Shallow-water models for water waves

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June 20, 2020

Abstract

These lecture notes were used during graduate courses taught in 2019 and 2020. They deal with the motion of surface gravity waves, that is the evolution under the effect of gravity of a layer of a fluid (typically water) delimited above by a free surface. The emphasis is on the derivation and rigorous justification, starting from the irrotational full Euler equations (also known as the water waves system), of shallow water models, specifically the Saint-Venant and Green–Naghdi equations.

These notes might be useful for someone seeking a briefer (since less detailed) exposition than in David Lannes' book, [Lan13]. It also includes a detailed proof of the well-posedness theory for the Green–Naghdi equations sketched in [FI15]. They shall be embedded in a larger project "soon".

Contents

1 The "master" equations

Much of the material of this section is borrowed from [Lan13], and concision has been pursued.

1.1 The full Euler equations

Let us introduce the equations which will serve as a reference for any other models in this manuscript. These equations are meant to predict the evolution of an infinite layer of a fluid (typically water) delimited above by a free surface and below by a rigid bottom under the effect of gravity. We will always assume that the upper surface and lower bottom of the fluid can be described through the graph of regular functions –so that no surging waves are allowed– and as such we can denote the domain of the fluid (see Figure 1) as

$$\Omega_t \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ -H + b(t, \boldsymbol{x}) < z < \zeta(t, \boldsymbol{x})\}.$$

As apparent in this definition, $\boldsymbol{x} \in \mathbb{R}^d$ with $d \in \{1, 2\}$ (when the dimension is prescribed we shall denote $\boldsymbol{x} = x$ when $d = 1$ and $\boldsymbol{x} = (x, y)$ when $d = 2$) is the horizontal variable and z the vertical variable. We represent the depth at rest by $H > 0$. We also denote the bottom topography and the surface deformation

$$\Gamma_{\text{bot}} \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = -H + b(t, \boldsymbol{x})\},$$
$$\Gamma_{\text{top}} \stackrel{\text{def}}{=} \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \zeta(t, \boldsymbol{x})\},$$

although we occasionally misname the surface deformation as the function ζ instead of Γ_{top} and the bottom topography as b instead of Γ_{bot}.

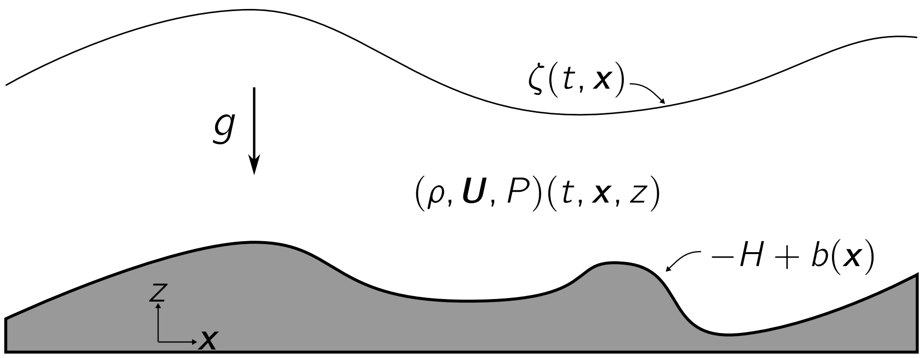


Figure 1: Sketch of the domain and notations

The main goal is to predict the evolution of the surface deformation, Γ_{top}, together with the velocity field inside the layer. To this aim, we introduce the following set of equations

$$\partial_t \rho + \nabla_{\boldsymbol{x},z} \cdot (\rho \boldsymbol{U}) = 0 \qquad \text{in } \Omega_t, \tag{1.1a}$$
$$\rho \partial_t \boldsymbol{U} + \rho(\boldsymbol{U} \cdot \nabla_{\boldsymbol{x},z})\boldsymbol{U} = -\nabla_{\boldsymbol{x},z} P - \rho g \mathbf{e}_z \qquad \text{in } \Omega_t, \tag{1.1b}$$
$$\operatorname{div} \boldsymbol{U} = 0 \qquad \text{in } \Omega_t, \tag{1.1c}$$
$$\partial_t \zeta = w - \nabla\zeta \cdot U \qquad \text{on } \Gamma_{\text{top}}, \tag{1.1d}$$
$$\partial_t b = w - \nabla b \cdot U \qquad \text{on } \Gamma_{\text{bot}}, \tag{1.1e}$$
$$P - P_{\text{atm}} = -\sigma \nabla \cdot \left(\frac{\nabla \zeta}{\sqrt{1 + |\nabla\zeta|^2}} \right) \qquad \text{on } \Gamma_{\text{top}}, . \tag{1.1f}$$

The first three equations are the Euler equations for incompressible fluids, and represent the *conservation of mass, of momentum and the incompressibility constraint.* The fourth one is called the *kinematic boundary condition* and corresponds to the assumption that no fluid particle shall cross the surface (in fact fluid particles at the surface are forever "trapped" at the surface). Similarly, the subsequent one is the *impermeability condition* ensuring that no fluid particle shall cross the bottom. We assume that the pressure jump at the surface is proportional to the mean curvature of the surface, with the constant σ denoting the *surface tension* coefficient. Finally, we impose that the density does not vanish on the fluid domain or on the boundaries and enforce the *non-cavitation assumption*, *i.e.* the depth of the layer nowhere vanishes.

Here, $\rho(t, \boldsymbol{x}, z)$ is the fluid mass density at time t and position $(\boldsymbol{x}, z)$. We denote by $\boldsymbol{U}(t, \boldsymbol{x}, z)$ the velocity of the fluid particle at time t and position $(\boldsymbol{x}, z)$. We denote $\nabla_{\boldsymbol{x},z}$ the $(d+1)$-dimensional gradient operator while $\nabla = \nabla_{\boldsymbol{x}}$ is the horizontal gradient operator. P denotes the pressure inside the fluid; it is not an unknown but rather the Lagrange multiplier associated with the incompressibility constraint, eq. (1.1c), and can be deduced from other unknowns at any time instant by solving the equation obtained when taking the divergence of eq. (1.1b). Finally, $P_{\rm atm}$ is the (prescribed) atmospheric pressure at the surface, g is the (constant) acceleration of gravity, and $\mathbf{e}_z$ is the vertical upward unit vector.

Additional assumptions Many assumptions were made so as to write eq. (1.1), and we shall add other important ones even before we move towards the derivation of asymptotic models. For instance we neglected the effects of compressibility, viscosity, and friction at the bottom. This is motivated by the fact that when considering a large body of water with relatively mild behavior, these effects are expected to have almost no contribution on the evolution of the flow. We have also neglected the Coriolis effect, as well as the curvature of earth. This assumption is valid provided we consider a body of water which is not too large. Hence our framework is restricted between two extremes, the rule of thumb being that we describe waves that a human eye can see (see [Lan13] for a more detailed and quantitative discussion). As our aim is to highlight only the relevant mechanism in the propagation of surface gravity waves, it makes sense to discard as early as possible any unnecessary complexities. In the same spirit, we shall discard the surface tension effects:

$$\sigma = 0, \ \ \textit{i.e. } P = P_{\rm atm} \qquad\qquad \text{on } \Gamma_{\rm top}. \tag{1.2a}$$

However it turns out that the surface tension component, although *a priori* negligible, has very important theoretical consequences for some problems because it strongly modifies the high-frequency behavior of the equations (in particular the linear group and phase velocity are no longer bounded and decreasing with the size of the wavenumber). This has strong consequences for instance when looking for traveling waves solutions, or for the well-posedness theory in the bilayer setting. This being said, we will use eq. (1.2a) for the sake of concision when deriving models; a version of the models with surface tension effects are always easy to deduce.

We will also assume thereafter that the bottom is time-independent:

$$\partial_t b = 0 \qquad\qquad \text{on } \Gamma_{\rm bot}, \tag{1.2b}$$

and that the atmospheric pressure at the surface is uniform in space,

$$\nabla P_{\rm atm} = 0 \qquad\qquad \text{on } \Gamma_{\rm top}. \tag{1.2c}$$

The above assumptions are made for concision and clarity, but again it is not difficult to add –at least formally– the effects of atmospheric or topographic changes in the models, which can be then used for studying the generation of waves in addition to their propagation. We refer for instance to [FI15, Mel] for such study.

Such is not the case concerning the following assumptions which will be very important for the mathematical analysis: we assume that there exist $\rho_\star$ and $H_\star$ positive constants such that

$$\rho \geq \rho_\star > 0 \qquad \text{in } \Omega_t, \tag{1.2d}$$

$$H + \zeta - b \geq H_\star > 0 \qquad \text{in } \mathbb{R}^d. \tag{1.2e}$$

The former assumption is quite natural in the oceanographic context, but not if eq. (1.1) is applied to the atmospheric motion. The non-cavitation assumption, eq. (1.2e), is much more stringent to our framework, and prevents any study near the shore, and in particular shoaling effects. We refer to [LM18, dP16] for some works dealing with this situation. We mention here that our unknowns, and in particular the surface deformation, ζ, will be assumed to vanish at infinity through finite energy assumptions:

$$|\zeta|, |\boldsymbol{U}| \to 0, \quad \rho(\cdot, z) \to \underline{\rho}(z) \qquad \text{as } |\mathbf{x}| \to \infty. \tag{1.2f}$$

The results extend to the periodic setting with almost no modifications (H is then the layer-depth average ensuring that ζ, b are mean-zero), but we expect they may also be extended to the more relevant Kato's uniformly local Sobolev spaces; see [ABZ16]. We will also assume sufficient regularity on all the variables so that all the identities above hold on the classical, pointwise sense.

Let us now introduce the two crucial (and arguable) additional assumptions from which the so-called water-waves system is derived. We shall, unless otherwise specified, assume that the fluid is homogeneous, *i.e.* there exists a constant $\rho_0 > 0$ such that

$$\rho \equiv \rho_0 \qquad \text{in } \Omega_t. \tag{1.2g}$$

This assumption needs only to be made initially in time, as it is automatically propagated for positive times thanks to the mass conservation, eq. (1.1a) and incompressibility constraint, eq. (1.1c).

Finally there is one last very important assumption: we shall restrict ourselves to irrotational (or potential) flows, namely

$$\operatorname{rot} \boldsymbol{U} = \mathbf{0} \qquad \text{in } \Omega_t. \tag{1.2h}$$

In the homogeneous setting, because all the forces in the right-hand side of eq. (1.1b) are potential, the irrotational assumption, eq. (1.2h) needs only to be set initially, and it is automatically propagated by the equations for positive times. Restricting the motions to homogeneous potential flows turns out to be an extremely rewarding assumption, as it allows to rewrite the entire set of equations as only two scalar evolution equations for unknowns depending only the time and horizontal space variables. This striking reduction, which is described in the following Section, should be seen as a warning that much of the diversity of the waves of the original system, eq. (1.1), has been discarded through the assumptions of eq. (1.2g) and eq. (1.2h).

1.2 The water-waves system

We shall rewrite in this section the full Euler system, eq. (1.1), in the homogeneous –eq. (1.2g)– and irrotational –eq. (1.2h)– framework as a simple-looking system of two scalar evolution equations. This is the so-called Zakharov-Craig-Sulem formulation which we will refer to simply as *the water-waves system*. The irrotationality assumption induces

$$\boldsymbol{U} = \nabla_{\boldsymbol{x},z} \Phi \qquad \text{in } \Omega_t, \tag{1.2h$'$}$$

where $\Phi(t, \boldsymbol{x}, z) \in \mathbb{R}$ is the velocity potential, defined by $\boldsymbol{U}$ up to a time-dependent additive constant. We can then rewrite the momentum equation and incompressibility constraint in terms of the velocity potential:

$$\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 = -\frac{1}{\rho_0}(P - P_{\text{atm}}) - gz \qquad \text{in } \Omega_t, \tag{1.1b$'$}$$

$$\Delta_{\boldsymbol{x},z} \Phi = 0 \qquad \text{in } \Omega_t, \tag{1.1c$'$}$$

The former is called the Bernoulli equation. As it has been obtained from an integration in space, it should include a time-dependent source term, which has been set to zero by choosing suitably the time-dependent additive constant in Φ. The latter equation is of course Laplace's equation, hence the potential is harmonic. With this in mind, we introduce its trace at the surface,

$$\psi(t, \boldsymbol{x}) \stackrel{\text{def}}{=} \Phi(t, \boldsymbol{x}, \zeta(t, \boldsymbol{x})),$$

and notice that the velocity potential, Φ, is uniquely determined (under reasonable hypotheses, see below) by the knowledge of (ζ, b, ψ) after solving

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi = 0 & \text{in } \Omega_t, \\ \Phi = \psi & \text{on } \Gamma_{\text{top}}, \\ \partial_z\Phi - \nabla b \cdot \nabla\Phi = 0 & \text{on } \Gamma_{\text{bot}}, \end{cases} \tag{1.3}$$

the last equation being provided by eq. (1.1e). The following result is standard in the theory of elliptic operators.

Proposition 1.1. *Let $(\zeta, b) \in W^{2,\infty}(\mathbb{R}^d)$ such that eq. (1.2e) holds. Then for any $\psi \in \mathring{H}^2(\mathbb{R}^d)$, there exists a unique $\Phi \in \mathring{H}^2(\Omega_t)$ strong solution to eq. (1.3).*

Following Craig, Sulem and Sulem [CSS92, CS93], it is then convenient to introduce the Dirichlet-to-Neumann operator:

Definition 1.2 (Dirichlet-to-Neumann operator)**.** *Under the assumptions of Proposition 1.1, the Dirichlet-to-Neumann operator*

$$\mathcal{G}[\zeta, b] : \begin{array}{ccc} \mathring{H}^2(\mathbb{R}^d) & \to & H^{1/2}(\mathbb{R}^d) \\ \psi & \mapsto & (\partial_z\Phi - \nabla\zeta \cdot \nabla\Phi)\big|_{z=\zeta} \end{array}$$

where $\Phi \in \mathring{H}^2(\Omega_t)$ is the solution to eq. (1.3) provided by Proposition 1.1, is well-defined and continuous. If, moreover, $\zeta, b, \psi \in \mathring{H}^{2+s_\star}(\mathbb{R}^d)$ with $s_\star > d/2$, then $\mathcal{G}[\zeta, b]\psi \in H^{s_\star}(\mathbb{R}^d) \subset \mathcal{C}^0(\mathbb{R}^d)$.

We provide a proof of these Propositions in Section 2. Sharper results are provided in [Lan13] together with a thorough description of many properties of the Dirichlet-to-Neumann operator. Let us collect some of them for future reference.

Proposition 1.3. *Under the assumptions of Proposition 1.1, for any $\psi_1, \psi_2 \in \mathring{H}^2(\mathbb{R}^d)$, the Dirichlet-to-Neumann operator satisfies the following*

- *Relation with layer-averaged velocity:*

$$\mathcal{G}[\zeta, b]\psi = -\nabla \cdot (h\overline{\boldsymbol{u}})$$

where $h = H + \zeta - b$ and $\overline{\boldsymbol{u}} \stackrel{\text{def}}{=} \frac{1}{h}\int_{-H+b}^{\zeta} \nabla\Phi \, \mathrm{d}z$. In particular, $\mathcal{G}[\zeta, b]\psi \in (\mathring{H}^2(\mathbb{R}^d))'$.

- *Symmetry*

$$\langle \psi_1, \mathcal{G}[\zeta, b]\psi_2 \rangle_{\mathring{H}^2-(\mathring{H}^2)'} = \langle \psi_2, \mathcal{G}[\zeta, b]\psi_1 \rangle_{\mathring{H}^2-(\mathring{H}^2)'}.$$

- *Positivity*

$$\langle \psi, \mathcal{G}[\zeta, b]\psi \rangle_{\mathring{H}^2-(\mathring{H}^2)'} \approx \big|\Lambda^{-1/2}\nabla\psi\big|^2_{L^2(\mathbb{R}^d)}.$$

- *Shape derivative:*

$$\mathrm{d}_\zeta\mathcal{G}[\zeta, b](\delta\zeta)\psi = -\mathcal{G}[\zeta, b]((\delta\zeta)\underline{w}) - \nabla \cdot ((\delta\zeta)\underline{\boldsymbol{u}})$$

where $\mathrm{d}_\zeta\mathcal{G}[\zeta, b](\delta\zeta)\psi$ is the derivative of the mapping $\zeta \mapsto \mathcal{G}[\zeta, b]\psi$ in the direction $\delta\zeta$, and we denote $\underline{w} = \frac{\mathcal{G}[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi}{1+|\nabla\zeta|^2}$ and $\underline{\boldsymbol{u}} = \nabla\psi - w\nabla\zeta$. One easily checks, by the above identity and chain rules, that $\boldsymbol{U}\big|_{z=\zeta} = (\nabla_{\boldsymbol{x},z}\Phi)\big|_{z=\zeta} = (\underline{\boldsymbol{u}}, \underline{w})$.

By the use of the chain rule, we can now rewrite the (trace at the surface of the) Bernoulli equation, eq. (1.1b′), as well as the kinematic boundary condition at the surface, eq. (1.1d):

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta, b]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \dfrac{(\mathcal{G}[\zeta, b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{cases} \tag{1.4}$$

We call the closed set of equations (1.4) the *water-waves system* in order to distinguish it from the *full Euler equations*, eq. (1.1). It is easy to see, following the lines above, that any sufficiently regular solution to the full Euler (or Bernoulli) equations, (1.1),(1.2), satisfies the water-waves system, eq. (1.4). The converse can also be verified. The analysis has been detailed even for mildly regular data –and in particular a very rough topography– in [ABZ13].

1.3 Variational structure

A remarkable property of the system (1.4) as put forward by Zakharov [Zak68] is its canonical Hamiltonian structure. Indeed, define the Hamiltonian as the total energy, summing up the potential and kinetic energies

$$\begin{aligned} \mathcal{H}(\zeta,\psi) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \psi\mathcal{G}[\zeta,b]\psi \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d}\int_{-H+b}^{\zeta} gz + \frac{1}{2}|\nabla_{\boldsymbol{x},z}\Phi|^2 \,\mathrm{d}z - \frac{1}{2}(H-b)^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Then one can show (using Proposition 1.3) that eq. (1.4) reads

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H} \\ \delta_\psi \mathcal{H} \end{pmatrix}$$

where $\delta_\zeta\mathcal{H}$ and $\delta_\psi\mathcal{H}$ denote the functional derivatives: for instance

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \lim_{\epsilon\to 0} \frac{\mathcal{H}(\zeta,\psi+\epsilon\varphi)-\mathcal{H}(\zeta,\psi)}{\epsilon} = \int_{\mathbb{R}^d} (\delta_\psi\mathcal{H})\varphi \,\mathrm{d}\boldsymbol{x}.$$

We may associate a Lagrangian formalism to the Hamiltonian structure: define

$$\mathcal{L}_Z = \int_{\mathbb{R}^d} \psi\partial_t\zeta \,\mathrm{d}\boldsymbol{x} - \mathcal{H}(\zeta,\psi).$$

Then we see that the water-waves system, eq. (1.4) follows from Hamilton's principle

$$\delta \int_{t_0}^{t_1} \mathcal{L}_Z \,\mathrm{d}t = 0.$$

One may notice that, using the conservation of mass as a constraint, one can rewrite the Lagrangian as the *difference* between the kinetic and the potential energies.

The Hamiltonian formulation is quite handy to quickly derive asymptotic models, which then enjoy by construction a Hamiltonian structure as well. We shall however not follow this path due to the formal nature of such a derivation, and we refer to [CD12, CDM17] for more details.

One of the nice outcomes of the Hamiltonian structure is that it relates, through Noether's theorem, symmetry groups and conserved quantities of the system.

Group symmetries Some relevant group symmetries are as follows. If (ζ,ψ) is a solution to eq. (1.4), then for any $\theta\in\mathbb{R}$, $(\zeta^\theta,\psi^\theta)$ also satisfies eq. (1.4), where

- Variation of base level for the velocity potential
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta,\psi+\theta\big)(t,\boldsymbol{x}).$$
- Horizontal translation along the direction $\boldsymbol{v}$ (in the flat bottom case)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta,\psi\big)(t,\boldsymbol{x}-\theta\boldsymbol{v}).$$
- Time translation
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta,\psi\big)(t-\theta,\boldsymbol{x}).$$
- Galilean boost along the direction $\boldsymbol{v}$ (in the flat bottom case)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta,\psi+\theta\boldsymbol{v}\cdot\boldsymbol{x}\big)(t,\boldsymbol{x}-\theta\boldsymbol{v}t).$$
- Horizontal rotation (in dimension $d=2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp\cdot\nabla b=0$)
$$\big(\zeta^\theta,\psi^\theta\big)(t,\boldsymbol{x}) \stackrel{\text{def}}{=} \big(\zeta,\psi\big)(t,R_\theta\boldsymbol{x})$$
where R_θ is the rotation matrix of angle θ.

Preserved quantities We have the following related preserved quantities.

- Mass
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}=0, \qquad \mathcal{Z}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\zeta\,\mathrm{d}\boldsymbol{x}.$$
- Horizontal impulse (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}=0, \qquad \mathcal{I}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\zeta\nabla\psi\,\mathrm{d}\boldsymbol{x}.$$
- Total energy
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}=0.$$
- Horizontal coordinate of mass centroid times mass (in the flat bottom case)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}=\int_{\mathbb{R}^d}\zeta\nabla\psi\,\mathrm{d}\boldsymbol{x}, \qquad \mathcal{C}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\zeta\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}.$$
- Angular impulse (in dimension $d=2$ and for a rotation-invariant bottom, $\boldsymbol{x}^\perp\cdot\nabla b=0$)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}=0, \qquad \mathcal{A}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\zeta\boldsymbol{x}^\perp\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}.$$
where $(x,y)^\perp\stackrel{\text{def}}{=}(-y,x)$.

The horizontal impulse and horizontal momentum are directly related after integration by parts: for instance in dimension $d=1$
$$\mathcal{M}\stackrel{\text{def}}{=}\int_{\mathbb{R}^d}\int_{-H+b}^{\zeta}\nabla\Phi\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x}=\mathcal{I}+\lim_{x\to+\infty}\int_{-H+b}^{\zeta}\Phi\,\mathrm{d}z-\lim_{x\to-\infty}\int_{-H+b}^{\zeta}\Phi\,\mathrm{d}z,$$
and the latter terms are time-independent (but do not necessarily vanish) as a consequence of the Bernoulli equation and our boundary conditions. The quantities are preserved in a stronger sense: their integrand satisfies a *conservation law*, which we do not write out explicitly. We let the reader refer to [BO82] for a full account on symmetry groups and conserved quantities of the full Euler system.

1.4 Linear analysis

The system (1.4) linearized around the trivial solution, $(\zeta = 0, \psi = 0)$ –and hence also around $(\zeta = 0, \psi = \boldsymbol{v} \cdot \boldsymbol{x})$, where $\boldsymbol{v} \in \mathbb{R}^d$ is constant, by Galilean invariance– is explicitly solvable in the flat-bottom case. Indeed, setting $\zeta = \epsilon\zeta_0, \psi = \epsilon\psi_0$ and $b = 0$, keeping only first-order terms in terms of the small ϵ, one is left with the system

$$\begin{cases} \partial_t\zeta_0 - \mathcal{G}[0,0]\psi_0 = 0, \\ \partial_t\psi_0 + g\zeta_0 = 0 \end{cases} \tag{1.5}$$

where $\mathcal{G}[0,0]\psi_0 = (\partial_z\Phi_0)\,|_{z=0}$ and Φ_0 is the unique solution to

$$\begin{cases} \Delta_{\boldsymbol{x},z}\Phi_0 = 0 & \text{in } \mathbb{R}^d \times (-H, 0), \\ \Phi_0 = \psi_0 & \text{on } \mathbb{R}^d \times \{0\}, \\ \partial_z\Phi_0 = 0 & \text{on } \mathbb{R}^d \times \{-H\}, \end{cases} \tag{1.6}$$

Remark 1.4. *One can give a rigorous ground of the above vague statement and rigorously justify eq. (1.5) as an asymptotic model in the small-amplitude regime, similarly to the shallow-water asymptotic models studied thereafter.*

One can "explicitly" solve the Laplace problem, eq. (1.6), using the Fourier transform:

$$\Phi_0 = \frac{\cosh((z+H)|D|)}{\cosh(H|D|)}\psi_0$$

and hence

$$\mathcal{G}[0,0]\psi_0 = |D|\tanh(H|D|)\psi.$$

Here and thereafter, we use the convention for the Fourier multiplier operator defined by

$$\widehat{f(D)\varphi}(\boldsymbol{\xi}) = f(\boldsymbol{\xi})\widehat{\varphi}(\boldsymbol{\xi}),$$

where the Fourier transform is applied only on the horizontal variable. Another standard notation is $|D| = (-\Delta)^{1/2}$.

Now returning to the time-evolution equations, eq. (1.5), we deduce the dispersion relation

$$\omega(\boldsymbol{\xi})^2 = g|\boldsymbol{\xi}|\tanh(H|\boldsymbol{\xi}|)$$

and the expressions

$$\begin{pmatrix} \widehat{\zeta}_0(t,\boldsymbol{\xi}) \\ \widehat{\psi}_0(t,\boldsymbol{\xi}) \end{pmatrix} = \exp(L_0 t)\begin{pmatrix} \widehat{\zeta}_0(0,\boldsymbol{\xi}) \\ \widehat{\psi}_0(0,\boldsymbol{\xi}) \end{pmatrix} = \begin{pmatrix} \cos(|\omega(\boldsymbol{\xi})|t) & \frac{|\omega(\boldsymbol{\xi})|}{g}\sin(|\omega(\boldsymbol{\xi})|t) \\ -\frac{g}{|\omega(\boldsymbol{\xi})|}\sin(|\omega(\boldsymbol{\xi})|t) & \cos(|\omega(\boldsymbol{\xi})|t) \end{pmatrix}\begin{pmatrix} \widehat{\zeta}_0(0,\boldsymbol{\xi}) \\ \widehat{\psi}_0(0,\boldsymbol{\xi}) \end{pmatrix}.$$

The *phase velocity*,

$$c_p \overset{\text{def}}{=} \pm\frac{\omega(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \pm\sqrt{gH}\Big(\frac{\tanh(H|\boldsymbol{\xi}|)}{H|\boldsymbol{\xi}|}\Big)^{1/2}$$

represents the traveling velocity of the phase of a plane wave with wavenumber $\boldsymbol{\xi}$. A relevant other velocity is the *group velocity*,

$$c_g \overset{\text{def}}{=} \pm\nabla_{\boldsymbol{\xi}}(\omega(\boldsymbol{\xi})) = \pm\sqrt{gH}\left(\frac{1}{2}\Big(\frac{\tanh(H|\boldsymbol{\xi}|)}{H|\boldsymbol{\xi}|}\Big)^{1/2} + \frac{\operatorname{sech}^2(H|\boldsymbol{\xi}|)}{2}\Big(\frac{H|\boldsymbol{\xi}|}{\tanh(H|\boldsymbol{\xi}|)}\Big)^{1/2}\right)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},$$

which represents the traveling velocity of the energy of a wavepacket about wavenumber $\boldsymbol{\xi}$.

We can infer the large-time behavior of the solution, at least in dimension $d = 1$, through the stationary phase theorem on oscillatory integrals; see *e.g.* [Ste93]. Using the explicit expression, for any $c \in \mathbb{R}$, and initial data such that $(\zeta_0(0,\cdot), |\omega|(\cdot)\widehat{\psi}_0(0,\cdot)) \in L^1(\mathbb{R})^2$, we have

$$\zeta(t,ct) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{i(c\xi-\omega(\xi))t}\Big(\widehat{\zeta}_0(0,\xi) + i\frac{\omega(\xi)}{g}\widehat{\psi}_0(0,\xi)\Big) + e^{i((c\xi+\omega(\xi))t}\Big(\widehat{\zeta}_0(0,\xi) - i\frac{\omega(\xi)}{g}\widehat{\psi}_0(0,\xi)\Big)\,\mathrm{d}\xi$$

where we denote $\omega(\xi) = \mathrm{sgn}(\xi)(g|\xi|\tanh(H|\xi|))^{1/2}$, and use a standard convention for the Fourier transform. We deduce that the following holds for sufficiently decaying and regular initial data.

i. For any $c \in (-\infty, -\sqrt{gH}) \cup (\sqrt{gH}, +\infty)$, one has for any $n \in \mathbb{N}$,

$$|\zeta|(t,ct) = \mathcal{O}(t^{-n}).$$

ii. For any $c \in (-\sqrt{gH}, \sqrt{gH})$, one has

$$|\zeta|(t,ct) \sim_{t\to\infty} \frac{1}{4\pi}(2!)^{1/2}\Gamma(\tfrac{3}{2})|A(\xi_c)|\big(|\omega''(\xi_c)|\,t\big)^{-\frac{1}{2}}$$

where ξ_c is defined by the relation $c = \omega'(\xi_c)$ and $A(\xi_c) \stackrel{\text{def}}{=} \widehat{\zeta}_0(0,\xi_c) + \mathrm{sgn}(c)i\frac{\omega(\xi_c)}{g}\widehat{\psi}_0(0,\xi_c)$; unless $A(\xi_c) = 0$ in which case the decay is at least $\mathcal{O}(t^{-1})$.

iii. If $c \in \{-\sqrt{gH}, \sqrt{gH}\}$, one has

$$|\zeta|(t,ct) \sim_{t\to\infty} \frac{1}{4\pi}(3!)^{\frac{1}{3}}\Gamma(\tfrac{4}{3})|A(0)|\big(H^2\sqrt{gH}\,t\big)^{-\frac{1}{3}} \approx a((H^2/L^2)\sqrt{gH}/Lt)^{-\frac{1}{3}},$$

with $A(0) \stackrel{\text{def}}{=} \lim_{\xi\to 0} A(\xi)$ (notice we require regularity only on $\xi\widehat{\psi}_0(0,\xi)$); unless $A(0) = 0$, in which case the decay is at least $\mathcal{O}(t^{-\frac{2}{3}})$. The last approximation is meant in a loose sense, where we set $A(0) \approx aL$. This allows to hint at the timescale for which dispersive mechanisms have a bearing on the behavior of the flow, which is large compared with the time period of long waves, $T \stackrel{\text{def}}{=} L/\sqrt{gH}$, when $H^2/L^2 \ll 1$.

Above, Γ is the Euler Gamma function: $\Gamma(s) \stackrel{\text{def}}{=} \int_0^{+\infty} \tau^{s-1}e^{-\tau}\,\mathrm{d}\tau$.

1.5 Non-dimensionalization

We discussed previously the relevance of neglecting some effects (viscosity, friction, *etc.*) based on vague comments on the typical scales of the setting. These comments can be made quantitative after scaling the variables so as to extract the relevant dimensionless parameters which allow to measure the respective strength of various mechanisms. This step is also of tremendous importance to our goal since we shall motivate asymptotic models based on a smallness assumption of such a dimensionless parameter. The following scaling appears naturally after solving explicitly the linearized system around the trivial solution, eq. (1.5). We set

$$\mathbf{x} = \frac{x}{L} \quad ; \quad z = \frac{z}{H} \quad ; \quad t = t\frac{\sqrt{gH}}{L}$$

and

$$\zeta = \frac{\zeta}{a_{\rm top}} \quad ; \quad b = \frac{b}{a_{\rm bot}} \quad ; \quad \Phi = \Phi\frac{H}{a_{\rm top}L\sqrt{gH}}.$$

In these formulae, we introduced a typical horizontal wavelength denoted L as well as $a_{\rm top}$ (resp. $a_{\rm bot}$) denoting the typical amplitude of the surface deformation (resp. bottom topography). We

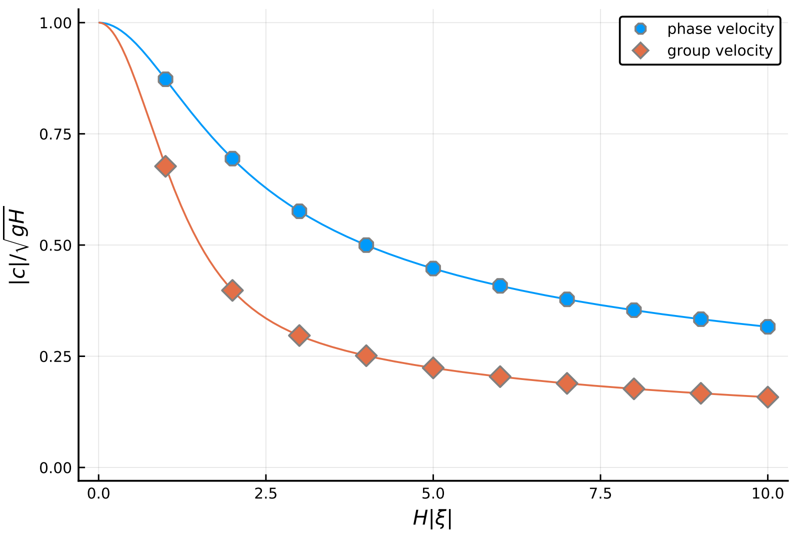


Figure 2: Phase velocity, $c_p(\xi)$ and group velocity, $c_g(\xi)$.

also recognize $c_0 \stackrel{\text{def}}{=} \sqrt{gH}$ which is the celerity of infinitesimally long and small waves, and $T = L/c_0$ their time period. With this scaling and introducing the dimensionless parameters

$$\varepsilon = \frac{a_{\text{top}}}{H} \quad ; \quad \beta = \frac{a_{\text{bot}}}{H} \quad ; \quad \mu = \frac{H^2}{L^2},$$

the dimensional water-waves system, eq. (1.4), becomes

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon \dfrac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = 0, \end{cases} \tag{1.7}$$

where we define the (dimensionless) Dirichlet-to-Neumann operator as

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = (\partial_z\Phi - \mu\varepsilon(\nabla\zeta)\cdot\nabla\Phi)\big|_{z=\varepsilon\zeta}$$

where Φ is the unique solution to

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0 & \text{in } \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} \ : \ -1+\beta b < z < \varepsilon\zeta\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta\}, \\ \partial_z\Phi - \mu\beta\nabla b\cdot\nabla\Phi = 0 & \text{on } \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} \ : \ z = -1+\beta b\}. \end{cases}$$

Now all the variables except for the dimensionless parameters are typically of size $\mathcal{O}(1)$.[1] It is clear from the above that ε measures the strength of the nonlinear effects in the systems, while β measures the magnitude of topography effects. Finally the parameter μ is the so-called shallowness parameter. A small value of the shallowness parameter amounts to assuming that most of the energy of the wave is located at low frequencies, and that in some sense "derivatives of the unknowns are

[1] More precisely, our results will be valid uniformly for data in a given ball around the origin of a normed space —typically Sobolev-based spaces with a given index of regularity— and the dependency with respect to the scales of the setting will be measured only through the two dimensionless parameters, ε and μ. Of course, describing a whole set of functions using only a handful of parameters is quite restrictive, and as a consequence our results offer only a rough description of the behavior of solutions.

smaller than the unknowns". We have also found in the previous section that smallness of the shallowness parameter is related to the weakness of dispersive effects.

As a rule of thumb, typical values of these dimensionless parameters in the context of coastal oceanography range as

$$\varepsilon \in [0, 0.1] \quad ; \quad \beta \in [0, 0.5] \quad ; \quad \mu \in (0, 0.01].$$

Our models will be derived from the assumption that

$$\mu \ll 1 \quad ; \quad \varepsilon, \beta = \mathcal{O}(1) \quad ,$$

although (most of) our results will hold for any triple of parameters $(\mu, \varepsilon, \beta)$ in the following set.

Definition 1.5 (Shallow-water asymptotic regime)**.** *Given $\mu^\star > 0$, we let*

$$\mathcal{P}_{\rm SW} = \big\{(\mu, \varepsilon, \beta) \ : \ \mu \in (0, \mu^\star], \ \varepsilon \in [0, 1], \ \beta \in [0, 1]\big\}.$$

Remark 1.6. *Keeping track of the surface tension effects would yield*

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon \dfrac{(\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = \frac{1}{\rm Bo}\nabla\cdot\left(\frac{\nabla\zeta}{\sqrt{1+\mu\varepsilon^2|\nabla\zeta|^2}}\right), \end{cases} \tag{1.8}$$

where the Bond number, $\mathrm{Bo} = \frac{\rho_0 g L^2}{\sigma}$, measures the ratio of gravity forces over capillary forces.

Our choice of scaling has been done having in mind the applications to coastal oceanography, and in particular the range $\mu \in (0, \mu^\star]$. Another natural framework is that of deep water $\mu \in [\mu_\star, \infty)$, for which the natural scaling is

$$\mathbf{x} = \frac{X}{L} \quad ; \quad z = \frac{Z}{L} \quad ; \quad t = t\frac{\sqrt{gL}}{L}$$

and

$$\zeta = \frac{\zeta}{a_{\rm top}} \quad ; \quad b = \frac{b}{a_{\rm bot}} \quad ; \quad \Phi = \Phi\frac{1}{a_{\rm top}\sqrt{gL}}.$$

With this scaling the dimensionless problem becomes

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta, \beta b]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \epsilon \dfrac{(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta, \beta b]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\epsilon^2|\nabla\zeta|^2)} = \frac{1}{\rm Bo}\nabla\cdot\left(\frac{\nabla\zeta}{\sqrt{1+\epsilon^2|\nabla\zeta|^2}}\right), \end{cases} \tag{1.9}$$

where we introduce a convenient new dimensionless parameter,

$$\epsilon = \varepsilon\sqrt{\mu} = \frac{a_{\rm top}}{L},$$

representing the typical steepness *of the wave. Notice that in the limit $\mu \to \infty$, one has*

$$\lim_{\mu\to\infty} \frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\frac{\epsilon}{\sqrt{\mu}}\zeta, \beta b]\psi = \mathcal{G}^\infty[\epsilon\zeta, \beta b]\psi = \left(\partial_z\Phi^\infty - \epsilon(\nabla\zeta)\cdot\nabla\Phi^\infty\right)\big|_{z=\epsilon\zeta}$$

where Φ is the unique solution to

$$\begin{cases} \Delta\Phi^\infty + \partial_z^2\Phi^\infty = 0 & \text{in } \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} \ : \ -\infty < z < \epsilon\zeta\}, \\ \Phi^\infty = \psi & \text{on } \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} \ : \ z = \epsilon\zeta\}, \\ \partial_z\Phi^\infty \to 0 & \text{as } z \to -\infty. \end{cases}$$

Hence eq. (1.9) *also makes sense in the infinite-layer framework* ($\mu = \infty$)*, and in this case any reference to the depth* H *has disappeared, as it should.*

If one wants to cover the full range of values $\mu \in (0, \infty)$*, we may as in [Lan13] use the scaling*

$$\mathbf{x} = \frac{X}{L} \quad ; \quad z = \frac{Z}{L} \quad ; \quad t = t\frac{\sqrt{gL\tanh(H/L)}}{L}$$

(the scaling in z *is somehow immaterial and appears only in the Laplace problem) and*

$$\zeta = \frac{\zeta}{a_{\rm top}} \quad ; \quad b = \frac{b}{a_{\rm bot}} \quad ; \quad \Phi = \Phi\frac{1}{a_{\rm top}\sqrt{gL/\tanh(H/L)}}.$$

This yields

$$\begin{cases} \partial_t\zeta - \frac{1}{\tanh(\sqrt{\mu})}\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon\sqrt{\mu}}{2\tanh(\sqrt{\mu})}|\nabla\psi|^2 - \frac{\varepsilon\sqrt{\mu}}{\tanh(\sqrt{\mu})}\frac{(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\sqrt{\mu}\nabla\zeta\cdot\nabla\psi)^2}{2(1+\mu\varepsilon^2|\nabla\zeta|^2)} = \frac{1}{\rm Bo}\nabla\cdot\Big(\frac{\nabla\zeta}{\sqrt{1+\mu\varepsilon^2|\nabla\zeta|^2}}\Big). \end{cases} \tag{1.10}$$

1.6 Well-posedness

Let us give only a rough statement. The full version can be found in [Lan13, Theorem 4.16].

Theorem 1.7. *Let* $d \in \{1,2\}$*,* $\mu^\star > 0$*,* $h_\star > 0$*,* $a_\star > 0$*,* $M^\star \geq 0$*, and* $N \geq 5$*. There exists* $C, T > 0$ *and an operator* $\mathfrak{a} : H^{N+1} \times \mathring{H}^{N+1} \to \mathbb{R}$ *such that for any* $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$*, for any* $(\zeta_0, \psi_0) \in H^{N+1} \times \mathring{H}^{N+1}$ *and* $b \in L^\infty \cap \mathring{H}^{N+1}$ *satisfying*

$$M \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^{N+1}} + \big|\varepsilon\nabla\psi_0\big|_{H^N} + \big|\beta b\big|_{L^\infty} + \big|\beta\nabla b\big|_{H^N} \leq M^\star$$

and

$$\inf_{\mathbb{R}^d}\big(1 + \varepsilon\zeta_0 - \beta b\big) \geq h_\star \quad \text{and} \quad \mathfrak{a}(\zeta_0, \psi_0) \geq a_\star,$$

there exists a unique $(\zeta, \psi) \in \mathcal{C}^0([0, T/M); H^N(\mathbb{R}^d) \times \mathring{H}^{N-1/2}(\mathbb{R}^d))$ *classical solution to eq.* (1.7) *with initial data* $(\zeta, \psi)\,|_{t=0} = (\zeta_0, \psi_0)$*. Moreover, one has*

$$\big\|\zeta\big\|_{L^\infty(0,T/M;H^N)} + \big\|\nabla\psi\big\|_{L^\infty(0,T/M;H^{N-3/2})} \leq C\big(\big|\zeta_0\big|_{H^{N+1}} + \big|\nabla\psi_0\big|_{H^N}\big).$$

Remark 1.8. *The loss of derivatives in the statement is only apparent: we can define an energy functional and a corresponding finctional space which is propagated by the flow, and we can prove in fact the well-posedness in the sense of Hadamard (*i.e. *with the continuity of the flow with respect to the initial data). Theorem 1.7 relies on (delicate) energy estimates, viewing the water-waves system as a quasilinear system, in the same way we will treat the Green-Naghdi system (and obviously the Saint-Venant system) later on. The operator* $\mathfrak{a}$ *naturally arises as a hyperbolicity condition, but is also physically motivated: it is equivalent to the Rayleigh-Taylor criterion, namely* $\inf_{\mathbb{R}^d}(-\partial_z P\,\big|_{z=\varepsilon\zeta}) > 0$*. The water-waves system is ill-posed if this criterion is violated [Ebi88]. The Rayleigh-Taylor criterion is automatically satisfied as soon as we restrict* $\mathcal{P}_{\rm SW}$ *to* $\varepsilon\mu$ *or* $\varepsilon^2\beta\mu$ *is sufficiently small; see [Lan13, §4.3.5].*

Remark 1.9. *Theorem 1.7 is only one of the well-posedness results on the water-waves system, and not the sharpest one. Alazard, Burq and Zuily (*e.g. *[ABZ14]) have extracted the paradifferential structure of the water-waves system, which allowed to considerably lower the regularity threshold for which the well-posedness hold. More recently, an impressive body of literature has been dedicated to the study of the large-time behavior of solutions, such as global or almost-global existence results, scattering or modified scattering for small initial data, depending on the dimension* d*, the domain* $\mathbb{R}^d$ *or* $\mathbb{T}^d$*, the presence of surface tension,* etc.*. However, all these delicate results rely on the dispersive nature of the system and hence typically do not hold uniformly with respect to the parameter* $\mu \ll 1$*.*

2 The Laplace problem and Dirichlet-to-Neumann operator

We provide here a brief account on some essential results concerning the Laplace problem underlying the Dirichlet-to-Neumann operator. Indeed, crucial estimates on and approximations of the Dirichlet-to-Neumann operators follow from related estimates on the Laplace problem. The latter are obtained following standard tools of elliptic problems, with special attention to the dependence with respect to the boundary of the domain —since it stands for a variable of the time-evolution problem— and and to dimensionless parameters. Again, most of the material of this section is given with more details and sharper estimates in [Lan13, Ch. 2&3]. However, significant modifications have been made so as to provide as simple proofs as possible.

Recall the (scaled) Dirichlet-to-Neumann operator is defined for sufficiently smooth data as

$$\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi = (\partial_z\Phi - \mu\varepsilon(\nabla\zeta)\cdot\nabla\Phi)\big|_{z=\varepsilon\zeta}$$

where Φ is the unique solution (see below) to

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega = \{(\mathbf{x},z)\in\mathbb{R}^{d+1} \ : \ -1+\beta b < z < \varepsilon\zeta\}, \\ \Phi = \psi & \text{on } \Gamma_{\rm top} = \{(\mathbf{x},z)\in\mathbb{R}^{d+1} \ : \ z = \varepsilon\zeta\}, \\ \partial_z\Phi - \mu\beta\nabla b\cdot\nabla\Phi = 0 & \text{on } \Gamma_{\rm bot} = \{(\mathbf{x},z)\in\mathbb{R}^{d+1} \ : \ z = -1+\beta b\}. \end{cases} \tag{2.1}$$

In this whole section, we drop any reference to the (freezed) time variable. We shall also assume

Assumption 2.1. *We have $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ and satisfy*

$$\forall \mathbf{x}\in\mathbb{R}^d, \qquad h(\mathbf{x}) = 1+\varepsilon\zeta(\mathbf{x}) - \beta b(\mathbf{x}) \geq h_\star > 0.$$

2.1 Flattening the domain

It is convenient to change variables so as to rewrite the constant-coefficient Laplace equations in a variable domain as variable-coefficient equations in a fixed domain; here the strip $\mathcal{S} \stackrel{\rm def}{=} \mathbb{R}^d\times(-1,0)$. We choose here the most obvious diffeomorphism for simplicity, since we are not too concerned by regularity issues. [2] Let us define

$$\Sigma : \begin{array}{ccc} \mathcal{S} & \to & \Omega \\ (\mathbf{x},z) & \mapsto & \big(\ \mathbf{x}\,,\ (1+\varepsilon\zeta(\mathbf{x})-\beta b(\mathbf{x}))z + \varepsilon\zeta(\mathbf{x})\ \big) \end{array}.$$

Of course this defines a diffeomorphism from the strip, $\mathcal{S}$, to the fluid domain, Ω, by Assumption 2.1. For sufficiently regular $\Phi, \psi, \mathrm{R}, r_{\rm bot}$ satisfying

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = \mathrm{R} & \text{in } \Omega, \\ \Phi = \psi & \text{on } \Gamma_{\rm top}, \\ \partial_z\Phi - \mu\beta\nabla b\cdot\nabla\Phi = r_{\rm bot} & \text{on } \Gamma_{\rm bot}, \end{cases} \tag{2.2}$$

we have that $\mathit{\Phi} \stackrel{\rm def}{=} \Phi\circ\Sigma$, and $R \stackrel{\rm def}{=} \mathrm{R}\circ\Sigma$ satisfies

$$\begin{cases} \frac{1}{\partial_z\sigma}\nabla^\mu_{\mathbf{x},z}\cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\mathit{\Phi} = R & \text{in } \mathbb{R}^d\times(-1,0), \\ \mathit{\Phi} = \psi & \text{on } \mathbb{R}^d\times\{0\}, \\ \mathbf{e}_{d+1}\cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\mathit{\Phi} = r_{\rm bot} & \text{on } \mathbb{R}^d\times\{-1\}. \end{cases} \tag{2.3}$$

where we denote $\nabla^\mu_{\mathbf{x},z} = (\sqrt{\mu}\nabla, \partial_z)^\top$, $\mathbf{e}_{d+1}$ is the unit (upward) vector in the vertical direction, $\sigma(\mathbf{x},z) \stackrel{\rm def}{=} (1+\varepsilon\zeta(\mathbf{x})-\beta b(\mathbf{x}))z+\varepsilon\zeta(\mathbf{x})$ and

$$P(\Sigma) \stackrel{\rm def}{=} \begin{pmatrix} (\partial_z\sigma)\mathrm{Id}_d & -\sqrt{\mu}\nabla\sigma \\ -\sqrt{\mu}\nabla\sigma^\top & \frac{1+\mu|\nabla\sigma|^2}{\partial_z\sigma} \end{pmatrix}.$$

[2] see [Lan13] or [Igu09] for more involved —regularizing— diffeomorphisms which are useful for obtaining optimal regularity estimates. The latter ones are crucial when studying the well-posedness of the water-waves problem, but not so much for deriving asymptotic models since we allow losses of derivatives.

That the two problems are equivalent for sufficiently regular solutions can be straightforwardly checked by chain rules. It also holds true for less regular, *variational solutions*, from which the elliptic theory can be built on. To this aim, it is convenient to substract the trace of the velocity potential, ψ, to the solutions, so as to work with the following functional spaces: $H^1_{0,\rm top}(\Omega)$ the completion of $\mathcal{D}(\Omega\cup\Gamma_{\rm bot})$ in $H^1(\Omega)$, and $H^1_{0,\rm top}(\mathcal{S})$ the completion of $\mathcal{D}(\mathbb{R}^d\times[-1,0))$ in $H^1(\mathcal{S})$. Notice that the above closures could equivalently use the Beppo-Levi (semi) norm $\big\|\bullet\big\|_{\mathring{H}^1}\stackrel{\rm def}{=}\big\|\nabla_{\mathbf{x},z}\bullet\big\|_{L^2}$ thanks to Poincaré's inequality: for any $\phi\in\mathcal{D}(\Omega\cup\Gamma_{\rm bot})$,

$$\big\|\phi\big\|_{L^2(\Omega)}^2=\iint_\Omega|\phi(\mathbf{x},z)|^2\,\mathrm{d}z\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^d}\int_{-1+\beta b(\mathbf{x})}^{\epsilon\zeta(\mathbf{x})}\Big|\int_z^{\varepsilon\zeta(\mathbf{x})}\partial_z\phi(\mathbf{x},z')\,\mathrm{d}z'\Big|^2\,\mathrm{d}z\,\mathrm{d}\mathbf{x}$$
$$\leq\Big(\sup_{\mathbb{R}^d}(1+\varepsilon\zeta-\beta b)\Big)^2\big\|\partial_z\phi\big\|_{L^2(\Omega)}^2.\qquad(2.4)$$

We shall also make use of the following trace formula:

$$\big|\phi\big|_{z=-1+\beta b}\big|_{L^2(\mathbb{R}^d)}^2\leq\int_{\mathbb{R}^d}\int_{-1+\beta b(\mathbf{x})}^{\epsilon\zeta(\mathbf{x})}\partial_z\big(\phi(\mathbf{x},z)^2\big)\,\mathrm{d}z\,\mathrm{d}\mathbf{x}\leq\big\|\phi\big\|_{L^2(\Omega)}^2+\big\|\partial_z\phi\big\|_{L^2(\Omega)}^2.\qquad(2.5)$$

By a density argument, eq. (2.4) and eq. (2.5) hold for any $\phi\in H^1_{0,\rm top}(\Omega)$, and obviously replacing the domain Ω with the strip $\mathcal{S}$. The latter is easily extended to $\phi\in H^1(\Omega)$ using a smooth truncation function.

Definition 2.2 (Variational solutions)**.** *Let $\psi\in\mathring{H}^1(\mathbb{R}^d)$ and ζ,b satisfying Assumption 2.1. We say that Φ is a variational solution to eq. (2.1) if there exists $\widetilde{\Phi}\in H^1_{0,\rm top}(\Omega)$ such that $\Phi=\psi+\widetilde{\Phi}$ and for any $\varphi\in H^1_{0,\rm top}(\Omega)$,*

$$\iint_\Omega\nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}\cdot\nabla^\mu_{\mathbf{x},z}\varphi\,\mathrm{d}z\,\mathrm{d}\mathbf{x}=-\iint_\Omega\mu\nabla\psi\cdot\nabla\varphi.$$

Let additionally $\mathbf{R}\in H^1(\mathcal{S})$. We say that $\mathit{\Phi}$ is a variational solution to eq. (2.3) with remainder terms $(\partial_z\sigma)R=\nabla^\mu_{\mathbf{x},z}\cdot\mathbf{R}$ and $r_{\rm bot}=\mathbf{e}_{d+1}\cdot\mathbf{R}\big|_{z=-1}$ if there exists $\widetilde{\mathit{\Phi}}\in H^1_{0,\rm top}(\mathcal{S})$ such that $\mathit{\Phi}=\psi+\widetilde{\mathit{\Phi}}$ and for any $\varphi\in H^1_{0,\rm top}(\mathcal{S})$,

$$\iint_{\mathcal{S}}\nabla^\mu_{\mathbf{x},z}\widetilde{\mathit{\Phi}}\cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=\iint_{\mathcal{S}}\big(\mathbf{R}-P(\Sigma)\nabla^\mu_{\mathbf{x},z}\psi\big)\cdot\nabla^\mu_{\mathbf{x},z}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

In the formula above we identified $\mathbf{x}\mapsto\psi(\mathbf{x})\in\mathring{H}^1(\mathbb{R}^d)$ and $(\mathbf{x},z)\mapsto\psi(\mathbf{x})\in\mathring{H}^1(\Omega)$ or $\mathring{H}^1(\mathcal{S})$.

Remark 2.3. *We have specified a particular form of remainder terms because these are the ones appearing in the proof of Proposition 2.5, below. We could treat in a similar way more general remainder terms, which would then be useful to justify asymptotic models, as in [Lan13]. We however use a slightly different path, and Proposition 2.5 will be used* in fine *only with $\mathbf{R}=\mathbf{0}$.*

Remark 2.4 (Assumption on the bottom topography)**.** *In the following, we choose to work with regular but not asymptotically flat topography,* i.e. *$b\in W^{n,\infty}(\mathbb{R}^d)$. All the results below are valid (with the same proof) for less regular but square-integrable $b\in H^n(\mathbb{R}^d)$, and may be refined to $b\in L^\infty(\mathbb{R}^d)\cap\mathring{H}^n(\mathbb{R}^d)$.*

2.2 The Laplace problem

The following result guarantees the existence and uniqueness of the above variational solutions, consistency with respect to strong and classical solutions for sufficiently regular data, as well as very useful regularity estimates. These results are not sharp; see [Lan13, Sect. 2] for a much more thorough analysis.

Proposition 2.5. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. We have the following.*

i. For any $(\mu,\varepsilon,\beta) \in \mathcal{P}_{\rm SW}$ and $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and any $\psi \in \mathring{H}^1(\mathbb{R}^d)$, there exists a unique variational solution, Φ, to eq. (2.1). Setting additionally $\mathbf{R} \in H^1(\mathcal{S})^{d+1}$, there exists a unique variational solution, $\mathit{\Phi}$, to eq. (2.3). If $\mathbf{R} = \mathbf{0}$, one has $\mathit{\Phi} = \Phi \circ \Sigma$.

If, moreover, $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ and $\psi \in \mathring{H}^2(\mathbb{R}^d)$, then $\nabla_{\mathbf{x},z}\mathit{\Phi} \in H^1(\mathcal{S})$ and $\mathit{\Phi}$ is a strong solution to eq. (2.3), i.e. *the identities hold in $L^2(\mathcal{S}) \times \mathring{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.*

ii. Let $M \geq 0$. There exists $C > 0$ such that for any $(\mu,\varepsilon,\beta) \in \mathcal{P}_{\rm SW}$ and any $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and $\big|\varepsilon\zeta\big|_{W^{1,\infty}} + \big|\beta b\big|_{W^{1,\infty}} \leq M$, for any $\mathbf{R} \in H^1(\mathcal{S})^{d+1}$, the variational solution to eq. (2.3) satisfies

$$\big\|\nabla^\mu_{\mathbf{x},z}\mathit{\Phi}\big\|_{L^2(\mathcal{S})} \leq C\big(\big\|\mathbf{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{L^2(\mathbb{R}^d)}\big).$$

iii. Let $k \in \mathbb{N}^\star$, $M \geq 0$. There exists C_k such that for any $(\mu,\varepsilon,\beta) \in \mathcal{P}_{\rm SW}$, $\zeta \in H^{\max\{1+k,2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{1+k,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

for any $\psi \in \mathring{H}^{1+k}(\mathbb{R}^d)$ and $\mathbf{R} \in H^1(\mathcal{S})^{d+1}$ such that $\Lambda^k \in H^1(\mathcal{S})^{d+1}$, the strong solution to eq. (2.3) satisfies $\Lambda^k\nabla^\mu_{\mathbf{x},z}\mathit{\Phi} \in L^2(\mathcal{S})$ and

$$\begin{aligned}\big\|\Lambda^k\nabla^\mu_{\mathbf{x},z}\mathit{\Phi}\big\|_{L^2(\mathcal{S})} &\leq C_k\big(\big\|\Lambda^k\mathbf{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^k}\big)\\ &\quad + C_k\Big\langle\big(\big|\varepsilon\nabla\zeta\big|_{H^k} + \big|\beta\nabla b\big|_{W^{k,\infty}}\big)\big(\big\|\Lambda^{1+s_\star}\mathbf{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}\big|\nabla\psi\big|_{H^{1+s_\star}}\big)\Big\rangle_{k>1+s_\star}.\end{aligned}$$

Moreover, for $k \geq 1 + s_\star$ and if $\mathbf{R} = \mathbf{0}$, it holds that $\mathit{\Phi} \in \mathcal{C}^2(\mathcal{S})$ and is a classical solution to eq. (2.3), i.e. *the identities hold pointwise everywhere.*

Proof. The bilinear form

$$(\varphi_1,\varphi_2) \mapsto \iint_\Omega \mu\nabla\varphi_1\cdot\nabla\varphi_2 + \partial_z\varphi_1\partial_z\varphi_2\,{\rm d}\mathbf{x}\,{\rm d}z$$

is continuous and coercive on the Hilbert space $(H^1_{0,\rm top}(\Omega), \big\|\bullet\big\|_{H^1})$ by Poincaré's inequality (2.4). Using that $(\mathbf{x},z) \mapsto \psi(\mathbf{x}) \in \mathring{H}^1(\Omega)$, the linear form

$$\varphi \mapsto -\iint_\Omega \mu\nabla\psi\cdot\nabla\varphi\,{\rm d}\mathbf{x}\,{\rm d}z,$$

is well-defined and continuous. Hence the existence and uniqueness of a variational solution to eq. (2.2) follows by Lax-Milgram theorem.

The existence and uniqueness of a variational solution to eq. (2.3) is obtained in the same way, using that $P(\Sigma)$ is coercive thanks to Assumption 2.1.

In order to check that the variational solutions correspond, namely that $\mathit{\Phi} = \Phi\circ\Sigma$ when $\mathbf{R} = \mathbf{0}$, it suffices to change variables in the integrals, since $(\nabla^\mu_{\mathbf{x},z}\Phi)\circ\Sigma = (J^{-1}_{\Sigma,\mu})^\top\nabla^\mu_{\mathbf{x},z}\mathit{\Phi}$, and

$$P(\Sigma) = \det(J_{\Sigma,\mu})(J^{-1}_{\Sigma,\mu})(J^{-1}_{\Sigma,\mu})^\top \quad \text{with} \quad (J^{-1}_{\Sigma,\mu})^\top = \begin{pmatrix} {\rm Id}_d & \frac{-\sqrt{\mu}\nabla\sigma}{\partial_z\sigma} \\ \mathbf{0}^\top & \frac{1}{\partial_z\sigma}\end{pmatrix}.$$

The estimate of item ii is a obtained by using the test function $\varphi = \widetilde{\mathit{\Phi}}$ in the variational identity, and the uniform coercivity of $P(\Sigma)$.

Now we prove that $\widetilde{\Phi} \in H^2(\mathcal{S})$ if $\psi \in \mathring{H}^2(\mathbb{R}^d)$ and $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$. For $h > 0$ and $\mathbf{e} \in \mathbb{R}^d$, let

$$\widetilde{\Phi}_{h\mathbf{e}} \stackrel{\text{def}}{=} (D_{h\mathbf{e}}\widetilde{\Phi}) : (\mathbf{x}, z) \mapsto \frac{\widetilde{\Phi}(\mathbf{x} + h\mathbf{e}, z) - \widetilde{\Phi}(\mathbf{x}, z)}{h}.$$

We have, using the test function $-D_{-h\mathbf{e}}D_{h\mathbf{e}}\widetilde{\Phi} \in H^1_{0,\text{top}}(\mathcal{S})$,

$$\iint_{\mathcal{S}} \nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}_{h\mathbf{e}} \cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}_{h\mathbf{e}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = +\iint_{\mathcal{S}} D_{h\mathbf{e}}\big(\mathbf{R} - P(\Sigma)\nabla^\mu_{\mathbf{x},z}\psi\big) \cdot \nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}_{h\mathbf{e}} \,\mathrm{d}z\,\mathrm{d}\mathbf{x}$$
$$- \iint_{\mathcal{S}} \big[D_{h\mathbf{e}}, P(\Sigma)\big]\nabla^\mu_{\mathbf{x},z}\widetilde{\Phi} \cdot \nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}_{h\mathbf{e}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

Using the previous estimate yields

$$\big\|\nabla^\mu_{\mathbf{x},z}\widetilde{\Phi}_{h\mathbf{e}}\big\|_{L^2(\mathcal{S})} \le C(h_\star^{-1}, |\varepsilon\zeta|_{W^{2,\infty}}, |\beta b|_{W^{2,\infty}})\big(|\Lambda^1\mathbf{R}|_{L^2(\mathcal{S})} + \sqrt{\mu}|\nabla\psi|_{H^1(\mathbb{R}^d)}\big),$$

were we used that for any $v \in H^1(\mathbb{R}^d)$, $|v_{h\mathbf{e}}|_{L^2(\mathbb{R}^d)} = \big|\int_0^1 \mathbf{e} \cdot \nabla v(\mathbf{x} + hr\mathbf{e})\,\mathrm{d}r\big|_{L^2} \le |\mathbf{e} \cdot \nabla v|_{L^2(\mathbb{R}^d)}$, by Minkowski's inequality. By Poincaré's inequality, we obtain a bound on $\|\widetilde{\Phi}_{h\mathbf{e}}\|_{H^1(\mathcal{S})}$ which is independent of h. Hence since $H^1(\mathcal{S})$ is a reflexive Banach space, there exists $\Psi \in H^1(\mathcal{S})$ and a subsequence $(\widetilde{\Phi}_{h_n\mathbf{e}})_n$ with $h_n \searrow 0$ such that $\Phi_{h_n\mathbf{e}} \rightharpoonup \Psi$. By uniqueness of the limit in $L^2(\mathcal{S})$, we deduce that $\Psi = \mathbf{e} \cdot \nabla\widetilde{\Phi} \in H^1(\mathcal{S})$. The limit satisfies the inequality above, and the estimate of item iii holds for $k = 1$ and $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$. In order to control the derivative in the vertical variable, we decompose for any $\varphi \in \mathcal{D}(\mathcal{S})$,

$$\iint_{\mathcal{S}} \nabla^\mu_{\mathbf{x},z}\widetilde{\Phi} \cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \iint_{\mathcal{S}} \frac{1 + \mu|\nabla\sigma|^2}{\partial_z\sigma}(\partial_z\widetilde{\Phi})(\partial_z\varphi) + \nabla^\mu_{\mathbf{x},z}\widetilde{\Phi} \cdot P_0(\Sigma)\nabla^\mu_{\mathbf{x},z}\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

Thanks to the estimate above, we may integrate by parts in the horizontal variable and deduce that

$$\iint_{\mathcal{S}} \frac{1 + \mu|\nabla\sigma|^2}{\partial_z\sigma}(\partial_z\widetilde{\Phi})(\partial_z\varphi) \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \iint_{\mathcal{S}} F\varphi \,\mathrm{d}z\,\mathrm{d}\mathbf{x}$$

with $F = \nabla^\mu_{\mathbf{x},z} \cdot \Big(P_0(\Sigma)\nabla^\mu_{\mathbf{x},z}\widetilde{\Phi} - \mathbf{R} + P(\Sigma)\nabla^\mu_{\mathbf{x},z}\psi\Big) \in L^2(\mathcal{S})$. Hence $\frac{1+\mu|\nabla\sigma|^2}{\partial_z\sigma}\partial_z\widetilde{\Phi}$ is weakly differentiable in the vertical variable and $\partial_z\Big(\frac{1+\mu|\nabla\sigma|^2}{\partial_z\sigma}\partial_z\widetilde{\Phi}\Big) \in L^2(\mathcal{S})$. By the positivity and regularity of $\frac{1+\mu|\nabla\sigma|^2}{\partial_z\sigma}$, we deduce that $\partial_z\widetilde{\Phi}$ is weakly differentiable in the vertical variable and $\partial_z^2\widetilde{\Phi} \in L^2(\mathcal{S})$. In particular, the Laplace equation holds in $L^2(\mathcal{S})$. By the trace formula (2.5), the boundary condition in eq. (2.3) is well-defined (and satisfied) in $L^2(\mathbb{R}^d)$.

Let us now estimate higher order derivatives, by induction on $k \in \mathbb{N}$. For $k \ge 2$, let us define $\mathbf{k} \in (\mathbb{N}^\star)^d$ a multi-index such that $|\mathbf{k}| = k$, and $\partial^{\mathbf{k}} = \partial_{x_1}^{\mathbf{k}_1}\partial_{x_2}^{\mathbf{k}_2}$. Using that Φ is a strong solution to eq. (2.3), we may differentiate the identities and deduce that $\partial^{\mathbf{k}}\Phi$ is a distributional solution to

$$\begin{cases} \nabla_{\mathbf{x},z} \cdot P^\mu(\Sigma)\nabla_{\mathbf{x},z}\Phi_{\mathbf{k}} = \nabla^\mu_{\mathbf{x},z} \cdot \big(\partial^{\mathbf{k}}\mathbf{R} - [\partial^{\mathbf{k}}, P(\Sigma)]\nabla^\mu_{\mathbf{x},z}\Phi\big) & \text{in } \mathbb{R}^d \times (-1, 0), \\ \Phi_{\mathbf{k}} = \partial^{\mathbf{k}}\psi & \text{on } \mathbb{R}^d \times \{0\}, \\ \mathbf{e}_{d+1} \cdot P^\mu(\Sigma)\nabla_{\mathbf{x},z}\Phi_{\mathbf{k}} = \mathbf{e}_{d+1} \cdot \big(\partial^{\mathbf{k}}\mathbf{R} - [\partial^{\mathbf{k}}, P(\Sigma)]\nabla^\mu_{\mathbf{x},z}\Phi\big) & \text{on } \mathbb{R}^d \times \{-1\}. \end{cases} \tag{2.6}$$

Using the previsouly obtained estimate of item ii (*i.e.* with $k = 0$) and using the product and commutator estimates of Appendix A.1, we obtain that the variational solution satisfies

$$\big\|\nabla^\mu_{\mathbf{x},z}\Phi_{\mathbf{k}}\big\|_{L^2} \le C\Big(\big\|\Lambda^{k-1}\nabla^\mu_{\mathbf{x},z}\Phi\big\|_{L^2(\mathcal{S})} + \big\|\Lambda^k\mathbf{R}\big\|_{L^2(\mathcal{S})} + \sqrt{\mu}|\nabla\psi|_{H^k}\Big)$$
$$+ C\Big\langle\big(|\varepsilon\nabla\zeta|_{H^k} + |\beta\nabla b|_{W^{k,\infty}}\big)\big\|\Lambda^{s_\star}\nabla^\mu_{\mathbf{x},z}\Phi\big\|_{L^2(\mathcal{S})}\Big\rangle_{k>1+s_\star}.$$

with $C = C(h_\star^{-1}, \big|\varepsilon\zeta\big|_{H^{2+s_\star}}, \big|\beta b\big|_{W^{3+s_\star,\infty}})$. The desired estimates follow by induction, and one readily observes that the distributional and variational solutions must coincide, *i.e.* $\mathit{\Phi}_{\mathbf{k}} = \partial^{\mathbf{k}}\mathit{\Phi}$.

To conclude, we notice that for $k \geq 1 + s_\star$ and if $\mathbf{R} = \mathbf{0}$, using that $\mathit{\Phi}$ is a strong solution and the aforementioned decomposition and the trace formula (2.5), one obtains easily $\partial_z^2\mathit{\Phi} \in \mathcal{C}^0(\mathcal{S})$, so that $\mathit{\Phi} \in \mathcal{C}^2(\mathcal{S})$, and the Laplace equation holds in a classical sense. □

2.3 The Dirichlet-to-Neumann operator

We now apply what we have learned on the Laplace problem, to the Dirichlet-to-Neumann operator. We start with two handy identities.

Lemma 2.6. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and $\psi \in \mathring{H}^2(\mathbb{R}^d)$. Then we have*

$$\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = -\mu\nabla\cdot(h\overline{\mathbf{u}})$$

where $h = 1 + \varepsilon\zeta - \beta b$ and

$$\overline{\mathbf{u}} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla\Phi \,{\rm d}z = \frac{1}{h}\int_{-1}^{0} (\partial_z\sigma)\nabla\mathit{\Phi} - (\nabla\sigma)\partial_z\mathit{\Phi}\,{\rm d}z,$$

recalling that Φ is the (strong) solution to eq. (2.1), $\mathit{\Phi} = \Phi\circ\Sigma$ where $\Sigma(\mathbf{x}, z) = (\mathbf{x}, \sigma(\mathbf{x}, z))$ and $\sigma(\mathbf{x}, z) \stackrel{\rm def}{=} (1 + \varepsilon\zeta(\mathbf{x}) - \beta b(\mathbf{x}))z + \varepsilon\zeta(\mathbf{x})$.

Proof. Using that Φ is a strong solution to eq. (2.1), we test the equation against $\varphi(\mathbf{x}, z) = \varphi(\mathbf{x})$ with $\varphi \in \mathcal{D}(\mathbb{R}^d)$. It follows, using Green's identity,

$$\begin{aligned}0 = \int_\Omega (\mu\Delta\Phi + \partial_z^2\Phi)\varphi &= -\int_{\mathcal{S}} \mu\nabla\Phi\cdot\nabla\varphi + \partial_z\Phi\partial_z\varphi\,{\rm d}z\,{\rm d}\mathbf{x} + \int_{\mathbb{R}^d} \left((\partial_z\Phi - \mu\varepsilon(\nabla\zeta)\cdot\nabla\Phi)\,\varphi\right)\big|_{z=\varepsilon\zeta}\,{\rm d}\mathbf{x}\\ &= -\mu\int_{\mathbb{R}^d}\left(\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi\,{\rm d}z\right)\cdot\nabla\varphi\,{\rm d}\mathbf{x} + \int_{\mathbb{R}^d} (\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi)\varphi\,{\rm d}\mathbf{x},\end{aligned}$$

and the result follows from integration by parts. The identity with $\mathit{\Phi}$ follows by the change of variables on the integral above. □

Lemma 2.7. *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $\mu^\star > 0$, and $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$. Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and $\psi \in \mathring{H}^2(\mathbb{R}^d)$. Then $\mathit{\Phi}$ the (strong) solution to eq. (2.3) with $\mathbf{R} = \mathbf{0}$ satisfies*

$$\mathit{\Phi} + \mu\ell[\varepsilon\zeta, \beta b]\mathit{\Phi} = \psi$$

where, denoting $h = 1 + \varepsilon\zeta - \beta b$,

$$\begin{aligned}\ell[\varepsilon\zeta, \beta b]\mathit{\Phi}(\cdot, z) \stackrel{\rm def}{=} &\int_z^0 \big(h(\nabla\sigma)\cdot(\nabla\mathit{\Phi}) - |\nabla\sigma|^2(\partial_z\mathit{\Phi})\big)(\cdot, z')\,{\rm d}z'\\ &- h\int_z^0\int_{-1}^{z'} \nabla\cdot\big((\partial_z\sigma)(\nabla\mathit{\Phi}) - (\nabla\sigma)(\partial_z\mathit{\Phi})\big)(\cdot, z'')\,{\rm d}z''\,{\rm d}z'.\end{aligned}$$

Proof. Denoting $\mathit{\Psi} = \mathit{\Phi} + \mu\ell[\varepsilon\zeta, \beta b]\mathit{\Phi}$, we have that $\mathit{\Psi}, \partial_z\mathit{\Psi}, \partial_z^2\mathit{\Psi} \in L^2(\mathcal{S})$. By direct algebraic computations, one checks that

$$\ell[\varepsilon\zeta, \beta b]\mathit{\Phi} = -z\,\beta\nabla b\cdot(h\nabla\mathit{\Phi} - \beta\nabla b\partial_z\mathit{\Phi})\,\big|_{z=-1} - h\int_z^0\int_{-1}^{z'} \nabla_{\mathbf{x},z}\cdot P_1(\Sigma)\nabla_{\mathbf{x},z}\mathit{\Phi}(\cdot, z'')\,{\rm d}z''\,{\rm d}z'$$

with $P_1(\Sigma) = \begin{pmatrix} (\partial_z\sigma){\rm Id}_d & -\nabla\sigma \\ -\nabla\sigma^\top & \frac{|\nabla\sigma|^2}{\partial_z\sigma}\end{pmatrix}$. Using that $P(\Sigma) = \frac{1}{h}\begin{pmatrix} 0_d & \mathbf{0} \\ \mathbf{0}^\top & 1\end{pmatrix} + \mu P_1(\Sigma)$, and that $\mathit{\Phi}$ is a strong solution to eq. (2.3), one readily checks that $\partial_z^2\mathit{\Psi} = 0$ on $\mathcal{S}$, $\partial_z\mathit{\Psi}\,\big|_{z=-1} = 0$, and $\mathit{\Psi}\,\big|_{z=0} = \psi$. □

Proposition 2.8. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$ and $k \in \mathbb{N}^\star$. Then there exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $\zeta \in H^{\max\{k+3,1+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{3+k,\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and*

$$\big|\varepsilon\zeta\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{1+s_\star,\infty}} \leq M,$$

any $\mathit{\Psi} \in \mathring{H}^1(\mathcal{S})$ such that $\Lambda^{k+2}\nabla_{\mathbf{x},z}\mathit{\Psi} \in L^2(\mathcal{S})$, one has $\Lambda^k \ell[\varepsilon\zeta, \beta b]\mathit{\Psi} \in H^1(\mathcal{S})$ and

$$\begin{aligned}\big\|\Lambda^k\nabla_{\mathbf{x},z}\big(\ell[\varepsilon\zeta,\beta b]\mathit{\Psi}\big)\big\|_{L^2(\mathcal{S})} &\leq C\big\|\Lambda^{k+2}\nabla_{\mathbf{x},z}\mathit{\Psi}\big\|_{L^2(\mathcal{S})}\\ &\quad + C\Big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big\|\Lambda^{s_\star}\nabla_{\mathbf{x},z}\mathit{\Psi}\big\|_{L^2(\mathcal{S})}\Big\rangle_{k+2>s_\star}.\end{aligned}$$

Proof. The result follows directly from the definition and product estimates in Appendix A.1. □

It is now simple to deduce approximate expressions of $\overline{\mathbf{u}}$ with arbitrary precision in terms of powers of μ.

Proposition 2.9. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0,1,2\}$. There exists C_n such that the following holds. For any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $\zeta \in H^{\max\{k+2n+1,2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+2n+1,2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and*

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\begin{array}{ccc}\mathring{H}^{k+1}(\mathbb{R}^d) & \to & H^k(\mathbb{R}^d)\\ \psi & \mapsto & \overline{\mathbf{u}} = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi(\cdot,z)\,{\rm d}z\end{array}$$

where Φ is the unique solution to eq. (2.1), is well-defined and continuous, and the following holds.

- *If $n = 0$, then $\overline{\mathbf{u}} \in H^k(\mathbb{R}^d)$ is well-defined and one has, denoting $h = 1 + \varepsilon\zeta - \beta b$,*

$$\big|h\overline{\mathbf{u}}\big|_{H^k} \leq C_0\left(\big|\nabla\psi\big|_{H^k} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+1}} + \big|\beta b\big|_{W^{k+1,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k>1+s_\star}\right).$$

- *If $n = 1$, then one has additionally*

$$\big|h\overline{\mathbf{u}} - h\nabla\psi\big|_{H^k} \leq C_1\,\mu\left(\big|\nabla\psi\big|_{H^{k+2}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\right).$$

- *If $n = 2$, then one has additionally*

$$\begin{aligned}&\big|h\overline{\mathbf{u}} - h\nabla\psi + \mu h\mathcal{T}[h,\beta\nabla b]\nabla\psi\big|_{H^k}\\ &\qquad\leq C_2\,\mu^2\left(\big|\nabla\psi\big|_{H^{k+4}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\right),\end{aligned}$$

  where we define

$$\mathcal{T}[h,\beta\nabla b]\mathbf{u} \stackrel{\rm def}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u}) + \frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\mathbf{u}\big) - h^2(\beta\nabla b)\nabla\cdot\mathbf{u}\Big) + \beta^2(\nabla b\cdot\mathbf{u})\nabla b. \quad (2.7)$$

Proof. By Lemma 2.6, we have the identity

$$h\overline{\mathbf{u}} = \int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi\,{\rm d}z = \int_{-1}^{0}(\partial_z\sigma)\nabla\mathit{\Phi} - (\nabla\sigma)\partial_z\mathit{\Phi}\,{\rm d}z,$$

where $\varPhi$ is the solution to eq. (2.3) with $\mathbf{R} = \mathbf{0}$. The result for $n = 0$ follows from product estimates in Appendix A.1 and Proposition 2.5. Plugging above the identity in Lemma 2.7, we obtain

$$h\bar{\mathbf{u}} = h\nabla\psi - \mu\int_{-1}^{0}(\partial_z\sigma)\nabla(\ell[\varepsilon\zeta,\beta b]\varPhi) - (\nabla\sigma)\partial_z(\ell[\varepsilon\zeta,\beta b]\varPhi)\,\mathrm{d}z.$$

The result for $n = 1$ is deduced, using additionally Proposition 2.8. Plugging again the identity in Lemma 2.7 in the identity above yields the result for $n = 2$, using that

$$\begin{aligned}\ell[\varepsilon\zeta,\beta b]\psi &= \int_z^0 \big(h((1+z')\nabla h + \beta\nabla b)\cdot(\nabla\psi)\,\mathrm{d}z' - h\int_z^0\int_{-1}^{z'}\nabla\cdot(h\nabla\psi)\,\mathrm{d}z''\,\mathrm{d}z' \\ &= (z+\frac{z^2}{2})h^2\nabla\cdot\nabla\psi + zh\beta(\nabla b)\cdot(\nabla\psi)\end{aligned}$$

and hence, after tedious computations,

$$\int_{-1}^{0} h\nabla(\ell[\varepsilon\zeta,\beta b]\psi) - (\nabla\sigma)\partial_z(\ell[\varepsilon\zeta,\beta b]\psi)\,\mathrm{d}z = h\mathcal{T}[h,\beta\nabla b]\nabla\psi.$$

This concludes the proof. □

The following result is an obvious consequence of Proposition 2.9 and the identity of Lemma 2.6.

Proposition 2.10. *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $M \geq 0$, $k \in \mathbb{N}^\star$, $n \in \{0,1,2\}$. There exists C_n such that the following holds. For any $(\mu,\varepsilon,\beta) \in \mathcal{P}_{\rm SW}$, any $\zeta \in H^{\max\{k+2n+1,2+s_\star\}}(\mathbb{R}^d)$ and $b \in W^{\max\{k+2n+1,2+s_\star\},\infty}(\mathbb{R}^d)$ satisfying Assumption 2.1 and*

$$\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\beta b\big|_{W^{2+s_\star,\infty}} \leq M,$$

the operator

$$\mathcal{G}^\mu[\varepsilon\zeta,\beta b]: \begin{array}{ccc} \mathring{H}^{k+1}(\mathbb{R}^d) & \to & H^{k-1}(\mathbb{R}^d) \\ \psi & \mapsto & (\partial_z\Phi - \mu\varepsilon(\nabla\zeta)\cdot\nabla\Phi)\big|_{z=\varepsilon\zeta}\end{array}$$

where Φ is the unique solution to eq. (2.1), is well-defined and continuous, and the following holds.

- *If $n = 0$, then*

$$\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi\big|_{H^{k-1}} \leq C_0\left(\big|\nabla\psi\big|_{H^k} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+1}} + \big|\beta b\big|_{W^{k+1,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k>1+s_\star}\right).$$

- *If $n = 1$, then*

$$\begin{aligned}&\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi)\big|_{H^{k-1}} \\ &\qquad\leq C_1\,\mu\left(\big|\nabla\psi\big|_{H^{k+2}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+3}} + \big|\beta b\big|_{W^{k+3,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+2>1+s_\star}\right).\end{aligned}$$

- *If $n = 2$, then, denoting $h = 1+\varepsilon\zeta-\beta b$ and $\mathcal{T}$ as in eq. (2.7)*

$$\begin{aligned}&\big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot(h\nabla\psi) - \mu\nabla\cdot\big(h\mathcal{T}[h,\beta\nabla b]\nabla\psi\big)\big|_{H^{k-1}} \\ &\qquad\leq C_2\,\mu^2\left(\big|\nabla\psi\big|_{H^{k+5}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^{k+5}} + \big|\beta b\big|_{W^{k+5,\infty}}\big)\big|\nabla\psi\big|_{H^{1+s_\star}}\big\rangle_{k+4>1+s_\star}\right).\end{aligned}$$

The strategy for deriving asymptotic models for the water-waves system, eq. (1.7), is now fairly obvious: we simply plug the truncated expansion at the desired order (in terms of powers of μ) in the system, and withdraw any negligible contribution. The result above allows to rigorously justifies such an approximation in the sense of consistency. The first-order system we obtain this way is the so-called Saint-Venant system, studied in Section 3. One can obtain at next order the Green-Naghdi system, studied in Section 4.

3 The Saint-Venant system

We now introduce the simplest fully nonlinear shallow-water model, namely the Saint-Venant system. It is obtained straightforwardly plugging the approximation

$$\frac{1}{\mu}\mathcal{G}^{\mu}[\varepsilon\zeta,\beta b]\psi = -\nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi) + \mathcal{O}(\mu).$$

stemming from Proposition 2.10 into the water-waves system, eq. (1.7), and withdrawing all terms of size $\mathcal{O}(\mu)$. One obtains the system

$$\begin{cases} \partial_t\zeta + \nabla\cdot((1+\varepsilon\zeta-\beta b)\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 = 0. \end{cases} \tag{3.1}$$

One usually rewrites eq. (3.1) using a velocity variable

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big((1+\varepsilon\zeta-\beta b)\mathbf{u}\big) = 0, \\ \partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0}. \end{cases} \tag{3.2}$$

System (3.2) is obtained immediately from eq. (3.1), taking the gradient of the second equation and setting $\mathbf{u} = \nabla\psi$. It is also valid as a $\mathcal{O}(\mu)$ approximation of the evolution equations for $(\zeta, \overline{\mathbf{u}})$ where $\overline{\mathbf{u}}$ is the layer-averaged horizontal velocity,

$$\overline{\mathbf{u}} = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi\,\mathrm{d}z,$$

in which case the first equation, representing the conservation of mass, is exactly satisfied by solutions of the water-waves system eq. (1.7) (by Lemma 2.6).

System (3.1) (or rather eq. (3.2)) is the prototype of hyperbolic quasilinear systems, the strong hyperbolicity being guaranteed by the non-cavitation assumption, $h > 0$; see below. In fact, in the flat-bottom case, $\beta b \equiv 0$, the Saint-Venant system corresponds to the isentropic, *compressible* Euler equation for ideal gases with the pressure law $p(\rho) \propto \rho^2$ (identifying ρ with h).

Using physical variables, system (3.1) reads

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \mathbf{0}, \end{cases}$$

with $h = H + \zeta - b$. By analogy with the compressible Euler equation, one recognizes here that the "sound speed" of long surface gravity waves in a layer of depth H is $c = \sqrt{gH}$.

3.1 Hyperbolicity

As mentioned above, eq. (3.2) is a quasilinear system of first-order evolution equations, *i.e.* can be written under the form

$$\partial_t\mathbf{U} + \sum_{i=1}^{d} A_i(\mathbf{U})\partial_{x_i}\mathbf{U} = F(t,x,\mathbf{U}), \tag{3.3}$$

where $\mathbf{U} : \mathbb{R}^+\times\mathbb{R}^d \to \mathbb{R}^n$ (here, $n = 1+d$) represents the unknowns, and where $A_i : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ –$\mathcal{M}_n(\mathbb{R})$ denotes $n\times n$ square matrices with real coefficients– and $F : \mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^n \to \mathbb{R}^n$ are given and smooth.

We shall not recall the rich theory of such systems (see for instance [BGS07, Mét08]), but recall a few facts adapted to our particular system, and which in particular allow to obtain the well-posedness and stability results Theorem 3.3 and Theorem 3.6.

Recall the *principal symbol* of eq. (3.3) is

$$A(\mathbf{U},\boldsymbol{\xi}) \stackrel{\text{def}}{=} \sum_{i=1}^{d} \boldsymbol{\xi} A_i(\mathbf{U})$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. One can check that for eq. (3.2), the characteristic equation

$$\det(A(\mathbf{U},\boldsymbol{\xi}) - \lambda \mathrm{Id}_{1+d}) = 0$$

admits $d+1$ real solutions for any $\boldsymbol{\xi} \neq \mathbf{0}$ as soon as $h = 1 + \varepsilon\zeta > 0$:

$$\lambda_\delta = \varepsilon(\mathbf{u}\cdot\boldsymbol{\xi}) + \delta\sqrt{h|\boldsymbol{\xi}|^2}$$

where $\delta \in \{-1, 1\}$ if $d = 1$ and $\delta \in \{-1, 0, 1\}$ if $d = 2$. Because all the eigenvalues of $A(\mathbf{U},\boldsymbol{\xi})$ are distinct for any for $\boldsymbol{\xi} \neq \mathbf{0}$ and any $\mathbf{U} \in \mathbb{R}^{1+d}_{h>0} \stackrel{\text{def}}{=} \{(\zeta, \mathbf{u}) \in \mathbb{R}^{1+d} \ : \ 1 + \varepsilon\zeta > 0\}$, the Saint-Venant system is *strictly hyperbolic*.

As a consequence, the symbol is smoothly diagonalizable with real eigenvalues, which in turn allows to construct a symbolic symmetrizer, namely $S : (\mathbf{U},\boldsymbol{\xi}) \in \mathbb{R}^{1+d}_{h>0} \times \mathbb{R}^d \setminus \{\mathbf{0}\} \to \mathcal{M}_{1+d}(\mathbb{R})$, smooth and homogeneous of degree 0 in the second variable such that for all $(\mathbf{U},\boldsymbol{\xi}) \in \mathbb{R}^{1+d}_{h>0} \times \mathbb{R}^d \setminus \{\mathbf{0}\}$, $S(\mathbf{U},\boldsymbol{\xi})$ is symmetric and definite positive, and $S(\mathbf{U},\boldsymbol{\xi})A(\mathbf{U},\boldsymbol{\xi})$ is symmetric.

In our case, it is in fact easy to see that the system is *symmetric-hyperbolic* in the sense of Friedrichs, namely we can exhibit a (non-symbolic) explicit symmetrizer $S \in \mathcal{C}^\infty(\mathbb{R}^{1+d}_{h>0}, \mathcal{M}_{1+d}(\mathbb{R}))$ such that for all $\mathbf{U} \in \mathbb{R}^{1+d}_{h>0}$, $S(\mathbf{U})$ is symmetric and definite positive, and for all $i \in \{1, \dots, d\}$, $S(\mathbf{U})A_i(\mathbf{U})$ is symmetric. An example of such symmetrizer is

$$S(\mathbf{U}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^\top & h\mathrm{Id}_d \end{pmatrix}, \qquad \mathbf{U} = (\zeta, \mathbf{u}), \ h = 1 + \varepsilon\zeta.$$

In other words, the Saint-Venant system is symmetric if one multiplies the second equation with the depth.

3.2 Hamiltonian structure

System (3.1) inherits a canonical Hamiltonian structure from the water-waves system (see Section 1.3):

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H}_{\mathrm{SV}} \\ \delta_\psi \mathcal{H}_{\mathrm{SV}} \end{pmatrix}$$

with

$$\mathcal{H}_{\mathrm{SV}}(\zeta, \psi) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + (1 + \varepsilon\zeta - \beta b)|\nabla\psi|^2 \ \mathrm{d}\mathbf{x}.$$

In fact, one could have (formally) derived the Saint-Venant system by plugging the approximation $\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta, \beta b]\psi = -\nabla\cdot(h\nabla\psi) + \mathcal{O}(\mu)$ directly into the water-waves Hamiltonian functional, and derive the Saint-Venant system from Hamilton's principle on

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \zeta\partial_t\psi \,\mathrm{d}\mathbf{x} + \mathcal{H}_{\mathrm{SV}} \,\mathrm{d}t.$$

The Saint-Venant enyoys the same symmetry groups as the water-waves system (again, see Section 1.3), and consistent preserved quantities, in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{SV}} = 0, \qquad \text{where} \quad \mathcal{Z} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta \,\mathrm{d}\mathbf{x}\,, \ \mathcal{I} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \zeta\nabla\psi \,\mathrm{d}\mathbf{x}.$$

Written with the velocity variable $\mathbf{u}$, the system still enjoys a (non-canonical) symplectic form (see *e.g.* [She90]). In dimension $d=2$, one has

$$\partial_t \begin{pmatrix} \zeta \\ u_x \\ u_y \end{pmatrix} = -\begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & -q \\ \partial_y & q & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H} \\ \delta_{u_x}\mathcal{H} \\ \delta_{u_y}\mathcal{H} \end{pmatrix}.$$

where $q = \varepsilon\frac{\operatorname{curl}\mathbf{u}}{h} = \varepsilon\frac{\partial_x u_y - \partial_y u_x}{1+\varepsilon\zeta-\beta b}$ and (misusing notations)

$$\mathcal{H}_{\rm SV}(\zeta,\mathbf{u}) \stackrel{\rm def}{=} \frac12\int_{\mathbb{R}^d} \zeta^2 + (1+\varepsilon\zeta-\beta b)|\mathbf{u}|^2 \, {\rm d}\mathbf{x}.$$

Of course, in our situation, $q\equiv 0$ if $\mathbf{u}=\nabla\psi$, but it turns out the Saint-Venant system is also relevant for non-potential flows; see [CL14, CL15] for a rigorous justification. Within this formalism, one can check that the time and space invariance of the Hamiltonian yield the conservation of total energy and momentum,

$$\frac{\rm d}{{\rm d}t}\mathcal{H}_{\rm SV} = 0 \quad ; \quad \frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^d}(1+\varepsilon\zeta-\beta b)\mathbf{u}\,{\rm d}\mathbf{x} = \mathbf{0},$$

while Casimir invariants are, for any function C,

$$\frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^d} hC(q)\,{\rm d}\mathbf{x} = 0,$$

which yields the conservation of mass –with $C(q)=1$– and circulation –with $C(q)=q$– as special cases. It is of course straightforward to derive conservation laws associated with any of these preserved quantities.

3.3 Riemann invariants

In dimension $d=1$, as any system of two balance laws, the Saint-Venant system enjoys a basis of Riemann invariants. The Riemann invariants are explicit in this case: setting $r_\pm = \varepsilon u \pm 2\sqrt{1+\varepsilon\zeta}$,[3] the system (3.2) (with $\beta b\equiv 0$ for simplicity) is equivalent to

$$\begin{cases} \partial_t r_+ + \frac{3r_+ + r_-}{4}\partial_x r_+ = 0, \\ \partial_t r_- + \frac{3r_- + r_+}{4}\partial_x r_- = 0. \end{cases} \tag{3.4}$$

Notice that $\frac{3r_+ + r_-}{4} = \varepsilon u + \sqrt{1+\varepsilon\zeta}$ and $\frac{3r_- + r_+}{4} = \varepsilon u - \sqrt{1+\varepsilon\zeta}$, consistently with the hyperbolicity discussion above.

The diagonal formulation, eq. (3.4), allows to construct *simple waves*, *i.e.* solutions of the form

$$(r_+, r_-) = \mathbf{R}(\theta(t,x))$$

where θ is a scalar function. For instance, any sufficiently regular solution to eq. (3.4) with initial data satisfying $\varepsilon u\,|_{t=0} = 2\sqrt{1+\varepsilon\zeta\,|_{t=0}} - 2$ satisfies $r_-\equiv -2$, from which we deduce $r_+ \equiv 2+2\varepsilon u$, where $u(t,x)$ satisfies the inviscid Burgers (or Hopf) equation

$$\partial_t u + \big(1+\frac{3\varepsilon}{2}u\big)\partial_x u = 0. \tag{3.5}$$

[3] This nonlinear change of variables is a diffeomorphism as long as we restrict data in the hyperbolicity domain, $\mathbb{R}^2_{h>0} = \{(\zeta,u), 1+\varepsilon\zeta>0\}$. That any solution in $W^{1,\infty}([0,T)\times\mathbb{R}^2)$ cannot leave the hyperbolicity domain if it is included in the hyperbolicity domain at time $t=0$ can be deduced from the existence of Riemann invariants, see *e.g.* [Daf10]. This can however be seen in a simpler way and in a greater generality (in dimension $d=2$ and with non-flat bottom) as a consequence of the conservation of mass, by noticing that $\frac{\rm d}{{\rm d}t}h(t,\mathbf{x}(t)) = -(h\nabla\cdot\mathbf{u})(t,\mathbf{x}(t))$ where $\mathbf{x}'(t) = \mathbf{u}(t,\mathbf{x}(t))$, and hence $\inf_{\mathbb{R}^d} h(t,\cdot) \geq \inf_{\mathbb{R}^d} h(t,\cdot)\exp(-|\mathbf{u}|_{W^{1,\infty}} t) > 0$ if $\inf_{\mathbb{R}^d} h(t,\cdot) > 0$.

Conversely, any solution to the above equation provides a particular solution to eq. (3.6) by setting $(r_-, r_+) = (2 + 2\varepsilon u, -2)$, or equivalently a solution to eq. (3.2) with original variables by setting $\zeta = \varepsilon^{-1}\Big(\big(1 + \frac{\varepsilon}{2}u\big)^2 - 1\Big) = u + \frac{\varepsilon}{4}u^2$.

Equation (3.5) may be solved by the hodograph transform, or the characteristics method. Assume $u \in W^{1,\infty}((0,T)\times\mathbb{R})$ is a solution to (3.5) and define $v_{x_0}(t) \stackrel{\text{def}}{=} u(t, x_{x_0}(t))$ where $x_{x_0}(t)$ is defined by the initial condition $x_{x_0}(t) = x_0$ and the differential equation $x'_{x_0}(t) = 1 + \frac{3\varepsilon}{2}u(t, x(t))$. Notice that $v'_{x_0}(t) = \partial_t u(t, x_{x_0}(t)) + x'_{x_0}(t)\partial_x u = 0$. We deduce $v_{x_0}(t) = v_{x_0}(0) = u(0, x_0)$ and $x_{x_0}(t) = x_0 + (1 + \frac{3\varepsilon}{2}u(0, x_0))t$. In other words, the solution is constant along the characteristics defined by $x_{x_0}(t)$ (for any $x_0 \in \mathbb{R}$) and the characteristics are straight lines. This allows to define and describe solutions as long as two characteristics do not cross, namely for any $t \in (0,T)$, there does not exists $x_0 \neq x_1 \in \mathbb{R}$ such that

$$x_0 + \big(1 + \frac{3\varepsilon}{2}u(0, x_0)\big)t = x_1 + \big(1 + \frac{3\varepsilon}{2}u(0, x_1)\big)t \quad \Longleftrightarrow \quad \frac{u(0, x_1) - u(0, x_0)}{x_1 - x_0} = -\frac{2}{3\varepsilon t}.$$

Hence we see that for any initial data $u(t = 0, \cdot) = u_0 \in W^{1,\infty}(\mathbb{R}^d)$, the solution described above (which is unique) exists on the time domain $[0, T)$ where $T^\star = -\frac{2}{3\varepsilon}(\inf_{\mathbb{R}^d} u_0')^{-1}$ with the convention $T^\star = \infty$ if $\inf_{\mathbb{R}^d} u_0' \geq 0$. In the situation where $\inf_{\mathbb{R}^d} u_0' < 0$ (in particular for any non-trivial u_0 such that $u_0 \to 0$ as $|x| \to \infty$), there exists indeed a singularity formation as $t \to T^\star$: since the solution remains bounded but $\inf_{\mathbb{R}^d} \partial_x u(t, \cdot) \to -\infty$ as $t \nearrow T^\star$, we say that a shock occurs.

Let us now go back to the system case, eq. (3.4). Each of the Riemann invariants, $r_\pm$, is constant along characteristics curves: by

$$\frac{\mathrm{d}}{\mathrm{d}t} r_\pm(t, x_{\pm,x_0}(t)) = 0, \quad x_{\pm,x_0}(0) = x_0, \quad x'_{\pm,x_0}(t) = \frac{1}{4}(3r_\pm + r_\mp)(t, x_{\pm,x_0}(t)).$$

However the characteristics curves are no longer straight lines, unless one of the Riemann invariants is constant. Still we can infer the behavior of the solution for instance if we assume that the initial data $(\zeta(t = 0, \cdot), u(t = 0, \cdot) = (\zeta_0, u_0)$ has compact support

$$\operatorname{supp}(\zeta_0, u_0) \subset (-L, L)$$

and is sufficiently small so that there exists $r \in (0, 1)$ such that

$$r_{+,0} \stackrel{\text{def}}{=} \varepsilon u_0 + 2\sqrt{1 + \varepsilon\zeta_0} \in (2 - r, 2 + r) \quad \text{and} \quad r_{-,0} \stackrel{\text{def}}{=} \varepsilon u_0 - 2\sqrt{1 + \varepsilon\zeta_0} \in (-2 - r, -2 + r).$$

Because the Riemann invariants are constant along characteristics, we have (as long as the solution remains regular)

$$\frac{3r_+ + r_-}{4} \in (1 - r, 1 + r) \quad \text{and} \quad \frac{3r_- + r_+}{4} \in (-1 - r, -1 + r).$$

As a consequence, we have that

$$r_+(t, x) \equiv 2 \text{ if } x \leq -L + (1 - r)t \quad \text{and} \quad r_-(t, x) \equiv -2 \text{ if } x \geq L - (1 - r)t.$$

If the initial data is sufficiently small in order to ensure that no shock formation occurs before $T_\star = \frac{L}{1-r}$, we can afterwards decompose the flow as the superposition of two simple waves: for any $t \geq T_\star$,

$$\begin{cases} (r_+, r_-)(t, x) = (2 + 2\varepsilon u_+(t, x), -2) \text{ and } \partial_t u_+ + \big(1 + \frac{3\varepsilon}{2}u_+\big)\partial_x u_+ = 0 & \text{if } x \geq 0, \\ (r_+, r_-)(t, x) = (2, -2 + 2\varepsilon u_-(t, x)) \text{ and } \partial_t u_- - \big(1 - \frac{3\varepsilon}{2}u_-\big)\partial_x u_- = 0 & \text{if } x \leq 0. \end{cases}$$

In particular, a shock inevitably occurs after sufficiently large time.

3.4 Rigorous justification

Thanks to the results of Section 2, and in particular Proposition 2.10, it is now straightforward to justify eq. (3.1) in the sense of *consistency*.

Theorem 3.1 (Consistency). *Let $d, s_\star \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. Let $s \in \mathbb{N}$ and $M^\star \geq 0$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $b \in W^{\max\{s+4,2+s_\star\},\infty}(\mathbb{R}^d)$, any $T > 0$ and any $(\zeta, \psi) \in L^\infty(0, T; H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water-waves system, eq. (1.7), satisfying Assumption 2.1 uniformly for $t \in (0, T)$ and*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi\big|_{H^{1+s_\star}}\big)(t) + \big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\nabla\psi) = \mu\, r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 = \mu\, r_2, \end{cases}$$

with

$$\big\|(r_1, r_2)\big\|_{L^\infty(0,T;H^s\times H^{s+2})} \leq C \operatorname*{ess\,sup}_{t\in[0,T)} \big(\big|\zeta\big|_{H^{s+4}} + \big|\nabla\psi\big|_{H^{s+3}}\big)(t).$$

Proof. The Proposition is an immediate consequence of Proposition 2.10 with $n = 1$ and $k = s+1$ for the first equation, and with $n = 1$ and $k = s+3$ and $k = 1+s_\star$ (with the product and composition estimates of Appendix A.1) for the second equation. □

Remark 3.2. *One could also prove that, provided that $\partial_t\zeta, \partial_t\nabla\psi$ is sufficiently regular –or infering the regularity from the water-waves system, eq. (1.7)–, then $\overline{\mathbf{u}}$ defined as in Proposition 2.9, satisfies*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{\mathbf{u}}) = 0, \\ \partial_t\overline{\mathbf{u}} + \nabla\zeta + \varepsilon(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = \mu\,\mathbf{r}, \end{cases}$$

with a uniform bound on $\big\|\mathbf{r}\big\|_{L^\infty(0,T;H^s)}$. The first equation, i.e. *the conservation of mass, is an exact identity by Lemma 2.6. The second equation follows from Proposition 2.9 as well as an estimate on*

$$\big\|\partial_t\nabla\psi - \partial_t\overline{\mathbf{u}}\big\|_{L^\infty(0,T;H^s)},$$

which would be obtained by applying **??** *to $\partial_t\Phi$ the strong solution to*

$$\begin{cases} \frac{1}{\partial_z\sigma}\nabla^\mu_{\mathbf{x},z}\cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\partial_t\Phi = -\frac{1}{\partial_z\sigma}\nabla^\mu_{\mathbf{x},z}\cdot\big[\partial_t, P(\Sigma)\big]\nabla^\mu_{\mathbf{x},z}\Phi & \text{in } \mathbb{R}^d\times(-1,0), \\ \partial_t\Phi = \partial_t\psi & \text{on } \mathbb{R}^d\times\{0\}, \\ \mathbf{e}_{d+1}\cdot P(\Sigma)\nabla^\mu_{\mathbf{x},z}\partial_t\Phi = -\mathbf{e}_{d+1}\cdot\big[\partial_t, P(\Sigma)\big]\nabla^\mu_{\mathbf{x},z}\Phi & \text{on } \mathbb{R}^d\times\{-1\}, \end{cases}$$

and differentiating (with respect to time) the identity provided by Lemma 2.6 and Lemma 2.7:

$$\overline{\mathbf{u}} = \nabla\psi - \frac{\mu}{h}\int_{-1}^0 (\partial_z\sigma)\nabla(\ell[\varepsilon\zeta,\beta b]\Phi) - (\nabla\sigma)\partial_z(\ell[\varepsilon\zeta,\beta b]\Phi)\,\mathrm{d}z.$$

The consistency alone is not sufficient to provide a full justification of the Saint-Venant system eq. (3.2). Fortunately, classical results on hyperbolic systems (see *e.g.* [BGS07, Mét08]) provide the well-posedness theory and stability estimates which allow a stronger notion of justification.

Theorem 3.3 (Local well-posedness). *Let $d \in \mathbb{N}^\star$, $h_\star > 0$, $s > 1 + d/2$ and $M^\star > 0$. There exists $T > 0$ and $C > 0$ such that for any $\varepsilon, \beta \in (0, 1)$, any $b \in W^{s+1,\infty}(\mathbb{R}^d)$, and any $(\zeta_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ satisfying Assumption 2.1 and*

$$M \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^s} + \big|\varepsilon\mathbf{u}_0\big|_{H^s} + \big|\beta b\big|_{W^{s+1,\infty}} \leq M^\star,$$

there exists a unique $(\zeta, \mathbf{u}) \in \mathcal{C}^0([0, T/M]; H^s(\mathbb{R}^d)^{1+d}) \cap \mathcal{C}^1([0, T/M]; H^{s-1}(\mathbb{R}^d)^{1+d})$ *classical solution to the Saint-Venant system, eq.* (3.2)*, with initial data* $(\zeta, \mathbf{u})\,|_{t=0} = (\zeta_0, \mathbf{u}_0)$*. Moreover,*

$$\forall t \in [0, T/M], \quad \big\|\zeta\big\|_{L^\infty(0,t;H^s)} + \big\|\mathbf{u}\big\|_{L^\infty(0,t;H^s)} \leq C \times \Big(\big|\zeta_0\big|_{H^s} + \big|\mathbf{u}_0\big|_{H^s}\Big)$$

and $\inf_{(t,\mathbf{x})\in[0,T/M]\times\mathbb{R}^d} \big(1 + \varepsilon\zeta(t, \mathbf{x})\big) \geq h_\star/2$.

Remark 3.4. *Uniqueness in Theorem 3.3 allows to define* $T_{\max}$ *the supremum of* $T > 0$ *such that the Cauchy problem has a solution* $(\zeta, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^s(\mathbb{R}^d)^{1+d}) \cap \mathcal{C}^1([0, T]; H^{s-1}(\mathbb{R}^d)^{1+d})$ *which remains in the hyperbolic domain,* $\inf_{(t,\mathbf{x})\in[0,T]\times\mathbb{R}^d} \big(1 + \varepsilon\zeta(t, \mathbf{x})\big) > 0$*. The use of tame estimates as in Remark A.12 yields the following blowup criterion:*

$$T_{\max} < \infty \quad \Longrightarrow \quad \lim_{t \nearrow T_{\max}} \Big(\big\|\zeta\big\|_{L^\infty(0,t;W^{1,\infty})} + \big\|\mathbf{u}\big\|_{L^\infty(0,t;W^{1,\infty})}\Big) \to \infty,$$

since the hyperbolicity criterion remains satisfied; see footnote 3. In particular, for given initial data, $T_{\max}$ *and the maximal solution do not depend on the choice of the regularity index,* $s > 1 + d/2$.

Remark 3.5. *In order to prove the well-posedness of the Cauchy problem associated with eq.* (3.6) *in the sense of Hadamard, one should state the continuity of the flow map*

$$\varphi^t : (\zeta_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d} \mapsto (\zeta(t, \cdot), \mathbf{u}(t, \cdot)) \in H^s(\mathbb{R}^d)^{1+d}.$$

While this property holds true, it is not significant for our purposes, where we are happy to ask an extra derivative on the initial data to ensure that the flow-map is Lipschitz. This result is a particular case of the stability property, Theorem 3.6, below.

Theorem 3.6 (Stability)**.** *Let* $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $s \in \mathbb{N}$, $M^\star \geq 0$, $n \stackrel{\rm def}{=} \max\{s+1, 1+s_\star\}$. *There exists* $C > 0$ *such that for any* $\varepsilon, \beta \in (0, 1)$*, for any* $b \in W^{n,\infty}(\mathbb{R}^d)$*, for any* $T^\star > 0$ *and* $(\zeta^0, \mathbf{u}^0) \in L^\infty(0, T^\star; H^n(\mathbb{R}^d)^{1+d})$ *satisfying the Saint-Venant system, eq.* (3.2)*, as well as any* $(\zeta, \mathbf{u}) \in L^\infty(0, T^\star; H^n(\mathbb{R}^d)^{1+d})$ *satisfying*

$$\left\{ \begin{array}{l} \partial_t\zeta + \nabla \cdot \big((1 + \varepsilon\zeta - \beta b)\mathbf{u}\big) = r, \\ \partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{r}, \end{array} \right. \tag{3.6}$$

with $(r, \mathbf{r}) \in L^1(0, T^\star; H^s(\mathbb{R}^d)^{1+d})$*, and assuming that* $h = 1 + \varepsilon\zeta - \beta b$ *and* $h_0 = 1 + \varepsilon\zeta_0 - \beta b$ *satisfy Assumption 2.1 uniformly for* $t \in (0, T^\star)$ *and*

$$M \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{t\in[0,T^\star]} \big(\big|(\varepsilon\zeta, \varepsilon\mathbf{u})\big|_{H^n} + \big|(\varepsilon\zeta^0, \varepsilon\mathbf{u}^0)\big|_{H^n}\big)(t) + \big|\beta b\big|_{W^{n,\infty}} \leq M_\star,$$

one has, for any $t \in (0, T^\star)$*,*

$$\begin{aligned} \big\|\zeta - \zeta^0\big\|_{L^\infty(0,t;H^s)} + \big\|\mathbf{u} - \mathbf{u}^0\big\|_{L^\infty(0,t;H^s)} \leq C e^{CMt}\Big(\big|\zeta - \zeta^0\big|_{H^s}(t = 0) + \big|\mathbf{u} - \mathbf{u}^0\big|_{H^s}(t = 0)\Big) \\ + C \int_0^t e^{CM(t-\tau)} \big(\big|r\big|_{H^s} + \big|\mathbf{r}\big|_{H^s}\big)(\tau)\, \mathrm{d}\tau\,. \end{aligned}$$

The following result is a direct consequence of Theorem 3.1, Theorem 3.3 and Theorem 3.6.

Theorem 3.7 (Convergence)**.** *Let* $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $s \in \mathbb{N}$ *and* $M^\star \geq 0$*. There exists* $T > 0$ *and* $C > 0$ *such that for any* $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$*, any* $b \in W^{\max\{s+4,2+s_\star\},\infty}(\mathbb{R}^d)$*, any* $T^\star > 0$ *and any* $(\zeta, \psi) \in \mathcal{C}^0([0, T^\star]; H^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+4,2+s_\star\}}(\mathbb{R}^d))$ *classical solution to the water-waves system, eq.* (1.7)*, satisfying Assumption 2.1 uniformly on* $[0, T^\star)$ *and*

$$M \stackrel{\rm def}{=} \sup_{t\in[0,T^\star]} \big(\big|\varepsilon\zeta\big|_{H^{\max\{s+1,2+s_\star\}}} + \big|\varepsilon\nabla\psi\big|_{H^{\max\{s+1,1+s_\star\}}}\big)(t) + \big|\beta b\big|_{W^{\max\{s+4,2+s_\star\},\infty}} \leq M^\star,$$

there exists a unique $(\zeta_{\rm SV},\mathbf{u}_{\rm SV})\in\mathcal{C}^0([0,T/M);H^{\max\{s+1,1+s_\star\}}(\mathbb{R}^d)^{1+d})$ *classical solution to the Saint-Venant system* (3.2) *with initial data* $\zeta_{\rm SV}\big|_{t=0}=\zeta\big|_{t=0}$, $\mathbf{u}_{\rm SV}\big|_{t=0}=\nabla\psi\big|_{t=0}$ *and one has for any* $t\in(0,\min\{T^\star,T/M\}]$,

$$\big\|\zeta-\zeta_{\rm SV}\big\|_{L^\infty(0,t;H^s)}+\big\|\nabla\psi-\mathbf{u}_{\rm SV}\big\|_{L^\infty(0,t;H^s)}\leq C\,\mu\,t\left(\big\|\zeta\big\|_{L^\infty(0,t;H^{s+4})}+\big\|\nabla\psi\big\|_{L^\infty(0,t;H^{s+3})}\right).$$

Remark 3.8. *Theorem 3.1, Theorem 3.3, Theorem 3.6 and hence Theorem 3.7 hold assuming that* $b\in L^\infty(\mathbb{R}^d)\cap\mathring{H}^n(\mathbb{R}^d)$ *instead of* $b\in W^{n,\infty}(\mathbb{R}^d)$ *(with* $n(s,s_\star)$ *provided in the statement), replacing occurences of* $\big|\beta b\big|_{W^{n,\infty}}$ *with* $\big|\beta b\big|_{L^\infty}+\big|\beta\nabla b\big|_{H^{n-1}}$.

Remark 3.9. *That sufficiently smooth solutions to the water-waves system exist on the timescale of Theorem 3.7 is provided by Theorem 1.7.*

4 The Green-Naghdi system

We now introduce a second-order fully nonlinear shallow-water model, known as the Serre or the Green-Naghdi system. It would be tempting to plug into the water-waves system, eq. (1.7) the second-order approximation stemming from Proposition 2.10, namely

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi=-\nabla\cdot(h\nabla\psi)+\mu\nabla\cdot\big(h\mathcal{T}[h,\beta\nabla b]\nabla\psi\big)+\mathcal{O}(\mu^2),$$

where we recall that $h=1+\varepsilon\zeta-\beta b$ and

$$\mathcal{T}[h,\beta\nabla b]\mathbf{u}\stackrel{\rm def}{=}\frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u})+\frac{1}{2h}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\mathbf{u}\big)-h^2(\beta\nabla b)\nabla\cdot\mathbf{u}\Big)+\beta^2(\nabla b\cdot\mathbf{u})\nabla b.$$

However the system one would obtain woud suffer from strong instabilities at high frequencies,[4] due to the fact that the operator $\mathbf{u}\mapsto h\mathbf{u}-\mu h\mathcal{T}[h,\beta\nabla b]\mathbf{u}$ is not positive definite. As a consequence, we prefer the following alternative approximation:

$$\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi=-\nabla\cdot(h({\rm Id}+\mu\mathcal{T}[h,\beta\nabla b])^{-1}\nabla\psi)+\mathcal{O}(\mu^2).$$

Plugging this expansion into eq. (1.7), using Lemma 2.6 and withdrawing $\mathcal{O}(\mu^2)$ terms yields

$$\begin{cases}\partial_t\zeta+\nabla\cdot(h\mathbf{u})=0,\\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}|\mathbf{u}|^2=\mu\varepsilon\big(\mathcal{R}[h,\mathbf{u}]+\mathcal{R}_b[h,\beta\nabla b,\mathbf{u}]\big),\end{cases}\tag{4.1}$$

where $\mathbf{u}$ is deduced from (ζ,ψ) after solving the equation [5]

$$h\nabla\psi\;=\;h\mathbf{u}+\mu h\mathcal{T}[h,\beta\nabla b]\mathbf{u}\stackrel{\rm def}{=}\mathfrak{T}[h,\beta\nabla b]\mathbf{u}.\tag{4.2}$$

and

$$\mathcal{R}[h,\mathbf{u}]\stackrel{\rm def}{=}\frac{\mathbf{u}}{3h}\cdot\nabla(h^3\nabla\cdot\mathbf{u})+\frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2,$$
$$\mathcal{R}_b[h,\beta\nabla b,\mathbf{u}]\stackrel{\rm def}{=}-\frac{1}{2}\left(\frac{\mathbf{u}}{h}\cdot\nabla\big(h^2(\beta\nabla b\cdot\mathbf{u})\big)+h(\beta\nabla b\cdot\mathbf{u})\nabla\cdot\mathbf{u}+(\beta\nabla b\cdot\mathbf{u})^2\right).$$

[4]in the sense that the linearized system around the trivial solution $(\zeta=0,\psi=0)$, explicitly solvable in Fourier space in the flat-bottom setting, exhibits unstable modes whose amplitude grows exponentially and arbitrarily rapidly for large frequencies. We shall not write down this system, but the interested reader may find it in [Whi67, (10)-(11)] and [CG94, (14)-(15)] (in the one-dimension and flat-bottom situation).

[5]Notice that by Proposition 2.9, we have

$$\mathbf{u}=\overline{\mathbf{u}}+\mathcal{O}(\mu^2),\qquad\overline{\mathbf{u}}=\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\Phi(\cdot,z)\,{\rm d}z,$$

where Φ is the unique solution to **??**.

Taking the gradient of the second equation, one can check (see [DI18] for details) that the system can be written with only differential operators in terms of the unknowns ζ and $\mathbf{u}$, namely

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\mathbf{u}) = 0, \\ \big(\mathrm{Id} + \mu\mathcal{T}[h,\beta b]\big)\partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\varepsilon\big(\mathcal{Q}[h,\mathbf{u}] + \mathcal{Q}_b[h,\beta\nabla b,\mathbf{u}]\big) = \mathbf{0}, \end{cases} \tag{4.3}$$

where

$$\mathcal{Q}[h,\mathbf{u}] \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\Big),$$
$$\mathcal{Q}_b[h,\beta\nabla b,\mathbf{u}] \stackrel{\text{def}}{=} \frac{\beta}{2h}\Big(\nabla\big(h^2(\mathbf{u}\cdot\nabla)^2 b\big) - h^2\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\nabla b\Big) + \beta^2\big((\mathbf{u}\cdot\nabla)^2 b\big)\nabla b.$$

An even more compact formulation, if one removes the constraint of considering first-order evolution equations, is the following:

$$\begin{cases} \partial_t(\zeta - b) + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\mathcal{P}[h,\beta b,\mathbf{u}] = \mathbf{0}, \end{cases} \tag{4.4}$$

where

$$\mathcal{P}[h,\beta b,\mathbf{u}] \stackrel{\text{def}}{=} \frac{1}{h}\nabla\Big(h^2\big(\frac{\ddot{h}}{3} + \frac{\beta\ddot{b}}{2}\big)\Big) + \beta\big(\frac{\ddot{h}}{2} + \beta\ddot{b}\big)\nabla b$$

and where we denote $\dot{h} = \partial_t h + \mathbf{u}\cdot\nabla h$, $\ddot{h} = \partial_t\dot{h} + \mathbf{u}\cdot\nabla\dot{h}$, and similarly $\dot{b}, \ddot{b}$. The above formulation remains valid when the bottom has a prescribed but non-trivial time-dependent evolution; see [FI15].

Using physical variables (recall Section 1.5), system (4.4) reads

$$\begin{cases} \partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \\ \partial_t\boldsymbol{u} + g\nabla(h+b) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathcal{P}[h,b,\boldsymbol{u}] = \mathbf{0}. \end{cases}$$

with $h = H + \zeta - b$.

The dispersion relation associated with (4.4) (in the flat-bottom case, $b \equiv 0$) is

$$\omega(\boldsymbol{\xi})\Big(\omega(\boldsymbol{\xi})^2 - \frac{gH|\boldsymbol{\xi}|^2}{1 + \frac{1}{3}H^2|\boldsymbol{\xi}|^2}\Big) = 0.$$

The solution $\omega(\boldsymbol{\xi}) = 0$ corresponds to the propagation of the "vorticity", $\mathrm{curl}(\boldsymbol{u} + \mathcal{T}[h,\nabla b]\boldsymbol{u})$, and is irrelevant to potential flows as the vorticity vanishes. The remaining modes approximate the ones of the water-waves system when $H|\boldsymbol{\xi}| \ll 1$ (see Figure 3), and the large-time behavior discussion of Section 1.4 applies.

4.1 Hamiltonian structure

As the Saint-Venant system, eq. (4.1) inherits a canonical Hamiltonian structure from the water-waves system: Hamilton's principle on

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\zeta\partial_t\psi\,\mathrm{d}\mathbf{x} + \mathcal{H}_{\mathrm{GN}}\,\mathrm{d}t.$$

with

$$\mathcal{H}_{\mathrm{GN}}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 + (h\nabla\psi)\cdot\mathfrak{T}[h,\beta\nabla b]^{-1}(h\nabla\psi)\ \mathrm{d}\mathbf{x}$$

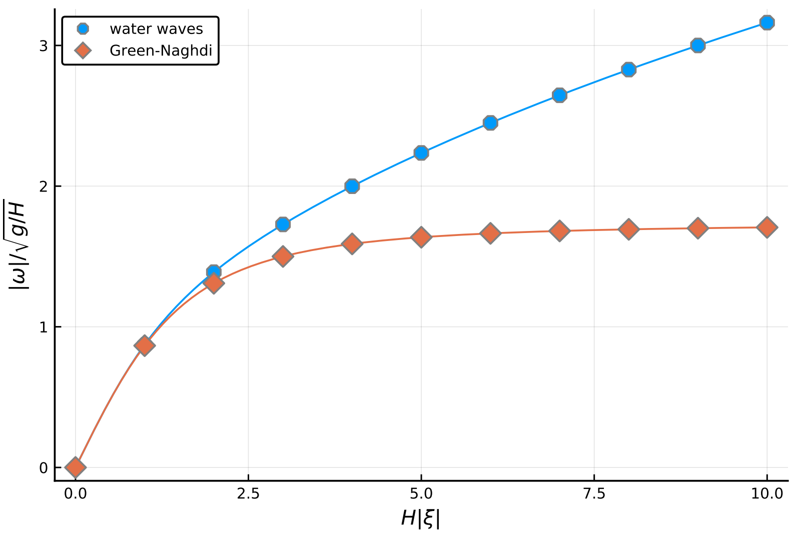


Figure 3: Wavenumbers, $|\omega|(|\boldsymbol{\xi}|)$, corresponding to the water-waves and the Green-Naghdi systems.

yields

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H}_{\rm GN} \\ \delta_\psi \mathcal{H}_{\rm GN} \end{pmatrix},$$

which corresponds to eq. (4.1). We may hence follow the discussion of Section 1.3.

We can check that, written with the velocity variable $\mathbf{u}$, eq. (4.3) still enjoys a (non-canonical) symplectic form. Indeed, the system turns out to be equivalent to (compare with eq. (4.1))

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0, \\ \big(\partial_t + \varepsilon \mathbf{u}^\perp \operatorname{curl}\big)\mathbf{v} + \nabla\zeta + \frac{\varepsilon}{2}\nabla(|\mathbf{u}|^2) = \mu\varepsilon\nabla\big(\mathcal{R}[h,\mathbf{u}] + \mathcal{R}_b[h,\beta\nabla b,\mathbf{u}]\big), \end{cases} \tag{4.5}$$

where we denote $h\mathbf{v} = \mathfrak{T}[h,\beta\nabla b]\mathbf{u}$, $\operatorname{curl}\mathbf{v} \stackrel{\rm def}{=} \partial_x v_y - \partial_y v_x$ and $\mathbf{u}^\perp \stackrel{\rm def}{=} (-u_y, u_x)$. Notice that the term $\varepsilon\mathbf{u}^\perp \operatorname{curl}\mathbf{v}$ is somewhat artificial (in dimension $d = 1$, this term should be dropped) since $\mathbf{v} = \nabla\psi + \mathcal{O}(\mu^2)$, and contrarily to the Saint-Venant case, we do not expect that the system is still relevant outside of the irrotational setting. Yet it allows to obtain the exact same symplectic form as the Saint-Venant system, and the conclusions still apply. In dimension $d = 2$, one has

$$\partial_t \begin{pmatrix} \zeta \\ v_x \\ v_y \end{pmatrix} = - \begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & -q \\ \partial_y & q & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta \mathcal{H} \\ \delta_{v_x} \mathcal{H} \\ \delta_{v_y} \mathcal{H} \end{pmatrix}.$$

where $q = \varepsilon\frac{\operatorname{curl}\mathbf{v}}{h} = \varepsilon\frac{\partial_x v_y - \partial_y v_x}{1+\varepsilon\zeta-\beta b}$ and (misusing notations)

$$\mathcal{H}_{\rm GN}(\zeta,\mathbf{v}) \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + (h\mathbf{v}) \cdot \mathfrak{T}[h,\beta\nabla b]^{-1}(h\mathbf{v}) \, \mathrm{d}\mathbf{x}$$

Within this formalism, one can check that the time and space invariance of the Hamiltonian yield the conservation of total energy and horizontal impulse,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\rm GN} = 0 \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \zeta\mathbf{v}\,\mathrm{d}\mathbf{x} = \mathbf{0},$$

while Casimir invariants are, for any function C,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} hC(q)\,\mathrm{d}\mathbf{x},$$

which yields the conservation of mass – with $C(q)=1$– and circulation –with $C(q)=q$– as special cases.

4.2 Traveling Waves

In the unidimensional ($d=1$) and flat bottom ($b\equiv 0$) framework, the Green-Naghdi system, eq. (4.3), enjoys an explicit family of traveling waves,

$$(\zeta,\psi)(t,x)=(\zeta_c,\psi_c)(x-ct),\qquad \lim_{|x|\to\infty}|(\zeta_c,\psi_c')|(x)=0.$$

Denoting $h_c=1+\varepsilon\zeta_c$ and $\psi_c'=u_c-\frac{\mu}{3h_c}\big(h_c^3u_c'\big)'$ and plugging the above Ansatz into eq. (4.1) yields

$$\begin{cases} -c\zeta_c'+(h_cu_c)'=0,\\ -c\Big(u_c-\frac{\mu}{3h_c}\big(h_c^3u_c'\big)'\Big)'+\zeta_c'+\frac{\varepsilon}{2}(u_c^2)'=\mu\varepsilon\Big(\frac{u_c}{3h_c}(h_c^3u_c')'+\frac{1}{2}-(h_c^2u_c')^2)\Big)', \end{cases}$$

We may now integrate and, using the vanishing condition at infinity to set the integration constant, we deduce from the first equation

$$-c\zeta_c+h_cu_c=0$$

and using this identity into the second equation yields

$$h_c-1-\frac{c^2}{2}\frac{h_c^2-1}{h_c^2}=\mu c^2\big(\frac{-1}{3h_c^2}(h_ch_c')'+\frac{1}{2h_c^2}(h_c')^2\big).$$

Multiplying with h_c' and once again integrating yields

$$(h_c-1)^2(c^2-h_c)=\frac{\mu c^2}{3}(h_c')^2.$$

Since $h_c\to 1$ as $|x|\to\infty$, the differential equation has a real solution only if $c>1$, in which case there exists a unique (up to translations) solution given by

$$h_c(x)=1+(c^2-1)\,\mathrm{sech}^2\left(\sqrt{\frac{3(c^2-1)}{4c^2}}\frac{x-x_\star}{\sqrt{\mu}}\right),\qquad \varepsilon\zeta_c=h_c-1,\ \varepsilon u_c=\frac{c(h_c-1)}{h}.$$

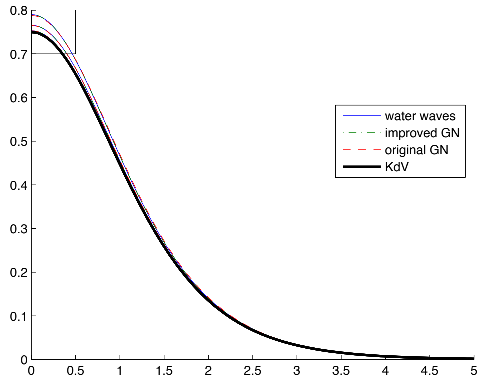
This explicit solution was provided as early as in [Ser53]. It should be compared with the infamous solitary wave solutions to the (right-going) Korteweg-de Vries equation

$$\partial_t\zeta_{\rm KdV}+\partial_x\zeta_{\rm KdV}+\frac{3\varepsilon}{4}\partial_x\big(\zeta_{\rm KdV}^2\big)+\frac{\mu}{6}\partial_x^3\zeta_{\rm KdV},$$

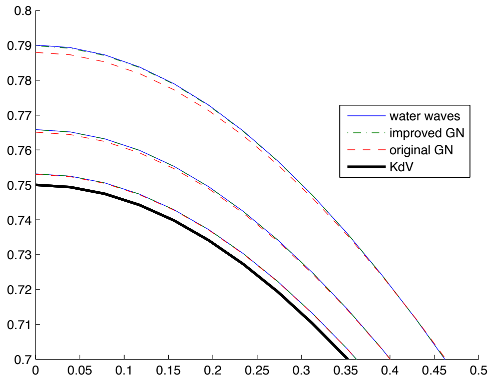
namely $\zeta_{\rm KdV}(t,x)=\zeta_{c,{\rm KdV}}(x-ct)$ with

$$\varepsilon\zeta_{c,{\rm KdV}}(x)=2(c-1)\,\mathrm{sech}^2\left(\sqrt{\frac{6(c-1)}{4}}\frac{x-x_\star}{\sqrt{\mu}}\right).$$

By Theorem 4.6 the above solutions provide good approximations of the traveling waves of the exact water-waves equations, eq. (1.7), when $c-1\approx\varepsilon\approx\mu\ll 1$, that is in the weakly nonlinear regime. See Figure 4 for a comparison of the solitary waves at a given velocity.



(a) Rescaled solitary waves for $c = 1.025,\ 1.01,\ 1.002$.



(b) Close-up.

Figure 4: Comparison of the solutions of the KdV and Green-Naghdi models and the water waves system (the latter is numerically computed). The waves are rescaled so that the Korteweg-de Vries solution does not depend on c. Consistently, we set $\varepsilon = \mu = 1$. See [DNW18] for more details.

Based on the Hamiltonian structure of the Green-Naghdi system, we may interpret traveling waves solutions to (4.3) as critical points to the functional $\mathcal{H}_{\rm GN}(\zeta, v) - c\mathcal{I}_{\rm GN}(\zeta, v)$, *i.e.*

$$c\zeta = \delta_v \mathcal{H}_{\rm GN}(\zeta, v) \quad \text{ and } \quad cv = \delta_\zeta \mathcal{H}_{\rm GN}(\zeta, v),$$

where $\mathcal{I}_{\rm GN}(\zeta, v) \stackrel{\rm def}{=} \int_{\mathbb{R}^d} \zeta v$ and $v = h^{-1}\mathfrak{T}[h, 0]u$. However these critical points are neither minimizers nor maximizers of the functional, and hence give no direct information to the stability of the solutions. The linear stability is investigated in [Li01, Li02]. In [DNW18], a constrained minimization problem is introduced which allows to prove the existence –together with a weak notion of stability– of solitary waves for a larger class of equations, including (4.3).

4.3 Rigorous justification

We provide a rigorous justification of (4.1) and (4.3). In order for these equations to make sense as evolution equations, one needs first to ensure that $\mathfrak{T}$ is invertible. As a matter of fact, robust and quantitative information on the operator, $\mathfrak{T}$, and its inverse, will be crucial in our proofs. To this aim, we first introduce some relevant functional spaces.

$$X^n \stackrel{\rm def}{=} \{\mathbf{u} \in L^2(\mathbb{R}^d)^d \ : \ \big|\mathbf{u}\big|_{X^n}^2 \stackrel{\rm def}{=} \sum_{|\mathbf{k}|=0}^{n} \big|\partial^{\mathbf{k}}\mathbf{u}\big|_{L^2}^2 + \mu\big|\partial^{\mathbf{k}}\nabla\cdot\mathbf{u}\big|_{L^2}^2 < \infty\},$$

$$Y^n \stackrel{\rm def}{=} \{\mathbf{v} \in (X^0)' \ : \ \big|\mathbf{v}\big|_{Y^n}^2 \stackrel{\rm def}{=} \sum_{|\mathbf{k}|=0}^{n} \big|\partial^{\mathbf{k}}\mathbf{v}\big|_{(X^0)'}^2 < \infty\}.$$

It turns out that the operator $\mathfrak{T}[h, \beta\nabla b] : X^n \to Y^n$ is well-defined, one-to-one and onto provided that ζ, b are sufficiently regular and Assumption 2.1 holds. This result is detailed in Lemma A.21 and Lemma A.22.

Theorem 4.1 (Consistency)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$. Let $s \in \mathbb{N}$ and $M^\star \geq 0$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $b \in \mathring{H}^{\max\{s+6, 2+s_\star\}}(\mathbb{R}^d)$, any $T > 0$ and any $(\zeta, \psi) \in L^\infty(0, T; H^{\max\{s+6, 2+s_\star\}}(\mathbb{R}^d) \times \mathring{H}^{\max\{s+6, 2+s_\star\}}(\mathbb{R}^d)^2)$ classical solution to the water-waves* *system, eq.* (1.7)*, satisfying Assumption* 2.1 *uniformly for* $t \in (0,T)$ *and*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\varepsilon\zeta\big|_{H^{2+s_\star}} + \big|\varepsilon\nabla\psi\big|_{H^{1+s_\star}}\big)(t) + \big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}} \leq M^\star,$$

one has

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\mathbf{u}) = \mu^2\, r_1, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \mu\varepsilon(\mathcal{R}[h,\mathbf{u}] + \mathcal{R}_b[h,\beta\nabla b,\mathbf{u}]) = \mu^2\, r_2, \end{cases}$$

where we denote $h = 1+\varepsilon\zeta-\beta b$, $\mathbf{u} = \mathfrak{T}[h,\beta\nabla b]^{-1}(h\nabla\psi)$*, and one has*

$$\big\|(r_1,r_2)\big\|_{L^\infty(0,T;H^s\times H^{s+2})} \leq C \operatorname*{ess\,sup}_{t\in(0,T)} \big(\big|\zeta\big|_{H^{s+6}} + \big|\nabla\psi\big|_{H^{s+5}}\big)(t).$$

Proof. The control of r_1 is a obtained noticing that

$$\begin{aligned} \Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi &+ \nabla\cdot\big(h\mathfrak{T}[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\Big|_{H^s} \\ &\leq \Big|\frac{1}{\mu}\mathcal{G}^\mu[\varepsilon\zeta,\beta b]\psi + \nabla\cdot\big(h(\mathrm{Id}-\mu\mathcal{T}[h,\beta\nabla b])\nabla\psi\big)\Big|_{H^s} \\ &\qquad + \mu^2\big|\nabla\cdot\big(h\mathcal{T}[h,\beta\nabla b]\mathcal{T}[h,\beta\nabla b]\mathfrak{T}[h,\beta\nabla b]^{-1}(h\nabla\psi)\big)\big|_{H^s}. \end{aligned}$$

The first term is estimated by Proposition 2.10 (with $n=2$ and $k=s+1$), and we can use Lemma A.22 to estimate the second term. The control of r_2 is obtained using Proposition 2.10 with $n=1$ and $k=s+3$ as well as $k=1+s_\star$, together with Lemma A.22 and the product and composition estimates of Appendix A.1. ☐

The local well-posedness of system (4.1) (or, equivalently, system (4.3)) has been proved in [Li06] (dimension $d=1$, flat bottom), [Isr11] (dimension $d=1$), [ASL08] (existence and uniqueness of a solution "with loss of derivatives" through a Nash-Moser scheme), and [FI15, DI18] (general case). We provide a detailed proof based on [FI15] in Section 4.4.

Theorem 4.2 (Local well-posedness)**.** *Let* $d\in\mathbb{N}^\star$, $s_\star > d/2$ *and* $s\in\mathbb{N}$, $s\geq 1+s_\star$, $h_\star>0$ $\mu^\star>0$*, and* $M^\star\geq 0$*. There exist* $T>0$ *and* $C>0$ *such that the for any* $(\mu,\varepsilon,\beta)\in\mathcal{P}_{\rm SW}$*, any* $b\in W^{s+1,\infty}(\mathbb{R}^d)$*, and any* $(\zeta_0,\mathbf{u}_0)\in H^s(\mathbb{R}^d)\times X^s$ *satisfying Assumption* 2.1 *and*

$$M_0 \overset{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\mathbf{u}_0\big|_{X^{1+s_\star}} + \big|\beta b\big|_{W^{s,\infty}} \leq M^\star,$$

there exists a unique $(\zeta,\mathbf{u})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times X^s)\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times X^{s-1})$ *classical solution to the Green-Naghdi system, eq.* (4.3)*, with initial data* $(\zeta,\mathbf{u})\,|_{t=0} = (\zeta_0,\mathbf{u}_0)$*; and we have for any* $t\in[0,T/M_0]$

$$\big\|\zeta\big\|_{L^\infty(0,t;H^s)} + \big\|\mathbf{u}\big\|_{L^\infty(0,t;X^s)} \leq C\times\Big(\big|\zeta_0\big|_{H^s} + \big|\mathbf{u}_0\big|_{X^s}\Big).$$

Moreover, denoting $\mathbf{v} = h^{-1}\mathfrak{T}[\varepsilon\zeta,\beta\nabla b]\mathbf{u}$ *with* $h = 1+\varepsilon\zeta-\beta b$ *and* $\mathfrak{T}$ *defined in eq.* (4.2)*, we have that* $(\zeta,\mathbf{v})\in\mathcal{C}^0([0,T/M_0];H^s(\mathbb{R}^d)\times Y^s)\cap\mathcal{C}^1([0,T/M_0];H^{s-1}(\mathbb{R}^d)\times Y^{s-1})$ *is a classical solution to eq.* (4.1)*.*

Remark 4.3. *Uniqueness in Theorem* 4.2 *allows to define* $T_{\max}$ *the supremum of* $T>0$ *such that the Cauchy problem has a solution* $(\zeta,\mathbf{u})\in\mathcal{C}^0([0,T];H^s(\mathbb{R}^d)\times X^s)\cap\mathcal{C}^1([0,T];H^{s-1}(\mathbb{R}^d)\times X^{s-1})$*. We also have the blowup criterion*

$$T_{\max}<\infty \quad\Longrightarrow\quad \lim_{t\nearrow T_{\max}} \Big(\big\|\zeta\big\|_{L^\infty(0,t;H^{1+s_\star})} + \big\|\mathbf{u}\big\|_{L^\infty(0,t;X^{1+s_\star})}\Big) \to \infty,$$

since the hyperbolicity criterion remains satisfied; see footnote 3*. In particular, for given initial data,* $T_{\max}$ *and the maximal solution do not depend on the choice of the regularity index,* $s>1+d/2$*.*

Remark 4.4. *As discussed earlier, the continuity of the flow map*

$$\varphi^t : (\zeta_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d) \times X^s \mapsto (\zeta(t,\cdot), \mathbf{u}(t,\cdot)) \in H^s(\mathbb{R}^d) \times X^s$$

does hold (and can be obtained using the method of the proof), but is of little interest to our purposes.

Theorem 4.5 (Stability)**.** *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, and $M^\star \geq 0$, and denote $n \stackrel{\rm def}{=} \max\{s+1, 1+s_\star\}$. There exists $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $b \in W^{n,\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and $(\zeta^0, \mathbf{u}^0) \in L^\infty([0,T^\star); H^n(\mathbb{R}^d) \times X^n)$ satisfying the Green-Naghdi system, eq. (4.3), and any $(\zeta, \mathbf{u}) \in L^\infty([0,T^\star); H^n(\mathbb{R}^d) \times X^n)$ satisfying*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = r, \\ \big(\mathrm{Id} + \mu \mathcal{T}[h, \beta\nabla b]\big)\partial_t \mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\varepsilon\big(\mathcal{Q}[h,\mathbf{u}] + \mathcal{Q}_b[h, \beta\nabla b, \mathbf{u}]\big) = \mathbf{r}, \end{cases} \tag{4.6}$$

with $(r, \mathbf{r}) \in L^1(0, T^\star; H^s(\mathbb{R}^d) \times Y^s)$, and assuming that $h = 1 + \varepsilon\zeta - \beta b$ and $h_0 = 1 + \varepsilon\zeta_0 - \beta b$ satisfy Assumption 2.1 uniformly for $t \in [0, T^\star)$ and

$$M \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{t\in[0,T^\star)} \big(\big|(\varepsilon\zeta, \varepsilon\mathbf{u})\big|_{H^n\times X^n} + \big|(\varepsilon\zeta^0, \varepsilon\mathbf{u}^0)\big|_{H^n\times X^n}\big)(t) + \big|\beta b\big|_{W^{n,\infty}} \leq M^\star,$$

then one has for any $t \in (0, T^\star)$,

$$\big\|\zeta - \zeta^0\big\|_{L^\infty(0,t;H^s)} + \big\|\mathbf{u} - \mathbf{u}^0\big\|_{L^\infty(0,t;X^s)} \leq Ce^{CMt}\Big(\big|\zeta - \zeta^0\big|_{H^s}(t=0) + \big\|\mathbf{u} - \mathbf{u}^0\big\|_{X^s}(t=0)\Big) + C\int_0^t e^{CM(t-\tau)}\big(\big(|r|_{H^s} + |\mathbf{r}|_{Y^s}\big)(\tau)\,\mathrm{d}\tau\,.$$

The following result is a direct consequence of Theorem 4.1, Theorem 4.2 and Theorem 4.5.

Theorem 4.6 (Convergence)**.** *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $h_\star > 0$, $\mu^\star > 0$, $s \in \mathbb{N}$ and $M^\star \geq 0$. There exist $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $b \in W^{\max\{s+6,2+s_\star\},\infty}(\mathbb{R}^d)$, any $T^\star > 0$ and any $(\zeta, \psi) \in \mathcal{C}^0([0,T^\star]; H^{\max\{s+6,2+s_\star\}} \times \mathring{H}^{\max\{s+6,2+s_\star\}}(\mathbb{R}^d)^{1+d})$ solution to the water-waves system (1.7) and such that $h = 1 + \varepsilon\zeta - \beta b$ satisfies Assumption 2.1 uniformly for $t \in [0, T^\star]$ and*

$$M \stackrel{\rm def}{=} \sup_{t\in[0,T^\star]} \big(\big|\varepsilon\zeta\big|_{H^{\max\{s+1,2+s_\star\}}} + \big|\varepsilon\nabla\psi\big|_{H^{\max\{s+1,1+s_\star\}}}\big)(t) + \big|\beta b\big|_{W^{\max\{s+6,2+s_\star\},\infty}} \leq M^\star,$$

there exists a unique $(\zeta_{\rm GN}, \mathbf{u}_{\rm GN}) \in \mathcal{C}^0([0, T/M]; H^{\max\{s+1,1+s_\star\}}(\mathbb{R}^d) \times X^{\max\{s+1,1+s_\star\}})$ classical solution to the Green-Naghdi system (4.3) and $\big(\zeta_{\rm GN}\,|_{t=0}\,, \mathbf{u}_{\rm GN}\big)\,|_{t=0} = \big(\zeta, \mathfrak{T}[h, \beta\nabla b]^{-1}(h\nabla\psi)\big)\,|_{t=0}$ (see eq. (4.2)); and one has for any $t \in (0, \min\{T^\star, T/M\}]$,

$$\big\|\zeta - \zeta_{\rm GN}\big\|_{L^\infty(0,t;H^s)} + \big\|\nabla\psi - \mathbf{v}_{\rm GN}\big\|_{L^\infty(0,t;Y^s)} \leq C\,\mu^2\,t\,\big(\,\|\zeta\|_{L^\infty(0,t;H^{s+6})} + \|\nabla\psi\|_{L^\infty(0,t;H^{s+5})}\big)\,,$$

where we denote $\mathbf{v}_{\rm GN} = h_{\rm GN}^{-1}\mathfrak{T}[h_{\rm GN}, \beta\nabla b]\mathbf{u}_{\rm GN}$ and $h_{\rm GN} = 1 + \varepsilon\zeta_{\rm GN} - \beta b$.

4.4 Well-posedness

In this section we provide a proof of Theorem 4.2. The strategy mimics the standard "energy method" for hyperbolic first-order quasilinear systems [AG91, BGS07, Mét08], with specific adjustments due to the presence of high-order differential operators. In particular, we shall not use dispersive techniques such as Strichartz estimates (see for instance [Tao06]), consistently with the fact that our result should be robust enough to hold uniformly with respect to $\mu \in (0, \mu^\star)$.

Let us very roughly sketch a typical strategy concerning hyperbolic symmetrizable first-order quasilinear systems (such as the Saint-Venant system; see Section 3). Let us consider a system of the form

$$\mathcal{S}(\mathbf{U})\partial_t\mathbf{U}+\mathcal{S}_x(\mathbf{U})\partial_x\mathbf{U}+\mathcal{S}_y(\mathbf{U})\partial_y\mathbf{U}=0$$

where $\mathbf{U}(t,x,y)\in\mathbb{R}^n$ and $\mathcal{S},\mathcal{S}_x,\mathcal{S}_y$ are smooth functions with values into $n\times n$ symmetric matrices. The Picard iteration scheme consists in proving that we can define a sequence $\mathbf{U}_n$ by solving inductively the linearized system:

$$\mathcal{S}(\mathbf{U}_n)\partial_t\mathbf{U}_{n+1}+\mathcal{S}_x(\mathbf{U}_n)\partial_x\mathbf{U}_{n+1}+\mathcal{S}_y(\mathbf{U}_n)\partial_y\mathbf{U}_{n+1}=0,$$

and that the sequence converges (up to taking a subsequence) towards the desired solution of the nonlinear equation. Both the well-posedness of the Cauchy problem associated with the linearized system and the convergence result rely on robust *a priori* estimates, which can be derived as follows. Consider sufficiently smooth and decaying solutions of the system

$$\mathcal{S}(\underline{\mathbf{U}})\partial_t\mathbf{U}+\mathcal{S}_x(\underline{\mathbf{U}})\partial_x\mathbf{U}+\mathcal{S}_y(\underline{\mathbf{U}})\partial_y\mathbf{U}=0.$$

Testing the equation against $\mathbf{U}$, integrating by parts and using the symmetry of the matrices, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\big(\mathcal{S}(\underline{\mathbf{U}})\,\mathbf{U}\,,\,\mathbf{U}\big)_{L^2}\leq C(\big|\underline{\mathbf{U}}\big|_{W^{1,\infty}},\big|\partial_t\underline{\mathbf{U}}\big|_{L^\infty})\big|\mathbf{U}\big|_{L^2}^2.$$

In order to control higher-order derivatives, we may differentiate the system $\mathbf{k}$ times and test against $\partial^{\mathbf{k}}\mathbf{U}$. Using the gain of one derivative from commutator estimates (see Proposition A.9), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\big(\mathcal{S}(\underline{\mathbf{U}})\,\partial^{\mathbf{k}}\mathbf{U}\,,\,\partial^{\mathbf{k}}\mathbf{U}\big)_{L^2}\leq C(\big|\underline{\mathbf{U}}\big|_{H^{\max\{|\mathbf{k}|,1+s_\star\}}},\big|\partial_t\underline{\mathbf{U}}\big|_{H^{\max\{|\mathbf{k}|-1,s_\star\}}})\big|\mathbf{U}\big|_{H^{|\mathbf{k}|}}^2$$

where $s_\star>d/2$. By considering the above with any $\mathbf{k}\in\mathbb{N}^d$ such that $|\mathbf{k}|\in\{0,\dots,s\}$ where $s\in\mathbb{N}$, $s\geq 1+s_\star>1+d/2$, and provided that $\mathcal{S}(\underline{\mathbf{U}})$ is uniformly positive definite, we deduce a control of $\mathbf{U}\in L^\infty(0,T;H^s)$ (by Gronwall's estimate) and hence $\partial_t\mathbf{U}\in L^\infty(0,T;H^{s-1})$ (using the system of equations). This is our desired *a priori* energy estimate which, thanks to a regularization and a limiting procedure, eventually yield the well-posedness of the Cauchy problem for the linearized system. Because the control asked on the reference state $\underline{U}$ is the same as the control we provide on the solution, U, we may expect that the Picard iteration scheme converges, towards a solution of the nonlinear system.

When trying to adapt the strategy to the Green-Naghdi system, eq. (4.3), the main objection to robust energy estimates stem from the presence of nonlinear high-order differential operators, as the gain of one derivative due to commutator estimates is in principle insufficient to treat commutators as zero-order remainder terms. This problem is however only apparent, and a careful study of these high-order operators reveal that they are in fact of order one when considering the correct functional spaces, X^n and Y^n.

In Proposition 4.7, we extract the quasilinear structure of the Green-Naghdi system, which in this case is nothing but the linearized system about *constant* states, just as with hyperbolic systems. We then provide in Proposition 4.8 the key *a priori* energy estimates of the linearized system. Finally, we detail in Section 4.4.3 the proof, via a parabolic regularization of the equations, of the (local-in-time) existence and uniqueness of a solution to the Cauchy problem.

4.4.1 The quasilinear structure

Proposition 4.7. *Let $d\in\{1,2\}$, $s_\star\in\mathbb{N}$, $s_\star>d/2$, $\mathbf{k}\in\mathbb{N}^d$ be a non-trivial multi-index and denote $n=\max\{|\mathbf{k}|,1+s_\star\}$. Let $\mu^\star>0$, $T>0$ and $M^\star\geq 0$. Then there exists $C,\tilde{C}>0$ such that the following holds for any $(\mu,\varepsilon,\beta)\in\mathcal{P}_{\rm SW}$, any $\zeta\in\mathcal{C}^0([0,T];H^n(\mathbb{R}^d))\cap\mathcal{C}^1([0,T];H^{n-1}(\mathbb{R}^d))$,*

$b \in W^{n+1,\infty}(\mathbb{R}^d)$ such that Assumption 2.1 holds uniformly on $[0,T)$ and any $\mathbf{u} \in \mathcal{C}^0([0,T];X^n) \cap \mathcal{C}^1([0,T];X^{n-1})$ such that system eq. (4.3) holds and

$$M \stackrel{\text{def}}{=} \big\|\varepsilon\zeta\big\|_{L^\infty(0,T;H^{1+s_\star})} + \big\|\varepsilon\mathbf{u}\big\|_{L^\infty(0,T;X^{1+s_\star})} + \big|\beta b\big|_{W^{n+1,\infty}} \leq M^\star.$$

Then $\zeta^{(\mathbf{k})} \stackrel{\text{def}}{=} \partial^{\mathbf{k}}\zeta$ and $\mathbf{u}^{(\mathbf{k})} \stackrel{\text{def}}{=} \partial^{\mathbf{k}}\mathbf{u}$ satisfy

$$\begin{cases} \partial_t\zeta^{(\mathbf{k})} + \varepsilon\mathbf{u}\cdot\nabla\zeta^{(\mathbf{k})} + h\nabla\cdot\mathbf{u}^{(\mathbf{k})} = r_{(\mathbf{k})}, \\ \big(\mathrm{Id} + \mu\mathcal{T}[h,\beta\nabla b]\big)\partial_t\mathbf{u}^{(\mathbf{k})} + \nabla\zeta^{(\mathbf{k})} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u}^{(\mathbf{k})} + \mu\varepsilon\mathcal{Q}[h,\beta\nabla b,\mathbf{u}]\mathbf{u}^{(\mathbf{k})} = \mathbf{r}_{(\mathbf{k})}, \end{cases} \tag{4.7}$$

with $h = 1 + \varepsilon\zeta - \beta b$ and (abusing notations)

$$\begin{aligned} \mathcal{Q}[h,\beta\nabla b,\mathbf{u}]\mathbf{u}^{(\mathbf{k})} &\stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}^{(\mathbf{k})})\big)\Big) \\ &+ \frac{\beta}{2h}\Big(\nabla\big(h^2(\mathbf{u}\cdot\nabla)(\mathbf{u}^{(\mathbf{k})}\cdot\nabla b)\big) - h^2\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}^{(\mathbf{k})})\big)\nabla b\Big) + \beta^2\big((\mathbf{u}\cdot\nabla)(\mathbf{u}^{(\mathbf{k})}\cdot\nabla b)\big)\nabla b, \end{aligned}$$

and where $r_{(\mathbf{k})}, \mathbf{r}_{(\mathbf{k})}$ enjoy the estimate

$$\big|r_{(\mathbf{k})}\big|_{L^2} + \big|\mathbf{r}_{(\mathbf{k})}\big|_{Y^0} \leq C\ M\ \left(\big|\zeta\big|_{H^{|\mathbf{k}|}} + \big|\mathbf{u}\big|_{X^{|\mathbf{k}|}}\right). \tag{4.8}$$

Moreover, for any $\tilde\zeta, \tilde{\mathbf{u}}$ satisfying the same assumptions and denoting $\tilde r_{(\mathbf{k})}, \tilde{\mathbf{r}}_{(\mathbf{k})}$ the corresponding residuals, one has

$$\begin{aligned} \big|r_{(\mathbf{k})} - \tilde r_{(\mathbf{k})}\big|_{L^2} + \big|\mathbf{r}_{(\mathbf{k})} - \tilde{\mathbf{r}}_{(\mathbf{k})}\big|_{Y^0} &\leq \tilde C\ M\ \left(\big|\zeta - \tilde\zeta\big|_{H^{|\mathbf{k}|}} + \big|\mathbf{u} - \tilde{\mathbf{u}}\big|_{X^{|\mathbf{k}|}}\right) \\ &+ \left\langle \tilde C\ M_{\mathbf{k}}\ \left(\big|\zeta - \tilde\zeta\big|_{H^{s_\star}} + \big|\mathbf{u} - \tilde{\mathbf{u}}\big|_{X^{s_\star}}\right)\right\rangle_{|\mathbf{k}|>1+s_\star}. \end{aligned} \tag{4.9}$$

with $M_{\mathbf{k}} \stackrel{\text{def}}{=} \big|\varepsilon\zeta\big|_{H^{|\mathbf{k}|}} + \big|\varepsilon\tilde\zeta\big|_{H^{|\mathbf{k}|}} + \big|\varepsilon\mathbf{u}\big|_{X^{|\mathbf{k}|}} + \big|\varepsilon\tilde{\mathbf{u}}\big|_{X^{|\mathbf{k}|}} + \beta\big|\nabla b\big|_{W^{n,\infty}}$.

Proof. Within this proof, we use the following convenient notation. We denote

$$\begin{aligned} a \sim_{L^2} b \quad &\Longleftrightarrow \quad a - b = r, \\ \mathbf{a} \sim_{Y^0} \mathbf{b} \quad &\Longleftrightarrow \quad \mathbf{a} - \mathbf{b} = \mathbf{r} \end{aligned}$$

with $\big|r\big|_{L^2}$ and $\big|\mathbf{r}\big|_{Y^0}$ satisfying (4.8) with $C = C(|\mathbf{k}|, \mu^\star, h_\star^{-1}, M^\star)$.

First equation. We start by applying the operator $\partial^{\mathbf{k}}$ to the first equation of eq. (4.3):

$$\partial_t\zeta^{(\mathbf{k})} + \partial^{\mathbf{k}}(h\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla h) = 0.$$

By Proposition A.9, Proposition A.15 and using the obvious continuous embedding $X^0 \subset L^2(\mathbb{R}^d)^d$, one finds

$$\partial^{\mathbf{k}}\nabla\cdot(h\mathbf{u}) \sim_{L^2} h\nabla\cdot\partial^{\mathbf{k}}\mathbf{u} + \varepsilon\mathbf{u}\cdot\nabla\partial^{\mathbf{k}}\zeta$$

as desired.

Second equation. Here we multiply eq. $(4.3)_2$ with h before applying the operator $\partial^{\mathbf{k}}$. Proceeding a above and using the dual continuous embedding $L^2(\mathbb{R}^d)^d \subset Y^0$, we have

$$\partial^{\mathbf{k}}\big(\varepsilon h(\mathbf{u}\cdot\nabla)\mathbf{u}\big) \sim_{Y^0} \varepsilon h(\mathbf{u}\cdot\nabla)\partial^{\mathbf{k}}\mathbf{u}.$$

Now we consider the contribution of

$$\mu\varepsilon h\mathcal{Q}[h,\mathbf{u}] = \frac{-\mu\varepsilon}{3}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\Big),$$

Thanks to the μ prefactor and using the non-uniform embeddings of Lemma A.20, the second term gives no contribution, and the first one satisfies

$$\frac{-\mu\varepsilon}{3}\partial^{\mathbf{k}}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u})\Big) \sim_{Y^0} \frac{-\mu\varepsilon}{3}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\partial^{\mathbf{k}}\nabla\cdot\mathbf{u})\Big).$$

We proceed similarly with the contribution of

$$\mu\varepsilon h\mathcal{Q}_b[h,\beta\nabla b,\mathbf{u}] = \frac{\mu\varepsilon\beta}{2}\Big(\nabla\big(h^2(\mathbf{u}\cdot\nabla)^2 b\big) - h^2\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\nabla b\Big) + \mu\varepsilon\beta^2 h\big((\mathbf{u}\cdot\nabla)^2 b\big)\nabla b$$

and deduce

$$\mu\varepsilon\partial^{\mathbf{k}}\Big(h\mathcal{Q}_b[h,\beta\nabla b,\mathbf{u}]\Big) \sim_{Y^0} \frac{\mu\varepsilon\beta}{2}\Big(\nabla\big(h^2(\mathbf{u}\cdot\nabla)(\partial^{\mathbf{k}}u\cdot\nabla b)\big) - h^2\big((\mathbf{u}\cdot\nabla)(\partial^{\mathbf{k}}\nabla\cdot\mathbf{u})\big)\nabla b\Big) + \mu\varepsilon\beta^2 h\big((\mathbf{u}\cdot\nabla)(\partial^{\mathbf{k}}\mathbf{u}\cdot\nabla b)\big)\nabla b.$$

There remains the contribution of

$$h\big(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b]\big)\partial_t\mathbf{u} = h\partial_t\mathbf{u} - \frac{\mu}{3}\nabla(h^3\nabla\cdot\partial_t\mathbf{u}) + \frac{\mu}{2}\Big(\nabla\big(h^2(\beta\nabla b)\cdot\partial_t\mathbf{u}\big) - h^2(\beta\nabla b)\nabla\cdot\partial_t\mathbf{u}\Big) + \mu h\beta^2(\nabla b\cdot\partial_t\mathbf{u})\nabla b.$$

Let us first notice that we have $\partial_t\mathbf{u}\in X^j$ for any $j\in\{0,\dots,\max\{s_\star,k-1\}\}$ by using eq. $(4.3)_2$. Indeed, applying the operator $\mathfrak{T}[h,\beta\nabla b]^{-1} = \big(h\big(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b]\big)\big)^{-1}$, we have, by Lemma A.22,

$$\big|\partial_t\mathbf{u}\big|_{X^j} \le C(|\mathbf{k}|,\mu^\star,h_\star^{-1},M^\star)\big(\big|\zeta\big|_{H^{j+1}} + \big|\mathbf{u}\big|_{X^{j+1}}\big).$$

We may then proceed as above to prove that

$$\partial^{\mathbf{k}}\Big(h\big(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b]\big)\partial_t\mathbf{u}\Big) \sim_{Y^0} h\big(\mathrm{Id}+\mu\mathcal{T}[h,\beta\nabla b]\big)\partial^{\mathbf{k}}\partial_t\mathbf{u}.$$

This concludes the proof of eq. (4.8). The proof of eq. (4.9) is obtained in the exact same way. □

4.4.2 A priori energy estimates

Proposition 4.8. *Let $d\in\{1,2\}$, $h_\star>0$, $\mu^\star>0$, $T>0$ and $M>0$. Then there exists $C>0$ such that for any $(\mu,\varepsilon,\beta)\in\mathcal{P}_{\rm SW}$, any $(\underline{\zeta},\underline{\mathbf{u}})\in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d)^{1+d})\cap W^{1,\infty}(0,T;L^\infty(\mathbb{R}^d)^{1+d})$ and $b\in W^{1,\infty}(\mathbb{R}^d)$ such that $\underline{h}=1+\varepsilon\underline{\zeta}-\beta b$, satisfies Assumption 2.1 uniformly for $t\in(0,T)$ and*

$$\big\|\varepsilon\underline{\zeta}\big\|_{L^\infty(0,T;W^{1,\infty})} + \big\|\varepsilon\partial_t\underline{\zeta}\big\|_{L^\infty(0,T;L^\infty)} + \big\|\varepsilon\underline{\mathbf{u}}\big\|_{L^\infty(0,T;W^{1,\infty})} + \big|\beta b\big|_{W^{1,\infty}} \le M$$

as well as any $(\zeta,\mathbf{u})\in L^\infty(0,T;H^1(\mathbb{R}^d)\times X^1)\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^d)\times X^0)$ satisfying

$$\begin{cases} \partial_t\zeta + \varepsilon\underline{\mathbf{u}}\cdot\nabla\zeta + \underline{h}\nabla\cdot\mathbf{u} = r, \\ \big(\mathrm{Id}+\mu\mathcal{T}[\underline{h},\beta\nabla b]\big)\partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\underline{\mathbf{u}}\cdot\nabla)\mathbf{u} + \mu\varepsilon\mathcal{Q}[\underline{h},\beta\nabla b,\underline{\mathbf{u}}]\mathbf{u} = \mathbf{r}, \end{cases} \tag{4.10}$$

with

$$\mathcal{Q}[\underline{h},\beta\nabla b,\underline{\mathbf{u}}]\mathbf{u} \stackrel{\rm def}{=} \frac{-1}{3\underline{h}}\nabla\Big(h^3\big((\underline{\mathbf{u}}\cdot\nabla)(\nabla\cdot\mathbf{u})\big)\Big) + \frac{\beta}{2\underline{h}}\Big(\nabla\big(\underline{h}^2(\underline{\mathbf{u}}\cdot\nabla)(\mathbf{u}\cdot\nabla b)\big) - \underline{h}^2\big((\underline{\mathbf{u}}\cdot\nabla)(\nabla\cdot\mathbf{u})\big)\nabla b\Big) + \beta^2\big((\underline{\mathbf{u}}\cdot\nabla)(\mathbf{u}\cdot\nabla b)\big)\nabla b$$

and $(r, \mathbf{r}) \in L^\infty(0,T; L^2(\mathbb{R}^d)\times Y^0)$*, the following holds. Denoting*

$$\mathcal{E}(t) \overset{\rm def}{=} \frac12\int_{\mathbb{R}^d} |\zeta|^2 + \underline{h}|\mathbf{u}|^2 + \ \mu\,\underline{h}\mathcal{T}[\underline{h},\beta\nabla b]\mathbf{u}\cdot\mathbf{u}\,{\rm d}\mathbf{x},$$

we have

$$\frac{\rm d}{{\rm d}t}\mathcal{E} \le C\ M\ \mathcal{E} + C\ \big(\big|r\big|_{L^2} + \big|\mathbf{r}\big|_{Y^0}\big)\ \mathcal{E}^{1/2},$$

and as a consequence

$$\mathcal{E}^{1/2}(t) = \mathcal{E}^{1/2}(0)e^{CMt/2} + C\int_0^t e^{CM(t-\tau)/2}\big(\big|r\big|_{L^2} + \big|\mathbf{r}\big|_{Y^0}\big)(\tau)\,{\rm d}\tau.$$

Proof. We test the first equation of eq. (4.10) against ζ and the second against $\underline{h}\mathbf{u}$. It follows, after some algebra,

$$\frac{\rm d}{{\rm d}t}\mathcal{E} = \mathcal{I}_1 + \mu\mathcal{I}_2 + \mu\mathcal{I}_b + \mathcal{I}_r$$

with

$$\begin{aligned}
\mathcal{I}_1 &= \frac12\int_{\mathbb{R}^d} \varepsilon(\nabla\cdot\underline{\mathbf{u}})|\zeta|^2 + 2(\varepsilon\nabla\underline{\zeta} - \beta\nabla b)\cdot\mathbf{u}\zeta + \varepsilon\big(\partial_t\underline{\zeta} + \nabla\cdot(\underline{h}\,\underline{\mathbf{u}})\big)|\mathbf{u}|^2\,{\rm d}\mathbf{x},\\
\mathcal{I}_2 &= \frac{\varepsilon}{6}\int_{\mathbb{R}^d} \big(3\underline{h}^2\partial_t\underline{\zeta} + \nabla\cdot(\underline{h}^3\underline{\mathbf{u}})\big)(\nabla\cdot\mathbf{u})^2\,{\rm d}\mathbf{x},\\
\mathcal{I}_b &= \frac{\varepsilon}{2}\int_{\mathbb{R}^d} -\big(2\underline{h}\partial_t\underline{\zeta} + \nabla\cdot(\underline{h}^2\underline{\mathbf{u}})\big)(\beta\nabla b\cdot\mathbf{u})\nabla\cdot\mathbf{u} + \big(\partial_t\underline{\zeta} + \nabla\cdot(\underline{h}\,\underline{\mathbf{u}})\big)(\beta\nabla b\cdot\underline{\mathbf{u}})^2\,{\rm d}\mathbf{x},\\
\mathcal{I}_r &= \int_{\mathbb{R}^d} r\zeta + \underline{h}\mathbf{r}\cdot\mathbf{u}\,{\rm d}\mathbf{x}.
\end{aligned}$$

We deduce immediately, by Cauchy-Schwarz inequality,

$$\varepsilon|\mathcal{I}_1| + \varepsilon|\mathcal{I}_2| + \varepsilon|\mathcal{I}_b|\ \le\ C\ M\ \big(\big|\zeta\big|_{L^2}^2 + \big|\mathbf{u}\big|_{X^0}^2\big)$$

where $C = C(\mu^\star, \big|\varepsilon\partial_t\underline{\zeta}\big|, \big|\underline{h}\big|_{W^{1,\infty}}, \big|\varepsilon\underline{\mathbf{u}}\big|_{W^{1,\infty}}, \big|\beta\nabla b\big|_{L^\infty})$. In the same way, one has immediately

$$|\mathcal{I}_r| \le \big|r\big|_{L^2}\big|\zeta\big|_{L^2} + \big|\mathbf{r}\big|_{Y^0}\big|\underline{h}\mathbf{u}\big|_{X^0} \le \big|r\big|_{L^2}\big|\zeta\big|_{L^2} + C(\mu^\star, \big|\underline{h}\big|_{W^{1,\infty}})\big|\mathbf{r}\big|_{Y^0}\big|\mathbf{u}\big|_{X^0}.$$

There only remains to use Lemma A.21 to deduce

$$\big|\zeta\big|_{L^2}^2 + \big|\mathbf{u}\big|_{X^0}^2 \le C(h_\star^{-1})\mathcal{E},$$

and the result follows. □

4.4.3 Completion of the proof

The well-posedness result of Theorem 4.2 may be deduced, exploiting energy estimates similar to the ones derived above. Several strategies are known, including the use of regularizing operators as in [Mét09] or a duality method as in [AG91, BGS07]. However both these methods rely on pseudo-differential tools [6] which we have not introduced in these notes. Hence we will follow here the a parabolic regularization approach as advocated in [FI15]. The whole strategy is similar to that applied to the Navier-Stokes and Euler equations in [Kat72], as described in [Tao18].

[6] in fact in our framework we only need a generalization of the product and commutator estimates given in Appendix A.1 for Fourier multipliers such as $(\mathrm{Id}-\nu\Delta)^{-1/2}$, which follow from the Littlewood-Paley theory, *i.e.* dyadic decomposition of the frequency space.

Step 1: local existence for the regularized system. We introduce the regularized system

$$\begin{cases} \partial_t\zeta_\nu - \nu\Delta\zeta_\nu + \nabla\cdot(h_\nu\mathbf{u}_\nu) = 0 \\ h_\nu\big(\mathrm{Id}+\mu\mathcal{T}[h_\nu,\beta\nabla b]\big)(\partial_t\mathbf{u}_\nu - \nu\Delta\mathbf{u}_\nu) + h_\nu\nabla\zeta_\nu + h_\nu\varepsilon(\mathbf{u}_\nu\cdot\nabla)\mathbf{u}_\nu + \mu\varepsilon\underline{h}_\nu\mathcal{Q}[h_\nu,\beta\nabla b,\mathbf{u}_\nu] = \mathbf{0}, \end{cases}$$

where $h_\nu \stackrel{\text{def}}{=} 1+\varepsilon\zeta_\nu-\beta b$ and $\nu>0$ is a parameter which will eventually go to zero. By Lemma A.21, we may invert the operator $\mathfrak{T}[h,\beta\nabla b] = h_\nu\big(\mathrm{Id}+\mu\mathcal{T}[h_\nu,\beta\nabla b]\big)$ (for sufficiently regular data) and write the system under the abstract form

$$\partial_t\begin{pmatrix}\zeta_\nu\\ \mathbf{u}_\nu\end{pmatrix} - \nu\Delta\begin{pmatrix}\zeta_\nu\\ \mathbf{u}_\nu\end{pmatrix} = \mathbf{F}(\zeta_\nu,\mathbf{u}_\nu). \tag{4.11}$$

By Duhamel's formula, any solution $(\zeta_\nu,\mathbf{u}_\nu)\in C^0([0,T_\nu);H^s(\mathbb{R}^d)\times X^s)\cap C^1([0,T_\nu);H^{s-1}(\mathbb{R}^d)\times X^{s-1})$ satisfies

$$\begin{pmatrix}\zeta_\nu\\ \mathbf{u}_\nu\end{pmatrix}(t) = e^{\nu t\Delta}\begin{pmatrix}\zeta_\nu\\ \mathbf{u}_\nu\end{pmatrix}(0) + \int_0^t e^{\nu(t-s)\Delta}\mathbf{F}(\zeta_\nu,\mathbf{u}_\nu)(s)\,\mathrm{d}s. \tag{4.12}$$

Here, $e^{\nu t\Delta}$ is the Fourier multiplier (applied to all components)

$$\widehat{e^{\nu t\Delta}f}(\boldsymbol{\xi}) = e^{-\nu t|\boldsymbol{\xi}|^2}\widehat{f}(\boldsymbol{\xi}),$$

and we have, by Plancherel's formula for any $s\in\mathbb{R}$ and $t>0$

$$\big\|e^{\nu t\Delta}\big\|_{H^s\to H^s}\leq 1$$

and there exists $C_{s,s'}$, depending only on $s'-s\geq 0$, such that

$$\big\|e^{\nu t\Delta}\big\|_{H^s\to H^{s'}}\leq C_{s,s'}\big(1+(\nu t)^{-\frac{s'-s}{2}}\big).$$

Here we exhibited the regularizing effect of the heat operator. The estimate above indicates that we can gain regularity in space by using integrability in time: by Hölder's inequality, we have

$$\big\|e^{\nu t\Delta}\big\|_{L^p(0,T;H^s)\to L^{p'}(0,T;H^{s'})}\leq C_{s,s',p,q}$$

for any $1\leq p'<p\leq\infty$ if $\frac{s'-s}{2}<\frac{1}{p'}-\frac{1}{p}$.

Proposition 4.9. *Let $d\in\{1,2\}$, $h_\star>0$, $\mu^\star>0$ and $\nu>0$. Let $(\mu,\varepsilon,\beta)\in\mathcal{P}_{\rm SW}$, $s\in\mathbb{N}$, $s>d/2+1$ and $\zeta_0\in H^s(\mathbb{R}^d)$ and $b\in W^{s+1,\infty}(\mathbb{R}^d)$ be such that Assumption 2.1 holds, and $\mathbf{u}_0\in X^s$. Then there exists $T_\nu>0$ and $(\zeta_\nu,\mathbf{u}_\nu)\in C^0([0,T_\nu);H^s(\mathbb{R}^d)\times X^s)\cap L^2(0,T_\nu;H^{s+1}(\mathbb{R}^d)\times X^{s+1})$ solution to (4.12). Moreover, this solution is unique.*

Proof. Here we only need to use the standard Banach fixed point argument for

$$\Phi:\begin{pmatrix}\zeta_0\\ \mathbf{u}_\nu\end{pmatrix}\mapsto e^{\nu t\Delta}\begin{pmatrix}\zeta_0\\ \mathbf{u}_0\end{pmatrix} + \int_0^t e^{\nu(t-\tau)\Delta}\mathbf{F}(\zeta_\nu,\mathbf{u}_\nu)(\tau)\,\mathrm{d}\tau.$$

Consider $Z^s_{T_\nu} = C^0([0,T_\nu);H^s(\mathbb{R}^d)\times X^s)\cap L^2(0,T_\nu;H^{s+1}(\mathbb{R}^d)\times X^{s+1})$, endowed with the norm

$$\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^s_{T_\nu}} \stackrel{\text{def}}{=} \big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{L^\infty(0,T_\nu;H^s\times X^s)} + \nu^{1/2}\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{L^2(0,T_\nu;H^{s+1}\times X^{s+1})}.$$

Given $R>0$ and $h_\star>0$, we denote

$$B_{R,h_\star} = \Big\{(\zeta_\nu,\mathbf{u}_\nu)\in Z^s_{T_\nu}\ :\ \big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^s_{T_\nu}}\leq R,\ \inf_{t\in(0,T_\nu)} 1+\varepsilon\zeta_\nu-\beta b\geq h_\star\Big\}.$$

By product estimates, Proposition A.7, and Lemma A.22, we have for any $(\zeta_\nu,\mathbf{u}_\nu)\in B_{R,h_\star}$,

$$\big\|\mathbf{F}(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^{s-1}_{T_\nu}}\le C(R,h_\star^{-1}).$$

Thanks to the regularizing properties of the heat operator (using the energy method or by Plancherel's formula), there exists $C(T_\nu)$ such that

$$\big\|(e^{\nu t\Delta}\zeta_0,e^{\nu t\Delta}\mathbf{u}_0)\big\|_{Z^s_{T_\nu}}\le C(T_\nu)\big(\big|\zeta_\nu(0)\big|_{H^s}+\big|\mathbf{u}_\nu(0)\big|_{X^s}\big)$$

and $c_\nu=C(\nu,T_\nu)$ with $c_\nu\to 0$ as $T_\nu\to 0$ such that

$$\Big\|\int_0^t e^{\nu(t-\tau)\Delta}\mathbf{F}(\zeta_\nu,\mathbf{u}_\nu)(\tau)\,\mathrm{d}\tau\Big\|_{Z^s_{T_\nu}}\le c_\nu\big\|\mathbf{F}(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^{s-1}_{T_\nu}}.$$

Moreover, we have

$$\big\|\partial_t e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;L^\infty)}=\big\|\Delta e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;L^\infty)}\lesssim\big\|e^{\nu t\Delta}\zeta_0\big\|_{L^2(0,T;H^{s+1})}.$$

Hence we can choose R sufficiently large and then T_ν sufficiently small so that Φ is a contraction map on $B_{R,h_\star/2}$, and the result follows. □

Proposition 4.10. *Let $d\in\{1,2\}$, $h_\star>0$, $\mu^\star>0$ and $\nu>0$, $s_\star>d/2$. Let $(\mu,\varepsilon,\beta)\in\mathcal{P}_{\rm SW}$. Let $\zeta_0\in H^\infty(\mathbb{R}^d)$ and $b\in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 2.1 holds, and $\mathbf{u}_0\in H^\infty(\mathbb{R}^d)^d$. Then there exists a unique $T_{\max}>0$ and a unique $(\zeta_\nu,\mathbf{u}_\nu)\in\mathcal{C}^\infty([0,T_{\max});H^\infty(\mathbb{R}^d)^{1+d})$ maximal solution to (4.11). Moreover, if $T_{\max}<\infty$, then*

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^{1+s_\star})}+\big\|\mathbf{u}_\nu\big\|_{L^\infty(0,T^\star;X^{1+s_\star})}\to\infty\quad\text{or}\quad\inf_{\mathbb{R}^d}1+\varepsilon\zeta_\nu-\beta b\to 0.$$

Proof. By iterating Proposition 4.9, we have for any given $s\ge s_\star$, the existence and uniqueness of a maximal Cauchy development, *i.e.* $T^\star>0$ and a mild solution $(\zeta_\nu,\mathbf{u}_\nu)\in\mathcal{C}^0([0,T^\star);H^s(\mathbb{R}^d)\times X^s)$ such that if $T_\star<\infty$,

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^s)}+\big\|\mathbf{u}_\nu\big\|_{L^\infty(0,T^\star;X^s)}\to\infty\quad\text{or}\quad\inf_{\mathbb{R}^d}1+\varepsilon\zeta_\nu-\beta b\to 0.$$

By the uniqueness of the mild solution, the solutions do not depend on the regularity index, s, as long as their domain of existence coincide. In principle, the maximal time of existence, $T_\star$, may depend on the regularity index, s, we consider. Such is not the case thanks to the independent blowup criterion: if $T_\star<\infty$, then

$$\big\|\zeta_\nu\big\|_{L^\infty(0,T^\star;H^{1+s_\star})}+\big\|\mathbf{u}_\nu\big\|_{L^\infty(0,T^\star;X^{1+s_\star})}\to\infty\quad\text{or}\quad\inf_{\mathbb{R}^d}1+\varepsilon\zeta_\nu-\beta b\to 0.$$

This blowup criterion is obtained by contradiction and using tame product estimates of Proposition A.7, and Lemma A.22. Indeed, assuming that the above quantities are bounded (respectively from above and below), we find, following the steps of Proposition 4.9 that

$$\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}}\le C_1\big|(\zeta_\nu,\mathbf{u}_\nu)(\tau_1)\big|_{H^s\times X^s}+C_2\nu^{-1/2}(\tau_2-\tau_1)^{1/2}\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{L^\infty(\tau_1,\tau_2;H^s\times X^s)},$$

for any $0<\tau_1<\tau_2<T^\star$ and denoting

$$\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}}=\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{L^\infty(\tau_1,\tau_2;H^s\times X^s)}+\nu^{1/2}\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{L^2(\tau_1,\tau_2;H^{s+1}\times X^{s+1})}.$$

We can then choose τ_1 sufficiently large to absorb the second term of the right-hand side, and deduce an estimate on $\big\|(\zeta_\nu,\mathbf{u}_\nu)\big\|_{Z^s_{[\tau_1,\tau_2]}}$, uniform with respect to $\tau_2\in(\tau_1,T^\star)$, from which the contradiction follows.

There only remains to prove that the solution has the desired regularity (in time). Differentiating the Duhamel formula with respect to time, we have that $(\zeta_\nu,\mathbf{u}_\nu)$ satisfies eq. (4.11) in the sense of spacetime distributions, and hence $(\zeta_\nu,\mathbf{u}_\nu)\in\mathcal{C}^1([0,T^\star);H^\infty(\mathbb{R}^d)^{1+d})$. We may then iterate eq. (4.11) to obtain the desired regularity. □

Step 2: local existence for smooth solutions. In order to be able to construct (smooth) solutions to eq. (4.3) from (smooth) solutions to the parabolic regularization, eq. (4.11), we need to obtain uniform energy estimates.

Proposition 4.11. *Let $d \in \{1,2\}$, $h_\star > 0$, $\mu^\star > 0$, $s_\star > d/2$, $s \in \mathbb{N}$ and $M^\star \geq 0$. Then there exists $T > 0$ and $C > 0$ such that for any $\nu \in (0,1]$, any $(\mu,\varepsilon,\beta) \in \mathcal{P}_{\rm SW}$, any $\zeta_0 \in H^\infty(\mathbb{R}^d)$ and $b \in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 2.1 holds, and $\mathbf{u}_0 \in H^\infty(\mathbb{R}^d)^d$ such that*

$$M \stackrel{\rm def}{=} \big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\varepsilon\mathbf{u}_0\big|_{X^{1+s_\star}} + \big|\beta b\big|_{W^{\max\{s+1,2+s_\star+1\}.\infty}} \leq M^\star,$$

the unique maximal solution to eq. (4.3) provided by Proposition 4.10 satisfies $T^\star \geq T/M$ and for any $t \in [0,T/M)$,

$$\big|\zeta_\nu\big|_{H^s}(t) + \big|\mathbf{u}_\nu\big|_{X^s}(t) \leq C \times \big(\big|\zeta_0\big|_{H^s} + \big|\mathbf{u}_0\big|_{X^s}\big).$$

Proof. The estimate (and hence the lower bound on the maximal time of existence by the blowup criterion) follow from a priori energy estimates similar to Proposition 4.7 and Proposition 4.8. There are however additional terms to be taken care of. Let us just consider a handful of them. After testing the contribution

$$h_\nu\big(\mathrm{Id} + \mu\mathcal{T}[h_\nu,\beta\nabla b]\big)(\partial_t\partial^{\mathbf{k}}\mathbf{u}_\nu - \nu\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu)$$

against $\partial^{\mathbf{k}}\mathbf{u}_\nu$, we have to estimate the additional contributions

$$\big(h_\nu\big(\mathrm{Id} + \mu\mathcal{T}[h_\nu,\beta\nabla b]\big)(\partial_t\partial^{\mathbf{k}}\mathbf{u}_\nu - \nu\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu), \partial^{\mathbf{k}}\mathbf{u}_\nu\big)_{L^2}$$

Let us discard the terms stemming from the variable topography and concentrate on

$$-\nu\big(h_\nu\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu, \partial^{\mathbf{k}}\mathbf{u}_\nu\big)_{L^2} - \frac{\nu\mu}{3}\big(h_\nu^3\nabla\cdot\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu, \nabla\cdot\partial^{\mathbf{k}}\mathbf{u}_\nu\big)_{L^2}.$$

Now we have, assuming $d = 2$ and denoting $\mathbf{u}_\nu = (\mathbf{u}_{\nu,x}, \mathbf{u}_{\nu,y})$,

$$\begin{aligned} -\nu\big(h_\nu\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu, \partial^{\mathbf{k}}\mathbf{u}_\nu\big)_{L^2} &= \nu\int_{\mathbb{R}^d} h_\nu\big(|\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,x}|^2 + |\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,y}|^2\big)\,\mathrm{d}\mathbf{x} \\ &\quad + \nu(\nabla h_\nu\cdot\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,x}, \partial^{\mathbf{k}}\mathbf{u}_{\nu,x})_{L^2} + \nu(\nabla h_\nu\cdot\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,y}, \partial^{\mathbf{k}}\mathbf{u}_{\nu,y})_{L^2}, \end{aligned}$$

and hence

$$\begin{aligned} -\nu\big(h_\nu\Delta\partial^{\mathbf{k}}\mathbf{u}_\nu, \partial^{\mathbf{k}}\mathbf{u}_\nu\big)_{L^2} &- \frac{\nu}{2}\int_{\mathbb{R}^d} h\big(|\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,x}|^2 + |\nabla\partial^{\mathbf{k}}\mathbf{u}_{\nu,y}|^2\big)\,\mathrm{d}\mathbf{x} \\ &\leq \frac{\nu}{2}\int_{\mathbb{R}^d} h^{-1}\big(|\nabla h\cdot\partial^{\mathbf{k}}\mathbf{u}_{\nu,x}|^2 + |\nabla h\cdot\partial^{\mathbf{k}}\mathbf{u}_{\nu,y}|^2\big)\,\mathrm{d}\mathbf{x}. \end{aligned}$$

The second term is estimated similarly, using that

$$\nabla\cdot\Delta\mathrm{Id}\partial^{\mathbf{k}}\mathbf{u}_\nu = \Delta\nabla\cdot\partial^{\mathbf{k}}\mathbf{u}_\nu.$$

Using the boundedness of ν, we may then conclude as in Proposition 4.8. We obtain in fact the improved differential inequality

$$\mathcal{E}_s'(t) + \nu\mathcal{E}_{s+1}(t) \leq C\ M\ \mathcal{E}_s(t)$$

where

$$\mathcal{E}_s(t) \stackrel{\rm def}{=} \frac{1}{2}\sum_{|\mathbf{k}|\leq s}\int_{\mathbb{R}^d} |\partial^{\mathbf{k}}\zeta|^2 + h|\partial^{\mathbf{k}}\mathbf{u}|^2 + \ \mu\, h\mathcal{T}[h,\beta\nabla b]\partial^{\mathbf{k}}\mathbf{u}\cdot\partial^{\mathbf{k}}\mathbf{u}\ \mathrm{d}\mathbf{x} \approx \big|\zeta_\nu\big|_{H^s}^2 + \big|\mathbf{u}_\nu\big|_{X^s}^2.$$

However we need to ensure that Assumption 2.1 holds uniformly with respect to $t \in [0, T/M]$ (lowering $h_\star$ if necessary). This is obtained using that

$$|\zeta(t, \mathbf{x}) - \zeta(0, \mathbf{x})| \leq \big|\partial_t \zeta\big|_{L^1(0,T;L^\infty(\mathbb{R}^d))} \lesssim \big|\nu \nabla\zeta + \nabla \cdot (h\mathbf{u})\big|_{L^1(0,T;L^\infty(\mathbb{R}^d))}$$
$$\leq \nu\big|\nabla\zeta\big|_{L^1(0,T;H^{1+s_\star}(\mathbb{R}^d))} + \big|h\mathbf{u}\big|_{L^1(0,T;H^{1+s_\star}(\mathbb{R}^d))}$$

and the above differential inequality. □

Proposition 4.12. *Let $d \in \{1, 2\}$, $h_\star > 0$, $\mu^\star > 0$, $s_\star \in \mathbb{N}$, $s_\star > d/2$, $s \in \mathbb{N}$ and $M^\star \geq 0$. Then there exists $T > 0$ and $C > 0$ such that for any $(\mu, \varepsilon, \beta) \in \mathcal{P}_{\rm SW}$, any $\zeta_0 \in H^\infty(\mathbb{R}^d)$ and $b \in W^{\infty,\infty}(\mathbb{R}^d)$ be such that Assumption 2.1 holds, and $\mathbf{u}_0 \in H^\infty(\mathbb{R}^d)$ such that*

$$\big|\varepsilon\zeta_0\big|_{H^{1+s_\star}} + \big|\beta b\big|_{W^{\max\{1+s,2+s_\star\},\infty}} + \big|\beta b\big|_{L^\infty} + \big|\varepsilon\mathbf{u}_0\big|_{X^{1+s_\star}} \leq M,$$

there exists $(\zeta, \mathbf{u}) \in \mathcal{C}^\infty([0, T/M]; H^\infty(\mathbb{R}^d)^{1+d})$ solution to (4.3) and satisfying for any $t \in [0, T/M]$,

$$\big|\zeta\big|_{H^s}(t) + \big|\mathbf{u}\big|_{X^s}(t) \leq C \times \big(\big|\zeta_0\big|_{H^s} + \big|\mathbf{u}_0\big|_{X^s}\big).$$

Sketch of the proof. We introduce (ν_n) a sequence such that $\nu_n \searrow 0$. By Proposition 4.11, there exists C, T, *independent of* n and a sequence $(\zeta_{\nu_n}, \mathbf{u}_{\nu_n}) \in \mathcal{C}^\infty([0, T/M); H^\infty(\mathbb{R}^d)^{1+d})$, uniformly bounded and equicontinuous (by Sobolev embedding) and satisfying (as well as an arbitrary number of derivatives) the desired estimate. By weak compactness, there exists a converging subsequence. From Arzelá-Ascoli theorem, the convergence holds locally uniformly. Hence we can take limits and deduce that the limit satisfies (4.3). The desired bound is a direct consequence of the identical (uniform in n) estimate on $(\zeta_{\nu_n}, \mathbf{u}_{\nu_n})$. □

Step 3: Existence and uniqueness of classical solutions We are now in position to prove Theorem 4.2.

Proof of Theorem 4.2. We start with the uniqueness. Let us consider two classical solution to eq. (4.6), $(\zeta_1, \mathbf{u}_1)$ and $(\tilde\zeta_2, \tilde{\mathbf{u}}_2) \in \mathcal{C}^0([0, T/M_0]; H^s(\mathbb{R}^d) \times X^s) \cap \mathcal{C}^1([0, T/M_0]; H^{s-1}(\mathbb{R}^d) \times X^{s-1})$, with same initial data $(\zeta_i, \mathbf{u}_i)\,|_{t=0} = (\zeta_0, \mathbf{u}_0)$. By Proposition 4.7 (with $\mathbf{k} = \mathbf{0}$), we have that the difference $(\zeta, \mathbf{u}) \stackrel{\rm def}{=} (\zeta_2 - \zeta_1, \mathbf{u}_2 - \mathbf{u}_1)$ satisfies

$$\begin{cases} \partial_t\zeta + \varepsilon\mathbf{u}_2 \cdot \nabla\zeta + h_2\nabla \cdot \mathbf{u} = r, \\ \big(\mathrm{Id} + \mu\mathcal{T}[h_2, \beta\nabla b]\big)\partial_t\mathbf{u} + \nabla\zeta + \varepsilon(\mathbf{u}_2 \cdot \nabla)\mathbf{u} + \mu\varepsilon\mathcal{Q}[h_2, \beta\nabla b, \mathbf{u}_2]\mathbf{u} = \mathbf{r}, \end{cases}$$

where the right-hand side satisfies, using in particular eq. (4.9),

$$\big|r\big|_{L^2} + \big|\mathbf{r}\big|_{Y^0} \leq C(M)\, M\, \left(\big|\zeta\big|_{L^2} + \big|\mathbf{u}\big|_{X^0}\right),$$

where

$$M = \big\|\varepsilon\zeta_1\big\|_{L^\infty(0,t;H^{s_\star})} + \big\|\varepsilon\mathbf{u}_1\big\|_{L^\infty(0,t;X^{s_\star})} + \big\|\varepsilon\zeta_2\big\|_{L^\infty(0,t;H^{s_\star})} + \big\|\varepsilon\mathbf{u}_2\big\|_{L^\infty(0,t;X^{s_\star})} + \big|\beta\nabla b\big|_{H^{1+s_\star}}.$$

We can now use Proposition 4.8 to deduce that

$$\mathcal{E}(t) \stackrel{\rm def}{=} \frac{1}{2}\int_{\mathbb{R}^d} |\zeta|^2 + h_2|\mathbf{u}|^2 + \ \mu\, h_2\mathcal{T}[h_2, \beta\nabla b]\mathbf{u} \cdot \mathbf{u}\, {\rm d}\mathbf{x},$$

where $h_2 = 1 + \varepsilon\zeta_2 - \beta b$, satisfies

$$\frac{\rm d}{{\rm d}t}\mathcal{E} \leq C(M)\ M\ \mathcal{E},$$

and hence, since $\mathcal{E}(0)=0$, $\mathcal{E}\equiv 0$. By Lemma A.21, we deduce the uniqueness.

Now for the existence, we shall construct a solution as the limit of a Cauchy sequence of smooth solutions. The argument is classical and often refered to as the Bona-Smith technique [BS75] although it appeared already in the work of Kato [Kat72]. We introduce the one-parameter family of mollifiers: for any $\iota>0$,

$$\mathsf{J}^\iota \stackrel{\rm def}{=} \chi(\nu|D|), \qquad \chi(\xi)=\mathbf{1}_{|\xi|\leq 1}.$$

We shall use the following limits which follow by Plancherel's theorem and dominated convergence: for any $\zeta\in H^s(\mathbb{R}^d)$,

$$\big|\zeta-\mathsf{J}^\iota\zeta\big|_{H^s}+\iota^{-1}\big|\zeta-\mathsf{J}^\iota\zeta\big|_{H^{s-1}}+\iota\big|\mathsf{J}^\iota\zeta\big|_{H^{s+1}}\to 0, \tag{4.13a}$$

and for any $\mathbf{u}\in X^s$,

$$\big|\mathbf{u}-\mathsf{J}^\iota\mathbf{u}\big|_{X^s}+\iota^{-1}\big|\mathbf{u}-\mathsf{J}^\iota\mathbf{u}\big|_{X^{s-1}}+\iota\big|\mathsf{J}^\iota\mathbf{u}\big|_{X^{s+1}}\to 0. \tag{4.13b}$$

By Proposition 4.12 with initial data $(\zeta_0^\iota,\mathbf{u}_0^\iota)\stackrel{\rm def}{=}(\mathsf{J}^\iota\zeta_0,\mathsf{J}^\iota\mathbf{u}_0)\in H^\infty(\mathbb{R}^d)^{1+d}$ (and assuming at first that the bottom topography is smooth [7]), there exists $C,T>0$, *independent of* ι, and $(\zeta_\iota,\mathbf{u}_\iota)\in\mathcal{C}^\infty([0,T/M_0];H^\infty(\mathbb{R}^d)^{1+d})$ solution to (4.3) and satisfying for any $t\in[0,T/M_0)$,

$$\big|\zeta_\iota\big|_{H^s}(t)+\big|\mathbf{u}_\iota\big|_{X^s}(t)\leq C\big(\big|\mathsf{J}^\iota\zeta_0\big|_{H^s}+\big|\mathsf{J}^\iota\mathbf{u}_0\big|_{X^s}\big)\leq C\big(\big|\zeta_0\big|_{H^s}+\big|\mathbf{u}_0\big|_{X^s}\big). \tag{4.14}$$

We wish to prove that, given a decreasing sequence $\iota_n\to 0$, the constructed $(\zeta_{\iota_n},\mathbf{u}_{\iota_n})$ is a Cauchy sequence. Proceeding as above, we may estimate the difference between two solution

$$(\zeta_{m,n},\mathbf{u}_{m,n})\stackrel{\rm def}{=}(\zeta_{\iota_n}-\zeta_{\iota_m},\mathbf{u}_{\iota_n}-\mathbf{u}_{\iota_m})$$

by Proposition 4.7 and Proposition 4.8. One obtains, for any $\mathbf{k}\in\mathbb{N}^d$, $0\leq|\mathbf{k}|\leq s'$, and any $n>m$,

$$\frac{\rm d}{{\rm d}t}\mathcal{E}_{\mathbf{k}}\leq C_{\mathbf{k}}(M,C)M\ \mathcal{E}_{\mathbf{k}}+C_{\mathbf{k}}(M,C)\mathcal{E}_{\mathbf{k}}^{1/2}\big(\big|\zeta_{\iota_m}\big|_{H^{|\mathbf{k}|+1}}+\big|\mathbf{u}_{\iota_m}\big|_{X^{|\mathbf{k}|+1}}\big)\big(\big|\zeta_{m,n}\big|_{H^{s_\star}}+\big|\mathbf{u}_{m,n}\big|_{X^{s_\star}}\big),$$

with

$$\mathcal{E}_{\mathbf{k}}(t)\stackrel{\rm def}{=}\frac12\int_{\mathbb{R}^d}|\partial^{\mathbf{k}}\zeta_{m,n}|^2+h_n|\partial^{\mathbf{k}}\mathbf{u}_{m,n}|^2+\ \mu\, h_n(\mathcal{T}[h_m,\beta\nabla b]\partial^{\mathbf{k}}\mathbf{u}_{m,n})\cdot(\partial^{\mathbf{k}}\mathbf{u}_{m,n})\ {\rm d}\mathbf{x},$$

where $h_n=1+\varepsilon\zeta_{\iota_n}-\beta b_{\iota_n}$, and as a consequence, there exists $C'>0$ such that

$$\begin{aligned}\big|\zeta_{m,n}\big|_{H^s}+\big|\mathbf{u}_{m,n}\big|_{X^s}=C'\big(\big|\zeta_{m,n}\big|_{H^s}+\big|\mathbf{u}_{m,n}\big|_{X^s}\big)(t=0)&\\ +C't\big(\big|\zeta_{\iota_m}\big|_{H^{s+1}}+\big|\mathbf{u}_{\iota_m}\big|_{X^{s+1}}\big)\big(\big|\zeta_{m,n}\big|_{H^{s_\star}}+\big|\mathbf{u}_{m,n}\big|_{X^{s_\star}}\big).&\end{aligned}$$

Using eq. (4.14) and applying eq. (4.13), we deduce that

$$\limsup_{m,n\to\infty}\Big(\big\|\zeta_{m,n}\big\|_{L^\infty(0,T/M;H^s)}+\big\|\mathbf{u}_{m,n}\big\|_{L^\infty(0,T/M;X^s)}\Big)=0,$$

and hence the sequence strongly converges in $\mathcal{C}^0([0,T/M];H^s\times X^s)$ towards $(\zeta,\mathbf{u})$, satisfying the desired initial condition. We have that $(\partial_t\zeta_{\iota_n},\partial_t\mathbf{u}_{\iota_n})\rightharpoonup(\partial_t\zeta,\partial_t\mathbf{u})$ in the sense of distributions, and hence $(\zeta,\mathbf{u})$ is a solution to eq. (4.3) in the sense of distributions. It follows from Lemma A.22 that $(\partial_t\zeta,\partial_t\mathbf{u})\in\mathcal{C}^0([0,T/M];H^{s-1}\times X^{s-1})$ and hence we have constructed a classical solution. It satisfies the desired estimate by eq. (4.14). □

[7] In order to deal with non-smooth topographies, we may consider the sequence of solutions corresponding to the mollified topographies

$$b^\iota\stackrel{\rm def}{=}\rho_\iota\star b=\int_{\mathbb{R}^d}\frac{1}{\iota^d}\rho\big(\frac{\cdot-\mathbf{y}}{\iota}\big)b(\mathbf{y})\ {\rm d}\mathbf{y},$$

where ρ is smooth, non-negative with compact support, and $\big|\rho\big|_{L^1}=1$.

A Useful tools

A.1 Product and commutator estimates

We provide elementary proofs of product, commutator and composition estimates, based on the following Hölder, Hausdorff-Young, interpolation and Sobolev embedding inequalities. The price to pay for the simplicity of the proof is that results are restricted to Sobolev spaces with integer indices; adapting the proofs Sobolev spaces with real indices would require for instance the Littlewood-Paley technology; see [AG91]. We refer to [Lan13, Appendix B] for sharp results and relevant references.

A.1.1 Basic inequalities

Lemma A.1 (Hölder inequality). *Let $d, n \in \mathbb{N}^\star$, $n \geq 2$, $r, p_1, \dots, p_n \in (0, +\infty]$ such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}.$$

There exists $C > 0$ such that for any $f_i \in L^{p_i}(\mathbb{R}^d)$ $(i \in \{1, \dots, n\})$, $\prod_{i=1}^n f_i \in L^r(\mathbb{R}^d)$ and

$$\big|\prod_{i=1}^n f_i\big|_{L^r} \leq C \prod_{i=1}^n \big|f_i\big|_{L^{p_i}}.$$

Proof. The case $n = 2$ and $r = 1$ is the standard Hölder inequality following from Young's inequality, *i.e.* the concavity of the logarithm. The case $r \in (0, +\infty)$ is deduced applying the case $r = 1$ to $|f_i|^r \in L^{p_i/r}(\mathbb{R}^d)$, the case $r = \infty$ is obvious. The result for $n \geq 2$ follows by induction on n. □

Lemma A.2 (Hausdorff-Young inequality). *Let $d \in \mathbb{N}^\star$, $p \in [1, 2]$ and denote $q \in [2, +\infty]$ such that $p^{-1} + q^{-1} = 1$. There exists $C > 0$ such that for any $f \in L^p(\mathbb{R}^d)$, $\widehat{f} \in L^q(\mathbb{R}^d)$ and*

$$\big|\widehat{f}\big|_{L^q} \leq C \big|f\big|_{L^p}.$$

Proof. The case $p = 1$ is obvious from its integral representation, the case $p = 2$ is Parseval's theorem, and the case $1 < p < 2$ follows from Riesz-Thorin interpolation theorem. □

Lemma A.3 (Interpolation inequality). *Let $d \in \mathbb{N}^\star$ and $s, s_-, s_+ \in \mathbb{R}$ such that $s_- < s_+$ and $s_- \leq s \leq s_+$. There exists $C > 0$ such that for any $f \in H^{s_+}(\mathbb{R}^d)$,*

$$\big|f\big|_{H^s} \leq C \big|f\big|_{H^{s_-}}^{\theta} \big|f\big|_{H^{s_+}}^{1-\theta},$$

with $\theta = \frac{s_+ - s}{s_+ - s_-}$.

Proof. We have

$$\big|f\big|_{H^s}^2 \lesssim \int_{\mathbb{R}^d} |\widehat{f}|^2 \langle\cdot\rangle^{2s} \, d\mathbf{x} = \int_{\mathbb{R}^d} \left(|\widehat{f}| \langle\cdot\rangle^{s_-} \right)^{2\theta} \left(|\widehat{f}| \langle\cdot\rangle^{s_+} \right)^{2(1-\theta)} \, d\mathbf{x}$$

and we conclude by Hölder's inequality. □

Lemma A.4 (Sobolev embedding). *Let $d \in \mathbb{N}^\star$, and $p \in [2, +\infty]$ Let $s \in \mathbb{R}$ such that $s > d(\frac{1}{2} - \frac{1}{p})$. There exists $C > 0$ such that for any $f \in H^s(\mathbb{R}^d)$, $f \in L^p(\mathbb{R}^d)$ and*

$$\big|f\big|_{L^p} \leq C \big|f\big|_{H^s}.$$

Proof. We have with $q = \frac{p}{p-1}$ and $r = \frac{2(p-1)}{p-2}$ ($(q, r) = (2, \infty)$ if $p = 2$, and $(q, r) = (1, 2)$ if $p = \infty$)

$$\big|f\big|_{L^p} \lesssim \big|\widehat{f}\big|_{L^q} = \left(\int_{\mathbb{R}^d} |\widehat{f}|^q \langle\cdot\rangle^{sq} \langle\cdot\rangle^{-sq} \right)^{\frac{1}{q}} \lesssim \big|f\big|_{H^s} \big|\langle\cdot\rangle^{-sq}\big|_{L^r}^{\frac{1}{q}}$$

where we used Hausdorff-Young inequality applied to the inverse Fourier transform, and then Hölder's inequality. Notice $\langle\cdot\rangle^{-sq} \in L^r(\mathbb{R}^d)$ since $sqr > d$. □

A.1.2 Product, commutator and composition estimates

Proposition A.5. *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$, and $s_1, s_2 \in \mathbb{R}$ satisfying $s_1 \geq s$, $s_2 \geq s$ and $s_1 + s_2 > s + d/2$. There exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, then $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}}.$$

Proof. Let us first deal with the case $s = 0$. The case $s_1 = 0$ or $s_2 = 0$ is straightforward by Sobolev embedding, Lemma A.4. Otherwise there exists $p_i > 2$ such that $\frac{1}{2} - \frac{1}{p_i} < \frac{s_i}{d}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. The result follows from Hölder's inequality, Lemma A.1, and again Sobolev embedding.

We consider now the case $s \in \mathbb{N}^\star$. By Leibniz rule, we have for any $\mathbf{k} \in \mathbb{N}^d$ with $|\mathbf{k}| \leq s$:

$$\partial^{\mathbf{k}}(fg) = \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} (\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g).$$

we estimate each summand using the $s = 0$ case:

$$\big|(\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)\big|_{L^2} \lesssim \big|\partial^{\mathbf{i}} f\big|_{H^{s_1 - |\mathbf{i}|}}\big|\partial^{\mathbf{j}} g\big|_{H^{s_2 - |\mathbf{j}|}},$$

and the result follows. □

Corollary A.6. *Let $d \in \mathbb{N}^\star$ and $s \in \mathbb{N}$. The space $H^s(\mathbb{R}^d)$ is a Banach algebra as soon as $s > d/2$.*

Proposition A.7. *Let $d \in \mathbb{N}^\star$, $s \in \mathbb{N}$ and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max(\{s_\star, s\})}(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_\star}}\big|g\big|_{H^s} + C\big\langle\big|f\big|_{H^s}\big|g\big|_{H^{s_\star}}\big\rangle_{s > s_\star}$$

where we recall the notation $\langle C\rangle_{a>b} = \begin{cases} C & \text{if } a > b, \\ 0 & \text{otherwise.}\end{cases}$

Proof. We consider the Leibniz rule for any $\mathbf{k} \in \mathbb{N}^d$ with $|\mathbf{k}| \leq s$:

$$\partial^{\mathbf{k}}(fg) = \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} (\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)$$

and estimate each summand independently.

If $|\mathbf{k}| \leq s_\star$, we may apply Proposition A.5 with $s_1 = s_\star - |\mathbf{i}|$ and $s_2 = s - |\mathbf{j}|$, and deduce

$$\big|(\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}\big|g\big|_{H^s}.$$

Assume now $|\mathbf{k}| > s_\star$. If $|\mathbf{i}| \leq s_\star$, we have as above $\big|(\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}\big|g\big|_{H^s}$. If $|\mathbf{j}| \leq s_\star$, we obtain symmetrically $\big|(\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^s}\big|g\big|_{H^{s_\star}}$. In the remaining cases, we let $s_1, s_2 \in \mathbb{R}$ be such that $s_\star < |\mathbf{i}| \leq s_1 \leq |\mathbf{k}|$, $s_\star < |\mathbf{j}| \leq s_2 \leq |\mathbf{k}|$ and $s_1 + s_2 = |\mathbf{k}| + s_\star$. By Proposition A.5 and Lemma A.3, we deduce

$$\big|(\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g)\big|_{L^2} \lesssim \big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}} \lesssim \big|f\big|_{H^{s_\star}}^{\theta}\big|f\big|_{H^s}^{1-\theta}\big|g\big|_{H^{s_\star}}^{1-\theta}\big|g\big|_{H^s}^{\theta}$$

with $\theta = \frac{|\mathbf{k}| - s_1}{|\mathbf{k}| - s_\star} = \frac{s_2 - s_\star}{|\mathbf{k}| - s_\star}$. We conclude by Young's inequality. □

Remark A.8. *A close inspection on the proof shows that the result may be sharpened by assuming only $f \in L^\infty(\mathbb{R}^d) \cap \mathring{H}^{\max\{s_\star, s\}}(\mathbb{R}^d)$ and one has*

$$\big|fg\big|_{H^s} \leq C\big(\big|f\big|_{L^\infty} + \big|\nabla f\big|_{H^{s_\star - 1}}\big)\big|g\big|_{H^s} + C\big\langle\big|\nabla f\big|_{H^{s-1}}\big|g\big|_{H^{s_\star}}\big\rangle_{s > s_\star}.$$

Proposition A.9. *Let $d \in \mathbb{N}^\star$, $\mathbf{k} \in \mathbb{N}^d$, and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in \mathring{H}^{\max\{1+s_\star, |\mathbf{k}|\}}(\mathbb{R}^d)$ and $g \in H^{|\mathbf{k}|-1}(\mathbb{R}^d)$, $[\partial^{\mathbf{k}}, f]g \in L^2(\mathbb{R}^d)$ and*

$$\big|[\partial^{\mathbf{k}}, f]g\big|_{L^2} \le C\big|\nabla f\big|_{H^{s_\star}}\big|g\big|_{H^{|\mathbf{k}|-1}} + C\big\langle\big|\nabla f\big|_{H^{|\mathbf{k}|-1}}\big|g\big|_{H^{s_\star}}\big\rangle_{|\mathbf{k}|-1>s_\star}.$$

Proof. By Leibniz rule, we have

$$[\partial^{\mathbf{k}}, f]g = \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}, \mathbf{i}\neq \mathbf{0}} \binom{\mathbf{k}}{\mathbf{i}} (\partial^{\mathbf{i}} f)(\partial^{\mathbf{j}} g).$$

We then proceed as in the proof of Proposition A.7 with $\partial^{\mathbf{i}} f = \partial^{\tilde{\mathbf{i}}}\partial^{\mathbf{e}} f$, $\tilde{\mathbf{i}}+\mathbf{e} = \mathbf{i}$, $|\tilde{\mathbf{i}}|+|\mathbf{j}| = |\mathbf{k}|-1$. □

Proposition A.10. *Let $n, d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$, $H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ and*

$$\big|f^n\big|_{H^s} \le C\big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}.$$

Proof. The case $n = 1$ is trivial, and the case $n = 2$ is a particular case to Proposition A.7. We first notice

$$\big|f^n\big|_{L^2} \lesssim \big|f\big|_{L^\infty}^{n-1}\big|f\big|_{H^s} \lesssim \big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}$$

by Sobolev embedding, Lemma A.4.

Now consider $\mathbf{k} \in \mathbb{N}^d$ and $|\mathbf{k}| = s$. By the general Leibniz rule, we have to estimate

$$\big|\prod_{i=1}^n \partial^{\mathbf{j}_i} f\big|_{L^2}$$

where $\sum_{i=1}^n \mathbf{j}_i = \mathbf{k}$. We may consider without loss of generality that $|\mathbf{j}_i| \ge 1$ for any $i \in \{1, \dots, n\}$ thanks to the Sobolev embedding $H^{s_\star} \subset L^\infty$ as above. We have

$$\big|\prod_{i=1}^n \partial^{\mathbf{j}_i} f\big|_{L^2} \lesssim \prod_{i=1}^n \big|\partial^{\mathbf{j}_i} f\big|_{L^{p_i}} \lesssim \prod_{i=1}^n \big|\partial^{\mathbf{j}_i} f\big|_{H^{s_i}},$$

as soon as $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{2}$ (using Lemma A.1) and $s_i > \frac{d}{2}(1 - \frac{2}{p_i})$ (using Lemma A.4)

Let us first consider the case $s \le s_\star$. We choose $s_i = s_\star - |\mathbf{j}_i|$ for $i \in \{0, \dots, n-1\}$ and $s_n = s - |\mathbf{j}_n|$. Recall, since $n \ge 1$, that $1 \le |\mathbf{j}_i| \le s - 1$, and hence $1 \le s_i \le s_\star - 1$ for any $i \in \{1, \dots, n\}$. Then we set $p_i = 2\frac{s_\star}{s_\star - s_i} \in (2, +\infty)$, so that $s_i = s_\star(1 - \frac{2}{p_i}) > \frac{d}{2}(1 - \frac{2}{p_i})$, and $\sum_{i=1}^n \frac{2}{p_i} = n - \frac{1}{s_\star}\sum_{i=1}^n s_i = 1$, and the result follows.

Let us now consider the case $s > s_\star$. We choose s_i such that

$$\max(\{|\mathbf{j}_i|, s_\star\}) < s_i + |\mathbf{j}_i| < \min(\{s, s_\star + |\mathbf{j}_i|\}) \quad \text{and} \quad \sum_{i=1}^n s_i = (n-1)s_\star.$$

This is possible since $1 \le |\mathbf{j}_i| \le s - 1$ and hence one has $\max(\{|\mathbf{j}_i|, s_\star\}) < \min(\{s, s_\star + |\mathbf{j}_i|\})$ and $\sum_{i=1}^n \max(\{|\mathbf{j}_i|, s_\star\}) < (n-1)s_\star + s < \sum_{i=1}^n \min(\{s, s_\star + |\mathbf{j}_i|\})$. Then we set as above $p_i = 2\frac{s_\star}{s_\star - s_i}$. Finally, we use the interpolation estimate of Lemma A.3:

$$\big|\partial^{\mathbf{j}_i} f\big|_{H^{s_i}} \lesssim \big|f\big|_{H^s}^{\theta_i}\big|f\big|_{H^{s_\star}}^{1-\theta_i}$$

with $\theta_i = \frac{s_i + |\mathbf{j}_i| - s_\star}{s - s_\star}$. This completes the proof, remarking that $\sum_{i=1}^n \theta_i = 1$. □

Proposition A.11. *Let $d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. Let $F : \mathbb{R} \to \mathbb{R}$ such that $F \in \mathcal{C}^s(\mathbb{R})$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ satisfying*

$$\big|f\big|_{H^{s_\star}} \leq M,$$

then $F(f) \in H^s(\mathbb{R}^d)$ and

$$\big|F(f)\big|_{H^s} \leq C\big|f\big|_{H^s}.$$

Proof. By the Sobolev embedding, Lemma A.4, there exists $C > 0$, such that for any $f \in H^{s_\star}(\mathbb{R}^d)$, $\big|f\big|_{L^\infty} \leq C\big|f\big|_{H^{s_\star}}$. Hence we can define a closed interval, I such that for any $f \in H^{s_\star}(\mathbb{R}^d)$,

$$\operatorname{supp} f \subset I.$$

It follows from the mean value theorem that

$$\big|F(f)\big|_{L^2} = \big|F(f) - F(0)\big|_{L^2} \leq \sup_{x \in I} |F'(x)|\big|f\big|_{L^2}.$$

We now use Faà di Bruno's formula: for any $\mathbf{k} \in \mathbb{N}^d$ such that $|\mathbf{k}| = s$, we have

$$\partial^{\mathbf{k}} F(f) = \sum_{n=1}^{|\mathbf{k}|} \sum_{\substack{\mathbf{j}_1,\dots,\mathbf{j}_n \neq \mathbf{0} \\ \mathbf{j}_1 + \dots + \mathbf{j}_n = \mathbf{k}}} C_{\mathbf{j}_1,\dots,\mathbf{j}_n}\ F^{(n)}(f) \times \prod_{i=1}^n \partial^{\mathbf{j}_i} f.$$

We have $\big|F^{(n)}(f)\big|_{L^\infty} \leq \sup_{x\in I} |F^{(s)}(x)|$ and we estimate

$$\big|\prod_{i=1}^n \partial^{\mathbf{j}_i} f\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}^{n-1}\big|f\big|_{H^s}$$

as in the proof of Proposition A.10. □

Remark A.12. *The estimates can be sharpened by making use of the following Gagliardo-Nirenberg estimate (see* e.g. *[Tay97]): for any $s \in \mathbb{N}$, and $\mathbf{k} \in \mathbb{N}^d$ such that $|\mathbf{k}| \leq s$, and any $s_\star > d/2$, there exists $C > 0$ such that for any $u \in H^{\max\{s,s_\star\}}(\mathbb{R}^d)$,*

$$\big|\partial^{\mathbf{k}} f\big|_{L^{2s/|\mathbf{k}|}} \leq C\big|f\big|_{L^\infty}^{1-|\mathbf{k}|/s}\big|\nabla f\big|_{H^{s-1}}^{|\mathbf{k}|/s}.$$

We deduce, under the (respective) assumptions of Proposition A.7, Proposition A.9, Proposition A.10 and Proposition A.11,

$$\begin{aligned}
\big|fg\big|_{H^s} &\leq C\big(\big|f\big|_{L^\infty}\big|g\big|_{H^s} + \big|f\big|_{H^s}\big|g\big|_{L^\infty}\big),\\
\big|[\partial^{\mathbf{k}}, f]g\big|_{L^2} &\leq C\big(\big|\nabla f\big|_{L^\infty}\big|g\big|_{H^{|\mathbf{k}|-1}} + \big|\nabla f\big|_{H^{|\mathbf{k}|-1}}\big|g\big|_{L^\infty}\big),\\
\big|\partial^{\mathbf{k}}(fg) - f\partial^{\mathbf{k}} g - g\partial^{\mathbf{k}} f\big|_{L^2} &\leq C\big|\nabla f\big|_{L^\infty}\big|\nabla g\big|_{H^{|\mathbf{k}|-2}} + \big|\nabla f\big|_{H^{|\mathbf{k}|-2}}\big|\nabla g\big|_{L^\infty}\big),\\
\big|f^n\big|_{H^s} &\leq C\big|f\big|_{L^\infty}^{n-1}\big|f\big|_{H^s},\\
\big|F(f)\big|_{H^s} &\leq C(\big|f\big|_{L^\infty})\big|f\big|_{H^s}.
\end{aligned}$$

A.1.3 Estimates with non-decreasing functions

We sometimes need to deal with nonlinear estimates involving a non square-integrable function, typically when a non-trivial topography is taken into account. We extend the result of the previous section to this framework.

Lemma A.13 (Interpolation inequality)**.** *Let $d \in \mathbb{N}^\star$ and $s, s_-, s_+ \in \mathbb{N}$ such that $s_- < s_+$ and $s_- \le s \le s_+$. There exists $C > 0$ such that for any $f \in W^{s_+,\infty}(\mathbb{R}^d)$,*

$$\big|f\big|_{W^{s,\infty}} \le C\big|f\big|_{W^{s_-,\infty}}^{\theta}\big|f\big|_{W^{s_+,\infty}}^{1-\theta},$$

with $\theta = \frac{s_+-s}{s_+-s_-}$.

Proof. We use the identity, valid for any $\mathbf{e} \in \mathbb{N}^d$ such that $|\mathbf{e}| = 1$ and any $\lambda > 0$:

$$(-\partial^{\mathbf{e}} f)(\mathbf{x}) = \int_0^{+\infty} (\partial^{2\mathbf{e}} f - \lambda^2 f)(\mathbf{x} + s\mathbf{e})e^{-\lambda s}\, \mathrm{d}s + \lambda f(\mathbf{x}),$$

which follows by integration by parts. We deduce, with $\lambda = \big(2\big|\partial^{2\mathbf{e}} f\big|_{L^\infty}/\big|f\big|_{L^\infty}\big)^{1/2}$

$$\big|\partial^{\mathbf{e}} f\big|_{L^\infty} \le 2\sqrt{2}\big|f\big|_{L^\infty}\big|\partial^{2\mathbf{e}} f\big|_{L^\infty},$$

and the result is proved for $(s_-, s, s_+) = (0, 1, 2)$. One obtains the result for $(s_-, s, s_+) = (0, s, s_+)$ with any $0 < s < s_+$ by induction on $s_+ \ge 2$ (the equality cases $s = 0$ or $s = s_+$ being straightforward), and the general case is immediately deduced. □

Proposition A.14. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$ and $s \in \mathbb{N}$. There exists $C > 0$ such that for any $f \in W^{s,\infty}(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and*

$$\begin{aligned}\big|fg\big|_{H^s} &\le C\big(\big|f\big|_{W^{s,\infty}}\big|g\big|_{L^2} + \big|f\big|_{L^\infty}\big|g\big|_{H^s}\big)\\ &\le 2C\big|f\big|_{W^{s_\star,\infty}}\big|g\big|_{H^s} + C\big\langle\big|f\big|_{W^{s,\infty}}\big|g\big|_{H^{s_\star}}\big\rangle_{s>s_\star}.\end{aligned}$$

Proof. The result is an immediate consequence of Leibniz rule, Lemma A.3 and Lemma A.13, and Young's inequality. □

Proposition A.15. *Let $d \in \mathbb{N}^\star$, $s_\star > d/2$, $\mathbf{k} \in \mathbb{N}^d$. There exists $C > 0$ such that for any $f \in W^{|\mathbf{k}|,\infty}(\mathbb{R}^d)$ and $g \in H^{|\mathbf{k}|-1}(\mathbb{R}^d)$, $[\partial^{\mathbf{k}}, f]g \in L^2(\mathbb{R}^d)$ and*

$$\begin{aligned}\big|[\partial^{\mathbf{k}}, f]g\big|_{L^2} &\le C\big(\big|\nabla f\big|_{L^\infty}\big|g\big|_{H^{|\mathbf{k}|-1}} + \big|\nabla f\big|_{W^{|\mathbf{k}|-1,\infty}}\big|g\big|_{L^2}\big)\\ &\le 2C\big|\nabla f\big|_{W^{s_\star,\infty}}\big|g\big|_{H^{|\mathbf{k}|-1}} + C\big\langle\big|\nabla f\big|_{W^{|\mathbf{k}|-1,\infty}}\big|g\big|_{H^{s_\star}}\big\rangle_{|\mathbf{k}|-1>s_\star}.\end{aligned}$$

Proof. We use once again Leibniz rule, Lemma A.3 and Lemma A.13, and Young's inequality. □

Proposition A.16. *Let $d, s \in \mathbb{N}^\star$, and $s_\star > d/2$. Let $I \ni 0$ be a closed interval, $F : I \to \mathbb{R}$ such that $F \in \mathcal{C}^s(I)$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in W^{s,\infty}(\mathbb{R}^d)$ and $g \in H^{\max(\{s,s_\star\})}(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp}(f + g) \subset I \quad \text{and} \quad \big|f\big|_{L^\infty} + \big|g\big|_{H^{s_\star}} \le M,$$

and for any $\mathbf{k} \in \mathbb{N}^d$ such that $|\mathbf{k}| = s$, one has $\partial^{\mathbf{k}}(F(f + g)) = F_{\mathbf{k}} + G_{\mathbf{k}}$ with

$$\big|F_{\mathbf{k}}\big|_{L^\infty} \le C\big|g\big|_{W^{s,\infty}} \quad \text{and} \quad \big|G_{\mathbf{k}}\big|_{L^2} \le C\big(\big|f\big|_{H^s} + \big|g\big|_{W^{s,\infty}}\big).$$

Proof. By reasoning as in the proof of Proposition A.11, we are left with the estimate of a sum of products of the form

$$\left(\prod_{i=1}^{n_1} \partial^{\mathbf{j}_i} f\right)\left(\prod_{i=1}^{n_2} \partial^{\mathbf{j}_i} g\right).$$

where $n \in \{1, \dots, |\mathbf{k}|\}$, $n_1 + n_2 = n$, and $\mathbf{i}_i \neq \mathbf{0}$ ($i \in \{1, \dots, n_1\}$), $\mathbf{j}_j \neq \mathbf{0}$ ($j \in \{1, \dots, n_2\}$) are such that $\mathbf{i}_1 + \dots + \mathbf{i}_{n_1} + \mathbf{j}_1 + \dots + \mathbf{j}_{n_2} = \mathbf{k}$. We may separate between two cases, depending on whether $n_1 = 0$ or $n_1 \neq 0$. We estimate as in Proposition A.10, if $n_1 \in \mathbb{N}^\star$,

$$\big|\prod_{i=1}^{n_1} \partial^{\mathbf{j}_i} f\big|_{L^2} \lesssim \big|f\big|_{H^{s_\star}}^{n_1-1} \big|f\big|_{H^{|\mathbf{k}_1|}}$$

and similarly, if $n_2 \in \mathbb{N}^\star$,

$$\big|\prod_{i=1}^{n_2} \partial^{\mathbf{j}_i} g\big|_{L^\infty} \lesssim \big|g\big|_{L^\infty}^{n_2-1} \big|g\big|_{W^{|\mathbf{k}_2|,\infty}}.$$

We conclude by using Lemma A.3, Lemma A.13, and Young's inequality. □

Proposition A.17. *Let $d, s \in \mathbb{N}^\star$, $s_\star > d/2$ and $\mathbf{k} \in \mathbb{N}^d$. Let $I \ni 0$ be a closed interval, $F : I \to \mathbb{R}$ such that $F \in \mathcal{C}^s(I)$ and $F(0) = 0$. For any $M > 0$, there exists $C > 0$ such that for any $f \in W^{|\mathbf{k}|,\infty}(\mathbb{R}^d)$ and $g \in H^{\max(\{|\mathbf{k}|, s_\star+1\})}(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp}(f+g) \subset I \quad \text{and} \quad \big|f\big|_{W^{1,\infty}} + \big|g\big|_{H^{1+s_\star}} \leq M,$$

and for any $h \in H^{|\mathbf{k}|-1}(\mathbb{R}^d)$, one has $\big[\partial^{\mathbf{k}}, F(f+g)\big]h \in L^2(\mathbb{R}^d)$ and

$$\big|\big[\partial^{\mathbf{k}}, F(f+g)\big]h\big|_{L^2} \leq C\big(\big|h\big|_{H^{|\mathbf{k}|-1}} + \big|\nabla f\big|_{W^{|\mathbf{k}|-1,\infty}}\big|h\big|_{L^2} + \big\langle\big|\nabla g\big|_{H^{|\mathbf{k}|-1}}\big|h\big|_{H^{s_\star}}\big\rangle_{|\mathbf{k}|-1>s_\star}\big).$$

If, additionally, one has $\big|f\big|_{W^{1+s_\star,\infty}} + \big|g\big|_{H^{1+s_\star}} \leq M$, then we may write the above as

$$\big|\big[\partial^{\mathbf{k}}, F(f+g)\big]h\big|_{L^2} \leq C\big(\big|h\big|_{H^{|\mathbf{k}|-1}} + \big\langle\big(\big|\nabla f\big|_{W^{|\mathbf{k}|-1,\infty}} + \big|\nabla g\big|_{H^{|\mathbf{k}|-1}}\big)\big|h\big|_{H^{s_\star}}\big\rangle_{|\mathbf{k}|-1>s_\star}\big).$$

Proof. The result is obtained with a combination of the techniques used in Proposition A.9, Proposition A.15 and Proposition A.16. □

A.1.4 Estimates for functions on the flat strip

All the product, results concerning functions defined on $\mathbb{R}^d$ have counterparts for functions defined on the strip $\mathcal{S} = (-1, 0) \times \mathbb{R}^d$. For $f \in L^2(\mathcal{S})$, we denote for $s \in \mathbb{N}$, $\Lambda^s f = (\mathrm{Id} - \Delta)^{s/2} f$ where the differentiation applies to the horizontal variable $\mathbf{x} \in \mathbb{R}^d$, and remark

$$\|\Lambda^s f\|_{L^2(\mathcal{S})}^2 = \iint_{\mathcal{S}} |\Lambda^s f|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{-1}^0 \big|f(z,\cdot)\big|_{H^s(\mathbb{R}^d)}^2 \,\mathrm{d}z.$$

Let us write as an example a counterpart to the product estimate.

Proposition A.18. *Let $d, s \in \mathbb{N}^\star$, $s_\star > d/2$. There exists $C > 0$ such that for any $f \in H^{\max\{s_\star, s\}}(\mathbb{R}^d)$ and $g \in L^2(\mathcal{S})$ such that $\Lambda^s g \in L^2(\mathcal{S})$, one has $\Lambda^s(fg) \in L^2(\mathcal{S})$ and*

$$\|\Lambda^s(fg)\|_{L^2(\mathcal{S})} \leq C\big|f\big|_{H^{s_\star}}\|\Lambda^s g\|_{L^2(\mathcal{S})} + \big\langle\big|f\big|_{H^s}\|\Lambda^{s_\star} g\|_{L^2(\mathcal{S})}\big\rangle_{s>s_\star}.$$

Proof. We have, for any $\mathbf{k} \in \mathbb{N}^d$ such that $|\mathbf{k}| = s$,

$$\begin{aligned}
\|\partial^{\mathbf{k}}(fg)\|_{L^2(\mathcal{S})}^2 &= \iint_{\mathcal{S}} |\partial^{\mathbf{k}}(fg)|^2(\mathbf{x}, z)\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{-1}^0 \big|\partial^{\mathbf{k}}(fg)(\cdot, z)\big|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}z \\
&\lesssim \int_{-1}^0 \big|f\big|_{H^{s_\star}}^2 \big|g(\cdot, z)\big|_{H^s}^2 + \big\langle\big|f\big|_{H^s}^2 \big|g(\cdot, z)\big|_{H^{s_\star}}^2\big\rangle_{s_\star > k} \,\mathrm{d}z \\
&= \big|f\big|_{H^{s_\star}}^2 \|\Lambda^s g\|_{L^2(\mathcal{S})}^2 + \big\langle\big|f\big|_{H^s}^2 \|\Lambda^{s_\star} g\|_{L^2(\mathcal{S})}^2\big\rangle_{s_\star > k},
\end{aligned}$$

and the result follows. □

Remark A.19. *The result obviously generalizes to $z \mapsto f(z, \cdot) \in L^\infty(-1, 0; H^{\max\{s_\star, s\}}(\mathbb{R}^d))$.*

A.2 Additional tools for Section 4

We recall the functional spaces. For $d \in \{1,2\}$ and $s \in \mathbb{N}$, we define

$$X^s \overset{\text{def}}{=} \{\mathbf{u} \in L^2(\mathbb{R}^d)^d \ : \ \big|\mathbf{u}\big|_{X^s}^2 \overset{\text{def}}{=} \sum_{|\mathbf{k}|=0}^{s} \big|\partial^{\mathbf{k}}\mathbf{u}\big|_{L^2}^2 + \mu\big|\partial^{\mathbf{k}}\nabla\cdot\mathbf{u}\big|_{L^2}^2 < \infty\},$$

$$Y^s \overset{\text{def}}{=} \{\mathbf{v} \in (X^0)' \ : \ \big|\mathbf{v}\big|_{Y^s}^2 \overset{\text{def}}{=} \sum_{|\mathbf{k}|=0}^{s} \big|\partial^{\mathbf{k}}\mathbf{v}\big|_{(X^0)'}^2 < \infty\}.$$

Lemma A.20. *The continuous embeddings $H^{s+1}(\mathbb{R}^d)^d \subset X^s \subset H^s(\mathbb{R}^d)^d$ and $H^s(\mathbb{R}^d)^d \subset Y^s \subset H^{s-1}(\mathbb{R}^d)^d$ hold. The following inequalities hold as soon as the right-hand side is finite:*

$$\big|\mathbf{u}\big|_{H^s} \le \big|\mathbf{u}\big|_{X^s}, \qquad \big|\mathbf{u}\big|_{X^s} \lesssim \big|\mathbf{u}\big|_{H^{s+1}}, \tag{A.1}$$

$$\big|\mathbf{v}\big|_{H^{s-1}} \lesssim \big|\mathbf{v}\big|_{Y^s}, \qquad \big|\mathbf{v}\big|_{Y^s} \le \big|\mathbf{v}\big|_{H^s}. \tag{A.2}$$

We also have the non-uniform continuous embedding

$$\big|\nabla f\big|_{Y^s} \lesssim \frac{1}{\sqrt{\mu}}\big|f\big|_{H^s}, \qquad \big|\nabla\cdot\mathbf{u}\big|_{H^s} \lesssim \frac{1}{\sqrt{\mu}}\big|\mathbf{u}\big|_{X^s}. \tag{A.3}$$

Proof. The continuous embeddings $H^1(\mathbb{R}^d)^d \subset X^0 \subset L^2(\mathbb{R}^d)^d$ are straightforward, and the corresponding $L^2(\mathbb{R}^d)^d \subset Y^0 \subset H^{-1}(\mathbb{R}^d)^d$ follow by duality. The estimate (A.3) with $n=0$ is easily checked, as for any $\mathbf{u} \in X^0$,

$$\big|\big\langle \nabla f, \mathbf{u}\big\rangle_{(X^0)'-X^0}\big| = \big|\big(f, \nabla\cdot\mathbf{u}\big)_{L^2}\big| \le \frac{1}{\sqrt{\mu}}\big|f\big|_{L^2}\big|\mathbf{u}\big|_{X^0}.$$

The case $n \in \mathbb{N}^\star$ is reduced to the case $n=0$ by considering $\partial^{\mathbf{k}}\mathbf{u}, \partial^{\mathbf{k}}\mathbf{v}, \partial^{\mathbf{k}} f$ with $0 \le |\mathbf{k}| \le n$. □

Lemma A.21. *Let $h_\star > 0$, $\mu^\star > 0$ and $M > 0$. Then there exists $C > 0$ such that for any $(\varepsilon, \beta, \mu) \in \mathcal{P}_{\mathrm{SW}}$, any $b \in W^{1,\infty}$ and $h \in L^\infty$ satisfying Assumption 2.1 and*

$$\big|h\big|_{L^\infty} + \big|\beta\nabla b\big|_{L^\infty} \le M,$$

$\mathfrak{T}[h, \beta\nabla b] : X^0 \to (X^0)'$ is a well-defined topological isomorphism, and one has

$$\forall \mathbf{u}_1, \mathbf{u}_2 \in X^0, \qquad \big\langle \mathfrak{T}[h,\beta\nabla b]\mathbf{u}_1, \mathbf{u}_2\big\rangle_{(X^0)'-X^0} = \big\langle \mathfrak{T}[h,\beta\nabla b]\mathbf{u}_2, \mathbf{u}_1\big\rangle_{(X^0)'},$$

$$\forall \mathbf{u} \in X^0, \qquad \big|\big\langle \mathfrak{T}[h,\beta\nabla b]\mathbf{u}\big\rangle_{(X^0)'}\big| \le C\,\big|\mathbf{u}\big|_{X^0},$$

$$\forall \mathbf{v} \in (X^0)', \qquad \big|\mathfrak{T}[h,\beta\nabla b]^{-1}\mathbf{v}\big|_{X^0} \le C\,\big|\mathbf{v}\big|_{(X^0)'}.$$

Proof. We establish the estimates for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}, \mathbf{v} \in \mathcal{S}(\mathbb{R}^d)^d$ so that all the terms are well-defined, and the $((X^0)' - X^0)$ duality product coincides with the L^2 inner product. The result for less regular functions is obtained by density of $\mathcal{S}(\mathbb{R}^d)^d$ in X^0 and continuous linear extension.

By definition of $\mathfrak{T}$ in (4.2) and after integration by parts, one has

$$\big(\mathfrak{T}[h,\beta\nabla b]\mathbf{u}_1, \mathbf{u}_2\big)_{L^2} = \int_{\mathbb{R}^d} h\mathbf{u}_1\cdot\mathbf{u}_2 + \frac{\mu}{3}h^3(\nabla\cdot\mathbf{u}_1)(\nabla\cdot\mathbf{u}_2)$$
$$\qquad - \frac{\mu}{2}h^2\big((\nabla\cdot\mathbf{u}_2)(\beta\nabla b\cdot\mathbf{u}_1) + (\beta\nabla b\cdot\mathbf{u}_2)(\nabla\cdot\mathbf{u}_1)\big) + \mu h(\beta\nabla b\cdot\mathbf{u}_1)(\beta\nabla b\cdot\mathbf{u}_2),$$

from which the symmetry is evident. Applying Cauchy-Schwarz inequality, we have

$$\forall \mathbf{u}_1, \mathbf{u}_2 \in X^0, \quad \big|\big\langle \mathfrak{T}[h,\beta\nabla b]\mathbf{u}_1, \mathbf{u}_2\big\rangle_{(X^0)'-X^0}\big| \le C(\big|h\big|_{L^\infty}, \big|\beta\nabla b\big|_{L^\infty})\big|\mathbf{u}_1\big|_{X^0}\big|\mathbf{u}_2\big|_{X^0},$$

and the first estimate follows by duality. The second one is obtained when rewriting

$$\big(\mathfrak{T}[h,\beta\nabla b]\mathbf{u},\mathbf{u}\big)_{L^2}=\int_{\mathbb{R}^d} h|\mathbf{u}|^2+\frac{\mu}{12}h^3|\nabla\cdot\mathbf{u}|^2+\frac{\mu}{4}h|h\nabla\cdot\mathbf{u}-2\beta\nabla b\cdot\mathbf{u}|^2.$$

This shows that $\mathfrak{T}[h,\beta\nabla b]:X^0\to(X^0)'$ is continuous and coercive, so that the operator version of Lax-Milgram theorem ensures that $\mathfrak{T}[h,\beta\nabla b]$ is an isomorphism. The continuity of the inverse is a consequence of the coercivity of $\mathfrak{T}[h,\beta\nabla b]$:

$$\big|\mathbf{u}\big|_{X^0}^2\le C(h_\star^{-1})|\langle\mathfrak{T}[h,\beta b]\mathbf{u},\mathbf{u}\rangle_{(X^0)'}|\le\big|\mathfrak{T}[h,\beta\nabla b]\mathbf{u}\big|_{(X^0)'}\big|\mathbf{u}\big|_{X^0},$$

and setting $\mathbf{u}=\mathfrak{T}[h,\beta\nabla b]^{-1}\mathbf{v}$ above. □

Lemma A.22. *Let $d\in\mathbb{N}^\star$, $s\in\mathbb{N}^\star$, $s_\star>d/2$, and $h_\star>0$, $M>0$, $\mu^\star>0$. There exists $C>0$ such that for any $(\varepsilon,\beta,\mu)\in\mathcal{P}_{\rm SW}$, for any $b\in W^{\max\{s+1,2+s_\star\},\infty}(\mathbb{R}^d)$* [8] *and $\zeta\in H^{\max\{s,1+s_\star\}}(\mathbb{R}^d)$ satisfying Assumption 2.1 and*

$$\big|\varepsilon\zeta\big|_{H^{1+s_\star}}+\big|\beta b\big|_{W^{2+s_\star,\infty}}\le M,$$

the following holds.

- *For any $\mathbf{u}\in X^s$, $\mathfrak{T}[h,\beta\nabla b]\mathbf{u}\in Y^s$ and*

$$\big|\mathfrak{T}[h,\beta\nabla b]\mathbf{u}\big|_{Y^s}\le C\times\Big(\big|\mathbf{u}\big|_{X^s}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^s}+\big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathbf{u}\big|_{X^{s_\star}}\big\rangle_{s>s_\star}\Big).$$

- *For any $\mathbf{v}\in Y^s$. Then $\mathfrak{T}[h,\beta\nabla b]^{-1}\mathbf{v}\in X^s$ and*

$$\big|\mathfrak{T}[h,\beta\nabla b]^{-1}\mathbf{v}\big|_{X^s}\le C\times\Big(\big|\mathbf{v}\big|_{Y^s}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^s}+\big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathbf{v}\big|_{Y^{s_\star}}\big\rangle_{s>1+s_\star}\Big).$$

Proof. Let us denote for simplicity $\mathfrak{T}\stackrel{\rm def}{=}\mathfrak{T}[h,\beta\nabla b]$. We also introduce $\mathbf{k}\in\mathbb{N}^d$ such that $|\mathbf{k}|=s$. We have for any $\mathbf{u},\mathbf{w}\in\mathcal{S}(\mathbb{R}^d)^d$,

$$\begin{aligned}|\big(\partial^{\mathbf{k}}(\mathfrak{T}\mathbf{u}),\mathbf{w}\big)_{L^2}|=\big(\partial^{\mathbf{k}}(h\mathbf{u}),\mathbf{w}\big)_{L^2}&+\frac{\mu}{3}\big(\partial^{\mathbf{k}}(h^3\nabla\cdot\mathbf{u}),\nabla\cdot\mathbf{w}\big)_{L^2}\\&-\frac{\mu}{2}\big(\partial^{\mathbf{k}}(h^2(\beta\nabla b)\cdot\mathbf{u}),\nabla\cdot\mathbf{w}\big)_{L^2}-\frac{\mu}{2}\big(\partial^{\mathbf{k}}(h^2(\beta\nabla b)\nabla\cdot\mathbf{u}),\mathbf{w}\big)_{L^2}\\&+\mu\big(\partial^{\mathbf{k}}(h(\beta\nabla b)(\beta\nabla b)\cdot\mathbf{u}),\mathbf{w}\big)_{L^2}.\end{aligned}$$

Hence, by product estimates, *i.e.* Proposition A.7 and Proposition A.14, we have

$$|\big(\partial^{\mathbf{k}}(\mathfrak{T}\mathbf{u}),\mathbf{w}\big)_{L^2}|\le C(M)\Big(\big|\mathbf{u}\big|_{X^s}+\big\langle\big(\big|\varepsilon\zeta\big|_{H^s}+\big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathbf{u}\big|_{X^{s_\star}}\big\rangle_{s>s_\star}\Big)\big|\mathbf{w}\big|_{X^0}.$$

The first result is deduced by density and continuity arguments. Now, notice

$$\begin{aligned}|\big([\partial^{\mathbf{k}},\mathfrak{T}]\mathbf{u},\mathbf{w}\big)_{L^2}|=\big([\partial^{\mathbf{k}},h]\mathbf{u},\mathbf{w}\big)_{L^2}&+\frac{\mu}{3}\big([\partial^{\mathbf{k}},h^3]\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w}\big)_{L^2}\\&-\frac{\mu}{2}\big([\partial^{\mathbf{k}},h^2(\beta\nabla b)\cdot]\mathbf{u},\nabla\cdot\mathbf{w}\big)_{L^2}-\frac{\mu}{2}\big([\partial^{\mathbf{k}},h^2(\beta\nabla b)]\nabla\cdot\mathbf{u},\mathbf{w}\big)_{L^2}\\&+\mu\big([\partial^{\mathbf{k}},h(\beta\nabla b)(\beta\nabla b)\cdot]\mathbf{u},\mathbf{w}\big)_{L^2}.\end{aligned}$$

[8] the result holds as well assuming instead that $b\in L^\infty(\mathbb{R}^d)\cap H^{\max\{s+1,2+s_\star\},\infty}(\mathbb{R}^d)$, replacing $\big|\beta b\big|_{W^{2+s_\star,\infty}}$ with $\big|\beta\nabla b\big|_{H^{1+s_\star}}+\big|\beta b\big|_{L^\infty}$, and $\big|\beta\nabla b\big|_{W^{s,\infty}}$ with $\big|\beta\nabla b\big|_{H^s}$.

Using commutator estimates, Proposition A.9 and Proposition A.15, we deduce

$$\big|\big(\big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathbf{u},\mathbf{w}\big)_{L^2}\big| \le C(M)\Big(\big|\mathbf{u}\big|_{X^{s-1}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathbf{u}\big|_{X^{s_\star}}\big\rangle_{s>1+s_\star}\Big)\big|\mathbf{w}\big|_{X^0}.$$

By density and continuity arguments, we infer that for any $\mathbf{u}\in X^{s-1}$, $\big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathbf{u}\in(X^0)'$ and

$$\big|\big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathbf{u}\big|_{(X^0)'} \le C(M)\Big(\big|\mathbf{u}\big|_{X^{s-1}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathbf{u}\big|_{X^{s_\star}}\big\rangle_{s>1+s_\star}\Big).$$

Now, we make use of the identity

$$\big[\partial^{\mathbf{k}},\mathfrak{T}^{-1}\big]\mathbf{v} = -\mathfrak{T}^{-1}\big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathfrak{T}^{-1}\mathbf{v}.$$

Combining the above and by Lemma A.21, we find

$$\begin{aligned}
\big|\partial^{\mathbf{k}}(\mathfrak{T}^{-1}\mathbf{v})\big|_{X^0} &= \big|\mathfrak{T}^{-1}\partial^{\mathbf{k}}\mathbf{v} - \mathfrak{T}^{-1}\big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathfrak{T}^{-1}\mathbf{v}\big|_{X^0}\\
&\le C_0\big|\partial^{\mathbf{k}}\mathbf{v} - \big[\partial^{\mathbf{k}},\mathfrak{T}\big]\mathfrak{T}^{-1}\mathbf{v}\big|_{(X^0)'}\\
&\le C(M)\Big(\big|\mathbf{v}\big|_{Y^s} + \big|\mathfrak{T}^{-1}\mathbf{v}\big|_{X^{s-1}} + \big\langle\big(\big|\varepsilon\zeta\big|_{H^s} + \big|\beta\nabla b\big|_{W^{s,\infty}}\big)\big|\mathfrak{T}^{-1}\mathbf{v}\big|_{X^{s_\star}}\big\rangle_{s>1+s_\star}\Big).
\end{aligned}$$

The result follows by induction on s, and by density of $\mathcal{S}(\mathbb{R}^d)^d$ in Y^s. □

A.3 Notations

- The notation $a \lesssim b$ means that $a \le C_0\ b$, where C_0 is a nonnegative constant whose exact expression is of no importance. We denote by $C(\lambda_1,\lambda_2,\dots)$ a nonnegative constant depending on the parameters λ_1, λ_2,... and whose dependence on the λ_j is always assumed to be nondecreasing. Straightforward dependence with respect to other parameters may be omitted.
- We use the multi-index notation for multi-dimensional differentiation: for $\mathbf{k}=(k_1,k_2)\in\mathbb{N}^2$, $\partial^{(k_1,k_2)}f(x,y)=\partial_x^{k_1}\partial_y^{k_2}f(x,y)$ and $|\mathbf{k}|=k_1+k_2$. For $\mathbf{i},\mathbf{j}\in\mathbb{N}^2$, $\binom{\mathbf{i}}{\mathbf{j}}=\binom{i_1}{j_1}\binom{i_2}{j_2}$. If $\mathbf{k}\in\mathbb{N}$, then $\partial^{\mathbf{k}}=\partial^k$ is the standard differentiation operator.
- For $1\le p<\infty$ and $d\in\mathbb{N}$, we denote $L^p(\mathbb{R}^d)$ the Lebesgue spaces associated with the norm

  $$|f|_{L^p} = \left(\int_{\mathbb{R}^d}|f(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} < \infty.$$

  The real inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^d)$ is denoted by

  $$\big(f_1,f_2\big)_{L^2} = \int_{\mathbb{R}^d} f_1(\mathbf{x})f_2(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

  The space $L^\infty(\mathbb{R}^d)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm

  $$\big|f\big|_{L^\infty} = \operatorname{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d}|f(\mathbf{x})| < \infty.$$

- For $k\in\mathbb{N}$, we denote the Sobolev space $H^k(\mathbb{R}^d)$ the subspace of $L^2(\mathbb{R}^d)$ such that all weak derivatives of order k are square-integrable, endowed with

  $$\big|f\big|_{H^k}^2 \;=\; \sum_{|\mathbf{k}|\le k}\big|\partial^{\mathbf{k}}f\big|_{L^2}^2.$$

  Similarly, we denote by $W^{k,\infty}(\mathbb{R}^d)=\{f\in L^\infty(\mathbb{R}^d)\ :\ \forall 0\le|\mathbf{k}|\le k,\ \partial^{\mathbf{k}}f\in L^\infty(\mathbb{R}^d)\}$ endowed with its canonical norm, and $\mathcal{C}^k(\mathbb{R}^d)=\{f\in L^\infty(\mathbb{R}^d)\ :\ \forall 0\le|\mathbf{k}|\le k,\ \partial^{\mathbf{k}}f\in\mathcal{C}^0(\mathbb{R}^d)\}$, where $\mathcal{C}^0(\mathbb{R}^d)$ denotes the space of (scalar) continuous functions. We denote the Beppo Levi space $\mathring{H}^{k+1}(\mathbb{R}^d)=\{f\in L^2_{\rm loc}(\mathbb{R}^d)\ :\ \nabla f\in H^k(\mathbb{R}^d)^d\}$, endowed with the semi-norm

  $$\big|f\big|_{\mathring{H}^{k+1}} \;=\; \big|\nabla f\big|_{H^k}.$$

- We denote $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of smooth rapidly decreasing functions.
- For any real constant $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the Sobolev space obtained by completing $\mathcal{S}(\mathbb{R}^d)$ for the norm $|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty$, where Λ is the pseudo-differential operator (and Fourier multiplier) $\Lambda = (1 - \Delta)^{1/2}$.
- We use double bars for norms associated to functional spaces defined on $\Omega \subset \mathbb{R}^{d+1}$. For instance, square-integrable functions on Ω are endowed with the norm

$$\big\|\Phi\big\|_{L^2(\Omega)}^2 = \iint_\Omega |\Phi(\mathbf{x}, z)|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

  Sobolev and Beppo Levi spaces with integer indices $H^k(\Omega)$ and $\mathring{H}^{k+1}(\Omega)$ are defined as above.
- Given X any of the previously defined functional spaces, we denote by X' its topological dual, endowed with the norm $\big|\varphi\big|_{X'} = \sup\{|\varphi(f)| \ : \ \big|f\big|_X \leq 1\}$; and by $\langle \cdot, \cdot \rangle_{X'-X}$ the $(X' - X)$ duality brackets.
- For any function $u = u(t, x)$ defined on $[0, T) \times \mathbb{R}$ with $T > 0$, and any of the previously defined functional spaces, X, we denote $L^\infty(0, T; X)$ the space of functions such that $u(t, \cdot)$, taking values in the Banach space X, is essentially bounded for $t \in (0, T)$, and denote the associated norm

$$\big\|u\big\|_{L^\infty(0,T;X)} \ = \ \operatorname*{ess\,sup}_{t\in[0,T)} \big|u(t, \cdot)\big|_X \ < \ \infty.$$

  Spaces $L^p(0, T; X)$ for $p \in [1, \infty)$ are defined similarly. For $k \in \mathbb{N}$, $\mathcal{C}^k([0, T); X)$ denotes the space of X-valued continuous functions on $[0, T)$ with continuous derivatives up to order k.

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