j\boldsymbol{U}\big|^2_{H^{s-j}}.$$

Hence we can focus on proving inductively that

$$\lambda\big\|(\eta^{(m)} - h^{(m)})(0,\cdot)\big\|_{s,m} \leq M_m$$

with $M_m = C(h_\star^{-1}, M_0^\star)M_0$. Notice that the result for $m = 0$ is trivial and the result for $m = 1$ follows from setting $c^{(1)} = -h_0\nabla\cdot\boldsymbol{u}_0 + \frac{3}{2}\beta\boldsymbol{u}_0\cdot\nabla b$, as well as the identity

$$\partial_t(\eta - h) + \varepsilon\boldsymbol{u}\cdot\nabla(\eta - h) = \varepsilon(w - \frac{3}{2}\beta\boldsymbol{u}\cdot\nabla b + h\nabla\cdot\boldsymbol{u}), \tag{9.33}$$

Differentiating the above, and applying once again eq. (9.4) on the first-order time derivatives, we find that any solution satisfies

$$\partial_t^2(\eta - h) = \mathfrak{r}[\varepsilon\boldsymbol{U}, \beta\nabla b]\boldsymbol{U} + \lambda\mu\mathfrak{s}[\varepsilon\boldsymbol{U}, \beta\nabla b](\eta - h, \eta - h) - \lambda h\mathfrak{t}[h, \beta\nabla b](\eta - h) \tag{9.34}$$

where $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{t}$ are differential operators (in space) of order two. The large prefactor that $\lambda\mu$ in front of $\mathfrak{s}$ is compensated by the fact that this operator is quadratic in $\eta - h$, and hence we collect truly singular terms in the operator $\mathfrak{t}$, defined in eq. (9.18).

Rooting from (9.33) and (9.34), we now define

$$\mathfrak{C}_j(\boldsymbol{U}) \stackrel{\text{def}}{=} \partial_t^j\left(\mathfrak{r}[\varepsilon\boldsymbol{U}, \beta\nabla b]\boldsymbol{U} + \lambda\mu\mathfrak{s}[\varepsilon\boldsymbol{U}, \beta\nabla b](\eta - h, \eta - h)\right) - \lambda\big[\,\partial_t^j\,,\, h\mathfrak{t}[h, \beta\nabla b]\,\big](\eta - h)$$

and

$$\mathfrak{S}_m(\boldsymbol{U}) \stackrel{\text{def}}{=} \sum_{k=0}^{\lfloor m/2\rfloor}(-\lambda h\mathfrak{t}[h, \beta\nabla b])^k\mathfrak{C}_{m-2k}$$

so that any solution to eq. (9.4) satisfies for any $m \geq 2$

$$\partial_t^m(\eta - h) - \mathfrak{S}_{m-2}(\boldsymbol{U}) = \begin{cases}(-\lambda h\mathfrak{t}[h, \beta\nabla b])^{m/2}(\eta - h) & \text{if } m \text{ is even,}\\ (-\lambda h\mathfrak{t}[h, \beta\nabla b])^{(m-1)/2}\partial_t(\eta - h) & \text{if } m \text{ is odd.}\end{cases} \tag{9.35}$$

We deduce the following expression for $c^{(m)}$:

$$(-h_0\mathfrak{t}[h_0, \beta\nabla b])^{\lfloor m/2\rfloor}c^{(m)} = -\mathfrak{S}_{m-2}(\boldsymbol{U}_0^{(m-1)}) - (-\lambda h_0\mathfrak{t}[h_0, \beta\nabla b])^{\lfloor m/2\rfloor}s^{(m-2)} \tag{9.36}$$

where $\mathfrak{S}_{m-2}(\boldsymbol{U}_0^{(m-1)})$ is the differential operator of order m obtained when all time derivatives have been replaced by spatial derivatives through eq. (9.4), and

$$s^{(m)} \stackrel{\text{def}}{=} \begin{cases}\sum_{k=1}^{m/2}\lambda^{-k}c^{(2k)} & \text{if } m \text{ is even,}\\ \sum_{k=1}^{(m-1)/2}\lambda^{-k}\varepsilon c^{(2k+1)} - \sum_{k=1}^{(m+1)/2}\lambda^{-k}\varepsilon\boldsymbol{u}_0\cdot\nabla(\varepsilon c^{(2k)} + \frac{3}{2}\beta b) & \text{if } m \text{ is odd.}\end{cases}$$

Notice $c^{(m)}$ is well-defined by (9.36) and induction on m. We already expressed $c^{(1)}$ and the expression for $c^{(2)}$ stems from the fact that the contribution from $\mathfrak{s}[\varepsilon\boldsymbol{U}_0^{(1)}, \beta\nabla b]$ vanishes as $\eta_0^{(1)} = h_0$, and $\mathfrak{r}[\varepsilon\boldsymbol{U}_0^{(1)}, \beta\nabla b]\boldsymbol{U}_0^{(1)}$ yields the right-hand side of eq. (9.17).

9.7 Discussion and open questions

We have shown in Section 9.5 the relevance of the Favrie–Gavrilyuk system (9.4) for producing approximate solutions to the Green–Naghdi system—and ultimately the water waves equations. To this aim, we have exhibited the impact of the shallowness parameter, which may induce undesirable oscillations in space in the shallow water regime. In order to avoid these oscillations it was necessary in our proofs to suitably set the initial data for the augmented variables η, w in a stronger sense than the usual criterion in low Mach number limits (as we have a condition on as many time derivatives as the number of space derivatives we wish to control); and to essentially assume that the bottom topography is flat.

Of course the first question which comes to mind is whether these assumptions are necessary. It is quite possible that some structural properties that we have not unveiled so far could permit to obtain a good control on the solution and the handful time derivatives which ensure a good consistency property (see Theorem 9.3), assuming only the corresponding initial control on these quantities. Even for completely ill-prepared initial data, it is possible that the solution would converge weakly towards a solution having the desired property, just as in the weak compressible limit for Euler equations; see in particular [312] for the non-isentropic case having properties somewhat similar to the Favrie–Gavrilyuk system.

In the same way, it is not clear whether the strong assumption on the bottom topography is necessary. Notice that the bottom-topography contributions in eq. (9.12) have the necessary skew-symmetric structure (at least when $\partial_t b \equiv 0$); however the obtained singular operator is now space-dependent. Our strategy described in Section 9.6 fails when we replace J with $\mathsf{J}(b(\boldsymbol{x}))$ because space derivative do not commute anymore with the singular operator. This is however not hopeless, and we refer to [62] for a similar problem concerning the large-time behavior of small solutions to the Saint-Venant system with topography.

Valuable information on the above questions should be gained from numerical experiments. To my knowledge, the Favrie–Gavrilyuk system has been numerically implemented in the original work [181] with a particular focus on the incidence of the mesh size, in [249] where Perfectly Matched Layer (PML) boundary conditions are proposed, and in [68].[48] The first two works are restricted to the flat bottom case, while the latter extends the analysis to variable bottom topographies (and dimension $d = 2$). To my opinion, an in-depth numerical investigation on the convergence towards solutions to the Green–Naghdi system as $\lambda \to \infty$, correlated with the influence of the shallow water parameter, bottom topographies and the preparation of the initial data is yet to be accomplished.

[48] See also to [179, 180, 207, 41, 206] for numerical experiments on closely related systems, as discussed in Remark 9.1.

10 The Whitham–Green–Naghdi system

We now introduce a *fully dispersive* counterpart of the Green–Naghdi system introduced and studied in Section 8. By fully dispersive, we mean that the linearized system about the trivial equilibrium coincides exactly with the one of the water waves system, namely eq. (2.3). Because of its close relation with the Green–Naghdi system and the fully dispersive property akin to the Whitham equation (see Section v), we refer to the following model as the Whitham–Green–Naghdi system. We will argue that the full dispersion property is in fact just one consequence of a quantitative improvement on the precision of the Whitham–Green–Naghdi system (as an asymptotic model) with respect to the original Green–Naghdi system. To this aim we use the improved approximations obtained in Section 4.4 and specifically the following one stemming from Proposition 4.15:[49]

$$\frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot\big(h(\mathrm{Id}+\mu\mathcal{T}^{\mathsf{F}^{\mu}}[h,\beta\nabla b])^{-1}\nabla\psi\big)+\mathcal{O}(\mu^2(\varepsilon+\beta)), \tag{10.1}$$

where we recall the notations $h=1+\varepsilon\zeta-\beta b$ and

$$\mathcal{T}^{\mathsf{F}^{\mu}}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\mathrm{def}}{=} \frac{-1}{3h}\nabla\mathsf{F}^{\mu}(h^3\mathsf{F}^{\mu}\nabla\cdot\boldsymbol{u})+\frac{1}{2h}\Big(\nabla\mathsf{F}^{\mu}\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big)-h^2(\beta\nabla b)\mathsf{F}^{\mu}\nabla\cdot\boldsymbol{u}\Big)+(\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b)$$

and F^{μ} is the Fourier multiplier (see Definition III.1) defined by

$$\mathsf{F}^{\mu}=F(\sqrt{\mu}|D|),\qquad F(\xi)=\sqrt{\frac{3}{|\xi|^2}\Big(\frac{|\xi|}{\tanh(|\xi|)}-1\Big)}.$$

Plugging this expansion into eq. (2.7) and withdrawing $\mathcal{O}(\mu^2(\varepsilon+\beta))$ terms yields

$$\begin{cases}\partial_t\zeta+\nabla\cdot(h\boldsymbol{u})=0,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\boldsymbol{u}|^2=\mu\varepsilon\mathcal{R}^{\mathsf{F}^{\mu}}[h,\beta\nabla b,\boldsymbol{u}],\end{cases}\tag{10.2}$$

where

$$\begin{aligned}\mathcal{R}^{\mathsf{F}^{\mu}}[h,\beta\nabla b,\boldsymbol{u}] \stackrel{\mathrm{def}}{=} {}&\frac{\boldsymbol{u}}{3h}\cdot\nabla\mathsf{F}^{\mu}(h^3\mathsf{F}^{\mu}\nabla\cdot\boldsymbol{u})+\frac{1}{2}h^2(\mathsf{F}^{\mu}\nabla\cdot\boldsymbol{u})^2\\ &-\frac{1}{2}\Big(\frac{\boldsymbol{u}}{h}\cdot\nabla\mathsf{F}^{\mu}\big(h^2(\beta\nabla b\cdot\boldsymbol{u})\big)+h(\beta\nabla b\cdot\boldsymbol{u})\mathsf{F}^{\mu}\nabla\cdot\boldsymbol{u}+(\beta\nabla b\cdot\boldsymbol{u})^2\Big)\end{aligned}$$

and $\boldsymbol{u}$ is deduced from (ζ,ψ) after solving the equation

$$h\nabla\psi \;=\; h\boldsymbol{u}+\mu h\mathcal{T}^{\mathsf{F}^{\mu}}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\mathrm{def}}{=} \mathfrak{T}^{\mathsf{F}^{\mu}}[h,\beta\nabla b]\boldsymbol{u},$$

[49] The reasoning expressed in footnote 38 to discard the direct approximation

$$\frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta,\beta b]\psi=-\nabla\cdot(h\nabla\psi)+\mu\nabla\cdot\big(h\mathcal{T}^{\tilde{\mathsf{F}}^{\mu}}[h,\beta\nabla b]\nabla\psi\big)+\mathcal{O}(\mu^2(\varepsilon+\beta))$$

where

$$\tilde{\mathsf{F}}^{\mu}=\tilde{F}(\sqrt{\mu}|D|),\qquad \tilde{F}(\xi)=\sqrt{\frac{3}{|\boldsymbol{\xi}|^2}\Big(1-\frac{\tanh(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\Big)},$$

is no longer valid since the modal analysis of the resulting fully dispersive system coincides with the one of the water waves system. Yet a closer look indicates that some extra—but harmless—regularization, as exhibited in eq. (10.10), is needed to secure the well-posedness of the initial-value problem. A closely related system is derived and justified in the sense of consistency in [177, Section 3], and the rigorous well-posedness analysis could be carried out in the lines of [176]. It should be pointed out that these models have a great practical benefit—at least from the numerical point of view—with respect to eq. (10.2) from the fact that the inversion of an elliptic operator is not needed to consider the system as closed evolution equations; see discussion in Section 9.

and represents the vertically averaged horizontal velocity. [50]

Taking the gradient of the second equation one can rewrite the system in terms of the unknowns ζ and $\boldsymbol{u}$ alone as for the original Green–Naghdi system:

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\big(\boldsymbol{u} + \mu\mathcal{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u}\big) + \nabla\zeta + \varepsilon\nabla\big(\tfrac{1}{2}|\boldsymbol{u}|^2 - \mu\mathcal{R}^{\mathsf{F}^\mu}[h,\beta\nabla b,\boldsymbol{u}]\big) = \boldsymbol{0}. \end{cases} \tag{10.4}$$

However it is unclear that an analogue to the formulation provided in eq. (8.6)—and even less for eq. (8.7)—can be devised. Hence it appears that some part of the structure of the original Green–Naghdi system has been lost in our fully dispersive system. Yet, as we discuss in Section 10.1, the important canonical Hamiltonian structure corresponding to Zakharov's one on the water waves system remains.

Using physical variables (recall Section 2.4) and taking the gradient of the second equation in eq. (10.2), we obtain the ***Whitham–Green–Naghdi system***

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\big(\boldsymbol{u} + \mathcal{T}^{\mathsf{F}^{d^2}}[h,\nabla b]\boldsymbol{u}\big) + g\nabla(h+b) + \nabla(\tfrac{1}{2}|\boldsymbol{u}|^2 - \mathcal{R}^{\mathsf{F}^{d^2}}[h,\nabla b,\boldsymbol{u}]) = \boldsymbol{0}, \end{cases} \tag{10.5}$$

with $h = d + \zeta - b$. The above system has been derived first in [164], in the flat bottom situation, horizontal dimension $d = 1$, and bilayer framework (see Section 14.5).

10.1 Hamiltonian structure

As discussed above, the profound structure of the Green–Naghdi system is not entirely carried over the Whitham–Green–Naghdi system. Yet the main variational structure of the water waves equations is. Indeed, define the Hamiltonian functional

$$\mathscr{H}_{\rm WGN}(\zeta,\psi) \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + (h\nabla\psi)\cdot\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]^{-1}(h\nabla\psi)\ \mathrm{d}\boldsymbol{x},$$

which is obtained from plugging the approximation eq. (10.1) in the Hamiltonian functional of the water waves equations (see Section 2.2). Then Hamilton's principle on

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \zeta\partial_t\psi\,\mathrm{d}\boldsymbol{x} + \mathscr{H}_{\rm WGN}\,\mathrm{d}t. \tag{10.6}$$

yields

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_\zeta\mathscr{H}_{\rm WGN}\\ \delta_\psi\mathscr{H}_{\rm WGN}\end{pmatrix},$$

which corresponds to eq. (10.2). Hence the Whitham–Green–Naghdi system enjoys a canonical Hamiltonian structure, and we may follow the discussion of Section 2.2 with straightforward adjustments. In particular we have the following group symmetries and related preserved quantities by Noether's theorem.

[50] More precisely, by Proposition 4.14, we have

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \mathcal{O}(\mu^2(\varepsilon+\beta)), \qquad \overline{\boldsymbol{u}} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Phi(\cdot,z)\,\mathrm{d}z,$$

where Φ is the unique solution to eq. (4.1). This allows to recognize in the first equation of eq. (10.2) the conservation of mass. As for the second equation, we notice that denoting $w \stackrel{\rm def}{=} (\beta\nabla b)\cdot\boldsymbol{u} - h\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}$, eq. (10.2) can be written as

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\psi + \zeta + \varepsilon\boldsymbol{u}\cdot\nabla\psi - \frac{\varepsilon}{2}\boldsymbol{u}\cdot\boldsymbol{u} - \frac{\varepsilon\mu}{2}w^2 = 0, \end{cases} \tag{10.3}$$

which echoes the formulation of the water waves equations displayed in eq. (2.7').

10.2 Group symmetries and preserved quantities

Group symmetries If (ζ, ψ) is a solution to eq. (10.2), then for any $\theta \in \mathbb{R}$, $(\zeta^\theta, \psi^\theta)$ also satisfies eq. (10.2), where

- Variation of base level for the velocity potential:

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi + \theta\big)(t, \boldsymbol{x}).$$

- Horizontal translation along the direction $\boldsymbol{e} \in \mathbb{R}^d$ (in the flat bottom case)

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t, \boldsymbol{x} - \theta \boldsymbol{e}).$$

- Time translation

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t - \theta, \boldsymbol{x}).$$

- Galilean boost along the direction $\boldsymbol{e} \in \mathbb{R}^d$ (in the flat bottom case)

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi + \theta \boldsymbol{e} \cdot \boldsymbol{x}\big)(t, \boldsymbol{x} - \theta \boldsymbol{e} t).$$

- Horizontal rotation (in dimension $d = 2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp \cdot \nabla b = 0$)

$$\big(\zeta^\theta, \psi^\theta\big)(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta, \psi\big)(t, R_\theta \boldsymbol{x})$$

  where R_θ is the rotation matrix of angle θ.

Preserved quantities We have the following related preserved quantities.

- Excess of mass

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{Z} = 0, \qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \, \mathrm{d}\boldsymbol{x}.$$

- Horizontal impulse (in the flat bottom case)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{I} = 0, \qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \nabla \psi \, \mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$

- Total energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_{\mathrm{WGN}} = 0, \qquad \mathscr{H}_{\mathrm{WGN}} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + (h \nabla \psi) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$

- Horizontal coordinate of mass centroid times mass (in the flat bottom case)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{C} = \int_{\mathbb{R}^d} \zeta \nabla \psi \, \mathrm{d}\boldsymbol{x}, \qquad \mathscr{C} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$

- Angular impulse (in dimension $d = 2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp \cdot \nabla b = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{A} = 0, \qquad \mathscr{A} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \boldsymbol{x}^\perp \cdot \nabla \psi \, \mathrm{d}\boldsymbol{x}.$$

  where $(x, y)^\perp \stackrel{\text{def}}{=} (-y, x)$.

Notice also the following conserved quantity which is seemingly trivial in the formulation (10.2) but not in the formulation (10.5):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}=0, \qquad\qquad \mathscr{V}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\nabla\psi\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^d}\boldsymbol{u}+\mathcal{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.$$

Moreover, in the flat bottom case, we deduce from the above the conservation of a quantity corresponding to the horizontal momentum (recall footnote 50)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}_{\mathrm{WGN}}=0, \quad \mathscr{M}_{\mathrm{WGN}}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}h\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^d}h\nabla\psi\,\mathrm{d}\boldsymbol{x}+\frac{1}{3}\nabla\mathsf{F}^\mu(h^3\mathsf{F}^\mu\nabla\cdot\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}=\mathscr{V}+\mathscr{I}+0.$$

The quantities presented here are preserved in a stronger sense: their integrand satisfies a *conservation law*, which we do not write out explicitly.

10.3 Modal analysis

Linearizing eq. (10.5) (in the flat bottom case, $b\equiv 0$) about the trivial rest solution, that is setting $h=d+\epsilon\zeta^0$, $\boldsymbol{u}=\epsilon\boldsymbol{u}^0$ (we could add a constant horizontal velocity by Galilean invariance) and keeping only first order terms with respect to $\epsilon\ll 1$ yields the system

$$\begin{cases} \partial_t\zeta^0+d\nabla\cdot\boldsymbol{u}=0,\\ \partial_t\big(\boldsymbol{u}^0+\big(\frac{d|D|}{\tanh(d|D|)}-\mathrm{Id}\big)\frac{1}{|D|^2}\nabla\nabla\cdot\boldsymbol{u}^0\big)+g\nabla\zeta^0=\boldsymbol{0}.\end{cases}$$

Specializing to irrotational velocity fields as we should, and denoting $\nabla\psi^0\stackrel{\text{def}}{=}\frac{d|D|}{\tanh(d|D|)}\boldsymbol{u}^0$, we infer

$$\begin{cases} \partial_t\zeta^0-|D|\tanh(d|D|)\psi^0=0,\\ \partial_t\psi^0+g\zeta^0=0,\end{cases}$$

and we recognize the linearized water waves system, eq. (2.3). In particular, the dispersion relation

$$\omega(\boldsymbol{\xi})^2=gd|\boldsymbol{\xi}|\tanh(d|\boldsymbol{\xi}|)$$

coincides with the one corresponding to the water waves system, hence the "full dispersion" terminology. The reader can refer to Section ii and Section 2.3 for more information (modal analysis, large-time behavior of solutions, dispersive and Strichartz estimates) on the linearized water waves system.

10.4 Solitary waves

We cannot expect to find explicit expressions for solitary wave—or cnoidal—solutions to the Whitham–Green–Naghdi system, contrarily to the ones we obtained in Section 8.4 on the Green–Naghdi system. Yet we can prove the existence of such traveling wave solutions by means of a suitable minimization problem.

Indeed, seeking solutions to eq. (10.2) in the unidimensional ($d=1$) and flat bottom ($b\equiv 0$) framework satisfying

$$(\zeta,\psi)(t,x)=(\zeta_c,\psi_c)(x-ct), \qquad \lim_{|x|\to\infty}|(\zeta_c,\psi_c')|(x)=0,$$

denoting $h_c=1+\varepsilon\zeta_c$ and $\psi_c'=u_c-\frac{\mu}{3h_c}\partial_x\mathsf{F}^\mu\big(h_c^3\partial_x\mathsf{F}^\mu u_c\big)$ yields the following set of equations:

$$\begin{cases} -c\zeta_c+h_cu_c=0,\\ -c\Big(u_c-\frac{\mu}{3h_c}\partial_x\mathsf{F}^\mu\big(h_c^3\mathsf{F}^\mu\partial_xu_c\big)\Big)+\zeta_c+\frac{\varepsilon}{2}(u_c^2)=\mu\varepsilon\Big(\frac{u_c}{3h_c}\partial_x\mathsf{F}^\mu\big(h_c^3\mathsf{F}^\mu\partial_xu_c\big)+\frac{1}{2}(h_c^2\mathsf{F}^\mu\partial_xu_c)^2\Big).\end{cases}$$

Substituting the relation $u_c = c\frac{\zeta_c}{h_c} = c\frac{\zeta_c}{1+\varepsilon\zeta_c}$ stemming from the first equation into the second one yields the scalar problem

$$-c^2\Big(\tfrac{\zeta_c}{h_c} - \tfrac{\mu}{3h_c}\partial_x\mathsf{F}^\mu\big(h_c^3\mathsf{F}^\mu\partial_x\tfrac{\zeta_c}{h_c}\big)\Big) + \zeta_c + c^2\tfrac{\varepsilon}{2}\big(\tfrac{\zeta_c^2}{h_c^2}\big) = c^2\mu\varepsilon\Big(\tfrac{\zeta_c}{3h_c^2}\partial_x\mathsf{F}^\mu\big(h_c^3\mathsf{F}^\mu\partial_x\tfrac{\zeta_c}{h_c}\big) + \tfrac{1}{2}\big(h_c^2\mathsf{F}^\mu\partial_x\tfrac{\zeta_c}{h_c}\big)^2\Big). \tag{10.7}$$

Now, defining

$$\mathscr{E}(\zeta) = \int_{\mathbb{R}} \frac{\zeta^2}{1+\varepsilon\zeta} + \tfrac{\mu}{3}(1+\varepsilon\zeta)^3\big(\mathsf{F}^\mu\partial_x\tfrac{\zeta_c}{1+\varepsilon\zeta}\big)^2\,\mathrm{d}x$$

we recognize that the Euler-Lagrange equations associated with the minimization problem

$$\operatorname*{arg\,min}_{\zeta\in L^2_q} \mathscr{E}(\zeta)\ , \qquad L^2_q \overset{\text{def}}{=} \Big\{\zeta\in L^2(\mathbb{R}) \ :\ \big|\zeta\big|_{L^2}^2 = q\Big\}, \tag{10.8}$$

namely

$$\mathrm{d}\mathscr{E}(\zeta) = 2\alpha\zeta$$

where $\alpha\in\mathbb{R}$ is a Lagrange multiplier, is equivalent to eq. (10.7) with $\alpha = c^{-2}$.

Remark 10.1. *The existence of such minimization problem is in fact directly related with the variational structure of the equations given in Section 10.1. Indeed, from Hamilton's principle on the Lagrangian action given in eq. (10.6) we infer that solitary waves (ζ_c,ψ_c') are critical points of the functional $\mathscr{H}_{\rm WGN}(\zeta,\psi) - c\mathscr{I}(\zeta,\psi)$. However, as noticed (for the Green–Naghdi system) in [283], critical points are neither minimizers nor maximizers. This is why we first reduce the dimension of the problem. Using that for each fixed c and ζ, the functional $\psi\mapsto\mathscr{H}_{\rm WGN}(\zeta,\psi) - c\mathscr{I}(\zeta,\psi)$ has a unique critical point, $\psi_{c,\zeta}$, and substituting its expression in our functional, we recognize*

$$\mathscr{H}_{\rm WGN}(\zeta,\psi_{c,\zeta}) - c\mathscr{I}(\zeta,\psi_{c,\zeta}) = \big\|\zeta\big\|_{L^2}^2 + c^2\mathscr{E}(\zeta),$$

and observe that ζ_c is a critical point of the above functional if an only if $(\zeta_c,\psi_{c,\zeta_c})$ is a critical point of $\mathscr{H}_{\rm WGN}(\zeta,\psi) - c\mathscr{I}(\zeta,\psi)$. The constrained minimization problem eq. (10.8) is now just one simple step away.

Minimization problems such as eq. (10.8) may be attacked thanks to a powerful tool known as Lions' *concentration-compactness principle* [286]. Let us very roughly describe the strategy. A natural strategy for proving the existence of minimizers is to consider a minimizing sequence of functions, $\zeta_n\in L^2_q$ such that $\mathscr{E}(\zeta_n)\searrow I_q \overset{\text{def}}{=} \min_{\zeta\in D_q}\mathscr{E}(\zeta)$, and using compactness properties to infer that the sequence converges—at least up to extracting a subsequence—towards a true minimizer, ζ_q. The concentration-compactness principle allows to single-down the possible scenarios when considering such limits of sequences of functions with a given "mass" (here, ζ_n^2 being nonnegative functions with fixed $L^1(\mathbb{R})$-norm). Loosely speaking, the possible obstructions to compactness amount to:

- *Translation.* The elements could shift towards infinity at the left or right sides of the real line so that the sequence converges point-wisely to zero everywhere.
- *Dichotomy.* The elements could split into two components comprising a non-trivial portion of the mass and moving away one from each other.
- *Vanishing.* The elements could diffuse, spreading over the real line while the amplitude vanishes everywhere.
- *Concentration.* In the opposite direction, the elements could concentrate so as for instance to (weakly) converge towards a Dirac distribution.

In all these cases, the sequence do not converge—even weakly and up to the extraction of a subsequence —towards a function without a loss of mass. The first (translation) scenario is not a serious obstruction: as our functionals are invariant with respect to translations, one can always shift the sequence so that the center of mass is located at a given location on the real line. The concentration scenario is immediately ruled out if an upper bound (straightforwardly satisfied by any minimizing sequence) on the energy functional ensures a control in a functional space with sufficient regularity. Hence the scenario of all the mass roughly staying at the same location (up to translations) is the favorable scenario, hence the name *concentration-compactness*. There remains to prove that the "dichotomy" and "vanishing" scenarios cannot occur.

While there are standard strategies, none are so robust that they can be applied in a systematic fashion. In our specific framework, we encounter a severe difficulty from the fact that the energy functional is not sufficient to guarantee enough regularity on any minimizing sequence. To deal with this, borrowing the strategy from [171], it is proved in [165] that one can construct—using solutions of the much more compact periodic problem—a special minimizing sequence which has the desired property, and eventually converges.

We reproduce below the main results in [165].

Theorem 10.2. *Set $\varepsilon = \mu = 1$ and let $\nu > 1/2$ and $M > 0$. Define the set of minimizers $D^\nu_{q,M}$ as*

$$D^\nu_{q,M} \stackrel{\text{def}}{=} \operatorname*{arg\,min}_{\zeta \in H^\nu_{q,M}} \mathscr{E}(\zeta) \ , \qquad H^\nu_{q,M} \stackrel{\text{def}}{=} \Big\{ \zeta \in H^\nu(\mathbb{R}) \ : \ \big|\zeta\big|^2_{L^2} = q, \ \big|\zeta\big|_{H^\nu} < M \Big\}.$$

Then there exists $q_0 > 0$ such that for all $q \in (0, q_0)$, the following statements hold:

- *The set $D^\nu_{q,M}$ is nonempty and each element in $D^\nu_{q,M}$ solves the traveling wave equation (10.7), with $c^2 = \alpha^{-1} > 1$. Thus for any $\zeta \in D^\nu_{q,M}$, $\big(\zeta(x \pm ct), \psi'_\pm = \pm c\frac{\zeta}{1+\zeta}(x \pm ct)\big)$ is a supercritical solitary wave solution to (10.2).*
- *For any minimizing sequence for $\mathscr{E}$, $\{\zeta_n\}_{n\in\mathbb{N}}$ in $H^\nu_{q,M}$ such that $\sup_{n\in\mathbb{N}}\big\|\zeta_n\big\|_{H^\nu} < M$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that a subsequence of $\{\zeta_n(\cdot + x_n)\}_{n\in\mathbb{N}}$ converges (weakly in $H^\nu(\mathbb{R})$ and strongly in $H^s(\mathbb{R})$ for $s \in [0, \nu)$) to an element in $D^\nu_{q,M}$.*
- *Each $\zeta \in D^\nu_{q,M}$ belongs to $H^s(\mathbb{R})$ for any $s \geq 0$ and there exist constants $m^\star, M^\star > 0$ such that*
  $$\big||c| - 1 - \tfrac{3}{8} q^{\frac{2}{3}}\big| \leq m^\star q^{\frac{5}{6}}$$
  and, denoting $\xi_{\rm KdV}(x) = \frac{3}{4}\operatorname{sech}^2\left(\frac{3}{4}x\right)$
  $$\sup_{\zeta \in D^\nu_{q,M}} \inf_{x_0 \in \mathbb{R}} \big|q^{-\frac{2}{3}}\zeta(q^{-\frac{1}{3}}\cdot) - \xi_{\rm KdV}(\cdot - x_0)\big|_{H^1(\mathbb{R})} \leq M^\star q^{\frac{1}{6}}$$
  uniformly over $q \in (0, q_0)$ and $\zeta \in D^\nu_{q,M}$.

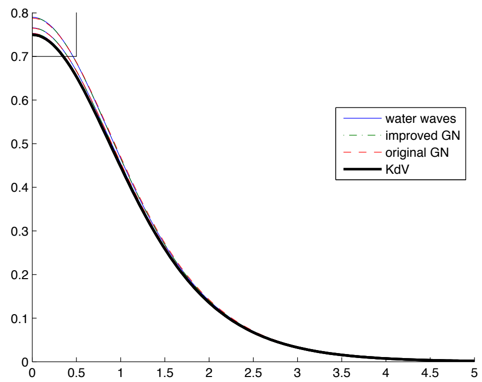
Remark 10.3. *We do not claim uniqueness up to spatial translation of the elements of the set $D^\nu_{q,M}$, even for q sufficiently small. Moreover, because $\mathcal{E}$ is* not *a quantity which is preserved by the flow of the Whitham–Green–Naghdi system, our construction offers no result on the dynamical stability of solitary waves, even measuring the distance of solutions to the set $D^\nu_{q,M}$.*

Remark 10.4. *Theorem 10.2 holds replacing $\mathsf{F}^\mu = F(\sqrt{\mu}|D|)$ where $F(\xi) = \sqrt{\frac{3}{|\xi|^2}\left(\frac{|\xi|}{\tanh(|\xi|)} - 1\right)}$ with any element in a wide class of* admissible *Fourier multipliers (see Definitions in [165]) ; for instance $\mathsf{F}^\mu = \mathrm{Id}$ corresponding to the original Green–Naghdi system.*

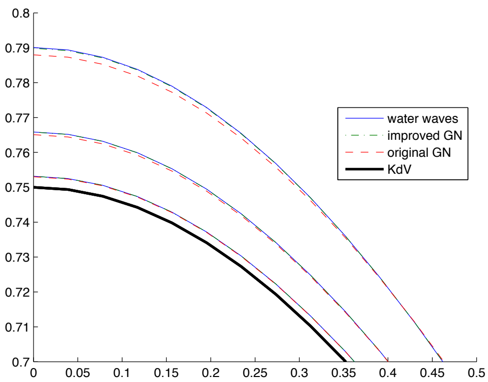
What is not shown in the result, but observed numerically, is that solitary wave solutions to the Whitham–Green–Naghdi system are closer (asymptotically at a distance $\mathcal{O}((c-1)^2)$ as $c \searrow 1$)

to the corresponding solutions to the water waves system than the solitary wave solutions to the Green–Naghdi system or the Korteweg–de Vries equation (both at a distance $\mathcal{O}(c-1)$ as $c \searrow 1$); see Figure 10.1. This is in complete agreement with the analysis provided in Section 10.5.

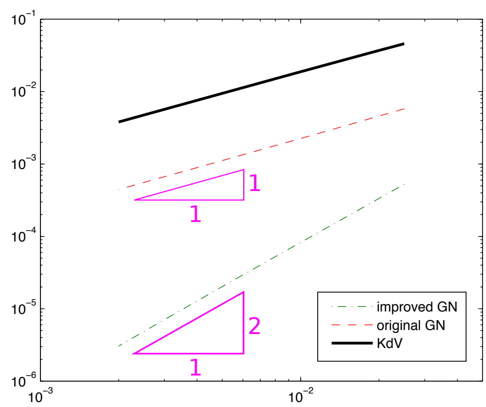
In the opposite direction, numerical experiments provided in [151] indicate the existence—as for the Green–Naghdi system—of solitary waves with arbitrarily large velocity and amplitude, and these waves appear to be dynamically stable, although in practice they become very sensitive to perturbations as $c-1$ gets large.



(a) Rescaled solitary waves for $c = 1.025$, 1.01, 1.002.



(b) Close-up.



(c) Convergence rate.

Figure 10.1: Comparison of the solutions of the KdV and Green–Naghdi models and the water waves system, taken from [165]. The waves are rescaled so that the Korteweg-de Vries solution does not depend on c. Consistently, we set $\varepsilon = \mu = 1$. The "improved" Green–Naghdi system is the Whitham–Green–Naghdi system and cannot be distinguished from the water waves solution. Figure 10.1c is a log-log plot of the normalized ℓ^2-norm of the error as a function of $c-1$.

10.5 Rigorous justification

In this section we discuss the complete rigorous justification of the Whitham–Green–Naghdi system, eq. (10.2) as an asymptotic model for the water waves system, eq. (2.7), in the shallow water regime (Definition III.2) that is for parameters in the set

$$\mathfrak{p}_{\mathrm{SW}} = \big\{(\mu,\varepsilon,\beta) \ : \ \mu \in (0,\mu^\star],\ \varepsilon \in [0,1],\ \beta \in [0,1]\big\}.$$

As already discussed several times in this manuscript, a complete justification follows from several results: (i) a *consistency* result stating that exact solutions to the water waves system satisfy approximately the Whitham–Green–Naghdi equations; (ii) a (local) *well-posedness* result on the initial-value problem for the Whitham–Green–Naghdi equations which should be uniform in the shallow water regime; and (iii) a *stability* result controlling the difference between an approximate and an exact solution to the Whitham–Green–Naghdi equations. Altogether, this yields the target *convergence* result which estimates the difference between solutions to the water waves system—which exist on the relevant timescale and satisfy the required bounds by Theorem 2.9—and the corresponding solutions to the Whitham–Green–Naghdi model.

It turns out that, because the Whitham–Green–Naghdi system, eq. (10.2), has the same structure as the Green–Naghdi system, eq. (8.2), the results obtained in Section 8.6 extend directly to our system, with straightforward modifications. For instance, the results in Section 8.6.1 extend to the operator $\mathfrak{T}^{\mathsf{F}^\mu}$—and its inverse—provided we replace the definitions of the functional spaces X^s_μ and Y^s_μ therein with

$$X^s_{\mathsf{F}^\mu} \stackrel{\text{def}}{=} \{\boldsymbol{u} \in L^2(\mathbb{R}^d)^d \ : \ \big|\boldsymbol{u}\big|^2_{X^s_{\mathsf{F}^\mu}} \stackrel{\text{def}}{=} \sum_{|\boldsymbol{k}|=0}^{s} \big|\partial^{\boldsymbol{k}}\boldsymbol{u}\big|^2_{L^2} + \mu\big|\partial^{\boldsymbol{k}}\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}\big|^2_{L^2} < \infty\},$$

$$Y^s_{\mathsf{F}^\mu} \stackrel{\text{def}}{=} \{\boldsymbol{v} \in (X^0_{\mathsf{F}^\mu})' \ : \ \big|\boldsymbol{v}\big|^2_{Y^s_{\mathsf{F}^\mu}} \stackrel{\text{def}}{=} \sum_{|\boldsymbol{k}|=0}^{s} \big|\partial^{\boldsymbol{k}}\boldsymbol{v}\big|^2_{(X^0_{\mathsf{F}^\mu})'} < \infty\}.$$

In particular—see Lemma 8.9 and Lemma 8.10—the operator $\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b] : X^s_{\mathsf{F}^\mu} \to Y^s_{\mathsf{F}^\mu}$ is well-defined, one-to-one and onto—and hence eq. (10.2) makes sense—provided that ζ, b are sufficiently regular and the non-cavitation assumption holds, that is

$$\forall \boldsymbol{x}\in\mathbb{R}^d, \qquad h(\boldsymbol{x}) = 1+\varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0. \tag{10.9}$$

This allows to obtain the following consistency result from the expansion of the Dirichlet-to-Neumann operator provided in Section 4.4.

Theorem 10.5 (Consistency)**.** *Let $d\in\mathbb{N}^\star$, $s_\star > d/2$, $h_\star>0$, $\mu^\star>0$. Let $s\in\mathbb{N}$ and $M^\star\geq 0$. There exists $C>0$ such that for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\mathrm{SW}}$, any $b\in W^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)$, any $T>0$ and any $(\zeta,\psi)\in L^\infty(0,T;H^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)\times \mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water waves equations, eq. (2.7), satisfying eq. (10.9) uniformly for $t\in(0,T)$ and*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\big) + \big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\boldsymbol{u}) = r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\boldsymbol{u}|^2 - \mu\varepsilon\mathcal{R}^{\mathsf{F}^\mu}[h,\beta\nabla b,\boldsymbol{u}] = r_2, \end{cases}$$

where we denote $h = 1+\varepsilon\zeta-\beta b$, $\boldsymbol{u} = \mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]^{-1}(h\nabla\psi)$, and one has for almost every $t\in(0,T)$

$$\big|r_1(t,\cdot)\big|_{H^s} \leq C\,\mu^2(\varepsilon M + \beta M_{\mathrm{b}})\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+5}}\big),$$
$$\big|r_2(t,\cdot)\big|_{H^{s+1}} \leq C\,\mu^2\varepsilon\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+5}}\big).$$

with $M \stackrel{\text{def}}{=} \big|\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}$ and $M_{\mathrm{b}} \stackrel{\text{def}}{=} \big|b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}$.

Proof. We first decompose

$$\mathsf{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u} = h\boldsymbol{u} + \mu\mathcal{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u} = h\mathsf{J}_0^\mu\boldsymbol{u} + h\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u}$$

where $\mathsf{J}_0^\mu \stackrel{\text{def}}{=} \frac{\sqrt{\mu}|D|}{\tanh(\sqrt{\mu}|D|)} = (\mathsf{I}_0^\mu)^{-1}$, $\mathsf{I}_0^\mu \stackrel{\text{def}}{=} \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and

$$\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\mathsf{F}^\mu((h^3-1)\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}) + \frac{h-1}{3h}\nabla\mathsf{F}^\mu(\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}) \\ + \frac{1}{2h}\Big(\nabla\mathsf{F}^\mu\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big) - h^2(\beta\nabla b)\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}\Big) + (\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b).$$

The control of r_1 then follows from

$$\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot\big(h\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\big|_{H^s} \\ \leq \big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot\big(h(\mathsf{I}_0^\mu - \mu\mathsf{I}_0^\mu\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\mathsf{I}_0^\mu)\nabla\psi\big)\big|_{H^s} \\ + \mu^2\big|\nabla\cdot\big(h\mathsf{I}_0^\mu\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\mathsf{I}_0^\mu\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\big|_{H^s}.$$

The control of the last term follows from Lemma 8.10 (adapted to our functional framework), the additional estimate (using the same notations as therein)

$$\big|\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u}\big|_{Y^s_{\mathsf{F}^\mu}} \leq C\times\Big((\big|\varepsilon\zeta\big|_{H^{s_\star}} + \big|\beta\nabla b\big|_{W^{s_\star,\infty}})\big|\boldsymbol{u}\big|_{X^s_{\mathsf{F}^\mu}} + \Big\langle(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}})\big|\boldsymbol{u}\big|_{X^{s_\star}_{\mathsf{F}^\mu}}\Big\rangle_{s>s_\star}\Big),$$

the (uniformly in μ) continuous embeddings $H^{s+1}(\mathbb{R}^d)^d \subset X^s_{\mathsf{F}^\mu} \subset H^s(\mathbb{R}^d)^d \subset Y^s_{\mathsf{F}^\mu} \subset H^{s-1}(\mathbb{R}^d)^d$, and the boundedness of I_0^μ in Sobolev spaces. As for the first term, we use Proposition 4.15 (with $n=2$ and $k=s+1$), and the identity

$$\nabla\cdot\Big(h\tfrac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi + \mu h\mathcal{T}[h,\beta\nabla b]\nabla\psi + \frac{\mu}{3}h\nabla\Delta\psi\Big) - \nabla\cdot\big(h(\mathsf{I}_0^\mu - \mu\mathsf{I}_0^\mu\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\mathsf{I}_0^\mu)\nabla\psi\big) \\ = \mu\nabla\cdot\big(h(\mathsf{I}_0^\mu\widetilde{\mathcal{T}}^{\mathsf{F}^\mu}[h,\beta\nabla b]\mathsf{I}_0^\mu - \widetilde{\mathcal{T}}^{\mathrm{Id}}[h,\beta\nabla b])\nabla\psi\big).$$

The last term is estimated as above using the fact that $\mu^{-1}(\mathsf{I}_0^\mu - \mathrm{Id}) : H^{s+2}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ and $\mu^{-1}(\mathsf{F}^\mu - \mathrm{Id}) : H^{s+2}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ are bounded, uniformly with respect to $\mu \in (0,\mu^\star]$.

As for the control of r_2, we can use the corresponding estimate Theorem 8.2, triangular inequality and an estimate on $\mu\varepsilon\mathcal{R}^{\mathsf{F}^\mu}[h,\beta\nabla b,\boldsymbol{u}] - \mu\varepsilon\mathcal{R}^{\mathrm{Id}}[h,\beta\nabla b,\boldsymbol{u}]$ which follows from the boundedness of $\mu^{-1}(\mathsf{F}^\mu - \mathrm{Id}) : H^{s+2}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$. □

Unfortunately, the analysis and energy estimates carried out in Section 8.6—outside of the results in Section 8.6.1—cannot be straightforwardly to our fully dispersive system, since the former applies to the formulation eq. (8.6) which does not have a direct analogous equivalent to eq. (10.4). Yet the analysis in [162] is carried on the formulation of the Green–Naghdi system analogous to eq. (10.4), and the analysis in [164] applies as a specific case to eq. (10.4) in the one-dimensional case $d=1$, flat bottom situation and with surface tension. This gives sufficient grounds to assert the following claims, of which only the level of regularity is questionable.

Conjecture 10.6 (Local well-posedness)**.** *Let $d\in\mathbb{N}^\star$, $s_\star > d/2$ and $s\in\mathbb{N}$, $s\geq 1+s_\star$, $h_\star > 0$, $\mu^\star > 0$, and $M^\star \geq 0$. There exist $T>0$ and $C>0$ such that the for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\mathrm{SW}}$, any $b\in W^{s+1,\infty}(\mathbb{R}^d)$, and any $(\zeta_0,\nabla\psi_0)\in H^s(\mathbb{R}^d)\times Y^s_{\mathsf{F}^\mu}$ satisfying eq. (10.9) and*

$$M_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\nabla\psi_0\big|_{Y^{1+s_\star}_{\mathsf{F}^\mu}} + \big|\beta b\big|_{W^{s,\infty}} \leq M^\star,$$

there exists a unique $(\zeta,\nabla\psi)\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times Y^s_{\mathsf{F}^\mu})\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times Y^{s-1}_{\mathsf{F}^\mu})$ *solution to the Whitham—Green–Naghdi system, eq.* (10.2)*, with initial data* $(\zeta,\psi)\,|_{t=0}=(\zeta_0,\psi_0)$*; and we have for any* $t\in[0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^s}+\big|\nabla\psi(t,\cdot)\big|_{Y^s_{\mathsf{F}^\mu}}+\big|\boldsymbol{u}(t,\cdot)\big|_{X^s_{\mathsf{F}^\mu}}\leq C\times\Big(\big|\zeta_0\big|_{H^s}+\big|\nabla\psi_0\big|_{Y^s_{\mathsf{F}^\mu}}\Big),$$

where we denote $\boldsymbol{u}=\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]^{-1}(h\nabla\psi)$ *with* $h=1+\varepsilon\zeta-\beta b$*, and eq.* (10.9) *holds with* $h_\star/2$*.*

Accompanying the well-posedness of the Cauchy problem (and in fact at the center of its proof) is a *stability* result (or rather conjecture), analogous to Theorem 8.6, which we shall not write down. From this, Theorem 10.5 and Conjecture 10.6 we conclude the rigorous justification of the Whitham–Green–Naghdi model as follows.

Conjecture 10.7 (Convergence)**.** *Let* $d\in\mathbb{N}^\star$*,* $s_\star>d/2$*,* $h_\star>0$*,* $\mu^\star>0$*,* $s\in\mathbb{N}$ *and* $M^\star\geq 0$*. There exist* $T>0$ *and* $C>0$ *such that for any* $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$*, any* $b\in W^{\max\{s+6,2+s_\star\},\infty}(\mathbb{R}^d)$*, any* $T^\star>0$ *and any* $(\zeta,\psi)\in\mathcal{C}^0([0,T^\star];H^{\max\{s+6,2+s_\star\}}\times\mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)^{1+d})$ *solution to the water waves equations* (2.7) *and such that* $h=1+\varepsilon\zeta-\beta b$ *satisfies Equation* (10.9) *uniformly for* $t\in[0,T^\star]$ *and*

$$M\stackrel{\rm def}{=}\sup_{t\in[0,T^\star]}\Big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}}+\big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{\max\{s+1,1+s_\star\}}}\Big)+\big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}\leq M^\star,$$

there exists a unique $(\zeta_{\rm WGN},\nabla\psi_{\rm WGN})\in\mathcal{C}^0([0,T/M];H^{\max\{s,1+s_\star\}}(\mathbb{R}^d)\times Y^{\max\{s,1+s_\star\}}_{\mathsf{F}^\mu})$ *strong solution to the Whitham–Green–Naghdi system* (10.2) *with initial data* $\big(\zeta_{\rm WGN},\psi_{\rm WGN}\big)\,|_{t=0}=\big(\zeta,\psi\big)\,|_{t=0}$*; and one has for any* $t\in(0,\min\{T^\star,T/M\}]$*,*

$$\big|(\zeta-\zeta_{\rm WGN})(t,\cdot)\big|_{H^s}+\big|(\nabla\psi-\nabla\psi_{\rm WGN})(t,\cdot)\big|_{Y^s_{\mathsf{F}^\mu}}\leq C\,\mu^2\,M\,t\,\big(\,\big\|\zeta\big\|_{L^\infty(0,t;H^{s+6})}+\big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+5})}\big)\,.$$

Remark 10.8. *It should be emphasized that the prefactor* $M=\mathcal{O}(\varepsilon+\beta)$ *in the last inequality in Conjecture 10.7 provides the quantitative gain with respect to the equivalent result on the Green–Naghdi system, Theorem 8.7. A striking consequence is that the solution to the Whitham–Green–Naghdi model remains close—in shallow water situations—to the corresponding solution to the water waves system over the full time interval* $t\in[0,\min\{T^\star,T/M\}]$*, specifically at a distance* $\mathcal{O}(\mu^2)$*. Comparatively, in the situation where* $\varepsilon+\beta\ll 1$*, the solution to the original Green–Naghdi–system is* a priori *useless for times* $t\gtrsim\mu^{-2}$*, due to second-order dispersive effects which are not captured by the model and arise before nonlinear effects. See Appendix I.5 for a numerical illustration of this feature.*

10.6 Boussinesq and Whitham–Boussinesq systems

Starting from the Green–Naghdi–Whitham system, eq. (10.2), and neglecting contributions of order $\mathcal{O}(\mu(\varepsilon+\beta))$, yields the much simpler-looking

$$\begin{cases}\partial_t\zeta+\nabla\cdot(\mathsf{I}^\mu_0\nabla\psi+(\varepsilon\zeta-\beta b)\nabla\psi)=0,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\nabla\psi|^2=0,\end{cases}$$

where $\mathsf{I}^\mu_0\stackrel{\rm def}{=}\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$. Yet there is little hope that the initial-value problem associated with the above system enjoys good well-posedness properties, and hence we will consider the following class of regularized systems

$$\begin{cases}\partial_t\zeta+\nabla\cdot\big(\mathsf{I}^\mu_0\nabla\psi+\mathsf{I}^\mu_1((\varepsilon\zeta-\beta b)\mathsf{I}^\mu_1\nabla\psi)\big)=0,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\mathsf{I}^\mu_1\nabla\psi|^2=0.\end{cases}\tag{10.10}$$

As soon as—roughly speaking—$\mathsf{I}_1^\mu = \mathrm{Id} + \mathcal{O}(\mu)$, the validity of eq. (10.10) as an asymptotic model does not suffer from introducing I_1^μ; see Theorem 10.13 below. When $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu$, the system corresponds to the one introduced in [143] (in the flat bottom situation), and thoroughly studied in [141, 146, 353, 142, 177]. Setting $\mathsf{I}_1^\mu = (\mathsf{I}_0^\mu)^{1/2}$ has the arguable advantages that the system maintains the quasilinear nature of the original water waves system (while the system is semilinear when $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu$), and that the last two terms in the first equation can be merged as a single term involving the depth $h = 1 + \varepsilon\zeta - \beta b$. All these systems belong to a class of so-called Whitham–Boussinesq systems[51] which can be loosely defined as fully dispersive modifications of standard Boussinesq models. Indeed, setting $\mathsf{I}_0^\mu = \mathsf{I}_1^\mu = (\mathrm{Id} - \frac{\mu}{3}\Delta)^{-1}$, and defining $\boldsymbol{u} = \mathsf{I}_0^\mu \nabla\psi$ we infer (using that $\boldsymbol{u}$ is by definition a gradient vector field)

$$\begin{cases} (\mathrm{Id} - \frac{\mu}{3}\Delta)\partial_t\zeta + (\mathrm{Id} - \frac{\mu}{3}\Delta)\nabla\cdot\boldsymbol{u} + \nabla\cdot\big((\varepsilon\zeta - \beta b)\boldsymbol{u}\big) = 0, \\ \partial_t(\boldsymbol{u} - \frac{\mu}{3}\nabla\nabla\cdot\boldsymbol{u}) + \nabla\zeta + \varepsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0}, \end{cases} \tag{10.11}$$

then we recognize the specific element of the "*abcd*" Boussinesq systems with $-a = b = d = \frac{1}{3}$ and $c = 0$; see Section iv.

Using physical variables (recall Section 2.4), eq. (10.10) yields a ***Whitham–Boussinesq system***:

$$\begin{cases} \partial_t h + \nabla\cdot\Big(\mathsf{I}_0^{d^2} d\boldsymbol{v} + \mathsf{I}_1^{d^2}\big((h-d)\mathsf{I}_1^{d^2}\boldsymbol{v}\big)\Big) = 0, \\ \partial_t\boldsymbol{v} + g\nabla(h+b) + \frac{1}{2}|\mathsf{I}_1^{d^2}\boldsymbol{v}|^2 = 0. \end{cases} \tag{10.12}$$

with $h = d + \zeta - b$ and $\boldsymbol{v} = \nabla\psi$; and eq. (10.11) yields a ***Boussinesq system***:

$$\begin{cases} \partial_t(h - \frac{d^2}{3}\Delta h) + \nabla\cdot\big(h\boldsymbol{u} - \frac{d^3}{3}\nabla\nabla\cdot\boldsymbol{u}\big) = 0, \\ \partial_t\big(\boldsymbol{u} - \frac{d^2}{3}\nabla\nabla\cdot\boldsymbol{u}\big) + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0}, \end{cases} \tag{10.13}$$

where $\nabla\psi = \boldsymbol{u} - \frac{d^2}{3}\nabla\nabla\cdot\boldsymbol{u}$ and $\boldsymbol{u}$ represents the vertically-averaged horizontal velocity.

10.6.1 Hamiltonian structure

It is easy to check that no matter the definition of the operators I_0^μ and I_1^μ (as long as they are independent of the variables, commute with spatial derivatives, and are symmetric with respect to the $L^2(\mathbb{R}^d)$ inner-product), eq. (10.10) enjoys a canonical Hamiltonian structure. Defining the Hamiltonian functional

$$\mathscr{H}_{\mathrm{WB}}(\zeta,\psi) \stackrel{\mathrm{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + |(\mathsf{I}_0^\mu)^{1/2}\nabla\psi|^2 + (\varepsilon\zeta + \beta b)|\mathsf{I}_1^\mu\nabla\psi|^2 \,\mathrm{d}\boldsymbol{x},$$

Hamilton's principle on

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \zeta\partial_t\psi \,\mathrm{d}\boldsymbol{x} + \mathscr{H}_{\mathrm{WB}} \,\mathrm{d}t.$$

yields

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_\zeta\mathscr{H}_{\mathrm{WB}}\\ \delta_\psi\mathscr{H}_{\mathrm{WB}}\end{pmatrix},$$

which corresponds to eq. (10.2). From this and Noether's theorem we infer the relationship between the easily checked group symmetries and related preserved quantities listed in Section 2.2 or Section 10.2.

[51] among which other models have been introduced in [5, 326, 220, 143] (see also [134] for a second-order Whitham–Boussinesq equation), discussed and compared in [78, 258, 106, 143], and investigated in [241, 356, 409] for the well-posedness of the initial-value problem, [221] for finite-time wavebreaking, [344, 145, 142, 172] for the existence of small-amplitude and large-amplitude traveling waves, [220, 353, 131] for the modulational and high frequency (in)stability of periodic traveling waves.

10.6.2 Modal analysis

As the Whitham–Green–Naghdi system (see Section 10.3), the Boussinesq–Green–Naghdi system, eq. (10.10) is *fully dispersive*: the linearized system about the rest state coincides with one of the water waves system. The reader can refer to Section ii and Section 2.3 for more information (modal analysis, large-time behavior of solutions, dispersive and Strichartz estimates) on the linearized water waves system.

As for the specific Boussinesq system eq. (10.11), its linearized system coincides with the one of the Green–Naghdi system. Hence the reader can refer to the discussion in Section 8.3, and in particular Figure 8.1.

10.6.3 Solitary waves

In [145] the authors study the existence of solitary waves for systems of the form eq. (10.10) with $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu$, in the flat bottom case and dimension $d = 1$. Their results cover as particular cases the Whitham–Boussinesq equations introduced in [143], that is $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$, and the specific Boussinesq system eq. (10.11), obtained by setting $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu = (1 + \frac{\mu}{3}|D|^2)^{-1}$. Their strategy and results are very similar to the ones presented in Section 10.4.

Let me display the result—or rather a corollary—obtained very recently by Dinvay in [142]. Instead of relying on a concentration-compactness argument applied to a minimization problem, the strategy—initiated by Stefanov and Wright [389] for scalar equations— relies on perturbative methods and the implicit function theorem. Besides simplicity and robustness, a nice feature of the strategy is that it provides the uniqueness of (even) solitary waves with sufficiently small (yet supercritical) velocities. Here the results applies to systems of which eq. (10.10) (in the flat bottom case and dimension $d = 1$) are particular cases.

Theorem 10.9. *Set $\varepsilon = \mu = 1$, $\beta = 0$ and $d = 1$. Let $\mathsf{I}_0^\mu = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and $\mathsf{I}_1^\mu = I_1(\sqrt{\mu}|D|)$ with $I_1 \in W^{2,\infty}(\mathbb{R})$ such that $I_1(0) = I_1'(0) = 0$, and there exists*

i. $C_1 > c_1 > 0$ and $\alpha \in [0, 2)$ such that for any $\xi \in \mathbb{R}$, $c_1\langle\xi\rangle^{-2\alpha_1} \leq I_1(\xi) \leq C_1\langle\xi\rangle^{-\alpha_1}$;

ii. $C_2 > 0$ and $\alpha_2 < 1$ such that for any $\xi \in \mathbb{R}$, $|I_1'(\xi)| + |(\frac{1}{I_1})'(\xi)| < C_2\langle\xi\rangle^{-\alpha_2}$.

Then there exists $c_0 > 1$ such that for any $c \in (1, c_0)$, there exists a unique (ζ_c, ψ_c) such that $(\zeta_c, \mathsf{I}_1^\mu \partial_x\psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ are two even functions and $(\zeta, \psi) : (t, x) \in \mathbb{R} \times \mathbb{R} \mapsto (\zeta_c, \psi_c)(x - ct)$ satisfy eq. (10.10). Moreover, for any $s \geq 1$, there exists $M^\star > 0$ such that $(\zeta_c, \partial_x\psi_c) \in H^s(\mathbb{R})^2$, and, denoting $c = 1 + \frac{3}{8}\epsilon$ and $\xi_{\mathrm{KdV}}(x) = \frac{3}{4}\operatorname{sech}^2\left(\frac{3}{4}x\right)$,

$$\left|\epsilon^{-1}\partial_x\psi(\epsilon^{-1/2}\cdot) - \xi_{\mathrm{KdV}}\right|_{H^s(\mathbb{R})} + \left|\epsilon^{-1}\zeta(\epsilon^{-1/2}\cdot) - \xi_{\mathrm{KdV}}\right|_{H^s(\mathbb{R})} \leq M^\star\epsilon^{\max(\{2-\alpha_1, \frac{3-2\alpha_2}{4-2\alpha_2}\})}$$

uniformly over $c \in (1, c_0)$.

Remark 10.10. *The asymptotic when $c \searrow 1$ is consistent with that of Theorem 10.2. The theorem applies in particular when $\mathsf{I}_1^\mu = (\mathsf{I}_0^\mu)^\alpha$ with $\alpha \in [0, 2)$, in which case we can set $\alpha_1 = \alpha$ and $\alpha_2 = \alpha - 1$. The result in [142] does not apply to the Boussinesq case $\mathsf{I}_1^\mu = \mathsf{I}_0^\mu = (1 + \frac{\mu}{3}|D|^2)^{-1}$ as a particular case, but to other abcd-Boussinesq systems; see details therein.*

Remark 10.11. *Contrarily to the result of Stefanov and Wright [389], there is no information concerning the (spectral) stability of the constructed solitary waves in the case of systems.*

Remark 10.12. *There exist much more results concerning the existence (sometimes with explicit formula) and stability of solitary wave solutions to the general abcd Boussinesq systems introduced in Section iv. The interested reader can use [142, 88] as a starting point.*

10.6.4 Rigorous justification

Let us now discuss the complete rigorous justification of the Whitham–Boussinesq system, eq. (10.10) as an asymptotic model for the water waves system, eq. (2.7), in the shallow water regime that is (see Definition III.2) for parameters in the set

$$\mathfrak{p}_{\rm SW} = \big\{(\mu,\varepsilon,\beta)\ :\ \mu\in(0,\mu^\star],\ \varepsilon\in[0,1],\ \beta\in[0,1]\big\}.$$

In particular, we will clarify the improvement in accuracy obtained when choosing $\mathsf{I}_0^\mu \stackrel{\rm def}{=} \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ (which yields a model with the full dispersion property) rather than, say, $\mathsf{I}_0^\mu=\mathsf{I}_1^\mu=(\mathrm{Id}-\frac{\mu}{3}\Delta)^{-1}$, which yields a Boussinesq system.

Theorem 10.13 (Consistency)**.** *Let $d\in\mathbb{N}^\star$, $s_\star>d/2$, $h_\star>0$, $\mu^\star>0$. Let $s\in\mathbb{N}$, $\alpha\geq 0$ and $M^\star\geq 0$. Let $\mathsf{I}_0^\mu=\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and $\mathsf{I}_1^\mu=I_1(\sqrt{\mu}|D|)$ with $I_1\in L^\infty(\mathbb{R})$ such that $I_1(\xi)=1+\mathcal{O}(\xi^2)$. There exists $C>0$ such that for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$, any $b\in W^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)$, any $T>0$ and any $(\zeta,\psi)\in L^\infty(0,T;H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)\times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water waves equations, eq. (2.7), satisfying for all $t\in(0,T)$*

$$\forall \boldsymbol{x}\in\mathbb{R}^d,\qquad h(t,\boldsymbol{x})=1+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x})\geq h_\star>0 \tag{10.14}$$

uniformly for $t\in(0,T)$ and

$$\operatorname*{ess\,sup}_{t\in(0,T)}\big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{2+s_\star}}+\big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\big)+\big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}\leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta+\nabla\cdot\big(\mathsf{I}_0^\mu\nabla\psi+\mathsf{I}_1^\mu((\varepsilon\zeta-\beta b)\mathsf{I}_1^\mu\nabla\psi)\big)=r_1,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\mathsf{I}_1^\mu\nabla\psi|^2=r_2,\end{cases}$$

and one has for almost every $t\in(0,T)$

$$\begin{aligned}\big|r_1(t,\cdot)\big|_{H^s}&\leq C\,\mu(\varepsilon M+\beta M_{\rm b})\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}}+\big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big),\\ \big|r_2(t,\cdot)\big|_{H^{s+1}}&\leq C\,\mu\varepsilon\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}}+\big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big),\end{aligned}$$

with $M\stackrel{\rm def}{=}\big|\zeta(t,\cdot)\big|_{H^{2+s_\star}}+\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}$ and $M_{\rm b}\stackrel{\rm def}{=}\big|b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}$.

Proof. Let us first treat the first statement. For the control of r_1,

$$\begin{aligned}\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi+\nabla\cdot(\mathsf{I}_0^\mu\nabla\psi+\mathsf{I}_1^\mu((\varepsilon\zeta-\beta b)\mathsf{I}_1^\mu\nabla\psi))\Big|_{H^s}&\\ \leq\Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi+\nabla\cdot((1+\varepsilon\zeta-\beta b)\mathsf{I}_0^\mu\nabla\psi)\Big|_{H^s}&\\ +\big|\nabla\cdot(\mathsf{I}_1^\mu((\varepsilon\zeta-\beta b)\mathsf{I}_1^\mu\nabla\psi)-(\varepsilon\zeta-\beta b)(\mathsf{I}_1^\mu-\mathsf{I}^\mu)\nabla\psi)\big|_{H^s}&\end{aligned}$$

The first term in the right-hand side is estimated by Proposition 4.15 (with $n=1$ and $k=s+1$), and the estimate of last term follows from the boundedness of $\frac{1}{\mu}(\mathsf{I}_1^\mu-\mathrm{Id}):H^{s+2}(\mathbb{R}^d)\to H^s(\mathbb{R}^d)$ and $\mathsf{I}_1^\mu:H^s(\mathbb{R}^d)\to H^s(\mathbb{R}^d)$, for any $s\in\mathbb{R}$, being uniform with respect to $\mu\in(0,\mu^\star]$, and product estimates in Appendix II.

As for the control of r_2, we simply use Proposition 4.10 (with $n=0$ and $k=s+2$) to control $\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi\big|_{H^{s+1}}$, aforementioned product and composition estimates in Appendix II and boundedness of I_1^μ and $\frac{1}{\mu}(\mathsf{I}_1^\mu-\mathrm{Id})$. ☐

Theorem 10.14 (Consistency)**.** *Under the assumptions and using the notations of Theorem 10.13, denoting* $\widetilde{\mathsf{I}}_0^\mu \stackrel{\text{def}}{=} \widetilde{I}(\sqrt{\mu}|D|)$ *with* $\widetilde{I} \in L^\infty(\mathbb{R})$ *such that* $\widetilde{I}(\xi) = 1 - \frac{1}{3}\xi^2 + \mathcal{O}(\xi^4)$*, and assuming additionally that* $\psi(t,\cdot) \in \mathring{H}^{s+6}(\mathbb{R}^d)$ *for almost every* $t \in (0,T)$*, we have*

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(\widetilde{\mathsf{I}}_0^\mu\nabla\psi + \mathsf{I}_1^\mu((\varepsilon\zeta-\beta b)\mathsf{I}_1^\mu\nabla\psi)\big) = \widetilde{r}_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\mathsf{I}_1^\mu\nabla\psi|^2 = \widetilde{r}_2, \end{cases}$$

and one has for almost every $t \in (0,T)$

$$\big|\widetilde{r}_1(t,\cdot)\big|_{H^s} \le C\,\mu(\varepsilon M + \beta M_{\rm b})\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big) + C\,\mu^2\,\big|\nabla\psi(t,\cdot)\big|_{H^{s+5}},$$
$$\big|\widetilde{r}_2(t,\cdot)\big|_{H^{s+1}} \le C\,\mu\varepsilon\big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+4}} + \big|\nabla\psi(t,\cdot)\big|_{H^{s+3}}\big),$$

with $M \stackrel{\text{def}}{=} \big|\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\nabla\psi(t,\cdot)\big|_{H^{1+s_\star}}$ *and* $M_{\rm b} \stackrel{\text{def}}{=} \big|b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}$*.*

Proof. The result immediately follows from Theorem 10.13 and $\frac{1}{\mu^2}(\widetilde{\mathsf{I}}_0^\mu - \mathsf{I}_0^\mu) : H^{s+5}(\mathbb{R}^d) \to H^{s+1}(\mathbb{R}^d)$ being bounded uniformly with respect to $(0,\mu^\star]$, which is a direct consequence of the Taylor expansion $\frac{\tanh(\xi)}{\xi} = 1 - \frac{\mu}{3}\xi^2 + \mathcal{O}(\xi^4)$. □

Energy estimates for eq. (10.10) can be obtained for a wide class of Fourier multipliers, I_0^μ and I_1^μ, including all of the specific ones mentioned above. We require the following.

Assumption 10.15. *For* $\ell \in \{0,1\}$, $\mathsf{I}_\ell^\mu \stackrel{\text{def}}{=} I_\ell(\sqrt{\mu}|D|)$ *with real-valued even* $I_\ell \in W^{1,\infty}(\mathbb{R})$ *satisfying*

i. there exists $C_0 > 0$ *such that for any* $\xi \in \mathbb{R}$, $0 \le I_\ell(\xi) \le C_0$*;*

ii. there exists $C_1 > 0$ *such that for almost any* $\xi \in \mathbb{R}$, $(1+|\xi|)|I_\ell'|(\xi) \le C_1$*;*

iii. for any $\xi \in \mathbb{R}$*, one has* $I_1(\xi)^2 \le I_0(\xi)$*.*

We shall not detail the proof of the following results, which consists in extending the standard energy method for hyperbolic symmetrizable quasilinear systems (as detailed for instance in [310]) and more precisely the Saint-Venant equations—which incidentally are a particular case of eq. (10.10) with $\mathsf{I}_0^\mu = \mathsf{I}_1^\mu = \mathrm{Id}$—to the presence of Fourier multipliers as above, and refer to [176] for the complete proof (in the flat-bottom situation). Let me simply precise that

- item i and item ii allow to consider I_0^μ and I_1^μ as order-zero operators in Sobolev spaces, satisfying suitable product and commutator estimates;
- item iii allows to consider nonlinear terms as being of the same order as linear terms. We could relax the assumption to $I_1(\xi)^2 \le C_2 I_0(\xi)$ with some $C_2 > 0$, yet the hyperbolicity condition would be more stringent than the non-cavitation assumption, eq. (10.14).

In the following, we denote for $s \in \mathbb{N}$

$$Z^s_{\mathsf{I}_0^\mu} \stackrel{\text{def}}{=} \big\{\psi \in L^2_{\rm loc}(\mathbb{R}^d) \ : \ \big|(\mathsf{I}_0^\mu)^{1/2}\nabla\psi\big|_{H^s} < \infty\big\}.$$

Theorem 10.16 (Local well-posedness)**.** *Let* $d \in \mathbb{N}^\star$*,* $s_\star > d/2$ *and* $s \in \mathbb{N}$*,* $s \ge 1 + s_\star$*,* $h_\star > 0$ $\mu^\star > 0$*,* $M^\star \ge 0$ *and* $C_0, C_1 > 0$*. There exist* $T > 0$ *and* $C > 0$ *such that the for any* $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm SW}$*, any* I_0^μ *and* I_1^μ *satisfying Assumption 10.15, any* $b \in W^{s+1,\infty}(\mathbb{R}^d)$ *and any* $(\zeta_0,\psi_0) \in H^s \times Z^s_{\mathsf{I}_0^\mu}$ *such that eq. (10.14) holds and*

$$M_0 \stackrel{\text{def}}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon(\mathsf{I}_0^\mu)^{1/2}\nabla\psi_0\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{s,\infty}} \le M^\star,$$

there exists a unique $(\zeta,\psi) \in \mathcal{C}^0([0,T/M_0]; H^s(\mathbb{R}^d)\times Z^s_{\mathsf{I}_0^\mu}) \cap \mathcal{C}^1([0,T/M_0]; H^{s-1}(\mathbb{R}^d)\times Z^{s-1}_{\mathsf{I}_0^\mu})$ *solution to eq. (10.10), with initial data* $(\zeta,\psi)\,|_{t=0} = (\zeta_0,\psi_0)$*; and we have for any* $t \in [0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla\psi(t,\cdot)\big|_{H^s} \le C \times \Big(\big|\zeta_0\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla\psi_0\big|_{H^s}\Big).$$

Theorem 10.17 (Stability)**.** *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M^\star \geq 0$ and $C_0, C_1 > 0$, and denote $n_0 \stackrel{\rm def}{=} \max\{s, 1+s_\star\}$, $n \stackrel{\rm def}{=} \max\{s+1, 1+s_\star\}$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any I_0^μ and I_1^μ satisfying Assumption 10.15, any $b \in W^{n,\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and $(\zeta^0, \psi^0) \in \mathcal{C}^0([0,T^\star]; H^{n_0}(\mathbb{R}^d) \times Z^{n_0}_{\mathsf{I}^\mu})$ satisfying eq. (10.10), and any $(\zeta, \psi) \in L^\infty(0,T^\star; H^n(\mathbb{R}^d) \times Z^n_{\mathsf{I}_0^\mu})$ satisfying*

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(\mathsf{I}_0^\mu\nabla\psi + \mathsf{I}_1^\mu((\varepsilon\zeta - \beta b)\mathsf{I}_1^\mu\nabla\psi)\big) = r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\mathsf{I}_1^\mu\nabla\psi|^2 = r_2, \end{cases}$$

with $(r_1, r_2) \in L^1(0,T^\star; H^s(\mathbb{R}^d) \times Z^s_{\mathsf{I}_0^\mu})$, and assuming that $h = 1 + \varepsilon\zeta - \beta b$ and $h^0 = 1 + \varepsilon\zeta^0 - \beta b$ satisfy eq. (10.14) uniformly for $t \in [0, T^\star]$ and

$$M \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{t\in[0,T^\star]} \big(\big|(\varepsilon\zeta, \varepsilon(\mathsf{I}_0^\mu)^{1/2}\nabla\psi)(t,\cdot)\big|_{H^n\times H^n} + \big|(\varepsilon\zeta^0, \varepsilon(\mathsf{I}_0^\mu)^{1/2}\nabla\psi^0)(t,\cdot)\big|_{H^{n_0}\times H^{n_0}}\big) + \big|\beta b\big|_{W^{n,\infty}} \leq M^\star,$$

then one has for any $t \in (0, T^\star)$,

$$\begin{aligned} \big|(\zeta-\zeta^0)(t,\cdot)\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla(\psi-\psi^0)(t,\cdot)\big|_{H^s} &\leq Ce^{CMt}\Big(\big|(\zeta-\zeta^0)(0,\cdot)\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla(\psi-\psi^0)(0,\cdot)\big|_{H^s}\Big) \\ &\quad + C\int_0^t e^{CM(t-\tau)}\big(\big|r_1(\tau,\cdot)\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla r_2(\tau,\cdot)\big|_{H^s}\big)\,{\rm d}\tau\,. \end{aligned}$$

The following results are a direct consequence of Theorem 10.13, Theorem 10.14, Theorem 10.16 and Theorem 10.17.

Theorem 10.18 (Convergence)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $s \in \mathbb{N}$ and $M^\star \geq 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, any $b \in W^{\max\{s+4, 2+s_\star\},\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and any $(\zeta, \psi) \in \mathcal{C}^0([0,T^\star]; H^{\max\{s+4,2+s_\star\}} \times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^{1+d})$ solution to the water waves equations (2.7) and such that $h = 1 + \varepsilon\zeta - \beta b$ satisfies Equation (10.14) uniformly for $t \in [0, T^\star]$ and*

$$M \stackrel{\rm def}{=} \sup_{t\in[0,T^\star]} \big(\big|\varepsilon\zeta(t,\cdot)\big|_{H^{\max\{s+1,2+s_\star\}}} + \big|\varepsilon\nabla\psi(t,\cdot)\big|_{H^{\max\{s+1,1+s_\star\}}}\big) + \big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}} \leq M^\star,$$

there exists a unique $(\zeta_{\rm WB}, \psi_{\rm WB}) \in \mathcal{C}^0([0,T/M]; H^{\max\{s,1+s_\star\}}(\mathbb{R}^d) \times Z^{\max\{s,1+s_\star\}}_{\mathsf{I}_0^\mu})$ strong solution to the Whitham–Boussinesq system (10.10)—with $\mathsf{I}_0^\mu = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ and I_1^μ satisfying Assumption 10.15, for instance $\mathsf{I}_1^\mu = (\mathsf{I}_0^\mu)^\alpha$ with $\alpha \geq 1/2$—with initial data $\big(\zeta_{\rm WB}, \psi_{\rm WB}\big)\,|_{t=0} = \big(\zeta, \psi\big)\,|_{t=0}$; and one has for any $t \in (0, \min\{T^\star, T/M\}]$,

$$\big|(\zeta - \zeta_{\rm WB})(t,\cdot)\big|_{H^s} + \big|(\mathsf{I}_0^\mu)^{1/2}\nabla(\psi - \psi_{\rm WB})(t,\cdot)\big|_{H^s} \leq C\,\mu\,M\,t\,\big(\big\|\zeta\big\|_{L^\infty(0,t;H^{s+4})} + \big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+3})}\big)\,.$$

The following result is a direct analogue of Theorem 10.18, using that the Boussinesq system (10.11) is equivalent to eq. (10.10) with $\mathsf{I}_0^\mu = \mathsf{I}_1^\mu = ({\rm Id} - \frac{\mu}{3}\Delta)^{-1}$.

Theorem 10.19 (Convergence)**.** *Under the assumptions and using the notations of Theorem 10.18, there exists a unique $(\zeta_{\rm B}, \boldsymbol{u}_{\rm B}) \in \mathcal{C}^0([0,T/M]; H^{\max\{s,1+s_\star\}}(\mathbb{R}^d) \times H^{\max\{s+1,2+s_\star\}})$ strong solution to the Boussinesq system (10.11) with initial data $\big(\zeta_{\rm WB}, \boldsymbol{u}_{\rm WB}\big)\,|_{t=0} = \big(\zeta, ({\rm Id} - \frac{\mu}{3}\Delta)^{-1}\nabla\psi\big)\,|_{t=0}$; and if moreover $\nabla\psi(t,\cdot) \in L^1(0,T; H^{s+5})$, one has for any $t \in (0, \min\{T^\star, T/M\}]$,*

$$\begin{aligned} &\big|(\zeta - \zeta_{\rm B})(t,\cdot)\big|_{H^s} + \big|({\rm Id} - \tfrac{\mu}{3}\Delta)^{-\frac{1}{2}}\nabla(\psi - \psi_{\rm B})(t,\cdot)\big|_{H^s} \\ &\qquad \leq C\,\mu\,M\,t\,\big(\big\|\zeta\big\|_{L^\infty(0,t;H^{s+4})} + \big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+3})}\big) + C\,\mu^2\int_0^t \big|\nabla\psi(\tau,\cdot)\big|_{H^{s+5}}\,{\rm d}\tau \end{aligned}$$

where we denote $\nabla\psi_{\rm B} = \boldsymbol{u} - \frac{\mu}{3}\nabla\nabla\cdot\boldsymbol{u}$.

Remark 10.20. *Comparing Theorem 10.13 with Theorem 10.14, or Theorem 10.18 with Theorem 10.19 allows to recognize clearly the gain of considering Whitham–Boussinesq systems (that is eq.* (10.10) *with* $\mathsf{I}_0^\mu \stackrel{\text{def}}{=} \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$*) rather than a standard Boussinesq system (that is eq.* (10.11) *or, equivalently, eq.* (10.10) *with* $\mathsf{I}_0^\mu = \mathsf{I}_1^\mu = (\mathrm{Id} - \frac{\mu}{3}\Delta)^{-1}$*). Indeed, in situations with very weak nonlinearities and mild bottom topographies,* $\varepsilon + \beta \ll 1$*, solutions of the former but not of the latter remain close—in shallow water situations—to the corresponding solution to the water waves system over the full time interval* $t \in [0, \min\{T^\star, T/M\}]$*, specifically at a distance* $\mathcal{O}(\mu)$*. What is more, the control of the additional error term in the Boussinesq system requires more regularity on the data.*

10.7 Discussion and open questions

The open questions concerning the Green–Naghdi system we put forward in Section 8.7 arise maybe even more strongly for the Whitham–Green–Naghdi or Boussinesq systems. Recall (see Section v) that the fully dispersive Whitham equation has been introduced in view of reproducing wavebreaking and peaked traveling waves of extreme height; and that such solutions were recently proved to exist in [219, 173, 402, 374]. Unfortunately, the numerical experiments provided in [151] did not allow to produce finite-time singularities even in extreme situations, and smooth solitary waves with (apparently) arbitrarily large amplitude were exhibited. Hence the full dispersion property alone is not sufficient to explain the attractive features of the Whitham equation.

The question of large time existence of solutions in the situation of weak nonlinearities but strong bottom variations is also completely open. In view of offering models which remain relevant on the full time interval $t \in [0, T/\varepsilon]$, an additional direction of investigation would consist in the derivation (and rigorous justification) of models with improved precision in such situations; ideally $\mathcal{O}(\mu^2\varepsilon)$ (resp. $\mathcal{O}(\mu\varepsilon)$ for a Boussinesq-type) instead of $\mathcal{O}(\mu^2(\varepsilon+\beta))$ (resp. $\mathcal{O}(\mu(\varepsilon+\beta))$) as shown in Theorem 10.5 (resp. Theorem 10.13). In this case the system needs to be exact when $\varepsilon = 0$, and hence the linearized system about the trivial solution must be

$$\begin{cases} \partial_t\zeta^0 - \mathcal{G}[0,b]\psi^0 = 0, \\ \partial_t\psi^0 + g\zeta^0 = 0, \end{cases}$$

where $\mathcal{G}[0,b]\psi^0 = (\partial_z\Phi^0)\,|_{z=0}$ and Φ^0 is the unique solution to

$$\begin{cases} \Delta_{x,z}\Phi^0 = 0 & \text{in } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ -d + b(t,\boldsymbol{x}) < z < 0\}, \\ \Phi^0 = \psi^0 & \text{on } \mathbb{R}^d \times \{0\}, \\ \partial_z\Phi^0 = 0 & \text{on } \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} \ : \ z = -d + b(t,\boldsymbol{x})\}. \end{cases}$$

In view of eq. (10.12), a natural candidate as a Boussinesq–Whitham equation for strong topographies is

$$\begin{cases} \partial_t\zeta + \mathcal{G}[0,b]\psi + \nabla\cdot\big(I_1(\zeta I_1\nabla\psi)\big) = 0, \\ \partial_t\psi + \zeta + \frac{1}{2}|I_1\nabla\psi|^2 = 0. \end{cases} \tag{10.15}$$

where I_1 is a sufficiently regularizing near-identity operator. Obviously the operator $\mathcal{G}[0,b]$ is much more complicated than the Fourier multiplier $\mathcal{G}_0 = |D|\tanh(d|D|)$—or the corresponding operator in eq. (10.12), namely $\mathcal{G}_0 - \nabla\cdot(\mathsf{I}_1^{d^2} b \mathsf{I}_1^{d^2}\nabla\bullet)$—yet as a (positive) self-adjoint operator we can use spectral decomposition of the former in lieu of Fourier decomposition (see *e.g.* [24] for a description of the Fourier decomposition in the periodic framework, and [423, 118] for a study of the Bloch decomposition when the bottom topography is periodic) to devise for instance suitable definitions of I_1. The interested reader can refer to [404] for the derivation of a model similar to eq. (10.15) with $I_1 = \mathrm{Id}$ and where $\mathcal{G}_b$ is replaced by a *ad hoc* pseudo-differential approximation, and to [405] for a numerical comparison of spectral modes of the full operator $\mathcal{G}[0,b]$, a first order approximation based on the Taylor expansion of $\mathcal{G}[0,b]$ in the variable b, and the aforementioned *ad hoc* approximation.

CHAPTER D

Higher order models

Jésus a dit : « Que celui qui cherche ne cesse pas de chercher, jusqu'à ce qu'il trouve. Et quand il aura trouvé, il sera troublé ; quand il sera troublé, il sera émerveillé, et il régnera sur le Tout. »

— THOMAS L'APÔTRE, *évangile selon Thomas*

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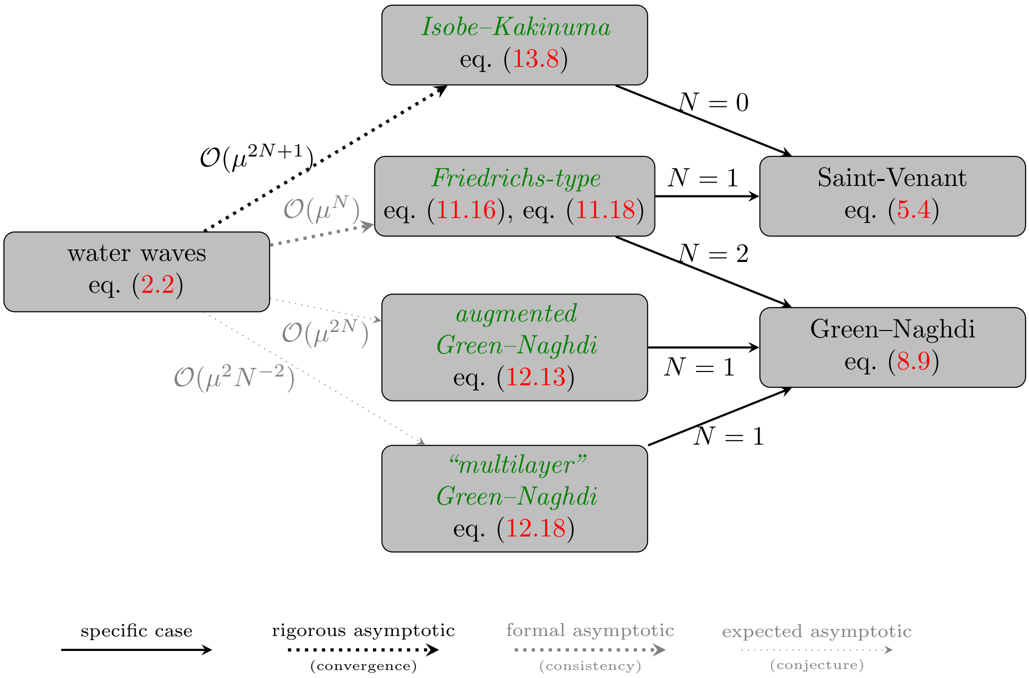


Figure D: Models in Chapter D (in *green*) and some filiations.

Foreword

In this chapter we introduce and discuss *higher order models* for the water waves system, building upon the Saint-Venant system (Section 5) and the Green–Naghdi system (Section 8). These are hierarchies of models, that is families of system depending on a parameter—always denoted N—which we call the rank of the model, of which the Saint-Venant and/or the Green–Naghdi system are typically the first rank elements. Recall that the Saint-Venant (resp. Green–Naghdi) system has been rigorously justified in Section 5.3 (resp. Section 8.5) as a shallow water model for the water waves system, in—roughly speaking—the following way: the size of the difference between solutions to the dimensionless water waves system and the corresponding solutions to the model equations grows proportionally to the size of the initial data with a prefactor bounded as $C\,\mu\,t$ (resp. $C\,\mu^2\,t$) over a relevant time interval (being of size inversely proportional to the size of the initial data), where μ is the shallow water parameter, and C depends on an upper bound on the size of the admissible initial data (together with a lower bound on the minimal depth of the layer, an upper bound on admissible values for μ, and the norms measuring the size of the data). In good cases we expect that a similar result holds for all elements in a hierarchy of models, with different prefactors $C_N\,\mu^{\alpha_N}\,t$. There are typically two situations:

i. the order as a shallow water model increases with N, that is $\alpha_N \to \infty$ as $N \to \infty$;

ii. the accuracy of the model improves with N, that is $C_N \to 0$ as $N \to \infty$.

In the latter but not in the former we can hope that the hierarchy provides a robust tool for the approximation of *any* (sufficiently regular) solution to the water waves system, and can be useful for instance to devise strategies for their numerical integration.

This chapter is decomposed into three sections, corresponding to three different strategies, each producing a variety of families of higher order models.[52]

In Section 11 we use an expansion due to Boussinesq [60] and Rayleigh [362] (see [140, §4.1] for discussion and other relevant references) of the velocity potential—as a solution to the Laplace problem—as a series involving powers of the shallow water parameter, μ. We are hence typically in the framework of the first aforementioned situation, and this section emphasizes its possible shortcomings. Among the different models which can be naturally constructed by this way—which we call Friedrichs-type systems in acknowledgment to his Appendix to [392]—we introduce explicitly two families of models: the as ***high order shallow water models***, eq. (11.16) and the ***extended Green–Naghdi models***, eq. (11.18). These models involve differential operators of increasing order as the rank of the model grows, which yields several complications. Firstly, half of these models suffer from very serious high frequency instabilities which prevent any hope as for the well-posedness of the Cauchy problem. But even in good cases, it is expected that for fixed initial data the solutions to the systems—if they exist—do not converge towards the corresponding solution to the water waves system, as $N \to \infty$. This can be seen in particular when studying the dispersion relation of the models, which converge towards the dispersion relation of the water waves system only for wavenumbers in a finite-size neighborhood of the origin.

In Section 12 we set up a Galerkin dimension reduction strategy to a reformulation of the Laplace problem, to devise the approximate formula for the velocity potential—or, more precisely, the horizontal velocity. As a second step, the usual procedure consists in using this approximate formula in the Hamiltonian functional of the water waves system, and express the model as the canonical Hamiltonian equations associated with the approximate Hamiltonian. This procedure produces a different model for any (reasonable) choice of subspace of real-valued functions of the fluid domain used in the Galerkin method. Natural examples of such spaces in the shallow water framework are functions of the form

$$\Phi(t, \boldsymbol{x}, z) = \sum_{i=1}^{N} \phi_i(t, \boldsymbol{x})\, \Psi_i(\boldsymbol{x}, z) \qquad (\star)$$

where $\phi_i(t, \boldsymbol{x})$ are variable unknowns of the resulting model, which is characterized by the choice of the vertical distribution, $\{\Psi_i\}_{i\in\{1,\dots,N\}}$. In Section 12.1.3 we explore the outcome of vertical distribution defined, following the finite element method, as piecewise polynomials in the vertical variable, z. We particularly emphasize two families of models (respectively playing with the degrees of the polynomials and the number of elements in the vertical discretization): the ***augmented Green–Naghdi models***, eq. (12.13) and the ***"multilayer" Green–Naghdi models***, eq. (12.18). In each case, the system consists in two evolution equations coupled with a system of differential equations of order two mimicking the Laplace problem. The first family is a higher order shallow water hierarchy comparable to the models of the preceding section, yet instead of involving high order differential operators, the size of the system of differential equations grows with the rank, N. The second family has different properties, akin to the second situation described above. The term "multilayer" stems from the fact that the models can be interpreted as resulting from the vertical discretization of the fluid layer in N prescribed—typically proportional—sublayers.

In Section 13 we describe the strategy referred in [261, 260] as "variational" (see [354] for an overview of related earlier and subsequent works). Of course the preceding strategy was also vari-

[52]The list is by no means complete. In particular it lacks *spectral methods* based on expansions with respect to the steepness parameter, $\epsilon = \varepsilon\sqrt{\mu}$, initiated in [147, 411, 129] (see *e.g.* [276, 376, 414, 343] for a detailed account and comparisons). Among them the strategy brought to light by Craig and Sulem in [129], consisting in expanding the Dirichlet-to-Neumann operator, $\mathcal{G}^\mu[\varepsilon\zeta]$, along the variable $\varepsilon\zeta$, is particularly elegant and effective. Contrarily to the models introduced in this chapter, the family of models involve Fourier multipliers in addition to differential operators; see [95] for discussion, references and the explicit display of the models up to fifth order. This method has been extended and successfully employed in many situations (see [208] and references therein), despite the claim—based on numerical experiments and formal arguments—in [17] that the Cauchy problem associated with any of the systems in the family is ill-posed in Sobolev spaces.

ational in nature, and we argue in Section 13.3 that the two strategies in fact differ only by the choice of the variational formulation of the Laplace problem. Yet in the latter, we plug directly the decomposition ($\star$) into Luke's Lagrangian action for the water waves system, and let Hamilton's principle do all the work in one single step. The outcome is surprising at first, as we obtain an overdetermined/underdetermined composite system of N evolution equations for the surface deformation, ζ, and only one evolution equation for $(\phi_1, \ldots, \phi_N)$. Yet as shown in Section 13.2 the systems can in fact be written—as in the above hierarchies—under a canonical Hamiltonian formulation of two evolution equations coupled with a system of differential equations of order two. Again each choice of the vertical distribution, $\{\Psi_i\}_{i\in\{1,\ldots,N\}}$, yields a different model. We only quickly mention the "multilayer" systems and instead focus on the shallow water system, eq. (13.8), named the ***Isobe–Kakinuma models*** in reference to [233, 239]. Indeed the latter benefit from a rigorous justification theory, thanks to the work of Iguchi and collaborators, which we report in Section 13.6.

11 The Boussinesq–Rayleigh expansion, and Friedrichs-type systems

In this section we follow a natural—and historical—route to derive high order shallow water models with formally arbitrary precision. The procedure and resulting models have been described in a number of works; see for instance [140, 418, 294] for fairly extensive accounts, and [302, 96] for more recent works.

11.1 Expansion procedure

Let us now describe the procedure which yields asymptotic expansions that we use to derive and justify the asymptotic models discussed in this section. We first describe the velocity potential, Φ—defined from the variables ζ, b and ψ as the solution to eq. (2.8)—through a formal series in powers of μ. Following Boussinesq [60] and Rayleigh [362][53] that is making use of the following relations stemming from eq. (2.8)

$$\mu \int_{-1+\beta b}^{z} \int_{-1+\beta b}^{z'} \Delta_{\boldsymbol{x}} \Phi(\cdot, z'')\, \mathrm{d}z''\, \mathrm{d}z' + \Phi(\cdot, z) - \Phi\,\big|_{z=-1+\beta b} - (z+1-\beta b)\big(\partial_z \Phi\big)\,\big|_{z=-1+\beta b} = 0$$

and the relations from chain rule and impermeability of the bottom

$$\begin{aligned} \nabla\big(\Phi\,\big|_{z=-1+\beta b}\big) &= \big(\nabla_{\boldsymbol{x}}\Phi\big)\,\big|_{z=-1+\beta b} + (\beta\nabla b)\big(\partial_z\Phi\big)\,\big|_{z=-1+\beta b}\,, \\ \big(\partial_z\Phi\big)\,\big|_{z=-1+\beta b} &= \mu(\beta\nabla b)\cdot\big(\nabla_{\boldsymbol{x}}\Phi\big)\,\big|_{z=-1+\beta b}\,, \end{aligned}$$

we find

$$\Phi(\boldsymbol{x}, z, t) = \sum_{n \geq 0} (z+1-\beta b)^n \phi_n(\boldsymbol{x}, t) \tag{11.1}$$

where $\phi_0 \stackrel{\text{def}}{=} \Phi\,\big|_{z=-1+\beta b}$ is the trace of the velocity potential at the bottom,

$$\phi_1 = \big(\partial_z\Phi\big)\,\big|_{z=-1+\beta b} = \mu\frac{(\beta\nabla b)\cdot(\nabla\phi_0)}{1+\mu\,|\beta\nabla b|^2}$$

and ϕ_n $(n \geq 2)$ is given by the recursion relation

$$\phi_{n+2} = -\mu\frac{\Delta\phi_n - 2(n+1)(\beta\nabla b)\cdot(\nabla\phi_{n+1}) - (n+1)(\beta\Delta b)\phi_{n+1}}{(n+1)(n+2)(1+\mu\,|\beta\nabla b|^2)}. \tag{11.2}$$

Remark 11.1. *The series in eq. (11.1) and subsequent ones are well-defined (let alone converge) only for a very restricted class of data—basically analytic functions—since more and more derivatives are involved in each successive summand. The formula in this section should be considered as formal series which, when truncated, offer approximate formula and eventually asymptotic models in a sense which is made precise in Section 11.5.*

[53] Lagrange [265] used a similar expansion, yet using the rest state instead of the bottom as the reference depth. The works of Boussinesq and Rayleigh is restricted to the flat bottom case; Mei and Le Méhauté [303] for instance extended the expansion to general topographies. The expansion produces equivalent results to the one we would obtain by extending the procedure developed in Section 4—based on Lemma 4.7—but has the advantage of being somewhat more explicit. Another important pioneering work is the one by Friedrichs which appears in the appendix of [392]. Friedrichs introduces the full Euler equations for homogeneous and potential flows with rescaled (dimensionless) variables, clearly expresses the role of the shallow water dimensionless parameter, μ—in contrast with *ad hoc* hypotheses such as the hydrostatic assumption—and sketches a procedure to derive models of arbitrary high order with respect to the shallow-water parameter, of which the Saint–Venant system is the first iterate. Yet Friedrichs does not pursue with the derivation of higher order models, and one could—depending on the choice of the velocity variable to describe higher order terms—obtain different models, including the two families of systems we describe in Section 11.2. Interestingly, Friedrichs also comments on the convergence of his expansion procedure; see footnote 27 therein. We shall use the terminology of Friedrichs-type models in place of the also-used Boussinesq models so as to avoid confusion with the first order long wave model described in Section iv.

From eq. (11.1) we infer

$$\psi = \Phi\big|_{z=\varepsilon\zeta} = \sum_{n\geq 0} h^n \phi_n, \tag{11.3}$$

$$\overline{\boldsymbol{u}} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi \,\mathrm{d}z = \sum_{n\geq 0} \frac{h^n}{n+1}\nabla\phi_n - (\beta\nabla b)\sum_{n\geq 1} h^{n-1}\phi_n, \tag{11.4}$$

$$\underline{\boldsymbol{u}} = \big(\nabla_{\boldsymbol{x}}\Phi\big)\big|_{z=\varepsilon\zeta} = \sum_{n\geq 0} h^n\nabla\phi_n - (\beta\nabla b)\sum_{n\geq 1} n h^{n-1}\phi_n, \tag{11.5}$$

$$\underline{w} = \frac{1}{\mu}\big(\partial_z\Phi\big)\big|_{z=\varepsilon\zeta} = \frac{1}{\mu}\sum_{n\geq 1} n h^{n-1}\phi_n, \tag{11.6}$$

where we use the notation $h = 1+\epsilon\zeta-\beta b$. Now we wish to express ϕ_0 in terms of the variables ζ, b and ψ. Once again, this can be done (formally) up to an arbitrarily high order. Expanding each ϕ_n as powers of μ and reorganizing expressions,[54] we rewrite eq. (11.3) as

$$\nabla\psi = \nabla\phi_0 + \sum_{n\geq 1}\mu^n \mathcal{V}_n[\epsilon\zeta,\beta b](\nabla\phi_0)$$

where $\mathcal{V}_n$ is a differential operator of order $2n$ acting on $(\varepsilon\zeta,\beta b,\phi_0)$ (and linear on the last variable), independent of μ. It follows

$$\nabla\phi_0 = \nabla\psi + \sum_{n\geq 1}\mu^n \mathcal{B}_n[\epsilon\zeta,\beta b](\nabla\psi) \tag{11.7}$$

where $\mathcal{B}_n$ $(n\geq 2)$ is the differential operator of order $2n$ given by the recursion relation

$$\mathcal{B}_n[\varepsilon\zeta,\beta b] \stackrel{\text{def}}{=} -\mathcal{V}_n[\varepsilon\zeta,\beta b] - \sum_{k=1}^{n-1}\mathcal{V}_k[\varepsilon\zeta,\beta b]\circ\mathcal{B}_{n-k}[\varepsilon\zeta,\beta b].$$

Plugging expansion (11.7) in (11.4),(11.5),(11.6) provides expansions of $\overline{\boldsymbol{u}},\underline{\boldsymbol{u}},\underline{w}$ in terms of $\nabla\psi$:

$$\overline{\boldsymbol{u}} = \sum_{n\geq 0}\mu^n\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi) \tag{11.8}$$

$$\underline{\boldsymbol{u}} = \sum_{n\geq 0}\mu^n\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi), \tag{11.9}$$

$$\underline{w} = \sum_{n\geq 0}\mu^n\mathcal{W}_n[\epsilon\zeta,\beta b](\nabla\psi), \tag{11.10}$$

where $\overline{\mathcal{U}}_n,\mathcal{U}_n$ are differential operators of order $2n$ and $\mathcal{W}_n$ is a differential operator of order $2n+1$.

In a similar fashion, [55] since $\overline{\mathcal{U}}_0[\epsilon\zeta,\beta b](\nabla\psi)=\nabla\psi$, we can use (11.8) to express

$$\nabla\psi = \overline{\boldsymbol{u}} + \sum_{n\geq 1}\mu^n\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}}) \tag{11.11}$$

[54]Here we remark also that ϕ_n for $n\geq 1$ depends only on $\beta\nabla b$ and $\nabla\phi_0$. Similarly, the dependency in $(\varepsilon\zeta,\beta b)$ of the differential operators below could be replaced by $(h,\beta\nabla b)$.

[55]We could also, equivalently, use eq. (11.4) to infer

$$\nabla\phi_0 = \overline{\boldsymbol{u}} + \sum_{n\geq 1}\mu^n\widetilde{\mathcal{B}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})$$

and infer eq. (11.11)–(11.12)–(11.13) from the above and eq. (11.3)–(11.5)–(11.6).

where $\widetilde{\mathcal{V}}_n$ $(n \geq 1)$ is the differential operator of order $2n$ given by the recursion relation

$$\widetilde{\mathcal{V}}_n[\varepsilon\zeta, \beta b] \stackrel{\text{def}}{=} -\overline{\mathcal{U}}_n[\varepsilon\zeta, \beta b] - \sum_{k=1}^{n-1} \overline{\mathcal{U}}_k[\varepsilon\zeta, \beta b] \circ \widetilde{\mathcal{V}}_{n-k}[\varepsilon\zeta, \beta b].$$

We infer immediately from the above and eq. (11.9)–(11.10)

$$\underline{\boldsymbol{u}} = \sum_{n \geq 0} \mu^n \widetilde{\mathcal{U}}_n[\epsilon\zeta, \beta b](\overline{\boldsymbol{u}}), \tag{11.12}$$

$$\underline{w} = \sum_{n \geq 0} \mu^n \widetilde{\mathcal{W}}_n[\epsilon\zeta, \beta b](\overline{\boldsymbol{u}}), \tag{11.13}$$

where $\widetilde{\mathcal{U}}_n$ is a differential operators of order $2n$ and $\widetilde{\mathcal{W}}_n$ is a differential operator of order $2n+1$.

First order expansions The above procedure can easily performed by computer algebra systems such as SageMath. Let us for the record provide the first order expansions provided by the above procedure, in the flat bottom case ($\beta b = 0$).

$$\begin{aligned}
\nabla\psi &= \nabla\phi_0 - \tfrac{\mu}{2}\nabla\big(h^2(\nabla\cdot\nabla)\phi_0\big) + \tfrac{\mu^2}{24}\nabla\big(h^4(\nabla\cdot\nabla)^2\phi_0\big) + \mathcal{O}(\mu^3) \\
\overline{\boldsymbol{u}} &= 1 - \tfrac{\mu}{6}h^2\nabla(\nabla\cdot\nabla)\phi_0 + \tfrac{\mu^2}{120}h^4\nabla(\nabla\cdot\nabla)^2\phi_0 + \mathcal{O}(\mu^3) \\
\boldsymbol{u} &= \nabla\phi_0 - \tfrac{\mu}{2}h^2\nabla(\nabla\cdot\nabla)\phi_0 + \tfrac{\mu^2}{24}h^4\nabla(\nabla\cdot\nabla)^2\phi_0 + \mathcal{O}(\mu^3) \\
w &= h(\nabla\cdot\nabla)\phi_0 - \tfrac{\mu}{6}h^3(\nabla\cdot\nabla)^2\phi_0 + \tfrac{\mu^2}{120}h^5(\nabla\cdot\nabla)^2\phi_0 + \mathcal{O}(\mu^3)
\end{aligned}$$

and in turn

$$\begin{aligned}
\overline{\boldsymbol{u}} = \nabla\psi &+ \tfrac{\mu}{2}\nabla\big(h^2(\nabla\cdot\nabla)\psi\big) - \tfrac{\mu}{6}h^2\nabla(\nabla\cdot\nabla)\psi \\
&- \tfrac{\mu^2}{24}\nabla\big(h^4(\nabla\cdot\nabla)^2\psi\big) + \tfrac{\mu^2}{4}\nabla\big(h^2(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big)\big) \\
&- \tfrac{\mu^2}{12}h^2\nabla(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big) + \tfrac{\mu^2}{120}h^4\nabla(\nabla\cdot\nabla)^2\psi + \mathcal{O}(\mu^3) \\
\boldsymbol{u} = \nabla\psi &- \tfrac{\mu}{2}h^2\nabla(\nabla\cdot\nabla)\psi + \tfrac{\mu}{2}\nabla\big(h^2(\nabla\cdot\nabla)\psi\big) \\
&+ \tfrac{\mu^2}{24}h^4\nabla\big((\nabla\cdot\nabla)^2\psi\big) - \tfrac{\mu^2}{4}h^2\nabla\big((\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big)\big) \\
&- \tfrac{\mu^2}{24}\nabla\big(h^4(\nabla\cdot\nabla)^2\psi\big) + \tfrac{\mu^2}{4}\nabla\big(h^2(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big)\big) + \mathcal{O}(\mu^3) \\
w = h(\nabla\cdot\nabla)\psi &+ \tfrac{\mu}{2}h(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big) - \tfrac{\mu}{6}h^3(\nabla\cdot\nabla)^2\psi \\
&- \tfrac{\mu^2}{24}h(\nabla\cdot\nabla)\big(h^4(\nabla\cdot\nabla)^2\psi\big) + \tfrac{\mu^2}{4}h(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\big(h^2(\nabla\cdot\nabla)\psi\big)\big) \\
&- \tfrac{\mu^2}{12}h^3(\nabla\cdot\nabla)^2\big(h^2(\nabla\cdot\nabla)\psi\big) + \tfrac{\mu^2}{120}h^5(\nabla\cdot\nabla)^3\psi + \mathcal{O}(\mu^3),
\end{aligned}$$

and

$$\begin{aligned}
\nabla\psi = \overline{\boldsymbol{u}} &+ \tfrac{\mu}{6}h^2\nabla\nabla\cdot\overline{\boldsymbol{u}} - \tfrac{\mu}{2}\nabla(h^2\nabla\cdot\overline{\boldsymbol{u}}) \\
&- \tfrac{\mu^2}{120}h^4\nabla(\nabla\cdot\nabla)\nabla\cdot\overline{\boldsymbol{u}} + \tfrac{\mu^2}{36}h^2\nabla\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big) \\
&- \tfrac{\mu^2}{12}\nabla\big(h^2\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big)\big) + \tfrac{\mu^2}{24}\nabla\big(h^4(\nabla\cdot\nabla)\nabla\cdot\overline{\boldsymbol{u}}\big) + \mathcal{O}(\mu^3), \\
\underline{\boldsymbol{u}} = \overline{\boldsymbol{u}} &- \tfrac{\mu}{3}h^2\nabla\nabla\cdot\overline{\boldsymbol{u}} + \tfrac{\mu^2}{30}h^4\nabla(\nabla\cdot\nabla)\nabla\cdot\overline{\boldsymbol{u}} - \tfrac{\mu^2}{18}h^2\nabla\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big) + \mathcal{O}(\mu^3), \\
w = h\nabla\cdot\overline{\boldsymbol{u}} &+ \frac{\mu}{6}h\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big) - \tfrac{\mu}{6}h^3(\nabla\cdot\nabla)\nabla\cdot\overline{\boldsymbol{u}} \\
&- \tfrac{\mu^2}{120}h\nabla\cdot\big(h^4\nabla(\nabla\cdot\nabla)\nabla\cdot\overline{\boldsymbol{u}}\big) + \tfrac{\mu^2}{36}h\nabla\cdot\big(h^2\nabla\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big)\big) \\
&- \tfrac{\mu}{36}h^3(\nabla\cdot\nabla)\nabla\cdot\big(h^2\nabla\nabla\cdot\overline{\boldsymbol{u}}\big) + \tfrac{\mu^2}{120}h^5(\nabla\cdot\nabla)^2\nabla\cdot\overline{\boldsymbol{u}} + \mathcal{O}(\mu^3).
\end{aligned}$$

Of course the right-hand sides can be written in a more compact way. In particular, neglecting $\mathcal{O}(\mu^2)$ contributions in the approximation of $\nabla\psi$ in terms of $\overline{\boldsymbol{u}}$, we recognize

$$\nabla\psi = \overline{\boldsymbol{u}} - \tfrac{\mu}{3h}\nabla(h^3\nabla\cdot\overline{\boldsymbol{u}}) + \mathcal{O}(\mu^2)$$

which is nothing but eq. (8.5) in the flat bottom situation. In the same way, using the relation (see Lemma 4.6)

$$\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = -\mu\nabla\cdot(h\overline{\boldsymbol{u}}), \tag{11.14}$$

we recognize

$$\begin{aligned}\overline{\boldsymbol{u}} &= \nabla\psi + \tfrac{\mu}{3h}\nabla\big(h^3\nabla\cdot\nabla\psi\big) + \mathcal{O}(\mu^2)\\ \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi &= -\nabla\cdot\big(h\nabla\psi\big) - \tfrac{\mu}{3}(\nabla\cdot\nabla)\big(h^3\nabla\cdot\nabla\psi\big) + \mathcal{O}(\mu^2)\end{aligned}$$

that is the first order approximations (again, in the flat bottom situation) stated in Proposition 4.9 and Proposition 4.10. As we mentioned before, the procedure used in their proof could be extended to obtain the above expansions at any order.

11.2 Two high order Friedrichs-type models

We can plug the formal series, eq. (11.8)-(11.9)-(11.10) into the water waves equations, eq. (2.7'). Denoting

$$(\overline{\boldsymbol{u}}_n,\underline{\boldsymbol{u}}_n,\underline{w}_n) \stackrel{\text{def}}{=} (\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi),\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi),\mathcal{W}_n[\epsilon\zeta,\beta b](\nabla\psi)),$$

using the relation (11.14) and truncating terms of order $\mathcal{O}(\mu^{N+1})$ yields

$$\begin{cases}\partial_t\zeta + \displaystyle\sum_{n=0}^{N}\mu^n\nabla\cdot\big(h\overline{\boldsymbol{u}}_n\big) = 0,\\ \partial_t\psi + \zeta + \epsilon\displaystyle\sum_{n=0}^{N}\mu^n\left(\underline{\boldsymbol{u}}_n\cdot\nabla\psi - \frac{1}{2}\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right) = 0.\end{cases} \tag{11.15}$$

Using physical variables (recall Section 2.4), we get

$$\begin{cases}\partial_t\zeta + \displaystyle\sum_{n=0}^{N}d^{2n}\nabla\cdot\big(h\overline{\boldsymbol{u}}_n\big) = 0,\\ \partial_t\psi + g\zeta + \displaystyle\sum_{n=0}^{N}d^{2n}\left(\underline{\boldsymbol{u}}_n\cdot\nabla\psi - \frac{1}{2}\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right) = 0,\end{cases} \tag{11.16}$$

where $(\overline{\boldsymbol{u}}_n,\underline{\boldsymbol{u}}_n,\underline{w}_n) \stackrel{\text{def}}{=} (\overline{\mathcal{U}}_n[d^{-1}\zeta,d^{-1}b](\nabla\psi),\mathcal{U}_n[d^{-1}\zeta,d^{-1}b](\nabla\psi),\mathcal{W}_n[d^{-1}\zeta,d^{-1}b](\nabla\psi))$. We shall refer to these systems as ***high order shallow water systems***.

Remark 11.2. *It is important to remark that we truncate the formal series after they have been plugged in the equations (2.7'). Alternatively, we could plug the truncated expansions of $\overline{\boldsymbol{u}}$, $\underline{\boldsymbol{u}}$ and $\underline{w}$ in eq. (2.7'). This would yield another family of systems, equivalent to eq. (11.15) in the sense of consistency, but which do not enjoy the Hamiltonian structure described in Section 11.3.*

Alternatively, we can use the formal series eq. (11.11)-(11.12)-(11.13) into eq. (2.7'). Denoting

$$(\boldsymbol{v}_n,\underline{\boldsymbol{u}}_n,\underline{w}_n) \stackrel{\text{def}}{=} (\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}}),\widetilde{\mathcal{U}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}}),\widetilde{\mathcal{W}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})),$$

and truncating terms of order $\mathcal{O}(\mu^{N+1})$ yields

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(h\underline{\boldsymbol{u}}\big) = 0, \\ \partial_t\left(\sum_{n=0}^{N}\mu^n \boldsymbol{v}_n\right) + \nabla\zeta + \epsilon\sum_{n=0}^{N}\mu^n\nabla\left(\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\boldsymbol{v}_{n-k} - \frac{1}{2}\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right) = \mathbf{0}. \end{cases} \tag{11.17}$$

Using physical variables, we get

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(h\underline{\boldsymbol{u}}\big) = 0, \\ \partial_t\left(d^{2n}\sum_{n=0}^{N}\boldsymbol{v}_n\right) + g\nabla\zeta + \sum_{n=0}^{N}d^{2n}\nabla\left(\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\boldsymbol{v}_{n-k} - \frac{1}{2}\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right) = \mathbf{0}, \end{cases} \tag{11.18}$$

where $(\boldsymbol{v}_n, \underline{\boldsymbol{u}}_n, \underline{w}_n) \stackrel{\text{def}}{=} (\widetilde{\mathcal{V}}_n[d^{-1}\zeta, d^{-1}b](\overline{\boldsymbol{u}}), \widetilde{\mathcal{U}}_n[d^{-1}\zeta, d^{-1}b](\overline{\boldsymbol{u}}), \widetilde{\mathcal{W}}_n[d^{-1}\zeta, d^{-1}b](\overline{\boldsymbol{u}}))$. Following the terminology in [301, 302], we will refer to these systems as the ***extended Green–Naghdi systems***.

Remark 11.3. *Setting $N = 0$ in eq. (11.15), we find the Saint-Venant system under formulation eq. (5.2). Setting $N = 1$ yields the system suffering from strong high frequency modal instabilities discussed in footnote 38. When $N = 2$ and in the flat bottom situation, we obtain [96, (60)–(61)].*

Setting $N = 0$ in eq. (11.17), we find the Saint-Venant system under formulation eq. (5.3). Setting $N = 1$ yields the Green–Naghdi system, eq. (8.6), or equivalently eq. (8.4) in footnote 39, provided we set $\nabla\psi$ from $(\varepsilon\zeta, \beta b, \overline{\boldsymbol{u}})$ according to eq. (8.5), that is $\nabla\psi = \overline{\boldsymbol{u}} + \mu h\mathcal{T}[h, \beta\nabla b]\overline{\boldsymbol{u}}$. When $N = 2$, the system suffers from strong high frequency modal instabilities; see Section 11.4. It is displayed explicitly, in the flat bottom situation, in [302, (2.5)–(3.3)]. The system for $N = 3$ is displayed explicitly, in the flat bottom one-dimensional situation, in [302, (2.5)–(3.21)].

Remark 11.4. *It is clear that we can produce different models by extending the procedure in Section 11.1 so as to produce expansions in terms of different velocity variables, such as $\nabla\phi_0$ or (following Nwogu [345]) the horizontal velocity evaluated at a given height. All these models will be different, and in particular will produce different dispersion relations.* [56] *Yet they will have the same nature, in that they involve differential operators of increasing order as the rank of the model in the family, N, grows. Additionally, manipulations equivalent to the "BBM trick" can be useful, and a long wave assumption of the form $\varepsilon + \beta = \mathcal{O}(\mu)$ is often employed. Given the such degrees of freedom, it is not a surprise that the literature on the subject is vast and cluttered. The interested reader can refer to [347, 140, 295, 256, 418, 51] and particularly [294] for a thorough account and extensive bibliographic references.*

11.3 Hamiltonian structure

The two models, eq. (11.15) and eq. (11.17), enjoy a Hamiltonian structure which is inherited from that of the water waves system; see Section 2.2.

The high order shallow water systems Let us define, based on Section 2.2 and Lemma 4.6,

$$\mathscr{H}_{\rm ho}^{N}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 + (1+\varepsilon\zeta-\beta b)(\nabla\psi)\cdot\left(\sum_{n=0}^{N}\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi)\right)\mathrm{d}\boldsymbol{x}.$$

[56] After the completion of this Section, the model using the horizontal velocity at the (flat) bottom has been derived and studied by Choi in [97]. This model has quite interesting properties compared with the ones presented in this Section, eq. (11.15) and eq. (11.17), in particular concerning its linear dispersion relation (see Section 11.4). Indeed, the model does not suffer from any modal instability and, moreover, its linear dispersion relation converges towards the one of the water waves system as the rank of the model, N, grows.

Then eq. (11.15) reads

$$\begin{cases} \partial_t \zeta - \delta_\psi \mathscr{H}_{\rm ho}^N = 0, \\ \partial_t \psi + \delta_\zeta \mathscr{H}_{\rm ho}^N = 0. \end{cases}$$

In order to check this claim, we rely on the following properties satisfied, for smooth and rapidly decaying data, by $(\overline{\boldsymbol{u}}_n, \underline{\boldsymbol{u}}_n, \underline{w}_n) \stackrel{\rm def}{=} (\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi), \underline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi), \underline{\mathcal{W}}_n[\epsilon\zeta,\beta b](\nabla\psi))$. We have

i. $\underline{\boldsymbol{u}}_0 = \nabla\psi$, $\underline{\boldsymbol{u}}_n = -\epsilon \underline{w}_{n-1}\nabla\zeta$ $(n\geq 1)$, and $\underline{w}_n - (\epsilon\nabla\zeta)\cdot\underline{\boldsymbol{u}}_n = -\nabla\cdot\big((1+\varepsilon\zeta-\beta b)\overline{\boldsymbol{u}}_n\big)$ $(n\geq 0)$;

ii. for any $n\geq 0$ and any smooth and rapidly decaying ψ_1, ψ_2,

$$\int_{\mathbb{R}^d} (1+\varepsilon\zeta-\beta b)(\nabla\psi_1)\cdot\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi_2)\,{\rm d}\boldsymbol{x} = \int_{\mathbb{R}^d} (1+\varepsilon\zeta-\beta b)(\nabla\psi_2)\cdot\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi_1)\,{\rm d}\boldsymbol{x};$$

iii. denoting $\mathcal{G}_{\rm ho}^N[\epsilon\zeta,\beta b]\psi \stackrel{\rm def}{=} -\nabla\cdot\big((1+\varepsilon\zeta-\beta b)\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi)\big)$, we have for any $n\geq 0$

$${\rm d}_\zeta\, \mathcal{G}_{\rm ho}^n[\epsilon\zeta,\beta b](\delta\zeta)\psi = -\sum_{k=0}^{n-1} \mathcal{G}_{\rm ho}^k[\epsilon\zeta,\beta b]\,\big((\delta\zeta)\underline{w}_{n-1-k}\big) - \nabla\cdot\left((\delta\zeta)\underline{\boldsymbol{u}}_n\right).$$

These properties are the direct counterparts of the identities satisfied by Φ the solution to the Laplace problem, eq. (2.8), (by chain rule and Proposition 2.3)

i. $\nabla\psi = (\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\varepsilon\zeta} + (\varepsilon\nabla\zeta)(\partial_z\Phi)\big|_{z=\varepsilon\zeta}$, and $\frac{1}{\mu}\mathcal{G}^\mu[\epsilon\zeta,\beta b]\psi = \frac{1}{\mu}(\partial_z\Phi)\big|_{z=\varepsilon\zeta} - (\varepsilon\nabla\zeta)\cdot(\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\varepsilon\zeta}$;

ii. $\int_{\mathbb{R}^d}\psi_1\mathcal{G}^\mu[\epsilon\zeta,\beta b]\psi_2\,{\rm d}\boldsymbol{x} = \int_{\mathbb{R}^d}\psi_2\mathcal{G}^\mu[\epsilon\zeta,\beta b]\psi_1\,{\rm d}\boldsymbol{x}$;

iii. ${\rm d}_\zeta\,\mathcal{G}^\mu[\epsilon\zeta,\beta b](\delta\zeta)\psi = -\mathcal{G}^\mu[\epsilon\zeta,\beta b]\,\big((\delta\zeta)(\partial_z\Phi)\big|_{z=\varepsilon\zeta}\big) - \mu\nabla\cdot\big((\delta\zeta)(\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\varepsilon\zeta}\big)$.

That they hold can be checked directly from the definitions of $\overline{\mathcal{U}}_n, \underline{\mathcal{U}}_n, \underline{\mathcal{W}}_n$; or by reasoning asymptotically from the expansions of the above identities as $\mu\searrow 0$ (making use of the rigorous analysis in Section 11.5). From the above identities we quickly infer, as desired,

$$\delta_\psi\mathscr{H}_{\rm ho}^N = \sum_{n=0}^N \mu^n\mathcal{G}_{\rm ho}^N[\epsilon\zeta,\beta b]\psi = -\sum_{n=0}^N \mu^n\nabla\cdot\big((1+\varepsilon\zeta-\beta b)\overline{\boldsymbol{u}}_n\big)$$

$$\delta_\zeta\mathscr{H}_{\rm ho}^N = \zeta + \frac{\varepsilon}{2}\sum_{n=0}^N \mu^n\left(\underline{\boldsymbol{u}}_n\cdot\nabla\psi + \underline{\boldsymbol{u}}_n\cdot\boldsymbol{u}_0 - \sum_{k=0}^n \underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right).$$

From the Hamiltonian structure we have—by Noether's theorem—a relation between group symmetries of the system and preserved quantities; see Section 2.2. In particular sufficiently regular and spatially localized solution to eq. (11.15) preserve the excess of mass,

$$\frac{\rm d}{{\rm d}t}\mathscr{Z} = 0, \qquad\qquad \mathscr{Z} \stackrel{\rm def}{=} \int_{\mathbb{R}^d}\zeta\,{\rm d}\boldsymbol{x},$$

horizontal impulse in the flat bottom case

$$\frac{\rm d}{{\rm d}t}\mathscr{I} = 0, \qquad\qquad \mathscr{I} \stackrel{\rm def}{=} \int_{\mathbb{R}^d}\zeta\nabla\psi\,{\rm d}\boldsymbol{x} \qquad (\text{if } \beta b\equiv 0),$$

and total energy

$$\frac{\rm d}{{\rm d}t}\mathscr{H}_{\rm ho}^N = 0.$$

We also have, obviously $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}=0$ where $\mathscr{V}=\int_{\mathbb{R}^d}\nabla\psi$ and we expect the preservation of horizontal momentum, in the flat bottom case:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}_{\mathrm{ho}}^N=0, \qquad \mathscr{M}=\int_{\mathbb{R}^d}\sum_{n=0}^N \mu^n h\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi)\,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b\equiv 0)$$

(and in fact that each of the summands, for $n\geq 1$, is a gradient vector field).

The extended Green–Naghdi system In the spirit of Section 8.1.1, we may define

$$\widetilde{\mathscr{H}}_{\mathrm{ho}}^N(\zeta,\psi)\stackrel{\mathrm{def}}{=}\frac12\int_{\mathbb{R}^d}\zeta^2+(1+\varepsilon\zeta-\beta b)(\nabla\psi)\cdot\mathcal{U}^{(N)}[\epsilon\zeta,\beta b](\nabla\psi)\,\mathrm{d}\boldsymbol{x}.$$

where $\mathcal{U}^{(N)}[\epsilon\zeta,\beta b](\nabla\psi)$ is formally defined by the relation

$$\mathcal{U}^{(N)}[\epsilon\zeta,\beta b](\nabla\psi)=\overline{\boldsymbol{u}}^{(N)} \quad \text{where} \quad \nabla\psi=\sum_{n=0}^N\mu^n\boldsymbol{v}_n, \qquad \boldsymbol{v}_n=\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}}^{(N)}),$$

or in other words

$$\Big(\sum_{n=0}^N\mu^n\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b]\Big)\circ\mathcal{U}^{(N)}[\epsilon\zeta,\beta b]=\mathrm{Id},$$

and infer similarly as above that eq. (11.17) reads

$$\begin{cases}\partial_t\zeta-\delta_\psi\widetilde{\mathscr{H}}_{\mathrm{ho}}^N=0,\\ \partial_t\psi+\delta_\zeta\widetilde{\mathscr{H}}_{\mathrm{ho}}^N=0,\end{cases}$$

making use of the identities $\nabla\psi=\sum_{n=0}^N\mu^n\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})$ and $\overline{\boldsymbol{u}}=\mathcal{U}^{(N)}[\epsilon\zeta,\beta b]\nabla\psi$.

Obviously the above is very formal since there is no reason to believe that, apart from very specific cases,[57] $\mathcal{U}^{(N)}$ should be well-defined. Matsuno exhibits in [302] a non-canonical Hamiltonian formulation of eq. (11.17) using the variables $(\zeta,\boldsymbol{m}\stackrel{\mathrm{def}}{=}h\overline{\boldsymbol{u}})$ and as such does not require to define $\mathcal{U}^{(N)}[\epsilon\zeta,\beta b]$. Yet the above formal canonical structure allows to highlight the relationship with the canonical Hamiltonian structure of the water waves system and to easily express preserved quantities satisfied by sufficiently regular and spatially localized solution to eq. (11.17), specifically the excess of mass,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z}=0, \qquad \mathscr{Z}\stackrel{\mathrm{def}}{=}\int_{\mathbb{R}^d}\zeta\,\mathrm{d}\boldsymbol{x},$$

horizontal impulse in the flat bottom case

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{I}}_{\mathrm{ho}}^N=0, \qquad \widetilde{\mathscr{I}}_{\mathrm{ho}}^N\stackrel{\mathrm{def}}{=}\int_{\mathbb{R}^d}\zeta\boldsymbol{v}^{(N)}\,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b\equiv 0),$$

and total energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{H}}_{\mathrm{ho}}^N=0, \qquad \widetilde{\mathscr{H}}_{\mathrm{ho}}^N=\frac12\int_{\mathbb{R}^d}\zeta^2+(1+\varepsilon\zeta-\beta b)\boldsymbol{v}^{(N)}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x},$$

where in the last two definitions we denote $\boldsymbol{v}^{(N)}\stackrel{\mathrm{def}}{=}\sum_{n=0}^N\mu^n\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{V}}_{\mathrm{ho}}^N=0, \qquad \widetilde{\mathscr{V}}_{\mathrm{ho}}^N=\int_{\mathbb{R}^d}\boldsymbol{v}^{(N)}\,\mathrm{d}\boldsymbol{x},$$

[57] among which the case $N=0$ (corresponding to the Saint-Venant system; Section 5) for which $\mathcal{U}^{(0)}[\epsilon\zeta,\beta b]=\mathrm{Id}$ and the case $N=1$ (corresponding to the Green–Naghdi system; Section 8) since $\mathcal{U}^{(1)}[\epsilon\zeta,\beta b]=(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b])^{-1}$.

which can be interpreted as the fact that $\boldsymbol{v}^{(N)}$ approximates to a gradient vector field, $\boldsymbol{v}^{(N)} \approx \nabla\psi$. It appears that for each $n \geq 1$, $h\widetilde{\mathcal{V}}_n[\epsilon\zeta, \beta b](\overline{\boldsymbol{u}})$ is a gradient vector field (the case $n = 1$ corresponds to the contribution $h\mathcal{T}[h, \beta\nabla b]$ in the Green–Naghdi system). Then we would infer in particular the preservation of horizontal momentum, in the flat bottom case:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M} = 0, \qquad \mathscr{M} = \int_{\mathbb{R}^d} h\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x} \qquad (\text{if } \beta b \equiv 0).$$

11.4 Modal analysis

The expansions described in Section 11.1 can be made fairly explicit in the flat bottom and linear setting, namely $\epsilon = \beta = 0$. In this case, we have

$$\phi(\boldsymbol{x}, z, t) = \sum_{n\geq 0} (z+1)^{2n} \frac{(-1)^n \mu^n}{(2n)!} (\nabla \cdot \nabla)^n \phi_0,$$

thus

$$\begin{aligned} \psi &= \sum_{n\geq 0} \frac{(-1)^n \mu^n}{(2n)!} (\nabla \cdot \nabla)^n \phi_0, \\ \overline{\boldsymbol{u}} &= \sum_{n\geq 0} \frac{(-1)^n \mu^n}{(2n+1)!} \nabla(\nabla \cdot \nabla)^n \phi_0, \end{aligned}$$

from which we infer

$$\begin{aligned} \nabla\phi_0 &= b_0 \nabla\psi + \sum_{n\geq 1} \mu^n b_n \nabla(\nabla \cdot \nabla)^n \psi, \\ \nabla\phi_0 &= \widetilde{b}_0 \overline{\boldsymbol{u}} + \sum_{n\geq 1} \mu^n \widetilde{b}_n \nabla(\nabla \cdot \nabla)^{n-1} \nabla \cdot \overline{\boldsymbol{u}}, \end{aligned}$$

where $b_0 = \widetilde{b}_0 = 1$ and $b_n, \widetilde{b}_n$ $(n \geq 1)$ are given by the recursion relation

$$b_n = -\sum_{k=1}^{n} \frac{(-1)^k}{(2k)!} b_{n-k}, \qquad \widetilde{b}_n = -\sum_{k=1}^{n} \frac{(-1)^k}{(2k+1)!} \widetilde{b}_{n-k}. \tag{11.19}$$

This yields

$$\begin{aligned} \overline{\boldsymbol{u}} &= c_0 \nabla\psi + \sum_{n\geq 1} \mu^n c_n \nabla(\nabla \cdot \nabla)^n \psi \\ \nabla\psi &= \widetilde{c}_0 \overline{\boldsymbol{u}} + \sum_{n\geq 1} \mu^n \widetilde{c}_n \nabla(\nabla \cdot \nabla)^{n-1} \nabla \cdot \overline{\boldsymbol{u}} \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} b_{n-k}, \qquad \widetilde{c}_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \widetilde{b}_{n-k}. \tag{11.20}$$

Hence we find that eq. (11.15) when linearized about the trivial solution[58] (that is setting $\varepsilon = \beta = 0$) yields

$$\begin{cases} \partial_t \zeta - \mathcal{G}^{\mu,N} \psi = 0, \\ \partial_t \psi + \zeta = 0, \end{cases} \tag{11.21}$$

[58]By Galilean invariance, linearizing about the constant solution $(\zeta = 0, \psi = \boldsymbol{u} \cdot \boldsymbol{x})$ yields only an additional advection term with velocity $\boldsymbol{u}$.

where $\mathcal{G}^{\mu,N}$ is the Fourier multiplier (see Definition III.1) defined by

$$\mathcal{G}^{\mu,N} \stackrel{\text{def}}{=} |D|^2 \sum_{n=0}^{N} \mu^n (-1)^n c_n |D|^{2n} \psi.$$

In particular we have the following dispersion relation associated with eq. (11.21):

$$\omega_{\text{ho}}^N(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 D_N(\sqrt{\mu}|\boldsymbol{\xi}|), \qquad D_N(\xi) \stackrel{\text{def}}{=} \sum_{n=0}^{N} (-1)^n c_n \xi^{2n}. \tag{11.22}$$

Similarly, eq. (11.17) when linearized about the trivial solution yields

$$\begin{cases} \partial_t \zeta + \nabla \cdot \overline{\boldsymbol{u}} = 0, \\ \partial_t \widetilde{\mathcal{V}}^{\mu,N} \overline{\boldsymbol{u}} + \nabla \zeta = 0, \end{cases} \tag{11.23}$$

where $\widetilde{\mathcal{V}}^{\mu,N}$ is the Fourier multiplier defined by

$$\widetilde{\mathcal{V}}^{\mu,N} \stackrel{\text{def}}{=} \mathrm{Id} + \sum_{n=1}^{N} \mu^n (-1)^n \widetilde{c}_n D |D|^{2(n-1)} D \cdot .$$

In particular we have the dispersion relation associated with eq. (11.23) in dimension $d = 2$:

$$\widetilde{\omega}_{\text{ho}}^N(\boldsymbol{\xi}) \left(\widetilde{\omega}_{\text{ho}}^N(\boldsymbol{\xi})^2 - \frac{|\boldsymbol{\xi}|^2}{\widetilde{D}_N(\sqrt{\mu}|\boldsymbol{\xi}|)} \right) = 0, \qquad \widetilde{D}_N(\xi) \stackrel{\text{def}}{=} \sum_{n=0}^{N} (-1)^n \widetilde{c}_n \xi^{2n}. \tag{11.24}$$

The solution $\widetilde{\omega}_{\text{ho}}^N(\boldsymbol{\xi}) = 0$ is a spurious mode stemming (when $d = 2$) from the fact that the second equation in eq. (11.17) describes an evolution through potential forces. We withdraw it from future discussions.

These dispersion relations should be compared with the one of the water waves system (when linearized about the rest state), namely

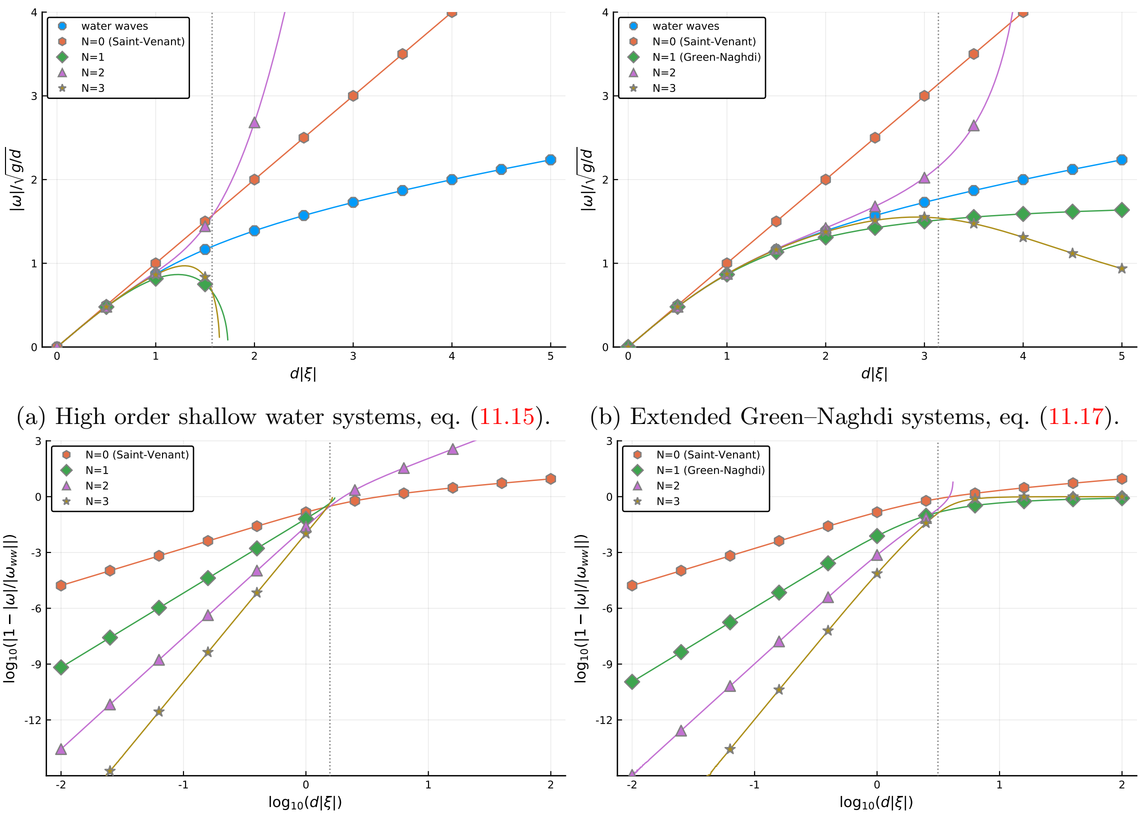
$$\omega_{\text{ww}}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{\tanh(\sqrt{\mu}|\boldsymbol{\xi}|)}{\sqrt{\mu}|\boldsymbol{\xi}|}. \tag{11.25}$$

Solutions to the above dispersion relations—eq. (11.22), (11.24) and (11.25)—are plotted in Figure 11.1. We explore in the following some properties of $D_N(\xi)$ and $\widetilde{D}_N(\xi)$ which prove and explain the behavior of the dispersion relations which can be witnessed in these figures.[59]

Lemma 11.5. *For any $N \in \mathbb{N}$, $D_N(\xi)$ is the Taylor expansion at order $2N$ of $\xi \mapsto \frac{\tanh(\xi)}{\xi}$, and $\widetilde{D}_N(\xi)$ is the Taylor expansion at order $2N$ of $\xi \mapsto \frac{\xi}{\tanh(\xi)}$ For instance,*

$$D_0(\xi) = 1 \quad ; \quad D_1(\xi) = 1 - \frac{1}{3}\xi^2 \quad ; \quad D_2(\xi) = 1 - \frac{1}{3}\xi^2 + \frac{2}{15}\xi^4 \quad \text{etc.}$$

[59]The are several important consequences to Proposition 11.6 concerning properties of eq. (11.15) (resp. eq. (11.17)) when linearized about the rest state. From item *i.* and eq. (11.22) (resp. eq. (11.24)) we see that the dispersion relation fits the one of the water-waves system, eq. (11.25), at order $\mathcal{O}(\mu^{N+1})$, *for sufficiently small wavenumbers.* From item *ii.* we see that for larger wavenumbers, the angular frequency predicted by the models fail to converge towards the one of the water waves system as N goes to infinity. This failure shows that the limitations in the results at the nonlinear level presented in Section 11.5 are not technical, but in fact a consequence of the detrimental behavior of high rank systems. From item *iii.* we see that when $N \geq 1$ is odd (resp. $N \geq 2$ even), the linearized system suffers from extremely strong instabilities at large wavenumbers. On the contrary, if $N \geq 0$ is even (resp. $N = 0$ or $N \geq 1$ odd), then the plane waves are stable for all wavenumbers. While it is true that it is possible to improve the behavior of Friedrichs-type models at the linear level by suitable choice of variables and/or ad hoc manipulations, the models that we obtain in that way typically do not enjoy the variational structure exhibited in Section 11.3, and still suffer from all the difficulties at the nonlinear level described in Section 11.5.



(a) High order shallow water systems, eq. (11.15).

(b) Extended Green–Naghdi systems, eq. (11.17).

(c) Same as (a), in log scale.

(d) Same as (b), in log scale.

Figure 11.1: In (a) and (b), non-trivial wave frequencies, $|\omega|(|\boldsymbol{\xi}|)$, given by the (rescaled) dispersion relations (11.22) and (11.24) corresponding to the (linearized about rest) high order Friedrichs-type models. The corresponding wave frequencies of the water waves system is given by eq. (11.25). In (c) and (d), the "error" is represented in log scale.
Wave frequencies with non-zero imaginary parts, corresponding to unstable modes, are not represented. The dotted vertical line delimits the domain of convergence as $N \to \infty$.

$$\widetilde{D}_0(\xi) = 1 \quad ; \quad \widetilde{D}_1(\xi) = 1 + \frac{1}{3}\xi^2 \quad ; \quad \widetilde{D}_2(\xi) = 1 + \frac{1}{3}\xi^2 - \frac{1}{45}\xi^4 \quad \text{etc.}$$

More precisely, we have

$$c_n = \frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!}, \qquad \widetilde{c}_n = \frac{2^{2n}B_{2n}}{(2n)!}$$

where B_n are the Bernoulli numbers: $B_0 = 1$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, etc.; and in particular

$$c_n = (-1)^n \frac{8}{\pi^2}\left(\frac{2}{\pi}\right)^{2n}(1+\mathcal{O}(2^{-2n})), \qquad \widetilde{c}_n = (-1)^{n-1}2\left(\frac{1}{\pi}\right)^{2n}(1+\mathcal{O}(2^{-2n})) \quad (n\to\infty).$$

Proof. In accordance with the Taylor expansion of the function sech at the origin, we find that

$$b_n = \frac{E_{2n}}{(2n)!}$$

where E_n is the Euler number. Indeed, one has $b_0 = E_0 = 1$ and (11.19) yields the recursion formula satisfied by E_{2n} [4, 23.1.7]:

$$(2n)!b_n = -\sum_{r=1}^{n}(-1)^r\left(\frac{(2n)!}{(2r)!}\right)(2n-2r)!b_{n-r}.$$

We deduce from (11.20) that $(-1)^n c_n$ is the $(2n+1)^{\text{th}}$ coefficient of the Taylor expansion of the function $\xi \mapsto \tanh(\xi)$ at the origin. The corresponding results for $\widetilde{D}_N(\xi)$ follows in a similar way, and are proved in [302].

The asymptotics as $n\to\infty$ are direct consequences of the formula $(-1)^n\frac{B_{2n}}{(2n!)} = \frac{2}{(2\pi)^{2n}}\zeta(2n)$ where ζ is the Riemann function. □

Proposition 11.6. *The polynomial D_N, defined in eq. (11.22), is even and satisfies the following.*

i. For any $r<\frac{\pi}{2}$, there exists $C>0$ such that for any $N\geq 0$ and $\xi\in[-r,r]$,

$$\left|D_N(\xi) - \frac{\tanh(\xi)}{\xi}\right| \leq C\big(\frac{2\xi}{\pi}\big)^{2N+2}.$$

ii. For any $|\xi|>\frac{\pi}{2}$, $(-1)^N D_N(\xi) \sim \frac{8}{\pi^2}\big(\frac{2\xi}{\pi}\big)^{2N} \to +\infty$ as $N\to\infty$.

iii. For any $N\geq 1$, $(-1)^N D_N(\xi) \sim (-1)^N c_N \xi^{2N} \to +\infty$ as $\xi\to\infty$.

iv. If $N\geq 0$ is even, then $D_N(\xi) \geq \frac{\tanh(\xi)}{\xi}$.

v. If $N\geq 1$ is odd, then $D_N(\xi) \leq \frac{\tanh(\xi)}{\xi}$.

The polynomial $\widetilde{D}_N$, defined in eq. (11.24), is even and satisfies the following properties.

i. For any $r<\pi$, there exists $C>0$ such that for any $N\geq 0$ and $\xi\in[-r,r]$,

$$\left|\frac{1}{\widetilde{D}_N(\xi)} - \frac{\tanh(\xi)}{\xi}\right| \leq C\big(\frac{\xi}{\pi}\big)^{2N+2}.$$

ii. For any $|\xi|>\pi$, $(-1)^{N-1}\widetilde{D}_N(\xi) \sim 2\big(\frac{\xi}{\pi}\big)^{2N} \to +\infty$ as $N\to\infty$.

iii. For any $N\geq 1$, $(-1)^{N-1}\widetilde{D}_N(\xi) \sim (-1)^{N-1}\widetilde{c}_N\xi^{2N} \to +\infty$ as $\xi\to\infty$.

iv. If $N \geq 0$ is even, then $\widetilde{D}_N(\xi) \leq \frac{\xi}{\tanh(\xi)}$.

v. If $N \geq 1$ is odd, then $\widetilde{D}_N(\xi) \geq \frac{\xi}{\tanh(\xi)}$, and hence $0 < \frac{1}{\widetilde{D}_N(\xi)} \leq \frac{\tanh(\xi)}{\xi}$.

Proof. The first three items are obvious consequences of Lemma 11.5.

For the last items, we will use a maximum principle. We remark that $\xi \tanh(\xi) = \phi_\xi'(0)$ where $\phi_\xi(z) \stackrel{\text{def}}{=} \tanh(\xi)\sinh(\xi z)\psi + \cosh(\xi z)$ is the unique solution to

$$\phi_\xi'' - \xi^2 \phi_\xi = 0 \quad ; \quad \phi_\xi'(-1) = 0 \quad ; \quad \phi_\xi(0) = 1.$$

We now define

$$\phi_{N,\xi}(z) \stackrel{\text{def}}{=} \sum_{n=0}^{N}\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} b_{n-k}(z+1)^{2k}(-1)^n \xi^{2n}.$$

Notice that $\phi_{N,\xi}'(-1) = 0$, $\phi_{N,\xi}(0) = \sum_{n=0}^{N}\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} b_{n-k}(-1)^n\xi^{2n} = 1$ by (11.19), and by (11.20)

$$\phi_{N,\xi}'(0) = \sum_{n=1}^{N}(-1)^n\xi^{2n}\sum_{k=1}^{n}\frac{(-1)^k}{(2k-1)!}b_{n-k} = \sum_{n=0}^{N-1}(-1)^n c_n \xi^{2n+2} = \xi^2 D_{N-1}(\xi).$$

Notice also

$$\phi_{N,\xi}''(z) = \xi^2\phi_{N,\xi}(z) - (-1)^N\xi^{2N+2}\sum_{k=0}^{N}\frac{(-1)^k}{(2k)!}b_{N-k}(z+1)^{2k}.$$

Let us admit for the moment that

$$\forall z \in [0,-1], \qquad \sum_{k=0}^{N}\frac{(-1)^k}{(2k)!}b_{N-k}(z+1)^{2k} \geq 0. \tag{11.26}$$

Summarizing, we have

$$(-1)^N(\phi_\xi - \phi_{N,\xi})'' - \xi^2(\phi_\xi - \phi_{N,\xi}) \geq 0; \quad (-1)^N(\phi_\xi - \phi_{N,\xi})'(-1) = 0 \quad ; \quad (-1)^N(\phi_\xi - \phi_{N,\xi})(0) = 0.$$

Let us assume for instance that N is odd, and, reasoning by contradiction, that $\phi_{N,\xi}'(0) - \phi_\xi'(0) < 0$. Denote $z_\star$ the maximal value $z_\star \in [-1,0]$ for which $\phi_{N,\xi}'(z) - \phi_\xi'(z) \geq 0$. We have $z_\star < 0$ by continuity, and since $\phi_{N,\xi}(0) - \phi_\xi(0) = 0$, one has $\phi_{N,\xi}(z) - \phi_\xi(z) > 0$ for $z \in (z_\star, 0)$. However, since $\phi_{N,\xi}''(z) - \phi_\xi''(z) \geq \xi^2(\phi_{N,\xi}(z) - \phi_\xi(z)) > 0$, we see that $\phi_{N,\xi}'(z_\star) - \phi_\xi'(z_\star) \leq \phi_{N,\xi}'(0) - \phi_\xi'(0)$, which is a contradiction. Hence we proved, when N is odd, that

$$0 \leq \phi_{N,\xi}'(0) - \phi_\xi'(0) = \xi^2 D_{N-1}(\xi) - \xi\tanh(\xi).$$

Of course, the same reasoning yields $\xi^2 D_{N-1}(\xi) \leq \xi\tanh(\xi)$ when N is even.

Let us now show eq. (11.26). Denote

$$P_N(Z) \stackrel{\text{def}}{=} \sum_{k=0}^{N}\frac{(-1)^k}{(2k)!}b_{N-k}Z^{2k}, \qquad Q_N(Z) \stackrel{\text{def}}{=} \sum_{k=0}^{N}\frac{(-1)^k}{(2k+1)!}b_{N-k}Z^{2k+1}$$

so that $P_N' = -Q_{N-1}$, and $Q_N' = P_N$. Using the above as well as the boundary values $P_N(1) = 0$ (by eq. (11.19)) and $Q_N(0) = 0$, it is straightforward to show by induction ($P_0 = 1$) that for any $N \in \mathbb{N}$, P_N is non-increasing, Q_N is non-decreasing, and $P_N(z) \geq 0, Q_N(z) \geq 0$ for any $z \in [0,1]$. This concludes the proof of the first part of Proposition 11.6.

The second part follows by the same techniques, in particular noticing that $\frac{\xi}{\tanh(\xi)} = \phi'_\xi(0)$ where $\widetilde{\phi}_\xi(z) \stackrel{\text{def}}{=} \frac{1}{\tanh(\xi)}\sinh(\xi z)\psi + \cosh(\xi z)$ is the unique solution to

$$\widetilde{\phi}''_\xi - \xi^2\widetilde{\phi}_\xi = 0 \quad ; \quad \widetilde{\phi}_\xi(-1) = 0 \quad ; \quad \widetilde{\phi}_\xi(0) = 1,$$

and introducing

$$\widetilde{\phi}_{N,\xi}(z) \stackrel{\text{def}}{=} \sum_{n=0}^{N}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\widetilde{b}_{n-k}(z+1)^{2k+1}(-1)^n\xi^{2n}.$$

The rest of the proof is left to the reader. □

11.5 Rigorous justification

In this section, we discuss the rigorous justification of the high order models, eq. (11.15) and eq. (11.17), as asymptotic models for the water waves system, eq. (2.7). The key ingredient is a rigorous statement for the formal expansions, through a quantitative estimate of the remainder term, generalizing Proposition 4.9 and Proposition 4.10.

As always, our results hold in the shallow water regime, *i.e.* for parameters in the set

$$\mathfrak{p}_{\rm SW} = \big\{(\mu,\varepsilon,\beta) \ : \ \mu \in (0,\mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1]\big\}.$$

Lemma 11.7. *Let $d \in \mathbb{N}^\star$ and $s_\star > d/2$. Let $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. There exists $C > 0$ such that for any $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm SW}$, and for any $\zeta \in H^{\max(\{s_\star,k+2n+1\})}(\mathbb{R}^d)$ and any $b \in W^{\max(\{s_\star,k+2n+1\}),\infty}(\mathbb{R}^d)$ satisfying*

$$\big|\varepsilon\zeta\big|_{H^{s_\star}} + \big|\beta b\big|_{W^{s_\star,\infty}} \leq M,$$

the operators $\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b] : \mathring{H}^{k+2n+1} \to H^k$, $\mathcal{U}_n[\epsilon\zeta,\beta b] : \mathring{H}^{k+2n+1} \to H^k$, $\mathcal{W}_n[\epsilon\zeta,\beta b] : \mathring{H}^{k+2n+2} \to H^k$ defined through the procedure described in Section 11.1, are well-defined and continuous, and

$$\big|\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi)\big|_{H^k} \leq C\left(\big|\nabla\psi\big|_{H^{k+2n}} + \langle\big(\big|\varepsilon\zeta\big|_{H^{k+2n}} + \big|\beta b\big|_{W^{k+2n,\infty}}\big)\big|\nabla\psi\big|_{H^{s_\star}}\rangle_{k+2n>s_\star}\right),$$
$$\big|\mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi)\big|_{H^k} \leq C\left(\big|\nabla\psi\big|_{H^{k+2n}} + \langle\big(\big|\varepsilon\zeta\big|_{H^{k+2n}} + \big|\beta b\big|_{W^{k+2n,\infty}}\big)\big|\nabla\psi\big|_{H^{s_\star}}\rangle_{k+2n>s_\star}\right),$$
$$\big|\overline{\mathcal{W}}_n[\epsilon\zeta,\beta b](\nabla\psi)\big|_{H^k} \leq C\left(\big|\nabla\psi\big|_{H^{k+2n+1}} + \langle\big(\big|\varepsilon\zeta\big|_{H^{k+2n+1}} + \big|\beta b\big|_{W^{k+2n+1,\infty}}\big)\big|\nabla\psi\big|_{H^{s_\star}}\rangle_{k+2n+1>s_\star}\right).$$

Proof. From the procedure described in Section 11.1, we infer that $\overline{\mathcal{U}}_n$, $\mathcal{U}_n$ and $\mathcal{W}_n$ can be written as the sum of contributions of the dorm

$$P_1\partial^{\alpha_1}\big(P_2\partial^{\alpha_2}\big(\cdots\big(P_k\partial^{\alpha_k}\nabla\psi\big)\cdots\big)\big)$$

where for any $j \in \{1,\dots,k\}$, $P_j = P_j(\varepsilon\zeta,\beta b)$ is a polynomial in its variables and $\alpha_j \in \mathbb{N}^d$ is a multiindex, with

$$\sum_j |\alpha_j| = \begin{cases} 2n \text{ for } \overline{\mathcal{U}}_n, \mathcal{U}_n, \\ 2n+1 \text{ for } \mathcal{W}_n. \end{cases}$$

The result follows from the product estimates in Proposition II.7 and Proposition II.14, the interpolation estimates in Lemma II.3 and Lemma II.13, and Young's inequality. □

Lemma 11.8. *There exists $p \in \mathbb{N}$ such that the following holds. Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $s \geq 0$, $N \in \mathbb{N}$. There exists $C > 0$ such that for any $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm SW}$, any $\zeta \in H^{\max(\{s+4N+p,2+s_\star\})}(\mathbb{R}^d)$ and $b \in W^{\max(\{s+4N+p,2+s_\star\}),\infty}(\mathbb{R}^d)$ satisfying*

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad h(\boldsymbol{x}) \stackrel{\text{def}}{=} 1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0 \tag{11.27}$$

and

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

one has

$$\begin{aligned}\big|\overline{\boldsymbol{u}} - \textstyle\sum_{n=0}^{N} \mu^n \overline{\boldsymbol{u}}_n\big|_{H^s} + \big|\underline{\boldsymbol{u}} - \textstyle\sum_{n=0}^{N} \mu^n \underline{\boldsymbol{u}}_n\big|_{H^s} + \big|\underline{w} - \textstyle\sum_{n=0}^{N} \mu^n \underline{w}_n\big|_{H^s} \\ \leq C\ \mu^{N+1} \Big(\big|\nabla\psi\big|_{H^{s+4N+p}} + (\big|\varepsilon\zeta\big|_{H^{s+4N+p}} + \big|\beta b\big|_{W^{s+4N+p,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big)\end{aligned}$$

with $(\ \overline{\boldsymbol{u}}_n\ ,\ \underline{\boldsymbol{u}}_n\ ,\ \underline{w}_n\ ) \stackrel{\rm def}{=} (\ \overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi)\ ,\ \mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi)\ ,\ \mathcal{W}_n[\epsilon\zeta,\beta b](\nabla\psi)\ )$ *and*

$$(\ \overline{\boldsymbol{u}}\ ,\ \underline{\boldsymbol{u}}\ ,\ \underline{w}\ ) \stackrel{\rm def}{=} \left(\ \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi(\cdot,z)\,{\rm d}z\ ,\ (\nabla_{\boldsymbol{x}}\Phi)(\cdot,\varepsilon\zeta)\ ,\ \frac{1}{\mu}(\partial_z\Phi)(\cdot,\varepsilon\zeta)\ \right)$$

where Φ is the strong solution to the Laplace problem eq. (2.8) (see Proposition 4.5).

Proof. We prove the result for $s = k \in \mathbb{N}$, the general case being an obvious consequence. Let us introduce, for $N \in \mathbb{N}$,

$$\Phi_{{\rm app},N}(\boldsymbol{x},z) = \sum_{n=0}^{2N+2} (z+1-\beta b)^n \phi_n(\boldsymbol{x}),$$

where

$$\phi_0 = \psi + \sum_{n=1}^{N} \mu^n \mathcal{B}_n[\epsilon\zeta,\beta b](\nabla\psi) \quad ; \quad \phi_1 = \mu\frac{(\beta\nabla b)\cdot(\nabla\phi_0)}{1+\mu\,|\beta\nabla b|^2},$$

with $\mathcal{B}_n[\epsilon\zeta,\beta b]$ defined in Section 11.1; and ϕ_n $(n \geq 2)$ is given by the recursion relation

$$\phi_{n+2} = -\mu\frac{\Delta\phi_n - 2(n+1)(\beta\nabla b)\cdot(\nabla\phi_{n+1}) - (n+1)(\beta\Delta b)\phi_{n+1}}{(n+1)(n+2)(1+\mu\,|\beta\nabla b|^2)}.$$

By proceeding as in Lemma 11.7 we have for any $k \in \mathbb{N}$ and $s_\star > d/2$

$$\big|\nabla\phi_0\big|_{H^k} \leq C\ \Big(\big|\nabla\psi\big|_{H^{k+2N}} + (\big|\varepsilon\zeta\big|_{H^{k+2N}} + \big|\beta b\big|_{W^{k+2N,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big)$$

where, here and thereafter, C depends uniquely on $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}$, $N \in \mathbb{N}$, and changes from line to line. By induction on $n \in \{0,\dots,2N+2\}$, we infer

$$\big|\phi_n\big|_{H^k} \leq \mu^{\lfloor (n+1)/2\rfloor}\ C\ \Big(\big|\nabla\psi\big|_{H^{k+2N+n-1}} + (\big|\varepsilon\zeta\big|_{H^{k+2N+n-1}} + \big|\beta b\big|_{W^{k+2N+n-1,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big).$$

By construction, we find that

$$\mu\Delta_{\boldsymbol{x}}\Phi_{{\rm app},N} + \partial_z^2\Phi_{{\rm app},N} = (z+1-\beta b)^{2N+1} r_{1,N} + (z+1-\beta b)^{2N+2} r_{2,N} \tag{11.28}$$

with

$$\begin{aligned} r_{1,N} &= \mu\Delta\phi_{2N+1} - 2\mu(2N+2)(\beta\nabla b)\cdot(\nabla\phi_{2N+2}) - \mu(2N+2)(\beta\Delta b)\phi_{2N+2},\\ r_{2,N} &= \mu\Delta\phi_{2N+2}.\end{aligned}$$

From the above, we infer

$$\big|r_{1,N}\big|_{H^k} + \big|r_{2,N}\big|_{H^k} \leq \mu^{N+2}\ C\ \Big(\big|\nabla\psi\big|_{H^{k+4N+3}} + (\big|\varepsilon\zeta\big|_{H^{k+4N+3}} + \big|\beta b\big|_{W^{k+4N+3,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big).$$

Moreover, we have

$$(\partial_z\Phi_{{\rm app},N} - \mu(\beta\nabla b)\cdot\nabla_{\boldsymbol{x}}\Phi_{{\rm app},N})\big|_{z=-1+\beta b} = \phi_1 - \mu\beta\nabla b\cdot(\nabla\phi_0 - (\beta\nabla b)\phi_1) = 0. \tag{11.29}$$

Finally, denoting $\mathcal{V}_n[\epsilon\zeta,\beta b]$ as in Section 11.1 and proceeding as in Lemma 11.7 we have,

$$\Phi_{\mathrm{app},N}\,\big|_{z=\epsilon\zeta} = \sum_{n=0}^{2N+1}(1+\varepsilon\zeta-\beta b)^n\phi_n = \phi_0 + \sum_{n=1}^{N}\mu^n\mathcal{V}_n[\epsilon\zeta,\beta b](\nabla\phi_0) + r_N$$

with

$$\big|r_N\big|_{H^k} \le \mu^{N+1}\, C\,\Big(\big|\nabla\psi\big|_{H^{k+4N}} + (\big|\varepsilon\zeta\big|_{H^{4N}} + \big|\beta b\big|_{W^{4N,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big),$$

and therefore, using the identity $\phi_0 = \psi + \sum_{n=1}^{2N}\mu^n\mathcal{B}_n[\epsilon\zeta,\beta b](\nabla\psi)$ and the recursion formula

$$\mathcal{B}_n[\varepsilon\zeta,\beta b] \stackrel{\mathrm{def}}{=} -\mathcal{V}_n[\varepsilon\zeta,\beta b] - \sum_{k=1}^{n-1}\mathcal{V}_k[\varepsilon\zeta,\beta b]\circ\mathcal{B}_{n-k}[\varepsilon\zeta,\beta b],$$

we find

$$\Phi_{\mathrm{app},N}\,\big|_{z=\epsilon\zeta} = \psi + \sum_{n=N+1}^{2N}\sum_{k=n-N}^{N}\mu^n\mathcal{V}_n[\epsilon\zeta,\beta b]\circ\mathcal{B}_{n-k}[\epsilon\zeta,\beta b](\nabla\psi) + r_N.$$

Proceeding once again as in Lemma 11.7, we deduce

$$\big\|\Phi_{\mathrm{app},N}\,\big|_{z=\epsilon\zeta} - \psi\big\|_{H^k} \le \mu^{N+1}\, C\,\Big(\big|\nabla\psi\big|_{H^{k+4N}} + (\big|\varepsilon\zeta\big|_{H^{4N}} + \big|\beta b\big|_{W^{4N,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big). \tag{11.30}$$

We have proved that $\Phi_{\mathrm{app},N}$ satisfies approximately the Laplace problem, eq. (2.8). The problem is slightly different from the one studied in Proposition 4.5 since the remainder terms in the bulk of the fluid domain and on the Neumann (impermeability) condition at the bottom do not correspond; see Definition 4.2. However the proof is straightforwardly adapted to this framework—see [268, Lemma 3.43]—and we find that there exists $p\in\mathbb{N}$ such that

$$\big\|\Lambda^k\nabla^\mu_{\boldsymbol{x},z}(\mathit{\Phi}_{\mathrm{app},N}-\mathit{\Phi})\big\|_{L^2(\mathcal{S})} \le \mu^{N+\frac32}\, C\,\Big(\big|\nabla\psi\big|_{H^{k+4N+p}}+(\big|\varepsilon\zeta\big|_{H^{k+4N+p}}+\big|\beta b\big|_{W^{k+4N+p,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big)$$

where we denote $\nabla^\mu_{\boldsymbol{x},z} \stackrel{\mathrm{def}}{=} (\sqrt{\mu}\nabla,\partial_z)^\top$, $\Lambda \stackrel{\mathrm{def}}{=} (\mathrm{Id}-\Delta_{\boldsymbol{x}})^{1/2}$, $\mathcal{S} \stackrel{\mathrm{def}}{=} \mathbb{R}^d\times(-1,0)$ and $\mathit{\Phi}_{\mathrm{app},N} = \Phi_{\mathrm{app},N}\circ\Sigma$ and $\mathit{\Phi} = \Phi\circ\Sigma$ with $\Sigma : (\boldsymbol{x},z)\in\mathcal{S}\mapsto \big(\boldsymbol{x},(1+\varepsilon\zeta(\boldsymbol{x})-\beta b(\boldsymbol{x}))z+\varepsilon\zeta(\boldsymbol{x})\big)$. It follows that for any $\alpha\in\mathbb{N}^d$ such that $|\alpha|\le k$,

$$\big\|\nabla^\mu_{\boldsymbol{x},z}\partial^\alpha(\Phi_{\mathrm{app},N}-\Phi)\big\|_{L^2(\Omega)} \le \mu^{N+\frac32}\, C\,\Big(\big|\nabla\psi\big|_{H^{k+4N+p}}+(\big|\varepsilon\zeta\big|_{H^{k+4N+p}}+\big|\beta b\big|_{W^{k+4N+p,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big)$$

Denoting

$$\overline{\boldsymbol{u}}_{\mathrm{app},N} \stackrel{\mathrm{def}}{=} \frac1h\int_{-1+\beta b}^{\epsilon\zeta}\nabla_{\boldsymbol{x}}\phi_{\mathrm{app},N}\,\mathrm{d}z,\quad \underline{\boldsymbol{u}}_{\mathrm{app},N} \stackrel{\mathrm{def}}{=} (\nabla_{\boldsymbol{x}}\phi_{\mathrm{app},N})\,\big|_{z=\epsilon\zeta}\,,\quad \underline{w}_{\mathrm{app},N} \stackrel{\mathrm{def}}{=} (\partial_z\phi_{\mathrm{app},N})\,\big|_{z=\epsilon\zeta}\,,$$

we deduce, augmenting p if necessary,

$$\begin{aligned}\big|\overline{\boldsymbol{u}}-\overline{\boldsymbol{u}}_{\mathrm{app},N}\big|_{H^k} + \big|\underline{\boldsymbol{u}}-\underline{\boldsymbol{u}}_{\mathrm{app},n}\big|_{H^k} + \big|\underline{w}-\underline{w}_{\mathrm{app},n}\big|_{H^k}&\\ \le C\,\mu^{N+1}\,\Big(\big|\nabla\psi\big|_{H^{k+4N+p}} + (\big|\varepsilon\zeta\big|_{H^{k+4N+p}} + \big|\beta b\big|_{W^{k+4N+p,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big)&\end{aligned}$$

where we use Cauchy–Schwarz inequality for the first contribution, the trace inequality eq. (4.5) for the second and together with eq. (11.28)–eq. (11.29) for the third.

The last step consists in showing that

$$\begin{aligned}\big|\overline{\boldsymbol{u}}_{\mathrm{app},N} - \textstyle\sum_{n=0}^N\mu^n\overline{\boldsymbol{u}}_n\big|_{H^k} + \big|\underline{\boldsymbol{u}}_{\mathrm{app},n} - \sum_{n=0}^N\mu^n\underline{\boldsymbol{u}}_n\big|_{H^k} + \big|\underline{w}_{\mathrm{app},n} - \sum_{n=0}^N\mu^n\underline{w}_n\big|_{H^k}&\\ \le C\,\mu^{N+1}\,\Big(\big|\nabla\psi\big|_{H^{k+4N+p}} + (\big|\varepsilon\zeta\big|_{H^{k+4N+p}} + \big|\beta b\big|_{W^{k+4N+p,\infty}})\big|\nabla\psi\big|_{H^{s_\star}}\Big).&\end{aligned}$$

This follows from the definitions of the operators $\overline{\mathcal{U}}_n[\epsilon\zeta,\beta b]$, $\mathcal{U}_n[\epsilon\zeta,\beta b]$ and $\mathcal{W}_n[\epsilon\zeta,\beta b]$, given in eq. (11.8)–(11.9)–(11.10), proceeding as we have done to prove eq. (11.30). □

Remark 11.9. *We observe a loss of $4N+p$ derivatives (or $4N+p-1$ for the variable $\underline{w}$) between the regularity of the data and the control of the approximation. This should be compared with Proposition 4.9 in which we observe a loss of "only" $2N+\tilde{p}$. It is clear that we could extend the analysis in Section 4 to obtain approximations at any order (see [268, Proposition 3.37]). Remarking that the two expansions must coincide, we infer that the above result does hold with a loss of $2N+\tilde{p}$ derivatives, which is optimal (up to the choice of $\tilde{p}$) since the next term in the approximation is given through a differential operator of order $2N+2$ (or $2N+3$ for the variable $\underline{w}$) which does not vanish identically, as we can see from the linear analysis in Section 11.4.*

The following consistency result is an obvious consequence of Lemma 11.7 and Lemma 11.8.

Proposition 11.10 (Consistency). *There exists $p \in \mathbb{N}$ such that the following holds.*

Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M^\star \geq 0$, $s \geq 0$, $N \in \mathbb{N}^\star$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$, for any $b \in W^{\max\{s+4N+p, 2+s_\star\}}(\mathbb{R}^d)$, for any $T > 0$ and for any $(\zeta, \psi) \in L^\infty(0,T; H^{\max\{s+4N+p,2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+4N+p,2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water waves equations, eq. (2.7'), satisfying eq. (11.27) uniformly for $t \in (0,T)$ and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left(\left|\varepsilon\zeta(t,\cdot)\right|_{H^{s_\star+2N}} + \left|\nabla\psi(t,\cdot)\right|_{H^{s_\star+2N}}\right) + \left|\beta b\right|_{W^{\max\{s+4N+p,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta + \sum_{n=0}^{N} \mu^n \nabla\cdot\left(h\overline{\boldsymbol{u}}_n\right) = r_{1,N}, \\ \partial_t\psi + \zeta + \epsilon\sum_{n=0}^{N}\mu^n\left(\underline{\boldsymbol{u}}_n\cdot\nabla\psi - \frac{1}{2}\sum_{k=0}^{n}\underline{\boldsymbol{u}}_k\cdot\underline{\boldsymbol{u}}_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\underline{w}_k\underline{w}_{n-1-k}\right) = r_{2,N}, \end{cases}$$

where we denote

$$(\ \overline{\boldsymbol{u}}_n\ ,\ \underline{\boldsymbol{u}}_n\ ,\ \underline{w}_n\ ) \stackrel{\rm def}{=} (\ \overline{\mathcal{U}}_n[\epsilon\zeta,\beta b](\nabla\psi)\ ,\ \mathcal{U}_n[\epsilon\zeta,\beta b](\nabla\psi)\ ,\ \mathcal{W}_n[\epsilon\zeta,\beta b](\nabla\psi)\ )$$

and one has for almost every $t \in (0,T)$

$$\begin{aligned} \left|r_{1,N}(t,\cdot)\right|_{H^s} &\leq C\,\mu^{N+1}\left(\left|\zeta(t,\cdot)\right|_{H^{s+4N+p}} + \left|\nabla\psi(t,\cdot)\right|_{H^{s+4N+p}}\right), \\ \left|r_{2,N}(t,\cdot)\right|_{H^{s+1}} &\leq C\,\mu^{N+1}\varepsilon\left|\nabla\psi(t,\cdot)\right|_{H^{s_\star+2N}}\left(\left|\zeta(t,\cdot)\right|_{H^{s+4N+p}} + \left|\nabla\psi(t,\cdot)\right|_{H^{s+4N+p}}\right). \end{aligned}$$

Remark 11.11. *We have provided the consistency of the water waves system with respect to the first model, eq. (11.15). We could provide the consistency of the water waves system with respect to the second model, eq. (11.17), along the same lines. Yet we have in this case an additional step, due to the fact that the variables of the model are $(\zeta, \overline{\boldsymbol{u}})$ instead of $(\zeta, \nabla\psi)$. Recall the latter is uniquely defined from the former and we have regularity estimates, by Proposition 4.9. In addition to controlling the differences*

$$\left|\underline{\boldsymbol{u}} - \textstyle\sum_{n=0}^N \mu^n\widetilde{\mathcal{U}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})\right|_{H^s} \quad \text{and} \quad \left|\underline{w} - \textstyle\sum_{n=0}^N \mu^n\widetilde{\mathcal{W}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})\right|_{H^s},$$

where $\widetilde{\mathcal{U}}_n$ and $\widetilde{\mathcal{W}}_n$ are defined in eq. (11.12)–(11.13), we also need to control

$$\left|\partial_t\left(\nabla\psi - \textstyle\sum_{n=0}^N \mu^n\widetilde{\mathcal{V}}_n[\epsilon\zeta,\beta b](\overline{\boldsymbol{u}})\right)\right|_{H^s}$$

where $\widetilde{\mathcal{V}}_n$ is defined in eq. (11.11). While the former would be obtained similarly as in Lemma 11.8, the latter requires a specific analysis basically amounting to considering the Laplace problem satisfied the time derivative of the velocity potential, $\partial_t\Phi$; see Remark 5.2 for a similar remark with a little bit more details.

In this manuscript, convergence results follow from stability and the well-posedness of the Cauchy problems. Except in particular cases already treated (specifically in Section 5.3 and Section 8.5) these are open problems, which are discussed in the following section. We could use the opposite reasoning and prove that regular solutions to the model systems, eq. (11.15) or eq. (11.17) (although in the latter we face the issue of reconstructing the velocity potential from the knowledge of the layer-averaged velocity), satisfy the water waves system up to a small remainder; and then refer to [268, Theorem 4.18] to deduce that the corresponding water waves solution remains close on the relevant time interval. Of course, while the set of regular solutions to the models may not be trivial even when lacking a robust well-posedness theory (it may contain solitary waves and analytic solutions for instance), the outcome is much less appealing than the ones we obtained so far.

11.6 Discussion and open questions

Let us quote Matsuno [302, §6], on the extended Green–Naghdi model.

> *There are a number of interesting problems associated with the extended GN equations that are worthy of further study. In conclusion, we list some of them.*
>
> *(i) The identification of physically relevant models among various extended GN equations.*
>
> *(ii) The effect of higher-order dispersion on the wave characteristics in comparison with that predicted by other asymptotic models like Boussinesq equations.*
>
> *(iii) Numerical computations of the initial value problems as well as solitary and periodic wave solutions.*
>
> *(iv) The justification of the asymptotic models by means of the rigorous mathematical analysis.*

Let us discuss these points. As we mentioned above there is so far no well-posedness result concerning the initial-value problem for systems eq. (11.15) (unless $N = 0$ corresponding to the Saint-Venant system; see Theorem 5.3) or systems eq. (11.17) (unless again $N = 0$, or $N = 1$ corresponding to the Green–Naghdi system; see Theorem 8.3). As mentioned in Section 11.4, the cases $N \geq 1$ odd for eq. (11.15) and $N \geq 2$ even for eq. (11.17) are hopeless, due to high frequency modal instabilities. The remaining cases are open problems. Such results, if they hold, typically rely on very delicate cancellations since high order differential operators are involved,[60] and/or on manipulations akin to the "BBM trick" to derive equivalent models with favorable properties; see [52, 77, 252] for Boussinesq-type models and [253, 251] for the full justification of a modified extended Green–Naghdi system built from eq. (11.17) with $N = 2$.

As we have mentioned in Section 11.4, and as is apparent in Figure 11.1, augmenting the rank, N, in the family of systems fails to improve the approximation (at least for the models we exhibited; see however footnote 56), even in the linear framework, as soon as moderate-to-high (spatial) frequencies account for a significant proportion of the energy. We expect that for a given (initial) datum, there is a critical $N_\star$ such that the prediction of the model improves with growing rank $N \leq N_\star$, and then quickly deteriorates. From Figure 11.1 one can imagine that higher rank systems will be relevant only for waves with very large wavelength; or, if we fix smooth data and allow the parameter $\mu \in (0, \mu^\star]$ to vary, for very small values of μ.

In the same way, assuming that solitary waves solutions to our systems do exist—there is no rigorous result to my knowledge—we do expect that they provide very good approximations for solitary waves solutions to the water waves system with very small (yet supercritical) velocities, since the latter are long waves; but we do not expect that increasing the rank of the model allows to

[60] Systems (11.17) appear more favorable since—as in our analysis of the Green–Naghdi (and obviously the Saint-Venant) system—the system can be interpreted as quasilinear of order 1 when using relevant functional spaces.

widen the velocity interval for which solitary wave solutions to the system provide a fair description of solitary wave solutions to the water waves system.

Shortly put, we expect that augmenting the rank of the model may improve its accuracy for flows with very large wavelength, but we cannot hope that they improve their domain of validity.

In their review [294], Madsen and Fuhrman report in many details the long history which led to enhanced Friedrichs-type (therein called Boussinesq-type) models with improved performance at moderate frequencies. Most often, due to the complexity of the models, one relies on the help of numerical simulations to compare the behavior of models. Conveniently, the next chapter in the same book [427], by Zang, Fang and Liu, provides an example of such analysis.

There is much more to discuss on Friedrichs-type high order models than the author can digest or report, and it should be clearly stated that the present Section only scratches the surface. Yet it is my opinion that the difficulties we describe are inherently linked to the method of derivation, and first and foremost to the use of the Boussinesq–Rayleigh expansion, eq. (11.1). We present in Sections 12 and 13 two approaches which appear much more promising.

12 The Galerkin method, and the augmented and "multilayer" Green–Naghdi systems

In this section we shall derive formally a large class of models for the water waves system, eq. (2.7), using as main tool the finite element method applied to a convenient reformulation of the Laplace equation, eq. (2.8). The classes of models we can construct in this way is huge, as it depends on the choice of the subspace defining the dimension reduction. As a matter of fact, the procedure provides a natural framework for the complete numerical discretization of the water waves system. We provide two class of examples representing two natural strategies in which the subspace is tensorized in the vertical and horizontal space variables. In the first one, we mimic the Boussinesq–Rayleigh shallow water expansion of the preceding Section, eq. (11.1); yet the outcome are models with completely different features, and in particular high order differential operators are replaced by a system of differential equations of order two. In the second strategy, the subspace is defined through an artificial discretization in "layers" in the vertical variable. Once again the models involve a system of differential equations of order two, in place of the Dirichlet-to-Neumann operator. Both models can be interpreted as special cases of the finite element method for which the basis functions are piecewise polynomials in the vertical variable. Interestingly, the Green–Naghdi system studied in Section 8 is a special case for both families.

12.1 Derivation of the models

12.1.1 Reformulation of the Laplace problem

In this section, we drop any reference to the time variable which acts as a parameter. We work with the following integral equation satisfied by the velocity potential, as a solution to the Laplace problem, eq. (2.8) (see Lemma 4.7)

$$\Phi + \mu\ell[\varepsilon\zeta, \beta b]\Phi = \psi$$

where

$$\big(\ell[\varepsilon\zeta,\beta b]\Phi\big)(\cdot,z) \stackrel{\text{def}}{=} -\int_z^{\varepsilon\zeta}\left(-(\beta\nabla b)\cdot(\nabla_{\boldsymbol{x}}\Phi)\big|_{z=-1+\beta b} + \int_{-1+\beta b}^{z'}\Delta_{\boldsymbol{x}}\Phi(\cdot,z'')\,\mathrm{d}z''\right)\mathrm{d}z'$$

More precisely, applying the horizontal gradient operator, we shall consider the relation

$$\nabla_{\boldsymbol{x}}\Phi(\boldsymbol{x},z) \ + \ \mu\left(\mathcal{L}[\varepsilon\zeta,\beta b]\nabla_{\boldsymbol{x}}\Phi\right)(\boldsymbol{x},z) \ = \ \nabla\psi(\boldsymbol{x})$$

were

$$\big(\mathcal{L}[\varepsilon\zeta,\beta b]U\big)(\cdot,z) \stackrel{\text{def}}{=} -\nabla_{\boldsymbol{x}}\left(\int_z^{\varepsilon\zeta}\left(-(\beta\nabla b)\cdot U\big|_{z=-1+\beta b} + \int_{-1+\beta b}^{z'}\nabla_{\boldsymbol{x}}\cdot U(\cdot,z'')\,\mathrm{d}z''\right)\mathrm{d}z'\right). \tag{12.1}$$

Note $\mathcal{L}[\varepsilon\zeta,\beta b]$ is a symmetric operator in $L^2(\Omega)$ where $\Omega = \{(\boldsymbol{x},z)\in\mathbb{R}^{d+1} : -1+\beta b(\boldsymbol{x}) < z < \varepsilon\zeta(\boldsymbol{x})\}$: for sufficiently regular and localized functions,

$$\begin{aligned}\big(\mathcal{L}[\varepsilon\zeta,\beta b]U \ , \ V\big)_{L^2(\Omega)} &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_{-1+\beta b(\boldsymbol{x})}^{\varepsilon\zeta(\boldsymbol{x})}(\mathcal{L}[\varepsilon\zeta,\beta b]U)\cdot V\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^d}\int_{-1+\beta b(\boldsymbol{x})}^{\varepsilon\zeta(\boldsymbol{x})}\left(\nabla_{\boldsymbol{x}}\cdot\left(\int_{-1+\beta b(\boldsymbol{x})}^{z}U(\boldsymbol{x},z')\,\mathrm{d}z'\right)\right)\left(\nabla_{\boldsymbol{x}}\cdot\left(\int_{-1+\beta b(\boldsymbol{x})}^{z}V(\boldsymbol{x},z')\,\mathrm{d}z'\right)\right)\mathrm{d}z\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Obviously, $\mathcal{L}[\varepsilon\zeta,\beta b]$ is positive: $\big(\mathcal{L}[\varepsilon\zeta,\beta b]U\ ,\ U\big)_{L^2(\Omega)} \geq 0$. It is hence natural to consider the space

$$X_\mu(\Omega) \stackrel{\rm def}{=} \Big\{ U \in L^2(\Omega)^d\ :\ \big\|\nabla_{\boldsymbol{x}} \cdot \big(\int_{-1+\beta b(\boldsymbol{x})}^{\cdot} U(\cdot,z')\,{\rm d}z' \big)\big\|_{L^2(\Omega)} \Big\}$$

endowed with the topology induced by the inner-product

$$\langle U\ , V\rangle_{X_\mu} \stackrel{\rm def}{=} \big(U\ ,\ V\big)_{L^2(\Omega)} + \mu\big(\nabla_{\boldsymbol{x}} \cdot \big(\int_{-1+\beta b}^{\cdot} U(\cdot,z')\,{\rm d}z' \big)\ ,\ \nabla_{\boldsymbol{x}} \cdot \big(\int_{-1+\beta b}^{\cdot} V(\cdot,z')\,{\rm d}z' \big)\big)_{L^2(\Omega)}. \tag{12.2}$$

we can apply Riesz representation Lemma to infer that $\mathrm{Id} + \mu\mathcal{L}[\varepsilon\zeta,\beta b] : X_\mu \to (X_\mu)'$ is invertible. For $\psi \in \mathring{H}^1(\mathbb{R}^d)$, we may define $\tilde\psi : (\boldsymbol{x},z) \in \Omega \mapsto \psi(\boldsymbol{x}) \in H^1(\Omega)^{d+1} \subset (X_\mu)'$, and the Laplace problem may be rewritten equivalently (dropping the tilde) as

$$\nabla_{\boldsymbol{x}}\Phi = \big(\mathrm{Id} + \mu\mathcal{L}[\varepsilon\zeta,\beta b]\big)^{-1}\nabla\psi. \tag{12.3}$$

More precisely, we define the following notion of variational solutions.

Definition 12.1 (Variational solutions)**.** *Let $\psi \in \mathring{H}^1(\mathbb{R}^d)$ and $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying*

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \qquad h(\boldsymbol{x}) = 1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0.$$

We say that $U \in X_\mu(\Omega)$ is a variational solution to eq. (12.3) if for any $V \in X_\mu(\Omega)$,

$$\langle U\ ,\ V\rangle_{X_\mu} = \big(\nabla\psi\ ,\ V\big)_{L^2(\Omega)} = \int_{\mathbb{R}^d} (\nabla\psi) \cdot \Big(\int_{-1+\beta b}^{\varepsilon\zeta} V(\cdot,z)\,{\rm d}z \Big)\,{\rm d}\boldsymbol{x}$$

where $\langle\cdot,\cdot\rangle_{X_\mu}$ is defined in eq. (12.2). In the formula above we identified $\boldsymbol{x} \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\mathbb{R}^d)$ and $(\boldsymbol{x},z) \mapsto \psi(\boldsymbol{x}) \in \mathring{H}^1(\Omega)$.

Remark 12.2. *It is interesting to pause and compare our formulation with the Dirichlet-to-Neumann operator. We have (see Lemma 4.6)*

$$\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = -\mu\nabla\cdot\Big(\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_{\boldsymbol{x}}\Phi(\cdot,z)\,{\rm d}z\Big) = -\mu\nabla\cdot\Big(\int_{-1+\beta b}^{\varepsilon\zeta} \big(\mathrm{Id} - \mu\mathcal{L}[\varepsilon\zeta,\beta b]\big)^{-1}\nabla\psi\Big). \tag{12.4}$$

We can therefore relate properties of the Dirichlet-to-Neumann operator (see Proposition 2.3) with properties of $(\mathrm{Id} - \mu\mathcal{L}[\varepsilon\zeta,\beta b])^{-1}$. For instance,

$$\begin{aligned}\big\langle \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi_1,\psi_2\big\rangle_{(\mathring{H}^1)'-\mathring{H}^1} &= \int_{\mathbb{R}^d}\Big(\int_{-1+\beta b}^{\varepsilon\zeta}\Big(\big(\mathrm{Id} - \mu\mathcal{L}[\varepsilon\zeta,\beta b]\big)^{-1}\nabla\psi_1\Big)(\cdot,z)\,{\rm d}z\Big)\cdot\nabla\psi_2\,{\rm d}\boldsymbol{x}\\ &= \Big(\big(\mathrm{Id} - \mu\mathcal{L}[\varepsilon\zeta,\beta b]\big)^{-1}\nabla\psi_1\ ,\ \nabla\psi_2\Big)_{L^2(\Omega)}.\end{aligned}$$

Moreover, the relation above shows that solving eq. (12.3) is sufficient to infer $\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi$, and hence every contribution in the water waves system, eq. (2.7).

12.1.2 Galerkin dimension reduction

We now apply the Galerkin method to the variational formulation of eq. (12.3). Define $Y \subset X_\mu(\Omega)$ a subspace and $\Pi_Y : X_\mu(\Omega) \to Y$ the corresponding orthogonal projection: $\Pi_Y^2 = \Pi_Y = \Pi_Y{}^\star$. Then we define an approximate solution to eq. (12.3) as the variational solution to

$$\big(\mathrm{Id} + \mu\Pi_Y\mathcal{L}[\varepsilon\zeta,\beta b]\Pi_Y\big)U_Y = \nabla\psi,$$

that is U_Y such that $U_Y - \nabla\psi \in Y$ and

$$\forall V \in Y, \qquad \langle U_Y\ ,\ V\rangle_{X_\mu} = \big(\nabla\psi\ ,\ V\big)_{L^2(\Omega)}. \tag{12.5}$$

where we recall that $\langle\cdot,\cdot\rangle_{X_\mu}$ is defined in eq. (12.2).

Remark 12.3. *By standard properties of the Galerkin method we have, denoting U the exact solution defined in Definition 12.1 and U_Y the approximate solution defined above,*

$$\big\|U-U_Y\big\|_{L^2(\Omega)}^2 \leq \langle\, U-U_Y \ ,\ U-U_Y\,\rangle_{X_\mu} \leq \inf_{V\in Y}\langle\, U-V \ ,\ U-V\,\rangle_{X_\mu}.$$

Once U_Y has been defined, we can plug this formula in eq. (12.4) to infer an approximation of the Dirichlet-to-Neumann operator, and in turn an approximate model for the water waves system, eq. (2.7). In this last step, it is beneficial to preserve the canonical Hamiltonian structure of the system, and to derive the model through an approximate Hamiltonian: we define

$$\mathscr{H}_Y(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + (\nabla\psi)\cdot\Big(\int_{-1+\beta b}^{\varepsilon\zeta} U_Y(\cdot,z)\,\mathrm{d}z\Big)\,\mathrm{d}\boldsymbol{x} \tag{12.6}$$

and consider

$$\begin{cases} \partial_t\zeta - \delta_\psi\mathscr{H}_Y = 0,\\ \partial_t\psi + \delta_\zeta\mathscr{H}_Y = 0.\end{cases} \tag{12.7}$$

By Noether's theorem (see Section 2.2), the system preserves mass, energy and horizontal impulse:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_Y = 0, \quad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\zeta\,\mathrm{d}\boldsymbol{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \quad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\zeta\nabla\psi\,\mathrm{d}\boldsymbol{x} \quad (\text{if } \beta b\equiv 0).$$

The procedure above yields a model for any choice of the subspace $Y\subset X_\mu(\Omega)$. We provide in the following subsection a few revealing examples, where subspaces Y are set—complying with the finite element method—as the superposition of piecewise polynomial functions in the variable z.

12.1.3 The augmented and "multilayer" Green–Naghdi systems

i. If we set $\Pi_0 = 0$ the null operator, then the procedure above yields $U_0 = \nabla\psi$,

$$\mathscr{H}^0_{\rm SV}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 + (1+\varepsilon\zeta-\beta b)|\nabla\psi|^2\,\mathrm{d}\boldsymbol{x}$$

and

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big((1+\varepsilon\zeta-\beta b)\nabla\psi\big) = 0,\\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 = 0.\end{cases}$$

In other words, we recover the Saint-Venant system, eq. (5.2).

ii. If we set Π_1 as the layer-averaging operator

$$\Pi_1 : U \mapsto \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta} U(\cdot,z')\,\mathrm{d}z',$$

then direct computations show that for any sufficiently regular U,

$$\Pi_1\mathcal{L}[\varepsilon\zeta,\beta b]\Pi_1 U = \mathcal{T}[h,\beta\nabla]\boldsymbol{u}.$$

where we denote $\boldsymbol{u}=\Pi_1 U$, $h = 1+\varepsilon\zeta-\beta b$ and $\mathcal{T}[h,\beta\nabla]$ has been defined in eq. (4.7):

$$\mathcal{T}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u}) + \frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big) - h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}\Big) + (\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b).$$

From this we infer $U_1 \stackrel{\text{def}}{=} (\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla])^{-1}\nabla\psi = \mathfrak{T}^\mu[h,\beta\nabla b]^{-1}(h\nabla\psi)$ with $\mathfrak{T}^\mu = h\,\mathrm{Id}+\mu h\mathcal{T}$,[61]

$$\mathscr{H}^1_{\rm GN}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 + (h\nabla\psi)\cdot\mathfrak{T}^\mu[h,\beta\nabla b]^{-1}(h\nabla\psi)\,\mathrm{d}\boldsymbol{x}$$

and the associated canonical Hamiltonian equations, eq. (12.7), are the Green–Naghdi equations, eq. (8.2); see Section 8.1.1.

[61] It is interesting to notice how the symmetry of $\mathfrak{T}^\mu$ in $L^2(\mathbb{R}^d)$ relates to the symmetry of $\Pi_1\mathcal{L}[\varepsilon\zeta,\beta b]\Pi_1$ in $L^2(\Omega)$.

iii. Building upon the previous examples, and motivated by the Boussinesq–Rayleigh expansion, eq. (11.1), we set $N \in \mathbb{N}^\star$, and $p_1, \cdots, p_N$ nonnegative integers with $p_1 < \cdots < p_N$ and set

$$Y_N = \left\{ U \in X_\mu(\Omega) \ : \ U(\boldsymbol{x}, z) = \sum_{i=1}^N \boldsymbol{u}_i(\boldsymbol{x})\,(z + 1 - \beta b(\boldsymbol{x}))^{p_i},\ \boldsymbol{u}_i \in X^0_\mu \right\}$$

where we denote, as in Section 8.5,

$$X^0_\mu \stackrel{\rm def}{=} \{\boldsymbol{u} \in L^2(\mathbb{R}^d)^d \ : \ \big|\boldsymbol{u}\big|_{X^0_\mu}^2 \stackrel{\rm def}{=} \big|\boldsymbol{u}\big|_{L^2}^2 + \mu\big|\nabla\cdot\boldsymbol{u}\big|_{L^2}^2 < \infty\}.$$

Solving eq. (12.5) yields

$$U_N = \sum_{i=1}^N \boldsymbol{u}_i(\boldsymbol{x})\,(z+1-\beta b(\boldsymbol{x}))^{p_i}$$

where $(\boldsymbol{u}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$ satisfies for any $(\boldsymbol{v}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$,

$$\sum_{i=1}^N\sum_{j=1}^N \int_{\mathbb{R}^d} \Bigg[\frac{h^{p_i+p_j+1}}{p_i+p_j+1}\boldsymbol{u}_i\cdot\boldsymbol{v}_j + \mu\frac{h^{p_i+p_j+3}}{(p_i+1)(p_j+1)(p_i+p_j+3)}(\nabla\cdot\boldsymbol{u}_i)(\nabla\cdot\boldsymbol{v}_j)$$
$$+ \mu\frac{h^{p_i+p_j+2}}{(p_j+1)(p_i+p_j+2)}(-\beta\nabla b\cdot\boldsymbol{u}_i)(\nabla\cdot\boldsymbol{v}_j) + \mu\frac{h^{p_i+p_j+2}}{(p_i+1)(p_i+p_j+2)}(\nabla\cdot\boldsymbol{u}_i)(-\beta\nabla b\cdot\boldsymbol{v}_j)$$
$$+ \mu\frac{h^{p_i+p_j+1}}{p_i+p_j+1}(-\beta\nabla b\cdot\boldsymbol{u}_i)(-\beta\nabla b\cdot\boldsymbol{v}_j)\Bigg]\,{\rm d}\boldsymbol{x} = \sum_{j=1}^N\int_{\mathbb{R}^d}\frac{h^{p_j+1}}{p_j+1}(\nabla\psi)\cdot\boldsymbol{v}_j\,{\rm d}\boldsymbol{x},$$

where $h = 1 + \varepsilon\zeta - \beta b$. The above can be written as a system of linear differential equations

$$\forall j \in \{1,\dots,N\}, \qquad \sum_{i=1}^N \mathcal{L}^\mu_{ji}[h,\beta\nabla b]\boldsymbol{u}_i = \frac{h^{p_j+1}}{p_j+1}\nabla\psi \tag{12.8}$$

where for any $i,j \in \{1,\dots,N\}$,

$$\mathcal{L}^\mu_{ji}[h,\beta\nabla b]\boldsymbol{u}_i \stackrel{\rm def}{=} \frac{h^{p_i+p_j+1}}{p_i+p_j+1}\boldsymbol{u}_i - \mu\nabla\left(\frac{h^{p_i+p_j+3}}{(p_i+1)(p_j+1)(p_i+p_j+3)}(\nabla\cdot\boldsymbol{u}_i)\right)$$
$$+\mu\nabla\left(\frac{h^{p_i+p_j+2}}{(p_j+1)(p_i+p_j+2)}(\beta\nabla b\cdot\boldsymbol{u}_i)\right) - \mu\frac{h^{p_i+p_j+2}}{(p_i+1)(p_i+p_j+2)}(\nabla\cdot\boldsymbol{u}_i)(\beta\nabla b)$$
$$+\mu\frac{h^{p_i+p_j+1}}{p_i+p_j+1}(\beta\nabla b\cdot\boldsymbol{u}_i)(\beta\nabla b). \tag{12.9}$$

If $N = 1$ and $p_1 = 0$, we recover the Green–Naghdi equations. Otherwise we put

$$\mathscr{H}^{(p_1,\dots,p_N)}_{\rm aGN}(\zeta,\psi) \stackrel{\rm def}{=} \frac12\int_{\mathbb{R}^d}\zeta^2 + \sum_{i=1}^N\frac{h^{p_i+1}}{p_i+1}\boldsymbol{u}_i\cdot\nabla\psi\,{\rm d}\boldsymbol{x}$$

where we recall that $(\boldsymbol{u}_i)_{i\in\{1,\dots,N\}}$ is determined from $(\varepsilon\zeta,\beta b,\psi)$ by eq. (12.8). Because

$$\psi \mapsto \mathcal{G}^{(p_1,\dots,p_N)}_{\rm aGN}[\varepsilon\zeta,\beta b]\psi \stackrel{\rm def}{=} -\nabla\cdot\left(\sum_{i=1}^N\frac{h^{p_i+1}}{p_i+1}\boldsymbol{u}_i\right)$$

is symmetric for the $L^2(\mathbb{R}^d)$ inner product, we have

$$\delta_\psi\mathscr{H}^{(p_1,\dots,p_N)}_{\rm aGN}(\zeta,\psi) = \mathcal{G}^{(p_1,\dots,p_N)}_{\rm aGN}[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot\left(\sum_{i=1}^N\frac{h^{p_i+1}}{p_i+1}\boldsymbol{u}_i\right). \tag{12.10}$$

We infer the functional derivative with respect to ζ as follows. Denote for all $i,j\in\{1,\dots,N\}$,

$$D_h\mathcal{L}^{\mu}_{ji}[h,\beta\nabla b](\eta)\boldsymbol{u}_j \stackrel{\text{def}}{=} \eta h^{p_i+p_j}\big(\boldsymbol{u}_i+(\beta\nabla b\cdot\boldsymbol{u}_i)(\beta\nabla b)\big)-\mu\nabla\left(\eta\frac{h^{p_i+p_j+2}}{(p_i+1)(p_j+1)}(\nabla\cdot\boldsymbol{u}_i)\right) + \mu\nabla\left(\eta\frac{h^{p_i+p_j+1}}{p_j+1}(\beta\nabla b\cdot\boldsymbol{u}_i)\right)-\mu\eta\frac{h^{p_i+p_j+1}}{p_i+1}(\nabla\cdot\boldsymbol{u}_i)(\beta\nabla b),$$

so that $\eta\mapsto D_h\mathcal{L}^{\mu}_{ji}[h,\beta\nabla b](\eta)$, is the Fréchet derivative of $\mathcal{L}^{\mu}_{ji}[h,\beta\nabla b]$ with respect to h. Then, denoting $(\boldsymbol{u}_i[\varepsilon\zeta,\beta\nabla b]\psi)_{i\in\{1,\dots,N\}}$ the solutions to eq. (12.8) we have for all $j\in\{1,\dots,N\}$

$$\sum_{i=1}^{N}\mathcal{L}^{\mu}_{ji}[h,\beta\nabla b]D_\zeta\boldsymbol{u}_i[h,\beta\nabla b](\eta)\psi+\varepsilon\sum_{i=1}^{N}D_h\mathcal{L}^{\mu}_{ji}[h,\beta\nabla b](\eta)\boldsymbol{u}_i[\varepsilon\zeta,\beta\nabla b]\psi=\varepsilon\eta h^{p_j}\nabla\psi.$$

From this and the fact that $(\mathcal{L}^{\mu}_{ji})^\star=\mathcal{L}^{\mu}_{ij}$ with $(\mathcal{L}^{\mu}_{ij})^\star$ the adjoint operator in $L^2(\mathbb{R}^d)$, we infer

$$\delta_\zeta\mathscr{H}^{(p_1,\dots,p_N)}_{\mathrm{aGN}}(\zeta,\psi)=\zeta+\varepsilon\sum_{i=1}^{N}h^{p_i}\boldsymbol{u}_i\cdot\nabla\psi-\frac{\varepsilon}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{Q}^{\mu}_{ij}[h,\beta\nabla b](\boldsymbol{u}_i,\boldsymbol{u}_j) \tag{12.11}$$

with, for all $i,j\in\{1,\dots,N\}$,

$$\mathcal{Q}^{\mu}_{ij}[h,\beta\nabla b](\boldsymbol{u},\boldsymbol{v}) \stackrel{\text{def}}{=} h^{p_i+p_j}\big(\boldsymbol{u}\cdot\boldsymbol{v}+(\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b\cdot\boldsymbol{v})\big)+\mu\frac{h^{p_i+p_j+2}}{(p_i+1)(p_j+1)}(\nabla\cdot\boldsymbol{v})(\nabla\cdot\boldsymbol{u}) -\mu\frac{h^{p_i+p_j+1}}{p_j+1}(\nabla\cdot\boldsymbol{v})(\beta\nabla b\cdot\boldsymbol{u})-\mu\frac{h^{p_i+p_j+1}}{p_i+1}(\nabla\cdot\boldsymbol{u})(\beta\nabla b\cdot\boldsymbol{v}).$$

Summing up, we consider

$$\left\{\begin{array}{l}\partial_t\zeta-\delta_\psi\mathscr{H}^{(p_1,\dots,p_N)}_{\mathrm{aGN}}(\zeta,\psi)=0,\\ \partial_t\psi+\delta_\zeta\mathscr{H}^{(p_1,\dots,p_N)}_{\mathrm{aGN}}(\zeta,\psi)=0.\end{array}\right. \tag{12.12}$$

where $\delta_\psi\mathscr{H}^{(p_1,\dots,p_N)}_{\mathrm{aGN}}(\zeta,\psi)$, $\delta_\zeta\mathscr{H}^{(p_1,\dots,p_N)}_{\mathrm{aGN}}(\zeta,\psi)$ are given by eq. (12.10)–(12.11) and, therein, $(\boldsymbol{u}_i)_{i\in\{1,\dots,N\}}\in(X^0_\mu)^N$ are defined as the solutions to eq. (12.8).

Remark 12.4. *It is interesting to compare with the models introduced in Section 11, in which the space Y_N plays a key role, but where the variables $(\boldsymbol{u}_i)_{i\in\{0,\dots,N\}}$ are explicitly computed from the Boussinesq–Rayleigh expansion eq. (11.1), instead of being characterized as solutions to a system of differential equations as above. We see that, when increasing the rank of the model, N, the size of the system grows in the latter strategy, while the order of the differential operators at stake grows in the former strategy.*

Using physical variables (recall Section 2.4), eq. (12.12) yields the ***augmented Green–Naghdi systems***

$$\left\{\begin{array}{l}\partial_t\zeta+\nabla\cdot\left(\sum_{i=1}^N\frac{h^{p_i+1}}{p_i+1}\boldsymbol{u}_i\right)=0,\\ \partial_t\psi+g\zeta+\sum_{i=1}^N h^{p_i}\boldsymbol{u}_i\cdot\nabla\psi-\frac12\sum_{i=1}^N\sum_{j=1}^N\mathcal{Q}^1_{ij}[h,\nabla b](\boldsymbol{u}_i,\boldsymbol{u}_j)=0,\end{array}\right. \tag{12.13}$$

where $h(t,\boldsymbol{x})=d+\zeta(t,\boldsymbol{x})-b(\boldsymbol{x})$, $\mathcal{L}_{ij}$ and $\mathcal{Q}_{ij}$ are defined above and $(\boldsymbol{u}_i)_{i\in\{1,\dots,N\}}$ are the solutions to

$$\forall j\in\{1,\dots,N\},\qquad \sum_{i=1}^{N}\mathcal{L}^1_{ji}[h,\nabla b]\boldsymbol{u}_i=\frac{h^{p_j+1}}{p_j+1}\nabla\psi.$$

iv. We now define a projection corresponding to a “multilayer” discretization of the flow.[62] Set $N \in \mathbb{N}^\star$ and
$$-1+\beta b(\boldsymbol{x}) = \eta_0(\boldsymbol{x}) < \eta_1(\boldsymbol{x}) < \cdots < \eta_N(\boldsymbol{x}) = \varepsilon\zeta(\boldsymbol{x}).$$
Then we denote
$$\Pi_N : U \mapsto \sum_{i=1}^N \overline{\boldsymbol{u}}_i(\boldsymbol{x})\mathbf{1}_{(\eta_{i-1}(\boldsymbol{x}),\eta_i(\boldsymbol{x}))}(z), \qquad \overline{\boldsymbol{u}}_i \stackrel{\text{def}}{=} \frac{1}{\eta_i-\eta_{i-1}}\int_{\eta_{i-1}}^{\eta_i} U(\cdot,z)\,\mathrm{d}z$$
where $\mathbf{1}_{(a,b)}(z)=1$ if $z\in(a,b)$, and 0 otherwise. Solving eq. (12.5) (in the flat bottom case, $\beta b\equiv 0$, for simplicity) yields
$$U_N = \sum_{i=1}^N \overline{\boldsymbol{u}}_i(\boldsymbol{x})\mathbf{1}_{(\eta_{i-1}(\boldsymbol{x}),\eta_i(\boldsymbol{x}))}(z)$$
where $(\overline{\boldsymbol{u}}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$ satisfies for any $(\overline{\boldsymbol{v}}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$,
$$\int_{\mathbb{R}^d}\sum_{i=1}^N h_i\overline{\boldsymbol{u}}_i\cdot\overline{\boldsymbol{v}}_i\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^d}\sum_{i=1}^N\int_{\eta_{i-1}}^{\eta_i}\Big(\textstyle\sum_{j=1}^{i-1}\nabla\cdot(h_j\overline{\boldsymbol{u}}_j)+\nabla\cdot((z-\eta_{i-1})\overline{\boldsymbol{u}}_i)\Big)$$
$$\times\Big(\textstyle\sum_{k=1}^{i-1}\nabla\cdot(h_k\overline{\boldsymbol{v}}_k)+\nabla\cdot((z-\eta_{i-1})\overline{\boldsymbol{v}}_i)\Big)\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x} = \textstyle\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\psi)\cdot(h_i\overline{\boldsymbol{v}}_i)\,\mathrm{d}\boldsymbol{x}.$$
where we denote $h_i=\eta_i-\eta_{i-1}$ for $i\in\{1,\dots,N\}$. The above can be written as a system of linear differential equations for $(\overline{\boldsymbol{u}}_i)_{i\in\{1,\dots,N\}}\in(X^0_\mu)^N$:
$$\forall i\in\{1,\dots,N\},\qquad h_i\overline{\boldsymbol{u}}_i+\mu h_i\mathcal{T}[h_i,\nabla\eta_{i-1}]\overline{\boldsymbol{u}}_i+\mu\mathcal{I}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)=h_i\nabla\psi \tag{12.14}$$
where we recall the notation
$$\mathcal{T}[h,\nabla\eta]\boldsymbol{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u}) + \frac{1}{2h}\Big(\nabla\big(h^2(\nabla\eta)\cdot\boldsymbol{u}\big)-h^2(\nabla\eta)\nabla\cdot\boldsymbol{u}\Big)+(\nabla\eta\cdot\boldsymbol{u})(\nabla\eta)$$
and introduce
$$\mathcal{I}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N) \stackrel{\text{def}}{=} -h_i(\nabla\eta_{i-1})\cdot\Big(\textstyle\sum_{j=1}^{i-1}\nabla\cdot(h_j\overline{\boldsymbol{u}}_j)\Big)+h_i\sum_{j=i+1}^N\nabla\left(h_j(\nabla\eta_{j-1})\cdot\overline{\boldsymbol{u}}_j\right)$$
$$-\tfrac12\nabla\Big(h_i^2\textstyle\sum_{j=1}^{i-1}\nabla\cdot(h_j\overline{\boldsymbol{u}}_j)\Big)-\frac{h_i}{2}\sum_{j=i+1}^N\nabla\left(h_j^2\nabla\cdot\overline{\boldsymbol{u}}_j\right)-h_i\sum_{k=i+1}^N\sum_{j=1}^{k-1}\nabla\left(h_k\nabla\cdot(h_j\overline{\boldsymbol{u}}_j)\right).$$
When $N=1$, $\mathcal{I}[\eta_0,\eta_1]\overline{\boldsymbol{u}}_1=\mathbf{0}$ and, denoting $\overline{\boldsymbol{u}}=\overline{\boldsymbol{u}}_1$ and $h=h_1=1+\varepsilon\zeta-\beta b$, eq. (12.14) reads
$$\overline{\boldsymbol{u}}+\mu\mathcal{T}[h,\beta\nabla b]\overline{\boldsymbol{u}}=\nabla\psi.$$
We recognize once again the Green–Naghdi system. This is of course consistent with the fact that the projection Π_N for $N=1$ is the layer-averaging operator. Otherwise we put
$$\mathscr{H}^{(\eta_0,\dots,\eta_N)}_{\text{mGN}}(\zeta,\psi) \stackrel{\text{def}}{=} \frac12\int_{\mathbb{R}^d}\zeta^2+\sum_{i=1}^N h_i\overline{\boldsymbol{u}}_i\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}$$
where $(\overline{\boldsymbol{u}}_i)_{i\in\{1,\dots,N\}}$ is determined from $(-1+\beta b=\eta_0,\eta_1,\dots,\eta_N=\varepsilon\zeta,\psi)$ by eq. (12.14). As above we remark that
$$\psi\mapsto\mathcal{G}^{(\eta_0,\dots,\eta_N)}_{\text{mGN}}[\varepsilon\zeta,\beta b]\psi \stackrel{\text{def}}{=} -\nabla\cdot\left(\sum_{i=1}^N h_i\overline{\boldsymbol{u}}_i\right)$$

[62] Here and thereafter, the “multilayer” terminology comes with quote marks. It should not be confused with the bilayer or multilayer framework studied in Section 3 and Section 6 where layers are defined by the density stratification. Here the layers simply correspond to a discretization strategy along the vertical variable. In particular, particles of fluid freely cross the “interfaces”.

is symmetric for the $L^2(\mathbb{R}^d)$ inner product, and hence

$$\delta_\psi \mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi) = \mathcal{G}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot\Big(\sum_{i=1}^N h_i\overline{\boldsymbol{u}}_i\Big). \tag{12.15}$$

Proceeding as for the augmented Green–Naghdi systems, we find

$$\delta_\zeta \mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi) = \zeta + \varepsilon\sum_{i=1}^N \tfrac{\partial h_i}{\varepsilon\partial\zeta}\big(\overline{\boldsymbol{u}}_i\cdot\nabla\psi - \tfrac12|\overline{\boldsymbol{u}}_i|^2\big) - \frac{\mu\varepsilon}{2}\sum_{i=1}^N \mathcal{Q}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N) \tag{12.16}$$

where $\mathcal{Q}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)$ is defined by

$$\int_{\mathbb{R}^d}\overline{\boldsymbol{u}}_i\cdot D_\zeta\big(h_i\mathcal{T}[h_i,\nabla\eta_{i-1}]\overline{\boldsymbol{u}}_i + \mathcal{I}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)\big)(\eta)\,{\rm d}\boldsymbol{x} \\ = \varepsilon\int_{\mathbb{R}^d}\eta\mathcal{Q}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)\,{\rm d}\boldsymbol{x},$$

where $\eta\mapsto D_\zeta\big(h_i\mathcal{T}[h_i,\nabla\eta_{i-1}]\overline{\boldsymbol{u}}_i + \mathcal{I}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)\big)(\eta)$ is the Fréchet derivative of the function with respect to ζ, the variables $\overline{\boldsymbol{u}}_i$ ($i\in\{1,\dots,N\}$) being fixed. The formula is explicit but not very nice, in particular if η_i depends on ζ for $i\in\{1,\dots,N\}$. A typical example is

$$\eta_i(t,\boldsymbol{x}) \stackrel{\rm def}{=} -1+\beta b(\boldsymbol{x}) + \ell_i(1+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x})), \qquad 0=\ell_0<\ell_1<\dots<\ell_N=1.$$

To summarize, we consider

$$\begin{cases} \partial_t\zeta - \delta_\psi\mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi)=0,\\ \partial_t\psi + \delta_\zeta\mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi)=0. \end{cases} \tag{12.17}$$

where $\delta_\psi\mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi)$, $\delta_\zeta\mathscr{H}_{\rm mGN}^{(\eta_0,\dots,\eta_N)}(\zeta,\psi)$ are given by eqs. (12.15) and (12.16) and where $(\overline{\boldsymbol{u}}_i)_{i\in\{1,\dots,N\}}\in(X^0_\mu)^N$ are solutions to eq. (12.14). Using physical variables (recall Section 2.4), eq. (12.17) yields the ***"multilayer" Green–Naghdi systems***

$$\begin{cases} \partial_t\zeta + \nabla\cdot\Big(\sum_{i=1}^N h_i\overline{\boldsymbol{u}}_i\Big)=0,\\ \partial_t\psi + g\zeta + \sum_{i=1}^N\frac{\partial h_i}{\partial\zeta}\big(\overline{\boldsymbol{u}}_i\cdot\nabla\psi - \frac12|\overline{\boldsymbol{u}}_i|^2\big) - \frac12\sum_{i=1}^N\mathcal{Q}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N)=0, \end{cases} \tag{12.18}$$

where $h_i=\eta_i-\eta_{i-1}$ with $-d+b(\boldsymbol{x})=\eta_0(\zeta(t,\boldsymbol{x}))<\eta_1(\zeta(t,\boldsymbol{x}))<\dots<\eta_N(\zeta(t,\boldsymbol{x}))=\zeta(t,\boldsymbol{x})$, $\mathcal{T}$, $\mathcal{I}_i$, and $\mathcal{Q}_i$ are defined above and $(\overline{\boldsymbol{u}}_i)_{i\in\{1,\dots,N\}}$ are the solutions to

$$\forall i\in\{1,\dots,N\},\qquad h_i\overline{\boldsymbol{u}}_i + h_i\mathcal{T}[h_i,\nabla\eta_{i-1}]\overline{\boldsymbol{u}}_i + \mathcal{I}_i[\eta_0,\dots,\eta_N](\overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_N) = h_i\nabla\psi.$$

v. More generally, we can associate a model to

$$Y=\Big\{U\in X_\mu(\Omega)\ :\ U(\boldsymbol{x},z)=\sum_{i=1}^N \boldsymbol{u}_i(\boldsymbol{x})\,\Psi_i(\boldsymbol{x},z),\ \boldsymbol{u}_i\in X^0_\mu\Big\}$$

for any given (linearly independent and sufficiently regular) choice of the vertical distribution, $\{\Psi_i\}_{i\in\{1,\dots,N\}}$, possibly depending on $(\varepsilon\zeta,\beta b)$. Solving eq. (12.5) yields

$$U_N=\sum_{i=1}^N\boldsymbol{u}_i(\boldsymbol{x})\,\Psi_i(\boldsymbol{x},z)$$

where $(\boldsymbol{u}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$ satisfies for any $(\boldsymbol{v}_i)_{i\in\{1,\dots,N\}} \in (X^0_\mu)^N$,

$$\sum_{i=1}^N\sum_{j=1}^N \int_{\mathbb{R}^d} a_{ij}\boldsymbol{u}_i\cdot\boldsymbol{v}_j + \mu\, b_{ij}(\nabla\cdot\boldsymbol{u}_i)(\nabla\cdot\boldsymbol{v}_j) \\ + \mu\,(\boldsymbol{c}_{ij}\cdot\boldsymbol{u}_i)(\nabla\cdot\boldsymbol{v}_j) + \mu(\nabla\cdot\boldsymbol{u}_i)(\boldsymbol{c}_{ji}\cdot\boldsymbol{v}_j) \\ + \mu\,\boldsymbol{u}_i\cdot M_{ij}\boldsymbol{v}_j\,\mathrm{d}\boldsymbol{x} = \sum_{j=1}^N\int_{\mathbb{R}^d} d_j(\nabla\psi)\cdot\boldsymbol{v}_j\,\mathrm{d}\boldsymbol{x},$$

with

$$\begin{aligned}
a_{ij} &= \int_{-1+\beta b}^{\varepsilon\zeta} \Psi_i(\cdot,z)\Psi_j(\cdot,z)\,\mathrm{d}z\ ,\\
b_{ij} &= \int_{-1+\beta b}^{\varepsilon\zeta}\left(\int_{-1+\beta b}^{z}\Psi_i(\cdot,z')\,\mathrm{d}z'\right)\left(\int_{-1+\beta b}^{z}\Psi_j(\cdot,z')\,\mathrm{d}z'\right)\mathrm{d}z\ ,\\
\boldsymbol{c}_{ij} &= \int_{-1+\beta b}^{\varepsilon\zeta}\left(\int_{-1+\beta b}^{z}\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z')\,\mathrm{d}z'\right)\left(\int_{-1+\beta b}^{z}\Psi_j(\cdot,z')\,\mathrm{d}z'\right)\mathrm{d}z\ ,\\
M_{ij} &= \int_{-1+\beta b}^{\varepsilon\zeta}\left(\int_{-1+\beta b}^{z}\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z')\,\mathrm{d}z'\right)\left(\int_{-1+\beta b}^{z}\nabla_{\boldsymbol{x}}^\top\Psi_j(\cdot,z')\,\mathrm{d}z'\right)\mathrm{d}z\ ,\\
d_j &= \int_{-1+\beta b}^{\varepsilon\zeta}\Psi_j(\cdot,z)\,\mathrm{d}z\ .
\end{aligned}$$

We infer the system of linear differential equations

$$\forall j\in\{1,\dots,N\},\qquad \sum_{i=1}^N \mathcal{L}^\mu_{ji}\boldsymbol{u}_i = d_j\nabla\psi \tag{12.19}$$

where for any $i,j\in\{1,\dots,N\}$,

$$\mathcal{L}^\mu_{ji}\boldsymbol{u}_i \stackrel{\text{def}}{=} a_{ji}\boldsymbol{u}_i - \mu\nabla\left(b_{ji}(\nabla\cdot\boldsymbol{u}_i)\right) - \mu\nabla(\boldsymbol{c}_{ij}\cdot\boldsymbol{u}_i) + \mu(\nabla\cdot\boldsymbol{u}_i)\boldsymbol{c}_{ji} + \mu M_{ji}\boldsymbol{u}_i.$$

We put

$$\mathscr{H}^{(\Psi_1,\dots,\Psi_N)}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 + \sum_{i=1}^N d_i\boldsymbol{u}_i\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}$$

and consider

$$\begin{cases} \partial_t\zeta - \delta_\psi\mathscr{H}^{(\Psi_1,\dots,\Psi_N)} = 0,\\ \partial_t\psi + \delta_\zeta\mathscr{H}^{(\Psi_1,\dots,\Psi_N)} = 0.\end{cases} \tag{12.20}$$

with

$$\delta_\psi\mathscr{H}^{(\Psi_1,\dots,\Psi_N)}(\zeta,\psi) = -\nabla\cdot(d_i\boldsymbol{u}_i)\,, \tag{12.21}$$

$$\delta_\zeta\mathscr{H}^{(\Psi_1,\dots,\Psi_N)}(\zeta,\psi) = \zeta + \varepsilon\sum_{i=1}^N \tfrac{\partial d_i}{\varepsilon\partial\zeta}\boldsymbol{u}_i\cdot\nabla\psi - \frac{\varepsilon}{2}\sum_{i=1}^N \mathcal{Q}^\mu_i(\boldsymbol{u}_1,\dots,\boldsymbol{u}_N), \tag{12.22}$$

where $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)$ is determined by eq. (12.19) and $\mathcal{Q}^\mu_i(\boldsymbol{u}_1,\dots,\boldsymbol{u}_N)$ is defined by

$$\int_{\mathbb{R}^d}\boldsymbol{u}_i\cdot\sum_{j=1}^N D_\zeta\big(\mathcal{L}^\mu_{ij}\boldsymbol{u}_j\big)(\eta)\,\mathrm{d}\boldsymbol{x} = \varepsilon\int_{\mathbb{R}^d}\eta\mathcal{Q}^\mu_i(\boldsymbol{u}_1,\dots,\boldsymbol{u}_N)\,\mathrm{d}\boldsymbol{x},$$

where $\eta\mapsto D_\zeta\big(\mathcal{L}^\mu_{ij}\boldsymbol{u}_j\big)(\eta)$ is the Fréchet derivative of $\mathcal{L}^\mu_{ij}\boldsymbol{u}_j$ with respect to ζ, the variable $\boldsymbol{u}_j$ being fixed. Of course the formulae depend heavily on the choice of the vertical distribution, $\{\Psi_i\}_{i\in\{1,\dots,N\}}$, and its dependency with respect to $\varepsilon\zeta$.

12.2 Modal analysis

In this section we study the dispersion relation associated to some augmented and "multilayer" Green–Naghdi systems, introduced in Item iii and Item iv above. The findings are illustrated in Figure 12.1. Recall that the water waves system, when linearized about the rest state solution and in the flat bottom situation reads (see Section 2.3)

$$\begin{cases} \partial_t\zeta^0 + \frac{1}{\sqrt{\mu}}|D|\tanh(\sqrt{\mu}|D|)\psi^0 = 0, \\ \partial_t\psi^0 + \zeta^0 = 0, \end{cases}$$

which yields the dispersion relation

$$\omega_{\rm ww}(\boldsymbol{\xi})^2 = \tfrac{1}{\sqrt{\mu}}|\boldsymbol{\xi}|\tanh(\sqrt{\mu}|\boldsymbol{\xi}|).$$

The corresponding linearized augmented Green–Naghdi system reads

$$\begin{cases} \partial_t\zeta^0 + \sum_{i=1}^N \frac{1}{p_i+1}\nabla\cdot\boldsymbol{u}_i^0 = 0, \\ \partial_t\psi^0 + \zeta^0 = 0, \end{cases}$$

where $(\boldsymbol{u}_i^0)_{i\in\{1,\dots,N\}}$ is determined by solving the system

$$\forall j\in\{1,\dots,N\},\qquad \sum_{i=1}^N \tfrac{1}{p_i+p_j+1}\boldsymbol{u}_i^0 - \mu\tfrac{1}{(p_i+1)(p_j+1)(p_i+p_j+3)}\nabla(\nabla\cdot\boldsymbol{u}_i^0) = \tfrac{1}{p_j+1}\nabla\psi.$$

The linearized "multilayer" Green–Naghdi system with $\eta_i = -1+\beta b+\ell_i(1+\varepsilon\zeta-\beta b)$ for $0=\ell_0<\ell_1<\dots<\ell_N=1$ read

$$\begin{cases} \partial_t\zeta^0 + \sum_{i=1}^N(\ell_i-\ell_{i-1})\nabla\cdot\overline{\boldsymbol{u}}_i^0 = 0, \\ \partial_t\psi^0 + \zeta^0 = 0, \end{cases}$$

where $(\overline{\boldsymbol{u}}_i^0)_{i\in\{1,\dots,N\}}$ is determined by solving the system

$$\forall i\in\{1,\dots,N\},\qquad \overline{\boldsymbol{u}}_i^0 - \tfrac{\mu}{3}(\ell_i-\ell_{i-1})^2\nabla(\nabla\cdot\overline{\boldsymbol{u}}_i^0) - \tfrac{\mu}{2}\textstyle\sum_{j=1}^{i-1}(\ell_i-\ell_{i-1})(\ell_j-\ell_{j-1})\nabla(\nabla\cdot\overline{\boldsymbol{u}}_j^0)$$
$$- \tfrac{\mu}{2}\textstyle\sum_{j=i+1}^N(\ell_j-\ell_{j-1})^2\nabla(\nabla\cdot\overline{\boldsymbol{u}}_j^0) - \mu\sum_{k=i+1}^N\sum_{j=1}^{k-1}(\ell_k-\ell_{k-1})(\ell_j-\ell_{j-1})\nabla(\nabla\cdot\overline{\boldsymbol{u}}_j^0) = \nabla\psi.$$

When choosing "layers" with equal depth, $\ell_i = i/N$ for $i\in\{0,\dots,N\}$, the above reads simply

$$\forall i\in\{1,\dots,N\},\qquad \overline{\boldsymbol{u}}_i^0 - \tfrac{\mu}{3N^2}\nabla(\nabla\cdot\overline{\boldsymbol{u}}_i^0) - \tfrac{\mu}{2N^2}\textstyle\sum_{j\neq i}\nabla(\nabla\cdot\overline{\boldsymbol{u}}_j^0) - \mu\sum_{j=1}^N\frac{N-\max(\{i,j\})}{N^2}\nabla(\nabla\cdot\overline{\boldsymbol{u}}_j^0) = \nabla\psi.$$

The augmented Green–Naghdi model Recall that for $N=1$ and $p_1=0$, the model presented in Item iii (in fact in Item ii) corresponds to the Green–Naghdi equations. Accordingly, $\boldsymbol{u}_1^0 = \big(\mathrm{Id}-\frac{\mu}{3}\nabla\nabla\cdot\big)^{-1}\nabla\psi = \nabla(\mathrm{Id}-\frac{\mu}{3}\Delta)^{-1}\psi$, and the dispersion relation (see Section 8.3) is

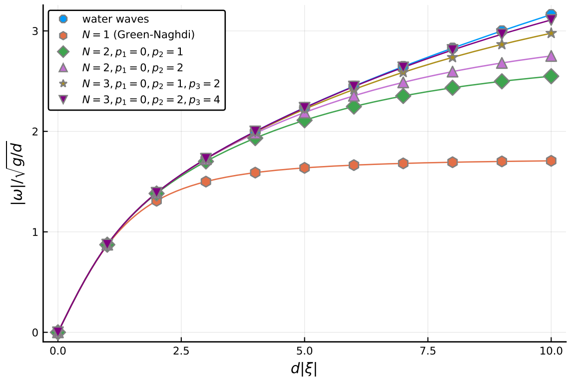
$$\omega_{\rm GN}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2\frac{1}{1+\frac{\mu}{3}|\boldsymbol{\xi}|^2}.$$

When $N=2$, and $(p_1,p_2)=(0,1)$, we find

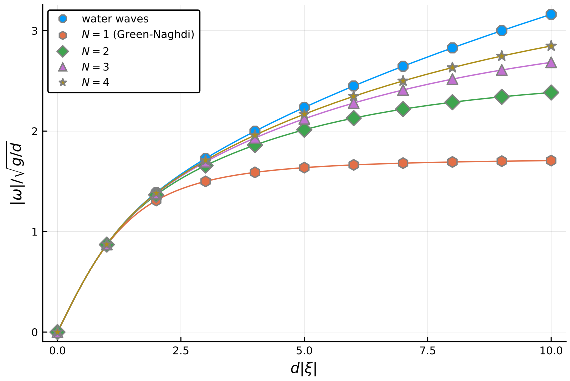
$$\boldsymbol{u}_1^0 = \nabla\frac{240+36\mu\Delta}{240-104\mu\Delta+3\mu^2\Delta^2}\psi,\qquad \boldsymbol{u}_2^0 = \nabla\frac{-120\mu\Delta}{240-104\mu\Delta+3\mu^2\Delta^2}\psi,$$

and the corresponding dispersion relation

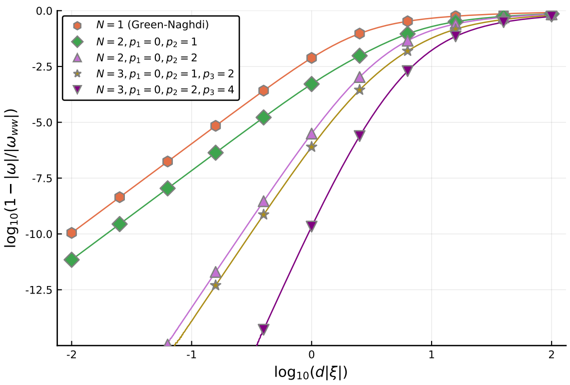
$$\omega_{\rm aGN}^{(0,1)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2\frac{240+24\mu|\boldsymbol{\xi}|^2}{240+104\mu|\boldsymbol{\xi}|^2+3\mu^2|\boldsymbol{\xi}|^4}.$$



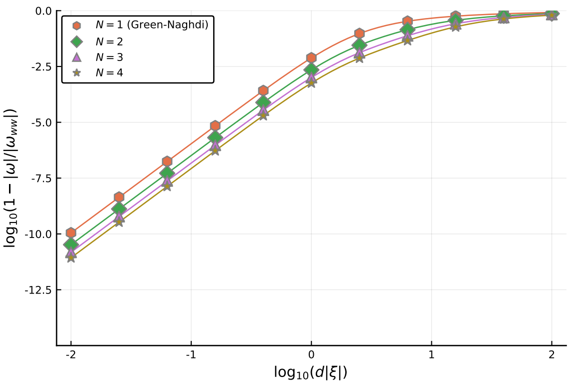
(a) Augmented Green–Naghdi systems.



(b) “Multilayer” Green–Naghdi systems.



(c) Same as (a), in log scale.



(d) Same as (b), in log scale.

Figure 12.1: In (a) and (b), wave frequencies, $|\omega|(|\boldsymbol{\xi}|)$, given by the dispersion relations corresponding to the (linearized about rest) augmented and “multilayer” Green–Naghdi models, respectively. In (c) and (d), the “error” is represented in log scale.

When $N=2$, and $(p_1,p_2)=(0,2)$, we find

$$\boldsymbol{u}_1^0=\nabla\frac{210+15\mu\Delta}{210-90\mu\Delta+2\mu^2\Delta^2}\psi,\qquad \boldsymbol{u}_2^0=\nabla\frac{-105\mu\Delta}{210-90\mu\Delta+2\mu^2\Delta^2}\psi,$$

and the corresponding dispersion relation

$$\omega_{\mathrm{aGN}}^{(0,1)}(\boldsymbol{\xi})^2=|\boldsymbol{\xi}|^2\frac{105+10\mu|\boldsymbol{\xi}|^2}{105+45\mu|\boldsymbol{\xi}|^2+\mu^2|\boldsymbol{\xi}|^4}.$$

Let us observe the small wavenumber Taylor series

$$\begin{aligned}
\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\big(1-\frac{1}{3}\mu|\boldsymbol{\xi}|^2+\frac{2}{15}\mu^2|\boldsymbol{\xi}|^4-\frac{17}{315}\mu^3|\boldsymbol{\xi}|^6+\frac{62}{2835}\mu^4|\boldsymbol{\xi}|^8+\mathcal{O}(\mu^5|\boldsymbol{\xi}|^{10})\big),\\
\omega_{\mathrm{GN}}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\big(1-\frac{1}{3}\mu|\boldsymbol{\xi}|^2+\frac{1}{9}\mu^2|\boldsymbol{\xi}|^4+\mathcal{O}(\mu^6|\boldsymbol{\xi}|^6)\big),\\
\omega_{\mathrm{aGN}}^{(0,1)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\big(1-\frac{1}{3}\mu|\boldsymbol{\xi}|^2+\frac{19}{144}\mu^2|\boldsymbol{\xi}|^4+\mathcal{O}(\mu^6|\boldsymbol{\xi}|^6)\big),\\
\omega_{\mathrm{aGN}}^{(0,2)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\big(1-\frac{1}{3}\mu|\boldsymbol{\xi}|^2+\frac{2}{15}\mu^2|\boldsymbol{\xi}|^4-\frac{17}{315}\mu^3|\boldsymbol{\xi}|^6+\frac{241}{11025}\mu^4|\boldsymbol{\xi}|^8+\mathcal{O}(\mu^5|\boldsymbol{\xi}|^{10})\big).
\end{aligned}$$

Hence we see that augmenting the rank of the model does not necessarily improve the small wavenumber behavior of the model by an order of magnitude—yet observe that $\frac{19}{144}=0.132$ is closer to $\frac{2}{15}\approx 0.133$ than $\frac{1}{9}\approx 0.111$. The approximation produced by the model with $N=2$, and $(p_0,p_1)=(0,2)$ is excellent. In fact $\omega_{\mathrm{aGN}}^{(0,2)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(4,4)$ to $\omega_{\mathrm{ww}}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$, so it is in some sense the best possible approximation with polynomials of such degrees.

When $N=3$, and $(p_1,p_2,p_3)=(0,1,2)$, we find the dispersion relation

$$\begin{aligned}
\omega_{\mathrm{aGN}}^{(0,1,2)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{6300+780\mu|\boldsymbol{\xi}|^2+15\mu^2|\boldsymbol{\xi}|^4}{6300+2880\mu|\boldsymbol{\xi}|^2+135\mu^2|\boldsymbol{\xi}|^4+\mu^3|\boldsymbol{\xi}|^6}\\
&=\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2-|\boldsymbol{\xi}|^2\big(\frac{1}{396900}\mu^4|\boldsymbol{\xi}|^8+\mathcal{O}(\mu^5|\boldsymbol{\xi}|^{10})\big).
\end{aligned}$$

When $N=3$, and $(p_1,p_2,p_3)=(0,2,4)$, we find the dispersion relation

$$\begin{aligned}
\omega_{\mathrm{aGN}}^{(0,2,4)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{10395+1260\mu|\boldsymbol{\xi}|^2+21\mu^2|\boldsymbol{\xi}|^4}{10395+4725\mu|\boldsymbol{\xi}|^2+210\mu^2|\boldsymbol{\xi}|^4+\mu^3|\boldsymbol{\xi}|^6}\\
&=\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2-|\boldsymbol{\xi}|^2\big(\frac{1}{1404728325}\mu^6|\boldsymbol{\xi}|^{12}+\mathcal{O}(\mu^7|\boldsymbol{\xi}|^{14})\big).
\end{aligned}$$

We see that the precision of $\omega_{\mathrm{aGN}}^{(0,1,2)}(\boldsymbol{\xi})^2$ is in par with the precision of $\omega_{\mathrm{aGN}}^{(0,2)}(\boldsymbol{\xi})^2$ in order of magnitude, although $\frac{62}{2835}-\frac{241}{11025}\approx 1.0\,10^{-5}$ and $\frac{1}{396900}\approx 2.5\,10^{-6}$. $\omega_{\mathrm{aGN}}^{(0,2,4)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(6,6)$ to $\omega_{\mathrm{ww}}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$, which motivates the following conjecture.

Conjecture 12.5. *For any $N\in\mathbb{N}^\star$ and setting $p_i=2(i-1)$ for $i\in\{1,\dots,N\}$, $\omega_{\mathrm{aGN}}^{(p_1,\dots,p_N)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(2N,2N)$ to $\omega_{\mathrm{ww}}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$. In particular [4, 4.5.70]*

$$\frac{\omega_{\mathrm{aGN}}^{(p_1,\dots,p_N)}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}=\cfrac{1}{1+\cfrac{\mu|\boldsymbol{\xi}|^2}{3+\cfrac{\mu|\boldsymbol{\xi}|^2}{5+\cfrac{\mu|\boldsymbol{\xi}|^2}{\ddots+\cfrac{\mu|\boldsymbol{\xi}|^2}{4N-1}}}}}\to\frac{\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\quad(N\to\infty)$$

and $0\leq\frac{\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2-\omega_{\mathrm{aGN}}^{(p_1,\dots,p_N)}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\leq C_N\mu^{2N}|\boldsymbol{\xi}|^{4N}$, where C_N depends uniquely on N.

Remark 12.6. *By this conjecture we do not mean that setting $p_i=2(i-1)$ is necessarily the best choice in the nonlinear framework, and in particular in presence of a non-trivial bottom topography. The interested reader should refer to Section 13, specifically Section 13.6.*

The "multilayer" Green–Naghdi model Recall that for $N = 1$, the model presented in Item iv corresponds to the Green–Naghdi equations. Accordingly,

$$\overline{\boldsymbol{u}}_1^0 = \big(\mathrm{Id} - \frac{\mu}{3}\nabla\nabla\cdot\big)^{-1}\nabla\psi = \nabla(\mathrm{Id} - \frac{\mu}{3}\Delta)^{-1}\psi$$

and the corresponding dispersion relation is

$$\omega_{\rm GN}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{1}{1 + \frac{\mu}{3}|\boldsymbol{\xi}|^2}.$$

When $N = 2$, solving the differential equation and denoting $h_1 = \ell_1 - \ell_0 = \ell_1$ and $h_2 = \ell_2 - \ell_1 = 1 - \ell_1$ yields

$$\overline{\boldsymbol{u}}_1^0 = \nabla \frac{36 + \mu 6 h_2^2 \Delta}{36 - \mu(12h_1^2 + 12h_2^2 + 36h_1h_2)\Delta + \mu(4h_1^2h_2^2 + 3h_1h_2^3)\Delta^2}\psi,$$
$$\overline{\boldsymbol{u}}_2^0 = \nabla \frac{36 - \mu 12 h_1^2 \Delta - \mu 18 h_1 h_2 \Delta}{36 - \mu(12h_1^2 + 12h_2^2 + 36h_1h_2)\Delta + \mu(4h_1^2h_2^2 + 3h_1h_2^3)\Delta^2}\psi,$$

and as a consequence the dispersion relation

$$\omega_{\rm mGN}^{(\ell_0,\ell_1,\ell_2)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{36 + \mu 12 h_1 h_2 |\boldsymbol{\xi}|^2}{36 + \mu(12h_1^2 + 12h_2^2 + 36h_1h_2)|\boldsymbol{\xi}|^2 + \mu(4h_1^2h_2^2 + 3h_1h_2^3)|\boldsymbol{\xi}|^4}.$$

When $h_1 = h_2 = 1/2$, *i.e.* $\ell_1 = 1/2$, the above reduces to

$$\omega_{\rm mGN}^{(0,1/2,1)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{576 + 48\mu|\boldsymbol{\xi}|^2}{576 + 240\mu|\boldsymbol{\xi}|^2 + 7\mu^2|\boldsymbol{\xi}|^4} = |\boldsymbol{\xi}|^2\big(1 - \frac{1}{3}\mu|\boldsymbol{\xi}|^2 + \frac{73}{576}\mu^2|\boldsymbol{\xi}|^4 + \mathcal{O}(\mu^3|\boldsymbol{\xi}|^6)\big).$$

Hence we see that augmenting the rank of the model from $N = 1$ to $N = 2$ does not improve the small wavenumber behavior of the model by an order of magnitude. It is possible to tailor the choice of η_1 to improve the small wavenumber behavior, yet again not by an order of magnitude. The optimal choice—even allowing values $\ell_1 < 0$ or $\ell_1 > 1$—is $\ell_1 = \frac{1+\sqrt{17}}{8}$ for which the prefactor $\frac{73}{576} \approx 0.127$ in the Taylor expansion is replaced by $\frac{2155}{18432} + \frac{17}{6144}\sqrt{17} \approx 0.128$ which is only slightly closer to the desired $\frac{2}{15} \approx 0.133$.

In the following, we always set $\ell_i = i/N$ for $i \in \{1, \dots, N\}$. For $N \in \{3, 4\}$, we find

$$\begin{aligned}
\omega_{\rm mGN}^{(0,\frac13,\frac23,1)}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \frac{78732 + 8748\mu|\boldsymbol{\xi}|^2 + 135\mu^2|\boldsymbol{\xi}|^4}{78732 + 34992\mu|\boldsymbol{\xi}|^2 + 1539\mu^2|\boldsymbol{\xi}|^4 + 13\mu^3|\boldsymbol{\xi}|^6}\\
&= |\boldsymbol{\xi}|^2\big(1 - \frac{1}{3}\mu|\boldsymbol{\xi}|^2 + \frac{95}{729}\mu^2|\boldsymbol{\xi}|^4 + \mathcal{O}(\mu^3|\boldsymbol{\xi}|^6)\big),\\
\omega_{\rm mGN}^{(0,\frac14,\frac12,\frac34,1)}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \frac{84934656 + 10616832\mu|\boldsymbol{\xi}|^2 + 248832\mu^2|\boldsymbol{\xi}|^4 + 1344\mu^3|\boldsymbol{\xi}|^6}{84934656 + 38928384\mu|\boldsymbol{\xi}|^2 + 2045952\mu^2|\boldsymbol{\xi}|^4 + 27264\mu^3|\boldsymbol{\xi}|^6 + 97\mu^4|\boldsymbol{\xi}|^8}\\
&= |\boldsymbol{\xi}|^2\big(1 - \frac{1}{3}\mu|\boldsymbol{\xi}|^2 + \frac{1213}{9216}\mu^2|\boldsymbol{\xi}|^4 + \mathcal{O}(\mu^3|\boldsymbol{\xi}|^6)\big).
\end{aligned}$$

These values coincide with the ones found in [184], where convergence towards the dispersion relation of the water waves system is proved. Once again we observe that augmenting the rank of the model does not improve the small wavenumber behavior of the model by an order of magnitude with respect to the Green–Naghdi equation, but improves the prefactors in the Taylor expansion:

$$\frac{2}{15} - \frac{1}{9} \approx 2.2\,10^{-1}, \quad \frac{2}{15} - \frac{73}{576} \approx 6.5\,10^{-3}, \quad \frac{2}{15} - \frac{95}{729} \approx 3.0\,10^{-3}, \quad \frac{2}{15} - \frac{1213}{9216} \approx 1.7\,10^{-3}.$$

We observe a similar behavior in the next order coefficient in the Taylor series, which motivates the following conjecture.

Conjecture 12.7. *Setting $\ell_i = i/N$ for $i \in \{0, \dots, N\}$, one has for any $\boldsymbol{\xi} \in \mathbb{R}^d$,*

$$0 \leq \frac{\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2 - \omega_{\mathrm{mGN}}^{(\ell_0,\ell_1,\dots,\ell_N)}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2} \times N^2 \; = \; \mathcal{O}(\mu^2|\boldsymbol{\xi}|^4),$$

uniformly with respect to $N \in \mathbb{N}^\star$.

Remark 12.8. *While this conjecture appears less impressive than the corresponding one for the augmented Green–Naghdi model, this does not mean that the "multilayer" model cannot be relevant in the nonlinear framework, in particular in the presence of small wavelengths.*

12.3 Discussion and open questions

In this section we have merely sketched a way to derive in a systematic way families of models for the water waves equations. All the models obtained by this procedure (except trivial ones) have in common the fact that a system of linear differential equations of order two, playing the role of the Laplace problem, needs to be solved.

It is likely that some of the models, or at least closely resembling ones, have already appeared in the literature, in particular in the works reviewed in [294]. In particular, the works [292, 293] can be seen as a starting point—at least in this community—of high order shallow water models with low order differential operators, and a "multilayer" approach was introduced in [291, 290]. My motivation to study such models was triggered by the "multilayer" approach presented in [184]. Another closely related "multilayer" approach is also presented in [269, §3.6], together with a discussion and earlier references. To my knowledge it is the first time that the two types of models are presented in the same framework, and that their inherent variational structure (for potential flows) is brought to light.

A strategy for the numerical discretization of the "multilayer" model has been proposed in [372]. Yet a thorough investigation of the "multilayer" approach, balancing improved accuracy and computational cost of increasing the number of layers, is yet to be accomplished.

There are, to my knowledge, no rigorous results concerning the models presented in this section. In particular, motivated by Remark 12.3 and the modal analysis in Section 12.2, it is natural to ask whether the models obtained by our procedure can approach solutions to the water waves system with an arbitrary precision (augmenting the rank of the model, N, and hence the size of the inherent system of differential equations), and in this case to characterize the rate of convergence.

The existence (and properties) of solitary waves, or the ability to reproduce phenomena featuring small scales are also completely open. It would also be interesting to extend the models outside of the irrotational and/or homogeneous framework. All these questions are obviously way outside of the scope of the present document.

Let me conclude in a positive note: the Isobe–Kakinuma model, which presents the same features as the the ones described in this section and is discussed in the following one, do enjoy a rigorous analysis thanks to the work of Iguchi and collaborators.

13 The variational method and Isobe–Kakinuma systems

In this section we discuss a procedure to derive asymptotic models for the water waves system, using strongly the variational structure of the latter. The name "variational method" can be argued since in the previous section the procedure was already based on variational methods (see Section 12.1.2) first when using the Galerkin method of dimension reduction to a variational formulation of the Laplace problem for the velocity potential, and then when deriving the models as canonical Hamiltonian equations. Here we shall use the variational structure of the water waves system in a single step, using Luke's variational formulation [289] (see Section 2.2). In Section 13.3 we show that the method studied in this section can in fact be interpreted as being exactly the method studied the previous section, although using another variational formulation of the Laplace problem.

13.1 Derivation by Luke's Lagrangian

Let us recall Luke's variational formulation of the water waves system, eq. (2.2), as displayed in Section 2.2. Using dimensionless variables, eq. (2.2) can be interpreted as Hamilton's principle

$$\delta\mathscr{L} = 0 \tag{13.1}$$

where the Lagrangian action is

$$\mathscr{L} \stackrel{\text{def}}{=} \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \frac{\varepsilon}{2}\zeta^2 + \Big(\int_{-1+\beta b}^{\varepsilon\zeta} \partial_t\Phi + \frac{\varepsilon}{2\mu}(\partial_z\Phi)^2 + \frac{\varepsilon}{2}|\nabla_{\boldsymbol{x}}\Phi|^2\,\mathrm{d}z\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{13.2}$$

The variational procedure consists simply in replacing the velocity potential, Φ, by

$$\Phi^{\text{app}}_{\text{IK}}(t,\boldsymbol{x},z) \stackrel{\text{def}}{=} \sum_{i=0}^{N}\Psi_i(\boldsymbol{x},z)\phi_i(t,\boldsymbol{x}) \tag{13.3}$$

where $\{\Psi_i\}_{i=0,1,\dots,N}$ is a given—*i.e.* chosen by the designer and characterizing the resulting model—family of functions independent of time, and $\{\phi_i\}_{i=0,1,\dots,N}$ are unknowns functions independent the variable, z. We shall refer to the family $\{\Psi_i\}_{i=0,1,\dots,N}$ as the *vertical distribution.*

When replacing Φ with Φ^{app} in eq. (13.2) yields

$$\mathscr{L}^{\text{app}}_{\text{IK}} \stackrel{\text{def}}{=} \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \ell^{\text{app}}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{13.4}$$

where

$$\begin{aligned}\ell^{\text{app}}_{\text{IK}} \stackrel{\text{def}}{=} \frac{\varepsilon}{2}\zeta^2 + \sum_{i=0}^{N}(\partial_t\phi_i)\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_i(\cdot,z)\,\mathrm{d}z + \frac{\varepsilon}{2\mu}\sum_{i=0}^{N}\sum_{j=0}^{N}\phi_i\phi_j\int_{-1+\beta b}^{\varepsilon\zeta}\big((\partial_z\Psi_i(\cdot,z))(\partial_z\Psi_j)\big)(\cdot,z)\,\mathrm{d}z\\ + \frac{\varepsilon}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\Big(\phi_i\phi_j\int_{-1+\beta b}^{\varepsilon\zeta}\big((\nabla_{\boldsymbol{x}}\Psi_i)\cdot(\nabla_{\boldsymbol{x}}\Psi_j)\big)(\cdot,z)\,\mathrm{d}z + \phi_i(\nabla\phi_j)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_j\nabla_{\boldsymbol{x}}\Psi_i\big)(\cdot,z)\,\mathrm{d}z\\ + \phi_j(\nabla\phi_i)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\nabla_{\boldsymbol{x}}\Psi_j\big)(\cdot,z)\,\mathrm{d}z + (\nabla\phi_i)\cdot(\nabla\phi_j)\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\Psi_j\big)(\cdot,z)\,\mathrm{d}z\Big).\end{aligned}$$

Euler–Lagrange equations on eq. (13.4) yield[63]

$$\forall i\in\{0,1,\dots,N\},\qquad 0=\delta_{\varepsilon\phi_i}\mathscr{L}_{\rm IK}^{\rm app}=-\partial_t\zeta\Psi_i\big|_{z=\varepsilon\zeta}+\frac{1}{\mu}\sum_{j=0}^N\phi_j\int_{-1+\beta b}^{\varepsilon\zeta}\big((\partial_z\Psi_i)(\partial_z\Psi_j)\big)(\cdot,z)\,{\rm d}z$$
$$+\sum_{j=0}^N\left(\phi_j\int_{-1+\beta b}^{\varepsilon\zeta}\big((\nabla_{\boldsymbol{x}}\Psi_i)\cdot(\nabla_{\boldsymbol{x}}\Psi_j)\big)(\cdot,z)\,{\rm d}z+(\nabla\phi_j)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_j\nabla_{\boldsymbol{x}}\Psi_i\big)(\cdot,z)\,{\rm d}z\right)$$
$$-\sum_{j=0}^N\nabla\cdot\left(\phi_j\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\nabla_{\boldsymbol{x}}\Psi_j\big)(\cdot,z)\,{\rm d}z+(\nabla\phi_j)\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\Psi_j\big)(\cdot,z)\,{\rm d}z\right) \tag{13.5}$$

and

$$0=\delta_{\varepsilon\zeta}\mathscr{L}_{\rm IK}^{\rm app}=\sum_{i=0}^N(\partial_t\phi_i)\Psi_i\big|_{z=\varepsilon\zeta}+\zeta+\frac{\varepsilon}{2\mu}\left(\sum_{j=0}^N\phi_j(\partial_z\Psi_j)\big|_{z=\varepsilon\zeta}\right)^2$$
$$+\frac{\varepsilon}{2}\left|\sum_{j=0}^N(\nabla\phi_j)\Psi_j\big|_{z=\varepsilon\zeta}+\phi_j(\nabla_{\boldsymbol{x}}\Psi_j)\big|_{z=\varepsilon\zeta}\right|^2. \tag{13.6}$$

Equation (13.5)–(13.6) can be viewed as a system of $(N+2)$ differential equations for the unknowns $(\zeta,\phi_0,\phi_1,\dots,\phi_N)$, and define our models.

Remark 13.1. *Notice that the system consists in $(N+1)$ evolution equations for ζ and only one evolution equation for $(\phi_0,\phi_1,\dots,\phi_N)$, so that it is an overdetermined/underdetermined composite system. Its structure is hence very different from all the systems discussed so far—and in particular the water waves system, eq. (2.7)—but we will show in Section 13.2 how a standard system of two evolution equations (under canonical Hamiltonian form) can be recovered from the above.*

Based on the Boussinesq–Rayleigh expansion, eq. (11.1), it is natural to set $\{\Psi_i\}_{i=0,1,\dots,N}$ as

$$\forall i\in\{0,1,\dots,N\},\qquad \Psi_i(\boldsymbol{x},z)=\big(z+1-\beta b(\boldsymbol{x})\big)^{p_i}$$

where $p_0,p_1,\dots,p_N$ are non-negative integers satisfying by convention $0=p_0<p_1<\dots<p_N$. In this case eq. (13.5)–(13.6) read

$$\left\{\begin{array}{l} h^{p_i}\partial_t\zeta+\sum_{j=0}^N\nabla\cdot\left(\frac{h^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_j-\frac{p_j}{p_i+p_j}h^{p_i+p_j}\phi_j(\beta\nabla b)\right)\\ \quad +\sum_{j=0}^N\frac{p_i}{p_i+p_j}h^{p_i+p_j}(\nabla\phi_j)\cdot(\beta\nabla b)-\sum_{j=0}^N\frac{p_ip_j}{p_i+p_j-1}h^{p_i+p_j-1}(\mu^{-1}+|\beta\nabla b|^2)\phi_j\;=\;0\\ \hfill\forall i\in\{0,1,\dots,N\},\\ \sum_{j=0}^N h^{p_j}(\partial_t\phi_j)+\zeta\\ \quad+\frac{\varepsilon}{2}\left(\left|\sum_{j=0}^N h^{p_j}(\nabla\phi_j)-p_jh^{p_j-1}\phi_j(\beta\nabla b)\right|^2+\mu^{-1}\Big(\sum_{j=0}^N p_jh^{p_j-1}\phi_j\Big)^2\right)\;=\;0,\end{array}\right. \tag{13.7}$$

where we denote as usual $h\overset{\rm def}{=}1+\varepsilon\zeta-\beta b$ the (non-dimensionalized) depth of the layer, and use the convention $\frac{0}{0}=0$.

[63] We assume here and henceforth that the bottom topography is time-independent.

Using physical variables (recall Section 2.4), eq. (13.7) reads

$$\begin{cases} h^{p_i}\partial_t\zeta + \sum_{j=0}^N \nabla\cdot\Big(\frac{h^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}h^{p_i+p_j}\phi_j\nabla b\Big) \\ \quad + \sum_{j=0}^N \frac{p_i}{p_i+p_j}h^{p_i+p_j}(\nabla\phi_j)\cdot(\nabla b) - \sum_{j=0}^N \frac{p_ip_j}{p_i+p_j-1}h^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N\}, \\ \sum_{i=0}^N h^{p_i}(\partial_t\phi_i) + g\zeta \\ \quad +\frac12\Big(\Big|\sum_{i=0}^N h^{p_i}(\nabla\phi_i) - p_ih^{p_i-1}\phi_i(\nabla b)\Big|^2 + \Big(\sum_{i=0}^N p_ih^{p_i-1}\phi_i\Big)^2\Big) \;=\; 0, \end{cases} \tag{13.8}$$

where $h(t,\boldsymbol{x}) \stackrel{\text{def}}{=} d+\zeta(t,\boldsymbol{x})-b(\boldsymbol{x})$. When $N=0$ (and $p_0=0$), the system coincides with the Saint-Venant system (Section 5). While the general equations (13.5)–(13.6) are displayed in [232, 233, 238, 239, 240], the above choice of vertical distribution is systematically put forward. This is why we refer to (13.8) as the ***Isobe–Kakinuma model***.

Remark 13.2 (Other choices of vertical distributions)**.** *We have implicitly assumed in eq. (13.3) that the vertical distribution, $\{\Psi_i\}_{i=0,1,\dots,N}$, does not depend on the unknown variable, ζ. This restriction can be removed and we can set, as was done by Klopman, van Groesen, and Dingemans in [261] (see also [355] and references therein)*

$$\Phi^{\rm app}(t,\boldsymbol{x},z) \stackrel{\text{def}}{=} \sum_{i=0}^N \Psi_i(\boldsymbol{x},z,\varepsilon\zeta(t,\boldsymbol{x}))\phi_i(t,\boldsymbol{x}) \tag{13.9}$$

but in this case the equations must be modified to take into account for

$$\nabla_{\boldsymbol{x}}\Phi^{\rm app} = \sum_{i=0}^N \Psi_i\nabla\phi_i + \phi_i\nabla_{\boldsymbol{x}}\Psi_i + \varepsilon\phi_i(\partial_\zeta\Psi_i)\nabla\zeta,$$

where the last term is new.

This extended framework is interesting as it allows to choose the vertical distribution such that $\Psi_i(\boldsymbol{x},z,\varepsilon\zeta)\big|_{z=\varepsilon\zeta} = 0$ for $i=1,\dots,N$ in which case the equations replacing eq. (13.5)–(13.6) consist in a system of $2N$ time-independent linear differential equations allowing to determine $(\phi_1,\dots,\phi_N)$ from the knowledge of (ζ,ϕ_0), and two evolution equations for (ζ,ϕ_0). Moreover, this system is readily under the canonical Hamiltonian form; see the discussion in Section 13.2.

For instance, one can set

$$\Psi_i(\boldsymbol{x},z,\varepsilon\zeta) = (z-\varepsilon\zeta)^{p_i}$$

with non-negative integers $0=p_0<p_1<\dots<p_N$.

Another natural choice consists in defining the vertical distribution from solutions to the vertical Sturm–Liouville eigenproblem

$$\partial_z^2\Psi_i + k_n^2\Psi_i = 0, \qquad -1+\beta b(\boldsymbol{x}) < z < \varepsilon\zeta(\boldsymbol{x})$$

with appropriate boundary conditions at $z=\varepsilon\zeta$ and $z=-1+\beta b$. Based on a tweak on this vertical distribution—proposed in [28]—Athanassoulis and Papoutsellis were able to prove in [29] the rigorous convergence of $\Phi^{\rm app}$ towards sufficiently regular Φ (with sufficiently regular (ζ,b)) for a vertical distribution constructed in this way. Moreover, using the boundary condition $\partial_z\Psi_i\big|_{z=\varepsilon\zeta}=0$ for $i\in\{1,\dots,N\}$, the Hamiltonian equations derived in Section 13.2 are much simplified; see [355].

Yet another natural choice of vertical distribution, based on the finite element method and the "multilayer" approach introduced in Section 12.1.3, is

$$\Psi_i(\boldsymbol{x},z) = \begin{cases} \frac{\eta_{i-1}(\boldsymbol{x})-z}{\eta_{i-1}(\boldsymbol{x})-\eta_i(\boldsymbol{x})} & \text{if } \eta_i(\boldsymbol{x})\le z<\eta_{i-1}(\boldsymbol{x}), \\ \frac{z-\eta_{i+1}(\boldsymbol{x})}{\eta_i(\boldsymbol{x})-\eta_{i+1}(\boldsymbol{x})} & \text{if } \eta_{i+1}(\boldsymbol{x})<z\le\eta_i(\boldsymbol{x}), \\ 0 & \text{otherwise,} \end{cases}$$

where $-1+\beta b(\boldsymbol{x}) = \eta_N(\boldsymbol{x}) < \dots < \eta_1(\boldsymbol{x}) < \eta_0(\boldsymbol{x}) = \varepsilon\zeta(\boldsymbol{x})$.

13.2 Reformulation of the equations and Hamiltonian structure

In this section we reformulate system eq. (13.5)–(13.6) as two (canonical Hamiltonian) evolution equations, coupled with a system of differential equations approximating the Laplace problem. This analysis is based on [161]; see also [355, § 4.3].

We first notice that—by construction—the system eq. (13.5)–(13.6) benefits from a Lagrangian structure which, based on the relation with Zakharov's Hamiltonian structure put forward by Miles [316], can be written as follows. Set

$$\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{\phi}) \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \frac{1}{2}\zeta^2 + \Big(\int_{-1+\beta b}^{\varepsilon\zeta} \frac{1}{2\mu}(\partial_z\Phi^{\rm app}_{\rm IK})^2 + \frac{1}{2}|\nabla_{\boldsymbol{x}}\Phi^{\rm app}_{\rm IK}|^2\,{\rm d}z\Big)\,{\rm d}\boldsymbol{x} \tag{13.10}$$

where $\Phi^{\rm app}$ is defined from $\boldsymbol{\phi} \stackrel{\rm def}{=} (\phi_0,\dots,\phi_N)^\top$ by eq. (13.3). Then eq. (13.5)–(13.6) reads

$$\begin{pmatrix} 0 & -\boldsymbol{l}^\top \\ \boldsymbol{l} & {\rm O}_{N+1}\end{pmatrix}\begin{pmatrix}\partial_t\zeta \\ \partial_t\boldsymbol{\phi}\end{pmatrix} = \begin{pmatrix}\delta_\zeta\mathscr{E}^{\rm app}_{\rm IK} \\ \delta_{\boldsymbol{\phi}}\mathscr{E}^{\rm app}_{\rm IK}\end{pmatrix}. \tag{13.11}$$

where ${\rm O}_{N+1}$ is the $(N+1)\times(N+1)$ null matrix, and

$$\boldsymbol{\phi} \stackrel{\rm def}{=} (\phi_0,\phi_1,\dots,\phi_N)^\top, \qquad \boldsymbol{l} = \big(\Psi_0\,\big|_{z=\varepsilon\zeta}, \Psi_1\,\big|_{z=\varepsilon\zeta}, \dots, \Psi_N\,\big|_{z=\varepsilon\zeta}\big)^\top.$$

Let us describe how the above Hamiltonian equations can be put in canonical form. Let us first introduce convenient notations: we define

$$\begin{aligned}\mathcal{L}^\mu_{ij}[\varepsilon\zeta,\beta b]\varphi \stackrel{\rm def}{=} &-\nabla\cdot\Big((\nabla\varphi)\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\Psi_j\big)(\cdot,z)\,{\rm d}z\Big) + \nabla\cdot\Big(\varphi\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_i\nabla_{\boldsymbol{x}}\Psi_j\big)(\cdot,z)\,{\rm d}z\Big)\\ &-(\nabla\varphi)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\big(\Psi_j\nabla_{\boldsymbol{x}}\Psi_i\big)(\cdot,z)\,{\rm d}z + \varphi\int_{-1+\beta b}^{\varepsilon\zeta}\big(\tfrac{1}{\mu}(\partial_z\Psi_i)(\partial_z\Psi_j)+(\nabla_{\boldsymbol{x}}\Psi_i)\cdot(\nabla_{\boldsymbol{x}}\Psi_j)\big)(\cdot,z)\,{\rm d}z.\end{aligned} \tag{13.12}$$

We have that $\mathcal{L}_{ij}$ is a differential operator of order two, and $(\mathcal{L}^\mu_{ij})^\star = \mathcal{L}^\mu_{ji}$, with $(\mathcal{L}^\mu_{ij})^\star$ the adjoint operator in $L^2(\mathbb{R}^d)$. We now consider for $\boldsymbol{\varphi} \stackrel{\rm def}{=} (\varphi_0,\varphi_1,\dots,\varphi_N)^\top$ the following system of equations satisfied by solutions to (13.5):

$$\begin{cases}\Psi_0\,\big|_{z=\varepsilon\zeta}\sum_{j=0}^N\mathcal{L}^\mu_{ij}\varphi_j = \Psi_i\,\big|_{z=\varepsilon\zeta}\sum_{j=0}^N\mathcal{L}^\mu_{0j}\varphi_j \qquad \forall i\in\{1,\dots,N\}\\ \boldsymbol{l}\bullet\boldsymbol{\varphi} = \psi,\end{cases} \tag{13.13}$$

where we use $\bullet$ to denote the $(N+1)$-dimensional inner product. Provided that the family of functions $\{\Psi_j\}_{j=0,\dots,N}$ are sufficiently "nice",[64] the above system can be inverted and there exists a linear operator

$$\boldsymbol{S}[\varepsilon\zeta,\beta b]:\psi\mapsto\boldsymbol{\varphi}$$

where $\boldsymbol{\varphi}$ is the unique solution to eq. (13.13). Finally, we define

$$\mathscr{H}^{\rm app}_{\rm IK}(\zeta,\psi) \stackrel{\rm def}{=} \mathscr{E}^{\rm app}_{\rm IK}(\eta,\boldsymbol{S}[\varepsilon\zeta,\beta]\psi). \tag{13.14}$$

Notice first that, differentiating the last equation in eq. (13.13), we have that $\eta\mapsto D_\zeta\boldsymbol{S}[\varepsilon\zeta,\beta b](\eta)$, the Fréchet derivative of $\boldsymbol{S}[\varepsilon\zeta,\beta b]$ with respect to ζ, satisfies

$$\boldsymbol{l}\bullet\big(D_\zeta\boldsymbol{S}[\varepsilon\zeta,\beta b](\eta)\psi\big) + \eta\boldsymbol{l}'\bullet\boldsymbol{\varphi} = 0, \tag{13.15}$$

[64] By "nice" it is meant regularity and linear independence. See [341, Proposition 1.3] for the rigorous result when $\Psi_i(\boldsymbol{x},z) = (z+1-\beta b(\boldsymbol{x}))^{p_i}$ with $0=p_0<p_1<\dots<p_N$.

where $\boldsymbol{l}' = \big(\partial_z\Psi_0\,\big|_{z=\varepsilon\zeta}\,, \partial_z\Psi_1\,\big|_{z=\varepsilon\zeta}\,, \dots, \partial_z\Psi_N\,\big|_{z=\varepsilon\zeta}\,\big)^\top$. Then we have for sufficiently smooth data, assuming $\Psi_0\,\big|_{z=\varepsilon\zeta}\neq 0$,

$$\begin{aligned}
D_\psi\mathscr{H}^{\rm app}_{\rm IK}(\zeta,\psi)\varphi &= D_{\boldsymbol{\phi}}\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\boldsymbol{S}[\varepsilon\zeta,\beta b]\varphi\\
&= \big(\ \boldsymbol{\mathcal{L}}^\mu\ \boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ \boldsymbol{S}[\varepsilon\zeta,\beta b]\varphi\ \big)_{(L^2)^{N+1}}\\
&= \big(\ (\Psi_0\,\big|_{z=\varepsilon\zeta}\,)^{-1}\boldsymbol{l}\boldsymbol{\mathcal{L}}^\mu_0\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ \boldsymbol{S}[\varepsilon\zeta,\beta b]\varphi\ \big)_{(L^2)^{N+1}}\\
&= \big((\Psi_0\,\big|_{z=\varepsilon\zeta}\,)^{-1}\boldsymbol{\mathcal{L}}^\mu_0\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ \varphi\ \big)_{L^2}
\end{aligned}$$

where we used eq. (13.11), eq. (13.13) and denote $\boldsymbol{\mathcal{L}}^\mu = (\mathcal{L}^\mu_{ij})_{i,j\in\{0,1,\dots,N\}}$ and $\boldsymbol{\mathcal{L}}^\mu_0 = (\mathcal{L}^\mu_{0j})_{j\in\{0,1,\dots,N\}}$;

$$\begin{aligned}
D_\zeta\mathscr{H}^{\rm app}_{\rm IK}(\zeta,\psi)\eta &= D_\zeta\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\eta + D_{\boldsymbol{\phi}}\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)D_\zeta\boldsymbol{S}[\varepsilon\zeta,\beta b](\eta)\psi\\
&= \big(\ \delta_\zeta\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\ ,\ \eta\ \big)_{L^2}\ +\ \big(\ \boldsymbol{\mathcal{L}}^\mu\ \boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ D_\zeta\boldsymbol{S}[\varepsilon\zeta,\beta b](\eta)\psi\ \big)_{(L^2)^{N+1}}\\
&= \big(\ \delta_\zeta\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\ ,\ \eta\ \big)_{L^2}\ +\ \big(\ (\Psi_0\,\big|_{z=\varepsilon\zeta}\,)^{-1}\boldsymbol{l}\boldsymbol{\mathcal{L}}^\mu_0\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ D_\zeta\boldsymbol{S}[\varepsilon\zeta,\beta b](\eta)\psi\ \big)_{(L^2)^{N+1}}\\
&= \big(\ \delta_\zeta\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\ ,\ \eta\ \big)_{L^2}\ -\ \big((\Psi_0\,\big|_{z=\varepsilon\zeta}\,)^{-1}\boldsymbol{\mathcal{L}}^\mu_0\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ ,\ \eta\boldsymbol{l}'\bullet\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi\ \big)_{L^2}
\end{aligned}$$

where we used eq. (13.15) and recall $\boldsymbol{l}' = \big(\partial_z\Psi_0\,\big|_{z=\varepsilon\zeta}\,, \partial_z\Psi_1\,\big|_{z=\varepsilon\zeta}\,, \dots, \partial_z\Psi_N\,\big|_{z=\varepsilon\zeta}\,\big)^\top$. Hence

$$\begin{aligned}
&(\Psi_0\,\big|_{z=\varepsilon\zeta}\,)\delta_\psi\mathscr{H}^{\rm app}_{\rm IK}(\zeta,\psi) = \boldsymbol{\mathcal{L}}^\mu_0\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi = \delta_{\phi_0}\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi),\\
&(\Psi_0\,\big|_{z=\varepsilon\zeta}\,)\delta_\zeta\mathscr{H}^{\rm app}_{\rm IK}(\zeta,\psi) = (\Psi_0\,\big|_{z=\varepsilon\zeta}\,)\delta_\zeta\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi) - (\boldsymbol{l}'\bullet\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi)\delta_{\phi_0}\mathscr{E}^{\rm app}_{\rm IK}(\zeta,\boldsymbol{S}[\varepsilon\zeta,\beta b]\psi).
\end{aligned}$$

From this and eq. (13.11) we infer immediately that $(\eta,\phi_0,\phi_1,\dots,\phi_N)$ sufficiently regular solutions to the system eq. (13.5)–(13.6) satisfy

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_\zeta\mathscr{H}^{\rm app}_{\rm IK}\\ \delta_\psi\mathscr{H}^{\rm app}_{\rm IK}\end{pmatrix} \tag{13.16}$$

where $\psi \stackrel{\rm def}{=} \sum_{j=0}^N \phi_j\Psi_j\,\big|_{z=\varepsilon\zeta}$. Conversely, given a (ζ,ψ) a sufficiently regular solution to eq. (13.16), and defining $(\phi_0,\phi_1,\dots,\phi_N)^\top \stackrel{\rm def}{=} \boldsymbol{S}[\varepsilon\zeta,\beta]\psi$ the unique solution to eq. (13.13), then $(\eta,\phi_0,\phi_1,\dots,\phi_N)$ satisfy eq. (13.5)–(13.6).

Remark 13.3. *This formal procedure to derive eq.* (13.16)*, the canonical Hamiltonian formulation to eq.* (13.5)*–*(13.6)*, can be made rigorous; see [161] in the case* $\Psi_i(\boldsymbol{x},z) = (z+1-\beta b(\boldsymbol{x}))^{p_i}$ *with* $0 = p_0 < p_1 < \dots < p_N$*. In this case, eq.* (13.16) *reads explicitly*

$$\left\{\begin{aligned}
&\partial_t\zeta + \textstyle\sum_{j=0}^N\nabla\cdot\Big(\frac{h^{p_j+1}}{p_j+1}\nabla\phi_j - h^{p_j}\phi_j(\beta\nabla b)\Big)\ =\ 0\,,\\
&\partial_t\psi + \zeta + \varepsilon\Big(\textstyle\sum_{i=0}^N p_i h^{p_i-1}\phi_i\Big)\Big(\sum_{j=0}^N\nabla\cdot\Big(\frac{h^{p_j+1}}{p_j+1}\nabla\phi_j - h^{p_j}\phi_j(\beta\nabla b)\Big)\Big)\\
&\qquad +\tfrac{\varepsilon}{2}\bigg(\Big|\textstyle\sum_{j=0}^N h^{p_j}(\nabla\phi_j) - p_j h^{p_j-1}\phi_j(\beta\nabla b)\Big|^2 + \mu^{-1}\Big(\sum_{j=0}^N p_j h^{p_j-1}\phi_j\Big)^2\bigg)\ =\ 0\,,
\end{aligned}\right. \tag{13.17}$$

where $h = 1+\varepsilon\zeta-\beta b$ *and* $(\phi_0,\phi_1,\dots,\phi_N)$ *are the unique solutions to the system*

$$\left\{\begin{aligned}
&-h^{p_i}\textstyle\sum_{j=0}^N\nabla\cdot\Big(\frac{h^{p_j+1}}{p_j+1}\nabla\phi_j - h^{p_j}\phi_j(\beta\nabla b)\Big) + \sum_{j=0}^N\nabla\cdot\Big(\frac{h^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}h^{p_i+p_j}\phi_j(\beta\nabla b)\Big)\\
&\qquad + \textstyle\sum_{j=0}^N\frac{p_i}{p_i+p_j}h^{p_i+p_j}(\nabla\phi_j)\cdot(\beta\nabla b) - \sum_{j=0}^N\frac{p_ip_j}{p_i+p_j-1}h^{p_i+p_j-1}(\mu^{-1}+|\beta\nabla b|^2)\phi_j = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall i\in\{1,\dots,N\},\\
&\textstyle\sum_{i=0}^N h^{p_i}\phi_i = \psi.
\end{aligned}\right.$$

Remark 13.4 (Preserved quantities)**.** *A consequence of the above analysis is the fact that—by Noether's theorem, see Section 2.2—solutions to eq. (13.5)–(13.6) preserve the excess of mass, energy, and horizontal impulse in the flat bottom case:*

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathscr{Z} = \frac{\mathrm{d}}{\mathrm{dt}}\mathscr{E}^{\mathrm{app}}_{\mathrm{IK}} = 0, \quad \mathscr{Z} \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d} \zeta \,\mathrm{d}\boldsymbol{x}, \qquad \frac{\mathrm{d}}{\mathrm{dt}}\mathscr{I}^{\mathrm{app}}_{\mathrm{IK}} = 0, \quad \mathscr{I}^{\mathrm{app}}_{\mathrm{IK}} \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d} \zeta\nabla\psi \,\mathrm{d}\boldsymbol{x} \quad (\text{if } \beta b \equiv 0).$$

where $\psi \stackrel{\mathrm{def}}{=} \sum_{j=0}^N \phi_j \Psi_j \big|_{z=\varepsilon\zeta}$ *and* $\mathscr{E}^{\mathrm{app}}_{\mathrm{IK}}$ *is defined in eq. (13.10).*

Remark 13.5. *The above analysis is made trivial when* $\{\Psi_i\}_{i=1,\dots,N}$ *are chosen so that* $\Psi_0\big|_{z=\varepsilon\zeta} \neq 0$ *and* $\Psi_i\big|_{z=\varepsilon\zeta} = 0$ *for* $i = 1,\dots,N$*. Then eq. (13.11) reads*

$$\begin{pmatrix} 0 & -\Psi_0\big|_{z=\varepsilon\zeta} \\ \Psi_0\big|_{z=\varepsilon\zeta} & 0 \end{pmatrix} \begin{pmatrix} \partial_t\zeta \\ \partial_t\phi_0 \end{pmatrix} = \begin{pmatrix} \delta_\zeta\mathscr{E}^{\mathrm{app}} \\ \partial_{\phi_0}\mathscr{E}^{\mathrm{app}} \end{pmatrix}, \qquad \partial_{\phi_i}\mathscr{E}^{\mathrm{app}} = 0 \qquad \forall i = 1,\dots,N,$$

and, since $\psi \stackrel{\mathrm{def}}{=} \sum_{j=0}^N \phi_j \Psi_j \big|_{z=\varepsilon\zeta} = \phi_0\Psi_0\big|_{z=\varepsilon\zeta}$*,*

$$\mathscr{E}^{\mathrm{app}}(\eta,\phi_0,\phi_1,\dots,\phi_N) = \mathscr{E}^{\mathrm{app}}(\eta, \boldsymbol{S}[\varepsilon\zeta,\beta b](\phi_0\Psi_0\big|_{z=\varepsilon\zeta})) \stackrel{\mathrm{def}}{=} \mathscr{H}^{\mathrm{app}}(\zeta,\phi_0\Psi_0\big|_{z=\varepsilon\zeta}).$$

This description was put forward in [261]. Notice however that here we implicitly imply that the vertical distribution $\{\Psi_i\}_{i\in\{1,\dots,N\}}$ *depends on the unknown variable* ζ*. Hence eq. (13.5)–(13.6) are no longer valid (see Remark 13.2); yet the discussion in this section applies, with straightforward adjustments.*

13.3 Derivation through the Galerkin method

In this section we recover (13.19)–(13.20) following the Galerkin method described in Section 12.1, replacing the vertically integrated variational formulation to the Laplace problem given in Definition 12.1 with the one we originally introduced in Definition 4.2, and which we recall below.

Definition 13.6 (Variational solutions)**.** *Let* $\psi \in \mathring{H}^1(\mathbb{R}^d)$ *and* $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ *satisfying*

$$\forall \boldsymbol{x}\in\mathbb{R}^d, \qquad h(\boldsymbol{x}) = 1 + \varepsilon\zeta(\boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0.$$

We say that Φ *is a variational solution to eq. (4.1) if there exists* $\widetilde{\Phi} \in H^1_{0,\mathrm{top}}(\Omega)$ *such that* $\Phi = \psi + \widetilde{\Phi}$ *and for any* $\widetilde{\varphi} \in H^1_{0,\mathrm{top}}(\Omega)$*,*

$$\iint_\Omega \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}\cdot\nabla^\mu_{\boldsymbol{x},z}\widetilde{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = -\mu\iint_\Omega \nabla\psi\cdot\nabla_{\boldsymbol{x}}\widetilde{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z,$$

which we can rewrite, denoting $\varphi = \psi + \widetilde{\varphi}$*,*

$$\iint_\Omega \nabla^\mu_{\boldsymbol{x},z}\Phi\cdot\nabla^\mu_{\boldsymbol{x},z}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \mu\iint_\Omega \nabla_{\boldsymbol{x}}\Phi\cdot\nabla(\varphi\big|_{z=\varepsilon\zeta})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z,$$

In the formula above we identified $\boldsymbol{x}\mapsto\psi(\boldsymbol{x})\in\mathring{H}^1(\mathbb{R}^d)$ *and* $(\boldsymbol{x},z)\mapsto\psi(\boldsymbol{x})\in\mathring{H}^1(\Omega)$*.*

Setting

$$V_{(\Psi_0,\dots,\Psi_N)} \stackrel{\mathrm{def}}{=} \left\{\Phi \;:\; \Phi(\boldsymbol{x},z) = \sum_{i=0}^N \Psi_i(\boldsymbol{x},z)\phi_i(\boldsymbol{x}),\ \phi_i\in H^1(\mathbb{R}^d)\right\},$$

the Galerkin approximate solution is $\Phi^{\mathrm{app}} \in V_{(\Psi_0,\dots,\Psi_N)}$ the solution to

$$\forall\varphi\in V_{(\Psi_0,\dots,\Psi_N)}, \qquad \iint_\Omega \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}^{\mathrm{app}}\cdot\nabla^\mu_{\boldsymbol{x},z}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = -\iint_\Omega \mu\nabla\psi\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z. \tag{13.18}$$

Denoting

$$\Phi^{\mathrm{app}} \overset{\mathrm{def}}{=} \sum_{i=0}^{N} \Psi_i(\boldsymbol{x},z)\phi_i(\boldsymbol{x}) \qquad \text{and} \qquad \varphi \overset{\mathrm{def}}{=} \sum_{j=0}^{N} \Psi_j(\boldsymbol{x},z)\varphi_j(\boldsymbol{x}),$$

eq. (13.18) reads

$$\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{N}\int_{\mathbb{R}^d}\Bigg(\phi_i\varphi_j\Big(\int_{-1+\beta b}^{\varepsilon\zeta}\big(\partial_z\Psi_i(\cdot,z)\big)\big(\partial_z\Psi_j(\cdot,z)\big)\,\mathrm{d}z\Big)\\
&\quad+\mu\int_{-1+\beta b}^{\varepsilon\zeta}\big(\phi_i(\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z))+\Psi_i(\cdot,z)(\nabla\phi_i)\big)\cdot\big(\varphi_j(\nabla_{\boldsymbol{x}}\Psi_j(\cdot,z))+\Psi_j(\cdot,z)(\nabla\varphi_j)\big)\,\mathrm{d}z\Bigg)\,\mathrm{d}\boldsymbol{x}\\
&\quad=\mu\sum_{i=0}^{N}\sum_{j=0}^{N}\int_{\mathbb{R}^d}\Bigg(\phi_i\int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z)\,\mathrm{d}z+(\nabla\phi_i)\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_i(\cdot,z)\,\mathrm{d}z\Bigg)\cdot\nabla(\phi_j\Psi_j\big|_{z=\varepsilon\zeta})\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

which we can rewrite as the following system of differential equations:

$$\begin{aligned}
&\forall j\in\{0,\dots,N\},\qquad \sum_{i=0}^{N}\Bigg(\phi_i\Big(\int_{-1+\beta b}^{\varepsilon\zeta}\big(\partial_z\Psi_i(\cdot,z)\big)\big(\partial_z\Psi_j(\cdot,z)\big)\,\mathrm{d}z\Big)\\
&\quad+\mu\phi_i\int_{-1+\beta b}^{\varepsilon\zeta}(\nabla_{\boldsymbol{x}}\Psi_j(\cdot,z))\cdot(\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z))\,\mathrm{d}z+\mu(\nabla\phi_i)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_i(\cdot,z)(\nabla_{\boldsymbol{x}}\Psi_j(\cdot,z))\,\mathrm{d}z\\
&\quad-\mu\nabla\cdot\Big(\phi_i\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_j(\cdot,z)(\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z))\,\mathrm{d}z+(\nabla\phi_i)\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_j(\cdot,z)\Psi_i(\cdot,z)\,\mathrm{d}z\Big)\Bigg)\\
&\qquad=-\mu\Psi_j\big|_{z=\varepsilon\zeta}\sum_{i=0}^{N}\nabla\cdot\Bigg(\phi_i\int_{-1+\beta b}^{\varepsilon\zeta}\nabla_{\boldsymbol{x}}\Psi_i(\cdot,z)\,\mathrm{d}z+(\nabla\phi_i)\int_{-1+\beta b}^{\varepsilon\zeta}\Psi_i(\cdot,z)\,\mathrm{d}z\Bigg).
\end{aligned}$$

Owing to the fact that the right-hand sides are proportional to $\Psi_j\big|_{z=\varepsilon\zeta}$ for $j\in\{0,\dots,N\}$, we infer

$$\forall j\in\{1,\dots,N\},\qquad \Psi_0\,\big|_{z=\varepsilon\zeta}\sum_{i=0}^{N}\mathcal{L}^{\mu}_{ji}\phi_i=\Psi_j\,\big|_{z=\varepsilon\zeta}\sum_{i=0}^{N}\mathcal{L}^{\mu}_{0i}\phi_i, \tag{13.19}$$

where we use the notation $\mathcal{L}^{\mu}_{ij}$ as in eq. (13.12). Notice also that we have, by definition,

$$\psi\overset{\mathrm{def}}{=}\Phi^{\mathrm{app}}\,\big|_{z=\varepsilon\zeta}=\sum_{i=0}^{N}\Psi_i\,\big|_{z=\varepsilon\zeta}\ \phi_i. \tag{13.20}$$

Proceeding as in Section 12.1, we set

$$\begin{aligned}
\mathscr{H}^{\mathrm{app}}&\overset{\mathrm{def}}{=}\frac12\int_{\mathbb{R}^d}\zeta^2+(\nabla\psi)\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi^{\mathrm{app}}(\cdot,z)\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}\\
&=\frac12\int_{\mathbb{R}^d}\zeta^2+\psi\textstyle\sum_{i=0}^{N}\mathcal{L}^{\mu}_{0i}\phi_i\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where $\{\phi_i\}_{i\in\{0,\dots,N\}}$ is determined from (ζ,ψ) by solving the system (13.19)–(13.20). It is now straightforward to check that $\mathscr{H}^{\mathrm{app}}$ is characterized by eq. (13.14). In particular, by the analysis in Section 13.2, Hamilton's equations

$$\begin{cases}\partial_t\zeta-\delta_\psi\mathscr{H}^{\mathrm{app}}=0,\\ \partial_t\psi+\delta_\zeta\mathscr{H}^{\mathrm{app}}=0,\end{cases} \tag{13.21}$$

are equivalent to eq. (13.5)–(13.6).

13.4 Traveling waves

The existence and properties of solitary wave solutions to the Isobe–Kakinuma systems (in the flat bottom and one-dimensional situation) has been investigated by Colin and Iguchi in [111]. In this work, the authors prove the existence of a family of solitary wave solutions to eq. (13.7) for any sufficiently small supercritical velocity, behaving—as expected—asymptotically as long waves when the velocity converges towards the critical velocity of infinitely long waves.

Let us reproduce their result with our notations, below.

Theorem 13.7. *Set $\varepsilon = \mu = 1$, $\beta = 0$ and $d = 1$. Let $N \in \mathbb{N}^\star$ and $0 = p_0 < p_1 < \cdots < p_N$. There exists $c_0 > 1$ such that for any $c \in (1, c_0)$, there exists $(\zeta_c, \phi_{c,0}, \phi_{c,1}, \ldots, \phi_{N,c}) \in \mathcal{C}^\infty(\mathbb{R})$ such that $(\zeta_c, \phi'_{c,0})$ are even, $(\phi_{c,1}, \ldots, \phi_{N,c})$ are odd, and*

$$(\zeta, \phi_0, \phi_1, \ldots, \phi_N) : (t, x) \in \mathbb{R} \times \mathbb{R} \mapsto (\zeta_c, \phi_{c,0}, \phi_{c,1}, \ldots, \phi_{N,c})(x - ct)$$

satisfies eq. (13.7). Moreover, there exists $\gamma > 0$ such that for any $s \geq 0$, there exists $M^\star > 0$ such that $(\zeta_c, \phi'_{c,0}, \phi_{c,1}, \ldots, \phi_{N,c}) \in H^s(\mathbb{R})^{2+N}$, and, denoting $c = 1 + \frac{3}{8}\epsilon$ and $\xi_{\rm KdV}(x) = \frac{3}{4}\operatorname{sech}^2\left(\frac{3}{4}x\right)$,

$$\left|\epsilon^{-1}\zeta_c(\epsilon^{-1/2}\cdot) - \xi_{\rm KdV}((3\gamma)^{-1/2}\cdot)\right|_{H^s(\mathbb{R})} \leq M^\star \epsilon$$

uniformly over $c \in (1, c_0)$.

Remark 13.8. *The coefficient γ depends in the above statement depends uniquely on $p_1 < \cdots < p_N$, and appears to value $\gamma = \frac{1}{3}$ if $p_1 = 2$. The result leaves open the expected behavior, based on the modal analysis below and the consistency result in Section 13.6, that at least when $p_i = 2i$ for $i \in \{0, 1, \ldots, N\}$, the solitary wave solution approaches the corresponding solitary wave solution to the water waves system (see Section 2.6) with precision $\mathcal{O}(\epsilon^{2N})$, hence improving upon the precision of the corresponding solitary waves solution to the Green–Naghdi system (see Section 8.4) for any $N \in \mathbb{N}^\star$, and the one of the Whitham–Green–Naghdi system (see Section 10.4 and Remark 10.4) as soon as $N \geq 2$.*

Moreover, the authors study numerically the case of $N = 1$ and $p_0 = 0$, $p_1 = 2$, and find that there appears to exists a maximal value $c_\star \approx 1.26153$ such that as $c \nearrow c_\star$, the profiles of the solitary wave, ζ_c, converges towards a peaked profile. Hence—contrarily to the Green–Naghdi system, see Section 8.4—the Isobe–Kakinuma model is able to reproduce qualitatively the corresponding feature of the water waves system; see Section 2.6. However the numerically computed angle at the crest is approximately $2\pi \times 0.424$ (in radians) and hence greater than the one of the water waves system, that is $2\pi\frac{1}{3}$. It is conjectured that augmenting N, the solitary wave of extreme height of the Isobe–Kakinuma model will approach the solitary wave of extreme height of the water waves system.

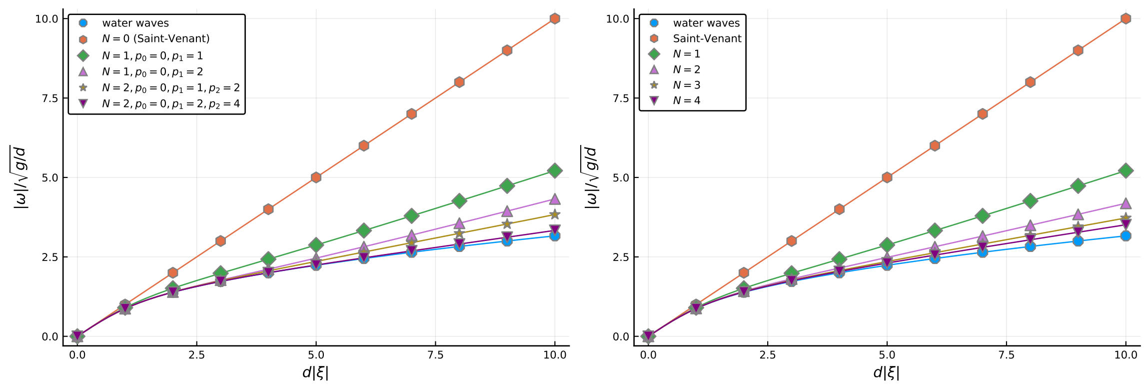
13.5 Modal analysis

In this section we compare the dispersion relation associated to some models described by eq. (13.5)–eq. (13.6) with the one associated with the water waves system, eq. (2.7). The results in this section are illustrated in Figure 13.1. Recall (see Section 2.3) that when linearized about the rest solution, $\zeta = 0$ and $\nabla\psi = \mathbf{0}$, and setting $\beta b = 0$, the water waves system reads

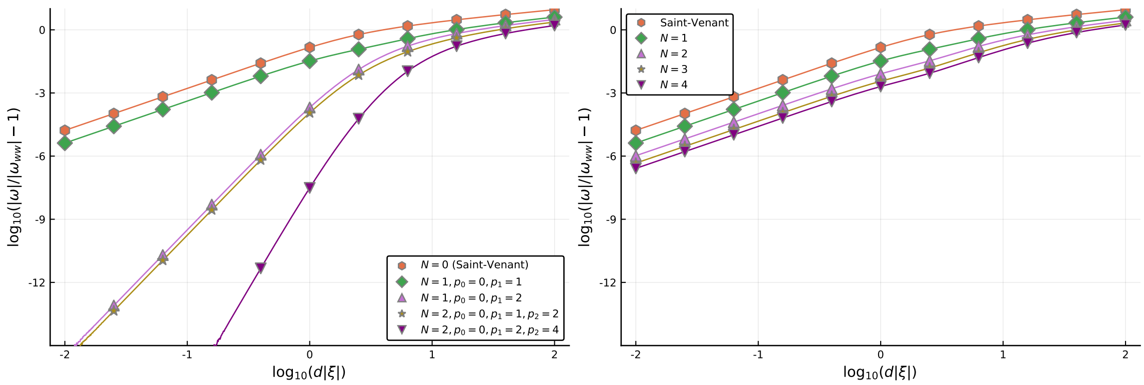
$$\begin{cases} \partial_t \zeta^0 + \frac{1}{\sqrt{\mu}}|D|\tanh(\sqrt{\mu}|D|)\psi^0 = 0, \\ \partial_t \psi^0 + \zeta^0 = 0, \end{cases}$$

which yields the dispersion relation

$$\omega_{\rm ww}(\boldsymbol{\xi})^2 = \frac{1}{\sqrt{\mu}}|\boldsymbol{\xi}|\tanh(\sqrt{\mu}|\boldsymbol{\xi}|).$$



(a) Isobe–Kakinuma model. (b) “Multilayer” variational model.



(c) Same as (a), in log scale. (d) Same as (b), in log scale.

Figure 13.1: In (a) and (b), wave frequencies, $|\omega|(|\boldsymbol{\xi}|)$, given by the dispersion relations corresponding to the (linearized about rest) Isobe–Kakinuma and “multilayer” variational models, respectively. In (c) and (d), the “error” is represented in log scale.

The Isobe–Kakinuma systems When linearizing about the rest state solution, the Isobe–Kakinuma system—under the formulation (13.17)—reads

$$\begin{cases} \partial_t \zeta^0 + \sum_{j=0}^N \frac{1}{p_j+1} \nabla \cdot \nabla \phi_j^0 = 0, \\ \partial_t \psi^0 + \zeta^0 = 0, \end{cases}$$

where $(\phi_0^0, \phi_1^0, \dots, \phi_N^0)$ are the unique solutions to the system

$$\begin{cases} \sum_{j=0}^N \frac{p_i}{(p_j+1)(p_i+p_j+1)} \mu \nabla \cdot \nabla \phi_j^0 + \frac{p_i p_j}{p_i+p_j-1} \phi_j^0 = 0 \qquad \forall i \in \{1, \dots, N\}, \\ \sum_{i=0}^N \phi_i^0 = \psi. \end{cases}$$

When $N = 0$ and $p_0 = 0$, the Isobe–Kakinuma model is simply the Saint-Venant system (Section 5), whose dispersion relation is

$$\omega_{\rm SV}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2.$$

When $N = 1$, $p_0 = 0$ and $p_1 = 1$, one has

$$\phi_0 = \frac{1 + \frac{1}{6}\mu \nabla \cdot \nabla}{1 - \frac{1}{3}\mu \nabla \cdot \nabla} \psi, \qquad \phi_1 = \frac{1}{2} \frac{-\mu \nabla \cdot \nabla}{1 - \frac{1}{3}\mu \nabla \cdot \nabla} \psi$$

and the corresponding dispersion relation

$$\omega_{\rm IK}^{(0,1)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{1 + \frac{1}{12}\mu |\boldsymbol{\xi}|^2}{1 + \frac{1}{3}\mu |\boldsymbol{\xi}|^2}.$$

When $N = 1$, $p_0 = 0$ and $p_1 = 2$, one has

$$\phi_0 = \frac{1 + \frac{1}{10}\mu \nabla \cdot \nabla}{1 - \frac{2}{5}\mu \nabla \cdot \nabla} \psi, \qquad \phi_1 = \frac{1}{2} \frac{-\mu \nabla \cdot \nabla}{1 - \frac{2}{5}\mu \nabla \cdot \nabla} \psi$$

and the corresponding dispersion relation

$$\omega_{\rm IK}^{(0,2)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2 \frac{1 + \frac{1}{15}\mu |\boldsymbol{\xi}|^2}{1 + \frac{2}{5}\mu |\boldsymbol{\xi}|^2}.$$

Observe the small wavenumber Taylor series

$$\begin{aligned} \omega_{\rm ww}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \big(1 - \frac{1}{3}\mu |\boldsymbol{\xi}|^2 + \frac{2}{15}\mu^2 |\boldsymbol{\xi}|^4 - \frac{17}{315}\mu^3 |\boldsymbol{\xi}|^6 + \frac{62}{2835}\mu^4 |\boldsymbol{\xi}|^8 + \mathcal{O}(\mu^5 |\boldsymbol{\xi}|^{10})\big), \\ \omega_{\rm IK}^{(0,1)}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \big(1 - \frac{1}{4}\mu |\boldsymbol{\xi}|^2 + \mathcal{O}(\mu^2 |\boldsymbol{\xi}|^2)\big), \\ \omega_{\rm IK}^{(0,2)}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \big(1 - \frac{1}{3}\mu |\boldsymbol{\xi}|^2 + \frac{2}{15}\mu^2 |\boldsymbol{\xi}|^4 - \frac{4}{75}\mu^3 |\boldsymbol{\xi}|^6 + \mathcal{O}(\mu^4 |\boldsymbol{\xi}|^8)\big), \\ \omega_{\rm GN}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \big(1 - \frac{1}{3}\mu |\boldsymbol{\xi}|^2 + \frac{1}{9}\mu^2 |\boldsymbol{\xi}|^4 + \mathcal{O}(\mu^6 |\boldsymbol{\xi}|^6)\big), \\ \omega_{\rm aGN}^{(0,2)}(\boldsymbol{\xi})^2 &= |\boldsymbol{\xi}|^2 \big(1 - \frac{1}{3}\mu |\boldsymbol{\xi}|^2 + \frac{2}{15}\mu^2 |\boldsymbol{\xi}|^4 - \frac{17}{315}\mu^3 |\boldsymbol{\xi}|^6 + \frac{241}{11025}\mu^4 |\boldsymbol{\xi}|^8 + \mathcal{O}(\mu^5 |\boldsymbol{\xi}|^{10})\big). \end{aligned}$$

As in Section 12.2, we see that $\omega_{\rm IK}^{(0,1)}$ is not more precise than the dispersionless angular frequency predicted by the Saint-Venant system, $\omega_{\rm SV}^2 = |\boldsymbol{\xi}|^2$. However the agreement when $N = 1$, $p_0 = 0$ and $p_1 = 2$, is excellent: between that of the Green–Naghdi system ($\omega_{\rm GN}$) and that of the corresponding augmented Green–Naghdi system ($\omega_{\rm aGN}^{(0,2)}$). This order of approximation is consistent with the degrees of the polynomials involved in the rational fraction. In fact $\omega_{\rm aGN}^{(0,2)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(4,2)$ to $\omega_{\rm ww}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$, so it is in some sense the best possible approximation with polynomials of such degrees.

When $N=2$, and $(p_0,p_1,p_2)=(0,1,2)$, we find the dispersion relation

$$\omega_{\rm IK}^{(0,1,2)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2\frac{720+72\mu|\boldsymbol{\xi}|^2+\mu^2|\boldsymbol{\xi}|^4}{720+312\mu|\boldsymbol{\xi}|^2+9\mu^2|\boldsymbol{\xi}|^4}$$
$$= \omega_{\rm ww}(\boldsymbol{\xi})^2 + |\boldsymbol{\xi}|^2\big(\frac{1}{2800}\mu^3|\boldsymbol{\xi}|^6+\mathcal{O}(\mu^4|\boldsymbol{\xi}|^8)\big).$$

When $N=2$, and $(p_1,p_2,p_3)=(0,2,4)$, we find the dispersion relation

$$\omega_{\rm IK}^{(0,2,4)}(\boldsymbol{\xi})^2 = |\boldsymbol{\xi}|^2\frac{945+105\mu|\boldsymbol{\xi}|^2+\mu^2|\boldsymbol{\xi}|^4}{945+420\mu|\boldsymbol{\xi}|^2+15\mu^2|\boldsymbol{\xi}|^4}$$
$$= \omega_{\rm ww}(\boldsymbol{\xi})^2 + |\boldsymbol{\xi}|^2\big(\frac{1}{9823275}\mu^5|\boldsymbol{\xi}|^{10}+\mathcal{O}(\mu^6|\boldsymbol{\xi}|^{12})\big).$$

We see that the precision of $\omega_{\rm IK}^{(0,1,2)}(\boldsymbol{\xi})^2$ is in par with the precision of $\omega_{\rm IK}^{(0,2)}(\boldsymbol{\xi})^2$ in order of magnitude, although $\frac{17}{315}-\frac{4}{75}\approx 6.3\,10^{-4}$ and $\frac{1}{2800}\approx 3.6\,10^{-4}$. $\omega_{\rm IK}^{(0,2,4)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(6,4)$ to $\omega_{\rm ww}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$. In fact we have the following result.

Proposition 13.9. *For any $N\in\mathbb{N}$ and setting $p_i=2i$ for $i\in\{0,\dots,N\}$, $\omega_{\rm IK}^{(p_0,p_1,\dots,p_N)}(\boldsymbol{\xi})^2$ is the Padé approximant of order $(2N+2,2N)$ to $\omega_{\rm ww}(|\boldsymbol{\xi}|)^2$ about $|\boldsymbol{\xi}|=0$. In particular,*

$$\frac{\omega_{\rm IK}^{(p_0,p_1,\dots,p_N)}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2} = \cfrac{1}{1+\cfrac{\mu|\boldsymbol{\xi}|^2}{3+\cfrac{\mu|\boldsymbol{\xi}|^2}{5+\cfrac{\mu|\boldsymbol{\xi}|^2}{\ddots+\frac{\mu|\boldsymbol{\xi}|^2}{4N+1}}}}} \to \frac{\omega_{\rm ww}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2} \quad (N\to\infty)$$

and $0\leq \frac{\omega_{\rm IK}^{(0,\dots,2N)}(\boldsymbol{\xi})^2-\omega_{\rm ww}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\leq C_N\mu^{2N+1}|\boldsymbol{\xi}|^{4N+2}$, where C_N depends uniquely on N.

The first part of the statement is stated and proved in [341, Theorem 2.2]. The second part is Lambert's continued fraction of tanh, see [4, 4.5.70]. Since all coefficients are positive, the truncated sequences $\omega_{\rm IK}^{(p_0,p_1,\dots,p_N)}(\boldsymbol{\xi})^2$ and $\omega_{\rm aGN}^{(p_1,\dots,p_N)}(\boldsymbol{\xi})^2$ in Conjecture 12.5 are adjacent.

Remark 13.10. *Compare with Conjecture 12.5. The Isobe–Kakinuma and augmented Green–Naghdi models realize a Padé approximant framing of the water waves dispersion relation. Again we do not mean that setting $p_i=2i$ is necessarily the best choice in the nonlinear framework, and in particular in presence of a nontrivial bottom topography; see Section 13.6.*

The Klopman, van Groesen and Dingemans systems We continue the analysis with other choices of vertical distribution, described in Remark 13.2. While we argued therein that vertical distributions of the form

$$\Phi^{\rm app}(t,\boldsymbol{x},z)\stackrel{\rm def}{=}\sum_{i=0}^{N}\Psi_i(\boldsymbol{x},z,\varepsilon\zeta(t,\boldsymbol{x}))\phi_i(t,\boldsymbol{x})$$

would modify equations eq. (13.5)–eq. (13.6), the changes are immaterial when we linearize about the rest state. This explains in particular why the dispersion relation of the "parabolic" model in [261] yields exactly $\omega_{\rm IK}^{(0,2)}(\boldsymbol{\xi})$, since the vertical distributions coincide with $\Psi_0(\boldsymbol{x},z)=1$, $\Psi_1(\boldsymbol{x},z)=(z+1)^2$ when $\varepsilon\zeta=0$. In the same way, the dispersion relation when $\Psi_i(\boldsymbol{x},z,\varepsilon\zeta)=(z-\varepsilon\zeta)^i\approx z^i$ for $i=0,1,\dots,N$—which are listed up to $N=5$ in [261, (5.21)]—fit with $\omega_{\rm IK}^{(0,1,\dots,N)}(\boldsymbol{\xi})$. The nonlinear equations, however, differ.

The "multilayer" systems We put $\eta_i = -1+\beta b+\ell_i(1+\varepsilon\zeta-\beta b)$ for given $0=\ell_N<\cdots<\ell_1<\ell_0=1$ and

$$\Psi_i(\boldsymbol{x},z,\varepsilon\zeta)=\begin{cases}\frac{\eta_{i-1}(\boldsymbol{x})-z}{\eta_{i-1}(\boldsymbol{x})-\eta_i(\boldsymbol{x})} & \text{if } \eta_i(\boldsymbol{x})\le z<\eta_{i-1}(\boldsymbol{x}),\\ \frac{z-\eta_{i+1}(\boldsymbol{x})}{\eta_i(\boldsymbol{x})-\eta_{i+1}(\boldsymbol{x})} & \text{if } \eta_{i+1}(\boldsymbol{x})<z\le\eta_i(\boldsymbol{x})\\ 0 & \text{otherwise.}\end{cases}$$

The linearized system about the rest state reads

$$\begin{cases}\partial_t\zeta^0+\mu^{-1}\frac{\phi_0^0-\phi_1^0}{\ell_0-\ell_1}-\frac{\ell_0-\ell_1}{3}\nabla\cdot\nabla\phi_0^0-\frac{\ell_0-\ell_1}{6}\nabla\cdot\nabla\phi_1^0=0,\\ \partial_t\phi_0^0+\zeta^0=0,\end{cases}$$

where $(\phi_1^0,\ldots,\phi_N^0)$ are the unique solutions to the system

$$\begin{cases}\frac{\phi_i^0-\phi_{i-1}^0}{\ell_{i-1}-\ell_i}+\frac{\phi_i^0-\phi_{i+1}^0}{\ell_i-\ell_{i+1}}-\frac{\ell_{i-1}-\ell_i}{6}\mu(\nabla\cdot\nabla)(2\phi_i^0+\phi_{i-1}^0)-\frac{\ell_i-\ell_{i+1}}{6}\mu(\nabla\cdot\nabla)(2\phi_i^0+\phi_{i+1}^0)=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall i\in\{1,\ldots,N-1\},\\ \frac{\phi_N^0-\phi_{N-1}^0}{\ell_{N-1}-\ell_N}-\mu\frac{\ell_{N-1}-\ell_N}{6}\mu(\nabla\cdot\nabla)(2\phi_N^0+\phi_{N-1}^0)=0.\end{cases}$$

The singularity as $\mu\searrow 0$ is only apparent as we can see in the explicit formula below. In fact, choosing $\Psi_0=1$ and $\{\Psi_i\}_{i\in\{1,\ldots,N\}}$ as above yields an equivalent system where the singularity vanishes (with the price to pay that the system to invert no longer enjoys the tridiagonal structure). When $N=1$ we have the dispersion relation

$$\omega^{(\ell_0,\ell_1)}(\boldsymbol{\xi})^2=(\ell_1-\ell_0)|\boldsymbol{\xi}|^2\frac{1+\frac{1}{12}\mu(\ell_1-\ell_0)^2|\boldsymbol{\xi}|^2}{1+\frac{1}{3}\mu(\ell_1-\ell_0)^2|\boldsymbol{\xi}|^2}$$

and since $\ell_0=1$ and $\ell_1=0$, we recover naturally

$$\omega^{(1,0)}(\boldsymbol{\xi})^2=\omega_{\rm IK}^{(0,1)}(\boldsymbol{\xi})^2=|\boldsymbol{\xi}|^2\frac{1+\frac{1}{12}\mu|\boldsymbol{\xi}|^2}{1+\frac{1}{3}\mu|\boldsymbol{\xi}|^2}=|\boldsymbol{\xi}|^2\big(1-\frac{1}{4}\mu|\boldsymbol{\xi}|^2+\mathcal{O}(\mu^2|\boldsymbol{\xi}|^2)\big).$$

When $N=2$ we find, denoting $h_1=\ell_0-\ell_1=1-\ell_1$ and $h_2=\ell_1-\ell_2=\ell_1$,

$$\begin{aligned}\omega^{(\ell_0,\ell_1,\ell_2)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{36+(3h_1^2+3h_2^2+9h_1h_2)\mu|\boldsymbol{\xi}|^2+h_1^2h_2^2\mu^2|\boldsymbol{\xi}|^4}{36+(12h_1^2+12h_2^2+36h_1h_2)\mu|\boldsymbol{\xi}|^2+(4h_1^2h_2^2+3h_1h_2^3)\mu^2|\boldsymbol{\xi}|^4}\\&=|\boldsymbol{\xi}|^2\big(1-\tfrac{1}{4}(h_1^2+h_2^2+3h_1h_2)\mu|\boldsymbol{\xi}|^2+\mathcal{O}(\mu^2|\boldsymbol{\xi}|^4)\big).\end{aligned}$$

The best result at low frequencies is obtained for $h_1=h_2=\frac{1}{2}$, and hence $(\ell_0,\ell_1,\ell_2)=(1,\frac{1}{2},0)$, for which

$$\begin{aligned}\omega^{(1,\frac{1}{2},0)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{576+60\mu|\boldsymbol{\xi}|^2+\mu^2|\boldsymbol{\xi}|^4}{576+240\mu|\boldsymbol{\xi}|^2+7\mu^2|\boldsymbol{\xi}|^4}\\&=|\boldsymbol{\xi}|^2\big(1-\frac{5}{16}\mu|\boldsymbol{\xi}|^2+\mathcal{O}(\mu^2|\boldsymbol{\xi}|^4)\big).\end{aligned}$$

In the following, we always set $\ell_i=1-i/N$ for $i\in\{0,1,\ldots,N\}$. For $N\in\{3,4\}$, we find

$$\begin{aligned}\omega^{(1,\frac{2}{3},\frac{1}{3},0)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{314928+37908\mu|\boldsymbol{\xi}|^2+864\mu^2|\boldsymbol{\xi}|^4+5\mu^3|\boldsymbol{\xi}|^4}{314928+139968\mu|\boldsymbol{\xi}|^2+6156\mu^2|\boldsymbol{\xi}|^4+52\mu^3|\boldsymbol{\xi}|^6}\\&=|\boldsymbol{\xi}|^2\big(1-\frac{35}{108}\mu|\boldsymbol{\xi}|^2+\mathcal{O}(\mu^2|\boldsymbol{\xi}|^4)\big),\\ \omega^{(1,\frac{3}{4},\frac{2}{4},\frac{1}{4},1)}(\boldsymbol{\xi})^2&=|\boldsymbol{\xi}|^2\frac{84934656+11059200\mu|\boldsymbol{\xi}|^2+304128\mu^2|\boldsymbol{\xi}|^4+2640\mu^3|\boldsymbol{\xi}|^6+7\mu^4|\boldsymbol{\xi}|^8}{84934656+38928384\mu|\boldsymbol{\xi}|^2+2045952\mu^2|\boldsymbol{\xi}|^4+27264\mu^3|\boldsymbol{\xi}|^6+97\mu^4|\boldsymbol{\xi}|^8}\\&=|\boldsymbol{\xi}|^2\big(1-\frac{21}{64}\mu|\boldsymbol{\xi}|^2+\mathcal{O}(\mu^2|\boldsymbol{\xi}|^4)\big).\end{aligned}$$

Hence we observe that the behavior of the dispersion relations is—unsurprisingly—very similar to the one of the "multilayer" Green–Naghdi model (in fact the polynomial at the denominator of the rational fraction is exactly the same); see Section 12.2. In particular, augmenting N does not improve the small wavenumber behavior of the model by an order of magnitude with respect to the Saint-Venant system, but improves the prefactors in the Taylor expansion:

$$(\frac{1}{3}-\frac{1}{4}) = 4\,(\frac{1}{3}-\frac{5}{16}) = 9\,(\frac{1}{3}-\frac{35}{108}) = 16\,(\frac{1}{3}-\frac{21}{64})$$

(the striking progression holds only approximately for the next order coefficient in the Taylor series). This motivates the following conjecture.

Conjecture 13.11. *Setting $\ell_i = 1 - i/N$ for $i \in \{0,\dots,N\}$, one has for any $\boldsymbol{\xi} \in \mathbb{R}^d$,*

$$0 \leq \frac{\omega^{(\ell_0,\ell_1,\dots,\ell_N)}(\boldsymbol{\xi})^2 - \omega_{\rm ww}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2} \times N^2 \;=\; \mathcal{O}(\mu|\boldsymbol{\xi}|^2),$$

uniformly with respect to $N \in \mathbb{N}^\star$.

This property appears to be the weakest of the high order models studied in Section 12.2 and Section 13.5. Again, this does not mean that the variational "multilayer" model cannot be relevant in the nonlinear framework, in particular in the presence of small wavelengths.

13.6 Rigorous justification

In this section we report on the rigorous justification of the Isobe–Kakinuma systems, eq. (13.7) as an asymptotic model for the water waves system, eq. (2.7), obtained in a series of work of Iguchi and collaborators, [335, 225, 341], culminating with [226]. Here we will restrict ourselves to the following vertical distribution:

$$\forall i \in \{0,1,\dots,N\}, \qquad \Psi_i(\boldsymbol{x},z) = \big(z+1-\beta b(\boldsymbol{x})\big)^{p_i}$$

where

$$\begin{cases} \forall i \in \{1,\dots,N\},\ p_i = 2i & \text{in the flat bottom case, } \beta b \equiv 0; \\ \forall i \in \{1,\dots,N\},\ p_i = i & \text{for variable bottom topographies.} \end{cases} \tag{13.22}$$

As a matter of fact the above restriction is not essential to the well-posedness result stated below, but to the following consistency result. As always, we consider the shallow water regime (Definition III.2) that is parameters in the set

$$\mathfrak{p}_{\rm SW} = \big\{(\mu,\varepsilon,\beta)\ :\ \mu \in (0,\mu^\star],\ \varepsilon \in [0,1],\ \beta \in [0,1]\big\}.$$

Before stating the results, it is convenient to recall the notations of Section 13.2. Defining

$$\boldsymbol{l}(h) \stackrel{\rm def}{=} \big(h^{p_0}, h^{p_1}, \dots, h^{p_N}\big)^\top$$

and, as in eq. (13.12),

$$\begin{aligned} \mathcal{L}^\mu_{ij}[h,\beta\nabla b]\varphi \stackrel{\rm def}{=} -\nabla\cdot\Big(\tfrac{h^{p_i+p_j+1}}{p_i+p_j+1}\nabla\varphi - \tfrac{p_j}{p_i+p_j}h^{p_i+p_j}\varphi(\beta\nabla b)\Big) \\ - \tfrac{p_i}{p_i+p_j}h^{p_i+p_j}(\nabla\varphi)\cdot(\beta\nabla b) + \tfrac{p_ip_j}{p_i+p_j-1}h^{p_i+p_j-1}(\mu^{-1}+|\beta\nabla b|^2)\varphi \end{aligned} \tag{13.23}$$

we can rewrite the Isobe–Kakinuma model eq. (13.7) compactly as

$$\begin{cases} -\boldsymbol{l}(h)\partial_t\zeta + \mathcal{L}^\mu[h,\beta\nabla b]\boldsymbol{\phi} = \boldsymbol{0}, \\ \boldsymbol{l}(h)\bullet\partial_t\boldsymbol{\phi} + \zeta + \frac{\varepsilon}{2}\big(|\boldsymbol{u}|^2 + \mu^{-1}w^2\big) = 0, \end{cases} \tag{13.24}$$

where we denote $h \stackrel{\text{def}}{=} 1+\varepsilon\zeta-\beta b$, $\boldsymbol{\phi} \stackrel{\text{def}}{=} (\phi_0,\phi_1,\dots,\phi_N)^\top$, $\mathcal{L}^\mu = (\mathcal{L}^\mu_{ij})_{i,j\in\{0,1,\dots,N\}}$ and

$$\boldsymbol{u} \stackrel{\text{def}}{=} \boldsymbol{l}(h)\bullet\nabla\boldsymbol{\phi} - (\boldsymbol{l}'(h)\bullet\boldsymbol{\phi})(\beta\nabla b), \qquad w \stackrel{\text{def}}{=} \boldsymbol{l}'(h)\bullet\boldsymbol{\phi}, \tag{13.25}$$

where we use $\bullet$ to denote the $(N+1)$-dimensional inner product. Notice

$$(\boldsymbol{u},w) = \left(\nabla_{\boldsymbol{x}}\Phi^{\text{app}}, \partial_z\Phi^{\text{app}}\right)\big|_{z=\varepsilon\zeta}, \qquad \Phi^{\text{app}} \stackrel{\text{def}}{=} \sum_{i=0}^N (z+1-\beta b)^{p_i}\phi_i.$$

Recall that any solution to eq. (13.24) must satisfy

$$\forall i\in\{1,\dots,N\}, \qquad \sum_{j=0}^N \mathcal{L}^\mu_{ij}\phi_j \;=\; h^{p_i}\sum_{j=0}^N \mathcal{L}^\mu_{0j}\phi_j. \tag{13.26}$$

In the following, we denote

$$X^s_\mu \stackrel{\text{def}}{=} \Big\{\boldsymbol{\phi} = (\phi_0,\phi_1,\dots,\phi_N)\in \mathring{H}^{s+1}(\mathbb{R}^d)\times H^{s+1}(\mathbb{R}^d)^N,$$
$$\big|\boldsymbol{\phi}\big|^2_{X^s_\mu} \stackrel{\text{def}}{=} \sum_{i=0}^N \big|\nabla\phi_i\big|^2_{H^s} + \sum_{j=1}^N \mu^{-1}\big|\phi_j\big|^2_{H^s} < \infty\Big\}.$$

Theorem 13.12 (Consistency)**.** *Let $d\in\mathbb{N}^\star$, $N\in\mathbb{N}$, $h_\star>0$, $\mu^\star>0$ and $M^\star\geq 0$. Let $s\in\mathbb{N}$ be such that $s\geq 4N+2$ and $s>d/2+2N+2$ in the flat bottom case, $\beta b\equiv 0$; $s\geq\max(\{3,4\lfloor N/2\rfloor+2\})$ and $s>d/2+2\lfloor N/2\rfloor+2$ otherwise. There exists $C>0$ such that for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\text{SW}}$, any $b\in W^{s+1,\infty}(\mathbb{R}^d)$, any $T>0$ and any $(\zeta,\boldsymbol{\phi})\in L^\infty(0,T;H^s(\mathbb{R}^d)\times X^s_\mu)$ solution to eq. (13.24) and satisfying*

$$\forall \boldsymbol{x}\in\mathbb{R}^d, \qquad h(t,\boldsymbol{x}) \stackrel{\text{def}}{=} 1+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x}) \geq h_\star > 0 \tag{13.27}$$

uniformly for $t\in(0,T)$ and

$$M \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t\in(0,T)}\left(\big|\varepsilon\zeta(t,\cdot)\big|_{H^s}\right) + \big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

then, denoting $\psi \stackrel{\text{def}}{=} \boldsymbol{l}(h)\bullet\boldsymbol{\phi}$, one has

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = r_2, \end{cases}$$

where, for almost every $t\in(0,T)$,

$$\big|r_1(t,\cdot)\big|_{H^{s-2\tilde N-4}} \leq C\,\mu^{1+\tilde N}\,\big|\nabla\psi(t,\cdot)\big|_{H^{s-1}},$$
$$\big|r_2(t,\cdot)\big|_{H^{s-2\tilde N-4}} \leq C\,\mu^{1+\tilde N}\varepsilon\big|\nabla\psi(t,\cdot)\big|^2_{H^{s-1}},$$

with $\tilde N = 2N$ in the flat bottom case, and $\tilde N = 2\lfloor N/2\rfloor$ otherwise.

Remark 13.13. *The statement of the consistency result—following [226, Theorem 2.2]—is in the opposite direction with respect to other consistency statements in this document, where solutions to the water waves system are shown to satisfy the model up to a small remainder terms. We favored the latter as long as stability results for the models were easier to prove and stronger than the corresponding stability result for the water waves system (see [268, Theorem 4.18]). This is no longer the case for the Isobe–Kakinuma model due to the special structure of the system, and specifically the fact that the hypersurface $t=0$ in the space-time $\mathbb{R}^d\times\mathbb{R}$ is characteristic.*

Remark 13.14. *The precision of order $\mathcal{O}(\mu^{1+\tilde{N}})$ with $\tilde{N} = 2N$ in the flat bottom case is striking when we compare with the expected precision of the Boussinesq–Rayleigh expansion, eq. (11.1), with the same number of terms, that is $\mathcal{O}(\mu^{1+N})$, and manifests the power of variational methods. As a side note, the approximate formula for the velocity potential*

$$\Phi^{\rm app} \stackrel{\rm def}{=} \sum_{i=0}^{N} \Psi_i(\boldsymbol{x}, z)\phi_i(\boldsymbol{x}) = \sum_{i=0}^{N} \big(z + 1 - \beta b(\boldsymbol{x})\big)^{p_i} \phi_i(\boldsymbol{x})$$

is not *precise at order $\mathcal{O}(\mu^{1+\tilde{N}})$ in the bulk of the fluid; but only $\int_{-1+\beta b}^{\varepsilon\zeta} \nabla\Phi^{\rm app}(\cdot, z)\,{\rm d}z$ is.*

Sketch of the proof. While giving the detailed proof is out of the scope of the present document, it is interesting to describe the main ideas. The key element comes from robust estimates on solutions to the system

$$\begin{cases} \sum_{j=0}^{N} \mathcal{L}^{\mu}_{ij}\phi_j \ - \ h^{p_i} \sum_{j=0}^{N} \mathcal{L}^{\mu}_{0j}\phi_j \ = \ \psi_i \qquad \forall i \in \{1, \dots, N\}, \\ \sum_j = 0^N h^{p_j}\phi_j = \psi_0. \end{cases} \tag{13.28}$$

This system mimics the Laplace problem studied in Section 4 (with remainder terms), and the following result should be compared with Proposition 4.5. In [226, §3] it is shown that for any $s > d/2 + 1$ and $k \in \{0, \pm 1, \dots, \pm s - 1\}$ under the assumptions

$$h \geq h_\star > 0, \quad \big|\varepsilon\zeta\big|_{H^s} + \big|\beta b\big|_{W^{s,\infty}} \leq M_\star,$$

for any $\boldsymbol{\psi} = (\psi_0, \psi_1, \dots, \psi_N) \in \mathring{H}^{k+1} \times (\mathring{H}^{k-1})^N$ there exists a unique $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N) \in \mathring{H}^{k+1} \times (H^{k+1})^N$ solution to the above and that

$$\big|\nabla\phi_0\big|_{H^k} + \sum_{j=1}^{N} \big(\big|\nabla\phi_j\big|_{H^k} + \mu^{-\frac{1}{2}}\big|\phi_j\big|_{H^k}\big) \leq C\big|\nabla\psi_0\big|_{H^k} + \sum_{j=1}^{N} \min\big(\{\big|\psi_j\big|_{H^{k-1}}, \mu^{\frac{1}{2}}\big|\psi_j\big|_{H^k}\}\big),$$

where C depends only on s, $h_\star$ and $M^\star$ (in particular it is uniform with respect to $(\mu, \varepsilon, \beta) \in \mathfrak{p}_{\rm SW}$). This results follows from rewriting the above system (substituting in the first N equations the formula for ϕ_0 induced by the last equation) as a symmetric N-by-N system of differential equations, and applying standard tools for elliptic problems, akin to the ones employed in Proposition 4.5. The first step, that is the coercivity of the linear system of differential operators, follows from recognizing the (positive) kinetic energy in eq. (13.10) in $(\mathcal{L}^\mu\boldsymbol{\phi}, \boldsymbol{\phi})_{L^2}$, that is

$$(\mathcal{L}^\mu\boldsymbol{\phi}, \boldsymbol{\phi})_{L^2} = \int_{\mathbb{R}^d}\int_{-1+\beta b}^{\varepsilon\zeta} \frac{1}{2\mu}(\partial_z\Phi^{\rm app}_{\rm IK})^2 + \frac{1}{2}|\nabla_{\boldsymbol{x}}\Phi^{\rm app}_{\rm IK}|^2\,{\rm d}z\,{\rm d}\boldsymbol{x}, \qquad \Phi^{\rm app} \stackrel{\rm def}{=} \sum_{i=0}^{N} \big(z + 1 - \beta b\big)^{p_i}\phi_i. \tag{13.29}$$

Here the non-cavitation assumption $h \geq h_\star > 0$ and the linear independence of the function $z \mapsto \big(z + 1 - \beta b(\boldsymbol{x})\big)^{p_i}$ play a crucial role. This quickly yields the case $k = 0$. Then product and commutator estimates in Appendix II allow to deduce by induction the cases $k = 1, \dots, s - 1$. Finally the cases $k = -1, \dots, -s + 1$ follow by duality.

As a second step, using the above estimates and separating first-order $\mathcal{O}(\mu^{-1})$ terms to up-to-second order $\mathcal{O}(1)$ terms in $(\mathcal{L}^\mu_{i\cdot} - h^{p_i}\mathcal{L}^\mu_{0\cdot})_{i\in\{1,\dots,N\}}$, and observing that the linear system induced by the former is non-singular, one deduces that when $\psi_1 = \dots = \psi_N = 0$ above, then provided $k + 2j - 1 \leq s - 1$

$$\begin{cases} \big|\phi_j\big|_{H^k} \leq C(M_\star)\mu^j\big|\nabla\psi_0\big|_{H^{k+2j-1}} & \text{in the flat bottom case,} \\ \big|\phi_{2j-1}\big|_{H^k} \leq C(M_\star)\mu^j\big|\nabla\psi_0\big|_{H^{k+2j-1}} & \text{otherwise.} \end{cases} \tag{13.30}$$

As one can see, here, the precise choice of the vertical distribution is essential.

Then we deduce the result from the previously proved bounds. One introduces as an intermediary approximate solution

$$\widetilde{\Phi}^{\rm app} \stackrel{\rm def}{=} \sum_{i=0}^{2N+2} \Psi_i(\boldsymbol{x},z)\widetilde{\phi}_i(\boldsymbol{x}) = \sum_{i=0}^{2N+2} \big(z+1-\beta b(\boldsymbol{x})\big)^{p_i}\widetilde{\phi}_i(\boldsymbol{x}),$$

where $\widetilde{\phi}_i$ is obtained by solving the augmented $(2N+3)$-by-$(2N+3)$ problem, eq. (13.26) and $\boldsymbol{l}(h)\bullet\widetilde{\boldsymbol{\phi}}=\psi$. Using the above estimates, one easily checks that the augmented approximate solution satisfies the Laplace problem, eq. (4.2), with $\mathcal{O}(\mu^{1+\tilde{N}})$ remainder terms. By Proposition 4.5 we have directly the desired control of the difference between Φ, the exact solution, and $\widetilde{\Phi}^{\rm app}$, and more importantly by Green's formula (see Lemma 4.6)

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi - \sum_{j=0}^{2N+2}\mathcal{L}^\mu_{ij}\widetilde{\phi}_j = \mathcal{O}(\mu^{2\tilde{N}+1}).$$

Now there remains to observing the key cancellation when considering $\sum_{j=0}^{2N+2}\mathcal{L}^\mu_{0j}\widetilde{\phi}_j - \sum_{j=0}^{2N}\mathcal{L}^\mu_{0j}\phi_j$ where ϕ_j are the solutions to the $(N+1)$-by-$(N+1)$ problem, eq. (13.26) and $\boldsymbol{l}(h)\bullet\boldsymbol{\phi}=\psi$. To this aim we use duality reasoning, set φ a smooth test function, denote $(\varphi_0,\varphi_1,\dots,\varphi_{2N+2})^\top$ the solution to the augmented $(2N+3)$-by-$(2N+3)$ problem, eq. (13.26) and $\sum_{i=0}^{2N+2}h^{p_i}\varphi_i=\varphi$. We have, using all the identities and that $(\mathcal{L}^\mu_{ij})^\star=\mathcal{L}^\mu_{ji}$ for the $L^2(\mathbb{R}^d)$ inner-product,

$$\begin{aligned}
\Big(\sum_{j=0}^{2N+2}\mathcal{L}^\mu_{0j}\widetilde{\phi}_j,\varphi\Big)_{L^2} &= \Big(\sum_{j=0}^{2N+2}\mathcal{L}^\mu_{0j}\widetilde{\phi}_j,\sum_{i=0}^{2N+2}h^{p_i}\varphi_i\Big)_{L^2} = \sum_{i=0}^{2N+2}\sum_{j=0}^{2N+2}\big(\mathcal{L}^\mu_{ij}\widetilde{\phi}_j,\varphi_i\big)_{L^2}\\
&= \sum_{i=0}^{2N+2}\sum_{j=0}^{2N+2}\big(\widetilde{\phi}_j,\mathcal{L}^\mu_{ji}\varphi_i\big)_{L^2} = \sum_{i=0}^{2N+2}\sum_{j=0}^{2N+2}\big(\widetilde{\phi}_j,h^{p_j}\mathcal{L}^\mu_{0i}\varphi_i\big)_{L^2} = \Big(\psi,\sum_{i=0}^{2N+2}\mathcal{L}^\mu_{0i}\varphi_i\Big)_{L^2}\\
&= \Big(\sum_{j=0}^{N}h^{p_j}\phi_j,\sum_{i=0}^{2N+2}\mathcal{L}^\mu_{0i}\varphi_i\Big)_{L^2} = \sum_{j=0}^{N}\Big(\phi_j,\sum_{i=0}^{2N+2}\mathcal{L}^\mu_{ji}\varphi_i\Big)_{L^2}\\
&= \sum_{j=0}^{N}\sum_{i=0}^{N}\big(\mathcal{L}^\mu_{ij}\phi_j,\varphi_i\big)_{L^2} + \sum_{j=0}^{N}\sum_{i=N+1}^{2N+2}\big(\mathcal{L}^\mu_{ij}\phi_j,\varphi_i\big)_{L^2}\\
&= \Big(\sum_{j=0}^{N}\mathcal{L}^\mu_{0j}\phi_j,\varphi\Big)_{L^2} - \Big(\sum_{j=0}^{N}\mathcal{L}^\mu_{0j}\phi_j,\sum_{i=N+1}^{2N+2}h^{p_i}\varphi_i\Big)_{L^2} + \sum_{j=0}^{N}\sum_{i=N+1}^{2N+2}\big(\mathcal{L}^\mu_{ij}\phi_j,\varphi_i\big)_{L^2}\\
&= \Big(\sum_{j=0}^{N}\mathcal{L}^\mu_{0j}\phi_j,\varphi\Big)_{L^2} + \sum_{j=0}^{N}\sum_{i=N+1}^{2N+2}\big((\mathcal{L}^\mu_{ij}-h^{p_i}\mathcal{L}^\mu_{0j})(\phi_j-\widetilde{\phi}_j),\varphi_i\big)_{L^2}\\
&\qquad + \sum_{j=N+1}^{2N+2}\sum_{i=N+1}^{2N+2}\big((\mathcal{L}^\mu_{ij}-h^{p_i}\mathcal{L}^\mu_{0j})\widetilde{\phi}_j,\varphi_i\big)_{L^2}
\end{aligned}$$

The last two terms are of size $\mathcal{O}(\mu^{1+\tilde{N}})$ because, $\varphi_i=\mathcal{O}(\mu^{1+\tilde{N}/2})$ for $i\in\{N+1,\dots,2N+2\}$ and $\phi_j-\widetilde{\phi}_j=\mathcal{O}(\mu^{1+\tilde{N}/2})$ for $j\in\{0,\dots,N\}$ and $\widetilde{\phi}_j=\mathcal{O}(\mu^{1+\tilde{N}/2})$ for $j\in\{N+1,\dots,2N+2\}$, by the previously obtained estimates. It is informative to revisit the above calculations in the light of the Galerkin interpretation described in Section 13.3, starting with

$$\Big(\sum_{j=0}^{2N+2}\mathcal{L}^\mu_{0j}\widetilde{\phi}_j,\varphi\Big)_{L^2} = \iint_\Omega \nabla^\mu_{\boldsymbol{x},z}\widetilde{\Phi}^{\rm app}\cdot\nabla^\mu_{\boldsymbol{x},z}\Psi^{\rm app}\,{\rm d}\boldsymbol{x}\,{\rm d}z = \Big(\widetilde{\phi}_j,\sum_{i=0}^{2N+2}\mathcal{L}^\mu_{0i}\varphi_i\Big)_{L^2}$$

where $\Psi^{\rm app} \stackrel{\rm def}{=} \sum_{i=0}^{2N+2} \Psi_i(\boldsymbol{x},z)\widetilde{\varphi}_i(\boldsymbol{x})$. In any case, we deduce the bound on r_0 the first component of $\boldsymbol{r}$ as desired. The bounds on the other ones follow immediately since $\boldsymbol{r} = r_0\boldsymbol{l}$. The bound on the last one is obtained in the same way, using $\boldsymbol{l}(h)\bullet\partial_t\boldsymbol{\phi} = \partial_t\psi - \varepsilon(\partial_t\zeta)\boldsymbol{l}'(h)\bullet\boldsymbol{\phi}$ and replacing $\partial_t\zeta$ with the formula from the first equations (that is, using the relation with the formulation eq. (13.17)), and a little algebra. $\square$

Theorem 13.15 (Local well-posedness)**.** *Let $d \in \mathbb{N}^\star$, $N \in \mathbb{N}$, $s \in \mathbb{N}$, $s > 1 + d/2$, $h_\star > 0$, $a_\star > 0$, $\mu^\star > 0$, and $M^\star \geq 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm SW}$, any $b \in W^{s+1,\infty}(\mathbb{R}^d)$, and any $(\zeta_0,\boldsymbol{\phi}_0) \in H^s(\mathbb{R}^d) \times X^s_\mu$ satisfying the relation eq. (13.26),*

$$h_0 \stackrel{\rm def}{=} 1 + \varepsilon\zeta_0 - \beta b \geq h_\star > 0, \qquad \mathfrak{a}_{\rm IK}\,\big|_{t=0} \geq a_\star > 0$$

where $\mathfrak{a}_{\rm IK}$ is defined in Remark 13.16 and

$$M_0 \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^s} + \big|\varepsilon\boldsymbol{\phi}\big|_{X^s_\mu} + \big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

there exists a unique $(\zeta,\boldsymbol{\phi}) \in \mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times X^s_\mu)$ solution to the Isobe–Kakinuma system, eq. (13.24), with initial data $(\zeta,\boldsymbol{\phi})\,|_{t=0} = (\zeta_0,\boldsymbol{\phi}_0)$; and we have for any $t \in [0,T/M_0]$

$$\big|\zeta(t,\cdot)\big|_{H^s} + \big|\boldsymbol{\phi}(t,\cdot)\big|_{X^s_\mu} \leq C \times \Big(\big|\zeta_0\big|_{H^s} + \big|\boldsymbol{\phi}_0\big|_{X^s_\mu}\Big)$$

and $\inf_{\mathbb{R}^d}(1+\varepsilon\zeta(t,\cdot)-\beta b) \geq h_\star/2$, $\inf_{\mathbb{R}^d}\mathfrak{a}_{\rm IK}(t,\cdot) \geq a_\star/2$.

Remark 13.16. *The scalar function $\mathfrak{a}_{\rm IK}$ plays the role of the Rayleigh–Taylor sign condition for the the water wave problem; see Theorem 2.9. It is defined as follows:*

$$\mathfrak{a}_{\rm IK} \stackrel{\rm def}{=} 1 + \varepsilon\boldsymbol{l}'(h)\bullet\partial_t\boldsymbol{\phi} + \varepsilon^2\boldsymbol{u}\cdot\big(\boldsymbol{l}'(h)\bullet\nabla\boldsymbol{\phi} - (\boldsymbol{l}''(h)\bullet\boldsymbol{\phi})(\beta\nabla b)\big) + \varepsilon^2\mu^{-1}w(\boldsymbol{l}''(h)\bullet\boldsymbol{\phi}),$$

where $(\boldsymbol{u},w)$ are defined by eq. (13.25). Its initial value, depending on $(\zeta_0,\boldsymbol{\phi}_0,b)$, is uniquely determined by differentiating with respect to time the relation eq. (13.26), substituting therein the value of $(\partial_t\zeta)\,|_{t=0}$ given by the first equation in eq. (13.24), and solving the resulting system of differential equations supplemented with the last equation in eq. (13.24)—which is of the form (13.28)—to determine uniquely $(\partial_t\boldsymbol{\phi})\,|_{t=0}$. Notice (and compare with Remark 2.10) the identity

$$\mathfrak{a}_{\rm IK} = -\partial_z P^{\rm app}\,\big|_{z=\varepsilon\zeta}\,, \qquad \partial_z P^{\rm app} \stackrel{\rm def}{=} -1 - \varepsilon\partial_z\big(\partial_t\Phi^{\rm app} + \tfrac{\varepsilon}{2}|\nabla_{\boldsymbol{x},z}\Phi^{\rm app}|^2\big).$$

We immediately deduce from the above and the solvability of the problem (13.28), another version of the theorem which is more suitable to the comparison with Theorem 2.9.

Corollary 13.17 (Local well-posedness)**.** *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, $s > 1+d/2$, $h_\star > 0$, $a_\star > 0$, $\mu^\star > 0$, and $M^\star \geq 0$. There exist $T > 0$ and $C > 0$ such that the for any $(\mu,\varepsilon,\beta) \in \mathfrak{p}_{\rm SW}$, any $b \in W^{s+2,\infty}(\mathbb{R}^d)$, and any $(\zeta_0,\psi_0) \in H^{s+1}(\mathbb{R}^d) \times \mathring{H}^{s+1}(\mathbb{R}^d)$ such that*

$$\inf_{\boldsymbol{x}\in\mathbb{R}^d}\big(1+\varepsilon\zeta_0-\beta b\big) \geq h_\star > 0, \qquad \inf_{\boldsymbol{x}\in\mathbb{R}^d}\mathfrak{a}_{\rm IK}\,|_{t=0} \geq a_\star > 0$$

where $\mathfrak{a}_{\rm IK}$ is defined in Remark 13.16 with $\boldsymbol{\phi}_0$ the unique solution to eq. (13.26) and $\boldsymbol{l}(h_0)\bullet\boldsymbol{\phi}_0 = \psi_0$,

$$M_0 \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^{s+1}} + \big|\nabla\psi_0\big|_{H^s} + \big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

there exists a unique $(\zeta,\psi) \in \mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times\mathring{H}^s(\mathbb{R}^d))$ solution to the Isobe–Kakinuma system under formulation eq. (13.17), with initial data $(\zeta,\psi)\,|_{t=0} = (\zeta_0,\psi_0)$; and we have for any

$t \in [0, T/M_0]$[65]

$$\big|\zeta(t,\cdot)\big|_{H^s} + \big|\nabla\psi(t,\cdot)\big|_{H^{s-1}} \leq C \times \Big(\big|\zeta_0\big|_{H^{s+1}} + \big|\nabla\psi_0\big|_{H^s}\Big).$$

and $\inf_{\mathbb{R}^d}(1+\varepsilon\zeta(t,\cdot)-\beta b) \geq h_\star/2$, $\inf_{\mathbb{R}^d} \mathfrak{a}_{\rm IK}(t,\cdot) \geq a_\star/2$.

Sketch of the proof. Let us quickly discuss the key elements of the proof of Theorem 13.15, which can be found in [341] and [226, §4]. It follows the energy method, as presented in Section 8.6, with a few tweaks given the structure of the equations. First we extract the quasilinear structure to the Isobe–Kakinuma system by differentiating several times the equations and neglecting order-zero terms (for our functional setting). By direct calculations and product and commutator estimates in Appendix II, we find that for $\boldsymbol{k} \in \mathbb{N}^d$ a multi index with $|\boldsymbol{k}| \leq s$, applying $\partial^{\boldsymbol{k}}$ to eq. (13.24) yields the system of equations

$$\begin{pmatrix} 0 & \boldsymbol{l}(h)^\top \\ -\boldsymbol{l}(h) & {\rm O}_{N+1} \end{pmatrix} \begin{pmatrix} (\partial_t + \varepsilon\boldsymbol{u}\cdot\nabla)\partial^{\boldsymbol{k}}\zeta \\ (\partial_t + \varepsilon\boldsymbol{u}\cdot\nabla)\partial^{\boldsymbol{k}}\boldsymbol{\phi} \end{pmatrix} + \begin{pmatrix} \mathfrak{a}_{\rm IK} & \boldsymbol{0}^\top \\ \boldsymbol{0} & \mathcal{L}^\mu[h,\beta\nabla b] \end{pmatrix} \begin{pmatrix} \partial^{\boldsymbol{k}}\zeta \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} r_{\boldsymbol{k}} \\ \boldsymbol{r}_{\boldsymbol{k}} \end{pmatrix},$$

where $\mathfrak{a}_{\rm IK}$ is defined as in Remark 13.16, and the remainder term $(r_{\boldsymbol{k}}, \boldsymbol{r}_{\boldsymbol{k}})$ is uniformly bounded in an appropriate space, and plays no role for the local-in-time existence and control of solutions. Then the key idea consists in using the special symmetric structure of the equations and test the above identity against

$$(\partial_t + \varepsilon\boldsymbol{u}\cdot\nabla)\begin{pmatrix} \partial_t\partial^{\boldsymbol{k}}\zeta \\ \partial_t\partial^{\boldsymbol{k}}\boldsymbol{\phi} \end{pmatrix}.$$

Then the contribution from the first term of the identity vanishes identically and from the second term arises (up to commutator terms) $\mathcal{E}_{\boldsymbol{k}}'(t)$ where $\mathcal{E}_{\boldsymbol{k}}$ is the suitable energy functional

$$\mathcal{E}_{\boldsymbol{k}}(t) \stackrel{\rm def}{=} \big(\ \mathfrak{a}_{\rm IK}\,\partial^{\boldsymbol{k}}\zeta(t,\cdot)\ ,\ \partial^{\boldsymbol{k}}\zeta(t,\cdot)\ \big)_{L^2} + \big(\ \mathcal{L}^\mu[h,\beta\nabla b]\partial^{\boldsymbol{k}}\boldsymbol{\phi}(t,\cdot)\ ,\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}(t,\cdot)\ \big)_{L^2}.$$

One of the difficulties stemming from the structure of the equations when compared with the situation of standard quasilinear systems discussed in Theorem 8.3 (where the hypersurface $t = 0$ in the space-time $\mathbb{R}^d \times \mathbb{R}$ is not characteristic, that is the matrix operator in front of time derivatives is invertible), is that one cannot deduce a control of time derivatives of the unknowns—which appear in the commutator and remainder terms—straightforwardly from the equations. Yet by combining the last equation in the Isobe–Kakinuma system, eq. (13.24), and the system obtained by time-differentiating the system of constraints, eq. (13.26), and substituting therein the value of $\partial_t\zeta$ given by the first equation in eq. (13.7), we obtain an elliptic system of differential equations for $\partial_t\boldsymbol{\phi}$ with the same structure as eq. (13.28), and from which we can infer the desired control of $\partial_t\boldsymbol{\phi}$, from the pointwise-in-time control of $(\zeta,\boldsymbol{\phi})$. As a matter of fact, this process must be realized twice as the control of $\partial_t a$ demands the control of $\partial_t^2\boldsymbol{\phi}$.

Another key ingredient consists in proving that there exists $0 < c \leq C < \infty$ such that

$$c\,\big(\big|\zeta\big|_{H^s}^2 + \big|\boldsymbol{\phi}\big|_{X^s_\mu}^2\big) \leq \sum_{|\boldsymbol{k}|=0}^{s} \mathcal{E}_{\boldsymbol{k}}(t) \leq C\,\big(\big|\zeta\big|_{H^s}^2 + \big|\boldsymbol{\phi}\big|_{X^s_\mu}^2\big).$$

The upper bound is easily found provided $\mathfrak{a}_{\rm IK}, \varepsilon\zeta, \beta b$ are sufficiently regular, and the lower bound follows follows from eq. (13.29)—using the non-cavitation assumption $h \geq h_\star > 0$ and the linear

[65] This estimate shows an apparent loss of one derivative between the regularity of the data at positive time, and the initial regularity. We can withdraw this loss of derivative by measuring the size of data with the functional

$$\mathcal{E}_s(t) \stackrel{\rm def}{=} \big|\zeta(t,\cdot)\big|_{H^s}^2 + \big|\nabla\psi(t,\cdot)\big|_{H^{s-1}}^2 + \sum_{|\boldsymbol{k}|=0}^{s} \big|\big(\partial^{\boldsymbol{k}}\nabla\psi - \varepsilon w\partial^{\boldsymbol{k}}\nabla\zeta\big)(t,\cdot)\big|_{L^2}^2,$$

which stems from considering $\boldsymbol{l}(h)\bullet(\partial^{\boldsymbol{k}}\nabla\boldsymbol{\phi})$ instead of $\partial^{\boldsymbol{k}}\nabla\psi = \partial^{\boldsymbol{k}}\nabla(\boldsymbol{l}(h)\bullet\boldsymbol{\phi})$. Here we recognize the natural energy functional in terms of Alinhac's good unknowns; see Remark 2.10.

independence of the function $z \mapsto \big(z+1-\beta b(\boldsymbol{x})\big)^{p_i}$—and, obviously, the hyperbolicity condition that $\mathfrak{a}_{\rm IK}$ is bounded from below by a positive constant.

Eventually, we obtain as usual a differential equation for the functionals $\mathcal{E}_{\boldsymbol{k}}$ (with $|\boldsymbol{k}|\in\{0,\dots,s\}$) from which—using Gronwall's estimate and the above equivalence—the desired *a priori* control of $(\zeta,\boldsymbol{\phi})_{H^s\times X^s_\mu}$ follows. The completion of the proof uses a parabolic regularization of the equations and a limiting procedure, as in Section 8.6.4. □

As consequence to Theorem 13.12, Corollary 13.17, the equivalence between the two formulations of the Isobe–Kakinuma model, eq. (13.7) (or eq. (13.24)) and eq. (13.17) and the stability result for the water waves system given in [268, Theorem 4.18], we have the following result.

Theorem 13.18 (Convergence)**.** *Let $d\in\mathbb{N}^\star$, $N\in\mathbb{N}$, $\mu^\star>0$, $s\in\mathbb{N}$, $h_\star>0$, $a_\star>0$ and $M^\star\geq 0$. There exist $m\in\mathbb{N}$, $T>0$ and $C>0$ such that for any $(\mu,\varepsilon,\beta)\in\mathfrak{p}_{\rm SW}$, any $b\in W^{s+m,\infty}(\mathbb{R}^d)$, any $T^\star>0$ and any $(\zeta_0,\psi_0)\in H^{s+m}\times\mathring{H}^{s+m}(\mathbb{R}^d)$ such that*

$$\inf_{\boldsymbol{x}\in\mathbb{R}^d}\big(1+\varepsilon\zeta_0-\beta b\big)\geq h_\star>0,\qquad \inf_{\boldsymbol{x}\in\mathbb{R}^d}\mathfrak{a}_{\rm IK}\,|_{t=0}\geq a_\star>0,\qquad \inf_{\boldsymbol{x}\in\mathbb{R}^d}\mathfrak{a}\,|_{t=0}\geq a_\star>0$$

where $\mathfrak{a}_{\rm IK}$ is defined in Remark 13.16 and $\mathfrak{a}$ is defined in Remark 2.10, and

$$M\stackrel{\rm def}{=}\big|\varepsilon\zeta_0\big|_{H^{s+m}}+\big|\varepsilon\nabla\psi_0\big|_{H^{s+m-1}}+\big|\beta b\big|_{W^{s+m,\infty}}\leq M^\star,$$

there exists a unique $(\zeta_{\rm ww},\psi_{\rm ww})$ classical solution to the water waves system, eq. (2.7), with initial data $\big(\zeta_{\rm ww},\psi_{\rm ww}\big)\,|_{t=0}=(\zeta_0,\psi_0)$; and a unique $(\zeta_{\rm IK},\psi_{\rm IK})$ classical solution to the Isobe–Kakinuma model, eq. (13.17), with initial data $\big(\zeta_{\rm IK},\psi_{\rm IK}\big)\,|_{t=0}=(\zeta_0,\psi_0)$; both defined on the time interval $[0,T/M]$ and one has for any $t\in[0,T/M]$,

$$\big|(\zeta_{\rm ww}-\zeta_{\rm IK})(t,\cdot)\big|_{H^s}+\big|(\nabla\psi_{\rm ww}-\nabla\psi_{\rm IK})(t,\cdot)\big|_{H^s}\leq C\,\mu^{1+\tilde N}\,t\,\big(\big|\zeta_0\big|_{H^{s+m}}+\big|\nabla\psi_0\big|_{H^{s+m-1}}\big)\,,$$

with $\tilde N=2N$ in the flat bottom case, and $\tilde N=2\lfloor N/2\rfloor$ otherwise.

Remark 13.19. *The "loss of derivatives", $m\in\mathbb{N}$ grows with N as $m=4N+p$ in the flat bottom case, and $m=2N+p$ otherwise, with some fixed $p\in\mathbb{N}$ a universal constant (if, say, $s>d/2$).*

13.7 Discussion and open questions

The main open question concerning the study of the Isobe–Kakinuma model, and more generally systems of the form (13.5)–(13.6) based on different vertical distributions, is their ability to reproduce (sufficiently regular) solutions to the water waves system to any prescribed accuracy by taking the rank of the model in the family, N, sufficiently large. It is clear that Theorem 13.18 is not satisfactory in that respect since—just as for the high order models in Section 11, see Proposition 11.10—it exhibits a loss of derivatives, m, which grows with N. Yet this important loss of derivative can be tracked back to the "second step" in the proof of Theorem 13.12, where smallness of the functions ϕ_j, measured in powers of the shallow water parameter (see eq. (13.30)), is deduced from the specific choice of the vertical distribution. The rest of the arguments appears fairly robust and does not require so much regularity. Hence we can hope that solutions to the Isobe–Kakinuma model—or to other models with the same structure—as provided by Corollary 13.17 do converge towards the corresponding solution to the water waves system as N grows, with precision $\mathcal{O}(N^{-\alpha})$ with some $\alpha>0$ (and maybe $\mathcal{O}(\mu^\beta N^{-\alpha})$ where $\beta>0$ grows with the number of derivatives that we tolerate to lose). This hope is supported by the modal analysis in Section 13.5. As far as I know there is essentially no rigorous result in that direction. Yet one should recall that the convergence of the expansion displayed in eq. (13.3) was proved for a certain choice of the vertical distribution in [355], and report that very satisfactory numerical results were obtained in [44, 421] (and references therein). Of utmost importance concerning this matter is to provide some clues as for the best choice of the vertical distribution in a given situation (shallow water *vs* deep water, weak *vs* strong nonlinearities, properties of the bottom topography, *etc.*).

Additionally, for fixed given $N \neq 0$, all the questions asked in Section 8.7 are completely open, and interesting as potentially giving valuable information on solutions to the water waves system. It can be noticed that the argument in the "small-amplitude, large-time dynamics" paragraph therein, concerning the order of the approximation to $\frac{1}{\mu} G^{\mu}[0, \beta b]$ given by the model is rather encouraging, and that we can hope that the result in [62] for the Saint-Venant system can be extended to the case $N \neq 0$.

CHAPTER E

Non-hydrostatic models for interfacial waves

Toujours vouloir tout essayer, et recommencer

— Michel Berger, *Le paradis blanc*

Contents

Foreword

We have introduced in Section 3 the equations extending the water waves system to the framework of interfacial waves between two layers of incompressible, homogeneous, inviscid and immiscible fluids with potential flows. The physical motivation for studying such systems is the reported (and ubiquitous) existence of coherent waves traveling at the sharp interface between, say, fresh and warm water above denser salted cold water. One can refer to *e.g.* [235, 213] for a small peek at the vast literature on the subject. The main features of these waves is that they have tremendously large amplitudes—sometimes of the order of magnitude of the layer itself—, very long wave length, and travel over very long distances. Hence the assumptions of the shallow water regime, and in

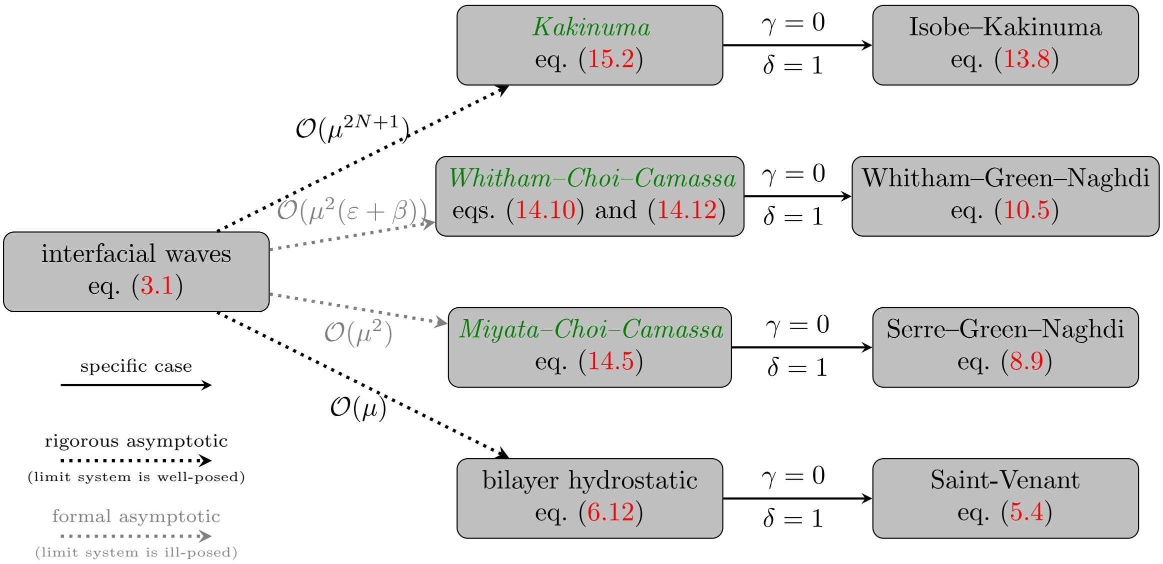


Figure E: Models in Chapter E (in *green*) and some filiations.

particular the fact that we do not impose any smallness assumption on the amplitude of the wave, is perfectly suited to the study of such waves. It is therefore very tempting to introduce asymptotic models for interfacial waves which are analogous to the asymptotic models for the water waves system. This was done in the hydrostatic framework in Section 6 and non-hydrostatic models are the subject of this chapter.

In addition to the physical motivation, there are interesting new features and challenges when studying interfacial waves. First and foremost, as we have seen in Section 3.4, three additional dimensionless parameters come into view, namely

$$\alpha = \frac{a_{\rm top}}{a_{\rm int}} \quad ; \quad \delta = \frac{d_1}{d_2} \quad ; \quad \gamma = \frac{\rho_1}{\rho_2}$$

respectively the amplitude ratio of the free surface and interface, the depth ratio between the two layers, and the density ratio. Hence there are plethora of interesting limits to consider. We will focus here on the framework which is the most similar to the one-layer case[66] and in particular we will assume that the two layers are of comparable depth, both small with respect to the typical horizontal wavelength of the flow. We have already discussed the relation between the limit of small density contrast, $\gamma \nearrow 1$ and the rigid-lid hypothesis, in Section 3.1.1, and in more details in the hydrostatic framework in Section 6.2. Somewhat inconsistently, we will restrict henceforth to the rigid-lid situation[67] without assuming the Boussinesq approximation, yet allowing γ to approach unity. To summarize, our results will hold for parameters in the following set.

Definition (Shallow water/Shallow water asymptotic regime)**.** *Given $\mu^\star, \delta_\star, \delta^\star > 0$, we let*

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} = \big\{(\mu,\varepsilon,\beta,\delta,\gamma) \ : \ \mu \in (0,\mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1], \ \delta \in [\delta_\star,\delta^\star], \gamma \in [0,1)\big\}.$$

As discussed in Section 3.3, one of the main striking difference between the water waves system and the corresponding interfacial waves system is the emergence of *Kelvin–Helmholtz instabilities*

[66] see *e.g.* [101, 54] for related studies in other physically relevant asymptotic regimes.
[67] see *e.g.* [99, 152, 153] for related studies in the free-surface framework.

in the latter. Recall that the provided modal analysis shows that large wavenumber modes are unstable, and that the exponential growth rate takes arbitrarily large values as the wavenumber goes to infinity. This explains why the initial-value problem associated with the nonlinear system is strongly ill-posed outside of the analytic framework as discussed in Section 3.5. This appears to contradict the fact that, as we said, large interfacial waves do exist and appear remarkably stable! An answer to this paradox has been given by Lannes in [267], by introducing interfacial tension effects: it is shown that well-posedness is restored, and more importantly the time of existence of solutions grows as $\mu \searrow 0$, consistently with the fact that the hydrostatic equations for interfacial waves are well-posed, as seen in Section 6. It should be emphasized however that interfacial tension is not physical at the interface between two miscible fluids such as fresh and salted water; here it plays the role of a regularizing operator acting mostly on the high (spatial) frequency component of the flow. The real physical explanation is that mixing occurs, yet on a very thin transition layer: the *pycnocline*. In the absence—to my knowledge—of a simple expression revealing the effective influence of such mixing in the equations, we shall discard any effect when deriving asymptotic models.

It should be emphasized however that the models studied in this manuscript behave very differently regarding Kelvin–Helmholtz instabilities. Indeed, as the derivation focuses on the low frequency (large wavelength) component of the flow, the high frequency behavior can be very dissimilar between the different models, and hence with respect to the original interfacial waves system. A key revelation of the forthcoming study is the following.

- The ***Miyata–Choi–Camassa*** model, which is analogous to the Green–Naghdi system (see Section 8) and studied in Section 14, overestimates Kelvin–Helmholtz instabilities.
- This unfortunate behavior can be corrected through artificial—but harmless for the precision (in the sense of consistency) of the asymptotic model—modifications, which naturally yields fully dispersive systems analogous to the ones presented in Section 10, named ***Whitham–Choi–Camassa***; or regularized system. This is shown in Section 14.5.
- The ***Kakinuma*** model, which extends the Isobe-Kakinuma model (see Section 13) to the bilayer framework and is studied in Section 15, inherently tames Kelvin–Helmholtz instabilities.

The latter model can be expected to be useful for understanding the propagation of long interfacial waves, focusing on the large-scale dynamics of the flow, and discarding small-scale effects as irrelevant. Once again, this should not blurry the fact that mixing do occur, and may in some circumstances play an important role on the large-scale dynamics. Models with the aim of tracking these effects—at least at first order—should use the continuously stratified Euler equations as a starting point. Yet as we have seen in Section 7, very little is known for this system in the shallow water regime. An important reference—in my opinion—dealing with long weakly dispersive internal (and not interfacial) waves is [138]. The Perspectives section in that reference supports and complements the present discussion.

14 The Miyata–Choi–Camassa system

We introduce the weakly dispersive fully nonlinear shallow water model, known in the literature as the Miyata–Choi–Camassa system, and introduced by Miyata [324, 325], Mal'tseva [297] and Choi and Camassa [99, 101], and which directly echoes the Green–Naghdi model for surface gravity waves introduced and studied in Section 8. To this aim, we start from the full bilayer interfacial waves system with rigid-lid, namely eq. (3.15). Proposition 4.10 yields

$$\frac{1}{\mu}\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1 = -\frac{1}{\mu}\mathcal{G}^{\mu}[-\varepsilon\zeta_2, 0]\psi_1 = \nabla\cdot\big(h_1(\mathrm{Id}+\mu\mathcal{T}[h_1,\mathbf{0}])^{-1}\nabla\psi_1\big) + \mathcal{O}(\mu^2) \tag{14.1}$$

$$\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2 = \frac{\delta}{\mu}\mathcal{G}^{\mu/\delta^2}[\varepsilon\delta\zeta,\beta\delta b]\psi = -\nabla\cdot\big(h_2(\mathrm{Id}+\mu\mathcal{T}[h_2,\beta\nabla b])^{-1}\nabla\psi\big) + \mathcal{O}(\mu^2/\delta^5), \tag{14.2}$$

where $h_1 = 1-\varepsilon\zeta_2$, $h_2 = \frac{1}{\delta}+\varepsilon\zeta_2-\beta b$ and

$$\mathcal{T}[h,\beta\nabla b]\boldsymbol{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\boldsymbol{u}) + \frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big) - h^2(\beta\nabla b)\nabla\cdot\boldsymbol{u}\Big) + (\beta\nabla b\cdot\boldsymbol{u})(\beta\nabla b).$$

Plugging these expansions into eq. (3.15) and withdrawing $\mathcal{O}(\mu^2)$ terms yields

$$\begin{cases} \partial_t\zeta_2 = \nabla\cdot(h_1\boldsymbol{u}_1) = -\nabla\cdot(h_2\boldsymbol{u}_2), \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_1|^2 - \mu\varepsilon\mathcal{R}[h_1,\mathbf{0},\boldsymbol{u}_1] = -\gamma^{-1}p_{\rm int} \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_2|^2 - \mu\varepsilon\mathcal{R}[h_2,\beta\nabla b,\boldsymbol{u}_2] = -p_{\rm int}, \end{cases} \tag{14.3}$$

where

$$\mathcal{R}[h,\beta\nabla b,\boldsymbol{u}] \stackrel{\text{def}}{=} \frac{\boldsymbol{u}}{3h}\cdot\nabla(h^3\nabla\cdot\boldsymbol{u}) + \frac{1}{2}h^2(\nabla\cdot\boldsymbol{u})^2, \\ -\frac{1}{2}\Big(\frac{\boldsymbol{u}}{h}\cdot\nabla\big(h^2(\beta\nabla b\cdot\boldsymbol{u})\big) + h(\beta\nabla b\cdot\boldsymbol{u})\nabla\cdot\boldsymbol{u} + (\beta\nabla b\cdot\boldsymbol{u})^2\Big).$$

and $\boldsymbol{u}_\ell$ ($\ell\in\{1,2\}$) is deduced from (ζ_2,ψ_1,ψ_2) after solving the equation

$$\begin{aligned} h_1\nabla\psi_1 &= h_1\boldsymbol{u}_1 + \mu h_1\mathcal{T}[h_1,\mathbf{0}]\boldsymbol{u}_1 \stackrel{\text{def}}{=} \mathfrak{T}^{\mu}[h_1,\mathbf{0}]\boldsymbol{u}_1, \\ h_2\nabla\psi_2 &= h_2\boldsymbol{u}_2 + \mu h_2\mathcal{T}[h_2,\beta\nabla b]\boldsymbol{u}_2 \stackrel{\text{def}}{=} \mathfrak{T}^{\mu}[h_2,\beta\nabla b]\boldsymbol{u}_2. \end{aligned}$$

As for the Green–Naghdi system, we infer a system written with only differential operators in terms of the unknowns ζ_2, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, namely

$$\begin{cases} \partial_t\zeta_2 = \nabla\cdot(h_1\boldsymbol{u}_1) = -\nabla\cdot(h_2\boldsymbol{u}_2), \\ \big(\mathrm{Id}+\mu\mathcal{T}[h_1,\mathbf{0}]\big)\partial_t\boldsymbol{u}_1 + \frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \varepsilon(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 + \mu\varepsilon\mathcal{Q}[h_1,\mathbf{0},\boldsymbol{u}_1] = -\gamma^{-1}\nabla p_{\rm int}, \\ \big(\mathrm{Id}+\mu\mathcal{T}[h_2,\beta\nabla b]\big)\partial_t\boldsymbol{u}_2 + \frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \varepsilon(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 + \mu\varepsilon\mathcal{Q}[h_2,\beta\nabla b,\boldsymbol{u}_2] = -\nabla p_{\rm int}, \end{cases} \tag{14.4}$$

where

$$\mathcal{Q}[h,\beta\nabla b,\boldsymbol{u}] \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}) - (\nabla\cdot\boldsymbol{u})^2\big)\Big), \\ + \frac{\beta}{2h}\Big(\nabla\big(h^2(\boldsymbol{u}\cdot\nabla)^2 b\big) - h^2\big((\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u}) - (\nabla\cdot\boldsymbol{u})^2\big)\nabla b\Big) + \beta\big((\boldsymbol{u}\cdot\nabla)^2 b\big)(\beta\nabla b).$$

We let the reader complete the family of equivalent formulations listed in Section 8, and conclude with the following compact formulation of the ***Miyata–Choi–Camassa system*** using physical variables (recall Section 3.4) and which one can recognize in [99, 101]:

$$\begin{cases} \partial_t h_1 + \nabla\cdot(h_1\boldsymbol{u}_1) = 0, \\ \partial_t h_2 + \nabla\cdot(h_2\boldsymbol{u}_2) = 0, \\ \rho_1\partial_t\boldsymbol{u}_1 + \rho_1(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 + \nabla p_{\rm int} + g\rho_1\nabla(h_2+b) + \mathcal{P}[h_1,0,\boldsymbol{u}_1] = \boldsymbol{0}, \\ \rho_2\partial_t\boldsymbol{u}_2 + \rho_2(\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 + \nabla p_{\rm int} + g\rho_2\nabla(h_2+b) + \mathcal{P}[h_2,b,\boldsymbol{u}_2] = \boldsymbol{0}, \end{cases} \tag{14.5}$$

with $h_1 = d_1 - \zeta_2$ and $h_2 = d_2 + \zeta_2 - b$, and

$$\mathcal{P}[h,b,\boldsymbol{u}] \stackrel{\rm def}{=} \frac{1}{h}\nabla\Big(h^2\big(\frac{\ddot h}{3}+\frac{\ddot b}{2}\big)\Big) + \Big(\frac{\ddot h}{2}+\ddot b\Big)(\nabla b)$$

and where we denote $\dot h = \partial_t h + \varepsilon\boldsymbol{u}\cdot\nabla h$, $\ddot h = \partial_t \dot h + \varepsilon\boldsymbol{u}\cdot\nabla\dot h$, and similarly $\dot b, \ddot b$.

Let us clarify the physical signification of these equations. The variable $\boldsymbol{u}_1$ (resp. $\boldsymbol{u}_2$) represents the layer-averaged horizontal velocity on the upper (resp. lower) layer, or at least a valid approximation thereof in our asymptotic regime. Hence the first two equations represent the conservation of masses. Discarding the contributions defined by $\mathcal{P}$, we recover the bilayer extension of the Saint-Venant system, given in eq. (6.12). Hence the additional terms defined by $\mathcal{P}$ represent a correction to the hydrostatic pressure, valid at first order for weakly dispersive flows. Recall that $\nabla p_{\rm int}$, which physically represents the pressure at the interface, can be seen as the Lagrange multiplier associated with the rigid-lid constraint: $\nabla\cdot(h_1\boldsymbol{u}_1) + \nabla\cdot(h_2\boldsymbol{u}_2) = -\partial_t(h_1+h_2) = \partial_t b$.

The Miyata–Choi–Camassa system can be derived in the free-surface framework, as in [99]. Yet in this case we cannot rely solely on Proposition 4.10 but use Proposition 4.23 as well; see [152]. In fact we can consider an arbitrary number of interfaces [94, 288], hence extending the multilayer hydrostatic equations, eq. (6.20), to weakly dispersive flows.

A final remark is that by setting $\gamma = 0$ and $\delta = 1$ in eq. (14.3) (respectively eq. (14.4)), we recognize the Green–Naghdi equations for (homogeneous with free-surface) water waves, namely eq. (8.2) (respectively eq. (8.6)).

14.1 Hamiltonian structure

We shall not dwell into the sophisticated variational structures analogous to the ones discussed in Section 8.1.1 for the Green–Naghdi system, but simply remark that the canonical Hamiltonian structure of the interfacial waves system (see Section 3.2) naturally extends to the Miyata–Choi–Camassa system.

More precisely define, according to the expansion of the Dirichlet-to-Neumann operators,

$$\mathscr{H}_{\rm MCC} \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d}(\delta+\gamma)\zeta_2^2 + (\psi_2-\gamma\psi_1)\nabla\cdot\big(h_2\mathfrak{T}^\mu[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big)\,{\rm d}\boldsymbol{x},$$

viewing $\mathscr{H}_{\rm MCC}$ as a functional for $(\zeta_2, \xi_2 \stackrel{\rm def}{=} \psi_2-\gamma\psi_1)$ and using the constraint

$$\nabla\cdot\big(h_1\mathfrak{T}^\mu[h_1,\boldsymbol{0}]^{-1}(h_1\nabla\psi_1)\big) + \nabla\cdot\big(h_2\mathfrak{T}^\mu[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big) = 0,$$

we find that solutions to the Miyata–Choi–Camassa equations eq. (14.3) satisfy the canonical Hamilton equations

$$\partial_t\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix} = \begin{pmatrix}\delta_{\xi_2}\mathscr{H}_{\rm MCC}\\ -\delta_{\zeta_2}\mathscr{H}_{\rm MCC}\end{pmatrix}.$$

As we have seen many times, associated with the Hamiltonian formulation and natural symmetry groups of the system are preserved quantities.

Related to the variation of base level for the velocity potentials we find the obvious conservation of the excess of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = 0, \qquad\qquad \mathscr{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta_2 \,\mathrm{d}\boldsymbol{x}.$$

From horizontal translation invariance (in the flat bottom case) we obtain the conservation of the horizontal impulse

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I} = 0, \qquad\qquad \mathscr{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta_2 \nabla\xi_2 \,\mathrm{d}\boldsymbol{x}. \qquad\qquad (\text{if } \beta b \equiv 0).$$

From time translation invariance we obtain the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{\mathrm{MCC}} = 0.$$

We also have the "trivial"—in the formulation (14.3) but not in the formulation (14.4)— conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}_1 = 0, \qquad\qquad \mathscr{V}_1 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \nabla\psi_1 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \boldsymbol{u}_1 + \mathcal{T}[h_1, \boldsymbol{0}]\boldsymbol{u}_1 \,\mathrm{d}\boldsymbol{x},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}_2 = 0, \qquad\qquad \mathscr{V}_2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \nabla\psi_2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \boldsymbol{u}_2 + \mathcal{T}[h_2, \beta\nabla b]\boldsymbol{u}_1 \,\mathrm{d}\boldsymbol{x}.$$

These conservation laws were already listed in [101], where the Hamiltonian structure was mentioned but not displayed.

14.2 Traveling waves

In the unidimensional ($d = 1$) and flat bottom ($b \equiv 0$) framework, the Miyata–Choi–Camassa system enjoys a semi-explicit family of solitary wave solutions that is satisfying

$$(\zeta_2, \boldsymbol{u}_1, \boldsymbol{u}_2)(t, x) = (\zeta_{2,c}, \boldsymbol{u}_{1,c}, \boldsymbol{u}_{2,c})(x - ct), \qquad \lim_{|x|\to\infty} |(\zeta_{2,c}, \boldsymbol{u}_{1,c}', \boldsymbol{c}_{2,c}')|(x) = 0.$$

Plugging this Ansatz into the equations (14.5) and after a few algebraic computations (see [101]) we are led to the following nonlinear ordinary differential equation:

$$(\zeta')^2 = C\frac{\zeta^2(\zeta - a_-)(\zeta - a_+)}{(\zeta - a_\star)}$$

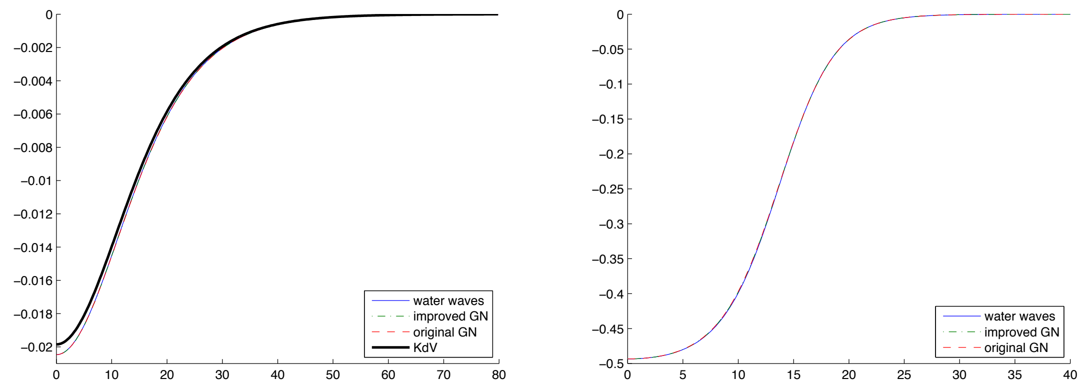
where we denote $\zeta_{2,c} = \zeta_{2,c}$ for simplicity, $C = \frac{3g(\rho_2-\rho_1)}{c^2(\rho_1 d_1^2 - \rho_2 d_2^2)}$, $a_\star = -d_1 d_2 \frac{\rho_1 d_1 + \rho_2 d_2}{\rho_1 d_1^2 - \rho_2 d_2^2}$, and $a_\pm$ are the two roots of the quadratic equation

$$\zeta^2 + (-\tfrac{c^2}{g} - d_1 + d_2)\zeta + d_1 d_2\big(\tfrac{c^2}{c_0^2} - 1\big) = 0$$

and c_0 is the velocity of infinitesimally long wave, that is $c_0^2 = \frac{g d_1 d_2(\rho_2-\rho_1)}{\rho_1 d_2 + \rho_2 d_1}$.

We deduce (again see [101] for the detailed analysis) first that, as in the one-layer case, solitary waves must be supercritical, that is $|c| > |c_0|$. But interesting new phenomena arise. First, on readily sees that in the situation where $\rho_1 d_1^2 = \rho_2 d_2^2$ is critical; in this situation no solitary wave can exists. Less straightforward is that $\rho_2 d_1^2 = \rho_1 d_2^2$ is also singular. Solitary waves are necessarily of depression type if $\rho_2 d_1^2 < \rho_1 d_2^2$, and of elevation type if $\rho_2 d_1^2 > \rho_1 d_2^2$, while the solution degenerates into the flat equilibrium as $\frac{\rho_2 d_1^2}{\rho_1 d_2^2} \to 1$. In the other singular limit, $\frac{\rho_1 d_1^2}{\rho_2 d_2^2} \to 1$, the solitary waves degenerate into a front-like solution (a bore).

The solitary wave solutions to the Miyata–Choi–Camassa system turn out to reproduce remarkably well what is observed in field observations, laboratory experiments and solutions to the full bilayer interfacial waves system; see Figure 14.1 for an illustration and [213] for a thorough discussion and more references.



(a) Small velocity, $c = 1.005$. (b) Large velocity, $c = 1.06065$.

Figure 14.1: Solitary wave solutions to the interfacial waves system (plain blue), Miyata–Choi–Camassa model (green,dash-dotted), fully dispersive counterpart (red, dashed) introduced in Section 14.5 and Korteweg-de Vries (thick black). The figure is taken from [165]. Dimensionless parameters are $\gamma = 1, \delta = 1/2$, $\mu = \varepsilon = 1$. The maximal amplitude predicted by the Miyata–Choi–Camassa model is $|a_{\max}| = 1/2$, corresponding to $c_{\max} = \sqrt{1 + 1/8} \approx 1.06066$.

14.3 Rigorous justification

The following result shows that the Miyata–Choi–Camassa system is an asymptotic model for the bilayer interfacial waves system with precision $\mathcal{O}(\mu^2)$, *in the sense of consistency*. We do not complete the rigorous justification—as was done for the Green–Naghdi system in Section 8.5—by well-posedness, stability and convergence results, because we expect from the modal analysis of Section 14.4 that the Miyata–Choi–Camassa system, as the interfacial waves system, is strongly ill-posed in functional spaces of finite regularity (although this result is yet open). This does not mean that the result below is empty, since non-trivial solutions to the interfacial waves equations (for instance solitary waves) do exist. Moreover the result is straightforwardly adapted to a situation where additional regularizing terms such as the contribution of interfacial tension are included.

Theorem 14.1 (Consistency)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $\delta^\star \geq \delta_\star > 0$, $s \in \mathbb{N}$ and $M^\star \geq 0$. There exists $C > 0$ such that for any*

$$(\mu, \varepsilon, \beta, \delta, \gamma) \in \mathfrak{p}_{\frac{\rm SW}{\rm SW}} \stackrel{\rm def}{=} \big\{(\mu, \varepsilon, \beta, \delta, \gamma) \ : \ \mu \in (0, \mu^\star],\ \varepsilon \in [0, 1],\ \beta \in [0, 1],\ \delta \in [\delta_\star, \delta^\star], \gamma \in [0, 1)\big\}.$$

any $b \in W^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)$, and $(\zeta, \psi_1, \psi_2) \in L^\infty(0, T; H^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)^2)$ solution to the interfacial waves equations, eq. (3.15), satisfying

$$\forall t \in [0, T], \quad \forall \boldsymbol{x} \in \mathbb{R}^d, \qquad \begin{cases} h_1(t, \boldsymbol{x}) \stackrel{\rm def}{=} 1 - \varepsilon\zeta(t, \boldsymbol{x}) \geq h_\star > 0, \\ h_2(t, \boldsymbol{x}) \stackrel{\rm def}{=} \delta^{-1} + \varepsilon\zeta(t, \boldsymbol{x}) - \beta b(\boldsymbol{x}) \geq h_\star > 0, \end{cases}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \big(\big|\varepsilon\zeta(t, \cdot)\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi_1(t, \cdot)\big|_{H^{1+s_\star}} + \big|\varepsilon\nabla\psi_2(t, \cdot)\big|_{H^{1+s_\star}}\big) + \big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta_2 - \nabla \cdot (h_1 \boldsymbol{u}_1) = r_1, \\ \partial_t\zeta_2 + \nabla \cdot (h_2 \boldsymbol{u}_2) = r_2, \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_1|^2 - \mu\varepsilon\mathcal{R}[h_1, \mathbf{0}, \boldsymbol{u}_1] = -\gamma^{-1}p_{\rm int} + r_3, \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_2|^2 - \mu\varepsilon\mathcal{R}[h_2, \beta\nabla b, \boldsymbol{u}_2] = -p_{\rm int} + r_4, \end{cases}$$

where we denote $\boldsymbol{u}_1 = \mathfrak{T}^\mu[h_1, \mathbf{0}]^{-1}(h_1 \nabla\psi_1)$, $\boldsymbol{u}_2 = \mathfrak{T}^\mu[h_2, \beta\nabla b]^{-1}(h_2\nabla\psi_2)$, *and one has for almost every* $t \in (0,T)$, *denoting* $M \stackrel{\rm def}{=} \big|\nabla\psi_1(t,\cdot)\big|_{H^{1+s_\star}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}$,

$$\big|(r_1(t,\cdot), r_2(t,\cdot))\big|_{(H^s)^2} \leq C\,\mu^2\left(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\right),$$
$$\big|(r_3(t,\cdot), r_4(t,\cdot))\big|_{(H^{s+1})^2} \leq C\,\mu^2\varepsilon M\left(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\right).$$

Proof. The proof is exactly the same as the one of Theorem 8.2—that is a direct consequence of Proposition 4.10, Lemma 8.10 and estimates in Sobolev spaces—once we remark the identities

$$\frac{1}{\mu}\mathcal{G}_1^\mu[\varepsilon\zeta_2]\psi_1 = -\frac{1}{\mu}\mathcal{G}^\mu[-\varepsilon\zeta_2, 0]\psi_1, \qquad \frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2, \beta b]\psi_2 = \frac{\delta}{\mu}\mathcal{G}^{\mu/\delta^2}[\delta\varepsilon\zeta, \delta\beta b]\psi_2.$$

Notice that the functional spaces for $p_{\rm int}$ and time derivatives have not been described but are immaterial at this stage. $\square$

Remark 14.2. *Our consistency result is lazy as we allow to satisfy the mass conservation equations up to a small remainder term. It is possible but more complicated to justify the equations eq. (14.4) with* $\boldsymbol{u}_\ell = \overline{\boldsymbol{u}}_\ell$ ($\ell \in \{1,2\}$) *the layer-averaged velocities, in which case the mass conservation identities hold exactly; see [163] for the rigorous analysis.*

14.4 Modal analysis

We now linearize eq. (14.3) about the constant shear solution[68] in the flat bottom case. Setting $\beta = 0$ and

$$(\zeta_2 = \epsilon\zeta^0, \psi_1 = \boldsymbol{u}_1\cdot\boldsymbol{x} + \epsilon\psi_1^0, \psi_2 = \boldsymbol{u}_2\cdot\boldsymbol{x} + \epsilon\psi_2^0)$$

and keeping only first-order terms in ϵ yields the following system

$$\begin{cases} \partial_t\zeta^0 = \nabla\cdot(-\zeta^0\boldsymbol{u}_1 + \boldsymbol{u}_1^0) = -\nabla\cdot(\zeta^0\boldsymbol{u}_2 + \delta^{-1}\boldsymbol{u}_2^0), \\ \partial_t\psi_1^0 + \frac{\delta+\gamma}{1-\gamma}\zeta^0 + \varepsilon\boldsymbol{u}_1\cdot\nabla\psi_1^0 = -\gamma^{-1}p_{\rm int} \\ \partial_t\psi_2^0 + \frac{\delta+\gamma}{1-\gamma}\zeta^0 + \varepsilon\boldsymbol{u}_2\cdot\nabla\psi_2^0 = -p_{\rm int}, \end{cases} \tag{14.6}$$

where $\boldsymbol{u}_\ell^0$ ($\ell \in \{1,2\}$) are defined by

$$\nabla\psi_1^0 \;=\; \boldsymbol{u}_1^0 - \frac{\mu}{3}\nabla\nabla\cdot\boldsymbol{u}_1^0, \qquad \nabla\psi_2^0 \;=\; \boldsymbol{u}_2^0 - \frac{\mu}{3\delta^2}\nabla\nabla\cdot\boldsymbol{u}_2^0.$$

Denoting $\psi^0 \stackrel{\rm def}{=} \psi_2^0 - \gamma\psi_1^0$ yields the identities

$$\begin{aligned} \nabla\Big(\big((\mathrm{Id} - \tfrac{\mu}{3}\Delta)^{-1} + \gamma\delta^{-1}(\mathrm{Id} - \tfrac{\mu}{3\delta^2}\Delta)^{-1}\big)\psi_1^0\Big) &= -\delta^{-1}(\mathrm{Id} - \tfrac{\mu}{3\delta^2}\Delta)^{-1}\nabla\psi^0 + (\boldsymbol{u}_1^0 + \delta^{-1}\boldsymbol{u}_2^0), \\ \nabla\Big(\big((\mathrm{Id} - \tfrac{\mu}{3}\Delta)^{-1} + \gamma\delta^{-1}(1 - \tfrac{\mu}{3\delta^2}\Delta)^{-1}\big)\psi_2^0\Big) &= (\mathrm{Id} - \tfrac{\mu}{3}\Delta)^{-1}\nabla\psi^0 + \gamma(\boldsymbol{u}_1^0 + \delta^{-1}\boldsymbol{u}_2^0). \end{aligned}$$

From this and the rigid-lid constraint

$$\nabla\cdot(\boldsymbol{u}_1^0 + \delta^{-1}\boldsymbol{u}_2^0) = -(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\nabla\zeta^0,$$

noticing that $\boldsymbol{u}_1^0$ and $\boldsymbol{u}_2^0$ are necessarily gradient vector fields from their definition, we infer the following constant-coefficient linearized equations:

$$\begin{cases} \partial_t\zeta^0 + \boldsymbol{c}_{\rm MCC}(D)\cdot\nabla\zeta^0 - b_{\rm MCC}(D)\psi^0 = 0, \\ \partial_t\psi^0 + a_{\rm MCC}(D)\zeta^0 + \boldsymbol{c}_{\rm MCC}(D)\cdot\nabla\psi^0 = 0, \end{cases}$$

[68] Recall the discussion in Section 3.3: contrarily to the homogeneous case with free-surface, one cannot invoke Galilean invariance to reduce the study to the linearization about the rest (no-shear) solution.

where

$$a_{\rm MCC}(D) \stackrel{\rm def}{=} (\delta+\gamma) - \frac{\frac{\gamma\delta}{\delta+\gamma}}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu}{3}|D|^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu}{3\delta^2}|D|^2)^{-1}} \frac{(\varepsilon(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot D)^2}{|D|^2},$$

$$b_{\rm MCC}(D) \stackrel{\rm def}{=} \frac{1}{\delta+\gamma} \frac{(1+\frac{\mu}{3}|D|^2)^{-1}(1+\frac{\mu}{3\delta^2}|D|^2)^{-1}}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu}{3}|D|^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu}{3\delta^2}|D|^2)^{-1}}|D|^2,$$

$$\boldsymbol{c}_{\rm MCC}(D) \stackrel{\rm def}{=} \frac{\frac{\delta}{\delta+\gamma}(1+\frac{\mu}{3}|D|^2)^{-1}\boldsymbol{u}_2 + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu}{3\delta^2}|D|^2)^{-1}\boldsymbol{u}_1}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu}{3}|D|^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu}{3\delta^2}|D|^2)^{-1}}.$$

Hence we have the dispersion relation

$$(\omega(\boldsymbol{\xi}) - \boldsymbol{c}_{\rm MCC}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2 = a_{\rm MCC}(\boldsymbol{\xi}) b_{\rm MCC}(\boldsymbol{\xi}), \tag{14.7}$$

which we recognize as a weakly dispersive approximation of the dispersion relation of the full (linearized) interfacial waves system, eq. (3.12). Hence the discussion from the latter in Section 3.3 applies word-to-word, and we realize that as soon as $\boldsymbol{u}_2 - \boldsymbol{u}_1 \neq \boldsymbol{0}$ Fourier modes with sufficiently large wavenumbers are exponentially amplified (*i.e. unstable*) with a growth rate taking arbitrarily large values.

This can be seen as a positive feature of the model, since it reproduces the *Kelvin–Helmholtz instabilities* which hold for the original interfacial waves system, and this was in fact pointed out in [101] (see also [287, 237]). This is however only qualitatively true. Since $a_{\rm MCC}(D)$ is always smaller than the corresponding coefficient appearing in the dispersion relation of the full (linearized) interfacial waves system, namely

$$a_{\rm WW}(D) \stackrel{\rm def}{=} (\delta+\gamma) - \frac{\frac{\gamma\delta}{\delta+\gamma}}{\frac{\delta}{\delta+\gamma}\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} + \frac{\gamma}{\delta+\gamma}\frac{\tanh(\sqrt{\mu/\delta^2}|D|)}{\sqrt{\mu/\delta^2}|D|}} \frac{(\varepsilon(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot D)^2}{|D|^2},$$

we conclude that the critical wavenumber (which exists by continuity and is unique by monotonicity) above which some modes are unstable is smaller for the Miyata–Choi–Camassa model than the one of the full system; see Figure 14.2a. Interestingly, the exponential growth rate is proportional to $|(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\boldsymbol{\xi}|$ in both cases, yet with *a priori* different prefactors.

In order to tame Kelvin–Helmholtz instabilities, it has been suggested to include surface tension effects, as in Section 3.1.3. Indeed, in that case—that is for eq. (3.10)—the Fourier multiplier $a_{\rm MCC}(D)$ is replaced with

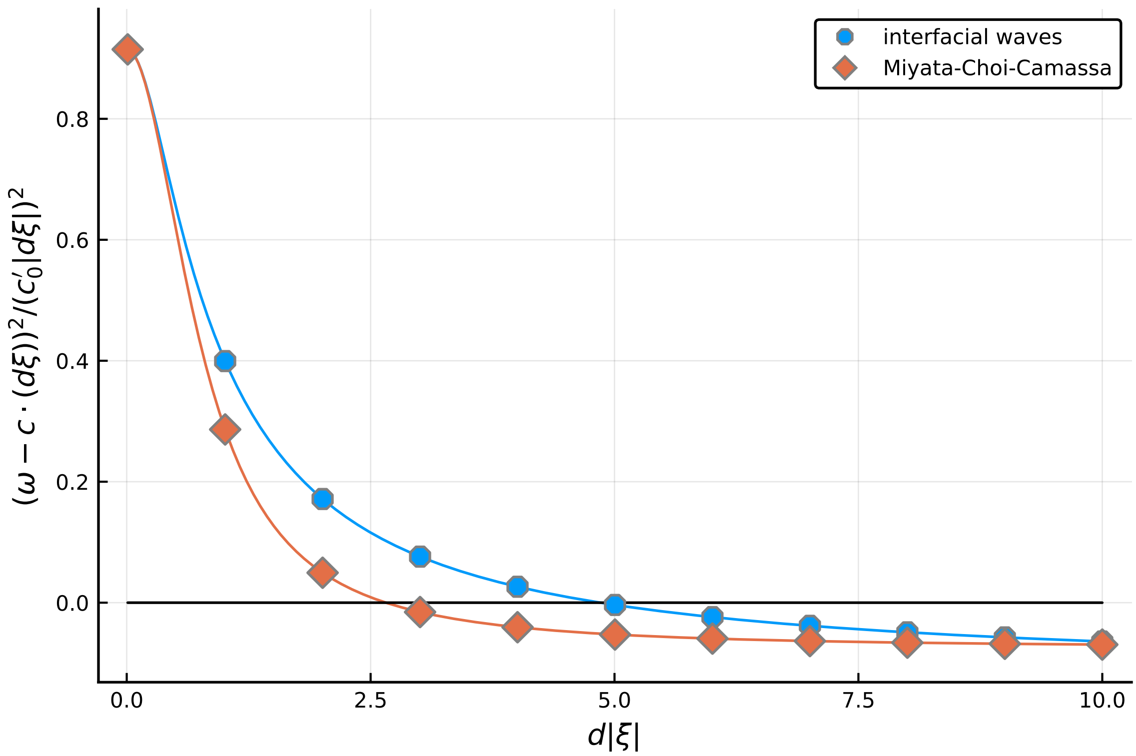
$$a_{\rm Bo}(D) = a(D) + \mu\frac{\delta+\gamma}{\rm Bo}|D|^2$$

where $\mathrm{Bo} \stackrel{\rm def}{=} \frac{g(\rho_2-\rho_1)d_1^2}{\sigma}$ is the *Bond number*. While for sufficiently small Bond numbers one can indeed suppress modal instabilities of the Miyata–Choi–Camassa model, one can check that the critical value below which this property holds is smaller for the Miyata–Choi–Camassa model than the one for the full interfacial waves system. Specifically, the dimensionless number corresponding to $\Upsilon \stackrel{\rm def}{=} \gamma^2\mu\varepsilon^4\,\mathrm{Bo}$ defined in Section 3.5 is

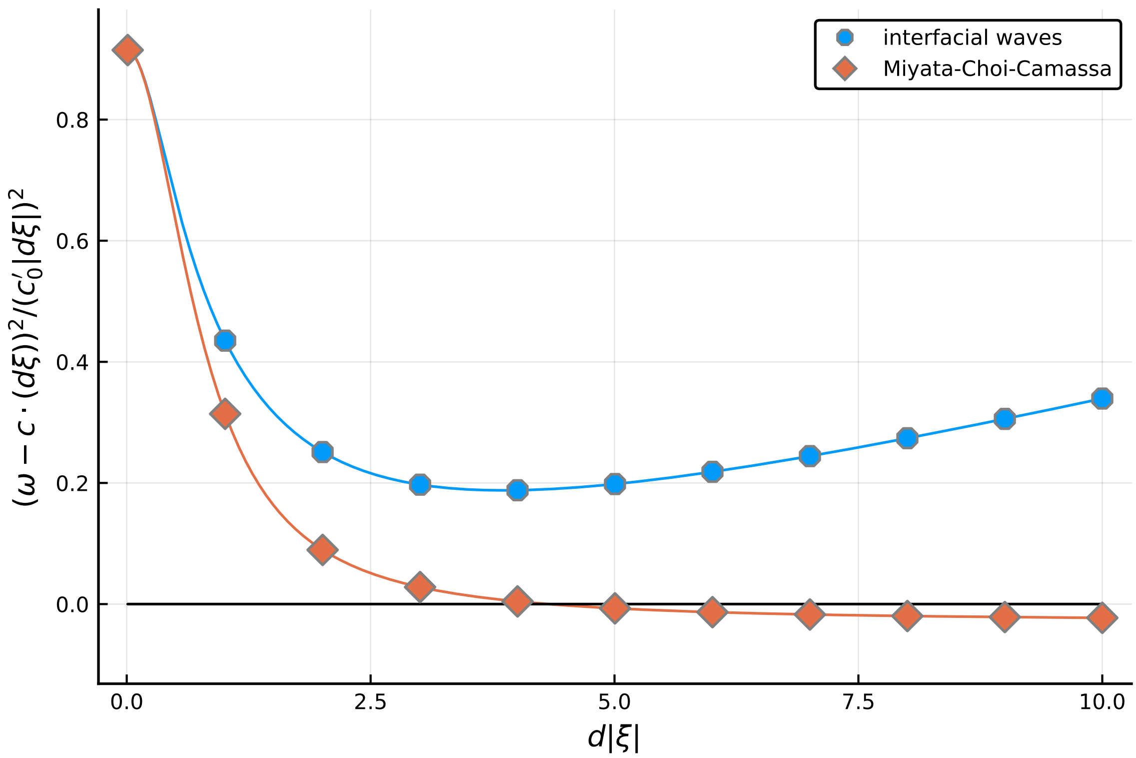
$$\Upsilon_{\rm MCC} \stackrel{\rm def}{=} \gamma\mu\varepsilon^2\,\mathrm{Bo}:$$

all modes are stable if $\Upsilon_{\rm MCC}$ is sufficiently small, while modes with arbitrarily large wavenumber are unstable is $\Upsilon_{\rm MCC}$ is too large. This should be compared with the fact that, as soon as $\mathrm{Bo} < \infty$, modes of the full interfacial waves system with sufficiently large wavenumbers are necessarily stable. See Figure 14.2b for an illustration.

From the above discussion (the reader can refer to [273] for an extended modal analysis) it is quite apparent that *the Miyata–Choi–Camassa model overestimates Kelvin–Helmholtz instabilities.*



(a) Without surface tension.



(b) With surface tension.

Figure 14.2: Dispersion relation. We plot $(\omega(\boldsymbol{\xi}) - \boldsymbol{c}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi})^2/|\boldsymbol{\xi}|^2 = a(\boldsymbol{\xi})b(\boldsymbol{\xi})/|\boldsymbol{\xi}|^2$ (with $\mu = 1$) for the full bilayer interfacial waves system and the Miyata–Choi–Camassa model; negative values indicate unstable modes. We set $\gamma = 0.9$, $\delta = \frac{1}{4}$, $\boldsymbol{u}_2 - \boldsymbol{u}_1 = \frac{1}{2}$. $\mathrm{Bo} = \infty$ in (a) and $\mathrm{Bo} = 15$ in (b).

14.5 Fully dispersive and regularized models

Many attempts have been made in order to "regularize" the Miyata–Choi–Camassa equations, that is proposing new models with the same precision as the original model, but which are not subject to Kelvin–Helmholtz instabilities, even without surface tension [342, 98, 58, 164, 273]. We shall not review all these attempts but concentrate on the one put forward in [164]. The main advantages of the latter is that the family of new models appears straightforwardly as a modification of the Miyata–Choi–Camassa equations that allows a lot of flexibility to tune the linear dispersion behavior (thus allowing to enforce either the full dispersion property or arbitrarily strong regularization properties) and by construction enjoys a canonical Hamiltonian structure. The main—and important—drawbacks is that it involves non-local operators (more precisely Fourier multipliers; see Definition III.1).

The fairly simple observation is that one can freely modify the linear dispersion behavior at large wavenumbers of the system without hurting its precision in the sense of consistency by inserting suitable near-identity Fourier multipliers. In order not to break the Hamiltonian structure in the process, we decide to introduce these operators in the Hamiltonian functional, and formally derive the equations from Hamilton's equations. The rigorous justification can be performed as a second step.

Recall (see Section 14.1) that the Miyata–Choi–Camassa equations eq. (14.3) satisfy the canonical Hamilton equations

$$\partial_t \begin{pmatrix} \zeta_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_{\xi_2}\mathscr{H}_{\rm MCC} \\ -\delta_{\zeta_2}\mathscr{H}_{\rm MCC} \end{pmatrix}.$$

with

$$\mathscr{H}_{\rm MCC} \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta_2^2 + (\psi_2-\gamma\psi_1)\nabla\cdot\big(h_2\mathfrak{T}^\mu[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big)\,{\rm d}\boldsymbol{x},$$

and $(\zeta_2, \xi_2 \stackrel{\rm def}{=} \psi_2-\gamma\psi_1)$, using the constraint

$$\nabla\cdot\big(h_1\mathfrak{T}^\mu[h_1,\mathbf{0}]^{-1}(h_1\nabla\psi_1)\big)+\nabla\cdot\big(h_2\mathfrak{T}^\mu[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big)=0,$$

Define now $\mathfrak{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]:\boldsymbol{u}\mapsto h\boldsymbol{u}+\mu h\mathcal{T}^{\mathsf{F}^\mu}[h,\beta\nabla b]\boldsymbol{u}$ with

$$\mathcal{T}^{\mathsf{F}^\mu}[h,\nabla b]\boldsymbol{u} \stackrel{\rm def}{=} \frac{-1}{3h}\nabla\mathsf{F}^\mu(h^3\mathsf{F}^\mu\nabla\cdot\boldsymbol{u})+\frac{1}{2h}\Big(\nabla\mathsf{F}^\mu\big(h^2(\nabla b)\cdot\boldsymbol{u}\big)-h^2(\nabla b)\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}\Big)+(\nabla b\cdot\boldsymbol{u})(\nabla b)$$

where $\mathsf{F}^\mu=F(\sqrt{\mu}|D|)$ is a self-adjoint (for the L^2 inner-product) Fourier multiplier—at this point it could be any self-adjoint operator commuting with spatial derivatives—to be defined later on. Then defining

$$\mathscr{H}_{\rm MCC}^{\mathfrak{F}_1^\mu,\mathfrak{F}_2^\mu} \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d}(\delta+\gamma)\zeta_2^2+(\psi_2-\gamma\psi_1)\nabla\cdot\big(h_2\mathfrak{T}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big)\,{\rm d}\boldsymbol{x}$$

and enforcing the constraint

$$\nabla\cdot\big(h_1\mathfrak{T}^{\mathsf{F}_1^\mu}[h_1,\mathbf{0}]^{-1}(h_1\nabla\psi_1)\big)+\nabla\cdot\big(h_2\mathfrak{T}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big)=0$$

we find that the canonical Hamilton equations

$$\partial_t\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_{\xi_2}\mathscr{H}_{\rm MCC}^{\mathsf{F}_1^\mu,\mathsf{F}_2^\mu}\\ -\delta_{\zeta_2}\mathscr{H}_{\rm MCC}^{\mathsf{F}_1^\mu,\mathsf{F}_2^\mu}\end{pmatrix}$$

read

$$\begin{cases} \partial_t\zeta_2=\nabla\cdot(h_1\boldsymbol{u}_1)=-\nabla\cdot(h_2\boldsymbol{u}_2),\\ \partial_t\xi_2+(\delta+\gamma)\zeta_2+\frac{\varepsilon}{2}\big(|\boldsymbol{u}_2|^2-\gamma|\boldsymbol{u}_1|^2\big)-\mu\varepsilon\big(\mathcal{R}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b,\boldsymbol{u}_2]-\gamma\mathcal{R}^{\mathsf{F}_1^\mu}[h_1,\mathbf{0},\boldsymbol{u}_1]\big)=0,\end{cases} \tag{14.8}$$

where

$$\mathcal{R}^{\mathsf{F}^\mu}[h,\beta\nabla b,\boldsymbol{u}] \stackrel{\text{def}}{=} \frac{\boldsymbol{u}}{3h}\cdot\nabla\mathsf{F}^\mu(h^3\mathsf{F}^\mu\nabla\cdot\boldsymbol{u}) + \frac{1}{2}h^2(\mathsf{F}^\mu\nabla\cdot\boldsymbol{u})^2 \\ -\frac{1}{2}\Big(\frac{\boldsymbol{u}}{h}\cdot\nabla\mathsf{F}^\mu\big(h^2(\beta\nabla b\cdot\boldsymbol{u})\big) + h(\beta\nabla b\cdot\boldsymbol{u})\mathsf{F}^\mu\nabla\cdot\boldsymbol{u} + (\beta\nabla b\cdot\boldsymbol{u})^2\Big)$$

and $\boldsymbol{u}_\ell$ ($\ell\in\{1,2\}$) are determined from (ζ_2,ξ_2) by solving the following identities

$$\begin{cases} \nabla\cdot\big(h_1\mathfrak{T}^{\mathsf{F}_1^\mu}[h_1,\mathbf{0}]^{-1}(h_1\nabla\psi_1)\big) + \nabla\cdot\big(h_2\mathfrak{T}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b]^{-1}(h_2\nabla\psi_2)\big) = 0, \\ \mathfrak{T}[h_1,\mathbf{0}]^{\mathsf{F}_1^\mu}\boldsymbol{u}_1 = h_1\nabla\psi_1\ , \qquad \mathfrak{T}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b]\boldsymbol{u}_2 = h_2\nabla\psi_2. \end{cases}$$

The above system is equivalent to the following one involving only the unknowns $(\zeta_2,\boldsymbol{u}_1,\boldsymbol{u}_2)$:

$$\begin{cases} \partial_t\zeta_2 = \nabla\cdot(h_1\boldsymbol{u}_1) = -\nabla\cdot(h_2\boldsymbol{u}_2), \\ \partial_t\big(\boldsymbol{u}_1 + \mu\mathcal{T}^{\mathsf{F}_1^\mu}[h_1,\mathbf{0}]\boldsymbol{u}_1\big) + \frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \nabla\big(\frac{\varepsilon}{2}|\boldsymbol{u}_1|^2 - \mu\varepsilon\mathcal{R}^{\mathsf{F}_1^\mu}[h_1,\mathbf{0},\boldsymbol{u}_1]\big) = -\gamma^{-1}\nabla p_{\rm int}, \\ \partial_t\big(\boldsymbol{u}_2 + \mu\mathcal{T}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b]\boldsymbol{u}_2\big) + \frac{\delta+\gamma}{1-\gamma}\nabla\zeta_2 + \nabla\big(\frac{\varepsilon}{2}|\boldsymbol{u}_2|^2 - \mu\varepsilon\mathcal{R}^{\mathsf{F}_2^\mu}[h_2,\beta\nabla b,\boldsymbol{u}_2]\big) = -\nabla p_{\rm int}, \end{cases} \tag{14.9}$$

where $p_{\rm int}$ is a Lagrange multiplier associated with the constraint $\nabla\cdot(h_1\boldsymbol{u}_1)+\nabla\cdot(h_2\boldsymbol{u}_2)=0$, and physically represents the pressure at the interface. In physical variables (recall Section 3.4), we get the ***modified Green–Naghdi systems***

$$\begin{cases} \partial_t h_1 + \nabla\cdot(h_1\boldsymbol{u}_1) = 0, \\ \partial_t h_2 + \nabla\cdot(h_2\boldsymbol{u}_2) = 0, \\ \rho_1\partial_t\big(\boldsymbol{u}_1 + \mathcal{T}^{\mathsf{F}_1^{d_1^2}}[h_1,\mathbf{0}]\boldsymbol{u}_1\big) + \rho_1\nabla\big(\frac{1}{2}|\boldsymbol{u}_1|^2 - \mathcal{R}^{\mathsf{F}_1^{d_1^2}}[h_1,\mathbf{0},\boldsymbol{u}_1]\big) + \nabla p_{\rm int} + g\rho_1\nabla(h_2+b) = \mathbf{0}, \\ \rho_2\partial_t\big(\boldsymbol{u}_2 + \mathcal{T}^{\mathsf{F}_2^{d_1^2}}[h_2,\nabla b]\boldsymbol{u}_2\big) + \rho_2\nabla\big(\frac{1}{2}|\boldsymbol{u}_2|^2 - \mathcal{R}^{\mathsf{F}_2^{d_1^2}}[h_2,\nabla b,\boldsymbol{u}_2]\big) + \nabla p_{\rm int} + g\rho_2\nabla(h_2+b) = \mathbf{0}, \end{cases} \tag{14.10}$$

with $h_1 = d_1-\zeta_2$ and $h_2 = d_2+\zeta_2-b$. Notice that in the examples below, $\mathsf{F}_2^{d_1^2} = \mathsf{F}_1^{d_2^2}$.

These equations have been introduced in [164]. By construction, they enjoy a canonical Hamiltonian structure and the discussion of Section 14.1 applies, *mutatis mutandis*.

A nice outcome of this Hamiltonian structure is that one can interpret solitary waves of the system (of which we cannot expect to obtain explicit constructions as in Section 14.2) as solutions of a constrained minimization problem, which can be studied by means of the standard concentration-compactness strategy; see the discussion in Section 10.4. This was pursued in [165], where the existence of small-amplitude and long-wavelength traveling wave solutions to eq. (14.9) (with slightly supercritical velocity) are obtained, for a wide class of Fourier multipliers F_1^μ, F_2^μ, including all of the ones discussed below. It is proved that the shape of these solitary waves resemble the explicit ones predicted by the Korteweg–de Vries equation in the long wave limit, for all choice of Fourier multipliers in the aforementioned class. Shortly put, the obtained result extends Theorem 10.2 to the bilayer framework. Numerical computations show that for larger values of the velocity (in the bilayer framework), solitary waves still agree remarkably well with the ones of the full interfacial wave system; see Figure 14.1.

We have yet said almost nothing about the choice of the Fourier multipliers. The first obvious remark is that if $\mathsf{F}_1^\mu = \mathsf{F}_2^\mu = \mathrm{Id}$, then eq. (14.9) is exactly the Miyata–Choi–Camassa eq. (14.4). Moreover, the Fourier multipliers appear only on dispersive terms, that is setting $\mu=0$ yields the bilayer hydrostatic equations eq. (6.7). Hence we expect that if for any $s\in\mathbb{N}$ and $\mu^\star>0$ there exists $m\in\mathbb{N}$ and $C_n>0$ such that

$$\forall\ell\in\{1,2\},\ \forall\mu\in(0,\mu^\star],\quad \big\|\mathsf{F}_\ell^\mu - \mathrm{Id}\big\|_{H^{s+m}\to H^s} \ \le\ \mu\, C_s,$$

then eq. (14.9) and eq. (14.4) are consistent with precision $\mathcal{O}(\mu^2)$, which is the precision of Miyata–Choi–Camassa model; see Theorem 14.1 in Section 14.3. In other words, models with such near-identity Fourier multipliers are as precise as the original Miyata–Choi–Camassa model. The rigorous statement for this claim (see Theorem 14.4 below) requires however to extend Section 8.6.1 to $\mathfrak{T}^{\mathsf{F}_\ell^\mu}$ ($\ell \in \{1,2\}$), which demands additional assumptions on the operators F_ℓ^μ; see discussion in Section 10.5. This yields the following natural class of Fourier multipliers.

Assumption 14.3. *For any $\ell \in \{1,2\}$, $\mathsf{F}_\ell^\mu \stackrel{\text{def}}{=} F_\ell(\sqrt{\mu}|D|)$ with $F_\ell \in L^\infty(\mathbb{R})$ real-valued and even such that $F_\ell(\xi) = 1 + \mathcal{O}(\xi^2)$.*

It is advisable (but non-necessary for our discussion) to add—as in Assumption 10.15—the requirement that $F_\ell \in W^{1,\infty}(\mathbb{R})$ and for almost any $\xi \in \mathbb{R}$, $F_\ell'(\xi) = \mathcal{O}((1+|\xi|)^{-1})$, as this provides commutator estimates suitable to the energy method.

Extending the modal analysis of Section 14.4 to the presence of the Fourier multipliers of the form $\mathsf{F}_1^\mu = F_1(\sqrt{\mu}|D|)$ and $\mathsf{F}_1^\mu = F_2(\sqrt{\mu}|D|)$, we obtain the dispersion relation

$$(\omega(\boldsymbol{\xi}) - \boldsymbol{c}_{\rm DIT}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi})^2 = a_{\rm DIT}(\boldsymbol{\xi}) b_{\rm DIT}(\boldsymbol{\xi}), \tag{14.11}$$

where

$$\begin{aligned}
a_{\rm DIT}(\boldsymbol{\xi}) &\stackrel{\text{def}}{=} (\delta+\gamma) - \frac{\frac{\gamma\delta}{\delta+\gamma}}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}} \frac{((\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2},\\
b_{\rm DIT}(\boldsymbol{\xi}) &\stackrel{\text{def}}{=} \frac{1}{\delta+\gamma}\frac{(1+\frac{\mu|\boldsymbol{\xi}|^2}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}}|\boldsymbol{\xi}|^2,\\
\boldsymbol{c}_{\rm DIT}(\boldsymbol{\xi}) &\stackrel{\text{def}}{=} \frac{\frac{\delta}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}\boldsymbol{u}_2 + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}\boldsymbol{u}_1}{\frac{\delta}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1} + \frac{\gamma}{\delta+\gamma}(1+\frac{\mu|\boldsymbol{\xi}|^2}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2)^{-1}}.
\end{aligned}$$

This yields two natural choices for the symbols F_1 and F_2.

Firstly, if we choose $\mathsf{F}_1^\mu = F_1(\sqrt{\mu}|D|)$ and $\mathsf{F}_2^\mu(\sqrt{\mu}|D|)$ with

$$F_1(\boldsymbol{\xi}) = \sqrt{\frac{3}{|\boldsymbol{\xi}|^2}\left(\frac{|\boldsymbol{\xi}|}{\tanh(|\boldsymbol{\xi}|)} - 1\right)}, \qquad F_2(\boldsymbol{\xi}) = F_1(\boldsymbol{\xi}/\delta), \tag{14.12}$$

so that

$$(1+\tfrac{\mu}{3}F_1(\sqrt{\mu}|\boldsymbol{\xi}|)^2|\boldsymbol{\xi}|^2)^{-1} = \frac{\tanh(\sqrt{\mu}|\boldsymbol{\xi}|)}{\sqrt{\mu}|\boldsymbol{\xi}|}, \qquad (1+\tfrac{\mu}{3\delta^2}F_2(\sqrt{\mu}|\boldsymbol{\xi}|)^2|\boldsymbol{\xi}|^2)^{-1} = \frac{\tanh(\sqrt{\mu/\delta^2}|\boldsymbol{\xi}|)}{\sqrt{\mu/\delta^2}|\boldsymbol{\xi}|}$$

then eq. (14.11) corresponds exactly to the dispersion relation of the full interfacial waves system; see eq. (3.12). In other words, with this specific choice the model defined by eq. (14.4) is *fully dispersive*. Consistently, the modal analysis shows strong instabilities of Kelvin–Helmholtz type for large wavenumbers. This fully dispersive system is the natural bilayer extension of the Whitham–Green–Naghdi model we introduced and studied in Section 10, and consistently we refer to it as the ***Whitham–Choi–Camassa model***. Its justification as an asymptotic model for the interfacial wave system with *improved precision* $\mathcal{O}(\mu^2(\varepsilon+\beta))$ in the sense of consistency is given in Theorem 14.4 below.

As a second choice, we can set $\mathsf{F}_1^\mu = F_1(\sqrt{\mu}|D|)$ and $\mathsf{F}_2^\mu(\sqrt{\mu}|D|)$ with

$$F_1(\boldsymbol{\xi}) = \frac{1}{(1+\theta|\boldsymbol{\xi}|^2)^{\alpha/2}}, \qquad F_2(\boldsymbol{\xi}) = F_1(\boldsymbol{\xi}/\delta)$$

with $\theta > 0$ and $\alpha \geq 1$. In this case the modal analysis shows that for $\boldsymbol{u}_2 - \boldsymbol{u}_1$ sufficiently small, the Kelvin–Helmholtz type instabilities are suppressed, that is all modes are stable. We refer to such systems as ***regularized Miyata–Choi–Camassa models***. As put forward in [164], setting $\alpha\theta = \frac{1}{15}$ allows to obtain for free that the precision of the model is enhanced at the linear level, that is its dispersion relation fits with the one of the full interfacial waves system at order $\mathcal{O}(\mu^3)$ instead of $\mathcal{O}(\mu^2)$.

We plot in Figure 14.3 the corresponding dispersion curves. The interested reader will find in [164] a nonlinear analysis (including the initial-value problem in the presence of surface tension) and numerical illustrations (with time integration).

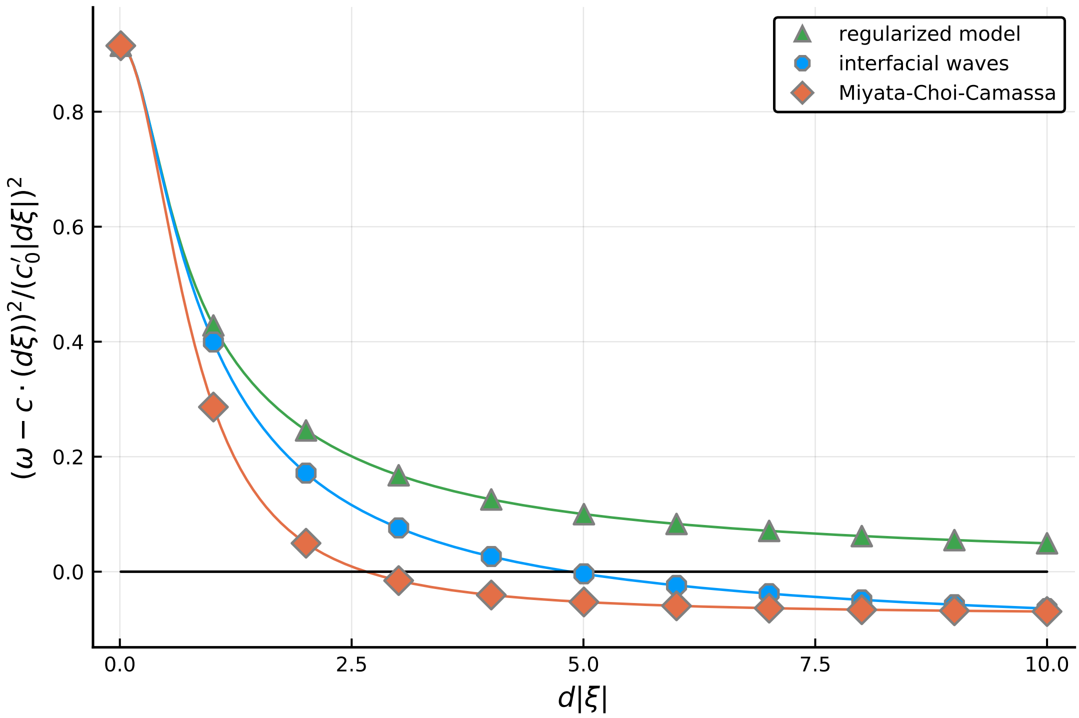


Figure 14.3: Dispersion relation. We plot $(\omega(\boldsymbol{\xi}) - \boldsymbol{c}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2/|\boldsymbol{\xi}|^2 = a(\boldsymbol{\xi})b(\boldsymbol{\xi})/|\boldsymbol{\xi}|^2$ (with $\mu = 1$) for the full bilayer interfacial waves system (and hence the fully dispersive model), the original Miyata–Choi–Camassa model and a regularized model; negative values indicate unstable modes. We set $\gamma = 0.9$, $\delta = \frac{1}{4}$, $\boldsymbol{u}_2 - \boldsymbol{u}_1 = \frac{1}{2}$. $\alpha = 1$ and $\theta = \frac{1}{15}$ for the regularized model.

We conclude this section by rigorously justifying the class of modified Miyata–Choi–Camassa models we introduced, in the sense of consistency. More precisely, we have the following

Theorem 14.4 (Consistency)**.** *Using the assumptions and notations of Theorem 14.1 (unless otherwise specified), and for any* F_1^μ *and* F_2^μ *satisfying Assumption 14.3, one has*

$$\begin{cases} \partial_t\zeta_2 - \nabla\cdot(h_1\boldsymbol{u}_1) = r_1, \\ \partial_t\zeta_2 + \nabla\cdot(h_2\boldsymbol{u}_2) = r_2, \\ \partial_t\psi_1 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_1|^2 - \mu\varepsilon\mathcal{R}^{\mathsf{F}_1^\mu}[h_1, \boldsymbol{0}, \boldsymbol{u}_1] = -\gamma^{-1}p_{\rm int} + r_3, \\ \partial_t\psi_2 + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}|\boldsymbol{u}_2|^2 - \mu\varepsilon\mathcal{R}^{\mathsf{F}_2^\mu}[h_2, \beta\nabla b, \boldsymbol{u}_2] = -p_{\rm int} + r_4, \end{cases}$$

where we denote $\boldsymbol{u}_1 = \mathfrak{T}^{\mathsf{F}_1^\mu}[h_1, \mathbf{0}]^{-1}(h_1\nabla\psi_1)$*,* $\boldsymbol{u}_2 = \mathfrak{T}^{\mathsf{F}_2^\mu}[h_2, \beta\nabla b]^{-1}(h_2\nabla\psi_2)$*, and one has for almost every* $t \in (0, T)$

$$\big|(r_1(t,\cdot), r_2(t,\cdot))\big|_{(H^s)^2} \leq C\,\mu^2\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\big),$$
$$\big|(r_3(t,\cdot), r_4(t,\cdot))\big|_{(H^{s+1})^2} \leq C\,\mu^2\varepsilon M\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\big),$$

with $M \stackrel{\rm def}{=} \big|\nabla\psi_1(t,\cdot)\big|_{H^{1+s_\star}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}$.

Moreover, if $\mathsf{F}_1^\mu = F_1(\sqrt{\mu}|D|)$ *and* $\mathsf{F}_2^\mu(\sqrt{\mu}|D|)$ *are defined by eq.* (14.12)*, then*

$$\big|(r_1(t,\cdot), r_2(t,\cdot))\big|_{(H^s)^2} \leq C\,\mu^2(\varepsilon\widetilde{M} + \beta M_{\rm b})\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\big),$$

with $\widetilde{M} \stackrel{\rm def}{=} \big|\zeta(t,\cdot)\big|_{H^{2+s_\star}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{1+s_\star}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{1+s_\star}}$ *and* $M_{\rm b} \stackrel{\rm def}{=} \big|b\big|_{W^{\max\{s+6,2+s_\star\},\infty}}$.

Proof. We begin with the first assertion. After Theorem 14.1 and using product estimates Proposition II.7, we only have to show that

$$\big|\boldsymbol{u}_\ell - \boldsymbol{u}_\ell^{\rm GN}\big|_{H^{s+1}} + \mu\big|\mathcal{R}[h_\ell, \mathbf{0}, \boldsymbol{u}_\ell] - \mathcal{R}[h_\ell, \mathbf{0}, \boldsymbol{u}_\ell^{\rm GN}]\big|_{H^{s+1}}$$
$$\leq \mu^2\,C\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\big)$$

where $\ell \in \{1,2\}$, $\boldsymbol{u}_1 = \mathfrak{T}^{\mathsf{F}_1^\mu}[h_1, \mathbf{0}]^{-1}(h_1\nabla\psi_1)$, $\boldsymbol{u}_2 = \mathfrak{T}^{\mathsf{F}_2^\mu}[h_2, \beta\nabla b]^{-1}(h_2\nabla\psi_2)$, $\boldsymbol{u}_1^{\rm GN} = \mathfrak{T}^{\rm Id}[h_1, \mathbf{0}]^{-1}(h_1\nabla\psi_1)$, and $\boldsymbol{u}_2^{\rm GN} = \mathfrak{T}^{\rm Id}[h_2, \beta\nabla b]^{-1}(h_2\nabla\psi_2)$. Yet from the identities

$$\mathfrak{T}^{\rm Id}[h_\ell, \beta\nabla b_\ell]^{-1} - \mathfrak{T}^{\mathsf{F}_\ell^\mu}[h_\ell, \beta\nabla b_\ell]^{-1} = \mathfrak{T}^{\rm Id}[h_\ell, \beta\nabla b_\ell]^{-1}(\mathfrak{T}^{\mathsf{F}_\ell^\mu} - \mathfrak{T}^{\rm Id})\mathfrak{T}^{\mathsf{F}_\ell^\mu}[h_\ell, \beta\nabla b_\ell]^{-1}$$

and

$$\mathsf{T}^{\mathsf{F}_\ell^\mu}[h_\ell, \beta\nabla b_\ell] - \mathsf{T}^{\rm Id}[h_\ell, \beta\nabla b_\ell] = -\frac{\mu}{3}\nabla(\mathsf{F}_\ell^\mu - {\rm Id})\big(h_\ell^3\mathsf{F}_\ell^\mu\nabla\cdot\boldsymbol{u}\big) - \frac{\mu}{3}\nabla\big(h_\ell^3(\mathsf{F}_\ell^\mu - {\rm Id})\nabla\cdot\boldsymbol{u}\big)$$
$$+ \frac{\mu}{2}\Big(\nabla(\mathsf{F}_\ell^\mu - {\rm Id})\big(h^2(\beta\nabla b)\cdot\boldsymbol{u}\big) - h^2(\beta\nabla b)(\mathsf{F}_\ell^\mu - {\rm Id})\nabla\cdot\boldsymbol{u}\Big)$$

we easily infer from Lemma 8.8 and Lemma 8.10 (which, as we said, can be extended to Fourier multipliers F_ℓ^μ satisfying Assumption 14.3) and the fact that for any $s \in \mathbb{R}$, $\mathsf{F}_\ell^\mu : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ and $\mu^{-1}(\mathsf{F}_\ell^\mu - {\rm Id}) : H^{s+2}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ are bounded, uniformly with respect to $\mu \in (0, \mu^\star]$, that

$$\big|\boldsymbol{u}_\ell - \boldsymbol{u}_\ell^{\rm GN}\big|_{H^{s+1}} \lesssim \mu^2\,\big(\big|\zeta(t,\cdot)\big|_{H^{s+6}} + \big|\nabla\psi_1(t,\cdot)\big|_{H^{s+5}} + \big|\nabla\psi_2(t,\cdot)\big|_{H^{s+5}}\big).$$

The corresponding estimate on $\mathcal{R}[h_\ell, \mathbf{0}, \boldsymbol{u}_\ell] - \mathcal{R}[h_\ell, \mathbf{0}, \boldsymbol{u}_\ell^{\rm GN}]$ follows from similar considerations.

As for the second assertion, we simply modify the proof of Theorem 14.1 by using Theorem 10.5 in lieu of Theorem 8.2. □

14.6 Bilayer Boussinesq systems and friends

In the same way the fully dispersive (Whitham–Choi–Camassa) model presented above is a natural bilayer extension of the Whitham–Green–Naghdi system for homogeneous free-surface flows, we can introduce generalizations of the Whitham–Boussinesq and Boussinesq systems introduced in Section 10.6, the latter being a specific case of a family of models generalizing the "*abcd*" Boussinesq systems introduced in Section iv. We refer to [373, 149] for studies on these "Boussinesq/Boussinesq systems" including the derivation and justification of the models, well-posedness of the Cauchy problem, existence of solitary waves and numerical investigations, as well as extensive lists of references. Among them, let me point out [54] for the derivation and justification of other bilayer models in different asymptotic regimes, and [155] for the rigorous justification of unidirectional (scalar) models such as the Korteweg–de Vries equation, among others. Not mentioned in these works is [116] which introduces the interesting square root depth ($\sqrt{D}$) model and its multilayer extension, bearing strong similarities with the Green–Naghdi model, Miyata–Choi–Camassa or multilayer Green–Naghdi counterparts, including variational structure and explicit solitary waves, but does not suffer from the high-frequency modal instabilities displayed in Section 14.4.

15 The Kakinuma systems

By following the procedure which led to the Isobe–Kakinuma model in Section 13, using the variational structure of the bilayer interfacial waves system with rigid-lid, eq. (3.15) (see Section 3.2), one obtains the following system of equations obtained by Kakinuma in [239, 240] (see also [338])

$$\begin{cases} h_1^{2i}\partial_t\zeta_2 - \sum_{j=0}^{N_1}\nabla\cdot\Big(\frac{h_1^{2i+2j+1}}{2i+2j+1}\nabla\phi_{1,j}\Big) + \sum_{j=0}^{N}\mu^{-1}\frac{4ij}{2i+2j-1}h_1^{2i+2j-1}\phi_{1,j} \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_1\}, \\ h_2^{p_i}\partial_t\zeta_2 + \sum_{j=0}^{N_2}\nabla\cdot\Big(\frac{h_2^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j}h_2^{p_i+p_j}\phi_{2,j}(\beta\nabla b)\Big) \\ \quad + \sum_{j=0}^{N_2}\frac{p_i}{p_i+p_j}h_2^{p_i+p_j}(\nabla\phi_{2,j})\cdot(\beta\nabla b) - \sum_{j=0}^{N_2}\frac{p_ip_j}{p_i+p_j-1}h_2^{p_i+p_j-1}(\mu^{-1}+|\beta\nabla b|^2)\phi_{2,j} \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_2\}, \\ \Big(\sum_{j=0}^{N_2}h_2^{p_j}(\partial_t\phi_{2,j}) - \gamma\sum_{j=0}^{N_1}h_1^{2j}(\partial_t\phi_{1,j})\Big) + (\delta+\gamma)\zeta_2 \\ \quad + \frac{\varepsilon}{2}\bigg(\Big|\sum_{j=0}^{N_2}h_2^{p_j}(\nabla\phi_{2,j}) - p_jh_2^{p_j-1}\phi_{2,j}(\beta\nabla b)\Big|^2 + \mu^{-1}\Big(\sum_{j=0}^{N_2}p_jh_2^{p_j-1}\phi_{2,j}\Big)^2\bigg) \\ \quad -\gamma\frac{\varepsilon}{2}\bigg(\Big|\sum_{j=0}^{N_1}h_1^{2j}(\nabla\phi_{1,j})\Big|^2 + \mu^{-1}\Big(\sum_{j=0}^{N_1}2jh_1^{2j-1}\phi_{1,j}\Big)^2\bigg) \;=\; 0, \end{cases} \tag{15.1}$$

where $N_1, N_2 \in \mathbb{N}$, $p_0, p_1, \dots, p_{N_2}$ are non-negative integers satisfying by convention

$$0 = p_0 < p_1 < \cdots < p_{N_2},$$

$h_1 \stackrel{\text{def}}{=} 1 - \varepsilon\zeta_2$ and $h_2 \stackrel{\text{def}}{=} \delta^{-1} + \varepsilon\zeta_2 - \beta b$ are the (non-dimensionalized) depth of the upper and lower layers, and we use the convention $\frac{0}{0} = 0$. When $N_1 = N_2 = 0$ (and $p_0 = 0$), the system coincides with the bilayer hydrostatic system, eq. (6.10).

Equivalently, the last equation in eq. (15.1) may be replaced with

$$\begin{cases} \sum_{j=0}^{N_1}h_1^{2j}(\partial_t\phi_{1,j}) + \frac{\delta+\gamma}{1-\gamma}\zeta_2 + \frac{\varepsilon}{2}\bigg(\Big|\sum_{j=0}^{N_1}h_1^{2j}(\nabla\phi_{1,j})\Big|^2 + \mu^{-1}\Big(\sum_{j=0}^{N_1}2jh_1^{2j-1}\phi_{1,j}\Big)^2\bigg) \;=\; -\gamma^{-1}p_{\rm int}, \\ \sum_{j=0}^{N_2}h_2^{p_j}(\partial_t\phi_{2,j}) + \frac{\delta+\gamma}{1-\gamma}\zeta_2 \\ \qquad + \frac{\varepsilon}{2}\bigg(\Big|\sum_{j=0}^{N_2}h_2^{p_j}(\nabla\phi_{2,j}) - p_ih_2^{p_j-1}\phi_{2,j}(\beta\nabla b)\Big|^2 + \mu^{-1}\Big(\sum_{j=0}^{N_2}p_jh_2^{p_j-1}\phi_{2,j}\Big)^2\bigg) = -p_{\rm int} \end{cases}$$

where $\nabla p_{\rm int}$ physically represents the pressure at the interface and can be seen as the Lagrange multiplier associated with the rigid-lid constraint which is obtained when taking the difference between the first two lines of eq. (15.1) (with $i = 0$):

$$\sum_{j=0}^{N_1}\nabla\cdot\Big(\frac{h_1^{2j+1}}{2j+1}\nabla\phi_{1,j}\Big) + \sum_{j=0}^{N_2}\nabla\cdot\Big(\frac{h_2^{p_j+1}}{p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_j}h_2^{p_j}\phi_{2,j}(\beta\nabla b)\Big) \;=\; 0.$$

Remark 15.1. *We chose to set $\tilde{p}_i = 2i$ in contributions of the upper-layer, consistently with the (flat) rigid-lid situation. The general case of non-flat rigid-lid, and $0 = \tilde{p}_0 < \tilde{p}_1 < \cdots < \tilde{p}_{N_1}$ is easily inferred. Obviously, one may obtain different systems by using different vertical distribution functions for the velocity potentials in the Lagrangian action displayed below; see Remark 13.2.*

Using physical variables (recall Section 3.4), eq. (15.1) yields the ***Kakinuma systems***

$$\begin{cases} h_1^{2i}\partial_t\zeta - \sum_{j=0}^{N_1}\nabla\cdot\Big(\frac{h_1^{2i+2j+1}}{2i+2j+1}\nabla\phi_j\Big) + \sum_{j=0}^{N_2}\frac{4ij}{2i+2j-1}h_1^{2i+2j-1}\phi_j \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_1\}, \\ h_2^{p_i}\partial_t\zeta + \sum_{j=0}^{N_2}\nabla\cdot\Big(\frac{h_2^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}h_2^{p_i+p_j}\phi_j\nabla b\Big) \\ \quad + \sum_{j=0}^{N_2}\frac{p_i}{p_i+p_j}h_2^{p_i+p_j}(\nabla\phi_j)\cdot(\nabla b) - \sum_{j=0}^{N_2}\frac{p_ip_j}{p_i+p_j-1}h_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_2\}, \\ \rho_2\Big\{\sum_{j=0}^{N_2}h_2^{p_j}(\partial_t\phi_{2,j}) + g\zeta_2 + \frac12\Big|\sum_{j=0}^{N_2}h_2^{p_j}(\nabla\phi_{2,j}) - p_jh_2^{p_j-1}\phi_{2,j}(\nabla b)\Big|^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac12\Big(\sum_{j=0}^{N_2}p_jh_2^{p_j-1}\phi_{2,j}\Big)^2\Big\} \\ -\rho_1\Big\{\sum_{j=0}^{N_1}h_1^{2j}(\partial_t\phi_{1,j}) + g\zeta_2 + \frac12\Big|\sum_{j=0}^{N_1}h_1^{2j}(\nabla\phi_{1,j})\Big|^2 + \frac12\Big(\sum_{j=0}^{N_1}2jh_1^{2j-1}\phi_{1,j}\Big)^2\Big\} \;=\; 0, \end{cases} \tag{15.2}$$

where $h_1(t,\boldsymbol{x}) \stackrel{\text{def}}{=} d_1 - \zeta(t,\boldsymbol{x})$ and $h_2(t,\boldsymbol{x}) \stackrel{\text{def}}{=} d_2 + \zeta(t,\boldsymbol{x}) - b(\boldsymbol{x})$.

15.1 Hamiltonian structure

As aforementioned, the Kakinuma model enjoys by construction a Lagrangian structure. More precisely, we can interpret eq. (15.1) as Euler–Lagrange equations,

$$\delta\mathscr{L}_{\mathrm{K}}^{\mathrm{app}} = 0$$

where the Lagrangian action is an approximation to the one corresponding to the full interfacial gravity waves system (see Section 3.2) and reads

$$\begin{aligned}\mathscr{L}_{\mathrm{K}}^{\mathrm{app}} \stackrel{\text{def}}{=} \int_{t_0}^{t_1}\int_{\mathbb{R}^d}\frac{\varepsilon}{2}(\delta+\gamma)\zeta_2^2 + \Big(\int_{-\delta^{-1}+\beta b}^{\varepsilon\zeta_2}\partial_t\Phi_2^{\mathrm{app}} + \frac{\varepsilon}{2\mu}(\partial_z\Phi_2^{\mathrm{app}})^2 + \frac{\varepsilon}{2}|\nabla_{\boldsymbol{x}}\Phi_2^{\mathrm{app}}|^2\,\mathrm{d}z\Big) \\ + \gamma\Big(\int_{\varepsilon\zeta_2}^{1}\partial_t\Phi_1^{\mathrm{app}} + \frac{\varepsilon}{2\mu}(\partial_z\Phi_1^{\mathrm{app}})^2 + \frac{\varepsilon}{2}|\nabla_{\boldsymbol{x}}\Phi_1^{\mathrm{app}}|^2\,\mathrm{d}z\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\end{aligned} \tag{15.3}$$

with

$$\Phi_{1,\mathrm{K}}^{\mathrm{app}}(t,\boldsymbol{x},z) \stackrel{\text{def}}{=} \sum_{i=0}^{N_1}(1-z)^{2i}\phi_{1,i}(t,\boldsymbol{x}), \qquad \Phi_{2,\mathrm{K}}^{\mathrm{app}}(t,\boldsymbol{x},z) \stackrel{\text{def}}{=} \sum_{i=0}^{N_2}(\delta^{-1}+z-\beta b)^{p_i}\phi_{2,i}(t,\boldsymbol{x}). \tag{15.4}$$

We have brought to light in Section 13.2 that the Isobe–Kakinuma systems could be equivalently written as two canonical Hamiltonian evolution equations, analogous to Zakharov's formulation of the water waves system [424]. The same feature holds for the Kakinuma systems, as shown in [159], and we show below how to equivalently write eq. (15.1) as two canonical Hamiltonian evolution equations, analogous to Benjamin and Bridges' formulation of the interfacial waves system [45]; see Section 3.2.

Let us first introduce convenient notations: we may rewrite eq. (15.1) as

$$\begin{cases} \boldsymbol{l}_1(h_1)\partial_t\zeta_2 + \mathcal{L}_1^{\mu}[h_1]\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(h_2)\partial_t\zeta_2 + \mathcal{L}_2^{\mu}[h_2,\beta\nabla b]\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ +\boldsymbol{l}_2(h_2)\bullet\partial_t\boldsymbol{\phi}_2 - \gamma\boldsymbol{l}_1(h_1)\bullet\partial_t\boldsymbol{\phi}_1 + (\delta+\gamma)\zeta_2 + \frac{\varepsilon}{2}\big(|\boldsymbol{u}_2|^2+\mu^{-1}w_2^2\big) - \gamma\frac{\varepsilon}{2}\big(|\boldsymbol{u}_1|^2+\mu^{-1}w_1^2\big) = 0, \end{cases} \tag{15.5}$$

where $\boldsymbol{l}_1 \stackrel{\text{def}}{=} (1, h_1^2, \dots, h^{2N_1})^\top$, $\boldsymbol{l}_2 \stackrel{\text{def}}{=} (h^{p_0}, h^{p_1}, \dots, h^{p_{N_2}})^\top$, $\bullet$ denotes the (N_1+1) or (N_2+1) inner-product, $\boldsymbol{\phi}_1 \stackrel{\text{def}}{=} (\phi_{1,0}, \phi_{1,1}, \dots, \phi_{1,N_1})^\top$, $\boldsymbol{\phi}_2 \stackrel{\text{def}}{=} (\phi_{2,0}, \phi_{2,1}, \dots, \phi_{2,N_2})^\top$, $\mathcal{L}_1^\mu[h_1] = (\mathcal{L}_{1,ij}^\mu[h_1])_{0\le i,j\le N_1}$ and $\mathcal{L}_2^\mu[h_2, \beta\nabla b] = (\mathcal{L}_{2,ij}^\mu[h_2, \beta\nabla b])_{0\le i,j\le N_2}$ with

$$\mathcal{L}_{1,ij}^\mu \varphi_{1,j} \stackrel{\text{def}}{=} -\nabla\cdot\Big(\frac{1}{2i+2j+1} h_1^{2i+2j+1}\nabla\varphi_{1,j}\Big) + \mu^{-1}\frac{4ij}{2i+2j-1} h_1^{2i+2j-1}\varphi_{1,j}, \tag{15.6}$$

$$\begin{aligned}\mathcal{L}_{2,ij}^\mu \varphi_{2,j} \stackrel{\text{def}}{=} &-\nabla\cdot\Big(\frac{1}{p_i+p_j+1} h_2^{p_i+p_j+1}\nabla\varphi_{2,j} - \frac{p_j}{p_i+p_j} H_2^{p_i+p_j}\varphi_{2,j}(\beta\nabla b)\Big)\\ &- \frac{p_i}{p_i+p_j} h_2^{p_i+p_j}\nabla b\cdot\nabla\varphi_{2,j} + \frac{p_i p_j}{p_i+p_j-1} h_2^{p_i+p_j-1}(\mu^{-1} + |\beta\nabla b|^2)\varphi_{2,j}.\end{aligned} \tag{15.7}$$

and $(\boldsymbol{u}_\ell, w_\ell) \stackrel{\text{def}}{=} (\nabla_{\boldsymbol{x}}\Phi_{1,\mathrm{K}}^{\mathrm{app}}, \partial_z\Phi_{1,\mathrm{K}}^{\mathrm{app}})\big|_{z=\varepsilon\zeta_2}$ for $\ell\in\{1,2\}$, that is

$$\boldsymbol{u}_1 \stackrel{\text{def}}{=} \sum_{i=0}^{N_1} h_1^{2i}\nabla\phi_{1,i}, \qquad \boldsymbol{u}_2 \stackrel{\text{def}}{=} \sum_{i=0}^{N_2}(h_2^{p_i}\nabla\phi_{2,i} - p_i h_2^{p_i-1}\phi_{2,i}(\beta\nabla b)), \tag{15.8}$$

$$w_1 \stackrel{\text{def}}{=} -\sum_{i=0}^{N_1} 2i h_1^{2i-1}\phi_{1,i} \qquad w_2 \stackrel{\text{def}}{=} \sum_{i=0}^{N_2} p_i h_2^{p_i-1}\phi_{2,i}. \tag{15.9}$$

Then defining

$$\begin{aligned}\mathscr{E}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^d} \frac12(\delta+\gamma)\zeta_2^2 + \Big(\int_{-\delta^{-1}+\beta b}^{\varepsilon\zeta_2} \frac{1}{2\mu}(\partial_z\Phi_2^{\mathrm{app}})^2 + \frac12|\nabla_{\boldsymbol{x}}\Phi_2^{\mathrm{app}}|^2\,\mathrm{d}z\Big)\\ &+ \gamma\Big(\int_{\varepsilon\zeta_2}^1 \frac{1}{2\mu}(\partial_z\Phi_1^{\mathrm{app}})^2 + \frac12|\nabla_{\boldsymbol{x}}\Phi_1^{\mathrm{app}}|^2\,\mathrm{d}z\Big)\,\mathrm{d}\boldsymbol{x}\\ &= \frac12\int_{\mathbb{R}^d}(\delta+\gamma)\zeta_2^2 + \gamma\boldsymbol{\phi}_1\bullet\mathcal{L}_1^\mu\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2\bullet\mathcal{L}_2^\mu\boldsymbol{\phi}_2\,\mathrm{d}\boldsymbol{x}\end{aligned} \tag{15.10}$$

we observe that eq. (15.5) reads

$$\begin{pmatrix} 0 & \gamma\boldsymbol{l}_1(h_1)^\top & -\boldsymbol{l}_2(h_2)^\top\\ -\gamma\boldsymbol{l}_1(h_1) & \mathrm{O} & \mathrm{O}\\ \boldsymbol{l}_2(h_2) & \mathrm{O} & \mathrm{O}\end{pmatrix}\partial_t\begin{pmatrix}\zeta_2\\ \boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{pmatrix} = \begin{pmatrix}\delta_{\zeta_2}\mathscr{E}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\\ \delta_{\boldsymbol{\phi}_1}\mathscr{E}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\\ \delta_{\boldsymbol{\phi}_2}\mathscr{E}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\end{pmatrix}. \tag{15.11}$$

which exhibits a first (non-canonical) Hamiltonian formulation of the Kakinuma model.

We now consider for $\boldsymbol{\varphi}_1 \stackrel{\text{def}}{=} (\varphi_{1,0}, \varphi_{1,1}, \dots, \varphi_{1,N_1})^\top$ and $\boldsymbol{\varphi}_2 \stackrel{\text{def}}{=} (\varphi_{2,0}, \varphi_{2,1}, \dots, \varphi_{2,N_2})^\top$ the following system of equations satisfied by solutions to (15.5):

$$\begin{cases}\sum_{j=0}^{N_1}\mathcal{L}_{1,ij}^\mu\varphi_{1,j} = h_1^{2i}\sum_{j=0}^{N_1}\mathcal{L}_{1,0j}^\mu\varphi_{1,j} & \forall i\in\{1,\dots,N_1\}\\ \sum_{j=0}^{N_2}\mathcal{L}_{2,ij}^\mu\varphi_{2,j} = h_2^{p_i}\sum_{j=0}^{N_2}\mathcal{L}_{2,0j}^\mu\varphi_{2,j} & \forall i\in\{1,\dots,N_2\}\\ \sum_{j=0}^{N_1}\mathcal{L}_{1,0j}^\mu\varphi_{1,j} + \sum_{j=0}^{N_2}\mathcal{L}_{2,0j}^\mu\varphi_{2,j} = 0\\ \boldsymbol{l}_2\bullet\boldsymbol{\varphi}_2 - \gamma\boldsymbol{l}_1\bullet\boldsymbol{\varphi}_1 = \xi_2.\end{cases} \tag{15.12}$$

It is shown in [159, Lemma 6.4] that assuming sufficient regularity on the bottom topography, b, as well as the non-cavitation assumption $h_1 = 1-\varepsilon\zeta_2 \ge h_\star > 0$ and $h_2 = \delta^{-1}+\varepsilon\zeta_2-\beta b \ge h_\star > 0$, then (ζ_2, ξ_2) in appropriate functional spaces defines $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$, solution to eq. (15.12), uniquely up to an additive constant of the form $(C, \gamma C)$ for $(\varphi_{1,0}, \varphi_{2,0})$. Since this constant plays no role in the subsequent analysis, we denote the solutions

$$\boldsymbol{\varphi}_1 \stackrel{\text{def}}{=} \boldsymbol{S}_1[\varepsilon\zeta, \beta b]\xi_2 \quad\text{and}\quad \boldsymbol{\varphi}_2 \stackrel{\text{def}}{=} \boldsymbol{S}_2[\varepsilon\zeta, \beta b]\xi_2$$

and finally

$$\mathscr{H}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2,\xi_2) \stackrel{\mathrm{def}}{=} \mathscr{E}_{\mathrm{K}}^{\mathrm{app}}(\zeta_2, \boldsymbol{S}_1[\varepsilon\zeta_2,\beta]\xi_2, \boldsymbol{S}_2[\varepsilon\zeta_2,\beta]\xi_2). \tag{15.13}$$

Then we can perform computations analogous to the ones in Section 13.2 and deduce that sufficiently regular solutions to the system eq. (15.1) satisfy

$$\partial_t \begin{pmatrix} \zeta_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{\zeta_2}\mathscr{H}_{\mathrm{K}}^{\mathrm{app}} \\ \delta_{\xi_2}\mathscr{H}_{\mathrm{K}}^{\mathrm{app}} \end{pmatrix} \tag{15.14}$$

where $\xi_2 \stackrel{\mathrm{def}}{=} \sum_{j=0}^{N_2} h_2^{p_j}\phi_{2,j} - \gamma\sum_{j=0}^{N_1} h_1^{2j}\phi_{1,j}$. Conversely, given (ζ_2,ξ_2) a sufficiently regular solution to eq. (15.14), and defining $\boldsymbol{\phi}_\ell = (\phi_{\ell,0},\phi_{\ell,1},\dots,\phi_{\ell,N_\ell})^\top \stackrel{\mathrm{def}}{=} \boldsymbol{S}_\ell[\varepsilon\zeta_2,\beta b]\xi_2$ for $\ell\in\{1,2\}$ as a solution to eq. (15.12), then $(\zeta_2,\phi_{1,0},\dots,\phi_{1,N_1},\phi_{2,0},\dots,\phi_{2,N_2})$ satisfies eq. (15.1).

The rigorous statements and proofs of the above claims can be found in [159, §8.2].

Remark 15.2 (Preserved quantities)**.** *A consequence of the above analysis is the fact that—by Noether's theorem—solutions to eq. (15.1) preserve the excess of mass, energy, and horizontal impulse in the flat bottom case:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Z} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{K}}^{\mathrm{app}} = 0, \quad \mathscr{Z} \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d}\zeta_2\,\mathrm{d}\boldsymbol{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}_{\mathrm{K}}^{\mathrm{app}} = 0, \quad \mathscr{I}_{\mathrm{K}}^{\mathrm{app}} \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d}\zeta_2\nabla\xi_2\,\mathrm{d}\boldsymbol{x} \quad (\textit{if } \beta b\equiv 0),$$

where $\xi_2 \stackrel{\mathrm{def}}{=} \sum_{j=0}^{N_2} h_2^{p_j}\phi_{2,j} - \gamma\sum_{j=0}^{N_1} h_1^{2j}\phi_{1,j}$ and $\mathscr{E}_{\mathrm{K}}^{\mathrm{app}}$ is defined in eq. (15.10). Conservation laws associated with these quantities are displayed in [159, §9].

15.2 Modal and stability analysis

The good behavior of the Kakinuma model can be seen from comparing the following modal and stability analysis in this section with that on the Miyata–Choi–Camassa model in Section 14.4.

15.2.1 Dispersion relation

As a first step we linearize eq. (15.1) about the rest state in the flat bottom case. Specifically, setting $\beta=\varepsilon=0$ in eq. (15.1) yields

$$\begin{cases} \partial_t\zeta_2^0 - \sum_{j=0}^{N_1}\frac{1}{2i+2j+1}\nabla\cdot\nabla\phi_{1,j}^0 + \sum_{j=0}^{N}\mu^{-1}\frac{4ij}{2i+2j-1}\phi_{1,j}^0 \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_1\}, \\ \partial_t\zeta_2^0 + \sum_{j=0}^{N_2}\frac{1}{p_i+p_j+1}\delta^{-p_j-1}\nabla\cdot\nabla\phi_{2,j}^0 - \sum_{j=0}^{N_2}\mu^{-1}\frac{p_ip_j}{p_i+p_j-1}\delta^{-p_j+1}\phi_{2,j}^0 \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i\in\{0,1,\dots,N_2\}, \\ \left(\sum_{j=0}^{N_2}\delta^{-p_j}(\partial_t\phi_{2,j}^0) - \gamma\sum_{j=0}^{N_1}(\partial_t\phi_{1,j}^0)\right) + (\delta+\gamma)\zeta_2^0 \;=\; 0, \end{cases} \tag{15.15}$$

Setting $\boldsymbol{\psi}_1^0 \stackrel{\mathrm{def}}{=} (\phi_{1,0}^0,\phi_{1,1}^0,\dots,\phi_{1,N_1}^0)^\top$ and $\boldsymbol{\psi}_2^0 \stackrel{\mathrm{def}}{=} (\delta^{-p_0}\phi_{2,0}^0,\delta^{-p_1}\phi_{2,1}^0,\dots,\delta^{-p_{N_2}}\phi_{2,N_1}^0)^\top$, denoting $\mathbf{1}$ the (N_1+1) or (N_2+1)-dimensional vector with coefficient $\mathbf{1}_j=1$,

$$A_{1,0} \stackrel{\mathrm{def}}{=} \left(\frac{1}{2i+2j+1}\right)_{0\le i,j\le N_1}, \qquad A_{1,1} \stackrel{\mathrm{def}}{=} \left(\frac{4ij}{2i+2j-1}\right)_{0\le i,j\le N_1}, \tag{15.16}$$

$$A_{2,0} \stackrel{\mathrm{def}}{=} \left(\frac{1}{p_i+p_j+1}\right)_{0\le i,j\le N_2}, \qquad A_{2,1} \stackrel{\mathrm{def}}{=} \left(\frac{p_ip_j}{p_i+p_j-1}\right)_{0\le i,j\le N_2}, \tag{15.17}$$

the above reads

$$\begin{pmatrix} 0 & -\gamma\mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & \mathrm{O} & \mathrm{O} \\ -\delta^{-1}\mathbf{1} & \mathrm{O} & \mathrm{O} \end{pmatrix} \partial_t \begin{pmatrix} \zeta_2^0 \\ \boldsymbol{\psi}_1^0 \\ \boldsymbol{\psi}_2^0 \end{pmatrix} = -\begin{pmatrix} \delta+\gamma & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & -A_{1,0}\Delta+\mu^{-1}A_{1,1} & \mathrm{O} \\ \mathbf{0} & \mathrm{O} & -\delta^{-2}A_{2,0}\Delta+\mu^{-1}A_{2,1} \end{pmatrix}.$$

Hence the dispersion relation of the linearized system is given by

$$\det\begin{pmatrix} \delta+\gamma & i\gamma\mathbf{1}^\top & -i\mathbf{1}^\top \\ -i\mathbf{1} & -A_{1,0}\Delta+\mu^{-1}A_{1,1} & \mathrm{O} \\ -\delta^{-1}i\mathbf{1} & \mathrm{O} & -\delta^{-2}A_{2,0}\Delta+\mu^{-1}A_{2,1} \end{pmatrix}=0$$

which we can expand as

$$\big(\,\gamma\,|\widetilde{A}_1|\,|A_2|\;+\;\delta^{-1}\,|\widetilde{A}_2|\,|A_1|\big)(\sqrt{\mu}\boldsymbol{\xi})\;\omega^2=(\delta+\gamma)\,\mu^{-1}\big(\,|A_1|\,|A_2|\,\big)(\sqrt{\mu}\boldsymbol{\xi}) \tag{15.18}$$

where

$$|A_1|(\boldsymbol{\xi})\overset{\text{def}}{=}\det\big(A_{1,1}+|\boldsymbol{\xi}|^2A_{1,0}\big),\qquad |\widetilde{A}_1|(\boldsymbol{\xi})\overset{\text{def}}{=}\det\begin{pmatrix}0 & \mathbf{1}^\top\\ -\mathbf{1} & A_{1,1}+|\boldsymbol{\xi}|^2A_{1,0}\end{pmatrix},$$

$$|A_2|(\boldsymbol{\xi})\overset{\text{def}}{=}\det\big(A_{1,1}+\delta^{-2}|\boldsymbol{\xi}|^2A_{1,0}\big)\qquad |\widetilde{A}_2|(\boldsymbol{\xi})\overset{\text{def}}{=}\det\begin{pmatrix}0 & \mathbf{1}^\top\\ -\mathbf{1} & A_{2,1}+\delta^{-2}|\boldsymbol{\xi}|^2A_{2,0}\end{pmatrix}.$$

Setting $\gamma=0$ and $\delta=1$, we recognize the one-layer with free surface situation. It is proved in [341, Proposition 2.1 and Theorem 2.2] that $|A_\ell|$ and $|\widetilde{A}_\ell|$ are positive for $\ell>0$, and there exists $c_{N_1},C_{N_1}>0$ depending only on $N_1\in\mathbb{N}$ such that

$$\left|\frac{|A_1|(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\boldsymbol{\xi})}\right|\geq c_{N_1},\qquad \left|\frac{|A_1|(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\boldsymbol{\xi})}-\frac{\tanh(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\right|\leq C_{N_1}|\boldsymbol{\xi}|^{4N_1+2}.$$

It follows that the dispersion relation, eq. (15.18), defines for any wave vector $\boldsymbol{\xi}\in\mathbb{R}^\star$, two opposite real angular frequencies $\omega_{\mathrm{K}}(\boldsymbol{\xi})$:

$$\frac{\omega_{\mathrm{K}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}=\left(\frac{\frac{|A_1|}{|\cdot|^2|\widetilde{A}_1|}\,\frac{|A_2|}{\delta^{-2}|\cdot|^2|\widetilde{A}_2|}}{\frac{\delta}{\delta+\gamma}\frac{|A_1|}{|\cdot|^2|\widetilde{A}_1|}+\frac{\gamma}{\delta+\gamma}\frac{|A_2|}{\delta^{-2}|\cdot|^2|\widetilde{A}_2|}}\right)(\sqrt{\mu}\boldsymbol{\xi}).$$

Moreover, recalling Section 3.3 and Section 3.4, we have the corresponding dispersion relation for the linearized interfacial waves system with rigid-lid:

$$\frac{\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}=\frac{\frac{\tanh(\sqrt{\mu}|\boldsymbol{\xi}|)}{\sqrt{\mu}|\boldsymbol{\xi}|}\,\frac{\tanh(\sqrt{\mu}\delta^{-1}|\boldsymbol{\xi}|)}{\sqrt{\mu}\delta^{-1}|\boldsymbol{\xi}|}}{\frac{\delta}{\delta+\gamma}\frac{\tanh(\sqrt{\mu}|\boldsymbol{\xi}|)}{\sqrt{\mu}|\boldsymbol{\xi}|}+\frac{\gamma}{\delta+\gamma}\frac{\tanh(\sqrt{\mu}\delta^{-1}|\boldsymbol{\xi}|)}{\sqrt{\mu}\delta^{-1}|\boldsymbol{\xi}|}}. \tag{15.19}$$

Hence we immediately infer the following.

Proposition 15.3. *For any $N\in\mathbb{N}$ and setting $N_2=N_1=N$ and $p_i=2i$ for $i\in\{0,1,\dots,N_2\}$, there exists a constant $C_N>0$, depending only on $N\in\mathbb{N}$ such that*

$$\left|\frac{\omega_{\mathrm{K}}(\boldsymbol{\xi})^2-\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\right|\leq C_N\mu^{2N+1}\big((1+\delta^{-1})|\boldsymbol{\xi}|\big)^{4N+2}$$

and for any fixed $\boldsymbol{\xi}\in\mathbb{R}^d$, $\omega_{\mathrm{K}}(\boldsymbol{\xi})^2\to\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2$ as $N\to\infty$.

Remark 15.4. *The good behavior for small wavenumbers does not extend to high wavenumbers, since for any $(N_1,N_2)\in\mathbb{N}^2$ and $0=p_0<p_1<\cdots<p_{N_2}$, there exists $c>0$ such that*

$$\forall\boldsymbol{\xi}\in\mathbb{R}^d,\qquad \left|\frac{\omega_{\mathrm{K}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\right|\;\geq\;c>0$$

while $\left|\frac{\omega_{\mathrm{ww}}(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\right|\to0$ as $|\boldsymbol{\xi}|\to\infty$. Yet this discrepancy can be seen as a good feature of the model, as shown by the stability analysis in the following section.

15.2.2 Stability analysis

Now we linearize eq. (15.1) about *any* solution $(\zeta_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$, in the flat bottom case ($\beta = 0$): plugging

$$(\widetilde{\zeta}_2 = \zeta_2 + \epsilon\dot{\zeta}, \widetilde{\boldsymbol{\phi}}_1 = \boldsymbol{\phi}_1 + \epsilon\dot{\boldsymbol{\phi}}_1, \widetilde{\boldsymbol{\phi}}_2 = \boldsymbol{\phi}_2 + \epsilon\dot{\boldsymbol{\phi}}_2)$$

in eq. (15.1) (where we denote as usual $\boldsymbol{\phi}_\ell = (\phi_{\ell,0}, \phi_{\ell,1}, \dots, \phi_{\ell,N_\ell})^\top$ and similarly with dots and tildes), keeping only first-order terms in ϵ and neglecting lower order terms in regularity:[69]

$$\begin{cases} \partial_t\dot{\zeta}_2 + \varepsilon\boldsymbol{u}_1 \cdot \nabla\dot{\zeta} - \sum_{j=0}^{N_1} \frac{h_1^{2j+1}}{2i+2j+1}\nabla\cdot\nabla\dot{\phi}_{1,j} &= 0 \qquad \forall i \in \{0,1,\dots,N_1\}, \\ \partial_t\zeta_2 + \varepsilon\boldsymbol{u}_2 \cdot \nabla\dot{\zeta} + \sum_{j=0}^{N_2} \frac{h_2^{p_j+1}}{p_i+p_j+1}\nabla\cdot\nabla\dot{\phi}_{2,j} &= 0 \qquad \forall i \in \{0,1,\dots,N_2\}, \\ \Big(\sum_{j=0}^{N_2} h_2^{p_j}(\partial_t\dot{\phi}_{2,j} + \varepsilon\boldsymbol{u}_2\cdot\nabla\dot{\phi}_{2,j}) - \gamma\sum_{j=0}^{N_1} h_1^{2j}(\partial_t\phi_{1,j} + \varepsilon\boldsymbol{u}_1\cdot\nabla\dot{\phi}_{1,j})\Big) + \mathfrak{a}_{\rm K}\dot{\zeta}_2 = 0, \end{cases} \tag{15.20}$$

where $h_1 \stackrel{\rm def}{=} 1 - \varepsilon\zeta_2$ and $h_2 \stackrel{\rm def}{=} \delta^{-1} + \varepsilon\zeta_2 - \beta b$, $\boldsymbol{u}_\ell$ and w_ℓ are defined in eq. (15.8)–(15.9), and

$$\begin{aligned} \mathfrak{a}_{\rm K} \stackrel{\rm def}{=} (\delta+\gamma) + \varepsilon\left(\sum_{j=0}^{N_1} 2jh_1^{2j-1}(\partial_t\phi_{1,j} + \varepsilon\boldsymbol{u}_1\cdot\nabla\phi_{1,j}) - \mu^{-1}\varepsilon w_1\sum_{j=0}^{N_1} 2j(2j-1)h_1^{2j-2}\phi_{1,j}\right) \\ + \varepsilon\left(\sum_{j=0}^{N_2} p_jh_2^{p_j-1}(\partial_t\phi_{2,j} + \varepsilon\boldsymbol{u}_2\cdot\nabla\phi_{2,j}) + \varepsilon(\mu^{-1}w_2 - \boldsymbol{u}_2\cdot(\beta\nabla b))\sum_{j=0}^{N_2} p_j(p_j-1)h_2^{p_j-2}\phi_{2,j}\right). \end{aligned} \tag{15.21}$$

If we set

$$\begin{cases} \dot{\boldsymbol{\psi}}_1 = (\dot{\phi}_{1,0}, h_1^2\dot{\phi}_{1,1}, \dots, h_1^{2N_1}\dot{\phi}_{1,N_1})^\top, \\ \dot{\boldsymbol{\psi}}_2 = (\dot{\phi}_{2,0}, h_2^{p_1}\dot{\phi}_{2,1}, \dots, h_2^{p_{N_2}}\dot{\phi}_{2,N_2})^\top, \end{cases}$$

and use the notations in Section 15.2.1, we can write eq. (15.20) compactly as

$$\begin{pmatrix} 0 & -\gamma\mathbf{1}^\top & \mathbf{1}^\top \\ h_1\mathbf{1} & \mathrm{O} & \mathrm{O} \\ -h_2\mathbf{1} & \mathrm{O} & \mathrm{O} \end{pmatrix} \partial_t \begin{pmatrix} \dot{\zeta}_2 \\ \dot{\boldsymbol{\psi}}_1 \\ \dot{\boldsymbol{\psi}}_2 \end{pmatrix} = - \begin{pmatrix} \mathfrak{a}_{\rm K} & -\gamma\mathbf{1}^\top(\varepsilon\boldsymbol{u}_1\cdot\nabla) & \mathbf{1}^\top(\varepsilon\boldsymbol{u}_2\cdot\nabla) \\ h_1\mathbf{1}(\varepsilon\boldsymbol{u}_1\cdot\nabla) & -h_1^2A_{1,0}\Delta & \mathrm{O} \\ -h_2\mathbf{1}(\varepsilon\boldsymbol{u}_2\cdot\nabla) & \mathrm{O} & -h_2^2A_{2,0}\Delta \end{pmatrix}.$$

If we now freeze the coefficients in the above, we arrive—after some linear algebra—to the following dispersion relation:

$$\frac{\gamma}{h_1\alpha_1}(\omega - \varepsilon\boldsymbol{u}_1\cdot\boldsymbol{\xi})^2 \; + \; \frac{1}{h_2\alpha_2}(\omega - \varepsilon\boldsymbol{u}_2\cdot\boldsymbol{\xi})^2 = \mathfrak{a}_{\rm K}|\boldsymbol{\xi}|^2. \tag{15.22}$$

where we denote, for $\ell \in \{1,2\}$,

$$\alpha_\ell \stackrel{\rm def}{=} \frac{\det A_{\ell,0}}{\det \widetilde{A}_{\ell,0}}, \qquad \widetilde{A}_{\ell,0} \stackrel{\rm def}{=} \begin{pmatrix} 0 & \mathbf{1}^\top \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}. \tag{15.23}$$

Solutions ω to the above dispersion relation are real for any wave vector $\boldsymbol{\xi} \in \mathbb{R}^d$ if and only if

$$\mathfrak{a}_{\rm K} - \frac{\frac{\gamma}{h_1\alpha_1}\frac{1}{h_2\alpha_2}}{\frac{\gamma}{h_1\alpha_1} + \frac{1}{h_2\alpha_2}}|\varepsilon\boldsymbol{u}_2 - \varepsilon\boldsymbol{u}_1|^2 \geq 0. \tag{15.24}$$

[69] Lower order terms do not play any role for the high frequency stability analysis, or short-time well-posedness of the nonlinear problem. They can be anticipated from the results in Section 13.6, and particularly Theorem 13.15: assuming $(\zeta_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2), (\dot{\zeta}_2, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2) \in \mathcal{C}([0,T]; X^s) \cap \mathcal{C}^1([0,T]; X^{s-1})$ with $s > 1 + d/2$ and

$$X^s \stackrel{\rm def}{=} H^s(\mathbb{R}^d) \times (\mathring{H}^{s+1}(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)^{N_1}) \times (\mathring{H}^{s+1}(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)^{N_2}),$$

we neglect all contributions bounded in $H^s(\mathbb{R}^d)$ (resp. $H^{s+1}(\mathbb{R}^d)$) for the first $(N_1+1) + (N_2+1)$ equations (resp. the last equation).

Otherwise there exists $\boldsymbol{\xi} \in \mathbb{R}^d$ with $|\boldsymbol{\xi}| = 1$ with associated wave frequencies $\omega(\boldsymbol{\xi}) = \omega_r(\boldsymbol{\xi}) \pm i\omega_i(\boldsymbol{\xi})$ with $\omega_r(\boldsymbol{\xi}), \omega_i(\boldsymbol{\xi}) \in \mathbb{R}$ and $\omega_i(\boldsymbol{\xi}) \neq 0$. By homogeneity, $\omega(\lambda\boldsymbol{\xi}) = \lambda\omega(\boldsymbol{\xi})$ for any $\lambda \in \mathbb{R}$, and we observe the high frequency instability of the problem.

Remark 15.5. *We have*

$$\mathfrak{a}_{\rm K} \stackrel{\rm def}{=} -\left(\partial_z(P_2^{\rm app} - P_1^{\rm app})\right)|_{z=\varepsilon\zeta}.$$

where $P_1^{\rm app}$ and $P_2^{\rm app}$ are approximate pressures in the upper and the lower layers calculated from Bernoulli's equations using the approximate velocity potentials that is,

$$P_\ell^{\rm app} = -\gamma^{2-\ell}\left(\varepsilon\partial_t\Phi_\ell^{\rm app} + \frac{\varepsilon^2}{2}\left(|\nabla\Phi_k^{\rm app}|^2 + \mu^{-1}(\partial_z\Phi_k^{\rm app})^2\right) + \tfrac{\delta+\gamma}{1-\gamma}z\right)$$

with $\Phi_\ell^{\rm app}$ defined in eq. (15.4) (with $\ell \in \{1,2\}$). Hence the stability criterion is a natural extension of the approximate Rayleigh–Taylor criterion of the Isobe–Kakinuma model in the free-surface homogeneous case (see Remark 13.16) to interfacial waves. In fact, setting $\gamma = 0$, we recover exactly the result of the free-surface homogeneous framework. Observe from eq. (15.21) that, contrarily to the case of the full interfacial waves system (recall the discussion in Section 3.3), the stability criterion is always satisfied for ε sufficiently small. Incidentally, the singular behavior as $\mu \searrow 0$ is only apparent, as a vigilant analysis shows that $\mu^{-1}w_\ell$ (for $\ell \in \{1,2\}$) is in fact bounded. It is an exercise of linear algebra (whose answer was kindly provided to me by T. Iguchi) to show that

$$\alpha_1 = \frac{1}{\sum_{j=0}^{N_1}(4j+1)}, \quad \alpha_2 = \frac{1}{\sum_{j=0}^{N_2}(2p_j+1)}.$$

Hence $\alpha_\ell \to 0$ as $N_\ell \to \infty$, so that the domain of stability eq. (15.24) shrinks as N_1 and N_2 grow. As a last remark, if $N_1 = N_2 = 0$, one has $\mathfrak{a}_{\rm K} = \delta + \gamma$ and $\alpha_1 = \alpha_2 = 1$, and we recover the hyperbolicity criterion for the bilayer hydrostatic system stated in Section 6.2.3.

Remark 15.6. *An alternative approach to the one developed in this section would consist—as in Section 14.4 and other modal analyses in this document—to linearize the Kakinuma systems, eq. (15.1), against constant shear solutions, that is*

$$\zeta_2 = 0, \quad \phi_{1,0} = \boldsymbol{u}_1 \cdot \boldsymbol{x}, \ \phi_{1,1} = \cdots = \phi_{1,N_1} = 0, \quad \phi_{2,0} = \boldsymbol{u}_2 \cdot \boldsymbol{x}, \ \phi_{2,1} = \cdots = \phi_{2,N_2} = 0$$

with $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ constant vectors. This yields the dispersion relation

$$(\omega(\boldsymbol{\xi}) - \boldsymbol{c}_{\rm K}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2 = a_{\rm K}(\boldsymbol{\xi})b_{\rm K}(\boldsymbol{\xi}), \tag{15.25}$$

with

$$a_{\rm K}(\boldsymbol{\xi}) \stackrel{\rm def}{=} (\delta+\gamma) - \frac{\frac{\gamma\delta}{\delta+\gamma}}{\frac{\delta}{\delta+\gamma}\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})} + \frac{\gamma}{\delta+\gamma}\frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})}}\frac{(\varepsilon(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2},$$

$$b_{\rm K}(\boldsymbol{\xi}) \stackrel{\rm def}{=} \frac{1}{\delta+\gamma}\frac{\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})}\frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})}}{\frac{\delta}{\delta+\gamma}\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})} + \frac{\gamma}{\delta+\gamma}\frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})}}|\boldsymbol{\xi}|^2,$$

$$\boldsymbol{c}_{\rm K}(\boldsymbol{\xi}) \stackrel{\rm def}{=} \frac{\frac{\delta}{\delta+\gamma}\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})}\boldsymbol{u}_2 + \frac{\gamma}{\delta+\gamma}\frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})}\boldsymbol{u}_1}{\frac{\delta}{\delta+\gamma}\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})} + \frac{\gamma}{\delta+\gamma}\frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})}}.$$

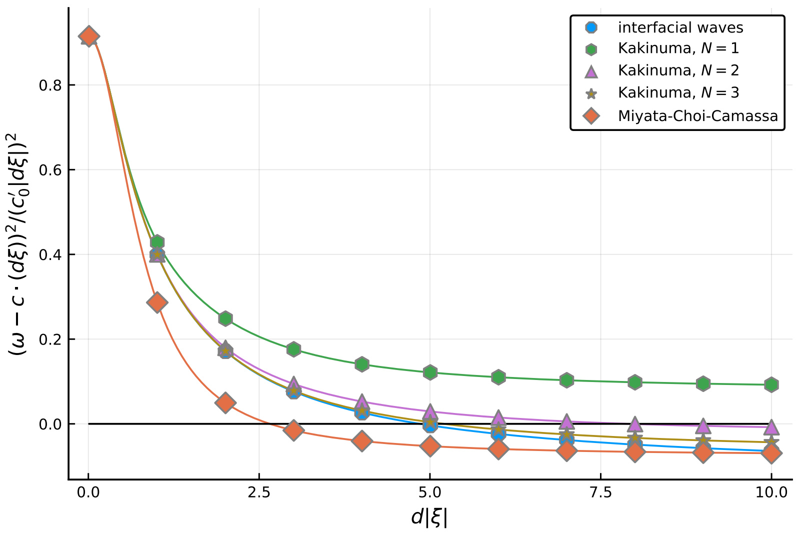
This approach does not allow to recover the nonlinear Rayleigh–Taylor criterion, $\mathfrak{a}_{\rm K}$ defined in eq. (15.21), but only its linear approximation, namely $\delta+\gamma$. On the plus side, it provides some information outside of the high frequency limit. Yet since

$$\frac{|A_1|(\sqrt{\mu}\boldsymbol{\xi})}{\mu|\boldsymbol{\xi}|^2|\widetilde{A}_1|(\sqrt{\mu}\boldsymbol{\xi})} \to \alpha_1 \quad \text{and} \quad \frac{|A_2|(\sqrt{\mu}\boldsymbol{\xi})}{\delta^{-2}\mu|\boldsymbol{\xi}|^2|\widetilde{A}_2|(\sqrt{\mu}\boldsymbol{\xi})} \to \alpha_2 \quad \text{as } |\boldsymbol{\xi}| \to \infty,$$

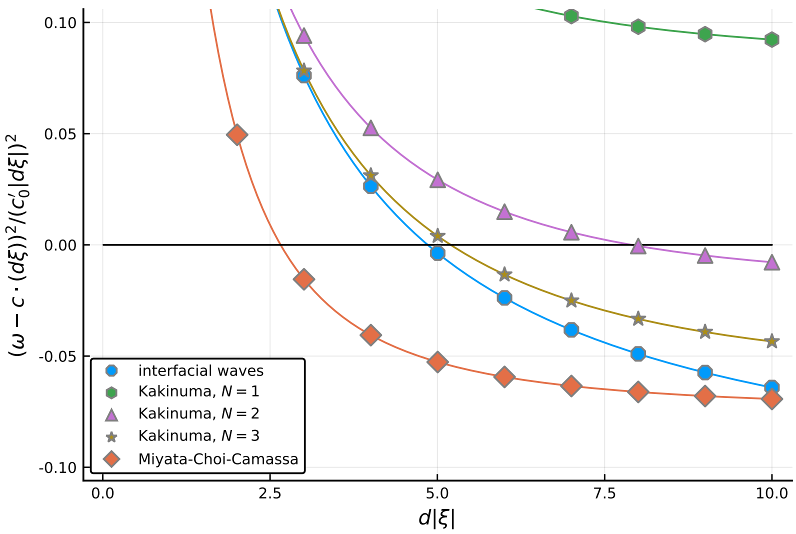
the outcome is matching: unstable modes with unbounded exponential growth arise if and only if

$$(\delta+\gamma) < \frac{\frac{\gamma\delta}{\delta+\gamma}}{\frac{\delta}{\delta+\gamma}\alpha_1 + \frac{\gamma}{\delta+\gamma}\alpha_2}(\varepsilon|\boldsymbol{u}_2-\boldsymbol{u}_1|)^2 = \frac{\frac{\gamma\delta}{\alpha_1\alpha_2}}{\frac{\gamma}{\alpha_1}+\frac{\delta}{\alpha_2}}(\varepsilon|\boldsymbol{u}_2-\boldsymbol{u}_1|)^2.$$

We plot the angular frequencies provided by eq. (15.25)*, and compare with the corresponding formula for the full interfacial waves system and Miyata–Choi–Camassa model in Figure 15.1. In the situation at stake, instabilities arise starting from* $N_1 = N_2 = N \geq 2$.



(a) Wide shot.



(b) Close-up.

Figure 15.1: Dispersion relation. We plot $(\omega(\boldsymbol{\xi}) - \boldsymbol{c}(\boldsymbol{\xi})\cdot\boldsymbol{\xi})^2/|\boldsymbol{\xi}|^2 = a(\boldsymbol{\xi})b(\boldsymbol{\xi})/|\boldsymbol{\xi}|^2$ (with $\mu = 1$) for the full bilayer interfacial waves system, Kakinuma systems and Miyata–Choi–Camassa model, as predicted by eq. (3.12), eq. (15.25) and eq. (14.7); negative values indicate unstable modes. We set $\gamma = 0.9$, $\delta = \frac{1}{4}$, $\boldsymbol{u}_2 - \boldsymbol{u}_1 = \frac{1}{2}$, $N_1 = N_2 = N$ and $p_i = 2i$ ($i \in \{0,1,\ldots,N\}$).

15.3 Rigorous justification

In this section we extend some results concerning the rigorous justification of the Isobe–Kakinuma systems as high order asymptotic models for the water waves system, which are displayed and discussed in Section 13.6, to the Kakinuma model for interfacial waves. These results are proved in [159, 160].

We denote $\mathfrak{p}_{\frac{\rm SW}{\rm SW}}$ the set of parameters

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} \stackrel{\rm def}{=} \big\{(\mu,\varepsilon,\beta,\delta,\gamma) \ : \ \mu\in(0,\mu^\star],\ \varepsilon\in[0,1],\ \beta\in[0,1],\ \delta\in[\delta_\star,\delta^\star],\gamma\in[0,1)\big\}$$

and we set (although this is not necessary for the well-posedness results stated afterwards)

$$\begin{cases} N_2=N_1=N \text{ and } \forall i\in\{1,\dots,N_2\},\ p_i=2i & \text{in the flat bottom case, } \beta b\equiv 0;\\ N_2=2N_1=2N \text{ and } \forall i\in\{1,\dots,N_2\},\ p_i=i & \text{for variable bottom topographies.}\end{cases} \tag{15.26}$$

Recall (see Section 15.1) that eq. (15.1) can be written compactly as

$$\begin{cases} \boldsymbol{l}_1(h_1)\partial_t\zeta_2+\mathcal{L}_1^\mu[h_1]\boldsymbol{\phi}_1=\mathbf{0},\\ -\boldsymbol{l}_2(h_2)\partial_t\zeta_2+\mathcal{L}_2^\mu[h_2,\beta\nabla b]\boldsymbol{\phi}_2=\mathbf{0},\\ \boldsymbol{l}_2(h_2)\bullet\partial_t\boldsymbol{\phi}_2-\gamma\boldsymbol{l}_1(h_1)\bullet\partial_t\boldsymbol{\phi}_1+(\delta+\gamma)\zeta_2+\frac{\varepsilon}{2}\big(|\boldsymbol{u}_2|^2+\mu^{-1}w_2^2\big)-\gamma\frac{\varepsilon}{2}\big(|\boldsymbol{u}_1|^2+\mu^{-1}w_1^2\big)=0,\end{cases} \tag{15.27}$$

where $\boldsymbol{l}_1 \stackrel{\rm def}{=} (1,h_1^2,\dots,h^{2N_1})^\top$, $\boldsymbol{l}_2 \stackrel{\rm def}{=} (h^{p_0},h^{p_1},\dots,h^{p_{N_2}})^\top$, $\bullet$ denotes the (N_1+1) or (N_2+1) inner-product, $\boldsymbol{\phi}_\ell \stackrel{\rm def}{=} (\phi_{\ell,0},\phi_{\ell,1},\dots,\phi_{\ell,N_\ell})^\top$ and $\mathcal{L}_\ell^\mu$, $\boldsymbol{u}_\ell$, w_ℓ are defined in eq. (15.6) to (15.9) (for $\ell\in\{1,2\}$).

By combining all but the last equations, we infer that solutions to eq. (15.27) must satisfy

$$\begin{cases} \sum_{j=0}^{N_1}\mathcal{L}_{1,ij}^\mu\phi_{1,j}=h_1^{2i}\sum_{j=0}^{N_1}\mathcal{L}_{1,0j}^\mu\phi_{1,j} & \forall i\in\{1,\dots,N_1\}\\ \sum_{j=0}^{N_2}\mathcal{L}_{2,ij}^\mu\phi_{2,j}=h_2^{p_i}\sum_{j=0}^{N_2}\mathcal{L}_{2,0j}^\mu\phi_{2,j} & \forall i\in\{1,\dots,N_2\}\\ \sum_{j=0}^{N_1}\mathcal{L}_{1,0j}^\mu\phi_{1,j}+\sum_{j=0}^{N_2}\mathcal{L}_{2,0j}^\mu\phi_{2,j}=0. \end{cases} \tag{15.28}$$

We can now state the justification of the Kakinuma model as a shallow water model for the interfacial waves system, eq. (3.15), in the sense of consistency.

Theorem 15.7 (Consistency). *Let $d\in\mathbb{N}^\star$, $N\in\mathbb{N}$, $h_\star>0$, $\mu^\star>0$, $\delta^\star\geq\delta_\star>0$ and $M^\star\geq 0$. Let $s\in\mathbb{N}$ be such that $s\geq 4(N+1)$ and $s>d/2+1$. There exists $C>0$ such that for any $(\mu,\varepsilon,\beta,\delta,\gamma)\in\mathfrak{p}_{\frac{\rm SW}{\rm SW}}$, any $b\in W^{s+1,\infty}(\mathbb{R}^d)$, $T>0$ and $(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ solution to eq. (15.27) and satisfying*

$$\forall \boldsymbol{x}\in\mathbb{R}^d,\quad h_1(t,\boldsymbol{x})\stackrel{\rm def}{=}1-\varepsilon\zeta(t,\boldsymbol{x})\geq h_\star>0 \qquad h_2(t,\boldsymbol{x})\stackrel{\rm def}{=}\delta^{-1}+\varepsilon\zeta(t,\boldsymbol{x})-\beta b(\boldsymbol{x})\geq h_\star>0 \tag{15.29}$$

uniformly for $t\in(0,T)$ and

$$M\stackrel{\rm def}{=}\operatorname*{ess\,sup}_{t\in(0,T)}\big(\big|\varepsilon\zeta_2(t,\cdot)\big|_{H^s}\big)+\big|\beta b\big|_{W^{s+1,\infty}}\leq M^\star,$$

then, denoting $\psi_2\stackrel{\rm def}{=}\boldsymbol{l}_2(h_2)\bullet\boldsymbol{\phi}_2$ and $\psi_1=\boldsymbol{l}_1(h_1)\bullet\boldsymbol{\phi}_1$, one has

$$\begin{cases} \partial_t\zeta_2-\frac{1}{\mu}\mathcal{G}_1^\mu[\varepsilon\zeta_2]\psi_1=r_1\\ \partial_t\zeta_2-\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2=r_2,\\ \partial_t(\psi_2-\gamma\psi_1)+(\delta+\gamma)\zeta_2+\Big(\frac{\varepsilon}{2}|\nabla\psi_2|^2-\mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2+\varepsilon\nabla\zeta_2\cdot\nabla\psi_2)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)}\Big)\\ \qquad\qquad -\gamma\Big(\frac{\varepsilon}{2}|\nabla\psi_1|^2-\mu\varepsilon\dfrac{(\frac{1}{\mu}\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1+\varepsilon\nabla\zeta_2\cdot\nabla\psi_1)^2}{2(1+\mu|\varepsilon\nabla\zeta_2|^2)}\Big)=r_0,\end{cases}$$

where $\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]$ *and* $\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]$ *are defined after eq.* (3.15) *and for almost every* $t\in(0,T)$*,*

$$\big|r_\ell(t,\cdot)\big|_{H^{s-4(N+1)}}\leq C\,\mu^{1+2N}\,\big|\nabla\psi_\ell(t,\cdot)\big|_{H^{s-1}}\qquad(\ell\in\{1,2\}),$$
$$\big|r_0(t,\cdot)\big|_{H^{s-4(N+1)}}\leq C\,\mu^{1+2N}\,\varepsilon\,\big(\big|\nabla\psi_2(t,\cdot)\big|_{H^{s-1}}^2+\gamma\big|\nabla\psi_1(t,\cdot)\big|_{H^{s-1}}^2\big),$$

Proof. The result is a direct consequence of Theorem 13.12 once we remark the identities

$$\frac{1}{\mu}\mathcal{G}_1^{\mu}[\varepsilon\zeta_2]\psi_1=-\frac{1}{\mu}\mathcal{G}^{\mu}[-\varepsilon\zeta_2,0]\psi_1,\qquad\frac{1}{\mu}\mathcal{G}_2^{\mu,\delta}[\varepsilon\zeta_2,\beta b]\psi_2=\frac{\delta}{\mu}\mathcal{G}^{\mu/\delta^2}[\delta\varepsilon\zeta,\delta\beta b]\psi_2$$

as well as corresponding scaling arguments on

$$\begin{aligned}\mathcal{L}_1^{\mu}[h_1]&=\mathcal{L}^{\mu}[h_1,\mathbf{0}], &\qquad \mathcal{L}_2^{\mu}[h_2,\beta\nabla b]&=\mathcal{L}^{\mu}[h_2,\beta\nabla b],\\ \boldsymbol{u}_1[h_1,\boldsymbol{\phi}_1]&=\boldsymbol{u}[h_1,\mathbf{0},\boldsymbol{\phi}_1], & \boldsymbol{u}_2[h_2,\beta\nabla b,\boldsymbol{\phi}_2]&=\boldsymbol{u}[h_2,\beta\nabla b,\boldsymbol{\phi}_2],\\ w_1[h_1,\boldsymbol{\phi}_1]&=-w[h_1,\mathbf{0},\boldsymbol{\phi}_1], & w_2[h_2,\boldsymbol{\phi}_2]&=w[h_2,\beta\nabla b,\boldsymbol{\phi}_2]\end{aligned}$$

where $\mathcal{L}^{\mu}$, $\boldsymbol{u}$ and w are defined in eq. (13.23) and eq. (13.25) (with slight but obvious misuse of notations as for the dependency with respect to N). □

Remark 15.8. *As for the Isobe–Kakinuma model (see Theorem 13.12 and the Remark afterwards), the statement of the consistency result is in the opposite direction with respect to other consistency statements in this document, and in particular Theorem 14.1, where solutions to the master equations are shown to satisfy the model up to a small remainder terms. One of the benefits from this choice is that there exists a large class of solutions to eq.* (15.27) *as in the above consistency statement, by the well-posedness result below. However it does not allow—directly—to compare these solutions with regular solutions of the interfacial waves system (such as solitary waves for instance); see [160] for such a statement. The next result is the last of this document.*

Theorem 15.9 (Local well-posedness)**.** *Let* $d\in\mathbb{N}^\star$*,* $N\in\mathbb{N}$*,* $s\in\mathbb{N}$*,* $s>1+d/2$*,* $h_\star>0$*,* $a_\star>0$*,* $\mu^\star>0$*,* $\delta^\star\geq\delta_\star>0$ *and* $M^\star\geq0$*. There exist* $T>0$*,* $C>0$ *such that for any* $(\mu,\varepsilon,\beta,\delta,\gamma)\in\mathfrak{p}_{\frac{\rm SW}{\rm SW}}$*, any* $b\in W^{s+2,\infty}(\mathbb{R}^d)$*, and any* $(\zeta_0,\boldsymbol{\phi}_{0,1},\boldsymbol{\phi}_{0,2})\in H^s(\mathbb{R}^d)\times X^s_\mu\times X^s_\mu$ *satisfying the compatibility condition eq.* (15.28)*, the non-cavitation assumptions*

$$h_{0,1}\stackrel{\rm def}{=}1-\varepsilon\zeta_0\geq h_\star>0,\quad h_{0,2}\stackrel{\rm def}{=}\delta^{-1}+\varepsilon\zeta_0-\beta b\geq h_\star>0,$$

the hyperbolicity condition

$$\mathfrak{a}_{\rm K}-\frac{\frac{\gamma}{h_{0,1}\alpha_1}\frac{1}{h_{0,2}\alpha_2}}{\frac{\gamma}{h_{0,1}\alpha_1}+\frac{1}{h_{0,2}\alpha_2}}|\varepsilon\boldsymbol{u}_2-\varepsilon\boldsymbol{u}_1|^2\geq a_\star>0 \tag{15.30}$$

where $\mathfrak{a}_{\rm K}$ *is defined in eq.* (15.21) *and* α_ℓ $(\ell\in\{1,2\})$ *in* (15.23)*, and*

$$M_0\stackrel{\rm def}{=}\big|\varepsilon\zeta_0\big|_{H^s}+\big|\varepsilon\gamma^{\frac12}\boldsymbol{\phi}_{0,1}\big|_{X^s_\mu}+\big|\varepsilon\boldsymbol{\phi}_{0,2}\big|_{X^s_\mu}+\big|\beta b\big|_{W^{s+2,\infty}}\leq M^\star,$$

there exists a unique $(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\in C^0([0,T/M_0];H^s(\mathbb{R}^d)\times X^s_\mu\times X^s_\mu)$ *solution to the Kakinuma system, eq.* (15.27)*, with initial data* $(\zeta_2,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\,|_{t=0}=(\zeta_0,\boldsymbol{\phi}_{0,1},\boldsymbol{\phi}_{0,2})$*; and for any* $t\in[0,T/M_0]$

$$\big|\zeta_2(t,\cdot)\big|_{H^s}+\big|\gamma^{\frac12}\boldsymbol{\phi}_1(t,\cdot)\big|_{X^s_\mu}+\big|\boldsymbol{\phi}_2(t,\cdot)\big|_{X^s_\mu}\leq C\times\Big(\big|\zeta_0\big|_{H^s}+\big|\gamma^{\frac12}\boldsymbol{\phi}_{0,1}\big|_{X^s_\mu}+\big|\boldsymbol{\phi}_{0,2}\big|_{X^s_\mu}\Big),$$

$\inf_{\mathbb{R}^d}(1-\varepsilon\zeta(t,\cdot))\geq\frac{h_\star}{2}$, $\inf_{\mathbb{R}^d}(\delta^{-1}+\varepsilon\zeta(t,\cdot)-\beta b)\geq\frac{h_\star}{2}$, $\inf_{\mathbb{R}^d}\Big(\mathfrak{a}_{\rm K}-\varepsilon^2\frac{\frac{\gamma}{h_1\alpha_1}\frac{1}{h_2\alpha_2}}{\frac{\gamma}{h_1\alpha_1}+\frac{1}{h_2\alpha_2}}|\boldsymbol{u}_2-\boldsymbol{u}_1|^2\Big)\geq\frac{a_\star}{2}$.

In the above, we denoted

$$X^s_\mu \stackrel{\text{def}}{=} \Big\{ \boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N) \in \mathring{H}^{s+1}(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)^N, \\ \big|\boldsymbol{\phi}\big|^2_{X^s_\mu} \stackrel{\text{def}}{=} \sum_{i=0}^{N} \big|\nabla\phi_i\big|^2_{H^s} + \sum_{j=1}^{N} \mu^{-1}\big|\phi_j\big|^2_{H^s} < \infty \Big\}.$$

where $N = N_1$ or $N = N_2$, depending on the size of the vector stake.

Remark 15.10. *As the formula for $\mathfrak{a}_{\mathrm{K}}$ involves time derivatives and the hypersurface $t = 0$ in the space-time $\mathbb{R}^d \times \mathbb{R}$ is characteristic, the definition of its initial value demands some clarifications. We infer $(\partial_t\boldsymbol{\phi}_1, \partial_t\boldsymbol{\phi}_2)\,|_{t=0}$ by differentiating eq. (15.28), replacing $(\partial_t\zeta)\,|_{t=0}$ therein using eq. (15.27) with $i = 0$, and solving the resulting system of differential equations supplemented with the last equation in eq. (15.27) (calling on [159, Lemma 6.4]).*

Remark 15.11. *The compatibility condition eq. (15.28) imposed on the initial data (and propagating for positive times) is not a limitation of the result. As discussed in Section 15.1, it is proved in [159, Lemma 6.4] that under the assumptions of Theorem 15.9 and for any $\xi_0 \in \mathring{H}^{s+1}(\mathbb{R}^d)$, there exists $(\boldsymbol{\phi}_{0,1}, \boldsymbol{\phi}_{0,2}) \in X^s_\mu \times X^s_\mu$ (unique up to an additive constant) solution to eq. (15.28) satisfying additionally $\boldsymbol{l}_2(h_{0,2}) \bullet \boldsymbol{\phi}_{0,2} - \gamma\boldsymbol{l}_1(h_{0,1}) \bullet \boldsymbol{\phi}_{0,1} = \xi_0$. Hence we could rewrite the above statement using as initial data (ζ_0, ξ_0) in a neighborhood of the origin in $H^s(\mathbb{R}^d) \times \mathring{H}^{s+1}(\mathbb{R}^d)$—in fact $H^{s+1}(\mathbb{R}^d) \times \mathring{H}^{s+1}(\mathbb{R}^d)$ to secure uniform estimates—as in Corollary 13.17.*

Sketch of the proof. The complete proof is provided in [159, 160]. The strategy of the proof is very similar to that of Theorem 13.15, yet there is an additional key ingredient in order to obtain the sharp hyperbolicity criteria $h_1 \geq h_\star > 0$, $h_2 \geq h_\star > 0$, and eq. (15.30).

When we extract the quasilinear structure to the Kakinuma system we find that for $\boldsymbol{k} \in \mathbb{N}^d$ a multi index with $|\boldsymbol{k}| \leq s$, solutions to eq. (15.27) satisfy the system

$$\mathcal{A}^\mu_1 \partial_t \begin{pmatrix} \partial^{\boldsymbol{k}}\zeta_2 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_1 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_2 \end{pmatrix} + \mathcal{A}^\mu_0 \begin{pmatrix} \partial^{\boldsymbol{k}}\zeta_2 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_1 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_2 \end{pmatrix} = \begin{pmatrix} r_{\boldsymbol{k}} \\ \boldsymbol{r}_{\boldsymbol{k},1} \\ \boldsymbol{r}_{\boldsymbol{k},2} \end{pmatrix} \tag{15.31}$$

where the remainder term $(r_{\boldsymbol{k}}, \boldsymbol{r}_{\boldsymbol{k},1}, \boldsymbol{r}_{\boldsymbol{k},2})$ is uniformly bounded in an appropriate space, and plays no role for the local-in-time existence and control of solutions; and, recalling the notations $h_1 = 1 - \varepsilon\zeta$, $h_2 = \delta^{-1} + \varepsilon\zeta - \beta b$, $\boldsymbol{l}_1(h_1) \stackrel{\text{def}}{=} (1, h_1^2, \dots, h_1^{2N_1})^\top$, $\boldsymbol{l}_2(h_2) \stackrel{\text{def}}{=} (h_2^{p_0}, h_2^{p_1}, \dots, h_2^{p_{N_2}})^\top$,

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\gamma\boldsymbol{l}_1(h_1)^\top & \boldsymbol{l}_2(h_2)^\top \\ \gamma\boldsymbol{l}_1(h_1) & \mathrm{O} & \mathrm{O} \\ -\boldsymbol{l}_2(h_2) & \mathrm{O} & \mathrm{O} \end{pmatrix}$$

and, defining $(\mathcal{L}^\mu_1, \mathcal{L}^\mu_2)$ by eq. (15.6)–(15.7), $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ by eq. (15.8) and $\mathfrak{a}_{\mathrm{K}}$ by eq. (15.21),

$$\mathcal{A}_0 \stackrel{\text{def}}{=} \begin{pmatrix} \mathfrak{a}_{\mathrm{K}} & -\gamma\boldsymbol{l}_1(h_1)^\top(\varepsilon\boldsymbol{u}_1\cdot\nabla) & \boldsymbol{l}_2(h_2)^\top(\varepsilon\boldsymbol{u}_2\cdot\nabla) \\ -\gamma(\varepsilon\boldsymbol{u}_1\cdot\nabla)^\star\boldsymbol{l}_1(h_1) & \gamma\mathcal{L}^\mu_1[h_1] & \mathrm{O} \\ (\varepsilon\boldsymbol{u}_2\cdot\nabla)^\star\boldsymbol{l}_2(h_2) & \mathrm{O} & \mathcal{L}^\mu_2[h_2, \beta\nabla b] \end{pmatrix}.$$

While this structure has the same symmetry properties as the one exhibited for the Isobe–Kakinuma model in the sketch of the proof of Theorem 13.15, it should be pointed out that the energy estimate inferred by testing the above against $(\partial_t\partial^{\boldsymbol{k}}\zeta_2, \partial_t\partial^{\boldsymbol{k}}\boldsymbol{\phi}_1, \partial_t\partial^{\boldsymbol{k}}\boldsymbol{\phi}_2)^\top$ is not satisfactory. Indeed, considering for simplicity the case of the Saint-Venant system, that is $N_1 = N_2 = 0$, we see that for any $\boldsymbol{U} \stackrel{\text{def}}{=} (\eta, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in \boldsymbol{X} \stackrel{\text{def}}{=} L^2(\mathbb{R}^d) \times \mathring{H}^1(\mathbb{R}^d) \times \mathring{H}^1(\mathbb{R}^d)$, and denoting $\boldsymbol{V} \stackrel{\text{def}}{=} (\eta, \nabla\boldsymbol{\varphi}_1, \nabla\boldsymbol{\varphi}_2)$, we have

$$\big\langle \boldsymbol{U}, \mathcal{A}_0\boldsymbol{U} \big\rangle_{\boldsymbol{X}-\boldsymbol{X}'} = \int_{\mathbb{R}^d} \boldsymbol{V} \cdot A_0\boldsymbol{V}\,\mathrm{d}\boldsymbol{x} \qquad \text{with} \qquad A_0 \stackrel{\text{def}}{=} \begin{pmatrix} \delta + \gamma & -\gamma\varepsilon\boldsymbol{u}_1^\top & \varepsilon\boldsymbol{u}_2^\top \\ -\gamma\varepsilon\boldsymbol{u}_1 & \gamma h_1\,\mathrm{Id}_d & \mathrm{O}_d \\ \varepsilon\boldsymbol{u}_2 & \mathrm{O}_d & h_2\,\mathrm{Id}_d \end{pmatrix}.$$

and the coercivity of the energy functional is equivalent to the positiveness of the matrix A_0. Since

$$\det A_0 = (h_1h_2)^d\big((\delta+\gamma) - \gamma\tfrac{|\varepsilon \boldsymbol{u}_1|^2}{h_1} - \tfrac{|\varepsilon \boldsymbol{u}_2|^2}{h_2}\big),$$

we see that coercivity may fail even when $\boldsymbol{u}_1 = \boldsymbol{u}_2$, in discrepancy with the hyperbolicity condition eq. (15.30) which for the Saint-Venant system reads simply (see Theorem 6.12)

$$(\delta+\gamma) - \gamma\frac{\varepsilon^2|\boldsymbol{u}_2-\boldsymbol{u}_1|^2}{h_1+\gamma h_2} \geq a_\star > 0.$$

However, we can write $A_0 = \widetilde{A}_0 + T_0$ with

$$\widetilde{A}_0 \stackrel{\rm def}{=} \begin{pmatrix} \delta+\gamma & \varepsilon\frac{\gamma h_1 \boldsymbol{v}^\top}{h_1+\gamma h_2} & \varepsilon\frac{\gamma h_2 \boldsymbol{v}^\top}{h_1+\gamma h_2} \\ \varepsilon\frac{\gamma h_1 \boldsymbol{v}}{h_1+\gamma h_2} & h_1\,\mathrm{Id}_d & \mathrm{O}_d \\ \varepsilon\frac{\gamma h_2 \boldsymbol{v}}{h_1+\gamma h_2} & \mathrm{O}_d & h_2\,\mathrm{Id}_d \end{pmatrix}, \quad T_0 \stackrel{\rm def}{=} \begin{pmatrix} 0 & -\gamma\varepsilon\boldsymbol{u}^\top & \varepsilon\boldsymbol{u}^\top \\ -\gamma\varepsilon\boldsymbol{u} & \mathrm{O}_d & \mathrm{O}_d \\ \varepsilon\boldsymbol{u} & \mathrm{O}_d & \mathrm{O}_d \end{pmatrix},$$

with $\boldsymbol{u} \stackrel{\rm def}{=} \frac{\gamma h_2\boldsymbol{u}_1 + h_1\boldsymbol{u}_2}{h_1+\gamma h_2}$, $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$. The contribution from T_0 can be treated as an advection term, and we have as desired

$$\det \widetilde{A}_0 = (h_1h_2)^d\big((\delta+\gamma) - \gamma\tfrac{\varepsilon^2|\boldsymbol{v}|}{h_1+\gamma h_2}\big).$$

Following this approach in the general case $N_1, N_2 \in \mathbb{N}^d$, we rewrite eq. (15.31) as

$$\mathcal{A}_1^\mu(\partial_t + \varepsilon\boldsymbol{u}\cdot\nabla)\begin{pmatrix}\partial^{\boldsymbol{k}}\zeta_2 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_1 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_2\end{pmatrix} + \widetilde{\mathcal{A}}_0^\mu\begin{pmatrix}\partial^{\boldsymbol{k}}\zeta_2 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_1 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_2\end{pmatrix} = \begin{pmatrix} r_{\boldsymbol{k}} \\ \widetilde{\boldsymbol{r}}_{\boldsymbol{k},1} \\ \widetilde{\boldsymbol{r}}_{\boldsymbol{k},2}\end{pmatrix} \tag{15.32}$$

where $\boldsymbol{u} \stackrel{\rm def}{=} \frac{\gamma h_2\alpha_2\boldsymbol{u}_1 + h_1\alpha_1\boldsymbol{u}_2}{h_1\alpha_1+\gamma h_2\alpha_2}$, with α_1, α_2 defined in eq. (15.23), $\mathcal{A}_1$ is as above and

$$\widetilde{\mathcal{A}}_0 \stackrel{\rm def}{=} \begin{pmatrix} \mathfrak{a}_{\rm K} & \gamma\theta_1\boldsymbol{l}_1(h_1)^\top(\varepsilon\boldsymbol{v}\cdot\nabla) & \theta_2\boldsymbol{l}_2(h_2)^\top(\varepsilon\boldsymbol{v}\cdot\nabla) \\ \gamma\theta_1(\varepsilon\boldsymbol{v}\cdot\nabla)^\star\boldsymbol{l}_1(h_1) & \gamma\mathcal{L}_1^\mu[h_1] & \mathrm{O} \\ \theta_2(\varepsilon\boldsymbol{v}\cdot\nabla)^\star\boldsymbol{l}_2(h_2) & \mathrm{O} & \mathcal{L}_2^\mu[h_2,\beta\nabla b] \end{pmatrix}$$

with $\boldsymbol{v} \stackrel{\rm def}{=} \boldsymbol{u}_2 - \boldsymbol{u}_1$, $\theta_1 \stackrel{\rm def}{=} \frac{h_1\alpha_1}{h_1\alpha_1+\gamma h_2\alpha_2}$, $\theta_2 \stackrel{\rm def}{=} \frac{\gamma h_2\alpha_2}{h_1\alpha_1+\gamma h_2\alpha_2}$. We will then obtain energy estimates by testing eq. (15.32) against

$$(\partial_t + \varepsilon\boldsymbol{u}\cdot\nabla)\begin{pmatrix}\partial^{\boldsymbol{k}}\zeta_2 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_1 \\ \partial^{\boldsymbol{k}}\boldsymbol{\phi}_2\end{pmatrix},$$

and using that the contribution from the first term vanishes identically. The last key ingredient consists in showing that, restricting to the flat bottom case for the sake of readability (the case of variable topography adds only lower order contributions) there exists $c, C > 0$ such that for any $\boldsymbol{U} \stackrel{\rm def}{=} (\eta, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in \boldsymbol{X} \stackrel{\rm def}{=} L^2(\mathbb{R}^d) \times (\mathring{H}^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)^{N_1}) \times (\mathring{H}^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)^{N_2})$ we have

$$c\big(|\eta|_{L^2}^2 + |\boldsymbol{\varphi}_1|_{X^0_\mu}^2 + |\boldsymbol{\varphi}_2|_{X^0_\mu}^2\big) \leq \big\langle \boldsymbol{U}, \mathcal{A}_0\boldsymbol{U}\big\rangle_{\boldsymbol{X}-\boldsymbol{X}'} \leq C\big(|\eta|_{L^2}^2 + |\boldsymbol{\varphi}_1|_{X^0_\mu}^2 + |\boldsymbol{\varphi}_2|_{X^0_\mu}^2\big).$$

After integrating by parts we find

$$\begin{aligned}\big\langle \boldsymbol{U}, \mathcal{A}_0\boldsymbol{U}\big\rangle_{\boldsymbol{X}-\boldsymbol{X}'} = {} & \big(\mathfrak{a}_{\rm K}\eta, \eta\big)_{L^2} \\ & + \gamma\big(h_1A_{1,0}D_1\nabla\boldsymbol{\varphi}_1, D_1\nabla\boldsymbol{\varphi}_1\big)_{L^2} + \gamma\mu^{-1}\big(h_1^{-1}A_{1,1}D_1\boldsymbol{\varphi}_1, D_1\boldsymbol{\varphi}_1\big)_{L^2} \\ & + \big(h_2A_{2,0}D_2\nabla\boldsymbol{\varphi}_2, D_2\nabla\boldsymbol{\varphi}_2\big)_{L^2} + \mu^{-1}\big(h_2^{-1}A_{2,1}D_2\boldsymbol{\varphi}_2, D_2\boldsymbol{\varphi}_2\big)_{L^2} \\ & + 2\varepsilon\big(\eta, \gamma\theta_1\boldsymbol{v}\cdot(\boldsymbol{1}\bullet D_1\nabla\boldsymbol{\varphi}_1) + \theta_2\boldsymbol{v}\cdot(\boldsymbol{1}\bullet D_2\nabla\boldsymbol{\varphi}_2)\big)_{L^2}\end{aligned}$$

where $A_{\ell,0}$ and $A_{\ell,1}$ (for $\ell \in \{1,2\}$) have been defined in eq. (15.16)–(15.17), and D_1 and D_2 are the diagonal matrices

$$D_1 = \operatorname{diag}(1, h_1^2, \dots, h_1^{2N_1}), \quad D_2 = \operatorname{diag}(h_2^{p_0}, h_2^{p_1}, \dots, h_2^{p_{N_2}}).$$

The upper bound is obvious by product estimates, and the lower bound is the crucial element. From the non-cavitation assumptions $h_1 \geq h_\star > 0$, $h_2 \geq h_\star > 0$, and using that when withdrawing their first (null) row and column $A_{1,1}$ and $A_{2,1}$ are definite positive, we infer immediately the L^2-control of $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$—except the first component. The remaining control follows from the lower bounds[70]

$$\forall \boldsymbol{\psi}_1 \in \mathbb{R}^{N_1+1},\ A_{1,0}\boldsymbol{\psi}_1 \bullet \boldsymbol{\psi}_1 \geq \alpha_1(\mathbf{1} \bullet \boldsymbol{\psi}_1)^2, \quad \forall \boldsymbol{\psi}_2 \in \mathbb{R}^{N_2+1},\ A_{2,0}\boldsymbol{\psi}_2 \bullet \boldsymbol{\psi}_2 \geq \alpha_2(\mathbf{1} \bullet \boldsymbol{\psi}_2)^2,$$

and the fact that the matrix

$$\begin{pmatrix} \mathfrak{a}_{\mathrm{K}} & -\gamma\theta_1|\boldsymbol{v}| & -\theta_2|\boldsymbol{v}| \\ -\gamma\theta_1|\boldsymbol{v}| & \gamma h_1\alpha_1 & 0 \\ -\theta_2|\boldsymbol{v}| & 0 & h_2\alpha_2 \end{pmatrix}$$

is positive-definite under the hyperbolicity criterion eq. (15.30), in addition to the non-cavitation assumptions.

From the above, we infer *a priori* energy estimates on smooth solutions satisfying the compatibility condition, hyperbolicity criterion and non-cavitation assumptions, as in the statement. The remaining of the proof is very similar to that of Theorem 13.15, and we conclude here. □

[70] In order to see this, set $\ell \in \{1,2\}$, introduce

$$\widetilde{A}_{\ell,0} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \mathbf{1}^\top \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}$$

and notice

$$(\widetilde{A}_{\ell,0})^{-1} = \begin{pmatrix} q_{\ell,0} & \boldsymbol{q}^\top \\ -\boldsymbol{q} & Q_{\ell,0} \end{pmatrix}$$

with $q_{\ell,0} = \frac{\det A_{\ell,0}}{\det \widetilde{A}_{\ell,0}} = \alpha_\ell$ and $Q_{\ell,0}$ is non-negative, since for any $\boldsymbol{\phi} \in \mathbb{R}^{N_\ell+1}$, denoting $\begin{pmatrix} \zeta \\ \boldsymbol{\psi} \end{pmatrix} = (\widetilde{A}_{\ell,0})^{-1}\begin{pmatrix} 0 \\ \boldsymbol{\phi} \end{pmatrix}$, we have

$$\boldsymbol{\phi} \bullet Q_{\ell,0}\boldsymbol{\phi} = \begin{pmatrix} 0 \\ \boldsymbol{\phi} \end{pmatrix} \bullet \begin{pmatrix} q_{\ell,0} & \boldsymbol{q}^\top \\ -\boldsymbol{q} & Q_{\ell,0} \end{pmatrix}\begin{pmatrix} 0 \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \zeta \\ \boldsymbol{\psi} \end{pmatrix} \bullet \begin{pmatrix} 0 & \mathbf{1}^\top \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}\begin{pmatrix} \zeta \\ \boldsymbol{\psi} \end{pmatrix} = \boldsymbol{\psi} \bullet A_{\ell,0}\boldsymbol{\psi} \geq 0,$$

where the positivity of $A_{\ell,0}$ follows from

$$\boldsymbol{\psi} \bullet A_{\ell,0}\boldsymbol{\psi} = \int_0^1 (\boldsymbol{\psi} \bullet \boldsymbol{z})^2 \, \mathrm{d}z, \qquad \boldsymbol{z} \stackrel{\text{def}}{=} (z^{p_0}, z^{p_1}, \dots, z^{p_{N_\ell}}).$$

Now, for any $\boldsymbol{\psi} \in \mathbb{R}^{N_\ell+1}$, we set $\eta \stackrel{\text{def}}{=} \mathbf{1} \bullet \boldsymbol{\psi}$, $\boldsymbol{\phi} \stackrel{\text{def}}{=} A_{\ell,0}\boldsymbol{\psi}$, and remark

$$\boldsymbol{\psi} \bullet A_{\ell,0}\boldsymbol{\psi} = \begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix} \bullet \begin{pmatrix} 0 & \mathbf{1}^\top \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}\begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \eta \\ \boldsymbol{\phi} \end{pmatrix} \bullet \begin{pmatrix} q_{\ell,0} & \boldsymbol{q}^\top \\ -\boldsymbol{q} & Q_{\ell,0} \end{pmatrix}\begin{pmatrix} \eta \\ \boldsymbol{\phi} \end{pmatrix} = q_{\ell,0}\eta^2 + \boldsymbol{\phi} \bullet Q_{\ell,0}\boldsymbol{\phi},$$

which gives the thesis.

Appendix

> This is the end [...] of our elaborate plans, the end of everything that stands, the end.
>
> — Jim Morrison, *The End*

Contents

I Numerics

I.1 Numerical schemes

In the forthcoming Appendix I.5 we present some numerical experiments, which have been generated using spectral methods, and the explicit four-step Runge–Kutta scheme for the time integration. In order to simplify the exposition, we will focus in this section on the use of such numerical scheme on the initial-value problem for the Whitham equation, eq. (x), which in dimensionless variables reads

$$\partial_t \zeta + \partial_x \big(\mathsf{L}(D)\zeta + \tfrac{3\varepsilon}{4}\zeta^2\big) = 0\,. \tag{I.1}$$

where $\mathsf{L}(D) = \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}$, and $\varepsilon, \mu > 0$ are dimensionless parameters. The discussion can be extended without any change to general class of self-adjoint linear Fourier multipliers $\mathsf{L}(D)$, and in particular to the Korteweg–de Vries equation, eq. (viii). When $\varepsilon \ll 1$ and solutions are computed over large times, or when the operator $\mathsf{L}(D)$ is of high order (*i.e.* the problem is *stiff*) as in the Korteweg–de Vries equation, it is beneficial to solve eq. (I.1) for $\eta \stackrel{\text{def}}{=} \exp(t\,\partial_x \mathsf{L})\zeta$, that is

$$\partial_t \eta + \tfrac{3\varepsilon}{4} \exp(t\,\partial_x \mathsf{L})\partial_x \big((\exp(-t\,\partial_x \mathsf{L})\eta)^2\big) = 0\,.$$

Solvers based on such representation are often called *exponential integrators*; see [330] for more details and references. We shall stick to eq. (I.1) for simplicity in this presentation.

I.1.1 Fourier spectral methods

There are many textbooks introducing spectral methods, *e.g.* [401] (I also like [419, Chapter V]), and only the basics shall be recalled here. In this section, time is freezed and we consider only the discretization in space.

Periodization Our first approximation consists in considering eq. (I.1) on a periodic domain of period $2\mathsf{L}$ rather than on the full real line. We apply a periodization operator, for instance

$$\zeta_{\rm p}(x) \stackrel{\text{def}}{=} \sum_{\ell\in\mathbb{Z}} \zeta(x + 2\mathsf{L}\ell).$$

In practice, we consider functions ζ rapidly decreasing (with $|x|$) and choose L sufficiently large so that ζ decays up to machine precision for $|x| \geq \mathsf{L}$, and so the difference between $\zeta_{\rm p}$ and ζ on $[-\mathsf{L}, \mathsf{L}]$ is immaterial.

The Fourier multiplier $\mathsf{L}(D)$ (respectively ∂_x), extended as an operator from $\mathcal{S}'(\mathbb{R})$ (the tempered distributions) to itself, maps periodic functions to periodic functions and acts by pointwise multiplication on the $k^{\rm th}$ coefficients of the Fourier series with $\sqrt{\frac{\tanh(\sqrt{\mu}\pi|k|/\mathsf{L})}{\sqrt{\mu}\pi|k|/\mathsf{L}}}$ (respectively $\frac{i\pi k}{\mathsf{L}}$). However, since $\mathsf{L}(D)$ is a nonlocal operator, $\mathsf{L}(D)\zeta_{\rm p} \neq (\mathsf{L}(D)\zeta)_{\rm p}$. Yet the symbol being smooth, its inverse Fourier transform (and derivatives), which we denote K,[71] is exponentially decaying, and hence

$$\mathsf{L}(D)(\zeta_{\rm p} - \zeta)(x) = \frac{1}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}^\star} \int_{\mathbb{R}} K(x-y)\zeta(y + 2\mathsf{L}\ell)\,{\rm d}y$$

is exponentially decaying on $[-\mathsf{L}, \mathsf{L}]$ with respect to L, provided that $\zeta(x)$ is exponentially decaying with $|x|$. As a consequence, for ζ exponentially decaying, the error made by the periodization procedure is exponentially decaying with L.

[71] More properties on the Whitham Kernel K (and its periodic counterpart) can be found in [173].

Fourier spectral methods On a problem with periodic boundary conditions, Fourier spectral methods consist in approaching solutions considering *finite* Fourier sums of the form

$$\zeta_{\mathrm{p}}(x) \approx \sum_{\mathsf{k}=-\lfloor \mathsf{N}/2 \rfloor}^{\lceil \mathsf{N}/2 \rceil -1} \mathsf{a}_{\mathsf{k}} e^{i \frac{\pi}{\mathsf{L}} \mathsf{k} x}. \tag{I.2}$$

In practice, we give ourselves the values $(\zeta(\mathsf{x}_{\mathsf{i}}))_{\mathsf{i}=1,\cdots,\mathsf{N}}$ at the regularly spaced *collocation points* $\mathsf{x}_{\mathsf{i}} \stackrel{\text{def}}{=} -\mathsf{L} + 2\mathsf{i}\frac{\mathsf{L}}{\mathsf{N}}$, and use the discrete Fourier transform (computed efficiently with a *Fast Fourier transform* (FFT)) to deduce the coefficients a_{k}. In that way, the coefficients a_{k} are related to the coefficients of the infinite Fourier series

$$\zeta_{\mathrm{p}}(x) = \sum_{k \in \mathbb{Z}} c_k e^{i \frac{\pi}{\mathsf{L}} k x} \tag{I.3}$$

through the relation

$$\mathsf{a}_{\mathsf{k}} = \sum_{j \in \mathbb{Z}} c_{\mathsf{k}+j\mathsf{N}}.$$

With an abuse of definition, we will still refer to a_{k} as "Fourier coefficients", and will identify a_{k} and $c_{\mathsf{k}+2j\mathsf{N}}$ for any $j \in \mathbb{Z}$. That a_{k} encompasses a full series of Fourier coefficient is of course unavoidable as $e^{i\frac{\pi}{\mathsf{L}}\mathsf{k}x}$ and $e^{i\frac{\pi}{\mathsf{L}}(\mathsf{k}+j\mathsf{N})x}$ are indistinguishable on the discrete grid $x \in \{\mathsf{x}_1, \cdots, \mathsf{x}_{\mathsf{N}}\}$, and is called *aliasing*. For functions in the Schwartz space, Fourier coefficients decrease exponentially fast, and hence the error between (I.2) and (I.3) is exponentially decaying with respect to $\delta\mathsf{x}^{-1} \propto \mathsf{N}/\mathsf{L}$; this is called *spectral accuracy* and is, together with the extraordinary efficiency of the FFT (whose cost grows theoretically as $\mathsf{N}\log(\mathsf{N})$), the main reason for the popularity of spectral methods. The simplicity of the resulting numerical codes is another incentive for using them.

After decomposition (I.2), the action of Fourier multipliers—and in particular differentiation—can be obtained *exactly* via multiplication on the discrete Fourier coefficients, and only nonlinear terms require some attention. The simplest way to approximately compute products is by pointwise multiplication on collocation points (which can be obtained from the discrete Fourier coefficients *via* discrete inverse Fourier transform), and then apply the discrete Fourier transform. Since

$$\zeta_{\mathrm{p}}^2(\mathsf{x}_{\mathsf{i}}) = \Big(\sum_{\mathsf{k}=-\lfloor \mathsf{N}/2 \rfloor}^{\lceil \mathsf{N}/2 \rceil -1} \mathsf{a}_{\mathsf{k}} e^{i\frac{\pi}{\mathsf{L}}\mathsf{k}\mathsf{x}_{\mathsf{i}}} \Big)^2 = \sum_{\mathsf{j}=-\lfloor \mathsf{N}/2 \rfloor}^{\lceil \mathsf{N}/2 \rceil -1} \sum_{\mathsf{l}=-\lfloor \mathsf{N}/2 \rfloor}^{\lceil \mathsf{N}/2 \rceil -1} \mathsf{a}_{\mathsf{j}}\mathsf{a}_{\mathsf{l}} e^{i\frac{\pi}{\mathsf{L}}(\mathsf{j}+\mathsf{l})\mathsf{x}_{\mathsf{i}}}$$

one obtains

$$\zeta_{\mathrm{p}}^2(x) \approx \sum_{\mathsf{k}=-\lfloor \mathsf{N}/2 \rfloor}^{\lceil \mathsf{N}/2 \rceil -1} \mathsf{b}_{\mathsf{k}} e^{i\frac{\pi}{\mathsf{L}}\mathsf{k}x} \quad \text{with} \quad \mathsf{b}_{\mathsf{k}} = \sum_{\mathsf{j}+\mathsf{l} \in \{\mathsf{k}-\mathsf{N},\mathsf{k},\mathsf{k}+\mathsf{N}\}} \mathsf{a}_{\mathsf{j}}\mathsf{a}_{\mathsf{l}}.$$

For $\mathsf{j},\mathsf{k},\mathsf{l} \in \{-\lfloor \mathsf{N}/2 \rfloor, \cdots, \lceil \mathsf{N}/2 \rceil - 1\}$, the contribution $\mathsf{a}_{\mathsf{j}}\mathsf{a}_{\mathsf{l}}$ such that $\mathsf{j}+\mathsf{l} \neq \mathsf{k}$ is a spurious effect from aliasing, and sometimes contributes to numerical instabilities. In order to suppress such terms, one can add a sufficient number of modes[72] (in the case of quadratic nonlinearity, the so-called Orszag's 3/2 rule [349]) with coefficients set to zero, so that $\mathsf{a}_{\mathsf{j}}\mathsf{a}_{\mathsf{l}} = 0$ when $|\mathsf{j}+\mathsf{l}-\mathsf{k}| = \mathsf{N}$ (with $\mathsf{j},\mathsf{k},\mathsf{l} \in \{-\lfloor \mathsf{N}/2 \rfloor, \cdots, \lceil \mathsf{N}/2 \rceil - 1\}$). The use of such dealiasing techniques separate (Galerkin) *spectral* approximations (since the error of the approximation is orthogonal to the expansion functions) from *pseudospectral* approximations [348]. The dealiasing technique deserves some comments. Firstly, it can be performed exactly only for power nonlinearities, and the number of additional coefficients grows with the power, as $(p+1)/2$ where p is the power. This prevents its use (again, if perfection is aimed at) for the second Whitham equation, eq. (ix), for instance. Secondly, it should be noticed

[72] In practice it is more convenient to use low-pass filters, that is set to zero extreme modes, so as to work with vectors with a fixed given length.

that the magnitude of the error generated by the spurious aliasing term is not necessarily greater (nor smaller) than the contributions omitted from the exact formula: by eq. (I.3),

$$\zeta_{\mathrm{p}}^2(x) = \big(\sum_{k\in\mathbb{Z}} c_k e^{i\frac{\pi}{\mathsf{L}}kx}\big)^2 = \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} c_j c_l e^{i\frac{\pi}{\mathsf{L}}(j+l)x}.$$

Finally, low-pass filters, Π, having discontinuous symbols, they do not enjoy good commutator properties (that is, $[\Pi,\zeta]$ is not regularizing of order -1); it is hence advisable for equations of quasilinear type to use regularized (at least Lipschitz) symbols. In practice, one can use when possible sufficiently many modes, N, so that discrete Fourier coefficients decrease to machine precision with room to spare and hence dealiasing is theoretically immaterial, and use dealiasing only when it appears necessary or beneficial.

In any case, we find that for data with exponentially decaying Fourier coefficients, the error made using the Fourier (pseudo-)spectral method is exponentially decaying with N/L (in addition to the error exponentially decaying with L from the periodization procedure).

I.1.2 Time integration

After discretization in space using the Fourier (pseudo-)spectral method, eq. (I.1) becomes a system of differential equations,

$$\frac{\mathrm{d}\mathsf{U}}{\mathrm{d}t} = \mathcal{F}(t,\mathsf{U}),$$

where $\mathsf{U}(t)$ is the N-dimensional vector of discrete Fourier coefficients of ζ_{p} at time t.

We numerically solve the initial-value problem using the explicit four-step Runge–Kutta method (RK4). Hence, given $\delta\mathsf{t}$ the timestep, we compute the approximate value of $\mathsf{U}(\mathsf{t}+\delta\mathsf{t})$ from the knowledge of $\mathsf{U}(\mathsf{t})$ with the formula

$$\mathsf{U}(\mathsf{t}+\delta\mathsf{t}) = \mathsf{U}(\mathsf{t}) + \tfrac{\delta\mathsf{t}}{6}\big(\mathsf{U}_1 + 2\mathsf{U}_2 + 2\mathsf{U}_3 + \mathsf{U}_4\big)$$

where

$$\begin{cases} \mathsf{U}_1 = \mathcal{F}(\mathsf{t},\mathsf{U}(\mathsf{t})),\\ \mathsf{U}_2 = \mathcal{F}(\mathsf{t}+\frac{\delta\mathsf{t}}{2},\mathsf{U}(\mathsf{t})+\frac{\delta\mathsf{t}}{6}\mathsf{U}_1),\\ \mathsf{U}_3 = \mathcal{F}(\mathsf{t}+\frac{\delta\mathsf{t}}{2},\mathsf{U}(\mathsf{t})+\frac{\delta\mathsf{t}}{2}\mathsf{U}_2),\\ \mathsf{U}_4 = \mathcal{F}(\mathsf{t}+\delta\mathsf{t},\mathsf{U}(\mathsf{t})+\delta\mathsf{t}\,\mathsf{U}_3). \end{cases}$$

The RK4 method is a fourth-order method, meaning that the local truncation error is on the order of $\mathcal{O}(\delta\mathsf{t}^5)$, while the total accumulated error is of the order of $\mathcal{O}(\delta\mathsf{t}^4)$.

Working with *stiff* problems (since operators at stake, and in particular space-differentiation are not bounded), the RK4 method suffers from numerical instabilities unless a Courant–Friedrichs–Levy (CFL) type smallness condition on the timestep is ensured (see [401, §10] or [419, §5.4]). This is not a strong issue in practice, since we aim at high accuracy on smooth functions, and that the exponential rate of spectral methods associated with the algebraic rate of the RK4 method will typically urge to use small time step-size/space intervals ratios anyway.

I.1.3 Code samples and validation

In this section we validate the numerical method by propagating in time a solitary wave for the Whitham equation, eq. (I.1). Given a velocity $c > 1$ (sufficiently small; see Figure vi), the solitary wave $\zeta(t,x) = \zeta_c(x-ct)$ satisfies

$$-c\zeta_c + \mathsf{L}(D)\zeta_c + \tfrac{3\varepsilon}{4}\zeta_c^2 = 0.$$

The profile ζ_c can be numerically obtained by using the Fourier spectral method, and solving the resulting system of nonlinear equations

$$\mathcal{G}(\mathsf{U}) = \mathbf{0}$$

by a standard Newton iteration,

$$\mathsf{U}^{(\mathsf{m}+1)} = \mathsf{U}^{(\mathsf{m})} - \delta\mathsf{U}^{(\mathsf{m})} \quad \text{with} \quad \mathrm{Jac}(\mathcal{G}(\mathsf{U}))|_{\mathsf{U}=\mathsf{U}^{(\mathsf{m})}}\delta\mathsf{U}^{(\mathsf{m})} = \mathcal{G}(\mathsf{U}^{(\mathsf{m})})$$

where $\mathsf{U}^{(\mathsf{m})}$ denotes the m^{th} iterate, and where $\mathrm{Jac}(\mathcal{G}(\mathsf{U}))$ is the Jacobian of $\mathcal{G}(\mathsf{U})$ with respect to $\mathsf{U} \in \mathbb{C}^{\mathsf{N}}$. Due to the translation invariance of the problem, the Kernel of the Jacobian of the continuous (infinite-dimensional) vector-field is non-empty when evaluated at the solution ζ_c, since $\partial_x\zeta_c$ is an element of the nullspace. It is hence advisable to add the corresponding spectral projection to $\mathrm{Jac}(\mathcal{G}(\mathsf{U}))$. The initial iterate $\mathsf{U}^{(0)}$ can be chosen from the explicit KdV solitary wave; see Section v.

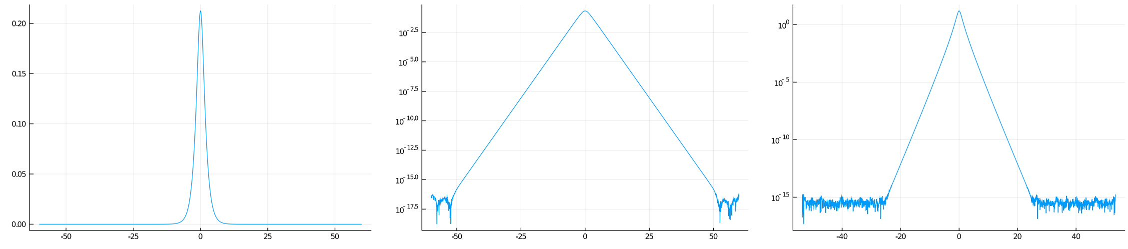
Our numerical illustrations have been performed using the Julia language [50]. We provide below a self-contained code, favoring conciseness and readability to performance.

First we generate the solitary wave solution to eq. (I.1) (with $\varepsilon = \mu = 1$) of velocity $c = 1.1$, using $\mathsf{N} = 2^{11}$ modes on the torus of half-period $\mathsf{L} = 60$. Figure I.1 represents the outcome. The alteration of values at collocation points is of the order of 10^{-16}, *i.e.* machine precision, when multiplying by two the number of modes, N, or both N and the half-period, L, indicating a maximal resolution of the real-line problem.

```
using FFTW,LinearAlgebra
" Compute the solitary wave of the Whitham equation with velocity c. "
function SolitaryWaveWhitham(;c=1.1,N=2^11,L=60)
    # Initialize
    dx  = 2*L/N; X = -L:dx:L-dx                  # mesh of collocation points
    dk  = pi/L; K = dk.* [0:N÷2-1 ; -N÷2:-1]     # Fourier wavenumbers
    FFT = exp.(-1im*K*(X.-X[1])')                # FFT as a matrix operator
    IFFT= exp.(1im*K*(X.-X[1])')/length(X)       # IFFT as a matrix operator
    Dx =  1im * K                                # Differentiation (symbol)
    LD = sqrt.(tanh.(K)./K); LD[1]=1             # Fourier multiplier L(D) (symbol)
    Z = 2*(c-1)*sech.(sqrt(3/2*(c-1))*X).^2      # Initial guess (KdV formula)
    # Solve G(Z) = 0
    for i in 1:10                                # Newton iteration with maximum 10 steps
        G = -c*Z+ifft(LD.*fft(Z))+3/4*Z.^2       # Compute G(Z)
        if norm(G)<10^(-15) break end            # Stop if |G(Z)| is below tolerance
        JacG = (IFFT*(Diagonal(LD)*FFT)          # Compute Jac(G(Z))
                +Diagonal(3/2*Z .-c))
        dxZ = ifft(Dx.*fft(Z));
        dZ = dxZ./norm(dxZ); Proj = dZ*dZ'       # Compute the projection onto span(∂Z)
        Z = Z - ( JacG + Proj ) \ G              # Compute next Newton iterate
    end
    return X,real.(Z),K,fft(Z)
end
```

Then we integrate in time eq. (I.1) starting with this solitary wave, using the pseudospectral method (without dealiasing) and explicit RK4 solver, with 10^5 time steps on $\mathsf{t} \in [0, 40]$. The comparison of the numerically computed solution and the exact solution (obtained from the previous function by a Fourier phase shift) is represented in Figure I.2. Augmenting the number of modes, N, or decreasing the time step-size, $\delta\mathsf{t}$, by a factor of two does not significantly improve (nor deteriorates) the accuracy.

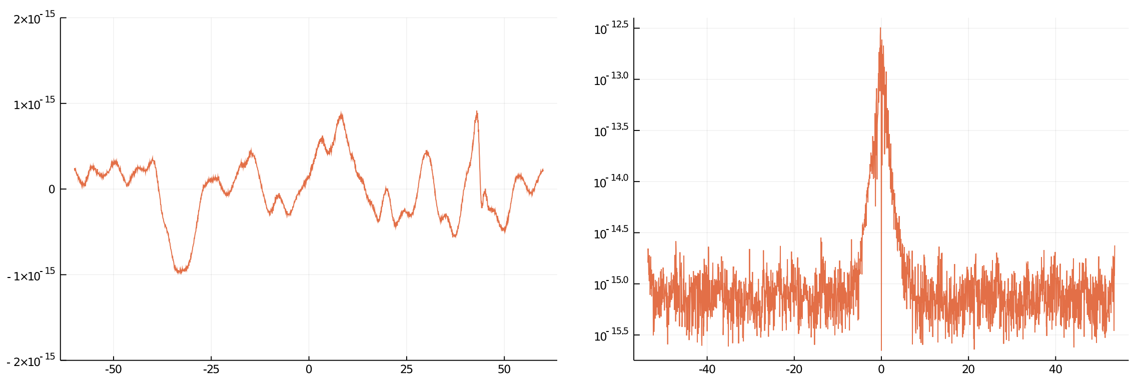
```
" Solve the initial-value problem for the Whitham equation with a solitary wave
   as initial data."
function SolveWhitham(;T=40,dt=0.0004)
    # Initialize mesh of collocation points X, Fourier wavenumbers K
    # and initial data (Z = values at collocation points, U = Fourier coefficients)
    X,Z,K,U=SolitaryWaveWhitham(c=1.1,N=2^11,L=60)
    Dx =  1im * K                                # Differentiation (symbol)
    LD = sqrt.(tanh.(K)./K); LD[1]=1             # Fourier multiplier L(D) (symbol)
```



(a) Physical space (b) Same as (a), in log scale (c) Discrete Fourier coefficients (modulus, in log scale)

Figure I.1: Computed solitary wave of eq. (I.1) with velocity $c = 1.1$.

```
    # Solve ∂t U = F(U)
    F(U) = -Dx.*LD.*U - 3/4*Dx.*fft(ifft(U).^2)
    for i in dt:dt:T                          # Iterate explicit solver RK4
        U1 = F( U )
        U2 = F( U + dt/2 * U1 )
        U3 = F( U + dt/2 * U2 )
        U4 = F( U + dt * U3 )
        U  = U + dt/6 * ( U1 + 2*U2 + 2*U3 + U4 )
    end
    return (X,real.(ifft(U)),K,U)
end
```



(a) Physical space (b) Fourier coefficients (modulus, in log scale)

Figure I.2: Difference between the computed solution at final time and the translated initial data.

I.2 The special case of the (Whitham–)Green–Naghdi system

Let us describe further on the numerical scheme we employ in Appendix I.5 later on when integrating in time the (one-dimensional with flat bottom) Green–Naghdi equations (see Section 8) and their fully dispersive counterparts (see Section 10), which read in dimensionless variables

$$\begin{cases} \partial_t \zeta + \partial_x (hu) = 0, \\ \partial_t v + \partial_x \big(\zeta + \varepsilon uv - \frac{\varepsilon}{2} u^2 - \frac{\mu\varepsilon}{2} h^2 (\partial_x \mathsf{F} u)^2 \big) = 0 \end{cases} \tag{I.4}$$

where $h = 1+\varepsilon\zeta$ and F is the Fourier multiplier operator with symbol $F(k) = \sqrt{\frac{3}{\mu|k|\tanh(\mu|k|)} - \frac{3}{\mu|k|^2}}$; and v and u are related through the elliptic equation

$$v = u - \frac{\mu}{3h}\partial_x\big(h^3\partial_x \mathsf{F}u\big). \tag{I.5}$$

Sufficiently regular solutions to the Green–Naghdi equations satisfy the above, replacing F by the identity. By Lemma 8.9, u is uniquely determined by eq. (I.5) from sufficiently regular (v,ζ) with $\inf_{\mathbb{R}}(1+\varepsilon\zeta) > 0$, and we can solve eq. (I.4) as evolution equations for (ζ, v). The presence of Fourier multipliers naturally leads to Fourier pseudospectral methods described above. As emphasized in Section 9,[73] one of the difficulties when integrating in time eq. (I.4) is that we are led to solve the elliptic problem eq. (I.5) at each time-step. However, it turns out to be not too costly—at least in the one-dimensional framework—to solve the elliptic problem at each time step while maintaining high resolution thanks to the efficiency of pseudospectral methods and of the Krylov subspace iterative method GMRES for solving the elliptic problem.

Let us now be more precise. We use the same Fourier pseudospectral approach as outlined in the previous section, i.e., we approximate the solution u, ζ via discrete Fourier transforms. With this spatial discretization, eq. (I.4) become finite-dimensional systems of ODEs coupled with a system of equations of the form

$$\begin{cases} \dfrac{\mathrm{d}\hat{\zeta}}{\mathrm{d}t} = \mathcal{G}_1(\hat{\zeta}, \hat{\mathsf{u}}), \\ \dfrac{\mathrm{d}\hat{\mathsf{v}}}{\mathrm{d}t} = \mathcal{G}_2(\hat{\zeta}, \hat{\mathsf{u}}, \hat{\mathsf{v}}), \\ \mathcal{M}[\hat{\zeta}]\hat{\mathsf{u}} = \hat{\mathsf{v}} \end{cases} \tag{I.6}$$

where $\hat{\zeta}(t), \hat{\mathsf{u}}(t), \hat{\mathsf{v}}(t)$ are the N-dimensional vectors of discrete Fourier coefficients, and $\mathcal{M}[\hat{\zeta}]$ is an N-by-N matrix. The two ODEs in system (I.6) are integrated with the standard explicit fourth order Runge–Kutta method. The system of linear equations in (I.6) is a convolution in the space of Fourier coefficients: the matrix $\mathcal{M}(\hat{\zeta})$ is constructed using (inverse) Fast Fourier Transform and multiplication in collocation points. As already mentioned, the inversion is performed with the very efficient Krylov approach GMRES [368] (typically using $\mathcal{M}[0]$, which is diagonal, as a preconditioner). This numerical method has been discussed and successfully employed in a variety of computationally challenging situations in [151].

I.3 The special case of the Isobe–Kakinuma systems

Under the formulation (13.17), the Isobe–Kakinuma model discussed in Section 13[74] has a structure similar to the one of the Green-Naghdi system described above, namely two scalar evolution equations involving variables which can be described as solutions of a system of differential equations. As such the strategy described in the previous section applies *mutatis mutandis*. More precisely we solve a systems of ODEs coupled with a system of equations of the form

$$\begin{cases} \dfrac{\mathrm{d}\hat{\zeta}}{\mathrm{d}t} = \mathcal{G}_1(\hat{\zeta}, \hat{\phi}_0', \hat{\phi}_1, \dots, \hat{\phi}_N), \\ \dfrac{\mathrm{d}\hat{\mathsf{v}}}{\mathrm{d}t} = \mathcal{G}_2(\hat{\zeta}, \hat{\phi}_0', \hat{\phi}_1, \dots, \hat{\phi}_N), \\ \mathcal{M}[\hat{\zeta}](\hat{\phi}_0', \hat{\phi}_1, \dots, \hat{\phi}_N) = \mathcal{L}\hat{\mathsf{v}} \end{cases} \tag{I.7}$$

where $\hat{\zeta}(t), \hat{\mathsf{v}}(t), \hat{\phi}_0'(t), \hat{\phi}_1(t), \dots, \hat{\phi}_N(t)$ denote the N-dimensional vectors of discrete Fourier coefficients, of the corresponding variables in (13.17) (because the variables ψ and ϕ_0 belong to Beppo Levi spaces and in view of satisfying periodic boundary conditions it is convenient to use the closed

[73] see also references therein for alternative approaches to the numerical integration of the Green–Naghdi system.
[74] In fact this structure is common to all the high order models presented in Section 12 and Section 13.

form of the equations in terms of $v \overset{\text{def}}{=} \partial_x\psi$ and $\phi_0' \overset{\text{def}}{=} \phi_0 - \psi$ instead), and $\mathcal{L}$ and $\mathcal{M}[\hat{\zeta}]$ are $(N+1)\mathtt{N}$-by-$(N+1)\mathtt{N}$ matrix (with N the rank of the Isobe–Kakinuma model). We can then follow exactly the strategy described in the previous section.

I.4 The special case of the water waves system

Numerical methods based on conformal mapping allow to solve very efficiently the time integration of the water waves system, eq. (2.2), in the framework of horizontal dimension $d = 1$. The idea of using conformal mapping to transform the Laplace problem, eq. (2.4) into an equivalent Laplace problem in a more convenient—and in particular fixed—domain can be found[75] as early as in [69, 350, 210] (the latter acknowledging the original idea in [263] on the linearized equations) and has been revisited several times,in particular in [169, 100, 426, 336, 285, 318, 406, 321] (see also [416, 284, 218, 222, 223, 6, 415] for the use of the conformal mapping in studies on the initial-value problem).

We briefly recall some principles below, following closely the exposition in [100], and in particular restricting ourselves to the flat bottom situation (in addition to dimension $d = 1$). The first step (after the non-dimensionalization, Section 2.4) consists in introducing suitable holomorphic coordinates, that is a map

$$(X,Y)[\varepsilon\zeta(t,\cdot)] : (\xi,\eta) \in \mathbb{R}\times(-\delta,0) \mapsto (x,y) \in \{(x,y)\in\mathbb{R}^2 \ : \ -1 < y < \varepsilon\zeta(t,x)\}$$

which is a conformal transformation when identifying the real plane and complex plane (and un-rescaling variables), *i.e.* is differentiable and satisfies the anisotropic Cauchy-Riemann equations

$$\partial_\xi X \ = \ \partial_\eta Y \quad ; \quad \mu\partial_\xi Y \ = \ -\partial_\eta X.$$

The function Y satisfies the Laplace problem

$$\begin{cases} \mu\partial_\xi^2 Y + \partial_\eta^2 Y = 0 & \text{in } \mathbb{R}\times(-\delta,0), \\ Y(t,\cdot,0) = \varepsilon\zeta(t,X(t,\cdot,0)), \\ Y(t,\cdot,-\delta) = -1. \end{cases} \tag{I.8}$$

Solving eq. (I.8) in Fourier space and denoting denoting $z(t,\xi) = \zeta(t,X(t,\xi,0))$, we find

$$Y(t,\cdot,\eta) \ = \ \frac{\eta}{\delta} + \varepsilon\frac{\sinh(\sqrt{\mu}|D|(\delta+\eta))}{\sinh(\sqrt{\mu}\delta|D|)} z(t,\cdot),$$

and the Cauchy-Riemann equations, yield

$$\partial_\xi X(t,\cdot,\eta) = \frac{1}{\delta} + \varepsilon\frac{\sqrt{\mu}|D|\cosh(\sqrt{\mu}|D|(\delta+\eta))}{\sinh(\sqrt{\mu}\delta|D|)} z(t,\cdot).$$

Up to here $\delta > 0$ is a free parameter, possibly depending on the time variable, $t \in \mathbb{R}$. In the periodic framework—which we will use henceforth unless otherwise noted—it is convenient to set the parameter so that the conformal map preserves the periodicity (with same period).[76] Using the above equation, we see that this can be done provided we choose

$$\delta(t) = 1 + \varepsilon\frac{1}{L}\int_{-L/2}^{L/2} z(t,\xi)\,\mathrm{d}\xi,$$

[75] in the context of the time integration of the evolution equations with general initial data. Holomorphic coordinates can also be used to compute or construct special solutions such as traveling waves [340, 279, 394, 381, 110, 32]; see also the recent [114, 167, 108, 144] and references therein.

[76] This discussion is of course essential in view of numerical simulations using Fourier (pseudo-)spectral methods.

where L is the wave period. Setting $\eta = 0$ in the above identity, we obtain

$$\partial_\xi X(t,\cdot,0) = \frac{1}{\delta} + \varepsilon\sqrt{\mu}|D|\operatorname{cotanh}(\sqrt{\mu}\delta|D|)\zeta(t,X(t,\cdot,0)), \tag{I.9}$$

which—after integration—implicitly defines $X(t,0,\cdot)$ up to a constant which can be arbitrarily set, and hence $z(t,\cdot) = \zeta(t,X(t,\cdot,0))$ and as a result the holomorphic coordinates $(X,Y)(t,\cdot,\cdot)$, from the knowledge of the surface deformation $\zeta(t,\cdot)$.

As a second step we rewrite the water waves system, eq. (2.7), using the holomorphic coordinates. Denoting Φ the solution to eq. (2.8) (with flat bottom and dimension $d = 1$), we have that $F(t,\xi,\eta) \overset{\text{def}}{=} \Phi(t,X(t,\xi,\eta),Y(t,\xi,\eta))$ satisfies

$$\begin{cases} \mu\Delta_\xi F + \partial_\eta^2 F = 0 & \text{in } \mathbb{R}\times(-\delta,0), \\ F(t,\cdot,0) = \psi(t,X(t,\cdot,0)), \\ (\partial_\eta F)(t,\cdot,-\delta) = 0, \end{cases}$$

which is immediately solvable in Fourier space. By using the chain rule we can rewrite the two equations in eq. (2.7) equivalently using the unknowns $X(t,\cdot,0)$, $Y(t,\cdot,0)$, and $f(t,\cdot) \overset{\text{def}}{=} \psi(t,X(t,\cdot,0))$. Yet these equations are not readily under the form of evolution equations for which $\{t\}\times\mathbb{R}$ are non-characteristic hypersurfaces in the space-time $\mathbb{R}\times\mathbb{R}$. It demands some clever manipulations and the help of analytic function theory to arrive at the following set of two evolution equations:

$$\begin{cases} \partial_t z - \frac{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)(\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f)}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2} + \mu\varepsilon(\partial_\xi z)\mathsf{H}_0^{\mu,\delta}\partial_\xi\big(\frac{\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2}\big) = \mu\varepsilon q_0\partial_\xi z, \\ \partial_t f + z + \frac{\varepsilon}{2}\frac{(\partial_\xi f)^2-\mu(\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f)^2}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2} + \mu\varepsilon(\partial_\xi f)\mathsf{H}_0^{\mu,\delta}\partial_\xi\big(\frac{\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2}\big) = \mu\varepsilon q_0\partial_\xi f, \end{cases} \tag{I.10}$$

where we define the following Fourier multipliers

$$\mathsf{G}_0^{\mu,\delta} \overset{\text{def}}{=} \sqrt{\mu}|D|\tanh(\sqrt{\mu}\delta|D|), \quad \mathsf{H}_0^{\mu,\delta} \overset{\text{def}}{=} \frac{\operatorname{cotanh}(\sqrt{\mu}\delta|D|)}{\sqrt{\mu}|D|}, \quad \mathsf{I}_0^{\mu,\delta} \overset{\text{def}}{=} \sqrt{\mu}|D|\operatorname{cotanh}(\sqrt{\mu}\delta|D|),$$

(we use the convention $\frac{\operatorname{cotanh}(0)}{0} = 0$, that is we first subtract the average before applying $\mathsf{H}_0^{\mu,\delta}$) and[77]

$$q_0 = \frac{1}{L}\int_{-L/2}^{L/2}(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)\mathsf{H}_0^{\mu,\delta}\partial_\xi\big(\frac{\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2}\big) + \varepsilon\frac{(\partial_\xi\zeta)(\frac{1}{\mu}\mathsf{G}_0^{\mu,\delta}f)}{(1+\varepsilon\mathsf{I}_0^{\mu,\delta}z)^2+\mu\varepsilon^2(\partial_\xi z)^2}\,\mathrm{d}\xi.$$

Remark I.1. *In the real-line setting, we set $\delta = 1$, and $q_0 = 0$, accordingly with the limit $L\to\infty$. The action of $\mathsf{H}_0^{\mu,\delta}\partial_\xi$ must be precised, since it is not a Fourier multiplier in the sense of Definition III.1, due to the singularity at wavenumber $\xi = 0$. Yet it applies to functions which can be decomposed as $\partial_\xi f_0 + f_1$ where $f_0\in L^2(\mathbb{R})$ and $f_1\in L^1(\mathbb{R})$ (we measure here spatial decay at infinity rather than regularity). The singularity is removed for the first contribution, and the Fourier transform of the second contribution is continuous at $\xi = 0$, hence its pointwise multiplication with $\frac{i\xi\operatorname{cotanh}(\sqrt{\mu}\delta|\xi|)}{\sqrt{\mu}|\xi|} = \frac{1}{\xi}\frac{i\xi}{\sqrt{\mu}\tanh(\sqrt{\mu}\xi)}$ defines a distribution via the Cauchy principal value, and its inverse Fourier transform is well-defined in $L^\infty(\mathbb{R})$.*

The numerical strategy for numerically solving eq. (2.7) for given initial data (ζ_0,ψ_0) then consists in the following steps:

[77] In [100], the authors choose to prescribe $q_0 = 0$ rather than the choice of the origin of the the X-coordinate in the physical domain (say $X(t,0,0) = 0$), which causes a (possibly time-dependent) spatial shift.

- First we solve eq. (I.9) with $\zeta = \zeta_0$, and infer the corresponding initial data

$$z\,|_{t=0} = \zeta_0(t, X(0,\cdot,0)) \quad \text{and} \quad f\,|_{t=0} = \psi_0(t, X(0,\cdot,0)).$$

  Numerically solving eq. (I.9) demands an iteration scheme[78] which requires to evaluate ζ_0 at any given location x, either through a specified formula or a reconstruction from its values at collocation points (*e.g.* through the Fourier spectral method). In the latter case each iteration costs $\mathcal{O}(\mathsf{N}^2)$ operations where N is the number of collocation points. Hence this step can be relatively costly, but needs to be performed only once.

- Then we integrate in time eq. (I.10). Since the equations involve only pointwise operations and Fourier multiplications (and extracting average values, that is the first Fourier coefficient), the equations—after differentiating the second one to consider periodic or spatially decaying functions—are perfectly suited to Fourier pseudospectral methods presented in **??** I.1.1. Thanks to the efficiency of the Fast Fourier Transform, each time-step of your favorite time-integration scheme will require $\mathcal{O}(\mathsf{N}\log(\mathsf{N}))$ operations.

- At prescribed time $t = T$, we can plot solutions—and in particular the surface deformation through the graph $\big(X(T,\cdot,0), z = \zeta(T, X(T,\cdot,0))\big)$—using again eq. (I.9) to infer $X(T,\cdot,0)$. Optionally, if we desire to recover the values of solutions in physical space, $(\zeta(T,\cdot), \psi(T,\cdot))$ at prescribed collocation points (to compare with the results of other models for instance), then we need to use a costly interpolation.[79]

Remark I.2. *It should be pointed out that the time integration of eq.* (I.10) *through Fourier pseudo-spectral methods turns out to be much more efficient than the corresponding time integration of the (Whitham–)Green–Naghdi system, eq.* (I.6) *or the Isobe–Kakinuma model, eq.* (I.7)*, due to the expensive operation of inverting $\mathcal{M}(\hat{\zeta})$ at each time step. This does not mean that the latter models are pointless! One should recall that the present strategy for solving the water waves system is limited to simple geometries, and in particular dimension $d = 1$.*[80] *Moreover, as pointed out in [18], the conformal mapping method suffers from "anti-resolution" for large-amplitude waves: the location of gridpoints after solving eq.* (I.9) *turn out to spread out near wave crests, which in practice demands the use of a much greater number of modes to resolve the flow accurately, even in smooth situations.*

I.5 Numerical experiments

In this section we discuss the results of fairly naive numerical experiments, as an illustration of the improved accuracy of solutions to the (fully dispersive) Whitham–Green–Naghdi system (WGN) and solutions to the Isobe–Kakinuma model (IK) with respect to solutions to the standard Serre–Green–Naghdi system (SGN), for approximating solutions to the water waves system. These experiments have been set up using numerical scripts written in the Julia language [50], and more specifically the package developed in collaboration with P. Navaro which available at `https://github.com/WaterWavesModels/WaterWaves1D.jl`. The numerical schemes for the time integration of the SGN and WGN systems, based on the strategy presented in Appendix I.2, has been written in collaboration with C. Klein and used in much more demanding situations in [151]. The numerical scheme for the time integration of the IK systems uses the same strategy, as displayed in Appendix I.3. Finally, the numerical scheme for the time integration of the water waves

[78] In the numerical experiments of Appendix I.5 we simply use the contraction mapping fixed point algorithm, since the Newton algorithm turns out to be less efficient.

[79] Alternatively, one may evaluate at the collocation points defined by $X(T,\cdot,0)$ some data defined on regularly-spaced collocation points, which requires $\mathcal{O}(\mathsf{N}^2)$ operations.

[80] The high order spectral methods presented in Chapter D, including the so-called spectral methods mentioned in footnote 52, are valuable substitutes in dimension $d = 2$. The interested reader can find in [414] a thorough numerical investigation and comparison—in dimension $d = 1$—along with important bibliographic references of several methods which extend to the framework of dimension $d = 2$.

system, following the strategy presented in Appendix I.4, is based on a preliminary version written by L. Emerald.

In these numerical simulations we will observe the evolution of an initial disturbance of the surface elevation, with zero velocity, depending on scales of the system determined by the dimensionless parameters ε and μ. We restrict ourselves to the flat bottom situation and dimension $d = 1$, and set

$$\zeta\,|_{t=0}\,(x) = \exp(-|x|^3), \qquad \psi\,|_{t=0}\, \equiv 0, \tag{I.11}$$

so that the initial disturbance is spatially localized but not smooth. We compute numerically the emerging solution to the water waves system, eq. (2.7), to the Serre–Green–Naghdi system, eq. (8.2), to the Whitham–Green–Naghdi system, eq. (10.2), and to the Isobe–Kakinuma model, eq. (13.7) with $N = 1$ and $(p_0, p_1) = (0, 2)$, for several values of ε and μ.

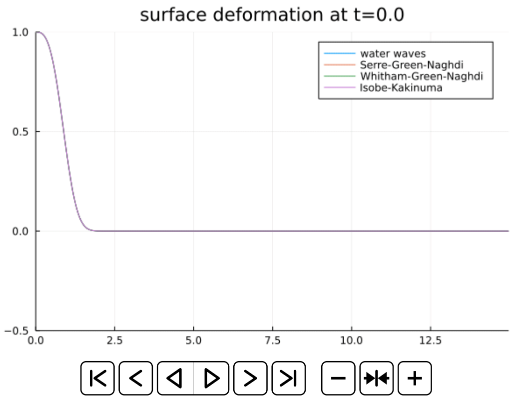
We use the pseudospectral method (without dealiasing) with $\mathtt{N} = 2^{10}$ collocation points on the torus of half-period $\mathtt{L} = 15$, and the standard explicit RK4 solver with 10^4 time steps on $\mathtt{t} \in [0, 10]$. The time evolution of the free surface predicted by the models and a snapshot of the difference between the water waves solution and the corresponding solutions of the weakly dispersive models for $\varepsilon \in \{0.5, 0.25, 0.125\}$ are represented in Figure I.3 ($\mu = 1$), Figure I.4 ($\mu = 0.1$), and Figure I.5 ($\mu = 0.01$). The corresponding ℓ^∞-norm of the differences (at collocation points defined by the water waves numerical solution) are aggregated in Table I.1.

Some comments are in order. Firstly the numbers in Table I.1 and features in Figures I.3 to I.5 do not vary significantly when the number collocation points or time steps are multiplied by two except when explicitly stated in the discussion below, so that the solutions can be considered as numerically resolved. We notice that spurious oscillations are visible in some plots, prominently Figures I.3b and I.4d. These appear to be due to aliasing issues on the numerical solver for the water waves system (the numerical solutions to other models do not exhibit such a noticeable rise of the high-frequency modes amplitudes, at least up to this timescale) associated with the fact that the initial data is not smooth, and hence Fourier modes with large wavenumbers retain relatively large amplitude. These oscillations disappear—or at least are tamed—when using dealiasing (according to Orszag's 3/2 rule), even when $N = 2^{11}$ collocation points are used (in fact in this case oscillations are no longer visible even without dealiasing). Additional spurious oscillations arise specifically on the numerical solution to the Isobe–Kakinuma model in Figure I.3b. A quick glance at the animation in Figure I.3a shows that the oscillations are artifacts generated when the numerical solution approaches what appears to be a singularity in the form of a gradient catastrophe (more precisely a plunging wave breaking). The computed solution cannot be considered as valid after this time, and is not computed (because the GMRES algorithm does not converge) when using $N = 2^{11}$ collocation points.

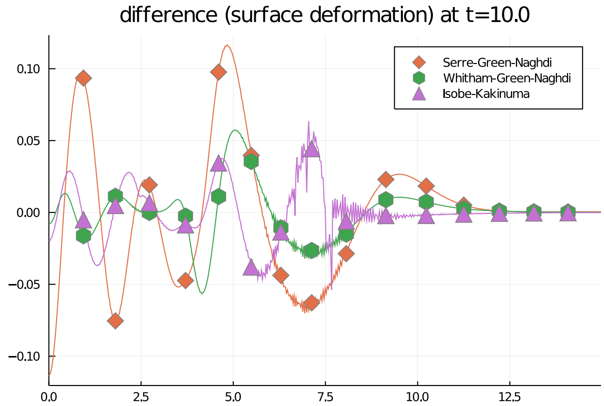
Finally the perceptive reader may be disappointed that the figures in Table I.1 do not quite fit the theory of convergence obtained in this manuscript, that is—roughly speaking—$\mathcal{O}(\mu^2 t)$ for the Serre–Green–Naghdi model, $\mathcal{O}(\mu^2 \varepsilon t)$ for the Whitham–Green–Naghdi model, and $\mathcal{O}(\mu^3 t)$ for the Isobe–Kakinuma model. This is partly explained by the fact that the theory is limited to the timescale $t = \mathcal{O}(1/\varepsilon)$. Indeed, when μ is small, the simulation is performed over a time range significantly larger than $1/\varepsilon$, and we observe the appearance of rapid modulations starting at the location where the Saint–Venant system would have produced a singularity. Even though the solutions to the dispersive equations remain smooth, steep gradients are produced which result in a large increase of Sobolev norms, that our theory is not able to digest. Roughly speaking, "μ ceases to be small". Notice in particular that when $\varepsilon = 1$, the accuracy of the numerical solution at time $\mathtt{t} = 10$ does not improve as μ decreases, and indeed the main error is seen to lie at the location of steep gradients. Yet this explanation does not explain some results when $\varepsilon\mathtt{t}$ is of order $\mathcal{O}(1)$. To my understanding, the situation is due to the fact that our initial data is not sufficiently regular. Using Gaussian initial data instead we do recover the expected accuracy for sufficiently small μ. This may serve as a pointer that the order of accuracy may not be the most desirable feature to look for in an asymptotic model, but that its robustness, that is its ability to produce fair results in a wide range of situations, is a key attribute.

	$t = 1$			$t = 10$		
	$\varepsilon = 0.5$	$\varepsilon = 0.25$	$\varepsilon = 0.125$	$\varepsilon = 0.5$	$\varepsilon = 0.25$	$\varepsilon = 0.125$
	Serre–Green–Naghdi					
$\mu = 1$	0.0666	0.0595	0.0538	0.116	0.137	0.133
$\mu = 0.1$	$9.08\,10^{-3}$	$7.86\,10^{-3}$	$7.36\,10^{-3}$	0.0441	0.0263	0.0279
$\mu = 0.01$	$7.05\,10^{-4}$	$3.14\,10^{-4}$	$2.74\,10^{-4}$	0.144	0.0242	$5.46\,10^{-3}$
	Whitham–Green–Naghdi					
$\mu = 1$	0.0201	0.0109	$5.58\,10^{-3}$	0.0573	0.0296	0.0133
$\mu = 0.1$	$3.81\,10^{-3}$	$1.7\,10^{-3}$	$8.63\,10^{-4}$	0.0204	$4.4\,10^{-3}$	$1.47\,10^{-3}$
$\mu = 0.01$	$2.6\,10^{-4}$	$7.98\,10^{-5}$	$3.8\,10^{-5}$	0.0909	$7.14\,10^{-3}$	$7.03\,10^{-4}$
	Isobe–Kakinuma					
$\mu = 1$	0.0148	0.0118	0.0105	0.0634	0.0444	0.0443
$\mu = 0.1$	$1.15\,10^{-3}$	$6.86\,10^{-4}$	$5.8\,10^{-4}$	$3.26\,10^{-3}$	$2.64\,10^{-3}$	$2.29\,10^{-3}$
$\mu = 0.01$	$5.35\,10^{-5}$	$4.0\,10^{-5}$	$3.74\,10^{-5}$	0.0112	$7.54\,10^{-4}$	$8.58\,10^{-5}$

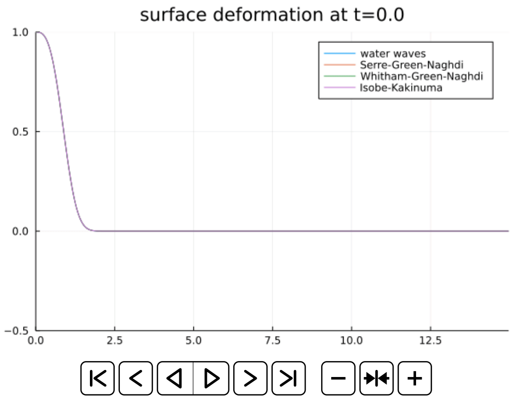
Table I.1: Errors produced by the Serre–Green–Naghdi, the Whitham–Green–Naghdi and the Isobe–Kakinuma models.



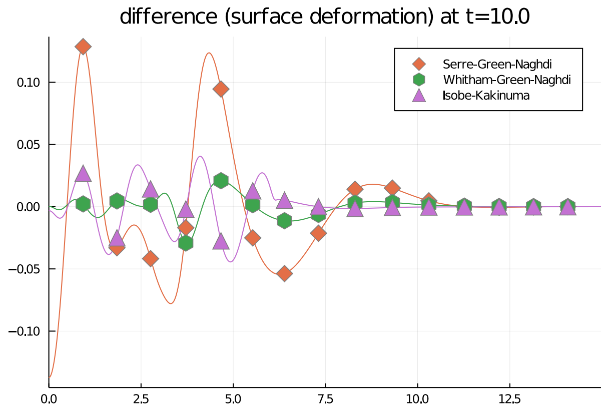
(a) Evolution in time. $\mu = 1$, $\varepsilon = 0.5$



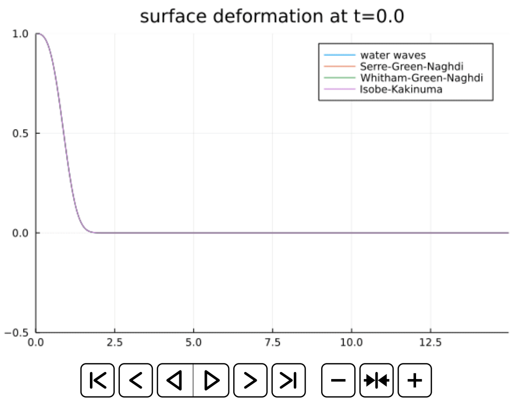
(b) Difference at time $\mathtt{t} = 10$. $\mu = 1$, $\varepsilon = 0.5$



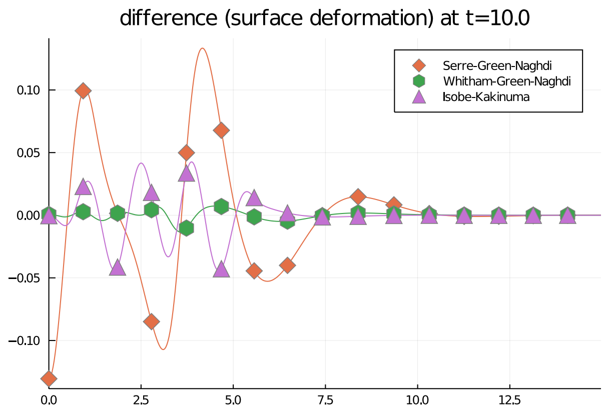
(c) Evolution in time. $\mu = 1$, $\varepsilon = 0.25$



(d) Difference at time $\mathtt{t} = 10$. $\mu = 1$, $\varepsilon = 0.25$

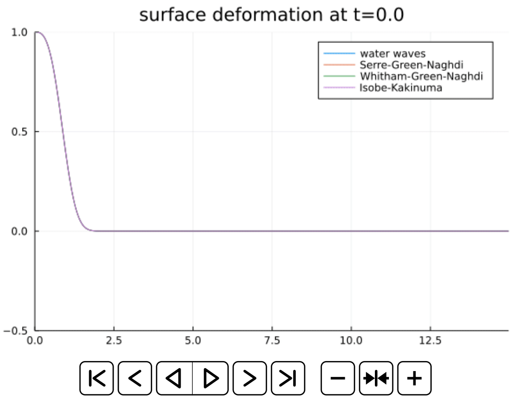


(e) Evolution in time. $\mu = 1$, $\varepsilon = 0.125$

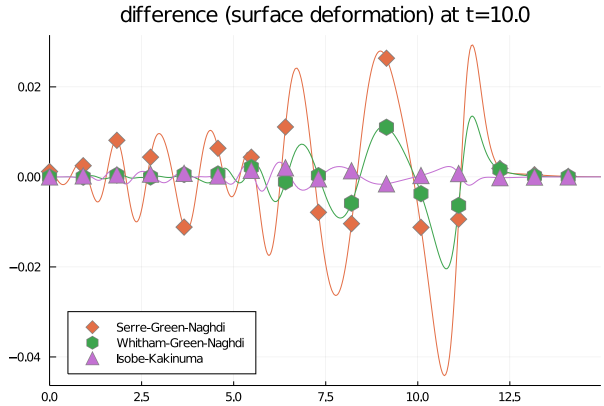


(f) Difference at time $\mathtt{t} = 10$. $\mu = 1$, $\varepsilon = 0.125$

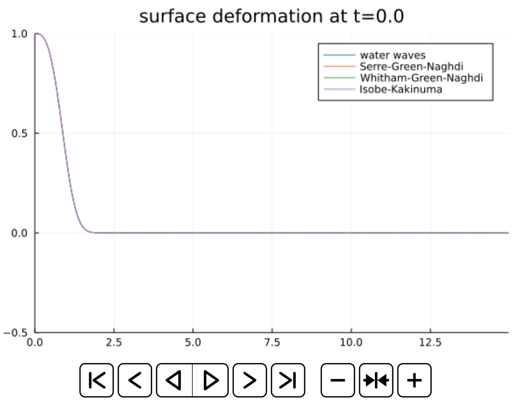
Figure I.3: Disintegration of a heap of water, eq. (I.11), according to the water waves system and the Serre–Green–Naghdi, Whitham–Green–Naghdi and Isobe–Kakinuma models in "deep" water situations ($\mu = 1$).



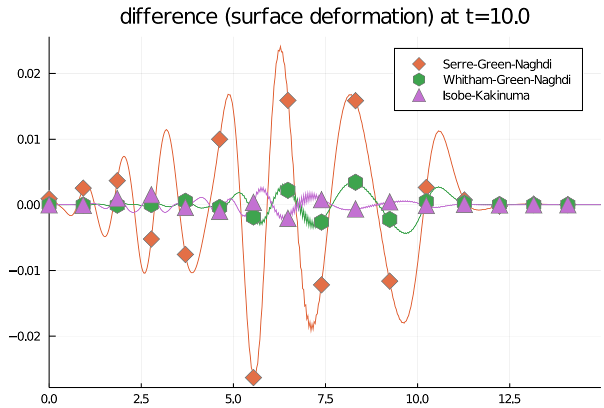
(a) Evolution in time. $\mu = 0.1$, $\varepsilon = 0.5$



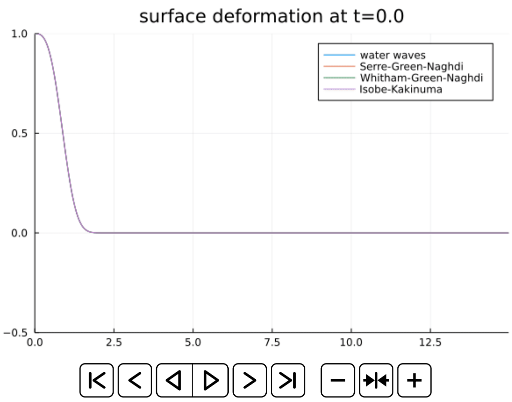
(b) Difference at time $\mathtt{t} = 10$. $\mu = 0.1$, $\varepsilon = 0.5$



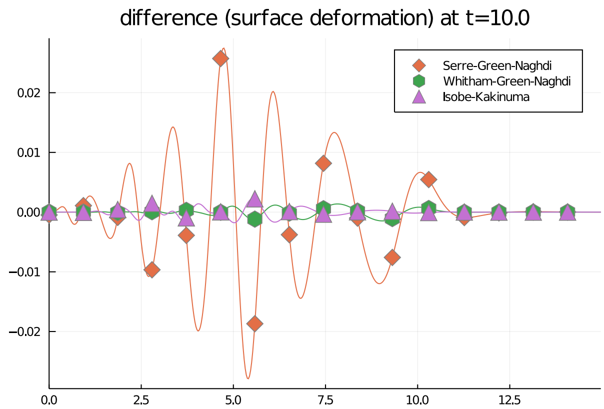
(c) Evolution in time. $\mu = 0.1$, $\varepsilon = 0.25$



(d) Difference at time $\mathtt{t} = 10$. $\mu = 0.1$, $\varepsilon = 0.25$

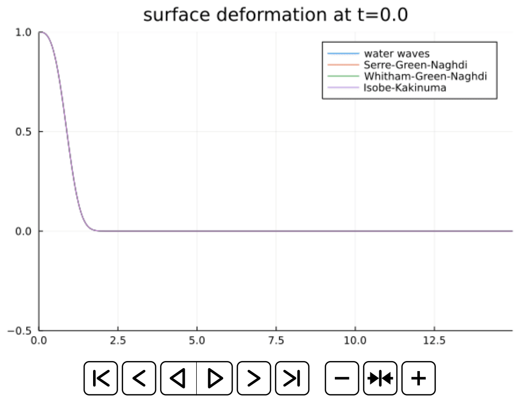


(e) Evolution in time. $\mu = 0.1$, $\varepsilon = 0.125$

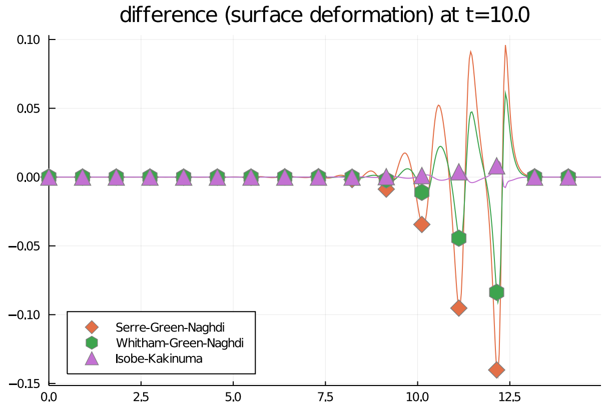


(f) Difference at time $\mathtt{t} = 10$. $\mu = 0.1$, $\varepsilon = 0.125$

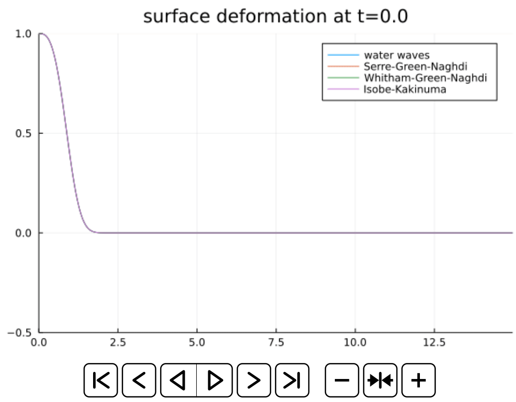
Figure I.4: Disintegration of a heap of water, eq. (I.11), according to the water waves system and the Serre–Green–Naghdi, Whitham–Green–Naghdi and Isobe–Kakinuma models in moderately shallow water situations ($\mu = 0.1$).



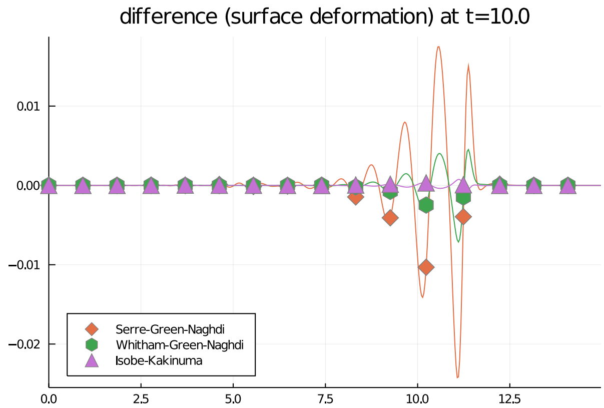
(a) Evolution in time. $\mu = 0.01$, $\varepsilon = 0.5$



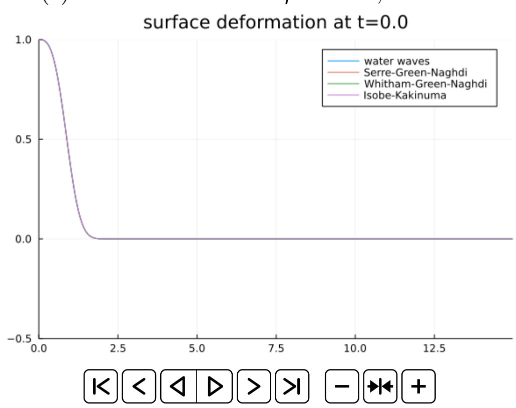
(b) Difference at time $\mathtt{t} = 10$. $\mu = 0.01$, $\varepsilon = 0.5$



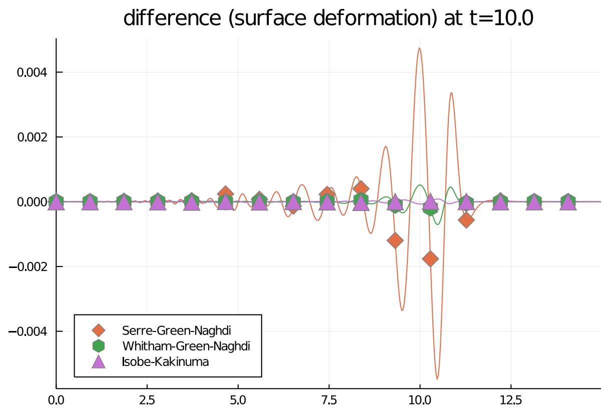
(c) Evolution in time. $\mu = 0.01$, $\varepsilon = 0.25$



(d) Difference at time $\mathtt{t} = 10$. $\mu = 0.01$, $\varepsilon = 0.25$



(e) Evolution in time. $\mu = 0.01$, $\varepsilon = 0.125$



(f) Difference at time $\mathtt{t} = 10$. $\mu = 0.01$, $\varepsilon = 0.125$

Figure I.5: Disintegration of a heap of water, eq. (I.11), according to the water waves system and the Serre–Green–Naghdi, Whitham–Green–Naghdi and Isobe–Kakinuma models in shallow water situations ($\mu = 0.01$).

II Technical tools

We provide elementary proofs of product, commutator and composition estimates, based on the following Hölder, Hausdorff-Young, interpolation and Sobolev embedding inequalities. The price to pay for the simplicity of the proof is that results are restricted to Sobolev spaces with integer indices; adapting proofs to Sobolev spaces with real indices would require for instance the Littlewood–Paley technology; see [13]. The reader can refer to [268, Appendix B] for sharp results and relevant references.

II.1 Basic inequalities

Lemma II.1 (Hölder inequality)**.** *Let $d, n \in \mathbb{N}^\star$, $n \geq 2$, $r, p_1, \dots, p_n \in (0, +\infty]$ such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}.$$

There exists $C > 0$ such that for any $f_i \in L^{p_i}(\mathbb{R}^d)$ $(i \in \{1, \dots, n\})$, $\prod_{i=1}^n f_i \in L^r(\mathbb{R}^d)$ and

$$\big|\prod_{i=1}^n f_i\big|_{L^r} \leq C \prod_{i=1}^n \big|f_i\big|_{L^{p_i}}.$$

Proof. The case $n = 2$ and $r = 1$ is the standard Hölder inequality following from Young's inequality, *i.e.* the concavity of the logarithm. The case $r \in (0, +\infty)$ is deduced applying the case $r = 1$ to $|f_i|^r \in L^{p_i/r}(\mathbb{R}^d)$, the case $r = \infty$ is obvious. The result for $n \geq 2$ follows by induction on n. $\square$

Lemma II.2 (Hausdorff-Young inequality)**.** *Let $d \in \mathbb{N}^\star$, $p \in [1, 2]$ and denote $q \in [2, +\infty]$ such that $p^{-1} + q^{-1} = 1$. There exists $C > 0$ such that for any $f \in L^p(\mathbb{R}^d)$, $\widehat{f} \in L^q(\mathbb{R}^d)$ and*

$$\big|\widehat{f}\big|_{L^q} \leq C\big|f\big|_{L^p}.$$

Proof. The case $p = 1$ is obvious from its integral representation, the case $p = 2$ is Parseval's theorem, and the case $1 < p < 2$ follows from Riesz–Thorin interpolation theorem. $\square$

Lemma II.3 (Interpolation inequality)**.** *Let $d \in \mathbb{N}^\star$ and $s, s_-, s_+ \in \mathbb{R}$ such that $s_- < s_+$ and $s_- \leq s \leq s_+$. There exists $C > 0$ such that for any $f \in H^{s_+}(\mathbb{R}^d)$,*

$$\big|f\big|_{H^s} \leq C\big|f\big|_{H^{s_-}}^{\theta}\big|f\big|_{H^{s_+}}^{1-\theta},$$

with $\theta = \frac{s_+ - s}{s_+ - s_-}$.

Proof. We have

$$\big|f\big|_{H^s}^2 \lesssim \int_{\mathbb{R}^d} |\widehat{f}|^2 \langle\cdot\rangle^{2s} \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \Big(|\widehat{f}|\langle\cdot\rangle^{s_-}\Big)^{2\theta} \Big(|\widehat{f}|\langle\cdot\rangle^{s_+}\Big)^{2(1-\theta)} \mathrm{d}\boldsymbol{x}$$

and we conclude by Hölder's inequality. $\square$

Lemma II.4 (Sobolev embedding)**.** *Let $d \in \mathbb{N}^\star$, and $p \in [2, +\infty]$ Let $s \in \mathbb{R}$ such that $s > d(\frac{1}{2} - \frac{1}{p})$. There exists $C > 0$ such that for any $f \in H^s(\mathbb{R}^d)$, $f \in L^p(\mathbb{R}^d)$ and*

$$\big|f\big|_{L^p} \leq C\big|f\big|_{H^s}.$$

Proof. We have with $q = \frac{p}{p-1}$ and $r = \frac{2(p-1)}{p-2}$ ($(q, r) = (2, \infty)$ if $p = 2$, and $(q, r) = (1, 2)$ if $p = \infty$)

$$\big|f\big|_{L^p} \lesssim \big|\widehat{f}\big|_{L^q} = \Big(\int_{\mathbb{R}^d} |\widehat{f}|^q \langle\cdot\rangle^{sq}\langle\cdot\rangle^{-sq}\Big)^{\frac{1}{q}} \lesssim \big|f\big|_{H^s}\big|\langle\cdot\rangle^{-sq}\big|_{L^r}^{\frac{1}{q}}$$

where we used Hausdorff-Young inequality applied to the inverse Fourier transform, and then Hölder's inequality. Notice $\langle\cdot\rangle^{-sq} \in L^r(\mathbb{R}^d)$ since $sqr > d$. $\square$

II.2 Product, commutator and composition estimates

Proposition II.5. *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, and $s_1, s_2 \in \mathbb{R}$ satisfying $s_1 \geq s$, $s_2 \geq s$ and $s_1 + s_2 > s + d/2$. There exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, then $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}}.$$

Proof. Let us first deal with the case $s = 0$. The case $s_1 = 0$ or $s_2 = 0$ is straightforward by Sobolev embedding, Lemma II.4. Otherwise there exists $p_i > 2$ such that $\frac{1}{2} - \frac{1}{p_i} < \frac{s_i}{d}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. The result follows from Hölder's inequality, Lemma II.1, and again Sobolev embedding.

We consider now the case $s \in \mathbb{N}^\star$. By Leibniz rule, we have for any $\boldsymbol{k} \in \mathbb{N}^d$ with $|\boldsymbol{k}| \leq s$:

$$\partial^{\boldsymbol{k}}(fg) = \sum_{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{i}} (\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g).$$

we estimate each summand using the $s = 0$ case:

$$\big|(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)\big|_{L^2} \lesssim \big|\partial^{\boldsymbol{i}} f\big|_{H^{s_1-|\boldsymbol{i}|}}\big|\partial^{\boldsymbol{j}} g\big|_{H^{s_2-|\boldsymbol{j}|}},$$

and the result follows. □

Corollary II.6. *Let $d \in \mathbb{N}^\star$ and $s \in \mathbb{N}$. The space $H^s(\mathbb{R}^d)$ is a Banach algebra as soon as $s > d/2$.*

Proposition II.7. *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$ and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max(\{s_\star, s\})}(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_\star}}\big|g\big|_{H^s} + C\big\langle\big|f\big|_{H^s}\big|g\big|_{H^{s_\star}}\big\rangle_{s>s_\star}$$

where we recall the notation $\langle C\rangle_{a>b} = \begin{cases} C & \text{if } a > b, \\ 0 & \text{otherwise.}\end{cases}$

Proof. We consider the Leibniz rule for any $\boldsymbol{k} \in \mathbb{N}^d$ with $|\boldsymbol{k}| \leq s$:

$$\partial^{\boldsymbol{k}}(fg) = \sum_{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{i}} (\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)$$

and estimate each summand independently.

If $|\boldsymbol{k}| \leq s_\star$, we may apply Proposition II.5 with $s_1 = s_\star - |\boldsymbol{i}|$ and $s_2 = s - |\boldsymbol{j}|$, and deduce

$$\big|(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}\big|g\big|_{H^s}.$$

Assume now $|\boldsymbol{k}| > s_\star$. If $|\boldsymbol{i}| \leq s_\star$, we have as above $\big|(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}\big|g\big|_{H^s}$. If $|\boldsymbol{j}| \leq s_\star$, we obtain symmetrically $\big|(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^s}\big|g\big|_{H^{s_\star}}$. In the remaining cases, we let $s_1, s_2 \in \mathbb{R}$ be such that $s_\star < |\boldsymbol{i}| \leq s_1 \leq |\boldsymbol{k}|$, $s_\star < |\boldsymbol{j}| \leq s_2 \leq |\boldsymbol{k}|$ and $s_1 + s_2 = |\boldsymbol{k}| + s_\star$. By Proposition II.5 and Lemma II.3, we deduce

$$\big|(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}} \lesssim \big|f\big|_{H^{s_\star}}^{\theta}\big|f\big|_{H^s}^{1-\theta}\big|g\big|_{H^{s_\star}}^{1-\theta}\big|g\big|_{H^s}^{\theta}$$

with $\theta = \frac{|\boldsymbol{k}|-s_1}{|\boldsymbol{k}|-s_\star} = \frac{s_2-s_\star}{|\boldsymbol{k}|-s_\star}$. We conclude by Young's inequality. □

Remark II.8. *A close inspection on the proof shows that the result may be sharpened by assuming only $f \in L^\infty(\mathbb{R}^d) \cap \mathring{H}^{\max\{s_\star, s\}}(\mathbb{R}^d)$ and one has*

$$\big|fg\big|_{H^s} \leq C\big(\big|f\big|_{L^\infty} + \big|\nabla f\big|_{H^{s_\star-1}}\big)\big|g\big|_{H^s} + C\big\langle\big|\nabla f\big|_{H^{s-1}}\big|g\big|_{H^{s_\star}}\big\rangle_{s>s_\star}.$$

Proposition II.9. *Let $d \in \mathbb{N}^\star$, $\boldsymbol{k} \in \mathbb{N}^d$, and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in \mathring{H}^{\max\{1+s_\star,|\boldsymbol{k}|\}}(\mathbb{R}^d)$ and $g \in H^{|\boldsymbol{k}|-1}(\mathbb{R}^d)$, $[\partial^{\boldsymbol{k}}, f]g \in L^2(\mathbb{R}^d)$ and*

$$\big|[\partial^{\boldsymbol{k}}, f]g\big|_{L^2} \leq C\big|\nabla f\big|_{H^{s_\star}}\big|g\big|_{H^{|\boldsymbol{k}|-1}} + C\big\langle\big|\nabla f\big|_{H^{|\boldsymbol{k}|-1}}\big|g\big|_{H^{s_\star}}\big\rangle_{|\boldsymbol{k}|-1>s_\star}.$$

Proof. By Leibniz rule, we have

$$[\partial^{\boldsymbol{k}}, f]g = \sum_{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{k},\boldsymbol{i}\neq\boldsymbol{0}} \binom{\boldsymbol{k}}{\boldsymbol{i}}(\partial^{\boldsymbol{i}} f)(\partial^{\boldsymbol{j}} g).$$

We then proceed as in the proof of Proposition II.7 with $\partial^{\boldsymbol{i}} f = \partial^{\tilde{\boldsymbol{i}}}\partial^{\boldsymbol{e}} f$, $\tilde{\boldsymbol{i}}+\boldsymbol{e} = \boldsymbol{i}$, $|\tilde{\boldsymbol{i}}|+|\boldsymbol{j}| = |\boldsymbol{k}|-1$. □

Proposition II.10. *Let $n, d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$, $H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ and*

$$\big|f^n\big|_{H^s} \leq C\big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}.$$

Proof. The case $n = 1$ is trivial, and the case $n = 2$ is a particular case to Proposition II.7. We first notice

$$\big|f^n\big|_{L^2} \lesssim \big|f\big|_{L^\infty}^{n-1}\big|f\big|_{H^s} \lesssim \big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}$$

by Sobolev embedding, Lemma II.4.

Now consider $\boldsymbol{k} \in \mathbb{N}^d$ and $|\boldsymbol{k}| = s$. By the general Leibniz rule, we have to estimate

$$\big|\prod_{i=1}^{n} \partial^{\boldsymbol{j}_i} f\big|_{L^2}$$

where $\sum_{i=1}^n \boldsymbol{j}_i = \boldsymbol{k}$. We may consider without loss of generality that $|\boldsymbol{j}_i| \geq 1$ for any $i \in \{1, \ldots, n\}$ thanks to the Sobolev embedding $H^{s_\star} \subset L^\infty$ as above. We have

$$\big|\prod_{i=1}^{n} \partial^{\boldsymbol{j}_i} f\big|_{L^2} \lesssim \prod_{i=1}^{n}\big|\partial^{\boldsymbol{j}_i} f\big|_{L^{p_i}} \lesssim \prod_{i=1}^{n}\big|\partial^{\boldsymbol{j}_i} f\big|_{H^{s_i}},$$

as soon as $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{2}$ (using Lemma II.1) and $s_i > \frac{d}{2}(1 - \frac{2}{p_i})$ (using Lemma II.4)

Let us first consider the case $s \leq s_\star$. We choose $s_i = s_\star - |\boldsymbol{j}_i|$ for $i \in \{0, \ldots, n-1\}$ and $s_n = s - |\boldsymbol{j}_n|$. Recall, since $n \geq 1$, that $1 \leq |\boldsymbol{j}_i| \leq s - 1$, and hence $1 \leq s_i \leq s_\star - 1$ for any $i \in \{1, \ldots, n\}$. Then we set $p_i = 2\frac{s_\star}{s_\star - s_i} \in (2, +\infty)$, so that $s_i = s_\star(1 - \frac{2}{p_i}) > \frac{d}{2}(1 - \frac{2}{p_i})$, and $\sum_{i=1}^n \frac{2}{p_i} = n - \frac{1}{s_\star}\sum_{i=1}^n s_i = 1$, and the result follows.

Let us now consider the case $s > s_\star$. We choose s_i such that

$$\max(\{|\boldsymbol{j}_i|, s_\star\}) < s_i + |\boldsymbol{j}_i| < \min(\{s, s_\star + |\boldsymbol{j}_i|\}) \quad \text{and} \quad \sum_{i=1}^{n} s_i = (n-1)s_\star.$$

This is possible since $1 \leq |\boldsymbol{j}_i| \leq s - 1$ and hence one has $\max(\{|\boldsymbol{j}_i|, s_\star\}) < \min(\{s, s_\star + |\boldsymbol{j}_i|\})$ and $\sum_{i=1}^n \max(\{|\boldsymbol{j}_i|, s_\star\}) < (n-1)s_\star + s < \sum_{i=1}^n \min(\{s, s_\star + |\boldsymbol{j}_i|\})$. Then we set as above $p_i = 2\frac{s_\star}{s_\star - s_i}$. Finally, we use the interpolation estimate of Lemma II.3:

$$\big|\partial^{\boldsymbol{j}_i} f\big|_{H^{s_i}} \lesssim \big|f\big|_{H^s}^{\theta_i}\big|f\big|_{H^{s_\star}}^{1-\theta_i}$$

with $\theta_i = \frac{s_i + |\boldsymbol{j}_i| - s_\star}{s - s_\star}$. This completes the proof, remarking that $\sum_{i=1}^n \theta_i = 1$. □

Proposition II.11. *Let $d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. Let $F : \mathbb{R} \to \mathbb{R}$ such that $F \in \mathcal{C}^s(\mathbb{R})$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ satisfying*

$$\big|f\big|_{H^{s_\star}} \leq M,$$

then $F(f) \in H^s(\mathbb{R}^d)$ and

$$\big|F(f)\big|_{H^s} \leq C\big|f\big|_{H^s}.$$

Proof. By the Sobolev embedding, Lemma II.4, there exists $C > 0$, such that for any $f \in H^{s_\star}(\mathbb{R}^d)$, $\big|f\big|_{L^\infty} \leq C\big|f\big|_{H^{s_\star}}$. Hence we can define a closed interval, I such that for any $f \in H^{s_\star}(\mathbb{R}^d)$,

$$\operatorname{supp} f \subset I.$$

It follows from the mean value theorem that

$$\big|F(f)\big|_{L^2} = \big|F(f) - F(0)\big|_{L^2} \leq \sup_{x\in I}|F'(x)|\big|f\big|_{L^2}.$$

We now use Faà di Bruno's formula: for any $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| = s$, we have

$$\partial^{\boldsymbol{k}} F(f) = \sum_{n=1}^{|\boldsymbol{k}|} \sum_{\substack{\boldsymbol{j}_1,\dots,\boldsymbol{j}_n \neq \boldsymbol{0} \\ \boldsymbol{j}_1+\dots+\boldsymbol{j}_n=\boldsymbol{k}}} C_{\boldsymbol{j}_1,\dots,\boldsymbol{j}_n}\ F^{(n)}(f) \times \prod_{i=1}^n \partial^{\boldsymbol{j}_i} f.$$

We have $\big|F^{(n)}(f)\big|_{L^\infty} \leq \sup_{x\in I}|F^{(s)}(x)|$ and we estimate

$$\big|\prod_{i=1}^n \partial^{\boldsymbol{j}_i} f\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}$$

as in the proof of Proposition II.10. $\square$

Remark II.12. *The estimates can be sharpened by making use of the following Gagliardo–Nirenberg estimate (see e.g. [399]): for any $s \in \mathbb{N}$, and $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| \leq s$, and any $s_\star > d/2$, there exists $C > 0$ such that for any $u \in H^{\max\{s,s_\star\}}(\mathbb{R}^d)$,*

$$\big|\partial^{\boldsymbol{k}} f\big|_{L^{2s/|\boldsymbol{k}|}} \leq C\big|f\big|_{L^\infty}^{1-|\boldsymbol{k}|/s}\big|\nabla f\big|_{H^{s-1}}^{|\boldsymbol{k}|/s}.$$

We deduce, under the (respective) assumptions of Propositions II.7, II.9, II.10 and II.11,

$$\begin{aligned}
\big|fg\big|_{H^s} &\leq C\big(\big|f\big|_{L^\infty}\big|g\big|_{H^s} + \big|f\big|_{H^s}\big|g\big|_{L^\infty}\big),\\
\big|[\partial^{\boldsymbol{k}}, f]g\big|_{L^2} &\leq C\big(\big|\nabla f\big|_{L^\infty}\big|g\big|_{H^{|\boldsymbol{k}|-1}} + \big|\nabla f\big|_{H^{|\boldsymbol{k}|-1}}\big|g\big|_{L^\infty}\big),\\
\big|\partial^{\boldsymbol{k}}(fg) - f\partial^{\boldsymbol{k}} g - g\partial^{\boldsymbol{k}} f\big|_{L^2} &\leq C\big(\big|\nabla f\big|_{L^\infty}\big|\nabla g\big|_{H^{|\boldsymbol{k}|-2}} + \big|\nabla f\big|_{H^{|\boldsymbol{k}|-2}}\big|\nabla g\big|_{L^\infty}\big),\\
\big|f^n\big|_{H^s} &\leq C\big|f\big|_{L^\infty}^{n-1}\big|f\big|_{H^s},\\
\big|F(f)\big|_{H^s} &\leq C\big(\big|f\big|_{L^\infty}\big)\big|f\big|_{H^s}.
\end{aligned}$$

II.3 Estimates with non-decreasing functions

We sometimes need to deal with nonlinear estimates involving a non square-integrable function, typically when a non-trivial topography is taken into account. We extend the result of the previous section to this framework.

Lemma II.13 (Interpolation inequality)**.** *Let $d \in \mathbb{N}^\star$ and $s, s_-, s_+ \in \mathbb{N}$ such that $s_- < s_+$ and $s_- \leq s \leq s_+$. There exists $C > 0$ such that for any $f \in W^{s_+,\infty}(\mathbb{R}^d)$,*

$$\big|f\big|_{W^{s,\infty}} \leq C\big|f\big|_{W^{s_-,\infty}}^{\theta}\big|f\big|_{W^{s_+,\infty}}^{1-\theta},$$

with $\theta = \frac{s_+ - s}{s_+ - s_-}$.

Proof. We use the identity, valid for any $\boldsymbol{e} \in \mathbb{N}^d$ such that $|\boldsymbol{e}| = 1$ and any $\lambda > 0$:

$$(-\partial^{\boldsymbol{e}} f)(\boldsymbol{x}) = \int_0^{+\infty} (\partial^{2\boldsymbol{e}} f - \lambda^2 f)(\boldsymbol{x} + s\boldsymbol{e})e^{-\lambda s}\,\mathrm{d}s + \lambda f(\boldsymbol{x}),$$

which follows by integration by parts. We deduce, with $\lambda = \big(2\big|\partial^{2\boldsymbol{e}} f\big|_{L^\infty}/\big|f\big|_{L^\infty}\big)^{1/2}$

$$\big|\partial^{\boldsymbol{e}} f\big|_{L^\infty} \leq 2\sqrt{2}\big|f\big|_{L^\infty}\big|\partial^{2\boldsymbol{e}} f\big|_{L^\infty},$$

and the result is proved for $(s_-, s, s_+) = (0, 1, 2)$. One obtains the result for $(s_-, s, s_+) = (0, s, s_+)$ with any $0 < s < s_+$ by induction on $s_+ \geq 2$ (the equality cases $s = 0$ or $s = s_+$ being straightforward), and the general case is immediately deduced. □

Proposition II.14. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$ and $s \in \mathbb{N}$. There exists $C > 0$ such that for any $f \in W^{s,\infty}(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and*

$$\begin{aligned}\big|fg\big|_{H^s} &\leq C\big(\big|f\big|_{W^{s,\infty}}\big|g\big|_{L^2} + \big|f\big|_{L^\infty}\big|g\big|_{H^s}\big)\\ &\leq 2C\big|f\big|_{W^{s_\star,\infty}}\big|g\big|_{H^s} + C\big\langle\big|f\big|_{W^{s,\infty}}\big|g\big|_{H^{s_\star}}\big\rangle_{s>s_\star}.\end{aligned}$$

Proof. The result is an immediate consequence of Leibniz rule, Lemma II.3 and Lemma II.13, and Young's inequality. □

Proposition II.15. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $\boldsymbol{k} \in \mathbb{N}^d$. There exists $C > 0$ such that for any $f \in W^{|\boldsymbol{k}|,\infty}(\mathbb{R}^d)$ and $g \in H^{|\boldsymbol{k}|-1}(\mathbb{R}^d)$, $[\partial^{\boldsymbol{k}}, f]g \in L^2(\mathbb{R}^d)$ and*

$$\begin{aligned}\big|[\partial^{\boldsymbol{k}}, f]g\big|_{L^2} &\leq C\big(\big|\nabla f\big|_{L^\infty}\big|g\big|_{H^{|\boldsymbol{k}|-1}} + \big|\nabla f\big|_{W^{|\boldsymbol{k}|-1,\infty}}\big|g\big|_{L^2}\big)\\ &\leq 2C\big|\nabla f\big|_{W^{s_\star,\infty}}\big|g\big|_{H^{|\boldsymbol{k}|-1}} + C\big\langle\big|\nabla f\big|_{W^{|\boldsymbol{k}|-1,\infty}}\big|g\big|_{H^{s_\star}}\big\rangle_{|\boldsymbol{k}|-1>s_\star}.\end{aligned}$$

Proof. We use once again Leibniz rule, Lemma II.3 and Lemma II.13, and Young's inequality. □

Proposition II.16. *Let $d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. Let $I \ni 0$ be a closed interval, $F : I \to \mathbb{R}$ such that $F \in \mathcal{C}^s(I)$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in W^{s,\infty}(\mathbb{R}^d)$ and $g \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp}(f + g) \subset I \quad \textit{and} \quad \big|f\big|_{L^\infty} + \big|g\big|_{H^{s_\star}} \leq M,$$

and for any $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| = s$, one has $\partial^{\boldsymbol{k}}(F(f + g)) = F_{\boldsymbol{k}} + G_{\boldsymbol{k}}$ with

$$\big|F_{\boldsymbol{k}}\big|_{L^\infty} \leq C\big|g\big|_{W^{s,\infty}} \quad \textit{and} \quad \big|G_{\boldsymbol{k}}\big|_{L^2} \leq C\big(\big|f\big|_{H^s} + \big|g\big|_{W^{s,\infty}}\big).$$

Proof. By reasoning as in the proof of Proposition II.11, we are left with the estimate of a sum of products of the form

$$\left(\prod_{i=1}^{n_1} \partial^{\boldsymbol{j}_i} f\right)\left(\prod_{i=1}^{n_2} \partial^{\boldsymbol{j}_i} g\right).$$

where $n \in \{1, \dots, |\boldsymbol{k}|\}$, $n_1 + n_2 = n$, and $\boldsymbol{i}_i \neq \boldsymbol{0}$ ($i \in \{1, \dots, n_1\}$), $\boldsymbol{j}_j \neq \boldsymbol{0}$ ($j \in \{1, \dots, n_2\}$) are such that $\boldsymbol{i}_1 + \dots + \boldsymbol{i}_{n_1} + \boldsymbol{j}_1 + \dots + \boldsymbol{j}_{n_2} = \boldsymbol{k}$. We may separate between two cases, depending on whether $n_1 = 0$ or $n_1 \neq 0$. We estimate as in Proposition II.10, if $n_1 \in \mathbb{N}^\star$,

$$\big|\prod_{i=1}^{n_1} \partial^{\boldsymbol{j}_i} f\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}^{n_1-1} \big|f\big|_{H^{|\boldsymbol{k}_1|}}$$

and similarly, if $n_2 \in \mathbb{N}^\star$,

$$\big|\prod_{i=1}^{n_2} \partial^{\boldsymbol{j}_i} g\big|_{L^\infty} \lesssim \big|g\big|_{L^\infty}^{n_2-1} \big|g\big|_{W^{|\boldsymbol{k}_2|,\infty}}.$$

We conclude by using Lemma II.3, Lemma II.13, and Young's inequality. $\square$

Proposition II.17. *Let $d, s \in \mathbb{N}^\star$, $s_\star > d/2$ and $\boldsymbol{k} \in \mathbb{N}^d$. Let $I \ni 0$ be a closed interval, $F : I \to \mathbb{R}$ such that $F \in \mathcal{C}^s(I)$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in W^{|\boldsymbol{k}|,\infty}(\mathbb{R}^d)$ and $g \in H^{\max(\{|\boldsymbol{k}|, s_\star+1\})}(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp}(f+g) \subset I \quad \text{and} \quad \big|f\big|_{W^{1,\infty}} + \big|g\big|_{H^{1+s_\star}} \leq M,$$

and for any $h \in H^{|\boldsymbol{k}|-1}(\mathbb{R}^d)$, one has $\big[\partial^{\boldsymbol{k}}, F(f+g)\big]h \in L^2(\mathbb{R}^d)$ and

$$\big|\big[\partial^{\boldsymbol{k}}, F(f+g)\big]h\big|_{L^2} \leq C\big(\big|h\big|_{H^{|\boldsymbol{k}|-1}} + \big|\nabla f\big|_{W^{|\boldsymbol{k}|-1,\infty}}\big|h\big|_{L^2} + \big\langle \big|\nabla g\big|_{H^{|\boldsymbol{k}|-1}}\big|h\big|_{H^{s_\star}}\big\rangle_{|\boldsymbol{k}|-1>s_\star}\big).$$

If, additionally, one has $\big|f\big|_{W^{1+s_\star,\infty}} + \big|g\big|_{H^{1+s_\star}} \leq M$, then we may write the above as

$$\big|\big[\partial^{\boldsymbol{k}}, F(f+g)\big]h\big|_{L^2} \leq C\big(\big|h\big|_{H^{|\boldsymbol{k}|-1}} + \big\langle\big(\big|\nabla f\big|_{W^{|\boldsymbol{k}|-1,\infty}} + \big|\nabla g\big|_{H^{|\boldsymbol{k}|-1}}\big)\big|h\big|_{H^{s_\star}}\big\rangle_{|\boldsymbol{k}|-1>s_\star}\big).$$

Proof. The result is obtained with a combination of the techniques used in Proposition II.9, Proposition II.15 and Proposition II.16. $\square$

II.4 Estimates for functions on the flat strip

All the product, results concerning functions defined on $\mathbb{R}^d$ have counterparts for functions defined on the strip $\mathcal{S} = (-1, 0) \times \mathbb{R}^d$. For $f \in L^2(\mathcal{S})$, we denote for $s \in \mathbb{N}$, $\Lambda^s f = (\operatorname{Id} - \Delta)^{s/2} f$ where the differentiation applies to the horizontal variable $\boldsymbol{x} \in \mathbb{R}^d$, and remark

$$\big\|\Lambda^s f\big\|_{L^2(\mathcal{S})}^2 = \iint_{\mathcal{S}} |\Lambda^s f|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \int_{-1}^0 \big|f(z,\cdot)\big|_{H^s(\mathbb{R}^d)}^2 \,\mathrm{d}z.$$

Let us write as an example a counterpart to the product estimate.

Proposition II.18. *Let $d, s \in \mathbb{N}^\star$, $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max\{s_\star, s\}}(\mathbb{R}^d)$ and $g \in L^2(\mathcal{S})$ such that $\Lambda^s g \in L^2(\mathcal{S})$, one has $\Lambda^s(fg) \in L^2(\mathcal{S})$ and*

$$\big\|\Lambda^s(fg)\big\|_{L^2(\mathcal{S})} \leq C\big|f\big|_{H^{s_\star}}\big\|\Lambda^s g\big\|_{L^2(\mathcal{S})} + \Big\langle \big|f\big|_{H^s}\big\|\Lambda^{s_\star} g\big\|_{L^2(\mathcal{S})}\Big\rangle_{s>s_\star}.$$

Proof. We have, for any $\boldsymbol{k} \in \mathbb{N}^d$ such that $|\boldsymbol{k}| = s$,

$$\begin{aligned}
\big\|\partial^{\boldsymbol{k}}(fg)\big\|_{L^2(\mathcal{S})}^2 &= \iint_{\mathcal{S}} |\partial^{\boldsymbol{k}}(fg)|^2(\boldsymbol{x}, z)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z = \int_{-1}^0 \big|\partial^{\boldsymbol{k}}(fg)(\cdot, z)\big|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}z \\
&\lesssim \int_{-1}^0 \big|f\big|_{H^{s_\star}}^2 \big|g(\cdot,z)\big|_{H^s}^2 + \Big\langle \big|f\big|_{H^s}^2 \big|g(\cdot,z)\big|_{H^{s_\star}}^2 \Big\rangle_{s_\star>k} \mathrm{d}z \\
&= \big|f\big|_{H^{s_\star}}^2 \big\|\Lambda^s g\big\|_{L^2(\mathcal{S})}^2 + \Big\langle \big|f\big|_{H^s}^2 \big\|\Lambda^{s_\star} g\big\|_{L^2(\mathcal{S})}^2 \Big\rangle_{s_\star>k},
\end{aligned}$$

and the result follows. $\square$

Remark II.19. *The result obviously generalizes to $z \mapsto f(z,\cdot) \in L^\infty(-1, 0; H^{\max\{s_\star, s\}}(\mathbb{R}^d))$.*

III Index of notations

- The notation $a \lesssim b$ means that $a \leq C_0\, b$, where C_0 is a nonnegative constant whose exact expression is of no importance. $a \nearrow b$ means $a \to b,\ a < b$ and $a \searrow b$ means $a \to b,\ a > b$. $a(\xi) = \mathcal{O}(b(\xi))$ means $|a|(\xi) \lesssim b(\xi)$ almost everywhere, with multiplicative constant independent of $\xi \in \mathbb{R}$. We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters λ_1, λ_2,... and whose dependence on the λ_j is generally assumed to be nondecreasing. Straightforward dependence with respect to other parameters may be omitted.

- For $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the largest integer smaller or equal to s; $\lceil s \rceil$ denotes the smallest integer larger or equal to s.

- Id is the identity operator. For $d \in \mathbb{N}^\star$, Id_d is the $d \times d$ identity matrix, O_d is the $d \times d$ null matrix, O a null matrix, $\mathbf{0}$ a null vector, $\boldsymbol{e}$ a unit vector, $\boldsymbol{e}_z$ the vertical unit vector.

- We use the multi-index notation for multi-dimensional differentiation: for $\boldsymbol{k} = (k_1, k_2) \in \mathbb{N}^2$, $\partial^{(k_1,k_2)} f(x,y) = \partial_x^{k_1} \partial_y^{k_2} f(x,y)$ and $|\boldsymbol{k}| = k_1 + k_2$. For $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^2$, $\binom{\boldsymbol{i}}{\boldsymbol{j}} = \binom{i_1}{j_1}\binom{i_2}{j_2}$. If $\boldsymbol{k} \in \mathbb{N}$, then $\partial^{\boldsymbol{k}} = \partial^k$ is the standard differentiation operator.

- For $1 \leq p < \infty$ and $d \in \mathbb{N}^\star$, we denote $L^p(\mathbb{R}^d)$ the Lebesgue spaces associated with the norm
$$|f|_{L^p} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} < \infty.$$
The real inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^d)$ is denoted by
$$\big(f_1, f_2\big)_{L^2} \overset{\text{def}}{=} \int_{\mathbb{R}^d} f_1(\boldsymbol{x}) f_2(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$
The space $L^\infty(\mathbb{R}^d)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm
$$\big|f\big|_{L^\infty} \overset{\text{def}}{=} \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |f(\boldsymbol{x})| < \infty.$$
We define similarly $L^p_{\text{loc}}(\mathbb{R}^d)$ the locally p-integrable functions such that the above holds when restricting the integration (or essential supremum) to $\boldsymbol{x} \in K$ for any $K \subset \mathbb{R}^d$ compact.

- For $k \in \mathbb{N}$, we denote the Sobolev space $H^k(\mathbb{R}^d)$ the subspace of $L^2(\mathbb{R}^d)$ such that all weak derivatives of order k are square-integrable, endowed with
$$\big|f\big|_{H^k}^2 \overset{\text{def}}{=} \sum_{|\boldsymbol{k}| \leq k} \big|\partial^{\boldsymbol{k}} f\big|_{L^2}^2.$$
Similarly, we denote by $W^{k,\infty}(\mathbb{R}^d) \overset{\text{def}}{=} \{f \in L^\infty(\mathbb{R}^d) \ : \ \forall 0 \leq |\boldsymbol{k}| \leq k,\ \partial^{\boldsymbol{k}} f \in L^\infty(\mathbb{R}^d)\}$ endowed with its canonical norm, and $\mathcal{C}^k(\mathbb{R}^d) \overset{\text{def}}{=} \{f \in L^\infty(\mathbb{R}^d) \ : \ \forall 0 \leq |\boldsymbol{k}| \leq k,\ \partial^{\boldsymbol{k}} f \in \mathcal{C}^0(\mathbb{R}^d)\}$, where $\mathcal{C}^0(\mathbb{R}^d)$ denotes the space of (scalar) continuous functions. We denote the Beppo Levi space $\mathring{H}^{k+1}(\mathbb{R}^d) \overset{\text{def}}{=} \{f \in L^2_{\text{loc}}(\mathbb{R}^d) \ : \ \nabla f \in H^k(\mathbb{R}^d)^d\}$, endowed with the semi-norm
$$\big|f\big|_{\mathring{H}^{k+1}} \overset{\text{def}}{=} \big|\nabla f\big|_{H^k}.$$

- We denote $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of smooth rapidly decreasing functions.

- For any real constant $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the Sobolev space obtained by completing $\mathcal{S}(\mathbb{R}^d)$ for the norm $|f|_{H^s} \overset{\text{def}}{=} |\Lambda^s f|_{L^2} < \infty$, where Λ is the Fourier multiplier (see Definition III.1) $\Lambda \overset{\text{def}}{=} (\mathrm{Id} - \Delta)^{1/2} = (\mathrm{Id} + |D|^2)^{1/2}$.

- We use double bars for norms associated to functional spaces defined on $\Omega \subset \mathbb{R}^{d+1}$. For instance, square-integrable functions on Ω are endowed with the norm

$$\big\|\Phi\big\|_{L^2(\Omega)}^2 \stackrel{\text{def}}{=} \iint_\Omega |\Phi(\boldsymbol{x},z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z.$$

  Sobolev and Beppo Levi spaces with integer indices $H^k(\Omega)$ and $\mathring{H}^{k+1}(\Omega)$ are defined as above.

- For functions defined on $\Omega \subset \mathbb{R}^{d+1} = \{(\boldsymbol{x},z) \ : \ \boldsymbol{x} = (x_1,\dots,x_d)\}$, we denote the gradients $\nabla_{\boldsymbol{x}} \stackrel{\text{def}}{=} (\partial_{x_1},\dots,\partial_{x_d})^\top$ and $\nabla_{\boldsymbol{x},z} \stackrel{\text{def}}{=} (\partial_{x_1},\dots,\partial_{x_d},\partial_z)^\top$ (where $\top$ is the transpose operator), and $\Delta_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{x}}$, $\Delta_{\boldsymbol{x},z} = \nabla_{\boldsymbol{x},z}\cdot\nabla_{\boldsymbol{x},z}$.

- Given X any of the previously defined functional spaces, we denote by X' its topological dual, endowed with the norm $\big|\varphi\big|_{X'} \stackrel{\text{def}}{=} \sup\{|\varphi(f)| \ : \ \big|f\big|_X \le 1\}$; and by $\langle\cdot,\cdot\rangle_{X'-X}$ the $(X'-X)$ duality brackets.

- For $T>0$ and any of the previously defined functional spaces, X, we denote $L^\infty(0,T;X)$ the space of functions such that $u(t,\cdot)$, taking values in the Banach space X, is essentially bounded for $t\in(0,T)$, and denote the associated norm

$$\big\|u\big\|_{L^\infty(0,T;X)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t\in(0,T)} \big|u(t,\cdot)\big|_X \ < \ \infty.$$

  Spaces $L^p(0,T;X)$ for $p\in[1,\infty)$ are defined similarly. For $k\in\mathbb{N}$, and I a real interval, $\mathcal{C}^k(I;X)$ denotes the space of X-valued continuous functions on I with continuous derivatives up to order k.

Definition III.1 (Fourier multipliers). *Let $F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be such that there exists $C>0$ and $m\in\mathbb{R}$ such that for almost every $\boldsymbol{\xi}\in\mathbb{R}^d$,*

$$|F(\boldsymbol{\xi})| \le C\langle\boldsymbol{\xi}\rangle^m.$$

For any $s\in\mathbb{R}$, we denote $\mathsf{F} = F(D) : H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$ the operator defined by $\widehat{\mathsf{F}g} = F\,\widehat{g}$, i.e.

$$\forall g\in\mathcal{S}(\mathbb{R}^d),\quad \forall\boldsymbol{x}\in\mathbb{R}^d,\qquad (\mathsf{F}g)(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}}\ F(\boldsymbol{\xi})\ g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\ .$$

The operator is continuously extended to $g\in H^s(\mathbb{R}^d)$ by the density of $\mathcal{S}(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$.

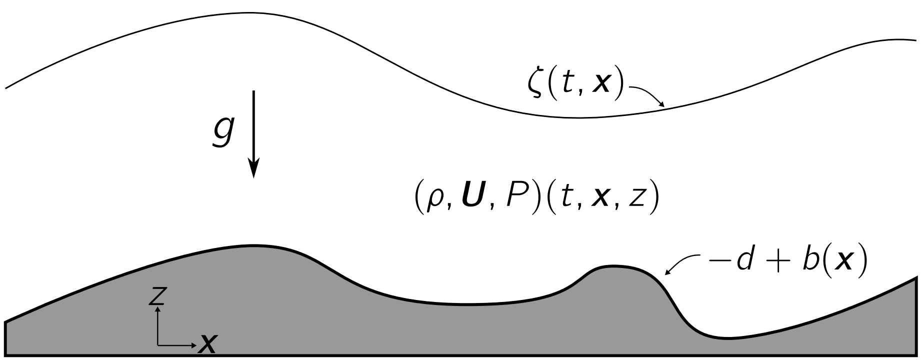


Figure III.1: Sketch of the fluid domain and notations.

For the sake of readability, we use different fonts for physical variables (*e.g.* $\mathsf{x}, \mathsf{z}, \zeta, \mathsf{u}, \Phi, \psi$) and corresponding dimensionless variables (*e.g.* $\boldsymbol{x}, z, \zeta, \boldsymbol{u}, \Phi, \psi$). Here are some key notations:

g is the gravitational acceleration.

d is the reference depth at rest.

$\mathsf{x} \in \mathbb{R}^d$ is the horizontal space variable, $\mathsf{z} \in \mathbb{R}$ the vertical space variable, $\mathsf{t} \in \mathbb{R}$ the time.

$\zeta(\mathsf{t}, \mathsf{x})$ represents the surface deformation. $b(\mathsf{x})$ represents the bottom topography.

$h(\mathsf{t}, \mathsf{x}) = d + \zeta(\mathsf{t}, \mathsf{x}) - b(\mathsf{x})$ is the height of the fluid layer at time t and position $\mathsf{x} \in \mathbb{R}^d$.

$\rho(\mathsf{t}, \mathsf{x}, \mathsf{z})$ is the fluid mass density at time t and position $(\mathsf{x}, \mathsf{z}) \in \mathbb{R}^{d+1}$.

$\mathsf{U}(\mathsf{t}, \mathsf{x}, \mathsf{z})$ is the flow velocity, U and w its horizontal and vertical components.

$P(\mathsf{t}, \mathsf{x}, \mathsf{z})$ is the pressure inside the fluid, $p_{\rm atm}(\mathsf{t}, \mathsf{x})$ the pressure at the surface.

When $\mathsf{U} = \nabla_{\mathsf{x},\mathsf{z}}\Phi$, $\psi(\mathsf{t}, \mathsf{x})$ is the trace of the velocity potential, Φ, at the surface.

$\mathsf{u}(\mathsf{t}, \mathsf{x})$ is a (typically layer-averaged) horizontal velocity.

$\mathsf{v}(\mathsf{t}, \mathsf{x})$ is (typically) a scaled tangent velocity at the surface $\nabla\psi(\mathsf{t}, \mathsf{x})$, or a shear velocity.

Asymptotic regimes are characterized as admissible sets of dimensionless parameters. These are defined, together with dimensionless versions of the above variables, in eq. (2.6) (in the one-layer case) and in eq. (3.13) (in the bilayer case).

μ measures the strength of dispersive effects.

ε measures the strength of nonlinear effects.

β measures the strength of topography effects.

Additionally, in the bilayer framework,

γ measures the density contrast.

α measures the size of surface deformations with respect to interface deformations.

δ is the upper layer depth to lower layer depth ratio.

Bo is the Bond number measuring the ratio of gravity forces over capillary forces.

Definition III.2 (Shallow water asymptotic regime)**.** *Given $\mu^\star > 0$, we let*

$$\mathfrak{p}_{\rm SW} \stackrel{\rm def}{=} \big\{(\mu, \varepsilon, \beta) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1]\big\}.$$

Definition III.3 (Long wave asymptotic regime)**.** *Given $\mu^\star > 0$ and $\theta > 0$, we let*

$$\mathfrak{p}_{\rm LW} \stackrel{\rm def}{=} \big\{(\mu, \varepsilon, \beta) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0, \theta\mu], \ \beta \in [0, \theta\mu]\big\}.$$

Definition III.4 (Shallow water/shallow water asymptotic regime)**.** *Given $\mu^\star, \delta_\star, \delta^\star > 0$, we let (in the free-surface framework)*

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} \stackrel{\rm def}{=} \big\{(\mu, \varepsilon, \beta, \alpha, \delta, \gamma) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1], \ \alpha \in (0,1], \ \delta \in [\delta_\star, \delta^\star], \gamma \in [0,1)\big\}$$

or (in the rigid-lid framework)

$$\mathfrak{p}_{\frac{\rm SW}{\rm SW}} \stackrel{\rm def}{=} \big\{(\mu, \varepsilon, \beta, \delta, \gamma) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0,1], \ \beta \in [0,1], \ \delta \in [\delta_\star, \delta^\star], \gamma \in [0,1)\big\}.$$

IV Asymptotic models: another reader's digest

Lannes offered a reader's digest of asymptotic models for the water waves system in [268, Appendix C]. Let us augment this digest with a brief a review of the main asymptotic models derived and discussed in the present monograph.

The ***full Euler equations***, eq. (1.1) page 3, are the "master" equations for (free-)surface gravity waves, from which all other models are subsequently derived. When restricting to homogeneous fluids and potential flows, the corresponding equations are called ***water waves system***, eq. (2.2) page 7. In the presence of two layers of homogeneous fluids with a free interface, we arrive at ***interfacial waves systems***, eq. (3.1) page 18, or eq. (3.5) page 19 (among other formulations) in the rigid-lid situation.

Hydrostatic equations can be derived as first order asymptotic models the shallow water limit. Starting from the full Euler equations we arrive at the ***hydrostatic Euler equations***, eq. (7.3) page 79 and, assuming additionally that the density is (continuously) stratified, eq. (7.10) page 82. Starting from the full Euler system we arrive at the ***Saint-Venant system***, eq. (5.4) page 48. Starting from the interfacial waves system we arrive at the ***bilayer hydrostatic systems***, eq. (6.3) page 54 in the free-surface case and eq. (6.12) page 63 in the rigid-lid case.

By keeping the next order terms in shallow water asymptotic expansions, we arrive at the ***(Serre–)Green–Naghdi system***, eq. (8.9) page 91 in the one-layer (homogeneous) framework, and at the ***Miyata–Choi–Camassa system***, eq. (14.5) page 209 in the bilayer framework (with rigid lid). These systems have fully dispersive counterparts, which we named respectively the ***Whitham–Green–Naghdi system***, eq. (10.5) page 133, and the ***Whitham–Choi–Camassa system***, eq. (14.10)–(14.12) page 216.

These models can be simplified in the long wave regime, that is assuming additional smallness on the data and bottom topography. One arrives in the one-layer (homogeneous) framework at the ***Boussinesq systems***, eq. (10.13) page 142—or eq. (vii) page ix—and fully dispersive counterparts, named ***Whitham–Boussinesq systems***, eq. (10.12) page 142. Obviously Boussinesq-type systems for interfacial waves system can be (and have been) derived, but are not discussed in this monograph.

By keeping an arbitrary number of terms in shallow water asymptotic expansions, we arrive at Friedrichs-type (sometimes called Boussinesq-type) systems, in particular the ***high order shallow water systems***, eq. (11.16) page 156, and the ***extended Green–Naghdi systems***, eq. (11.18) page 157. Other strategies yield different high order systems, such as the ***augmented Green–Naghdi system***, eq. (12.13) page 175 and the ***"multilayer" Green–Naghdi system***, eq. (12.18) page 177; as well as the ***Isobe–Kakinuma systems***, eq. (13.8) page 186. The latter has an extension to the bilayer framework (with rigid lid) which we named the ***Kakinuma systems***, eq. (15.2) page 221.

Some interplays between these models (and others) are represented in Figure B page 46, Figure C page 88, Figure D page 150, Figure E page 206, as well as Figure 6.2 page 68, Figure 7.1 page 80, and Figure 7.2 page 81.

Bibliography

[1] H. D. I. Abarbanel, D. D. Holm, J. E. Marsden, and T. S. Ratiu. Nonlinear stability analysis of stratified fluid equilibria. *Philos. Trans. Roy. Soc. London Ser. A*, 318(1543):349–409, 1986. (cit. p. 74)

[2] M. Abbott, H. Petersen, and O. Skovgaard. On the numerical modelling of short waves in shallow water. *J. Hydr. Res.*, 16(3):173–204, 1978. (cit. p. ix)

[3] M. B. Abbott, A. D. McCowan, and I. R. Warren. Accuracy of short-wave numerical models. *J. Hydr. Eng. ASCE*, 110(10):1287–1301, 1984. (cit. p. ix)

[4] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964. (cit. pp. 163, 181, and 194)

[5] P. Aceves-Sánchez, A. A. Minzoni, and P. Panayotaros. Numerical study of a nonlocal model for water-waves with variable depth. *Wave Motion*, 50(1):80–93, 2013. (cit. p. 142)

[6] A. Ai, M. Ifrim, and D. Tataru. Two dimensional gravity waves at low regularity II: Global solutions. arXiv preprint:2009.11513. (cit. pp. 15 and 240)

[7] G. B. Airy. Tides and waves. *Encycl. Metropolitana*, 5:291–396, 1845. (cit. p. v)

[8] N. Aïssiouene, M.-O. Bristeau, E. Godlewski, A. Mangeney, C. Parés Madroñal, and J. Sainte-Marie. A two-dimensional method for a family of dispersive shallow water models. *SMAI J. Comput. Math.*, 6:187–226, 2020. (cit. pp. 91 and 116)

[9] T. Alazard. A minicourse on the low Mach number limit. *Discrete Contin. Dyn. Syst. Ser. S*, 1(3):365–404, 2008. (cit. pp. 59, 68, 114, and 125)

[10] T. Alazard, N. Burq, and C. Zuily. The water-wave equations: from Zakharov to Euler. In *Studies in phase space analysis with applications to PDEs*, volume 84 of *Progr. Nonlinear Differential Equations Appl.*, pages 1–20. Birkhäuser/Springer, New York, 2013. (cit. p. 7)

[11] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198(1):71–163, 2014. (cit. p. 15)

[12] T. Alazard, N. Burq, and C. Zuily. Cauchy theory for the gravity water waves system with non-localized initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(2):337–395, 2016. (cit. p. 5)

[13] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Savoirs Actuels. InterEditions et Editions du CNRS, Paris, 1991. (cit. pp. 101, 107, and 248)

[14] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.*, 171(3):485–541, 2008. (cit. p. 14)

[15] B. Alvarez-Samaniego and D. Lannes. A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green-Naghdi equations. *Indiana Univ. Math. J.*, 57(1):97–131, 2008. (cit. pp. 91 and 99)

[16] D. M. Ambrose. Well-posedness of vortex sheets with surface tension. *SIAM J. Math. Anal.*, 35(1):211–244 (electronic), 2003. (cit. p. 28)

[17] D. M. Ambrose, J. L. Bona, and D. P. Nicholls. On ill-posedness of truncated series models for water waves. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 470(2166):20130849, 16, 2014. (cit. p. 151)

[18] D. M. Ambrose, R. Camassa, J. L. Marzuola, R. M. McLaughlin, Q. Robinson, and J. Wilkening. Numerical Algorithms for Water Waves with Background Flow over Obstacles and Topography. arXiv preprint:2108.01786. (cit. p. 242)

[19] D. M. Ambrose and N. Masmoudi. Well-posedness of 3D vortex sheets with surface tension. *Commun. Math. Sci.*, 5(2):391–430, 2007. (cit. p. 28)

[20] D. Ambrosi. Hamiltonian formulation for surface waves in a layered fluid. *Wave Motion*, 31(1):71–76, 2000. (cit. p. 21)

[21] C. J. Amick. Regularity and uniqueness of solutions to the Boussinesq system of equations. *J. Differential Equations*, 54(2):231–247, 1984. (cit. p. 113)

[22] C. J. Amick, L. E. Fraenkel, and J. F. Toland. On the Stokes conjecture for the wave of extreme form. *Acta Math.*, 148:193–214, 1982. (cit. p. 16)

[23] C. J. Amick and J. F. Toland. On solitary water-waves of finite amplitude. *Arch. Rational Mech. Anal.*, 76(1):9–95, 1981. (cit. p. 16)

[24] D. Andrade and A. Nachbin. A three-dimensional Dirichlet-to-Neumann operator for water waves over topography. *J. Fluid Mech.*, 845:321–345, 2018. (cit. p. 147)

[25] D. C. Antonopoulos, V. A. Dougalis, and D. E. Mitsotakis. On the well-posedness of the Galerkin semidiscretization of the periodic initial-value problem of the Serre equations. arXiv preprint:2107.04403. (cit. p. 116)

[26] L. Armi. The hydraulics of two flowing layers with different densities. *J. Fluid Mech.*, 163:27–58, 1986. (cit. p. 56)

[27] K. Asano. On the incompressible limit of the compressible Euler equation. *Japan J. Appl. Math.*, 4(3):455–488, 1987. (cit. p. 115)

[28] G. A. Athanassoulis and K. A. Belibassakis. A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions. *Journal of Fluid Mechanics*, 389:275–301, 1999. (cit. p. 186)

[29] G. A. Athanassoulis and C. E. Papoutsellis. Exact semi-separation of variables in waveguides with non-planar boundaries. *Proc. A.*, 473(2201):20170017, 18, 2017. (cit. p. 186)

[30] E. Audusse, M.-O. Bristeau, M. Pelanti, and J. Sainte-Marie. Approximation of the hydrostatic Navier-Stokes system for density stratified flows by a multilayer model: kinetic interpretation and numerical solution. *J. Comput. Phys.*, 230(9):3453–3478, 2011. (cit. p. 53)

[31] E. Audusse, M.-O. Bristeau, B. Perthame, and J. Sainte-Marie. A multilayer Saint-Venant system with mass exchanges for shallow water flows. Derivation and numerical validation. *ESAIM Math. Model. Numer. Anal.*, 45(1):169–200, 2011. (cit. p. 53)

[32] K. I. Babenko. Some remarks on the theory of surface waves of finite amplitude. *Dokl. Akad. Nauk SSSR*, 294(5):1033–1037, 1987. (cit. p. 240)

[33] H. Bae and R. Granero-Belinchón. Singularity formation for the Serre-Green-Naghdi equations and applications to abcd-Boussinesq systems. *Monatsh. Math.*, 198(3):503–516, 2022. (cit. p. 113)

[34] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer, 2011. (cit. pp. 70 and 115)

[35] P. G. Baines. A general method for determining upstream effects in stratified flow of finite depth over long two-dimensional obstacles. *J. Fluid Mech.*, 188:1–22, 1988. (cit. pp. 71 and 73)

[36] D. Bambusi. Hamiltonian studies on counter-propagating water waves. *Water Waves*, pages 1–35, 2020. (cit. p. x)

[37] C. Bardos and N. Besse. The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits. *Kinet. Relat. Models*, 6(4):893–917, 2013. (cit. p. 79)

[38] C. Bardos and N. Besse. Hamiltonian structure, fluid representation and stability for the Vlasov-Dirac-Benney equation. In *Hamiltonian partial differential equations and applications*, volume 75 of *Fields Inst. Commun.*, pages 1–30. Fields Inst. Res. Math. Sci., Toronto, ON, 2015. (cit. p. 79)

[39] R. Barros and W. Choi. On the hyperbolicity of two-layer flows. In *Frontiers of applied and computational mathematics*, pages 95–103. World Sci. Publ., Hackensack, NJ, 2008. (cit. p. 55)

[40] E. Barthélemy. Nonlinear shallow water theories for coastal waves. *Surveys in Geophysics*, 25(3-4):315–337, 2004. (cit. p. 91)

[41] C. Bassi, L. Bonaventura, S. Busto, and M. Dumbser. A hyperbolic reformulation of the Serre-Green-Naghdi model for general bottom topographies. *Comput. & Fluids*, 212:104716, 21, 2020. (cit. pp. 117 and 131)

[42] S. Bazdenkov, N. Morozov, and O. Pogutse. Dispersive effects in two-dimensional hydrodynamics. In *Soviet Physics Doklady*, volume 32, page 262, 1987. In Russian. (cit. p. 91)

[43] J. T. Beale. The existence of solitary water waves. *Comm. Pure Appl. Math.*, 30(4):373–389, 1977. (cit. p. 16)

[44] K. A. Belibassakis and G. A. Athanassoulis. A coupled-mode system with application to nonlinear water waves propagating in finite water depth and in variable bathymetry regions. *Coastal Engineering*, 58(4):337–350, 2011. (cit. p. 202)

[45] T. B. Benjamin and T. J. Bridges. Reappraisal of the Kelvin-Helmholtz problem. I. Hamiltonian structure. *J. Fluid Mech.*, 333:301–325, 1997. (cit. pp. 21, 22, 23, and 221)

[46] T. B. Benjamin and P. J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982. (cit. p. 9)

[47] D. J. Benney. Some properties of long nonlinear waves. *Studies in Appl. Math.*, 52(1):45–50, 1973. (cit. p. 79)

[48] G. S. Benton. The occurrence of critical flow and hydraulic jumps in a multi-layered fluid system. *Journal of Meteorology*, 11(2):139–150, 1954. (cit. p. 73)

[49] S. Benzoni-Gavage and D. Serre. *Multidimensional hyperbolic partial differential equations. First-order systems and applications.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. (cit. pp. 49, 51, 101, 107, and 121)

[50] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. Julia: a fresh approach to numerical computing. *SIAM Rev.*, 59(1):65–98, 2017. (cit. pp. 237 and 242)

[51] J. L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.*, 12(4):283–318, 2002. (cit. pp. ix and 157)

[52] J. L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity*, 17(3):925–952, 2004. (cit. p. 169)

[53] J. L. Bona, T. Colin, and D. Lannes. Long wave approximations for water waves. *Arch. Ration. Mech. Anal.*, 178(3):373–410, 2005. (cit. p. ix)

[54] J. L. Bona, D. Lannes, and J.-C. Saut. Asymptotic models for internal waves. *J. Math. Pures Appl. (9)*, 89(6):538–566, 2008. (cit. pp. 19, 63, 206, and 219)

[55] J. L. Bona and R. Smith. The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287):555–601, 1975. (cit. pp. 100 and 111)

[56] A.-S. Bonnet-Ben Dhia, M.-O. Bristeau, E. Godlewski, S. Imperiale, A. Mangeney, and J. Sainte-Marie. Pseudo-compressibility, dispersive model and acoustic waves in shallow water flows. preprint available at `https://hal.inria.fr/hal-02493518`. (cit. p. 117)

[57] A. Boonkasame and P. Milewski. The stability of large-amplitude shallow interfacial non-Boussinesq flows. *Stud. Appl. Math.*, 128(1):40–58, 2012. (cit. p. 65)

[58] A. Boonkasame and P. A. Milewski. A model for strongly nonlinear long interfacial waves with background shear. *Stud. Appl. Math.*, 133(2):182–213, 2014. (cit. p. 215)

[59] J. Boussinesq. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.*, 17(2):55–108, 1872. (cit. p. ix)

[60] J. Boussinesq. Addition au mémoire sur la théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire, etc. *J. Math. Pures Appl.*, 17(2):47–52, 1873. (cit. pp. 151 and 153)

[61] J. Boussinesq. Essai sur la théorie des eaux courantes. *Mém. présent. divers savants Acad. sci. Inst. Fr.*, 23:1–680, 1877. (cit. p. x)

[62] D. Bresch and G. Métivier. Anelastic limits for Euler-type systems. *Appl. Math. Res. Express. AMRX*, 2010(2):119–141, 2010. (cit. pp. 51, 76, 115, 131, and 203)

[63] D. Bresch and M. Renardy. Well-posedness of two-layer shallow water flow between two horizontal rigid plates. *Nonlinearity*, 24:1081–1088, 2011. (cit. pp. 67 and 68)

[64] T. J. Bridges, M. D. Groves, and D. P. Nicholls, editors. *Lectures on the theory of water waves*, volume 426 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2016. Papers from the talks given at the Isaac Newton Institute for Mathematical Sciences, Cambridge, July–August, 2014. (cit. pp. i and 16)

[65] M.-O. Bristeau, A. Mangeney, J. Sainte-Marie, and N. Seguin. An energy-consistent depth-averaged Euler system: derivation and properties. *Discrete Contin. Dyn. Syst. Ser. B*, 20(4):961–988, 2015. (cit. p. 91)

[66] G. Browning and H.-O. Kreiss. Problems with different time scales for nonlinear partial differential equations. *SIAM J. Appl. Math.*, 42(4):704–718, 1982. (cit. pp. 125, 128, and 129)

[67] B. Buffoni, M. D. Groves, S. M. Sun, and E. Wahlén. Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves. *J. Differential Equations*, 254(3):1006–1096, 2013. (cit. p. 16)

[68] S. Busto, M. Dumbser, C. Escalante, N. Favrie, and S. Gavrilyuk. On High Order ADER Discontinuous Galerkin Schemes for First Order Hyperbolic Reformulations of Nonlinear Dispersive Systems. *J. Sci. Comput.*, 87(2):48, 2021. (cit. pp. 120, 121, and 131)

[69] J. G. B. Byatt-Smith. An integral equation for unsteady surface waves and a comment on the Boussinesq equation. *J. Fluid Mech.*, 49:625–633, 1971. (cit. p. 240)

[70] R. Camassa, S. Chen, G. Falqui, G. Ortenzi, and M. Pedroni. An inertia 'paradox' for incompressible stratified Euler fluids. *J. Fluid Mech.*, 695:330–340, 2012. (cit. p. 70)

[71] R. Camassa, S. Chen, G. Falqui, G. Ortenzi, and M. Pedroni. Effects of inertia and stratification in incompressible ideal fluids: pressure imbalances by rigid confinement. *J. Fluid Mech.*, 726:404–438, 2013. (cit. p. 70)

[72] R. Camassa, G. Falqui, G. Ortenzi, M. Pedroni, and C. Thomson. Hydrodynamic models and confinement effects by horizontal boundaries. *J. Nonlinear Sci.*, 29(4):1445–1498, 2019. (cit. p. 5)

[73] R. Camassa, D. D. Holm, and C. D. Levermore. Long-time effects of bottom topography in shallow water. *Phys. D*, 98(2-4):258–286, 1996. Nonlinear phenomena in ocean dynamics (Los Alamos, NM, 1995). (cit. pp. 91 and 114)

[74] R. Camassa, D. D. Holm, and C. D. Levermore. Long-time shallow-water equations with a varying bottom. *J. Fluid Mech.*, 349:173–189, 1997. (cit. p. 114)

[75] C. Cao, J. Li, and E. S. Titi. Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion. *Comm. Pure Appl. Math.*, 69(8):1492–1531, 2016. (cit. p. 86)

[76] C. Cao, J. Li, and E. S. Titi. Strong solutions to the 3D primitive equations with only horizontal dissipation: near H^1 initial data. *J. Funct. Anal.*, 272(11):4606–4641, 2017. (cit. p. 86)

[77] R. A. Capistrano-Filho, F. A. Gallego, and A. F. Pazoto. On the well-posedness and large-time behavior of higher order Boussinesq system. *Nonlinearity*, 32(5):1852–1881, 2019. (cit. p. 169)

[78] J. D. Carter. Bidirectional Whitham equations as models of waves on shallow water. *Wave Motion*, 82:51–61, 2018. (cit. p. 142)

[79] J. D. Carter and R. Cienfuegos. The kinematics and stability of solitary and cnoidal wave solutions of the Serre equations. *Eur. J. Mech. B Fluids*, 30(3):259–268, 2011. (cit. pp. 97 and 114)

[80] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and J. Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. *Ann. of Math. (2)*, 178(3):1061–1134, 2013. (cit. p. 15)

[81] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and M. López-Fernández. Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves. *Ann. of Math. (2)*, 175(2):909–948, 2012. (cit. p. 15)

[82] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo, and J. Gómez-Serrano. Splash singularity for water waves. *Proc. Natl. Acad. Sci. USA*, 109(3):733–738, 2012. (cit. p. 15)

[83] A. Castro and D. Lannes. Fully nonlinear long-waves models in presence of vorticity. *J. Fluid Mech.*, 759:642–675, 2014. (cit. pp. 48 and 49)

[84] A. Castro and D. Lannes. Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity. *Indiana Univ. Math. J.*, 64(4):1169–1270, 2015. (cit. pp. 17 and 49)

[85] M. Cathala. *Problématiques d'analyse numérique et de modélisation pour écoulements de fluides environnementaux*. PhD thesis, Université Montpellier II, 2013. (cit. p. 15)

[86] M. Cathala. Asymptotic shallow water models with non smooth topographies. *Monatsh. Math.*, 179(3):325–353, 2016. (cit. p. 15)

[87] S. Chandrasekhar. *Hydrodynamic and hydromagnetic stability*. The International Series of Monographs on Physics. Clarendon Press, Oxford, 1961. (cit. p. 23)

[88] R. M. Chen and J. Jin. Global bifurcation of solitary waves to the Boussinesq *abcd* system. *J. Differential Equations*, 310:235–263, 2022. (cit. p. 143)

[89] R. M. Chen and S. Walsh. Continuous dependence on the density for stratified steady water waves. *Arch. Ration. Mech. Anal.*, 219(2):741–792, 2016. (cit. p. 74)

[90] R. M. Chen and S. Walsh. Orbital stability of internal waves. *Comm. Math. Phys.*, 391(3):1091–1141, 2022. (cit. p. 28)

[91] B. Cheng, Q. Ju, and S. Schochet. Three-scale singular limits of evolutionary PDEs. *Arch. Ration. Mech. Anal.*, 229(2):601–625, 2018. (cit. p. 125)

[92] C.-H. A. Cheng, D. Coutand, and S. Shkoller. On the motion of vortex sheets with surface tension in three-dimensional Euler equations with vorticity. *Comm. Pure Appl. Math.*, 61(12):1715–1752, 2008. (cit. p. 28)

[93] A. A. Chesnokov, G. A. El, S. L. Gavrilyuk, and M. V. Pavlov. Stability of shear shallow water flows with free surface. *SIAM J. Appl. Math.*, 77(3):1068–1087, 2017. (cit. p. 77)

[94] W. Choi. Modeling of strongly nonlinear internal gravity waves. In *Proceedings of 4thInternational Conference on Hydrodynamics, Yokohama, Japan*, pages 453–458, 2000. (cit. p. 209)

[95] W. Choi. Fifth-order nonlinear spectral model for surface gravity waves: From pseudo-spectral to spectral formulations (workshop on nonlinear water waves). *RIMS Kokyuroku*, 2109:47–60, 2019. (cit. p. 151)

[96] W. Choi. On Rayleigh expansion for nonlinear long water waves. *J. Hydrodyn.*, 31(6):1115–1126, 2019. (cit. pp. 153 and 157)

[97] W. Choi. High-order strongly nonlinear long wave approximation and solitary wave solution. *J. Fluid Mech.*, 945:Paper No. A15, 38, 2022. (cit. p. 157)

[98] W. Choi, R. Barros, and T.-C. Jo. A regularized model for strongly nonlinear internal solitary waves. *J. Fluid Mech.*, 629:73–85, 2009. (cit. p. 215)

[99] W. Choi and R. Camassa. Weakly nonlinear internal waves in a two-fluid system. *J. Fluid Mech.*, 313:83–103, 1996. (cit. pp. 206, 208, and 209)

[100] W. Choi and R. Camassa. Exact evolution equations for surface waves. *J. Eng. Mech.*, 125(7):756–760, 1999. (cit. pp. 240 and 241)

[101] W. Choi and R. Camassa. Fully nonlinear internal waves in a two-fluid system. *J. Fluid Mech.*, 396:1–36, 1999. (cit. pp. 206, 208, 209, 210, and 213)

[102] L. Chumakova, F. E. Menzaque, P. A. Milewski, R. R. Rosales, E. G. Tabak, and C. V. Turner. Shear instability for stratified hydrostatic flows. *Comm. Pure Appl. Math.*, 62(2):183–197, 2009. (cit. pp. 72 and 74)

[103] L. Chumakova, F. E. Menzaque, P. A. Milewski, R. R. Rosales, E. G. Tabak, and C. V. Turner. Stability properties and nonlinear mappings of two and three-layer stratified flows. *Stud. Appl. Math.*, 122(2):123–137, 2009. (cit. pp. 65 and 73)

[104] L. Chumakova and E. G. Tabak. Simple waves do not avoid eigenvalue crossings. *Comm. Pure Appl. Math.*, 63(1):119–132, 2010. (cit. p. 76)

[105] R. Cienfuegos, E. Barthélemy, and P. Bonneton. A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations. I. Model development and analysis. *Internat. J. Numer. Methods Fluids*, 51(11):1217–1253, 2006. (cit. p. 116)

[106] K. M. Claassen and M. A. Johnson. Numerical bifurcation and spectral stability of wavetrains in bidirectional Whitham models. *Stud. Appl. Math.*, 141(2):205–246, 2018. (cit. p. 142)

[107] D. Clamond and D. Dutykh. Practical use of variational principles for modeling water waves. *Phys. D*, 241(1):25–36, 2012. (cit. pp. 8, 91, and 93)

[108] D. Clamond and D. Dutykh. Accurate fast computation of steady two-dimensional surface gravity waves in arbitrary depth. *J. Fluid Mech.*, 844:491–518, 2018. (cit. p. 240)

[109] D. Clamond, D. Dutykh, and D. Mitsotakis. Conservative modified Serre-Green-Naghdi equations with improved dispersion characteristics. *Commun. Nonlinear Sci. Numer. Simul.*, 45:245–257, 2017. (cit. p. 93)

[110] E. D. Cokelet. Steep gravity waves in water of arbitrary uniform depth. *Philos. Trans. Roy. Soc. London Ser. A*, 286(1335):183–230, 1977. (cit. p. 240)

[111] M. Colin and T. Iguchi. Solitary wave solutions to the Isobe-Kakinuma model for water waves. *Stud. Appl. Math.*, 145(1):52–80, 2020. (cit. p. 191)

[112] A. Constantin. *Nonlinear water waves with applications to wave-current interactions and tsunamis*, volume 81 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. (cit. p. 16)

[113] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004. (cit. p. 17)

[114] A. Constantin and E. Varvaruca. Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.*, 199(1):33–67, 2011. (cit. p. 240)

[115] D. Córdoba and C. Fefferman. Water waves with or without surface tension. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pages 1329–1349. Springer, Cham, 2018. (cit. p. 15)

[116] C. J. Cotter, D. D. Holm, and J. R. Percival. The square root depth wave equations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 466(2124):3621–3633, 2010. (cit. p. 219)

[117] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985. (cit. p. 15)

[118] W. Craig, M. Gazeau, C. Lacave, and C. Sulem. Bloch theory and spectral gaps for linearized water waves. *SIAM J. Math. Anal.*, 50(5):5477–5501, 2018. (cit. p. 147)

[119] W. Craig and M. D. Groves. Hamiltonian long-wave approximations to the water-wave problem. *Wave Motion*, 19(4):367–389, 1994. (cit. p. 90)

[120] W. Craig and M. D. Groves. Normal forms for wave motion in fluid interfaces. *Wave Motion*, 31(1):21–41, 2000. (cit. p. 21)

[121] W. Craig, P. Guyenne, and H. Kalisch. Hamiltonian long-wave expansions for free surfaces and interfaces. *Comm. Pure Appl. Math.*, 58(12):1587–1641, 2005. (cit. pp. 21, 22, and 54)

[122] W. Craig, P. Guyenne, D. P. Nicholls, and C. Sulem. Hamiltonian long-wave expansions for water waves over a rough bottom. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2055):839–873, 2005. (cit. p. 15)

[123] W. Craig, P. Guyenne, and C. Sulem. Water waves over a random bottom. *J. Fluid Mech.*, 640:79–107, 2009. (cit. p. 15)

[124] W. Craig, P. Guyenne, and C. Sulem. Coupling between internal and surface waves. *Natural Hazards*, 57(3):617–642, 2010. (cit. p. 23)

[125] W. Craig, P. Guyenne, and C. Sulem. The surface signature of internal waves. *J. Fluid Mech.*, 710:277–303, 2012. (cit. p. 23)

[126] W. Craig, P. Guyenne, and C. Sulem. Internal waves coupled to surface gravity waves in three dimensions. *Commun. Math. Sci*, 13:893–910, 2015. (cit. p. 23)

[127] W. Craig, D. Lannes, and C. Sulem. Water waves over a rough bottom in the shallow water regime. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(2):233–259, 2012. (cit. p. 15)

[128] W. Craig and D. P. Nicholls. Travelling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.*, 32(2):323–359, 2000. (cit. p. 16)

[129] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73–83, 1993. (cit. pp. 6 and 151)

[130] W. Craig, C. Sulem, and P.-L. Sulem. Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity*, 5(2):497–522, 1992. (cit. p. 6)

[131] R. Creedon, B. Deconinck, and O. Trichtchenko. High-Frequency Instabilities of a Boussinesq–Whitham System: A Perturbative Approach. *Fluids*, 6(4):136, 2021. (cit. p. 142)

[132] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010. (cit. p. 65)

[133] O. Darrigol. The spirited horse, the engineer, and the mathematician: water waves in nineteenth-century hydrodynamics. *Arch. Hist. Exact Sci.*, 58(1):21–95, 2003. (cit. pp. i and ix)

[134] B. Deconinck and O. Trichtchenko. High-frequency instabilities of small-amplitude solutions of Hamiltonian PDEs. *Discrete Contin. Dyn. Syst.*, 37(3):1323–1358, 2017. (cit. p. 142)

[135] P. Degond and M. Tang. All speed scheme for the low Mach number limit of the isentropic Euler equations. *Commun. Comput. Phys.*, 10(1):1–31, 2011. (cit. p. 116)

[136] J.-M. Delort. Long time existence results for solutions of water waves equations. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 2241–2260. World Sci. Publ., Hackensack, NJ, 2018. (cit. p. 15)

[137] T. Deneke, A. Tesfahun, and T. Temesgen. Dispersive estimates for linearized water wave type equations in $\mathbb{R}^d$. arXiv preprint:2106.02717. (cit. pp. 10 and 11)

[138] B. Desjardins, D. Lannes, and J.-C. Saut. Normal mode decomposition and dispersive and nonlinear mixing in stratified fluids. *Water Waves*, 3(1):153–192, 2021. (cit. pp. 84 and 207)

[139] F. Dias and P. Milewski. On the fully-nonlinear shallow-water generalized Serre equations. *Phys. Lett., A*, 374(8):1049–1053, 2010. (cit. p. 116)

[140] M. W. Dingemans. *Water Wave Propagation Over Uneven Bottoms: Non-linear wave propagation*, volume 13 of *Advanced Series on Ocean Engineering*. World Scientific, Cornell Univ., Hollister Hall, 1997. (cit. pp. 151, 153, and 157)

[141] E. Dinvay. On well-posedness of a dispersive system of the Whitham-Boussinesq type. *Appl. Math. Lett.*, 88:13–20, 2019. (cit. p. 142)

[142] E. Dinvay. Travelling waves in the Boussinesq type systems. *J. Math. Pures Appl. (9)*, 163:1–10, 2022. (cit. pp. ix, 142, and 143)

[143] E. Dinvay, D. Dutykh, and H. Kalisch. A comparative study of bi-directional Whitham systems. *Appl. Numer. Math.*, 141:248–262, 2019. (cit. pp. 142 and 143)

[144] E. Dinvay and N. Kuznetsov. Modified Babenko's equation for periodic gravity waves on water of finite depth. *Quart. J. Mech. Appl. Math.*, 72(4):415–428, 2019. (cit. p. 240)

[145] E. Dinvay and D. Nilsson. Solitary wave solutions of a Whitham-Boussinesq system. *Nonlinear Anal. Real World Appl.*, 60:103280, 2021. (cit. pp. 142 and 143)

[146] E. Dinvay, S. Selberg, and A. Tesfahun. Well-Posedness for a Dispersive System of the Whitham–Boussinesq Type. *SIAM J. Math. Anal.*, 52(3):2353–2382, 2020. (cit. pp. 10, 11, and 142)

[147] D. G. Dommermuth and D. K. Yue. A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid Mech.*, 184:267–288, 1987. (cit. p. 151)

[148] V. A. Dorodnitsyn, E. I. Kaptsov, and S. V. Meleshko. Symmetries, conservation laws, invariant solutions and difference schemes of the one-dimensional Green-Naghdi equations. *J. Nonlinear Math. Phys.*, 28(1):90–107, 2021. (cit. pp. 94, 95, and 116)

[149] V. A. Dougalis, A. Duran, and L. Saridaki. Notes on numerical analysis and solitary wave solutions of Boussinesq/Boussinesq systems for internal waves. arXiv preprint:2012.07992. (cit. p. 219)

[150] V. A. Dougalis and D. E. Mitsotakis. Theory and numerical analysis of Boussinesq systems: a review. In *Effective computational methods for wave propagation*, volume 5 of *Numer. Insights*, pages 63–110. Chapman & Hall/CRC, Boca Raton, FL, 2008. (cit. p. x)

[151] V. Duchêne and C. Klein. Numerical study of the Serre-Green-Naghdi equations and a fully dispersive counterpart. *Discrete Contin. Dyn. Syst. Ser. B*, 27(10):5905–5933, 2022. (cit. pp. 113, 114, 116, 138, 147, 239, and 242)

[152] V. Duchêne. Asymptotic shallow water models for internal waves in a two-fluid system with a free surface. *SIAM J. Math. Anal.*, 42(5):2229–2260, 2010. (cit. pp. 18, 59, 206, and 209)

[153] V. Duchêne. Boussinesq/Boussinesq systems for internal waves with a free surface, and the KdV approximation. *ESAIM Math. Model. Numer. Anal.*, 46(1):145–185, 2012. (cit. p. 206)

[154] V. Duchêne. A note on the well-posedness of the one-dimensional multilayer shallow water model. hal preprint:00922045, 2013. (cit. p. 73)

[155] V. Duchêne. Decoupled and unidirectional asymptotic models for the propagation of internal waves. *Math. Models Methods Appl. Sci.*, 24(1):1–65, 2014. (cit. p. 219)

[156] V. Duchêne. On the rigid-lid approximation for two shallow layers of immiscible fluids with small density contrast. *J. Nonlinear Sci.*, 24(4):579–632, 2014. (cit. pp. 59, 68, and 71)

[157] V. Duchêne. The multilayer shallow water system in the limit of small density contrast. *Asymptot. Anal.*, 98(3):189–235, 2016. (cit. pp. 55, 59, 68, 71, 73, and 74)

[158] V. Duchêne. Rigorous justification of the Favrie-Gavrilyuk approximation to the Serre-Green-Naghdi model. *Nonlinearity*, 32(10):3772–3797, 2019. (cit. pp. 89, 116, and 121)

[159] V. Duchêne and T. Iguchi. A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part I: Structures and well-posedness. To appear in Ann. Inst. H. Poincaré C Anal. Non Linéaire. arXiv preprint:2103.12392. (cit. pp. 67, 221, 222, 223, 228, and 230)

[160] V. Duchêne and T. Iguchi. A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part II: Justification as a shallow-water approximation. arXiv preprint:2212.07117. (cit. pp. 228, 229, and 230)

[161] V. Duchêne and T. Iguchi. A Hamiltonian structure of the Isobe-Kakinuma model for water waves. *Water Waves*, 3(1):193–211, 2021. (cit. pp. 187 and 188)

[162] V. Duchêne and S. Israwi. Well-posedness of the Green-Naghdi and Boussinesq-Peregrine systems. *Ann. Math. Blaise Pascal*, 25(1):21–74, 2018. (cit. pp. 91, 99, 112, and 140)

[163] V. Duchêne, S. Israwi, and R. Talhouk. Shallow water asymptotic models for the propagation of internal waves. *Discrete Contin. Dyn. Syst. Ser. S*, 7(2):239–269, 2014. (cit. pp. 63, 67, and 212)

[164] V. Duchêne, S. Israwi, and R. Talhouk. A new class of two-layer Green-Naghdi systems with improved frequency dispersion. *Stud. Appl. Math.*, 137(3):356–415, 2016. (cit. pp. 133, 140, 215, 216, and 218)

[165] V. Duchêne, D. Nilsson, and E. Wahlén. Solitary Wave Solutions to a Class of Modified Green–Naghdi Systems. *J. Math. Fluid Mech.*, 20(3):1059–1091, 2018. (cit. pp. 98, 113, 137, 138, 211, and 216)

[166] A. Duran and F. Marche. Discontinuous-Galerkin discretization of a new class of Green-Naghdi equations. *Commun. Comput. Phys.*, 17(3):721–760, 2015. (cit. p. 116)

[167] D. Dutykh and D. Clamond. Efficient computation of steady solitary gravity waves. *Wave Motion*, 51(1):86–99, 2014. (cit. p. 240)

[168] D. Dutykh, D. Clamond, P. Milewski, and D. Mitsotakis. Finite volume and pseudo-spectral schemes for the fully nonlinear 1D Serre equations. *European J. Appl. Math.*, 24(5):761–787, 2013. (cit. p. 116)

[169] A. I. Dyachenko, E. A. Kuznetsov, M. Spector, and V. E. Zakharov. Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping). *Phys. Lett. A*, 221(1-2):73–79, 1996. (cit. p. 240)

[170] D. G. Ebin. Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids. *Comm. Partial Differential Equations*, 13(10):1265–1295, 1988. (cit. pp. 15 and 28)

[171] M. Ehrnström, M. D. Groves, and E. Wahlén. On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type. *Nonlinearity*, 25(10):2903–2936, 2012. (cit. p. 137)

[172] M. Ehrnström, M. A. Johnson, and K. M. Claassen. Existence of a highest wave in a fully dispersive two-way shallow water model. *Arch. Ration. Mech. Anal.*, 231(3):1635–1673, 2019. (cit. p. 142)

[173] M. Ehrnström and E. Wahlén. On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(6):1603–1637, 2019. (cit. pp. xi, 89, 147, and 234)

[174] G. A. El, R. H. J. Grimshaw, and N. F. Smyth. Unsteady undular bores in fully nonlinear shallow-water theory. *Phys. Fluids*, 18(2):027104, 17, 2006. (cit. p. 114)

[175] G. A. El, M. A. Hoefer, and M. Shearer. Expansion shock waves in regularized shallow-water theory. *Proc. A.*, 472(2189):20160141, 10, 2016. (cit. p. 113)

[176] L. Emerald. Local well-posedness result for a class of non-local quasi-linear systems and its application to the justification of Whitham-Boussinesq systems. arXiv preprint:2206.09213. (cit. pp. 132 and 145)

[177] L. Emerald. Rigorous derivation from the water waves equations of some full dispersion shallow water models. *SIAM J. Math. Anal.*, 53(4):3772–3800, 2021. (cit. pp. 38, 132, and 142)

[178] L. Emerald. Rigorous derivation of the Whitham equations from the water waves equations in the shallow water regime. *Nonlinearity*, 34(11):7470–7509, 2021. (cit. pp. x and 89)

[179] C. Escalante, M. Dumbser, and M. J. Castro. An efficient hyperbolic relaxation system for dispersive non-hydrostatic water waves and its solution with high order discontinuous Galerkin schemes. *J. Comput. Phys.*, 394:385–416, 2019. (cit. pp. 117 and 131)

[180] C. Escalante and T. Morales de Luna. A general non-hydrostatic hyperbolic formulation for Boussinesq dispersive shallow flows and its numerical approximation. *J. Sci. Comput.*, 83(3):Paper No. 62, 37, 2020. (cit. pp. 91, 117, and 131)

[181] N. Favrie and S. Gavrilyuk. A rapid numerical method for solving Serre-Green-Naghdi equations describing long free surface gravity waves. *Nonlinearity*, 30(7):2718–2736, 2017. (cit. pp. 89, 116, 121, and 131)

[182] Z. I. Fedotova, G. S. Khakimzyanov, and D. Dutykh. Energy equation for certain approximate models of long-wave hydrodynamics. *Russian J. Numer. Anal. Math. Modelling*, 29(3):167–178, 2014. (cit. p. 91)

[183] R. Feola and F. Giuliani. Quasi-periodic traveling waves on an infinitely deep fluid under gravity. arXiv preprint:2005.08280. (cit. p. 16)

[184] E. D. Fernández-Nieto, M. Parisot, Y. Penel, and J. Sainte-Marie. A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows. *Commun. Math. Sci.*, 16(5):1169–1202, 2018. (cit. pp. 91, 182, and 183)

[185] A. G. Filippini, S. Bellec, M. Colin, and M. Ricchiuto. On the nonlinear behaviour of Boussinesq type models: Amplitude-velocity vs amplitude-flux forms. *Coast. Eng.*, 99:109–123, 2015. (cit. p. ix)

[186] J.-P. Fouque, J. Garnier, and A. Nachbin. Shock structure due to stochastic forcing and the time reversal of nonlinear waves. *Phys. D*, 195(3-4):324–346, 2004. (cit. p. 15)

[187] K. O. Friedrichs and D. H. Hyers. The existence of solitary waves. *Comm. Pure Appl. Math.*, 7:517–550, 1954. (cit. p. 16)

[188] J. T. Frings. *An adaptive multilayer model for density-layered shallow water flows.* PhD thesis, Aachen University, 2012. (cit. pp. 73 and 76)

[189] H. Fujiwara and T. Iguchi. A shallow water approximation for water waves over a moving bottom. In *Nonlinear dynamics in partial differential equations*, volume 64 of *Adv. Stud. Pure Math.*, pages 77–88. Math. Soc. Japan, Tokyo, 2015. (cit. pp. 4, 91, 99, 107, and 122)

[190] I. Gallagher. Résultats récents sur la limite incompressible. *Astérisque*, 299:Exp. No. 926, vii, 29–57, 2005. Séminaire Bourbaki. Vol. 2003/2004. (cit. pp. 59, 68, 114, 115, and 125)

[191] T. Gallay. Stability of vortices in ideal fluids: the legacy of Kelvin and Rayleigh. In *Hyperbolic problems: theory, numerics, applications*, volume 10 of *AIMS Ser. Appl. Math.*, pages 42–59. Am. Inst. Math. Sci. (AIMS), Springfield, MO, [2020] ©2020. (cit. pp. 76 and 84)

[192] T. Gallay and D. Smets. On the linear stability of vortex columns in the energy space. *J. Math. Fluid Mech.*, 21(4):Paper No. 48, 27, 2019. (cit. p. 76)

[193] S. Gavrilyuk. Multiphase flow modeling via Hamilton's principle. In *Variational models and methods in solid and fluid mechanics*, volume 535 of *CISM Courses and Lect.*, pages 163–210. SpringerWienNewYork, Vienna, 2011. (cit. p. 93)

[194] S. Gavrilyuk and H. Gouin. A new form of governing equations of fluids arising from Hamilton's principle. *Internat. J. Engrg. Sci.*, 37(12):1495–1520, 1999. (cit. p. 93)

[195] S. Gavrilyuk, H. Kalisch, and Z. Khorsand. A kinematic conservation law in free surface flow. *Nonlinearity*, 28(6):1805–1821, 2015. (cit. pp. 91 and 95)

[196] S. Gavrilyuk, B. Nkonga, K.-M. Shyue, and L. Truskinovsky. Stationary shock-like transition fronts in dispersive systems. *Nonlinearity*, 33(10):5477–5509, 2020. (cit. pp. 97, 113, and 116)

[197] S. L. Gavrilyuk and V. M. Teshukov. Generalized vorticity for bubbly liquid and dispersive shallow water equations. *Contin. Mech. Thermodyn.*, 13(6):365–382, 2001. (cit. p. 93)

[198] P. R. Gent and J. C. Mcwilliams. Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.*, 20(1):150–155, 1990. (cit. p. 86)

[199] P. R. Gent, J. Willebrand, T. J. McDougall, and J. C. McWilliams. Parameterizing eddy-induced tracer transports in ocean circulation models. *Journal of Physical Oceanography*, 25(4):463–474, 1995. (cit. p. 86)

[200] B. J. Geurts and D. D. Holm. Leray and LANS-α modelling of turbulent mixing. *J. Turbul.*, 7:Paper 10, 33, 2006. (cit. p. 86)

[201] A. E. Gill. *Atmosphere-ocean dynamics*, volume 30 of *International geophysics series*. Academic Press, 1982. (cit. pp. 54 and 56)

[202] A. E. Green and P. M. Naghdi. A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.*, 78(02):237–246, 1976. (cit. p. 91)

[203] M. D. Groves and S.-M. Sun. Fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem. *Arch. Ration. Mech. Anal.*, 188(1):1–91, 2008. (cit. p. 16)

[204] B. Guelmame. On the blow-up scenario for some modified Serre–Green–Naghdi equations. Preprint available at https://hal.archives-ouvertes.fr/hal-03264688, 2021. (cit. p. 113)

[205] B. Guelmame. Global weak solutions of the Serre–Green–Naghdi equations with surface tension. Preprint available at https://hal.archives-ouvertes.fr/hal-03585433., 2022. (cit. p. 113)

[206] J.-L. Guermond, C. Kees, B. Popov, and E. Tovar. Hyperbolic relaxation technique for solving the dispersive Serre-Green-Naghdi equations with topography. *J. Comput. Phys.*, 450:Paper No. 110809, 16, 2022. (cit. pp. 91, 117, and 131)

[207] J.-L. Guermond, B. Popov, E. Tovar, and C. Kees. Robust explicit relaxation technique for solving the Green-Naghdi equations. *J. Comput. Phys.*, 399:108917, 17, 2019. (cit. pp. 117 and 131)

[208] P. Guyenne. HOS simulations of nonlinear water waves in complex media. In D. Henry, K. Kalimeris, E. Părău, J. M. Vanden-Broeck, and E. Wahlén, editors, *Nonlinear Water Waves*, Tutorials, Schools, and Workshops in the Mathematical Sciences, pages 53–69. Birkhäuser, Cham, 2019. (cit. p. 151)

[209] P. Guyenne, D. Lannes, and J.-C. Saut. Well-posedness of the Cauchy problem for models of large amplitude internal waves. *Nonlinearity*, 23(2):237–275, 2010. (cit. pp. 63, 67, and 68)

[210] J. Hamilton. Differential equations for long-period gravity waves on fluid of rapidly varying depth. *J. Fluid Mech.*, 83(2):289–310, 1977. (cit. pp. 15 and 240)

[211] S. V. Haziot, V. M. Hur, W. A. Strauss, J. F. Toland, E. Wahlén, S. Walsh, and M. H. Wheeler. Traveling water waves—the ebb and flow of two centuries. *Quart. Appl. Math.*, 80(2):317–401, 2022. (cit. pp. 16 and 28)

[212] M. W. Hecht, D. D. Holm, M. R. Petersen, and B. A. Wingate. Implementation of the LANS-α turbulence model in a primitive equation ocean model. *J. Comput. Phys.*, 227(11):5691–5716, 2008. (cit. p. 86)

[213] K. R. Helfrich and W. K. Melville. Long nonlinear internal waves. In *Annual review of fluid mechanics. Vol. 38*, pages 395–425. 2006. (cit. pp. 17, 205, and 210)

[214] D. D. Holm. Hamiltonian structure for two-dimensional hydrodynamics with nonlinear dispersion. *Phys. Fluids*, 31(8):2371–2373, 1988. (cit. p. 92)

[215] D. D. Holm and B. Long. Lyapunov stability of ideal stratified fluid equilibria in hydrostatic balance. *Nonlinearity*, 2(1):23–35, 1989. (cit. pp. 74 and 81)

[216] L. N. Howard. Note on a paper of John W. Miles. *J. Fluid Mech.*, 10:509–512, 1961. (cit. pp. 76 and 84)

[217] T. J. R. Hughes, T. Kato, and J. E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63(3):273–294, 1976. (cit. p. 68)

[218] J. K. Hunter, M. Ifrim, and D. Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346(2):483–552, 2016. (cit. p. 240)

[219] V. M. Hur. Wave breaking in the Whitham equation. *Adv. Math.*, 317:410–437, 2017. (cit. pp. xi, 89, and 147)

[220] V. M. Hur and A. K. Pandey. Modulational instability in a full-dispersion shallow water model. *Stud. Appl. Math.*, 142(1):3–47, 2019. (cit. p. 142)

[221] V. M. Hur and L. Tao. Wave breaking in a shallow water model. *SIAM J. Math. Anal.*, 50(1):354–380, 2018. (cit. p. 142)

[222] M. Ifrim and D. Tataru. Two dimensional water waves in holomorphic coordinates II: Global solutions. *Bull. Soc. Math. France*, 144(2):369–394, 2016. (cit. p. 240)

[223] M. Ifrim and D. Tataru. The lifespan of small data solutions in two dimensional capillary water waves. *Arch. Ration. Mech. Anal.*, 225(3):1279–1346, 2017. (cit. p. 240)

[224] T. Iguchi. A shallow water approximation for water waves. *J. Math. Kyoto Univ.*, 49(1):13–55, 2009. (cit. pp. 14, 29, and 101)

[225] T. Iguchi. Isobe-Kakinuma model for water waves as a higher order shallow water approximation. *J. Differential Equations*, 265(3):935–962, 2018. (cit. p. 196)

[226] T. Iguchi. A mathematical justification of the Isobe-Kakinuma model for water waves with and without bottom topography. *J. Math. Fluid Mech.*, 20(4):1985–2018, 2018. (cit. pp. 196, 197, 198, and 201)

[227] T. Iguchi, N. Tanaka, and A. Tani. On the two-phase free boundary problem for two-dimensional water waves. *Math. Ann.*, 309(2):199–223, 1997. (cit. p. 28)

[228] H. Inci. On a Lagrangian formulation of the 1D Green-Naghdi system. arXiv preprint:2111.06192. (cit. p. 99)

[229] H. Inci. On the local well-posedness of the 1D Green-Naghdi system over a nonflat bottom. arXiv preprint:2111.09681. (cit. p. 99)

[230] D. Ionescu-Kruse. Variational derivation of the Green-Naghdi shallow-water equations. *J. Nonlinear Math. Phys.*, 19(suppl. 1):1240001, 12, 2012. (cit. p. 91)

[231] C. O. Iselin. The influence of vertical and lateral turbulence on the characteristics of the waters at mid-depths. *Eos, Trans. AGU*, 20(3):414–417, 1939. (cit. p. 86)

[232] M. Isobe. A proposal on a nonlinear gentle slope wave equation. *Proc. Coast. Eng. Jpn. Soc. Civ. Eng.*, 41:1–5, 1994. [Japanese]. (cit. p. 186)

[233] M. Isobe. Time-dependent mild-slope equations for random waves. In *Proceedings of 24th International Conference on Coastal Engineering*, pages 285–299. ASCE, 1994. (cit. pp. 152 and 186)

[234] S. Israwi. Large time existence for 1D Green-Naghdi equations. *Nonlinear Analysis: Theory, Methods & Applications*, 74(1):81–93, 2011. (cit. pp. 91 and 99)

[235] C. R. Jackson. An atlas of internal solitary-like waves and their properties, 2004. http://www.internalwaveatlas.com/Atlas2_index.html. (cit. pp. 17 and 205)

[236] S. Jin. Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review. *Riv. Math. Univ. Parma (N.S.)*, 3(2):177–216, 2012. (cit. pp. 116 and 117)

[237] T.-C. Jo and W. Choi. Dynamics of strongly nonlinear internal solitary waves in shallow water. *Stud. Appl. Math.*, 109(3):205–227, 2002. (cit. p. 213)

[238] T. Kakinuma. [title in japanese]. *Proc. Coast. Eng. Jpn. Soc. Civ. Eng.*, 47:1–5, 2000. [Japanese]. (cit. p. 186)

[239] T. Kakinuma. A set of fully nonlinear equations for surface and internal gravity waves. In *Coastal Engineering V: Computer Modelling of Seas and Coastal Regions*, pages 225–234. WIT Press, 2001. (cit. pp. 152, 186, and 220)

[240] T. Kakinuma. A nonlinear numerical model for surface and internal waves shoaling on a permeable beach. In *Coastal engineering VI: Computer Modelling and Experimental Measurements of Seas and Coastal Regions*, pages 227–236. WIT Press, 2003. (cit. pp. 186 and 220)

[241] H. Kalisch and D. Pilod. On the local well-posedness for a full-dispersion Boussinesq system with surface tension. *Proc. Amer. Math. Soc.*, 147(6):2545–2559, 2019. (cit. p. 142)

[242] V. Kamotski and G. Lebeau. On 2D Rayleigh-Taylor instabilities. *Asymptot. Anal.*, 42(1-2):1–27, 2005. (cit. p. 28)

[243] T. Kano. Une théorie trois-dimensionnelle des ondes de surface de l'eau et le développement de Friedrichs. II. *J. Math. Kyoto Univ.*, 26(2):157–175, 1986. (cit. p. 15)

[244] T. Kano and T. Nishida. Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde. *J. Math. Kyoto Univ.*, 19(2):335–370, 1979. (cit. p. 15)

[245] T. Kano and T. Nishida. Water waves and Friedrichs expansion. In *Recent topics in nonlinear PDE (Hiroshima, 1983)*, volume 98 of *North-Holland Math. Stud.*, pages 39–57. North-Holland, Amsterdam, 1984. (cit. p. 15)

[246] E. I. Kaptsov, S. V. Meleshko, and N. F. Samatova. The one-dimensional green–naghdi equations with a time dependent bottom topography and their conservation laws. *Phys. Fluids*, 32(12):123607, 2020. (cit. p. 94)

[247] T. Kato. Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$. *J. Functional Analysis*, 9:296–305, 1972. (cit. pp. 107 and 111)

[248] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition. (cit. p. 76)

[249] M. Kazakova. *Dispersive models of ocean waves propagation: Numerical issues and modelling.* PhD thesis, Université Toulouse 3 Paul Sabatier, 2018. (cit. pp. 116, 121, and 131)

[250] P. Kelland. On the theory of waves. *Trans. R. Soc. Edinburgh*, 14:497–545, 1840. (cit. p. v)

[251] B. Khorbatly and S. Israwi. Full justification for the extended Green-Naghdi system for an uneven bottom with surface tension. Preprint hal-02994586. (cit. p. 169)

[252] B. Khorbatly, R. Lteif, S. Israwi, and S. Gerbi. Mathematical modeling and numerical analysis for the higher order Boussinesq system. *ESAIM Math. Model. Numer. Anal.*, 56(2):593–615, 2022. (cit. p. 169)

[253] B. Khorbatly, I. Zaiter, and S. Israwi. Derivation and well-posedness of the extended Green-Naghdi equations for flat bottoms with surface tension. *J. Math. Phys.*, 59(7):071501, 20, 2018. (cit. p. 169)

[254] P. D. Killworth. On the parameterization of eddy transfer part i. theory. *J. Mar. Res.*,, 55(6):1171–1197, 1997. (cit. p. 86)

[255] J. W. Kim, K. J. Bai, R. C. Ertekin, and W. C. Webster. A derivation of the Green-Naghdi equations for irrotational flows. *J. Engrg. Math.*, 40(1):17–42, 2001. (cit. p. 91)

[256] J. Kirby. Nonlinear ocean surface waves, 2004. (cit. p. 157)

[257] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, 34(4):481–524, 1981. (cit. p. 125)

[258] C. Klein, F. Linares, D. Pilod, and J.-C. Saut. On Whitham and related equations. *Stud. Appl. Math.*, 140(2):133–177, 2018. (cit. p. 142)

[259] R. Klein. Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics. I. One-dimensional flow. *J. Comput. Phys.*, 121(2):213–237, 1995. (cit. p. 116)

[260] G. Klopman. *Variational Boussinesq modelling of surface gravity waves over bathymetry*. PhD thesis, Univ. of Twente, 2010. (cit. p. 151)

[261] G. Klopman, B. van Groesen, and M. W. Dingemans. A variational approach to Boussinesq modelling of fully nonlinear water waves. *J. Fluid Mech.*, 657:36–63, 2010. (cit. pp. 151, 186, 189, and 194)

[262] D. J. Korteweg and G. De Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 5(39):422–443, 1895. (cit. p. x)

[263] G. Kreisel. Surface waves. *Quart. Appl. Math.*, 7(1):21–44, 1949. (cit. p. 240)

[264] I. Kukavica, R. Temam, V. C. Vicol, and M. Ziane. Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain. *J. Differential Equations*, 250(3):1719–1746, 2011. (cit. p. 83)

[265] J. L. de Lagrange. Mémoire sur la théorie du mouvement des fluides. In *Œuvres complètes, tome 4*, pages 695–748. Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Berlin, 1781. (cit. pp. iii and 153)

[266] H. Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, reprint of the 1932 sixth edition. edition, 1993. (cit. p. 23)

[267] D. Lannes. A Stability Criterion for Two-Fluid Interfaces and Applications. *Arch. Ration. Mech. Anal.*, 208(2):481–567, 2013. (cit. pp. 19, 20, 24, 28, and 207)

[268] D. Lannes. *The water waves problem*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics. (cit. pp. i, ii, x, 2, 4, 6, 13, 14, 15, 29, 30, 31, 101, 167, 168, 169, 197, 202, 248, and 257)

[269] D. Lannes. Modeling shallow water waves. *Nonlinearity*, 33(5):R1–R57, 2020. (cit. pp. i, x, and 183)

[270] D. Lannes and P. Bonneton. Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation. *Phys. Fluids*, 21(1):016601, 2009. (cit. p. 91)

[271] D. Lannes and F. Marche. A new class of fully nonlinear and weakly dispersive Green-Naghdi models for efficient 2D simulations. *J. Comput. Phys.*, 282:238–268, 2015. (cit. p. 116)

[272] D. Lannes and G. Métivier. The shoreline problem for the one-dimensional shallow water and Green-Naghdi equations. *J. Éc. polytech. Math.*, 5:455–518, 2018. (cit. p. 5)

[273] D. Lannes and M. Ming. The Kelvin-Helmholtz instabilities in two-fluids shallow water models. In *Hamiltonian partial differential equations and applications*, volume 75 of *Fields Inst. Commun.*, pages 185–234. Fields Inst. Res. Math. Sci., Toronto, ON, 2015. (cit. pp. 23, 24, 28, 213, and 215)

[274] M. A. Lavrent′ev. I. On the theory of long waves. II. A contribution to the theory of long waves. *Amer. Math. Soc. Translation*, 1954(102):53, 1954. (cit. p. 16)

[275] O. Le Métayer, S. Gavrilyuk, and S. Hank. A numerical scheme for the Green-Naghdi model. *J. Comput. Phys.*, 229(6):2034–2045, 2010. (cit. p. 116)

[276] D. Le Touzé. *Méthodes spectrales pour la modélisation non-linéaire d'écoulements a surface libre instationnaires*. PhD thesis, École Centrale de Nantess, 2003. (cit. p. 151)

[277] G. Lebeau. Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d. *ESAIM Control Optim. Calc. Var.*, 8:801–825 (electronic), 2002. A tribute to J. L. Lions. (cit. p. 28)

[278] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for models of shallow water in a basin with a varying bottom. *Indiana Univ. Math. J.*, 45(2):479–510, 1996. (cit. p. 114)

[279] T. Levi-Civita. Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.*, 93(1):264–314, 1925. (cit. pp. 16 and 240)

[280] J. Li and E. S. Titi. Recent advances concerning certain class of geophysical flows. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pages 933–971. Springer, Cham, 2018. (cit. p. 86)

[281] M. Li, P. Guyenne, F. Li, and L. Xu. High order well-balanced CDG-FE methods for shallow water waves by a Green-Naghdi model. *J. Comput. Phys.*, 257(part A):169–192, 2014. (cit. p. 116)

[282] Y. A. Li. Linear stability of solitary waves of the Green-Naghdi equations. *Comm. Pure Appl. Math.*, 54(5):501–536, 2001. (cit. p. 113)

[283] Y. A. Li. Hamiltonian structure and linear stability of solitary waves of the Green-Naghdi equations. *J. Nonlinear Math. Phys.*, 9(suppl. 1):99–105, 2002. Recent advances in integrable systems (Kowloon, 2000). (cit. pp. 92, 113, and 136)

[284] Y. A. Li. A shallow-water approximation to the full water wave problem. *Comm. Pure Appl. Math.*, 59(9):1225–1285, 2006. (cit. pp. 91, 99, and 240)

[285] Y. A. Li, J. M. Hyman, and W. Choi. A numerical study of the exact evolution equations for surface waves in water of finite depth. *Stud. Appl. Math.*, 113(3):303–324, 2004. (cit. p. 240)

[286] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984. (cit. p. 136)

[287] R. Liska and B. Wendroff. Analysis and computation with stratified fluid models. *J. Comput. Phys.*, 137(1):212–244, 1997. (cit. p. 213)

[288] P. L.-F. Liu and X. Wang. A multi-layer model for nonlinear internal wave propagation in shallow water. *J. Fluid Mech.*, 695:341–365, 2012. (cit. p. 209)

[289] J. C. Luke. A variational principle for a fluid with a free surface. *J. Fluid Mech.*, 27:395–397, 1967. (cit. pp. 8 and 184)

[290] P. Lynett and P. L.-F. Liu. Linear analysis of the multi-layer model. *Coastal Engineering*, 51(5-6):439–454, 2004. (cit. p. 183)

[291] P. Lynett and P. L.-F. Liu. A two-layer approach to wave modelling. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2049):2637–2669, 2004. (cit. p. 183)

[292] P. A. Madsen, H. B. Bingham, and H. Liu. A new Boussinesq method for fully nonlinear waves from shallow to deep water. *J. Fluid Mech.*, 462:1–30, 2002. (cit. p. 183)

[293] P. A. Madsen, H. B. Bingham, and H. A. Schäffer. Boussinesq-type formulations for fully nonlinear and extremely dispersive water waves: derivation and analysis. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459(2033):1075–1104, 2003. (cit. p. 183)

[294] P. A. Madsen and D. R. Fuhrman. High-order Boussinesq-type modelling of nonlinear wave phenomena in deep and shallow water. In Q. Ma, editor, *Advances in Numerical Simulation of Nonlinear Water Waves*, chapter 7, pages 245–285. World Scientific, 2010. (cit. pp. i, 153, 157, 170, and 183)

[295] P. A. Madsen and H. A. Schäffer. A review of boussinesq-type equations for gravity waves. *Advances in Coastal and Ocean Engineering*, 5:1–94, 1999. (cit. pp. ix and 157)

[296] N. Makarenko. A second long-wave approximation in the cauchy-poisson problem. *Dyn. Contin. Media*, 77:56–72, 1986. In Russian. (cit. p. 91)

[297] Z. L. Mal′tseva. Unsteady long waves in a two-layer fluid. *Dinamika Sploshn. Sredy*, (93-94):96–110, 1989. (cit. p. 208)

[298] J. E. Marsden and S. Shkoller. Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains. In *Topological methods in the physical sciences*, volume 359, pages 1449–1468. R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 2001. (cit. p. 86)

[299] N. Masmoudi. Examples of singular limits in hydrodynamics. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 195–275. Elsevier/North-Holland, Amsterdam, 2007. (cit. pp. 59, 68, 84, 114, and 125)

[300] N. Masmoudi and T. K. Wong. On the H^s theory of hydrostatic Euler equations. *Arch. Ration. Mech. Anal.*, 204(1):231–271, 2012. (cit. p. 84)

[301] Y. Matsuno. Hamiltonian formulation of the extended Green-Naghdi equations. *Phys. D*, 301/302:1–7, 2015. (cit. p. 157)

[302] Y. Matsuno. Hamiltonian structure for two-dimensional extended green–naghdi equations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 472(2190):20160127–, 2016. (cit. pp. 153, 157, 159, 163, and 169)

[303] C. C. Mei and B. Le Méhauté. Note on the equations of long waves over an uneven bottom. *J. Geophys. Res.*, 71:393–400, 1966. (cit. p. 153)

[304] A. Meister. Asymptotic single and multiple scale expansions in the low Mach number limit. *SIAM J. Appl. Math.*, 60(1):256–271, 2000. (cit. p. 116)

[305] B. Melinand. A mathematical study of meteo and landslide tsunamis: the Proudman resonance. *Nonlinearity*, 28(11):4037–4080, 2015. (cit. p. 4)

[306] B. Mésognon-Gireau. The singular limit of the Water-Waves equations in the rigid lid regime. arXiv preprint:1512.02424. (cit. p. 115)

[307] B. Mésognon-Gireau. The Cauchy problem on large time for a Boussinesq-Peregrine equation with large topography variations. *Adv. Differential Equations*, 22(7-8):457–504, 2017. (cit. p. 115)

[308] B. Mésognon-Gireau. The Cauchy problem on large time for the water waves equations with large topography variations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34(1):89–118, 2017. (cit. p. 15)

[309] B. Mésognon-Gireau. A dispersive estimate for the linearized water-waves equations in finite depth. *J. Math. Fluid Mech.*, 19(3):469–500, 2017. (cit. p. 115)

[310] G. Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008. (cit. pp. vii, 49, 51, 61, 101, and 145)

[311] G. Métivier. The mathematics of nonlinear optics. In *Handbook of differential equations: evolutionary equations. Vol. V*, Handb. Differ. Equ., pages 169–313. Elsevier/North-Holland, Amsterdam, 2009. (cit. p. 107)

[312] G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, 158(1):61–90, 2001. (cit. pp. 125 and 131)

[313] A. Mielke. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Methods Appl. Sci.*, 10(1):51–66, 1988. (cit. p. 16)

[314] J. Miles and R. Salmon. Weakly dispersive nonlinear gravity waves. *J. Fluid Mech.*, 157:519–531, 1985. (cit. pp. 90 and 91)

[315] J. W. Miles. On the stability of heterogeneous shear flows. *J. Fluid Mech.*, 10:496–508, 1961. (cit. pp. 76 and 84)

[316] J. W. Miles. On Hamilton's principle for surface waves. *J. Fluid Mech.*, 83(1):153–158, 1977. (cit. pp. 8 and 187)

[317] P. Milewski, E. Tabak, C. Turner, R. Rosales, and F. Menzaque. Nonlinear stability of two-layer flows. *Commun. Math. Sci.*, 2(3):427–442, 2004. (cit. p. 65)

[318] P. A. Milewski, J.-M. Vanden-Broeck, and Z. Wang. Dynamics of steep two-dimensional gravity–capillary solitary waves. *J. Fluid Mech.*, 664:466–477, 2010. (cit. p. 240)

[319] M. Ming and C. Wang. Water waves problem with surface tension in a corner domain I: A priori estimates with constrained contact angle. *SIAM J. Math. Anal.*, 52(5):4861–4899, 2020. (cit. p. 5)

[320] M. Ming and C. Wang. Water-waves problem with surface tension in a corner domain II: the local well-posedness. *Comm. Pure Appl. Math.*, 74(2):225–285, 2021. (cit. p. 5)

[321] D. Mitsotakis, D. Dutykh, and J. Carter. On the nonlinear dynamics of the traveling-wave solutions of the Serre system. *Wave Motion*, 70:166–182, 2017. (cit. pp. 116 and 240)

[322] D. Mitsotakis, D. Dutykh, and J. Carter. On the nonlinear dynamics of the traveling-wave solutions of the Serre system. *Wave Motion*, 70:166–182, 2017.

[323] D. Mitsotakis, C. Synolakis, and M. McGuinness. A modified Galerkin/finite element method for the numerical solution of the Serre-Green-Naghdi system. *Internat. J. Numer. Methods Fluids*, 83(10):755–778, 2017. (cit. p. 116)

[324] M. Miyata. An internal solitary wave of large amplitude. *La mer*, 23(2):43–48, 1985. (cit. p. 208)

[325] M. Miyata. Long internal waves of large amplitude. In *Nonlinear Water Waves: IUTAM Symposium*, pages 399–405, Tokyo, Aug. 1987. Springer. (cit. p. 208)

[326] D. Moldabayev, H. Kalisch, and D. Dutykh. The Whitham equation as a model for surface water waves. *Phys. D*, 309:99–107, 2015. (cit. p. 142)

[327] L. Molinet, R. Talhouk, and I. Zaiter. The classical Boussinesq system revisited. arXiv preprint:2001.11870. (cit. p. 113)

[328] R. Monjarret. Local well-posedness of the multi-layer shallow-water model with free surface. arXiv preprint:1411.2342. (cit. p. 74)

[329] R. Monjarret. Local well-posedness of the two-layer shallow water model with free surface. *SIAM J. Appl. Math.*, 75(5):2311–2332, 2015. (cit. pp. 55 and 60)

[330] H. Montanelli and N. Bootland. Solving periodic semilinear stiff PDEs in 1D, 2D and 3D with exponential integrators. *Math. Comput. Simulation*, 178:307–327, 2020. (cit. p. 234)

[331] R. Montgomery. The present evidence on the importance of lateral mixing processes in the ocean. *Bull. Amer. Meteor. Soc.*, 21(3):87–94, 1940. (cit. p. 86)

[332] G. Moon. *Analysis of Water Waves in the Presence of Geometry and Damping*. PhD thesis, University of North Carolina, Chapel Hill, 2022. (cit. p. 15)

[333] C.-D. Munz. Computational fluid dynamics and aeroacoustics for low Mach number flow. In *Hyperbolic partial differential equations (Hamburg, 2001)*, pages 269–320. Friedr. Vieweg, Braunschweig, 2002. (cit. p. 116)

[334] C.-D. Munz, S. Roller, R. Klein, and K. J. Geratz. The extension of incompressible flow solvers to the weakly compressible regime. *Comput. & Fluids*, 32(2):173–196, 2003. (cit. p. 116)

[335] Y. Murakami and T. Iguchi. Solvability of the initial value problem to a model system for water waves. *Kodai Math. J.*, 38(2):470–491, 2015. (cit. p. 196)

[336] A. Nachbin. A terrain-following Boussinesq system. *SIAM J. Appl. Math.*, 63(3):905–922, 2003. (cit. p. 240)

[337] A. Nachbin and K. Sø lna. Apparent diffusion due to topographic microstructure in shallow waters. *Phys. Fluids*, 15(1):66–77, 2003. (cit. p. 15)

[338] K. Nakayama and T. Kakinuma. Internal waves in a two-layer system using fully nonlinear internal-wave equations. *Internat. J. Numer. Methods Fluids*, 62(5):574–590, 2010. (cit. p. 220)

[339] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974. (cit. p. 15)

[340] A. I. Nekrasov. *Točnaya teoriya voln ustanovivšegocya vida na poverhnosti tyaželoĭ židkosti.* Izdat. Akad. Nauk SSSR, Moscow, 1951. (cit. p. 240)

[341] R. Nemoto and T. Iguchi. Solvability of the initial value problem to the Isobe-Kakinuma model for water waves. *J. Math. Fluid Mech.*, 20(2):631–653, 2018. (cit. pp. 187, 194, 196, 201, and 224)

[342] H. Y. Nguyen and F. Dias. A Boussinesq system for two-way propagation of interfacial waves. *Phys. D*, 237(18):2365–2389, 2008. (cit. p. 215)

[343] D. P. Nicholls. High-order perturbation of surfaces short course: boundary value problems. In *Lectures on the theory of water waves*, volume 426 of *London Math. Soc. Lecture Note Ser.*, pages 1–18. Cambridge Univ. Press, Cambridge, 2016. (cit. p. 151)

[344] D. Nilsson and Y. Wang. Solitary wave solutions to a class of Whitham-Boussinesq systems. *Z. Angew. Math. Phys.*, 70(3):Paper No. 70, 13, 2019. (cit. p. 142)

[345] O. Nwogu. Alternative form of boussinesq equations for nearshore wave propagation. *Journal of waterway, port, coastal, and ocean engineering*, 119(6):618–638, 1993. (cit. p. 157)

[346] M. Oliver. Classical solutions for a generalized Euler equation in two dimensions. *J. Math. Anal. Appl.*, 215(2):471–484, 1997. (cit. p. 114)

[347] P. J. Olver. Hamiltonian and non-Hamiltonian models for water waves. In *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, volume 195 of *Lecture Notes in Phys.*, pages 273–290. Springer, Berlin, 1984. (cit. p. 157)

[348] S. A. Orszag. On the elimination of aliasing in finite-difference schemes by filtering high-wavenumber components. *J. Atmos. Sci.*, 28:1074, 1971. (cit. p. 235)

[349] S. A. Orszag. Comparison of pseudospectral and spectral approximation. *Stud. Appl. Math.*, 51:253–259, 1972. (cit. p. 235)

[350] L. V. Ovsjannikov. To the shallow water theory foundation. *Arch. Mech. (Arch. Mech. Stos.)*, 26:407–422, 1974. (cit. pp. 15 and 240)

[351] L. V. Ovsjannikov. Cauchy problem in a scale of Banach spaces and its application to the shallow water theory justification. In *Applications of methods of functional analysis to problems in mechanics (Joint Sympos., IUTAM/IMU, Marseille, 1975)*, pages 426–437. Lecture Notes in Math., 503. 1976. (cit. p. 15)

[352] L. V. Ovsjannikov. Models of two-layered "shallow water". *Zh. Prikl. Mekh. i Tekhn. Fiz.*, (2):3–14, 180, 1979. (cit. p. 55)

[353] A. K. Pandey. The effects of surface tension on modulational instability in full-dispersion water-wave models. *Eur. J. Mech. B Fluids*, 77:177–182, 2019. (cit. p. 142)

[354] C. Papoutsellis. *Nonlinear water waves over varying bathymetry: theoretical and numerical study using variational methods*. PhD thesis, National Technical University of Athens, 2017. (cit. p. 151)

[355] C. E. Papoutsellis and G. A. Athanassoulis. A new efficient hamiltonian approach to the nonlinear water-wave problem over arbitrary bathymetry. arXiv preprint:1704.03276. (cit. pp. 186, 187, and 202)

[356] L. Pei and Y. Wang. A note on well-posedness of bidirectional Whitham equation. *Appl. Math. Lett.*, 98:215–223, 2019. (cit. p. 142)

[357] M. Petcu, R. M. Temam, and M. Ziane. Some mathematical problems in geophysical fluid dynamics. In *Handbook of numerical analysis. Vol. XIV. Special volume: computational methods for the atmosphere and the oceans*, volume 14 of *Handb. Numer. Anal.*, pages 577–750. Elsevier/North-Holland, Amsterdam, 2009. (cit. p. 78)

[358] J. P. A. Pitt, C. Zoppou, and S. G. Roberts. Behaviour of the Serre equations in the presence of steep gradients revisited. *Wave Motion*, 76:61–77, 2018. (cit. p. 116)

[359] P. I. Plotnikov. Proof of the Stokes conjecture in the theory of surface waves. *Stud. Appl. Math.*, 108(2):217–244, 2002. Translated from Dinamika Sploshn. Sredy No. 57 (1982), 41–76 [MR0752600 (85f:76036)]. (cit. p. 16)

[360] P. I. Plotnikov and J. F. Toland. Nash-Moser theory for standing water waves. *Arch. Ration. Mech. Anal.*, 159(1):1–83, 2001. (cit. p. 16)

[361] T. de Poyferré. A priori estimates for water waves with emerging bottom. *Arch. Ration. Mech. Anal.*, 232(2):763–812, 2019. (cit. p. 5)

[362] J. W. S. Rayleigh. On waves. *Philos. Mag.*, 1(5):251–271, 1876. (cit. pp. 151 and 153)

[363] M. Renardy. Ill-posedness of the hydrostatic Euler and Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 194(3):877–886, 2009. (cit. p. 84)

[364] G. L. Richard. An extension of the Boussinesq-type models to weakly compressible flows. *Eur. J. Mech. B Fluids*, 89:217–240, 2021. (cit. p. 117)

[365] P. Ripa. General stability conditions for a multi-layer model. *J. Fluid Mech.*, 222:119–137, 1991. (cit. p. 74)

[366] R. R. Rosales and G. C. Papanicolaou. Gravity waves in a channel with a rough bottom. *Stud. Appl. Math.*, 68(2):89–102, 1983. (cit. p. 15)

[367] A. Ruiz de Zárate, D. G. A. Vigo, A. Nachbin, and W. Choi. A higher-order internal wave model accounting for large bathymetric variations. *Stud. Appl. Math.*, 122(3):275–294, 2009. (cit. p. 15)

[368] Y. Saad and M. H. Schultz. GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.*, 7(3):856–869, 1986. (cit. p. 239)

[369] A. R. Said. A geometric proof of the quasi-linearity of the water-waves system. arXiv preprint:2002.02940. (cit. p. 15)

[370] B. de Saint-Venant. Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. *C.R. Acad. Sci. Paris*, 73:147–154, 1871. (cit. p. vii)

[371] R. Salmon. Hamiltonian fluid mechanics. *Annual Review of Fluid Mechanics*, 20(1):225–256, 1988. (cit. p. 91)

[372] C. E. Sánchez, E. D. Fernández-Nieto, T. M. de Luna, Y. Penel, and J. Sainte-Marie. Numerical Simulations of a Dispersive Model Approximating Free-Surface Euler Equations. *J. Sci. Comput.*, 89(3):Paper No. 55, 2021. (cit. p. 183)

[373] J.-C. Saut. *Asymptotic models for surface and internal waves*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2013. 29o Colóquio Brasileiro de Matemática. [29th Brazilian Mathematics Colloquium]. (cit. pp. i, 26, 69, and 219)

[374] J.-C. Saut and Y. Wang. The wave breaking for Whitham-type equations revisited. *SIAM J. Math. Anal.*, 54(2):2295–2319, 2022. (cit. pp. xi, 89, and 147)

[375] J.-C. Saut and L. Xu. Long time existence for a two-dimensional strongly dispersive Boussinesq system. *Comm. Partial Differential Equations*, 2021. (cit. pp. ix and x)

[376] H. A. Schäffer. Comparison of dirichlet–neumann operator expansions for nonlinear surface gravity waves. *Coastal Engineering*, 55(4):288–294, 2008. (cit. p. 151)

[377] T. Schneider, N. Botta, K. J. Geratz, and R. Klein. Extension of finite volume compressible flow solvers to multi-dimensional, variable density zero Mach number flows. *J. Comput. Phys.*, 155(2):248–286, 1999. (cit. p. 116)

[378] S. Schochet. Symmetric hyperbolic systems with a large parameter. *Comm. Partial Differential Equations*, 11(15):1627–1651, 1986. (cit. pp. 125, 127, 128, and 129)

[379] S. Schochet. The mathematical theory of low Mach number flows. *M2AN Math. Model. Numer. Anal.*, 39(3):441–458, 2005. (cit. pp. 59, 68, 114, and 125)

[380] M. E. Schonbek. Existence of solutions for the Boussinesq system of equations. *J. Differential Equations*, 42(3):325–352, 1981. (cit. p. 113)

[381] L. W. Schwartz. Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.*, 62(3):553–578, 1974. (cit. p. 240)

[382] F. J. Seabra-Santos, D. P. Renouard, and A. M. Temperville. Numerical and experimental study of the transformation of a solitary wave over a shelf or isolated obstacle. *J. Fluid Mech.*, 176:117–134, 3 1987. (cit. p. 91)

[383] F. Serre. Contribution à l'étude des écoulements permanents et variables dans les canaux. *La Houille Blanche*, (6):830–872, 1953. (cit. pp. 91 and 97)

[384] J. Shatah and C. Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008. (cit. p. 28)

[385] J. Shatah and C. Zeng. Local well-posedness for fluid interface problems. *Arch. Ration. Mech. Anal.*, 199(2):653–705, 2011. (cit. p. 28)

[386] T. G. Shepherd. Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics*, 32:287–338, 1990. (cit. p. 49)

[387] S. Shkoller and T. C. Sideris. Global existence of near-affine solutions to the compressible Euler equations. *Arch. Ration. Mech. Anal.*, 234(1):115–180, 2019. (cit. p. 5)

[388] B. Simon. Sturm oscillation and comparison theorems. In *Sturm-Liouville theory*, pages 29–43. Birkhäuser, Basel, 2005. (cit. p. 73)

[389] A. Stefanov and J. D. Wright. Small amplitude traveling waves in the full-dispersion Whitham equation. *J. Dynam. Differential Equations*, 32(1):85–99, 2020. (cit. p. 143)

[390] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. (cit. pp. vi and 11)

[391] A. L. Stewart and P. J. Dellar. Multilayer shallow water equations with complete coriolis force. part 3. hyperbolicity and stability under shear. *J. Fluid Mech.*, 723:289–317, 5 2013. (cit. p. 73)

[392] J. J. Stoker. The formation of breakers and bores. The theory of nonlinear wave propagation in shallow water and open channels. *Communications on Appl. Math.*, 1:1–87, 1948. (cit. pp. 15, 151, and 153)

[393] G. G. Stokes. On the theory of oscillatory waves. *Trans. Cambridge Philos. Soc.*, 8:441–455, 1847. (cit. p. xi)

[394] D. J. Struik. Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie. *Math. Ann.*, 95(1):595–634, 1926. (cit. pp. 16 and 240)

[395] C. Sturm. Mémoire sur les équations différentielles linéaires du second ordre. *J. Math. Pures Appl. (1)*, 1:106–186, 1834. (cit. p. 75)

[396] C. H. Su and C. S. Gardner. Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation. *J. Mathematical Phys.*, 10:536–539, 1969. (cit. p. 91)

[397] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis. (cit. pp. 11 and 101)

[398] T. Tao. Incompressible fluid equations, 2018. Blog posts available at https://terrytao.wordpress.com/category/teaching/254a-incompressible-fluid-equations/. (cit. p. 107)

[399] M. E. Taylor. *Partial differential equations. III Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997. (cit. p. 251)

[400] J. F. Toland. On the existence of a wave of greatest height and Stokes's conjecture. *Proc. Roy. Soc. London Ser. A*, 363(1715):469–485, 1978. (cit. p. 16)

[401] L. N. Trefethen. *Spectral methods in MATLAB*, volume 10 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. (cit. pp. 234 and 236)

[402] T. Truong, E. Wahlén, and M. H. Wheeler. Global bifurcation of solitary waves for the Whitham equation. *Math. Ann.*, 2021. (cit. pp. xi, 89, and 147)

[403] S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.*, 26(2):323–331, 1986. (cit. p. 115)

[404] R. M. Vargas-Magaña and P. Panayotaros. A Whitham-Boussinesq long-wave model for variable topography. *Wave Motion*, 65:156–174, 2016. (cit. p. 147)

[405] R. M. Vargas-Magaña, P. Panayotaros, and A. A. Minzoni. Linear modes for channels of constant cross-section and approximate Dirichlet–Neumann operators. *Water Waves*, 1:343–370, 2019. (cit. p. 147)

[406] C. Viotti, D. Dutykh, and F. Dias. The conformal-mapping method for surface gravity waves in the presence of variable bathymetry and mean current. *Procedia IUTAM*, 11:110–118, 2014. (cit. p. 240)

[407] F. d. M. Viríssimo and P. A. Milewski. Three-layer flows in the shallow water limit. *Stud. Appl. Math.*, 142(4):487–512, 2019. (cit. p. 76)

[408] F. d. M. Viríssimo and P. A. Milewski. Nonlinear stability of two-layer shallow water flows with a free surface. *Proc. A.*, 476(2236):20190594, 20, 2020. (cit. pp. 55 and 76)

[409] Y. Wang. Well-posedness to the cauchy problem of a fully dispersive boussinesq system. *J. Dynam. Differential Equations*, pages 1–12, 2020. (cit. p. 142)

[410] G. Wei, J. T. Kirby, S. T. Grilli, and R. Subramanya. A fully nonlinear Boussinesq model for surface waves. I. Highly nonlinear unsteady waves. *J. Fluid Mech.*, 294:71–92, 1995. (cit. p. 91)

[411] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, and R. L. Milton. A new numerical method for surface hydrodynamics. *J. Geophys. Res.*, 92:11803–11824, 1987. (cit. p. 151)

[412] G. B. Whitham. Variational methods and applications to water waves. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 299, 06 1967. (cit. pp. xi and 90)

[413] G. B. Whitham. *Linear and nonlinear waves*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication. (cit. p. xi)

[414] J. Wilkening and V. Vasan. Comparison of five methods of computing the Dirichlet-Neumann operator for the water wave problem. In *Nonlinear wave equations: analytic and computational techniques*, volume 635 of *Contemp. Math.*, pages 175–210. Amer. Math. Soc., Providence, RI, 2015. (cit. pp. 151 and 242)

[415] S. Wu. The quartic integrability and long time existence of steep water waves in 2D. arXiv preprint:2010.09117. (cit. p. 240)

[416] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997. (cit. p. 240)

[417] S. Wu. Mathematical analysis of vortex sheets. *Comm. Pure Appl. Math.*, 59(8):1065–1206, 2006. (cit. p. 28)

[418] T. Y. Wu. A unified theory for modeling water waves. *Adv. in Appl. Mech.*, 37:1–88, 2001. (cit. pp. 153 and 157)

[419] Z. Xu. *Asymptotic analysis and numerical analysis of the Benjamin-Ono equation*. PhD thesis, University of Michigan, 2010. (cit. pp. 234 and 236)

[420] Y. J. Yang, Y. C. Fang, T. Y. Tang, and S. R. Ramp. Convex and concave types of second baroclinic mode internal solitary waves. *Nonlin. Processes Geophys.*, 17(6):605–614, 2010. (cit. pp. 73 and 76)

[421] M. L. Yates and M. Benoit. Accuracy and efficiency of two numerical methods of solving the potential flow problem for highly nonlinear and dispersive water waves. *Internat. J. Numer. Methods Fluids*, 77(10):616–640, 2015. (cit. p. 202)

[422] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982. (cit. p. 15)

[423] J. Yu and L. N. Howard. Exact Floquet theory for waves over arbitrary periodic topographies. *J. Fluid Mech.*, 712:451–470, 2012. (cit. p. 147)

[424] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.*, 9:190–194, 1968. (cit. pp. 7 and 221)

[425] V. E. Zakharov. Benney equations and quasiclassical approximation in the inverse problem method. *Funktsional. Anal. i Prilozhen.*, 14(2):15–24, 1980. (cit. p. 79)

[426] V. E. Zakharov, A. I. Dyachenko, and O. A. Vasilyev. New method for numerical simulation of a nonstationary potential flow of incompressible fluid with a free surface. *Eur. J. Mech. B Fluids*, 21(3):283–291, 2002. (cit. p. 240)

[427] Z. L. Zou, K. Z. Fang, and Z. B. Liu. Inter-comparisons of different forms of higher-order Boussinesq equations. In Q. Ma, editor, *Advances in Numerical Simulation of Nonlinear Water Waves*, chapter 8, pages 287–323. World Scientific, 2010. (cit. p. 170)