Scattering and Localization Properties of Highly Oscillatory Potentials

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Abstract

We investigate scattering, localization and dispersive time-decay properties for the one-dimensional Schrödinger equation with a rapidly oscillating and spatially localized potential, $q_\epsilon = q(x, x/\epsilon)$, where $q(x, y)$ is periodic and mean zero with respect to y. Such potentials model a microstructured medium. Homogenization theory fails to capture the correct low-energy (k small) behavior of scattering quantities, *e.g.* the transmission coefficient, $t^{q_\epsilon}(k)$, as ϵ tends to zero. We derive an *effective potential well*, $\sigma^\epsilon_{\rm eff}(x) = -\epsilon^2\Lambda_{\rm eff}(x)$, such that $t^{q_\epsilon}(k) - t^{\sigma^\epsilon_{\rm eff}}(k)$ is small, uniformly for $k \in \mathbb{R}$ as well as in any bounded subset of a suitable complex strip. Within such a bounded subset, the scaled limit of the transmission coefficient has a universal form, depending on a single parameter, which is computable from the effective potential. A consequence is that if ϵ, the scale of oscillation of the microstructure potential, is sufficiently small, then there is a pole of the transmission coefficient (and hence of the resolvent) in the upper half plane, on the imaginary axis at a distance of order ϵ^2 from zero. It follows that the Schrödinger operator $H_{q_\epsilon} = -\partial_x^2 + q_\epsilon(x)$ has an L^2 bound state with negative energy situated a distance $\mathcal{O}(\epsilon^4)$ from the edge of the continuous spectrum. Finally, we use this detailed information to prove the local energy time-decay estimate: $\big|(1+|\cdot|)^{-3}e^{-itH_{q_\epsilon}}P_c\psi_0\big|_{L^\infty} \leq C\, t^{-1/2}\ \left(1+\epsilon^4\ \left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2 t\ \right)^{-1}\big|(1+|\cdot|^3)\psi_0\big|_{L^1}$, where P_c denotes the projection onto the continuous spectral part of H_{q_ϵ}.

1 Introduction

We investigate scattering and localization phenomena for the one-dimensional Schrödinger equation, $i\partial_t\psi = (-\partial_x^2 + V(x))\psi$, where V denotes a real-valued, rapidly oscillating and spatially localized potential. This equation governs the behavior of a quantum particle or, in the paraxial approximation of electromagnetics, waves in a medium with strong and rapidly varying inhomogeneities. We find interesting and subtle low energy behavior and study its consequences for scattering, localization and dispersive time-decay. Our results imply the existence of waveguide modes which display very short length-scale localization of light in photonic microstructures [4].

The scattering problem for the Schrödinger equation

$$\left(H_V\ -\ k^2\right)u\ =\ 0, \qquad H_V\ \equiv\ -\partial_x^2\ +\ V(x), \tag{1.1}$$

is the question of the scattered field in response to an incoming plane wave, e^{ikx}:

$$u(x;k)\ =\ \begin{cases} e^{ikx}\ +\ r^V(k)e^{-ikx}, & x\to-\infty\ , \\ t^V(k)e^{ikx}, & x\to+\infty\ . \end{cases} \tag{1.2}$$

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$t^V(k)$ and $r^V(k)$ are called reflection and transmission coefficients for the potential V; see section 2. Considered as a function of a complex variable, k, the transmission coefficient, $t^V(k)$, is meromorphic in the upper half $k-$plane, having possibly simple poles located on the positive imaginary axis. If $i\rho,\ \rho > 0$, is a pole of t^V then $E = -\rho^2$ is a discrete eigenvalue of H_V of multiplicity one.

In this paper, we are interested in the case where $V(x)$ is spatially localized and highly oscillatory. A class of potentials to which our results apply are potentials of the form:

$$V_\epsilon(x) \ = \ q_{\rm av}(x) + q(x, x/\epsilon), \quad \epsilon \ll 1 \ . \tag{1.3}$$

Here, $q_{\rm av}(x)$ denotes a spatially localized background average potential and $q(x,y)$ a potential which is spatially localized on the slow scale, x, and periodic and mean zero on the fast scale y:

$$q(x, y+1) \ = \ q(x,y), \qquad \text{and} \qquad \int_0^1 q(x,y) \ \ {\rm d}y \ = \ 0. \tag{1.4}$$

Thus,

$$q(x,y) \ = \ \sum_{j\neq 0} \ q_j(x) \ e^{2\pi i j y}. \tag{1.5}$$

More generally, our theory admits potentials which are aperiodic. For example, we allow for real-valued potentials:

$$q(x,y) \ = \ \sum_{j\neq 0} \ q_j(x) \ e^{2\pi i \lambda_j y}, \tag{1.6}$$

where $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a sequence of non-zero distinct frequencies for which there is a constant $\theta > 0$ such that

$$\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \theta > 0, \quad \inf_{j\in\mathbb{Z}} |\lambda_j| \geq \theta > 0. \tag{1.7}$$

That q is real-valued is imposed by:

$$\overline{q_j(x)} \ = \ q_{-j}(x), \qquad \lambda_{-j} \ = \ -\lambda_j, \qquad j \in \mathbb{Z}\setminus\{0\} \ . \tag{1.8}$$

We ask the following:
Question: What are the characteristics of solutions to the scattering problem (1.1), (1.2) *in the limit as ϵ tends to zero?*

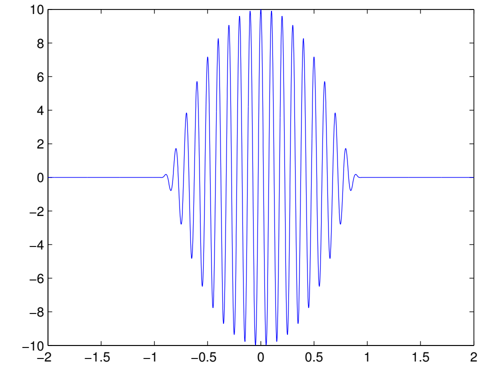
For fixed $k \neq 0$, this is the regime where averaging or homogenization theory applies; the leading order behavior in ϵ is governed by the average of V_ϵ over its fast variations. To simplify the present motivating discussion we consider the case where V_ϵ is periodic on the fast scale with vanishing mean, satisfying (1.4). Then, for any fixed $k \neq 0$, as $\epsilon \to 0$, we have

$$t^{V_\epsilon}(k) \to t^0(k) \equiv 1, \quad r^{V_\epsilon}(k) \to r^0(k) \equiv 0 \ ;$$

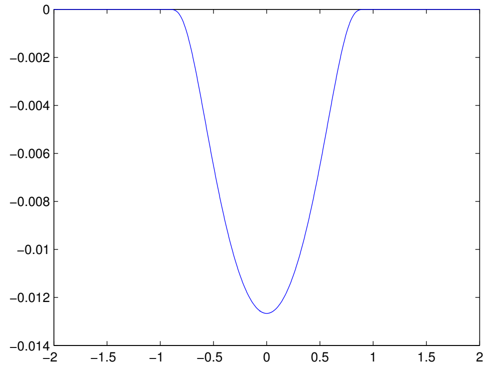
see [5], which contains very detailed asymptotic expansions of $t^{V_\epsilon}(k)$ for a general class of V_ϵ, admitting singularities. Very generally, as k tends to infinity, $t^V(k) \to 1$; the large k transmission behavior of $V_\epsilon(x)$ and its average, $q_{\rm av}(x)$, agree.

However, the low energy, $k \approx 0$, comparison between the scattering behavior for $q_{\rm av}(x) \equiv 0$ and $V_\epsilon(x)$ is far more subtle. First of all, the potential $V(x) \equiv 0$ has non-generic low energy behavior! Indeed, for *generic* localized potentials, V, $\lim_{k\to 0} t^V(k) = 0$; see the discussion of and references to genericity in Section 2. Thus we expect (and our analysis implies for small and non-zero ϵ) that $t^{V_\epsilon}(k) \to 0$ as $k \to 0$; see Corollary 3.4.

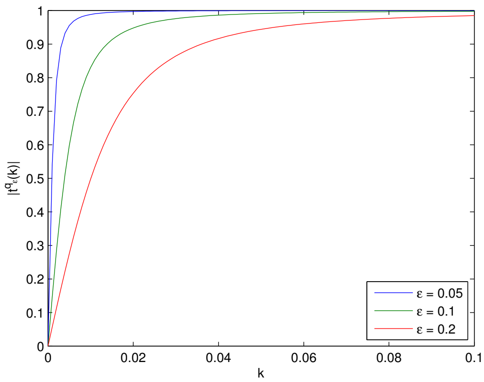
It follows that the convergence of $t^{V_\epsilon}(k)$, as ϵ tends to zero, to the *homogenized transmission coefficient* $t^{q_{\rm av}}(k) \equiv t^0(k) \equiv 1$ is non-uniform in a neighborhood of $k = 0$. Figure 1(c) displays



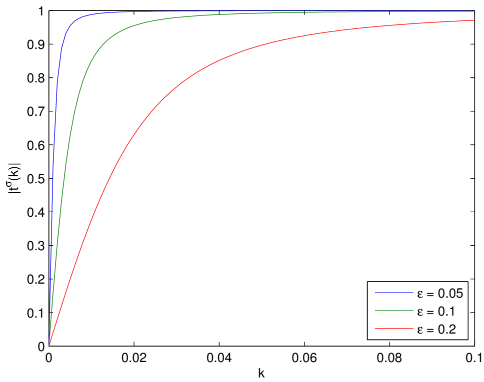
(a) $V_\epsilon(x) = \mathbf{1}_{[-1;1]}(x) 10 e^{\frac{-x^2}{1-x^2}} \cos(2\pi x/\epsilon)$



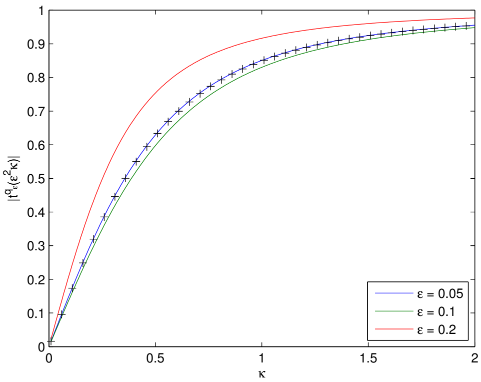
(b) $\sigma_{\text{eff}}^\epsilon(x) = \frac{-\epsilon^2}{8\pi^2} \mathbf{1}_{[-1;1]}(x) \Big(10 e^{\frac{-x^2}{1-x^2}} \cos\big(\frac{2\pi x}{\epsilon}\big) \Big)^2$



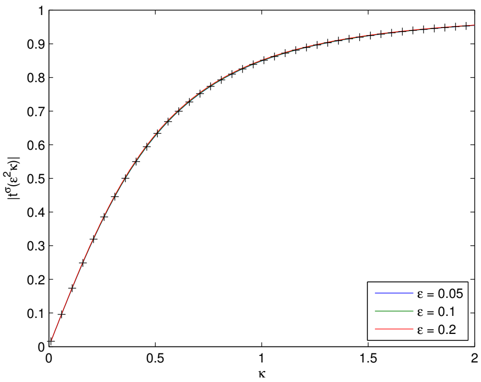
(c) $|t^{V_\epsilon}(k)|$, $k \in (0; 0.1)$, ϵ varying



(d) $\big|t^{\sigma_{\text{eff}}^\epsilon}(k)\big|$, $k \in (0; 0.1)$, ϵ varying



(e) $|t^{V_\epsilon}(\epsilon^2\kappa)|$, $\kappa \in (0; 2)$, ϵ varying



(f) $|t^{\sigma_{\text{eff}}^\epsilon}(\epsilon^2\kappa)|$, $\kappa \in (0; 2)$, ϵ varying

Figure 1: Plots of potentials $V_\epsilon(x)$, (a), and the corresponding effective potential $\sigma_{\text{eff}}^\epsilon(x)$, (b). Transmission coefficients $t^{V_\epsilon}(k)$, (c), and $t^{\sigma_{\text{eff}}^\epsilon}(k)$, (d). Plots (e) and (f) show convergence of scaled transmission coefficients $t^{V_\epsilon}(\epsilon^2\kappa)$ and $t^{\sigma_{\text{eff}}^\epsilon}(\epsilon^2\kappa)$ to the transmission coefficent $t^{\text{Dirac}}(\kappa) = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{\text{eff}}}$ associated with the Dirac delta potential well of mass $\int \Lambda_{\text{eff}}$. The cross markers in plots (e) and (f) correspond to values of $t^{\text{Dirac}}(\kappa)$.

plots of $t^{V_\epsilon}(k)$ for several successively smaller values of ϵ. *Underlying this non-uniformity is a subtle behavior of $t^{V_\epsilon}(k)$ in the complex plane and an interesting localization phenomenon, which we now explain.*

To fix ideas, stick with the case $q_{\rm av}(x) \equiv 0$ and thus, $H_{V_\epsilon} = H_{q_\epsilon}$, with $q_\epsilon(x) \equiv q(x, x/\epsilon)$. We comment below on the case where $q_{\rm av}$ is non-zero. We clarify the nature of low energy scattering by proving that there is an *effective potential well*:

$$\sigma^\epsilon_{\rm eff}(x) \ = \ -\epsilon^2 \Lambda_{\rm eff}(x), \tag{1.9}$$

such that

$$t^{q_\epsilon}(k) \ - t^{\sigma^\epsilon_{\rm eff}}(k) \to \ 0 \ \text{ as } \epsilon \to 0, \ \text{uniformly in } k \in \mathbb{R}; \tag{1.10}$$

see Theorem 4.1, Corollary 4.4, and Theorem 3.3, proved by a "normal form" type analysis in section 6. Here, $\Lambda_{\rm eff}(x)$ is a positive and localized function defined in terms of the Fourier expansion of the 2-scale potential, $q(x,y)$:

$$\Lambda_{\rm eff}(x) \ = \ -\frac{1}{(2\pi)^2} \sum_{j\neq 0} \frac{|q_j(x)|^2}{\lambda_j^2}. \tag{1.11}$$

For the periodic case, $q(x, y+1) = q(x,y)$, $\lambda_j = j, \ j \neq 0$ and $\Lambda_{\rm eff}$ is given by:

$$\Lambda_{\rm eff}(x) \ = \ -\frac{1}{(2\pi)^2} \sum_{j\neq 0} \frac{|q_j(x)|^2}{j^2} \ = \ -\left\langle -\partial_y^{-2} q(x,y), q(x,y) \right\rangle_{L^2(S^1_y)} \ . \tag{1.12}$$

This particular choice of effective potential well is anticipated by a formal two-scale homogenization expansion. An example of a mean zero potential $V_\epsilon(x) = q_\epsilon(x) = q(x, x/\epsilon)$ and the associated effective potential is displayed in Figures 1(a) and 1(b). A clue to the source of non-uniformity in k is offered by a result of Simon [13], applied to $\sigma^\epsilon_{\rm eff}$, which implies that for ϵ small, the operator $H_{\sigma^\epsilon_{\rm eff}}$, has a single negative eigenvalue:

$$E^{\sigma^\epsilon_{\rm eff}} \ = \ -\frac{\epsilon^4}{4} \left(\int_{\mathbb{R}} \Lambda_{\rm eff} \right)^2 \ + \ \mathcal{O}(\epsilon^6). \tag{1.13}$$

Since the eigenvalues of H_V are associated with poles of $t^V(k)$ located on the positive imaginary axis (section 2), the eigenvalue $E^{\sigma^\epsilon_{\rm eff}}$ is associated with a pole at

$$k^{\sigma^\epsilon_{\rm eff}}(\epsilon) \ = i \frac{\epsilon^2}{2} \ \left(\int_{\mathbb{R}} \Lambda_{\rm eff} \right) \ + \ \mathcal{O}(\epsilon^4) \tag{1.14}$$

The estimates of Theorem 3.3 and Corollary 3.5, comparing $t^{q_\epsilon}(k)$ to $t^{\sigma^\epsilon_{\rm eff}}(k)$, in a complex neighborhood of $k = 0$ for small ϵ, enable us to conclude, via Rouché's Theorem, that $t^{q_\epsilon}(k)$ has a pole $k^{q_\epsilon}(\epsilon) \approx k^{\sigma^\epsilon_{\rm eff}}(\epsilon)$. It follows that H_{q_ϵ} has a bound state, $u_{E_{q_\epsilon}}(x)$, with energy

$$E^{q_\epsilon} \ = \ -\frac{\epsilon^4}{4} \left(\int_{\mathbb{R}} \Lambda_{\rm eff} \right)^2 \ + \ \mathcal{O}(\epsilon^5) \ . \tag{1.15}$$

Moreover, $u_{E_{q_\epsilon}}(x) = \mathcal{O}\left(e^{-\sqrt{|E_{q_\epsilon}|}\ |x|} \right)$ as $|x| \to \infty$ (Corollary 3.7). Furthermore, by Corollary 3.6, there is a universal scaled limit depending on a single parameter, $\int_{\mathbb{R}} \Lambda_{\rm eff}$:

$$t^{q_\epsilon}(\epsilon^2 \kappa) \ \to \ t^\star \left(\kappa; \int_{\mathbb{R}} \Lambda_{\rm eff} \right) \ \equiv \ \frac{\kappa}{\kappa - \frac{i}{2} \int_{\mathbb{R}} \Lambda_{\rm eff}} \ \text{ as } \ \epsilon \to 0 \quad \text{for } \kappa \neq \frac{i}{2} \int_{\mathbb{R}} \Lambda_{\rm eff} \ .$$

Note that $t^\star\left(\kappa; \int_\mathbb{R} \Lambda_{\rm eff}\right)$ is the transmission coefficient for the Schrödinger operator with a Dirac-distribution potential well of total mass $\int_\mathbb{R} \Lambda_{\rm eff} > 0$:

$$H^\star \equiv -\partial_x^2 - \left(\int_\mathbb{R} \Lambda_{\rm eff}(\zeta)d\zeta\right) \times \delta(x).$$

Figures 1(e) and 1(f), as well as Figure 2, illustrate this behavior.

A further consequence concerns the large-time dispersive character of solutions to the time-dependent Schrödinger equation:

$$i\partial_t\psi = -\partial_x^2\psi + q(x,x/\epsilon)\psi, \quad \psi(0,x) = \psi_0 \ . \tag{1.16}$$

We have the following time-decay estimate (Theorem 5.1) for sufficiently localized initial conditions, ψ_0 , in the continuous spectral part of H_{q_ϵ}, *i.e.* $u_{E_{q_\epsilon}} \perp_{L^2} \psi_0$:

$$\left(1+|x|^3\right)^{-1} |\psi(x,t)| \leq \frac{C}{t^{1/2}} \frac{1}{1+\epsilon^4\left(\int_\mathbb{R}\Lambda_{\rm eff}\right)^2 t} \int_\mathbb{R} \left(1+|\zeta|^3\right)|\psi_0(\zeta)| \ {\rm d}y \ . \tag{1.17}$$

Therefore the effect of the oscillatory perturbation on the rate of dispersion is only seen on the time scale $t \gtrsim \epsilon^{-4}$.

The above results follow from the non-generic low energy behavior of the average potential $V \equiv 0$. Thus we ask:

Question: Are there non-trivial potentials, $V(x) \equiv q_{\rm av}(x)$, with low energy behavior analogous to $V \equiv 0$, such that $V_\epsilon = q_{\rm av}(x) + q_\epsilon(x)$ exhibits similar behavior?

The answer is yes! Such examples need to exhibit the behavior

$$|t^{q_{\rm av}}(k)| \to |t^{q_{\rm av}}(0)| \neq 0 \quad \text{as} \quad k \to 0.$$

How such non-generic behavior arises is discussed in section 3.2. The class of *reflectionless potentials*, for which one has $|t(k)| \equiv 1$ for all $k \in \mathbb{R}$, is a large family of such examples. Our main Theorem 3.3 holds for general $q_{\rm av}$, and shows that the low energy behavior is determined by the effective potential:

$$q_{\rm av}(x) + \sigma_{\rm eff}^\epsilon(x) = q_{\rm av}(x) - \epsilon^2\Lambda_{\rm eff}(x) \ .$$

Therefore, if $q_{\rm av}$ is a reflectionless potential, then $t^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(k)$ has a pole, $k^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\epsilon)$, situated on the positive imaginary axis, and of size $\mathcal{O}(\epsilon^2)$. An application of Rouché's Theorem yields that $t^{q_{\rm av}+q_\epsilon}(k)$, has a pole near $k^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\epsilon)$ and a bound state

$$E^{q_{\rm av}+q_\epsilon}(\epsilon) \approx E^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\epsilon) = \left[k^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\epsilon)\right]^2 < 0; \quad \text{see Corollary 3.8.}$$

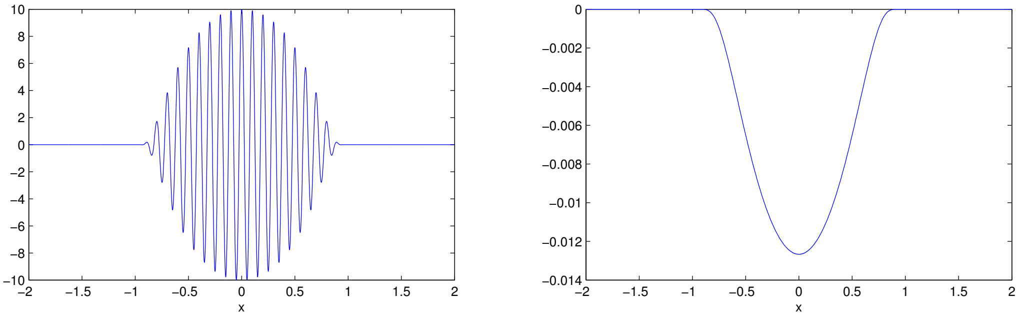
1.1 Outline of the paper

In section 2 we review the prerequisite one-dimensional scattering theory. Section 3 contains statements of our main results and is structured as follows:

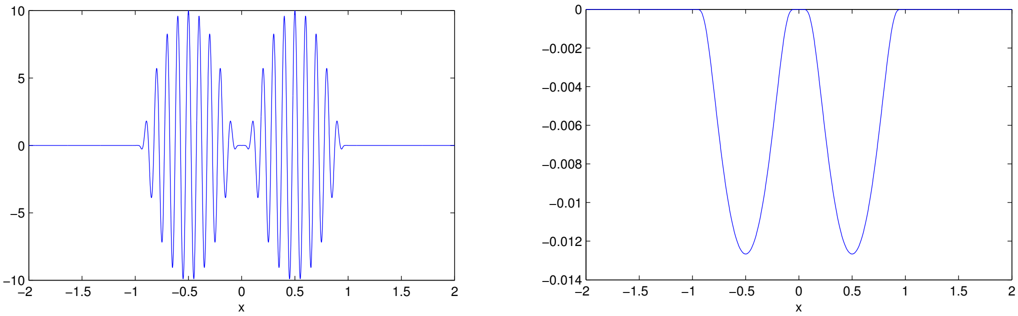
(1) Detailed hypotheses on the class of potentials: $V_\epsilon(x) = q_{\rm av}(x) + q(x,x/\epsilon)$ are given in Hypotheses **(V)** at the beginning of section 3.

(2) We consider the case where $q_{\rm av}$ is generic and the case where $q_{\rm av}$ is non-generic. As indicated above, the non-generic case, *i.e.* $q_{\rm av} \equiv 0$, is of greatest interest and we emphasize this case.

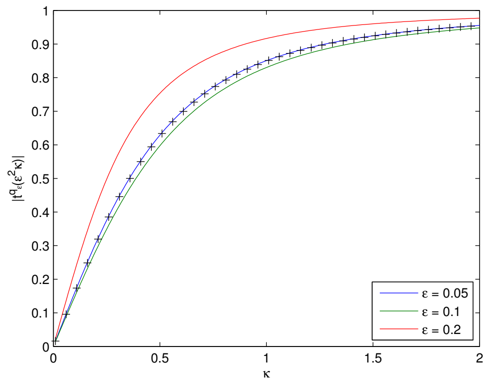
(3) For non-generic $q_{\rm av}$, Theorem 3.3 and Corollary 3.5 give precise estimates on the difference $t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(k)$, for k in a complex neighborhood of zero, and $\epsilon \to 0$.



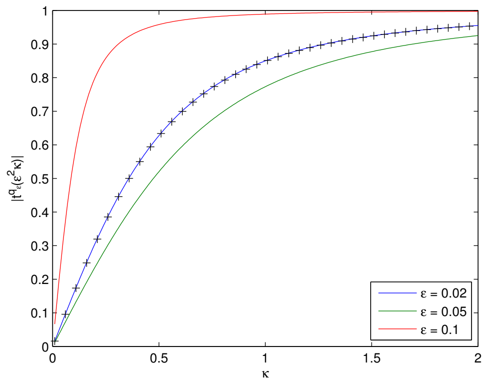
(a) $V_{1,\epsilon}(x) = \mathbf{1}_{[-1;1]}(x) 10 e^{\frac{-x^2}{1-x^2}} \cos\left(\frac{2\pi x}{\epsilon}\right)$, and effective potential, $\sigma_{1,\epsilon} \equiv -\epsilon^2 \Lambda_1$, $\epsilon = 0.1$



(b) $V_{2,\epsilon}(x) = 10\left(\mathbf{1}_{[-1;0]}(x) e^{\frac{-(2x+1)^2}{1-(2x+1)^2}} + \mathbf{1}_{[0;1]}(x) e^{\frac{-(2x-1)^2}{1-(2x-1)^2}}\right) \cos\left(\frac{2\pi x}{\epsilon}\right)$ and effective potential $\sigma_{2,\epsilon} \equiv -\epsilon^2 \Lambda_2$, $\epsilon = 0.1$



(c) $|t^{V_{1,\epsilon}}(\epsilon^2 \kappa)|$, $\kappa \in (0;2)$, ϵ varying



(d) $|t^{V_{2,\epsilon}}(\epsilon^2 \kappa)|$, $\kappa \in (0;2)$, ϵ varying

Figure 2: Plots (a) and (b) are of two mean zero potentials, $V_{1,\epsilon}$ and $V_{2,\epsilon}$ (left), and effective potentials $\sigma^{\epsilon}_{1,\text{eff}}$ and $\sigma^{\epsilon}_{2,\text{eff}}$ (right). Potentials chosen so that: $\int \Lambda_{1,\text{eff}} = \int \Lambda_{2,\text{eff}}$. Plots (c) and (d) illustrate universality of scaled limits: $t^{V_\epsilon}(\epsilon^2\kappa)$ and $t^{\sigma^{\epsilon}_{\text{eff}}}(\epsilon^2\kappa)$. The cross markers correspond to the scaled limit: $t^\star(\kappa) = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{1,\text{eff}}} = \frac{\kappa}{\kappa - \frac{i}{2}\int \Lambda_{2,\text{eff}}}$

(4) For $q_{\rm av}=0$, Corollary 3.6 gives a universal form of the scaled limit of $t^{q_{\rm av}+q_\epsilon}(\epsilon^2\kappa)$ as $\epsilon\to 0$. This limit depends on a single parameter, given by the integral of the effective potential.

(5) For $q_{\rm av}=0$, Corollary 3.7 states the potential $q_{\rm av}+q_\epsilon$, has a bound state with negative energy $\approx\mathcal{O}(\epsilon^4)$, near the edge of the continuous spectrum.

(6) In subsection 3.2 we present non-trivial (non-indentically zero) potentials, $q_{\rm av}$, which are non-generic, for which the above results for $q_{\rm av}\equiv 0$ also apply. We work out the details for "one-soliton" potentials $q_{{\rm av},\rho}(x)=-2\rho^2{\rm sech}^2(\rho(x-x_0))$, for which $H_{q_{{\rm av},\rho}}$ has exactly one negative eigenvalue at $E_0(\rho)=-\rho^2$ and continuous spectrum extending from zero to positive infinity. In this example, our result shows that $H_{q_{{\rm av},\rho}+q_\epsilon}$ has an eigenvalue of order $\mathcal{O}(\epsilon^4)$, which bifurcates from the edge of the continuous spectrum. Specifically,

$$E^{q_{\rm av}+q_\epsilon}\ \approx\ -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}}\tanh^2(y)\ \Lambda_{\rm eff}(y)\ \ {\rm d}y\ \right)^2; \tag{1.18}$$

compare with (1.15). A second eigenvalue is $\mathcal{O}(\epsilon^2)$ distant from $E_0(\rho)$.

(7) In subsection 3.3 we deal with the relatively simple case of highly oscillatory perturbations of a generic potential, $q_{\rm av}$.

In section 4, we combine our precise analysis for bounded k with the relatively simple analysis when $k\in\mathbb{R}$ is bounded away from zero, and obtain control on the difference $t^{q_\epsilon}(k)-t^{\sigma^\epsilon_{\rm eff}}(k)$, uniformly for $k\in\mathbb{R}$.
In section 5 our results on the high and low energy behavior of $t^{q_\epsilon}(k)$ are used to prove the local energy time-decay estimate (1.17); Theorem 5.1.
The proof of Theorem 3.3, and the emergence of the effective potential, $\sigma^\epsilon_{\rm eff}(x)$, are presented in section 6. Appendix A contains detailed estimates on Jost solutions for general localized potentials in an appropriate domain in the complex plane. Appendix B presents a discussion of the potential $q_{\rm av}(x)+\sigma^\epsilon_{\rm eff}(x)=q_{\rm av}(x)-\epsilon^2\Lambda_{\rm eff}(x)$.

1.2 Remarks on related work

(1) Detailed and rigorous asymptotic expansions of $t^{q_{\rm av}+q_\epsilon}(k)$ were derived in [5] by a method developed in [7]. In this work, singular potentials were also admitted. Potentials with singularities, *e.g.* jump discontinuities, Dirac delta singularities, give rise to interface-effects which require the inclusion of interface correctors, not captured by standard bulk homogenization theory, in the expansions. For generic potentials these expansions hold for any fixed $k\in\mathbb{R}$ and $\epsilon\downarrow 0$.

(2) As discussed, our results are related to those of Simon [13] on shallow depth potentials with negative or zero average. Our results can be viewed as a generalization to a larger class of perturbations, admitting high-contrast and rapidly oscillatory potentials, *i.e* potentials which converge only weakly to their mean.

(3) We conjecture, motivated by [13], that in dimension 2, there is a discrete eigenvalue which is exponentially small in ϵ; and that in dimension 3, there exists no bound state for ϵ sufficiently small.

(4) *E. Schrödinger meets P. Kapitza:* There is an interesting connection between our results and a phenomenon in Mechanics known as the *Kapitza Pendulum.* Very generally, this refers to the stabilization of an unstable equilibrium of a dynamical system through time-dependent parametric forcing, *i.e.* the stabilization of the classical inverted pendulum [8, 9].

1.3 Notation, norms and function spaces

Various norms are introduced in the analysis of the transmission coefficient, Jost solutions *etc.* These norms involve spatial weights of the potential which are algebraic, when we analyze scattering properties for $k \in \mathbb{R}$, and exponential, when we consider these properties for $k \in \mathbb{C}$. Our convention throughout is that spaces with algebraic spatial weights are denoted with calligraphic upper-case letters, *e.g.* $\mathcal{W}_\gamma^{k,p}$, and spaces with exponential spatial weights are denoted with ordinary upper-case Roman letters, *e.g.* $W_\beta^{k,p}$. The parameters γ and β define the spatial weight.

We denote by $\mathcal{L}^1_\gamma(\mathbb{R})$ the space of measurable functions g such that

$$\big|g\big|_{\mathcal{L}^1_\gamma} \;=\; \int_{\mathbb{R}} |g(x)|(1+|x|)^\gamma \ \mathrm{d}x \;<\; \infty.$$

The space of functions, g, whose derivatives up to order n are in $\mathcal{L}^1_\gamma$ is denoted $\mathcal{W}_\gamma^{n,1}$ and the associated norm is

$$\big|g\big|_{\mathcal{W}_\gamma^{n,1}} \;\equiv\; \sum_{l=0}^{n} \big|\partial^l g\big|_{\mathcal{L}^1_\gamma}.$$

For a fixed $\beta > 0$, we denote by L^∞_β the space of measurable functions g defined on $\mathbb{R}$ such that

$$\big|g\big|_{L^\infty_\beta} \;\equiv\; \big|e^{\beta\cdot} g\big|_{L^\infty} \;\equiv\; \text{ess sup}_{x\in\mathbb{R}}\ e^{\beta x}\,|g(x)| \;<\; \infty.$$

$W_\beta^{n,\infty}$ denotes the space of the functions g defined on $\mathbb{R}$, whose derivatives up to order n are in L^∞_β with associated norm

$$\big|g\big|_{W_\beta^{n,\infty}} \;\equiv\; \sum_{l=0}^{n} \big|\partial^l g\big|_{L^\infty_\beta}.$$

For a function, V, of the form

$$V(x,y) \;=\; q_{\rm av}(x) + q(x,y) \;=\; q_{\rm av}(x) \;+ \sum_{j\in\mathbb{Z}\setminus\{0\}} q_j(x)\ e^{2\pi i\lambda_j y},$$

we introduce the following norms:

exponentially weighted: $$\big| V \big| \;\equiv\; \big|q_{\rm av}\big|_{W_\beta^{1,\infty}} \;+ \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|q_j\big|_{W_\beta^{3,\infty}}\ ;$$

algebraically weighted: $$\big\| V \big\| \;\equiv\; \big|q_{\rm av}\big|_{\mathcal{W}_2^{1,1}} \;+ \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|q_j\big|_{\mathcal{W}_3^{3,1}}\ .$$

2 Review of 1d scattering theory

In this section we briefly review some of the basics of scattering theory for the one-dimensional Schrödinger equation:

$$\left(-\frac{d^2}{dx^2} \;+\; V(x) \;-\; k^2\right)\, u(x;k) \;=\; 0, \tag{2.1}$$

for localized potentials, $V(x)$, assumed to satisfy

$$V \in \mathcal{L}^1_2(\mathbb{R}) \;=\; \{V : (1+|x|)^2 V(x) \in L^1(\mathbb{R})\}.$$

In particular, in section 2.1 we discuss the Jost solutions, $f_\pm^V(x;k)$, and the reflection and transmission coefficients, $r_\pm^V(k)$ and $t^V(k)$. An extensive discussion can be found in [3], [11], [10]. Section 2.2 explains what is meant by a *generic potential.* Finally, in section 2.3 we introduce some important tools enabling us to compare the transmission coefficients of two different potentials. This is based on the Volterra integral equation for the Jost solution for a potential, V, viewed as a perturbation of a second potential, W.

2.1 The Jost solutions, and reflection and transmission coefficients

For $k \in \mathbb{R}$, introduce $f_\pm^V(x;k)$, the unique solutions of (2.1) with

$$f_\pm^V(x;k) \sim e^{\pm ikx}, \quad \text{as} \quad x \to \pm\infty. \tag{2.2}$$

Observe from the asymptotics as $x \to \infty$, we have $\mathcal{W}[f_+^V(\cdot;k), f_+^V(\cdot;-k)] = 2ik$, where $\mathcal{W}[h_1,h_2]$ denotes the Wronskian of functions $h_1(x)$ and $h_2(x)$:

$$\mathcal{W}[h_1,h_2] = h_1(x)h_2'(x) - h_2(x)h_1'(x). \tag{2.3}$$

Therefore, for $k \in \mathbb{R}\setminus\{0\}$, the set $\{f_+^V(x;k), f_+^V(x;-k)\}$ is a linearly independent set of solutions of (2.1).

The transmission coefficients, $t_\pm^V(k)$, and the reflection coefficients $r_\pm^V(k)$ are defined via the algebraic relations, among the Jost solutions $f_\pm^V(x;k)$:

$$f_+^V(x;k) \equiv \frac{r_+^V(k)}{t_+^V(k)} f_-^V(x;k) + \frac{1}{t_+^V(k)} f_-^V(x;-k), \tag{2.4}$$

$$f_-^V(x;k) \equiv \frac{r_-^V(k)}{t_-^V(k)} f_+^V(x;k) + \frac{1}{t_-^V(k)} f_+^V(x;-k). \tag{2.5}$$

One can check that $\mathcal{W}[f_+^V, f_-^V] = -2ik[t_-^V(k)]^{-1} = -2ik[t_+^V(k)]^{-1}$, and therefore we write

$$\mathcal{W}[f_+^V, f_-^V] = -\frac{2ik}{t^V(k)}, \tag{2.6}$$

with $t^V(k) \equiv t_-^V(k) = t_+^V(k)$. Furthermore, one has

$$\left|t^V(k)\right|^2 + \left|r_\pm^V(k)\right|^2 = 1, \quad k \in \mathbb{R}. \tag{2.7}$$

The Jost solutions, $f_\pm^V$, and scattering coefficients, t^V and $r_\pm^V$, can be analytically extended into the upper-half complex $k-$plane. Note that if k_1 is a pole of $t^V(k)$, with $\Im(k_1) > 0$, then $\mathcal{W}[f_+^V, f_-^V](k_1) = 0$. In this case, $f_+^V(x;k_1)$ and $f_-^V(x;k_1)$ are proportional and therefore decay exponentially as $x \to \pm\infty$. Thus, k_1^2 is an L^2-eigenvalue of H_V.

If the potential $V(x)$ is exponentially decaying as x tends to infinity, then the Jost solutions can be analytically extended into the lower half complex $k-$plane. More precisely, if $V \in L^\infty_\beta$ (see Section 1.3), then $f_\pm^V(x;k)$ are defined for $\Im(k) > -\beta/2$ as the unique solutions of the Volterra integral equations

$$\begin{aligned} f_+^V(x;k) &= e^{ikx} + \int_x^\infty \frac{\sin(k(y-x))}{k} V(y) f_+^V(y;k)\, \mathrm{d}y, \\ f_-^V(x;k) &= e^{-ikx} - \int_{-\infty}^x \frac{\sin(k(y-x))}{k} V(y) f_-^V(y;k)\, \mathrm{d}y. \end{aligned} \tag{2.8}$$

Detailed bounds on $f_\pm^V(x;k)$ and their derivatives are presented in Appendix A.

Finally, note the following consequences of $V(x)$ being real-valued, the uniqueness of the Jost solutions as defined above, and (2.4)–(2.5):

$$f_\pm^V(x;-\overline{k}) = \overline{f_\pm^V(x;k)}, \qquad t^V(-\overline{k}) = \overline{t^V(k)}, \qquad r_\pm^V(-\overline{k}) = \overline{r_\pm^V(k)}. \tag{2.9}$$

In particular, $f_\pm^V(x;0)$, $t^V(0)$, $r_\pm^V(0)$ are real.

2.2 Generic and non-generic potentials

Using the decay hypotheses of potential $V \in L^\infty_\beta$ and the method of [3], page 145, one can check that the transmission and reflection coefficients are well-defined by (2.4)–(2.5) for $|\Im(k)| < \beta/2$, and satisfy the important relations, which follow from (2.6) and (2.8):

$$\frac{1}{t^V(k)} = 1 - \frac{1}{2ik} I^V(k), \qquad \text{thus} \quad \mathcal{W}[f^V_+, f^V_-](k) = -2ik + I^V(k),$$

where $I^V(k) \equiv \int_{-\infty}^{\infty} V(y) e^{-iky} f^V_+(y;k)\, \mathrm{d}y$. Equivalently, one has

$$t^V(k) = -\frac{2ik}{\mathcal{W}[f^V_+, f^V_-](k)} = \frac{2ik}{2ik - I^V(k)}. \tag{2.10}$$

Recall that if $V(x) \equiv 0$, then $t^V(k) \equiv 1$. Moreover, if

$$I^V(0) = \mathcal{W}[f^V_+, f^V_-](0) = \int_{-\infty}^{\infty} V(y) f^V_+(y;0)\, \mathrm{d}y \neq 0, \tag{2.11}$$

then by continuity of $t^V(k)$ and (2.10), one has

$$t^V(0) = \lim_{k \to 0} t^V(k) = 0. \tag{2.12}$$

The case where (2.11) and therefore (2.12) holds is typical. Indeed, it has been shown in Appendix 2 of [14] that for a dense subset of $\mathcal{L}^1_1$, one has $I^V(0) \neq 0$; see also [3] and [10]. Thus we say that (2.11) and (2.12) holds *generically in the space of potentials.*

Definition 2.1 (Generic potentials)**.** *We say that a potential, V, is* generic *if one has $t^V(0) = 0$. Equivalently, V is generic if and only if*

$$\frac{k}{t^V(k)} \longrightarrow \frac{I^V(0)}{2i} \neq 0, \qquad \text{as } k \to 0.$$

Note that in the non-generic case, where $\mathcal{W}[f^V_+, f^V_-](0) = 0$, we have that Jost solutions $f^V_\pm(x;k)$ satisfy $f^V_\pm(x;0) \sim 1$ as $x \to \pm\infty$ and are multiples of one another. Thus, non-genericity is equivalent to the existence of a globally bounded solution of the Schrödinger equation at zero energy. Such states are sometimes referred to as zero energy resonances. The simplest example is $V \equiv 0$ where $f^0_\pm(x;k) = e^{\pm ikx}$ and $f^0_\pm(x;0) \equiv 1$.

2.3 Relations between $f^V_\pm$ and $f^W_\pm$ for general V and W

Our approach is based on associating with $V_\epsilon(x) = q_{\mathrm{av}}(x) + q_\epsilon(x)$ a more accurate (than q_{av}) minimal model or *normal form*, $V_{\epsilon,\mathrm{eff}}(x) = q_{\mathrm{av}}(x) + \sigma^\epsilon_{\mathrm{eff}}(x)$, of the asymptotic scattering properties for k bounded and $\epsilon \to 0$. An important tool will then be to compare the Jost solutions associated with the potential, $V = V_\epsilon$, with those of some family of potentials, $W = q_{\mathrm{av}} + \sigma$, parametrized by σ, which is to be determined. This section develops the necessary tools for this comparison.

In the Volterra equation (2.8) we write $f^V_\pm(x;k)$ as a perturbation of the states $e^{\pm ikx}$, which lie in the kernel of $-\partial_x^2 - k^2$. In the following proposition, we generalize this formula by viewing $f^V_\pm(x;k)$ as a perturbation of the Jost solutions $f^W_\pm(x;k)$ for the problem:

$$\left(-\frac{d^2}{dx^2} + W - k^2 \right) u = 0.$$

Proposition 2.2. *Let $V, W \in L^\infty_\beta$ and and let $f^V_\pm, f^W_\pm$ denote the associated Jost solutions. Then for $|\Im(k)| < \beta/2$, one has*

$$\begin{aligned} f^V_+(x;k) &= \alpha_+[V,W]\ f^W_+(x;k) + \beta_+[V,W]\ f^W_-(x;k) \\ f^V_-(x;k) &= \alpha_-[V,W]\ f^W_+(x;k) + \beta_-[V,W]\ f^W_-(x;k), \end{aligned} \tag{2.13}$$

with $\alpha_\pm[V,W](x;k)$ and $\beta_\pm[V,W](x;k)$ defined by

$$\alpha_+[V,W] \equiv 1 + \int_x^\infty \frac{f^W_-(V-W)f^V_+}{\mathcal{W}[f^W_+, f^W_-]}\ dy, \quad \beta_+[V,W] \equiv -\int_x^\infty \frac{f^W_+(V-W)f^V_+}{\mathcal{W}[f^W_+, f^W_-]}\ dy, \tag{2.14}$$

$$\alpha_-[V,W] \equiv -\int_{-\infty}^x \frac{f^W_-(V-W)f^V_-}{\mathcal{W}[f^W_+, f^W_-]}\ dy, \quad \beta_-[V,W] \equiv 1 + \int_{-\infty}^x \frac{f^W_+(V-W)f^V_-}{\mathcal{W}[f^W_+, f^W_-]}\ dy. \tag{2.15}$$

Equivalently, one has the Volterra equation

$$\begin{aligned} f^V_+(x;k) &= f^W_+(x;k) + \int_x^\infty \frac{f^W_+(x;k)f^W_-(y;k) - f^W_-(x;k)f^W_+(y;k)}{\mathcal{W}[f^W_+, f^W_-]}(V-W)f^V_+(y;k)\ dy, \\ f^V_-(x;k) &= f^W_-(x;k) - \int_{-\infty}^x \frac{f^W_+(x;k)f^W_-(y;k) - f^W_-(x;k)f^W_+(y;k)}{\mathcal{W}[f^W_+, f^W_-]}(V-W)f^V_-(y;k)\ dy. \end{aligned} \tag{2.16}$$

A very useful consequence is:

Corollary 2.3. *Let $V, W \in L^\infty_\beta$ and and let $f^V_\pm, f^W_\pm$ denote their respective associated Jost solutions. Then for $|\Im(k)| < \beta/2$, one has*

$$\mathcal{W}[f^V_+, f^V_-](k) = \mathcal{M}[V,W](k)\ \mathcal{W}[f^W_+, f^W_-](k)\ , \tag{2.17}$$

where $\mathcal{M}[V,W](x;k)$ is constant in x, and given by

$$\mathcal{M}[V,W](k) \equiv \alpha_+[V,W](x;k)\beta_-[V,W](x;k) - \alpha_-[V,W](x;k)\beta_+[V,W](x;k). \tag{2.18}$$

By (2.6)*, and taking the limit as $x \to -\infty$ of* (2.14) *and* (2.15) *in* (2.18)*, one has*

$$\frac{k}{t^V(k)} = \frac{k}{t^W(k)} - \frac{I^{[V,W]}(k)}{2i}, \quad \text{with} \quad I^{[V,W]}(k) \equiv \int_{-\infty}^\infty f^W_-(y;k)(V-W)(y)f^V_+(y;k)\ dy. \tag{2.19}$$

Remark 2.4. *The relation* (2.19)*, applied for $V = V_\epsilon$ and a judicious choice of W, is the point of departure for the proofs of our main results.*

Proof of Corollary 2.3. Equation (2.17) follows from substituting the expressions (2.13) into the definition of $\mathcal{W}[f^V_+, f^V_-]$, and using that $\alpha_+[V,W], \beta_+[V,W]$ satisfy the identity: $(\alpha_\pm)' f^W_+ + (\beta_\pm)' f^W_- = 0$; see (2.21) below.

To prove (2.19), we begin by making use of relation (2.6). One has

$$\frac{k}{t^V(k)} = -\frac{\mathcal{W}[f^V_+, f^V_-](k)}{2i}$$

We next relate $\mathcal{W}[f^V_+, f^V_-]$ to $\mathcal{W}[f^W_+, f^W_-]$ by substitution of the expressions (2.13) into the definition of $\mathcal{W}[f^V_+, f^V_-]$ and using (2.14) and (2.15) to obtain

$$\frac{k}{t^V(k)} = -\mathcal{M}[V,W](x,k)\frac{\mathcal{W}[f^W_+, f^W_-](k)}{2i} = \mathcal{M}[V,W](x,k)\frac{k}{t^W(k)}.$$

Now, since $V, W \in L^\infty_\beta$, the estimates of Lemma A.2 yield

$$\lim_{x\to-\infty} \beta_+[V,W](x) \;<\; \infty, \quad \lim_{x\to-\infty} \alpha_-[V,W](x) \;=\; 0 \quad \text{and} \quad \lim_{x\to-\infty} \beta_-[V,W](x) \;=\; 1.$$

Therefore,

$$\mathcal{M}[V,W](k) \;=\; \lim_{x\to-\infty} \alpha_+[V,W](x).$$

Therefore, one deduces from Proposition 2.2 that

$$\frac{k}{t^V(k)} \;=\; \frac{k}{t^W(k)} \lim_{x\to-\infty} \alpha_+[V,W] \;=\; \frac{k}{t^W(k)}\left(1 + \frac{I^{[V,W]}(k)}{\mathcal{W}[f_+^W, f_-^W](k)}\right) \;=\; \frac{k}{t^W(k)} \;-\; \frac{I^{[V,W]}(k)}{2i},$$

where $I^{[V,W]}(k)$ is given in (2.19). The proof of Corollary 2.3 is complete. □

Proof of Proposition 2.2. The integral equation governing a Jost solution for the potential V may be written relative to the potential W as follows. Start with the equation for $u_\pm = f_\pm^V$ written in the form:

$$(H_W - k^2)\, u \;=\; \left(-\frac{d^2}{dx^2} \;+\; W \;-\; k^2\right)\, u \;=\; (W - V)u. \tag{2.20}$$

Treating the right hand side of (2.20) as an inhomogeneous term, we now derive an equivalent integral equations for the Jost solutions . Thus, we seek solutions $u_\pm$ of (2.20), such that $u_\pm(x;k) \;\sim\; f_\pm^V(x;k),\quad x \to \pm\infty$ of the form

$$u(x,k) \;\equiv\; \alpha(x,k) f_+^W(x,k) \;+\; \beta(x,k) f_-^W(x,k), \quad \text{with} \quad \alpha' f_+^W \;+\; \beta' f_-^W \;=\; 0.$$

We obtain $u' \;=\; \alpha f_+^{W\prime} \;+\; \beta f_-^{W\prime}$, $u'' \;= \alpha' f_+^{W\prime} \;+\; \beta' f_-^{W\prime} \;+\; (W - k^2)u$ and eventually the following system for (α', β'):

$$\begin{cases} \alpha' f_+^W \;+\; \beta' f_-^W \;=\; 0 \\ \alpha' f_+^{W\prime} \;+\; \beta' f_-^{W\prime} \;=\; -\left(-\partial_x^2 \;+\; W \;-\; k^2\right)\, u \;=\; (V - W)u \end{cases} \tag{2.21}$$

Solving for α' and β' we have:

$$\alpha' \;=\; \frac{-f_-^W(x,k)(V(x) - W(x))u(x,k)}{\mathcal{W}[f_+^W, f_-^W](k)} \quad \text{and} \quad \beta' \;=\; \frac{f_+^W(x,k)(V(x) - W(x))u(x,k)}{\mathcal{W}[f_+^W, f_-^W](k)}.$$

The expressions for $\alpha_\pm$ and $\beta_\pm$ in (2.14) and (2.15) follow by integrating and imposing the asymptotic behavior of $u_\pm \sim f_\pm^V$ as $x \to \pm\infty$. In particular, one has $f_+^V(x;k) \sim f_+^W(x;k) \sim e^{ikx}$ when $x \to \infty$, and $f_-^V(x;k) \sim f_-^W(x;k) \sim e^{-ikx}$ when $x \to -\infty$. This completes the proof of Proposition 2.2. □

3 Convergence of $t^{q_\epsilon}(k)$ for $k \in \mathbb{C}$ and bifurcation of eigenvalues from the edge of the continuous spectrum

In this section we state our main results for the Schrödinger equation (1.1) with potential of the form:

$$V_\epsilon(x) \;=\; V(x, x/\epsilon). \tag{3.1}$$

Recall the exponentially weighted norms $\left|g\right|_{W_\beta^{n,\infty}}$ introduced in section 1.3. The potential $V(x,y)$ is assumed to satisfy the following precise hypotheses:

Hypotheses (V): *$V(x,y)$ is real-valued and of the form:*

$$V(x,y) \;=\; q_{\rm av}(x) + q(x,y) \;=\; q_{\rm av}(x) \;+\; \sum_{j\neq 0} q_j(x)\; e^{2\pi i\lambda_j y}\ . \tag{3.2}$$

There exist positive constants $\theta > 0$ and $\beta > 0$ such that the sequence of non-zero (distinct) frequencies $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies

$$\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \theta > 0, \qquad \inf_{j\in\mathbb{Z}\setminus\{0\}} |\lambda_j| \geq \theta > 0\ , \tag{3.3}$$

and the coefficients $\{q_j(x)\}_{j\in\mathbb{Z}}$, satisfy the decay and regularity assumptions

$$\left|\!\left| V \right|\!\right| \;\equiv\; \left|q_{\rm av}\right|_{W^{1,\infty}_\beta} \;+\; \sum_{j\in\mathbb{Z}\setminus\{0\}} \left|q_j\right|_{W^{3,\infty}_\beta} \;<\; \infty. \tag{3.4}$$

Remark 3.1. *If V satisfies Hypotheses* **(V)***, and $\sigma^\epsilon_{\mathit{eff}}$ is defined in* (1.9),(1.11)*, then $V_\epsilon \in L^\infty_\beta$, $q_{\rm av} + \sigma^\epsilon_{\mathit{eff}} \in W^{1,\infty}_\beta$ and $\sigma^\epsilon_{\mathit{eff}} \in W^{3,\infty}_\beta$, and there exists $C(\left|\!\left| V \right|\!\right|)$, independent of ϵ, such that*

$$\left|V_\epsilon\right|_{L^\infty_\beta} \;\leq\; C(\left|\!\left| V \right|\!\right|), \quad \left|q_{\rm av} + \sigma^\epsilon_{\mathit{eff}}\right|_{W^{1,\infty}_\beta} \;\leq\; C(\left|\!\left| V \right|\!\right|), \quad \left|\sigma^\epsilon_{\mathit{eff}}\right|_{W^{3,\infty}_\beta} \;\leq\; \epsilon^2 C(\left|\!\left| V \right|\!\right|).$$

Our approach is to study the Jost solutions, $f^{V_\epsilon}(x;k)$, and scattering coefficients, $t^{V_\epsilon}(k)$, $r^{V_\epsilon}_\pm(k)$, for ϵ sufficiently small $\epsilon \in [0,\epsilon_0)$, and for k in a complex neighborhood of zero. More precisely, we assume

Hypotheses (K): *We assume that the wave number, k, varies in K, a compact subset of $\mathbb{C}$ such that*

- *$K \subset \{k,\ |\Im(k)| < \alpha\}$, with $0 < \alpha < \beta/2$, and β is as in Hypotheses* **(V)***;*
- *K does not contain any pole of the transmission coefficient, $t^{q_{\rm av}}(k)$.*

It follows that $t^{q_{\rm av}}(k)$ is bounded, uniformly for $k \in K$, and we define

$$M_K \;\equiv\; \max\big(1\ ,\ \sup_{k\in K} |t^{q_{\rm av}}(k)|\big) \;<\; \infty. \tag{3.5}$$

Moreover, if $K \subset \mathbb{R}$, then $M_K = 1$; see (2.7).

Remark 3.2. *We can relax the spatial decay assumptions of Hypotheses* **(V)***, if we restrict Hypotheses* **(K)** *to the upper-half plane $\Im(k) \geq 0$. Our methods apply and only require sufficient algebraic decay of $V(x)$. Results of this kind for $k \in \mathbb{R}$ are presented in Section 4.*

We now state our main theorem and its important consequences.

Theorem 3.3 (Convergence of the transmission coefficient)**.**
Assume $V_\epsilon(x) = V(x, x/\epsilon)$ satisfies Hypotheses **(V)***, and $k \in K$ satisfies Hypotheses* **(K)***. Then there exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, $t^{q_{\rm av}+q_\epsilon}(k)$, the transmission coefficient of the scattering problem* (1.1)-(1.2) *with*

$$V_\epsilon(x) \;=\; q_{\rm av}(x) + q_\epsilon(x) \;=\; q_{\rm av}(x) + q(x, x/\epsilon),$$

is uniformly approximated by the transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, for

$$V^\epsilon_{\rm eff}(x) \;=\; q_{\rm av}(x) \;+\; \sigma^\epsilon_{\mathit{eff}}(x),$$

where $\sigma^\epsilon_{\mathit{eff}}(x)$ *denotes the* <u>*effective potential well*</u>*,*

$$\sigma^\epsilon_{\mathit{eff}}(x) \;\equiv\; -\epsilon^2\ \Lambda_{\rm eff}(x) \;\equiv\; -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{|q_j(x)|^2}{{\lambda_j}^2}. \tag{3.6}$$

Specifically, we have the estimate

$$\sup_{k\in K}\left|\ \frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(k)}\ -\ \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\ \right|\ \le\ \epsilon^3\ M_K\ C(\,\boldsymbol{|}\,V\,\boldsymbol{|}\,,\sup_{k\in K}|k|), \tag{3.7}$$

with $C(\,\boldsymbol{|}\,V\,\boldsymbol{|}\,)$ *a constant, independent of* ϵ*.*

The proof of Theorem 3.3 is given in section 6; we first present its consequences. A simple outcome of (3.7) and the genericity of $q_{\rm av}+\sigma^\epsilon_{\rm eff}$ for ϵ sufficiently small (which holds for $q_{\rm av}$ generic *and* non-generic; see Corollary B.2[1]) is:

Corollary 3.4. *Assume* $V_\epsilon = q_{\rm av}+q_\epsilon$ *satisfies Hypotheses* **(V)***. We allow* $q_{\rm av}$ *to be either generic or non-generic in the sense of Definition 2.1. Then, there exists* $\epsilon_0>0$ *such that for any* $0<\epsilon<\epsilon_0$*,* V_ϵ *is generic.*

Theorem 3.3 holds for both generic and non-generic potentials, $q_{\rm av}$. In the following section we explore consequences for the non-generic potential, $q_{av}(x)\equiv 0$, *i.e.* $V_\epsilon(x) = q(x,x/\epsilon)$, with $\int_0^1 q(x,y)\ {\rm d}y = 0$. In particular, we explain the non-uniformity localization phenomenon discussed in the Introduction. Results for non-trivial $q_{\rm av}(x)$ are developed in sections 3.2 and 3.3.

3.1 Results for mean-zero oscillatory potentials: $q_{\rm av}(x)\equiv 0$

The following corollary, comparing $t^{q_\epsilon}(k)$ and $t^{\sigma^\epsilon_{\rm eff}}(k)$, is a consequence of Theorem 3.3, and Lemma B.1.

Corollary 3.5. *Let* $q_{\rm av}\equiv 0$*, so that* $V_\epsilon(x) = q_\epsilon(x) = q(x,x/\epsilon)$*. Let* K *denote the compact set of Hypotheses* **(K)***. There exists* $\epsilon_0>0$ *such that if*

$$\left|k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}\right|\ge C\epsilon^\tau,\quad \tau<3,\quad k\in K,\quad 0<\epsilon<\epsilon_0, \tag{3.8}$$

then one has for $0<\epsilon<\epsilon_0$*,*

$$\left|\frac{t^{\sigma^\epsilon_{\mathit{eff}}}(k)}{t^{q_\epsilon}(k)}-1\right|\ =\ \mathcal{O}\big(\epsilon^{3-\tau}\big). \tag{3.9}$$

If in addition to (3.8)*, the following condition holds:*

$$\left|k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}\right|\ge C|k|,\quad k\in K,\quad 0<\epsilon<\epsilon_0$$

then one has for $0<\epsilon<\epsilon_0$*,*

$$\left|t^{\sigma^\epsilon_{\mathit{eff}}}(k)-t^{q_\epsilon}(k)\right|\ =\ \mathcal{O}\big(\epsilon^{3-\tau}\big),\quad \textit{and}\quad \left|t^{q_\epsilon}(k)\ -\ \frac{k}{k-\frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}}\right|\ =\ \mathcal{O}\big(\epsilon^{3-\tau}\big).$$

In particular, if $k=\epsilon^2\kappa$*, with* $\kappa\ne\kappa^\star\equiv-\frac{1}{2i}\int\Lambda_{\rm eff}$*, then for* $0<\epsilon<\epsilon_0$*,*

$$\left|t^{\sigma^\epsilon_{\mathit{eff}}}(\epsilon^2\kappa)-t^{q_\epsilon}(\epsilon^2\kappa)\right|\ =\ \mathcal{O}\left(\frac{\epsilon\ |\kappa|}{|\kappa-\kappa^\star|^2}\right)\ =\ \mathcal{O}(\epsilon),\quad \left|t^{q_\epsilon}(\epsilon^2\kappa)-\frac{\kappa}{\kappa-\frac{i}{2}\int_{-\infty}^{\infty}\Lambda_{\rm eff}}\right|\ =\ \mathcal{O}\big(\epsilon\big). \tag{3.10}$$

[1]Note that in the non-generic case, the condition $\int_{\mathbb{R}}\Lambda_{\rm eff}(y)(f_-^{q_{\rm av}}(y;0))^2\ {\rm d}y\ne 0$ is always satisfied. Indeed, $f_-^{q_{\rm av}}(\cdot;0)\in\mathbb{R}$ by (2.9), and is non-zero almost everywhere on the support of $\Lambda_{\rm eff}$.

Proof. Corollary B.2 of appendix B gives

$$\frac{k}{t^{\sigma^\epsilon_{\rm eff}}(k)} \;=\; k \;-\; \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} \Lambda_{\rm eff}(y)\ {\rm d}y \;+\; \mathcal{O}\big(\epsilon^4\big),\quad \epsilon \to 0\ , \tag{3.11}$$

uniformly for $k \in K$. By Theorem 3.3, one has

$$\frac{k}{t^{q_\epsilon}(k)} \;=\; k \;-\; \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} \Lambda_{\rm eff}(y)\ {\rm d}y \;+\; \mathcal{O}\left(\epsilon^3\right),\ \text{uniformly for } k \in K\ . \tag{3.12}$$

Expansions (3.11) and (3.12) imply straightforwardly (3.9)–(3.10). □

A direct consequence of Corollary 3.5 and the expansion of $t^{\sigma^\epsilon_{\rm eff}}$ implied by Lemma B.1, is the following result showing a universal scaled limit of t^{q_ϵ}, depending on the single parameter, $\int_{\mathbb{R}} \Lambda_{\rm eff}$.

Corollary 3.6 (Scaled limit of t^{q_ϵ})**.** *Let $k = \epsilon^2\kappa$, with $\kappa \neq \frac{i}{2}\ \int_{\mathbb{R}} \Lambda_{\rm eff}$. Then one has*

$$t^{q_\epsilon}(\epsilon^2\kappa)\ \to\ t^\star\Big(\kappa; \int_{\mathbb{R}} \Lambda_{\rm eff}\Big) \equiv \frac{\kappa}{\kappa - \frac{i}{2}\ \int_{\mathbb{R}} \Lambda_{\rm eff}}\quad \text{as}\ \ \epsilon \to 0, \tag{3.13}$$

where $t^\star(\kappa; m)$ is the transmission coefficient associated with the Schrödinger operator with attractive $\delta-$function potential well of total mass $m > 0$:

$$H_{-m\delta} = -\partial_X^2 - m\delta(X).$$

As observed in section 2, the poles of the transmission coefficient in the upper half $k-$plane, which must lie on the imaginary axis, correspond to the L^2 point eigenvalues. From our estimates on the transmission coefficient, $t^{q_\epsilon}(k)$, we further deduce the existence of a discrete eigenvalue near the edge of the continuous spectrum.

Corollary 3.7 (Edge bifurcation of point spectrum from the continuum)**.**
If ϵ if sufficiently small, then the transmission coefficient, $t^{q_\epsilon}(k)$ has a pole in the upper half plane at

$$k_\epsilon \;=\; i\ \frac{\epsilon^2}{2}\ \left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right) + \mathcal{O}(\epsilon^3)\ ,\quad \epsilon \to 0\ ,$$

and therefore H_{q_ϵ} has the simple eigenpair

$$E_\epsilon = k_\epsilon^2 = -\frac{\epsilon^4}{4}\ \left(\int_{\mathbb{R}} \Lambda_{\rm eff}\right)^2\ +\ \mathcal{O}(\epsilon^5),\quad \epsilon \to 0\ ,$$
$$u_{E_{q_\epsilon}}(x)\ =\ \mathcal{O}\left(e^{-\sqrt{|E_{q_\epsilon}|}\ |x|}\right),\quad |x| \gg 1\ .$$

Proof of Corollary 3.7: Let us recall Rouché's Theorem: Let f and g denote analytic functions, defined on an open set $A \subset \mathbb{C}$. Let γ denote a simple loop within A, which is homotopic to a point. If $|g(k) - f(k)| < |f(k)|$ for all $k \in \gamma$, then f and g have the same number of roots inside γ.

Now let

$$f(k)\ \equiv\ k\ -\ \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} \Lambda_{\rm eff}(y)\ {\rm d}y\ ,$$
$$g_1(k)\ =\ \frac{k}{t^{\sigma^\epsilon_{\rm eff}}(k)},\qquad g_2(k)\ =\ \frac{k}{t^{q_\epsilon}(k)},$$

and $\gamma\ =\ \{k\ :\ |k - \frac{i\epsilon^2}{2}\int \Lambda_{\rm eff}| = C\epsilon^3\} \subset K$. These functions are analytic in k; see [3] and our previous discussion. Theorem 3.3 and Corollary B.2 imply, respectively,

$$g_2(k) = f(k) + \mathcal{O}(\epsilon^3)\ \ \text{and}\ \ g_1(k) = f(k) +\ \mathcal{O}(\epsilon^4).$$

Therefore, there exist constants $a_K,\ b_K$, such that for $k \in \gamma$:

$$\big|f(k)\ -\ g_1(k)\big|\ \le\ a_K\epsilon^4,\qquad \big|f(k)\ -\ g_2(k)\big|\ \le\ b_K\epsilon^3,\qquad \text{and}\qquad |f(k)| = C\epsilon^3.$$

Taking ϵ sufficiently small and choosing C sufficiently large, Rouché's Theorem implies that both g_1 and g_2, have unique roots, poles of $t^{\sigma^\epsilon_{\rm eff}}$ and t^{q_ϵ}, in the set $\{k\ :\ |k - \frac{i\epsilon^2}{2}\int\Lambda_{\rm eff}| \le C\epsilon^3\}$. By self-adjointness, these poles lie on the positive imaginary axis. Corollary 3.7 now follows. □

3.2 Non-generic and non-zero $q_{\rm av}$; example of an oscillatory perturbation of a reflectionless potential

As seen above, for the case where $q_{\rm av}\equiv 0$ the transmission coefficient $t^{q_\epsilon}(k)$, does not converge to $t^0(k)\equiv 1$ uniformly in a neighborhood of $k=0$ and the obstruction to uniform convergence is the approach, as $\epsilon\to 0$, of a pole of $t^{q_\epsilon}(k)$ toward $k=0$. Such non-uniform convergence will occur whenever $t^{q_{\rm av}}(0)\neq 0$. By (2.10), (2.11), we can have $t^{q_{\rm av}}(0)\neq 0$ if and only if $\mathcal{W}[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}](0)=0$, the case where $q_{\rm av}$ is non-generic; see section 2.2.

One may construct non-generic potentials as follows. Let $v(x)$ denote a potential well, $v(x)\le 0$, say a square well, having one eigenstate and which is generic, *i.e.* $\mathcal{W}[f_+^v, f_-^v](0)\neq 0$ and therefore $t^v(0)=0$. Consider the one-parameter family of Schrödinger operators defined as $h_g=-\partial_x^2+gv(x),\ g\ge 1$. As g increases, new eigenvalues of h_g appear as g tranverses discrete values $g_1<g_2<\cdots$. These eigenvalues appear via the crossing of a pole of $t^{gv}(k)$ in the lower half $k-$plane, for $g<g_N$, into the upper half plane for $g>g_N$. For g equal to one of these transition values, g_N, one has $t^{g_Nv}(0)\neq 0$. Thus, $g_Nv(x)$ is a non-generic potential. Our analysis gives, for $q_{\rm av}$ taken to be any such non-generic potential, a precise description of the motion of the pole of $t^{q_{\rm av}+q_\epsilon}$ as it approaches $k=0$ for ϵ small.

The following corollary, the analogue of Corollaries 3.5 and 3.6, follows as in the case $q_{\rm av}\equiv 0$ from Theorem 3.3 and Lemma B.1.

Corollary 3.8 (Oscillatory perturbation of a reflectionless potential)**.**
Let $V_\epsilon(x)=q_{\rm av}+q_\epsilon(x)=q_{\rm av}+q(x,x/\epsilon)$ satisfy Hypotheses **(V)***, let $q_{\rm av}$ be reflectionless, and finally let $k\in K$ satisfy Hypotheses* **(K)***. Assume in addition that the following condition holds,*

$$\left|\frac{k}{t^{q_{\rm av}}(k)} - \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;k)\ dy\right| \ge C\min(|k|,\epsilon^\tau),\ \ \tau<3, \tag{3.14}$$

then one has for ϵ sufficiently small

$$\left|t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k) - t^{q_{\rm av}+q_\epsilon}(k)\right|\ =\ \mathcal{O}\big(\epsilon^{3-\tau}\big). \tag{3.15}$$

In particular, $k=\epsilon^2\kappa$ satisfies (3.14)*, and therefore, by Lemma B.1, there is a universal scaled limit of $t^{q_{\rm av}+q_\epsilon}(\epsilon^2\kappa)$:*

$$\begin{aligned} t^{q_{\rm av}+q_\epsilon}(\epsilon^2\kappa)\ &\to\ \frac{t^{q_{\rm av}}(0)\ \kappa}{\kappa-\frac{i}{2}t^{q_{\rm av}}(0)\ \int_{\mathbb{R}} f_-^{q_{\rm av}}(y;0)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;0)\ dy}\\ &=\ \frac{t^{q_{\rm av}}(0)\ \kappa}{\kappa-\frac{i}{2}(1+r_-^{q_{\rm av}}(0))\ \int_{\mathbb{R}}(f_-^{q_{\rm av}}(y;0))^2\ \Lambda_{\rm eff}(y)\ \ dy},\qquad \text{as}\quad \epsilon\to 0\end{aligned} \tag{3.16}$$

provided $\kappa\neq\kappa^\star\equiv\frac{i}{2}t^{q_{\rm av}}(0)\ \int_{\mathbb{R}} f_-^{q_{\rm av}}(y;0)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;0)\ dy.$ [2] *The last equality in* (3.16) *follows from* (2.4)*.*

[2]Note that $\kappa^\star$ lies in the positive imaginary axis. Indeed, $f_-^{q_{\rm av}}(\cdot;0)\in\mathbb{R}$ and $r_-(0)\in\mathbb{R}$ by (2.9), and one has $r_-(0)+1\ge 0$, since $|r_-(0)|\le 1$; see (2.7).

The transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)$ has a pole in the upper half plane at $k_{q_{\rm av}+\sigma^\epsilon_{\it eff}}$ the solution of the implicit equation:

$$k \;=\; i\frac{\epsilon^2}{2}\ t^{q_{\rm av}}(k)\ \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ \Lambda_{\rm eff}(y)\ f_+^{q_{\rm av}}(y;k)\ dy \;+\; \mathcal{O}(\epsilon^4). \tag{3.17}$$

It follows that $H_{q_{\rm av}+\sigma^\epsilon_{\it eff}}$ has an eigenvalue at $E^{\sigma^\epsilon_{\it eff}} = (k_{q_{\rm av}+\sigma^\epsilon_{\it eff}}(\epsilon))^2 < 0$. Finally, Lemma B.1 and an application of Rouché's Theorem imply that $t^{q_{\rm av}+q_\epsilon}(k)$, has a pole near $k^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(\epsilon)$, on the positive imaginary axis, and a bound state

$$E^{q_{\rm av}+q_\epsilon}(\epsilon) \approx\ E^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(\epsilon) \;=\; \left[k^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(\epsilon)\right]^2 \;<\; 0.$$

We now consider this result in the context of a particular family of potentials. Consider the family of operators $h(g) = -\partial_x^2 - g\ {\rm sech}^2(x)$. Let $g_N = N(N+1),\ N = 0,1,2,\dots$. For $g_N \le g < g_{N+1}$, the operator $h(g)$ has precisely N- bound states. At the transition values, $h(g_N)$ has a zero energy resonance and $t^{h(g_N)}(0) \neq 0$. The family of potentials for the values $g_N,\ N = 0,1,2,\dots$, are called *reflectionless potentials* for which $|t(k)| \;\equiv\; 1$ and $r_\pm(k) \;\equiv\; 0,\quad k \in \mathbb{R}$; see [1]. These potentials are well-known for their role as soliton solutions of the Korteweg-de Vries equation.

Consider the case of the one-soliton potential, corresponding to $N = 1$ in the above discussion. Here,

$$V_1(x) \;=\; -2\rho^2{\rm sech}^2(\rho(x-x_0)),\quad \text{where } x_0 \text{ satisfies } C = 2\rho\exp(2\rho x_0).$$

In this case, the transmission coefficient satisfies

$$\frac{1}{t^{V_1}(k)} \;=\; \lim_{x\to-\infty} f_+^{V_1}(x;k)e^{-ikx} \;=\; \frac{k-i\rho}{k+i\rho}.$$

As for the Jost solutions, one has (setting $x_0 = 0$ for simplicity)

$$f_+^{V_1}(x;k) = e^{ikx}\left(1 - \frac{2i\rho}{k+i\rho}\frac{e^{-x}}{e^x+e^{-x}}\right).$$

Since the V_1 is reflectionless, one has by (2.5)

$$f_-^{V_1}(x;k) = 0 + \frac{1}{t^{V_1}(k)} f_+^{V_1}(x;-k) \;=\; \frac{1}{t^{V_1}(k)} e^{-ikx}\left(1 - \frac{2i\rho}{-k+i\rho}\frac{e^{-x}}{e^x+e^{-x}}\right).$$

In this case, there exists a pole of $t^{V_1+\sigma^\epsilon_{\rm eff}}(k)$, and similarly a pole of $t^{V_1+q_\epsilon}(k)$, located around

$$\begin{aligned} k \;&=\; i\frac{\epsilon^2}{2}\int_{-\infty}^{\infty} t^{V_1}(0) f_-^{V_1}(y;0)\Lambda_{\rm eff}(y) f_+^{V_1}(y;0)\ {\rm d}y \;+\; \mathcal{O}(\epsilon^3)\ , \\ &=\; i\frac{\epsilon^2}{2}\int_{-\infty}^{\infty} \tanh^2(y)\Lambda_{\rm eff}(y)\ {\rm d}y \;+\; \mathcal{O}(\epsilon^3)\ ,\quad \epsilon\to 0. \end{aligned}$$

Finally, $H_{V_1+q_\epsilon}$ and $H_{V_1+\sigma^\epsilon_{\rm eff}}$ have a bound state with energy

$$E \;=\; -\frac{\epsilon^4}{4}\left(\int_{\mathbb{R}} \tanh^2(y)\Lambda_{\rm eff}(y)\ \ {\rm d}y\right)^2 \;+\; \mathcal{O}(\epsilon^5)\ ,\quad \epsilon\to 0.$$

3.3 Results for generic potentials, $q_{\rm av}$, and their highly oscillatory perturbations

In this section, we study the case where $q_{\rm av}$ is a generic potential in the sense of section 2. In this case $t^{q_{\rm av}+q_\epsilon}(k)$ converges uniformly to $t^{q_{\rm av}}(k)$ in a neighborhood of $k = 0$ [5]. More precise information is contained in the following Corollary, a direct consequence of Lemma B.1, and Theorem 3.3.

Corollary 3.9. *Let $V_\epsilon(x) = q_{\rm av}(x) + q_\epsilon(x) = q_{\rm av}(x) + q(x, x/\epsilon)$ satisfy Hypotheses* **(V)** *with $q_{\rm av}$ generic, and $k \in K$ satisfy Hypotheses* **(K)**. *Then for k and ϵ small enough, one has*

$$\left|t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)\right| \;\leq\; C_0|k|, \tag{3.18}$$

$$\left|t^{q_{\rm av}+q_\epsilon}(k)\right| \;\leq\; C_0|k|, \tag{3.19}$$

$$\left|t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)\right| \;\leq\; C_0\epsilon^3|k|, \tag{3.20}$$

with $C_0 = C_0(M_K)$, $M_K = \max(1, \sup_{k\in K}|t^{q_{\rm av}}(k)|)$.

Proof. In the case of generic potentials, $q_{\rm av}$, we know from [3] that there exists a constant $a_{q_{\rm av}}$ such that

$$t^{q_{\rm av}}(k) \;=\; a_{q_{\rm av}}k \;+\; o(k), \qquad \text{as } k \to 0.$$

It follows that for k sufficiently small, there exists a constant C_0 such that $\left|k\ (t^{q_{\rm av}}(k))^{-1}\right| \;\geq\; C_0 > 0$. Estimate (3.18) follows then straightforwardly from Lemma B.1, when ϵ is sufficiently small. Now, applying Theorem 3.3, one has

$$\begin{aligned}\left|t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k) - t^{q_{\rm av}+q_\epsilon}(k)\right| &= \left|\frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)} - \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\right|\left|\frac{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)\ t^{q_{\rm av}+q_\epsilon}(k)}{k}\right| \\ &\leq\; C_0\epsilon^3\left|t^{q_{\rm av}+q_\epsilon}(k)\right|.\end{aligned}$$

Estimate (3.19) and then (3.20) follow easily. This concludes the proof. □

4 Behavior of the transmission coefficient, uniformly in $k \in \mathbb{R}$

In this section we focus on the properties of $t^{q_\epsilon}(k)$, which hold uniformly in $k \in \mathbb{R}$. The results presented in section 2 are valid for $k \in \mathbb{R}$, and under the less stringent condition: $V \in \mathcal{L}^1_2(\mathbb{R}) = \{V : (1+|x|)^2V(x) \in L^1(\mathbb{R})\}$. Most of these results can be found in [3]. Our required bounds on the Jost solutions, $f^V_\pm$ are given in Lemma A.1.

Since k is constrained to the real axis, we find that we can relax the assumption of exponential decay on the potential $V_\epsilon = V(x, x/\epsilon)$.

Hypotheses (V'): *$V(x, y)$ is a real-valued potential of the form*

$$V(x,y) \;=\; q_{\rm av}(x) + q(x,y) \;=\; q_{\rm av}(x) \;+\; \sum_{j\neq 0} q_j(x)\ e^{2\pi i\lambda_j y},$$

such that the sequence of non-zero (distinct) frequencies $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies (3.3)*, and the coefficients $\{q_j(x)\}_{j\in\mathbb{Z}}$, satisfy the decay and regularity assumptions*

$$\|\, V \,\| \;\equiv\; |q_{\rm av}|_{\mathcal{W}^{1,1}_2} \;+\; \sum_{j\in\mathbb{Z}\setminus\{0\}} |q_j|_{\mathcal{W}^{3,1}_3} \;<\; \infty. \tag{4.1}$$

We first investigate the difference between the transmission coefficients $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, where $\sigma^\epsilon_{\rm eff}$ is defined as in Theorem 3.3. The proof of the following theorem is analogous to that of Theorem 3.3 (section 6). We omit the proof for the sake of brevity.

Theorem 4.1 (Transmission coefficient, $t^{V_\epsilon}(k)$, for $k \in \mathbb{R}$)**.** *Assume $V_\epsilon(x) = V(x, x/\epsilon)$ satisfies Hypotheses* **(V')**. *Assume $k \in \mathbb{R}$, $|k| \leq 1$. Then, the following holds:*

(1) There exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, $t^{q_{\rm av}+q_\epsilon}(k)$ is uniformly approximated by the transmission coefficient, $t^{q_{\rm av}+\sigma^\epsilon_{\it eff}}(k)$, for $H_{q_{\rm av}+\sigma^\epsilon_{\it eff}}$. Here $\sigma^\epsilon_{\it eff}(x)$ denotes the effective potential well defined in (3.6)*.*

Moreover, there is a constant $C(\| V \|)$, independent of ϵ and k, such that

(4.2)
$$\sup_{k\in\mathbb{R},\ |k|\leq 1}\left|\frac{k}{t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)}-\frac{k}{t^{q_{\rm av}+q_\epsilon}(k)}\ \right|\leq \epsilon^3\ C(\| V \|)\ \max(1,\sup_{k\in K}|t^{q_{\rm av}}(k)|)\leq \epsilon^3\ C(\| V \|).$$

(2) *Assume $q_{av}\equiv 0$, so that $H_{V_\epsilon}=-\partial_x^2+q(x,x/\epsilon)$, where $y\mapsto q(x,y)$ has mean zero. Then, applying* (4.2) *and Corollary B.2 we have*

$$t^{q_\epsilon}(k)\ =\ \frac{k}{k-\frac{i}{2}\ \epsilon^2\ \int_{\mathbb{R}}\Lambda_{\rm eff}\ +\ \mathcal{O}(\epsilon^3)} \tag{4.3}$$

In the following, we are able to control the difference between $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, for large real wave number, $|k|\geq 1$. This allows, in particular, control of the difference between $t^{q_{\rm av}+q_\epsilon}(k)$ and $t^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(k)$, when the averaged potential $q_{\rm av}\equiv 0$, *uniformly in $k\in\mathbb{R}$.*

Proposition 4.2. *Let $V_\epsilon\equiv V(x,x/\epsilon)\equiv q_{\rm av}+q_\epsilon$ with V satisfying Hypotheses* **(V')**, *and $\sigma^\epsilon(x)$ denote any potential for which*

$$\int|\sigma^\epsilon(y)|(1+|y|)\ dy\leq\epsilon^2\ C_\sigma$$

Then, for $k\in\mathbb{R}\setminus\{0\}$, one has

$$\left|\ t^{q_{\rm av}+q_\epsilon}(k)-t^{q_{\rm av}+\sigma^\epsilon}(k)\ \right|\ \leq\ \epsilon^2\ |k|^{-1}\ C(\| V \| ,C_\sigma)\ , \tag{4.4}$$

where $\| V \|$ is defined in (4.1).

Remark 4.3. *We shall apply this proposition to $\sigma^\epsilon(x)=\sigma^\epsilon_{\rm eff}(x)$, for which $C_\sigma=\mathcal{O}(\| V \|)$.*

Proof. Recall the identity (2.19), relating the transmission coefficients of *any potentials $V,W\in\mathcal{L}^1_2$*:

$$\frac{k}{t^V(k)}=\frac{k}{t^W(k)}-\frac{I^{[V,W]}(k)}{2i},\ \text{with}\ \ I^{[V,W]}(k)\equiv\int_{-\infty}^{\infty}f_-^W(y;k)(V-W)(y)f_+^V(y;k)\ {\rm d}y. \tag{4.5}$$

Since $t^{q_{\rm av}+q_\epsilon}-t^{q_{\rm av}+\sigma^\epsilon}=\big[t^{q_{\rm av}+q_\epsilon}-t^{q_{\rm av}}\big]\ +\ \big[t^{q_{\rm av}}-t^{q_{\rm av}+\sigma^\epsilon}\big]$, we estimate the two bracketed terms independently. We begin by comparing the transmission coefficients for $W\equiv q_{\rm av}$ and $V\ \equiv\ q_{\rm av}+\sigma^\epsilon$. We have by (4.5)

$$\frac{k}{t^{q_{\rm av}+\sigma^\epsilon}(k)}-\frac{k}{t^{q_{\rm av}}(k)}=-\frac{1}{2i}I^{[q_{\rm av}+\sigma^\epsilon,q_{\rm av}]}(k)=-\frac{1}{2i}\int_{-\infty}^{\infty}f_-^{q_{\rm av}}(y;k)\sigma^\epsilon(y)f_+^{q_{\rm av}+\sigma^\epsilon}(y;k)\ {\rm d}y. \tag{4.6}$$

Using the estimates of Lemma A.2, we obtain

$$\left|\ \int_{-\infty}^{\infty}f_-^{q_{\rm av}}(y;k)\ \sigma^\epsilon(y)\ f_+^{q_{\rm av}+\sigma^\epsilon}(y;k)\ {\rm d}y\ \right|\leq\epsilon^2 C_\sigma. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\left|\ t^{q_{\rm av}+\sigma^\epsilon}(k)-t^{q_{\rm av}}(k)\ \right|\ \leq\ \epsilon^2\ |k|^{-1}\ C_\sigma\ \left|t^{q_{\rm av}}(k)\ t^{q_{\rm av}+\sigma^\epsilon}(k)\right|\ . \tag{4.8}$$

Using the general relation $|t^V(k)|\leq 1$, for any $k\in\mathbb{R}$, (see (2.7)), we obtain

$$\left|\ t^{q_{\rm av}+\sigma^\epsilon}(k)-t^{q_{\rm av}}(k)\ \right|\ \leq\ \epsilon^2\ |k|^{-1}\ C_\sigma\ .$$

We now turn to the comparison of the transmission coefficients of $V \equiv q_{\rm av} + q_\epsilon$ and $W \equiv q_{\rm av}$. Proceeding similarly, we have

$$\begin{aligned} \frac{k}{t^{q_{\rm av}+q_\epsilon}(k)} - \frac{k}{t^{q_{\rm av}}(k)} &= -\frac{1}{2i} I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k), \quad \text{where} \\ I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k) &\equiv \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\ q_\epsilon(y)\ f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y\ . \end{aligned} \tag{4.9}$$

Two integrations by parts yield

$$\begin{aligned} I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k) &= \sum_{j\neq 0} \int_{-\infty}^{\infty} q_j(y) e^{2i\pi\lambda_j y/\epsilon} f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y \\ &= \sum_{j\neq 0} \left(\frac{-\epsilon}{2i\pi\lambda_j}\right)^2 \int_{-\infty}^{\infty} \partial_y^2 \big(q_j(y) f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\big) e^{2i\pi\lambda_j y/\epsilon}\ {\rm d}y. \end{aligned}$$

Using the estimates of Lemma A.1 and Hypotheses **(V')**, one sees that the integrand is bounded. Indeed, one has

$$\begin{aligned} \Big|\ I^{[q_{\rm av},q_{\rm av}+q_\epsilon]}(k)\ \Big| &\leq \sum_{j\neq 0} \left(\frac{\epsilon}{2\pi\lambda_j}\right)^2 \int_{-\infty}^{\infty} \big|\partial_y^2 \big(q_j(y) f_-^{q_{\rm av}}(y;k) f_+^{q_{\rm av}+q_\epsilon}(y;k)\big)\big|\ {\rm d}y \\ &\leq\ \epsilon^2 C\big(|q_{\rm av}|_{\mathcal{L}^1_2}\big) \sum_{j\neq 0} \Big[\int_{-\infty}^{\infty} |\partial_y^2 q_j(y)| \frac{(1+|y|)^2}{(1+|k|)^2}\ {\rm d}y\ +\ \int_{-\infty}^{\infty} |\partial_y q_j(y)| \frac{(1+|y|)^2}{1+|k|}\ {\rm d}y \\ &\quad + \int_{-\infty}^{\infty} |q_j(y)| (1+|y|)^2\ {\rm d}y \Big]\ \leq\ \epsilon^2 C\big(|q_{\rm av}|_{\mathcal{L}^1_2}\big) \sum_{j\neq 0} |q_j|_{\mathcal{W}^{1,1}_2}. \end{aligned}$$

Arguing as in (4.8), we deduce

$$\Big|\ t^{q_{\rm av}+q_\epsilon}(k) - t^{q_{\rm av}}(k)\ \Big|\ \leq\ \epsilon^2\ |k|^{-1}\ C(\ \|\ V\ \|\ )\ \big|t^{q_{\rm av}}(k)\ t^{q_{\rm av}+q_\epsilon}(k)\big|\ \leq\ \epsilon^2\ |k|^{-1}\ C(\ \|\ V\ \|\ ).$$

This completes the proof of Proposition 4.2. □

The following corollary follows from Theorem 4.1 and Proposition 4.2.

Corollary 4.4. *Let $V_\epsilon = q_\epsilon = q(x, x/\epsilon)$ $(q = 0)$ satisfy Hypotheses* **(V')**. *Then*

$$\sup_{k\in\mathbb{R}} \Big| t^{\sigma^\epsilon_{\it eff}}(k) - t^{q_\epsilon}(k) \Big|\ =\ \mathcal{O}(\epsilon), \quad \epsilon \to 0. \tag{4.10}$$

Proof. The behavior for k small is controlled as in Corollary 3.5. Conditions (3.8) and (3.10) hold in particular when we restrict to real wave numbers, $k \in \mathbb{R}$. Therefore, one sees from (3.11) and (3.12) that the difference between $t^{q_\epsilon}(k)$ and $t^{\sigma^\epsilon_{\rm eff}}(k)$ is small, uniformly for $|k| \leq 1$, $k \in \mathbb{R}$:

$$\sup_{k\in\mathbb{R},\ |k|\leq 1} \Big| t^{\sigma^\epsilon_{\rm eff}}(k) - t^{q_\epsilon}(k) \Big|\ \leq\ C \frac{\epsilon^3}{\epsilon^2 + |k|},$$

where $C = C(M_K)$, and $M_K = \max(1, \sup_{k\in\mathbb{R}} |t^0(k)|) = 1$. The difference is controlled for $|k| \geq 1$ by Proposition 4.2, and Corollary 4.4 follows. □

5 Detailed dispersive time decay of $\exp\left(-iH_{q_\epsilon}t\right)\psi_0$; the effect of a pole of $t^{q_\epsilon}(k)$ near $k=0$

In this section we use our detailed results on $t^{q_\epsilon}(k)$ to prove time decay estimates of the Schrödinger equation:

$$i\partial_t\psi \ = \ H_V\psi \ \equiv \ -\,\partial_x^2\psi \ + \ V(x)\psi, \quad \psi(0,x) \ = \ \psi_0 \ . \tag{5.1}$$

for initial conditions ψ_0, which are orthogonal to the bound states of H_{q_ϵ}.

Let $V \in \mathcal{L}^1_1$. Then, it is known that H_V has no singular-continuous spectrum, no positive (*embedded*) eigenvalues and its absolutely-continuous spectrum is $[0,\infty)$; see, for example, [3]. In general, H_V may have a finite number of negative eigenvalues that are simple: $E_N < \cdots < E_0 < 0$. We denote by u_j the eigenvector associated to the eigenvalue E_j, normalized to have L^2 norm equal to one. By the spectral theorem, the solution of (5.1) can be decomposed as follows:

$$\psi(x,t) \ = \ e^{-itH_V}\psi_0 \ = \ \sum_{j=0}^{N} e^{-itE_j}(\psi_0,u_j)u_j + e^{-itH_V}P_c\psi_0,$$

where P_c denotes the projection onto the continuous spectral subspace of H. $\exp(-itH_V)P_c\psi_0$ is a *scattering state* which spatially spreads and temporally decays: $\left|e^{-itH_V}P_c\psi_0\right|_{L^\infty_x} \to 0$ as $t\to\infty$. In the case $V(x)\equiv 0$, we have $\psi(x,t) = \exp(it\partial_x^2)\psi_0 = K_t \star \psi_0$, where $|K_t(x)| \le (4\pi t)^{-1/2}$. From this decay estimate it follows immediately that $\left|e^{-itH_0}P_c\psi_0\right|_{L^\infty_x} \ \le \ C\ |t|^{-1/2}\ \left|\psi_0\right|_{L^1}$. This $|t|^{-1/2}$ decay-rate is associated with the potential $V\equiv 0$ being non-generic. For generic potentials the decay estimate is more rapid: $\left|e^{-itH_V}P_c\psi_0\right|_{L^\infty_x} \ = \ \mathcal{O}(t^{-3/2})$; see [6], [12]. In [14, 2] the time-decay of spatially weighted L^2 norms is studied.

Question: What is the precise behavior of the $e^{-itH_{q_\epsilon}}P_c\psi_0$, when q_ϵ is a highly oscillatory potential: $q_\epsilon(x)\equiv q(x,x/\epsilon)$? In particular, what is the influence of the low-energy bound state induced by the effective potential well (equivalently, the complex pole of $t^{q_\epsilon}(k)$ near $k=0$) on the dispersive decay properties?

Using the preceding analysis we can prove:

Theorem 5.1 (Dispersive decay estimate for $\exp(-iH_{q_\epsilon}t)$).
Let $V_\epsilon \ = \ q_\epsilon(x) \ = \ q(x,x/\epsilon)$ satisfy Hypotheses **(V')** *with $q_{\rm av}\equiv 0$, and $\psi_0 \in \mathcal{L}^1_3$. There exists constants $C = C(\,\|\, V\,\|\,) > 0$ and $\epsilon_0>0$ such that for $0<\epsilon<\epsilon_0$,*

$$\left|(1+|x|)^{-3}\left(e^{-itH_{q_\epsilon}}P_c\psi_0\right)(t,x)\right| \ \le \ C\ \frac{1}{t^{1/2}}\ \frac{1}{1+\epsilon^4\ \left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2\ t}\ \left|\psi_0\right|_{\mathcal{L}^1_3}\ . \tag{5.2}$$

Remark 5.2. *We expect that an analogous result holds with $V_\epsilon = q_{\rm av}(x) + q(x,x/\epsilon)$, where $q_{\rm av}$ is any non-generic potential.*

Remark 5.3. *As a consequence of our proof, a decay estimate like* (5.2) *holds in the case of small potentials: $V\equiv\lambda Q$, with $\int Q\neq 0$ and λ sufficiently small:*

$$\left|(1+|x|)^{-3}\left(e^{-itH_{\lambda Q}}P_c\psi_0\right)(t,x)\right| \ \le \ C\ \frac{1}{t^{1/2}}\ \frac{1}{1+\lambda^2\ \left(\int_{\mathbb{R}}Q\right)^2\ t}\ \left|\psi_0\right|_{\mathcal{L}^1_3}.$$

Proof of Theorem 5.1. We follow the method of [6, 12]. In particular, the starting point of our analysis is the spectral theorem for H: $P_c\phi = \mathcal{F}^\star\mathcal{F}\phi$, with $\mathcal{F}$ and $\mathcal{F}^\star$ the distorted Fourier transform and its adjoint, bounded operators on L^2:

$$\begin{aligned}\mathcal{F} \quad &: \ \phi \mapsto \mathcal{F}[\phi](k) \equiv \int_{\mathbb{R}} \phi(x)\overline{\Psi(x,k)}\ \ {\rm d}x,\\ \mathcal{F}^\star \quad &: \ \Phi \mapsto \int_{-\infty}^{+\infty} \Phi(k)\Psi(x,k)\ \ {\rm d}k\end{aligned}$$

and

$$\Psi(x;k) \equiv \frac{1}{\sqrt{2\pi}} \begin{cases} t(k) f_+^{q_\epsilon}(x;k) & k \geq 0, \\ t(-k) f_-^{q_\epsilon}(x;-k) & k < 0. \end{cases}$$

The role of the transmission coefficient, $t^{q_\epsilon}(k)$ on the time-evolution on the continuous spectral part of H_{q_ϵ} is made explicit via the representation of $\psi_c(x,t) = P_c\psi(x,t)$:

$$\begin{aligned} \psi_c(t,x) &\equiv e^{-itH_{q_\epsilon}} P_c\psi_0 = \mathcal{F}^\star e^{-itk^2} \mathcal{F}\psi_0 \\ &= \frac{1}{2\pi} \int_0^\infty e^{-ik^2 t} |t^{q_\epsilon}(k)|^2 F(x;k) \, \mathrm{d}k, \end{aligned}$$

with

$$F(x;k) = \int_{-\infty}^\infty \left[f_+^{q_\epsilon}(x;k) \overline{f_+^{q_\epsilon}(y,k)} + f_-^{q_\epsilon}(x;k) \overline{f_-^{q_\epsilon}(y,k)} \right] \psi_0(y) \, \mathrm{d}y.$$

We next decompose $\psi_c(x,t)$ into its high and low frequency components, respectively, *i.e.* components respectively near and far away from the edge of the continuous spectrum. Introduce the smooth cutoff function χ defined by

$$\chi(k) \equiv 0 \quad \text{for} \quad |k| \geq 2k_0 \,, \qquad \chi(k) \equiv 1 \quad \text{for} \quad |k| \leq k_0.$$

Here, we set $k_0 = 1 + \left\| V \right\|$, motivated by the high frequency analysis of [12]. Using $\chi(k)$, we decompose into high and low energy components ψ_{high} and ψ_{low}:

$$\begin{aligned} \psi_c(t,x) &= \psi_{\text{low}}(t,x) + \psi_{\text{high}}(t,x) \\ &= \int_0^\infty \chi \, e^{-ik^2 t} |t^{q_\epsilon}(k)|^2 \, F(x;k) \frac{\mathrm{d}k}{2\pi} + \int_0^\infty (1-\chi) e^{-ik^2 t} \, |t^{q_\epsilon}(k)|^2 \, F(x;k) \frac{\mathrm{d}k}{2\pi}. \end{aligned} \tag{5.3}$$

ψ_{high}, can be estimated without regard to whether or not V is generic. We refer to Proposition 3 of [6] and Theorem 3.1 of [12], for the following estimate:

$$\left| (1+|x|)^{-1} \psi_{\text{high}} \right|_{L^\infty_x} = \left| (1+|x|)^{-1} e^{-itH_{q_\epsilon}} (1-\chi(H)) P_c \psi_0 \right|_{L^\infty_x} \leq C \, |t|^{-3/2} \left| \psi_0 \right|_{\mathcal{L}^1_1}, \tag{5.4}$$

where C depends on $\left| q_\epsilon \right|_{L^1_1}$ and is bounded, independent of ϵ.

To estimate the low energy component, ψ_{low}, we make use of estimates on the Jost solutions, $f_\pm^{q_\epsilon}(x;k)$ and use the precise behavior of $t^{q_\epsilon}(k)$ obtained in Corollary 4.4. We first obtain $\mathcal{O}(t^{-1/2})$-decay, uniformly for ϵ. In a second step, we obtain the precise behavior in the statement of Theorem 5.1, for ϵ small.

Let us decompose ψ_{low} into contributions from frequencies in the ranges:

$$0 \leq k \leq \frac{k_0}{\sqrt{t}} \quad \text{and} \quad \frac{k_0}{\sqrt{t}} \leq k \leq 2k_0.$$

In terms of the cutoff function, χ, we have:

$$\begin{aligned} \psi_{\text{low}} &= \frac{1}{2\pi} \int_0^\infty \chi(k\sqrt{t}) \chi(k) \, e^{-ik^2 t} |t^{q_\epsilon}(k)|^2 \, F(x;k) \, \mathrm{d}k \\ &+ \frac{1}{2\pi} \int_0^\infty (1-\chi(k\sqrt{t})) \chi(k) \, e^{-ik^2 t} |t^{q_\epsilon}(k)|^2 \, F(x;k) \, \mathrm{d}k \\ &= \psi_{\text{low}}^{(i)}(x,t) + \psi_{\text{low}}^{(ii)}(x,t) \end{aligned} \tag{5.5}$$

Straightforward estimate of $\psi_{\text{low}}^{(i)}$ gives:

$$\left| \psi_{\text{low}}^{(i)}(x,t) \right| \leq \frac{1}{2\pi} \int_0^{2k_0/\sqrt{t}} |t^{q_\epsilon}(k)|^2 \, F(x;k) \, \mathrm{d}k \leq \frac{k_0}{\pi} \frac{1}{t^{1/2}} \sup_{k \in \mathbb{R}} |F(x,k)| \,. \tag{5.6}$$

To estimate $\psi_{\rm low}^{(ii)}$, we integrate by parts:

$$\psi_{\rm low}^{(ii)}(x,t) \ = \ \frac{-1}{4\pi it}\int_0^\infty e^{-ik^2t}\partial_k\left((1-\chi(k\sqrt{t}))\chi(k)k^{-1}|t^{q_\epsilon}(k)|^2F(x;k)\right)\ {\rm d}k.$$

Note that there is no boundary contribution from $k=\infty$, since $\chi(k)$ is compactly supported, and no boundary contribution from $k=0$, since $|t^{q_\epsilon}(0)|=0$; q_ϵ is generic if ϵ is small enough, by Corollary 3.4.

Since $\chi(x,k)\equiv 0$ for $k\geq 2k_0$ and $1-\chi(k\sqrt{t})\equiv 0$ for $k\leq k_0/\sqrt{t}$, it follows that

$$\begin{aligned}\left|\psi_{\rm low}^{(ii)}(x,t)\right| &\leq \frac{C}{t}\ \int_{k_0/\sqrt{t}}^{2k_0}\left|\ |t^{q_\epsilon}(k)|^2F(x;k)\partial_k\left[\chi(k)\frac{1-\chi(k\sqrt{t})}{2ik}\right]\ \right| \ + \ \left|\frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\right|\ {\rm d}k\\ &\leq \frac{C}{t}\sup_{k\in\mathbb{R}}|F(x,k)|\int_{k_0/\sqrt{t}}^{2k_0}\sqrt{t}\frac{|\chi'(k\sqrt{t})|}{k}+\frac{1}{k^2}\ {\rm d}k+\frac{C}{t}\int_{k_0/\sqrt{t}}^{2k_0}\left|\frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\right|\ {\rm d}k.\end{aligned}$$

Note that

$$\sqrt{t}\ \int_{k_0/\sqrt{t}}^{2k_0}\ \frac{|\chi'(k\sqrt{t})|}{k}\ {\rm d}k=\sqrt{t}\ \int_{k_0}^{2k_0\sqrt{t}}\ \frac{|\chi'(z)|}{z}\ dz\ =\mathcal{O}(\sqrt{t}),$$

since $\chi'(z)$ vanishes near 0 and is of compact support. Therefore,

$$\left|\psi_{\rm low}^{(ii)}(x,t)\right| \ \leq \ \frac{C(1+k_0^{-1})}{t^{1/2}}\sup_{k\in\mathbb{R}}|F(x,k)| \ + \ \frac{C}{t}\int_{k_0/\sqrt{t}}^{2k_0}\left|\ \frac{\partial_k\left[|t^{q_\epsilon}(k)|^2F(x;k)\right]}{k}\ \right|\ {\rm d}k. \tag{5.7}$$

The estimates (5.6) and (5.7) are bounded thanks to uniform (in ϵ) control of $t^{q_\epsilon}(k)$, $F(x;k)$ and their $k-$derivatives, which are given in (5.18) and Lemma 5.4, below. It follows then from (5.5) that

$$\left|(1+|x|)^{-3}\psi_{\rm low}(x,t)\right| \ \leq \ C(\ \|\!|\ V\ |\!\|\ )\ \frac{1}{t^{1/2}}\ \left|\psi_0\right|_{\mathcal{L}^1_3}. \tag{5.8}$$

We now refine (5.8) by carefully considering the ϵ- dependence for ϵ small at $t\gg 1$. In order to achieve a $\mathcal{O}(t^{-3/2})$ estimate, we first integrate by parts:

$$\psi_{\rm low} \ = \ \frac{-1}{4\pi it}\int_0^\infty e^{-ik^2t}\partial_k\left(\chi(k)k^{-1}|t^{q_\epsilon}(k)|^2F(x;k)\right)\ {\rm d}k \ \equiv \ \frac{-1}{4\pi it}\int_0^\infty e^{-ik^2t}G(x;k)\ {\rm d}k.$$

Note again, as above, that there are no boundary contributions from $k=\infty$ or, for ϵ small, from $k=0$, by genericity of q_ϵ. We now decompose $\psi_{\rm low}$ further into contributions from frequencies in the ranges: $0\leq k\leq\frac{k_0}{\sqrt{t}}$ and $\frac{k_0}{\sqrt{t}}\leq k\ \leq 2k_0$. In terms of the cutoff function, χ, we have:

$$\begin{aligned}\psi_{\rm low} \ &= \ \frac{-1}{4\pi it}\int_0^\infty\chi(k\sqrt{t})e^{-ik^2t}G(x;k)\ {\rm d}k \ + \ \frac{-1}{4\pi it}\int_0^\infty(1-\chi(k\sqrt{t}))e^{-ik^2t}G(x;k)\ {\rm d}k\\ &= \ \psi_{\rm low}^{(1)}(x,t) \ + \ \psi_{\rm low}^{(2)}(x,t)\end{aligned} \tag{5.9}$$

Estimation of $\psi_{\rm low}^{(1)}$ gives:

$$\left|\psi_{\rm low}^{(1)}(x,t)\right|\leq\frac{1}{4\pi t}\int_0^{2k_0/\sqrt{t}}|G(x;k)|\ {\rm d}k\ \leq\ \frac{k_0}{\pi}\ \frac{1}{t^{3/2}}\ \sup_{k\in\mathbb{R}}|G(x;k)|\,. \tag{5.10}$$

To estimate $\psi_{\rm low}^{(2)}$, we subject it to one further integration by parts:

$$\psi_{\rm low}^{(2)}(x,t) \ = \ \frac{1}{4\pi t^2}\ \int_0^\infty e^{-ik^2t}\frac{\partial}{\partial k}\left[\frac{1-\chi(k\sqrt{t})}{2ik}G(x;k)\right]\ {\rm d}k.$$

Since $G(x;k) \equiv 0$ for $k \geq 2k_0$, it follows that

$$\left|\psi^{(2)}_{\rm low}(x,t)\right| \;\leq\; \frac{C}{t^2}\int_{k_0/\sqrt{t}}^{2k_0}\left|\; G(x;k)\frac{\partial}{\partial k}\left[\frac{1-\chi(k\sqrt{t})}{2ik}\right]\;\right| \;+\; \left|\frac{\partial_k G(x;k)}{k}\right| \;{\rm d}k$$
$$\leq\; \frac{C}{t^2}\sup_{k\in\mathbb{R}}|\;G(x;k)|\int_{k_0/\sqrt{t}}^{2k_0}\sqrt{t}\;\frac{|\chi'(k\sqrt{t})|}{k}+\frac{1}{k^2}\;{\rm d}k \;+\; \frac{C}{t^2}\int_{k_0/\sqrt{t}}^{2k_0}\left|\;\frac{\partial_k G(x;k)}{k}\;\right|\;{\rm d}k$$

Note again that

$$\sqrt{t}\int_{k_0/\sqrt{t}}^{2k_0}\frac{|\chi'(k\sqrt{t})|}{k}\;{\rm d}k = \sqrt{t}\int_{k_0}^{2k_0\sqrt{t}}\frac{|\chi'(z)|}{z}\;dz \;=\mathcal{O}(\sqrt{t}),$$

since $\chi'(z)$ vanishes near 0 and is of compact support. Therefore,

$$\left|\psi^{(2)}_{\rm low}(x,t)\right| \;\leq\; \frac{C(1+k_0^{-1})}{t^{3/2}}\sup_{k\in\mathbb{R}}|\;G(x;k)| \;+\; \frac{C}{t^2}\int_{k_0/\sqrt{t}}^{2k_0}\left|\;\frac{\partial_k G(x;k)}{k}\;\right|\;{\rm d}k \tag{5.11}$$

We now use the following two bounds, proved below, to complete our estimation of $\psi^{(1)}_{\rm low}(x,t)$ and $\psi^{(2)}_{\rm low}(x,t)$:

$$|G(x;k)| \leq C(\,\|\,V\,\|\,)\frac{1+|x|^2}{k^2+\epsilon^4(\int\Lambda_{\rm eff})^2} \leq C(\,\|\,V\,\|\,)\frac{1+|x|^2}{\epsilon^4(\int\Lambda_{\rm eff})^2}|\psi_0|_{\mathcal{L}^1_2}, \tag{5.12}$$

$$|\partial_k G(x;k)| \;\leq\; C(\,\|\,V\,\|\,)\frac{1+|x|^3}{k(k^2+\epsilon^4(\int\Lambda_{\rm eff})^2)}|\psi_0|_{\mathcal{L}^1_3}. \tag{5.13}$$

Using these bounds in (5.10) and (5.11), we obtain:

$$(1+|x|^2)^{-1}\;\left|\psi^{(1)}_{\rm low}(x,t)\right| \;\leq\; C(\,\|\,V\,\|\,)\;t^{-3/2}\;\frac{1}{\epsilon^4\left(\int_{\mathbb{R}}\Lambda_{\rm eff}\right)^2}\;|\psi_0|_{\mathcal{L}^1_2}\;; \tag{5.14}$$

and

$$(1+|x|^3)^{-1}\;\left|\psi^{(2)}_{\rm low}(x,t)\right| \;\leq\; C(\,\|\,V\,\|\,)\;t^{-2}\int_{k_0/\sqrt{t}}^{2k_0}\frac{1}{k^2(k^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2}\;{\rm d}k\,|\psi_0|_{\mathcal{L}^1_3}$$
$$\leq\; C(\,\|\,V\,\|\,)\;\frac{1}{k_0\;t^{1/2}}\int_1^{2\sqrt{t}}\frac{1}{l^2}\;\frac{dl}{k_0^2 l^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\;t}\;|\psi_0|_{\mathcal{L}^1_3}$$
$$\leq\; C(\,\|\,V\,\|\,)\;\frac{1}{k_0\;t^{1/2}}\;\frac{1}{k_0^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\;t}\int_1^{2\sqrt{t}}\frac{1}{l^2}\;{\rm d}l\;|\psi_0|_{\mathcal{L}^1_3}$$
$$\leq\; C(\,\|\,V\,\|\,)\;\frac{1}{k_0\;t^{1/2}}\;\frac{1}{k_0^2+\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2\;t}\;|\psi_0|_{\mathcal{L}^1_3}\;. \tag{5.15}$$

Finally, one has from (5.9), (5.14) and (5.15) the estimate

$$\left|(1+|x|)^{-3}\psi_{\rm low}(x,t)\right| \;\leq\; C(\,\|\,V\,\|\,)\;\frac{t^{-3/2}}{\epsilon^4\left(\int\Lambda_{\rm eff}\right)^2}\;|\psi_0|_{\mathcal{L}^1_3}. \tag{5.16}$$

Theorem 5.1 is a consequence of (5.4), (5.8) and (5.16).

We conclude the proof by establishing (5.12)-(5.13). This requires sharp estimates on the transmission coefficient and the Jost solutions, as well as their derivatives. These estimates are given in

Lemmata 3.6 and 3.9 of [2] for any generic V sufficiently decreasing at infinity. We shall adapt the estimates to $V_\epsilon \equiv V(x, x/\epsilon)$.

The estimates concerning the Jost solutions are uniform with respect to ϵ. In particular, one has from Lemma 3.6 of [2]:

$$\begin{aligned} \sup_{k\in\mathbb{R}} \left|\partial_k^j \left(e^{-ikx} f_+^{V_\epsilon}(x;k)\right)\right| \ &\leq \ C\big(\big|V_\epsilon\big|_{\mathcal{L}^1_3}\big)(1+\max(0,-x))^{j+1}, \\ \sup_{k\in\mathbb{R}} \left|\partial_k^j \left(e^{ikx} f_-^{V_\epsilon}(x;k)\right)\right| \ &\leq \ C\big(\big|V_\epsilon\big|_{\mathcal{L}^1_3}\big)(1+\max(0,x))^{j+1}, \quad j=0,1,2. \end{aligned} \tag{5.17}$$

Therefore,

$$|\partial_k^j F(x;k)| \ \leq \ C(\big|V_\epsilon\big|_{\mathcal{L}^1_3})(1+|x|^{j+1})\big|\psi_0\big|_{\mathcal{L}^1_{j+1}}, \qquad j=0,1,2. \tag{5.18}$$

Estimates (5.12)-(5.13) are now a direct consequence of the following Lemma, together with (5.18).

Lemma 5.4. *Let $V_\epsilon = V(x,x/\epsilon)$ satisfy Hypotheses* **(V')***, with $q_{\rm av}\equiv 0$. Then for ϵ small enough, one has*

$$\left|\partial_k^j t^{V_\epsilon}(k)\right| \ \leq \ C(\,\|\,V\,\|\,)\left|\frac{k^{1-j}}{k+\epsilon^2\int\Lambda_{\rm eff}}\right| ,$$

with $j=0,1,2$.

Proof of the Lemma. The estimate for $j = 0$ is a consequence of Corollary 4.4 with the estimate (B.2). Estimates on the derivatives are obtained by deriving identity (2.10) with respect to k. We recall

$$t^{V_\epsilon}(k) \ = \ \frac{2ik}{2ik - I^{V_\epsilon}(k)}, \qquad \text{where} \quad I^{V_\epsilon}(k) \ \equiv \ \int_{-\infty}^{\infty} V_\epsilon(y)e^{-iky} f_+^{V_\epsilon}(y;k)\ {\rm d}y,$$

so that

$$\partial_k t^{V_\epsilon}(k) \ = \ \frac{2i}{2ik - I^{V_\epsilon}(k)} - \frac{2ik(2i-\partial_k I^{V_\epsilon}(k))}{(2ik - I^{V_\epsilon}(k))^2} \ = \ \frac{t^{V_\epsilon}(k)}{k} - \frac{(t^{V_\epsilon}(k))^2(2i-\partial_k I^{V_\epsilon}(k))}{2ik}.$$

Using (5.17), one controls uniformly $\partial_k I^{V_\epsilon}(k)$, so that

$$\big|\partial_k t^{V_\epsilon}(k)\big| \ \leq \ \frac{|t^{V_\epsilon}(k)|}{k}(1+C|t^{V_\epsilon}(k)|) \ \leq \ C(\,\|\,V\,\|\,)\left|\frac{1}{k+\epsilon^2\int\Lambda_{\rm eff}}\right|.$$

The second derivative in k follows in the same way. □

6 The effective potential, $\sigma^\epsilon_{\rm eff}(x)$; proof of Theorem 3.3

As discussed in the introduction, for small $|k|$, $t^{q_{\rm av}+q_\epsilon}(k)$ is <u>not</u> uniformly approximated by the transmission coefficient of the homogenized (averaged) potential $q^{\rm av}(x) = \int_0^1 V(x,y)\ {\rm d}y$, for ϵ small. In this section we prove for k bounded that a uniform approximation can be achieved comparing $t^{q_{\rm av}+q_\epsilon}(k)$ to the transmission coefficient of an appropriate *effective potential well*:

$$\begin{aligned} V_\epsilon^{\rm eff}(x) \ &= \ q_{\rm av}(x) \ + \ \sigma^\epsilon_{\rm eff}(x), \quad \text{where} \\ \sigma^\epsilon_{\rm eff}(x) \ &\equiv \ -\epsilon^2\ \Lambda_{\rm eff}(x) \ \equiv \ -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2}. \end{aligned} \tag{6.1}$$

The point of departure for the analysis is the identity (2.19), with the choices $V = q_{\rm av} + q_\epsilon$ and $W = q_{\rm av} + \sigma$:

$$\frac{k}{t^{q_{\rm av}+q_\epsilon}(k)} - \frac{k}{t^{q_{\rm av}+\sigma}(k)} = \frac{i}{2}\, I^{[q_{\rm av}+q_\epsilon, q_{\rm av}+\sigma]}(k), \quad \text{with} \tag{6.2}$$

$$I^{[q_{\rm av}+q_\epsilon, q_{\rm av}+\sigma]}(k) \equiv \int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma}(y;k)\,(\, q_\epsilon(y) - \sigma(y))\ f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y. \tag{6.3}$$

Here, $\sigma(x)$ is unspecified and to be chosen so that $I^{[q_{\rm av}+q_\epsilon, q_{\rm av}+\sigma]}$ is sufficiently high order in ϵ. The main step in the proof is:

Proposition 6.1. *Let $V_\epsilon \equiv q_{\rm av}(x) + q(x, x/\epsilon)$ satisfy Hypotheses* **(V)***, and $k \in K$ satisfy Hypotheses* **(K)***. Define the effective potential $\sigma^\epsilon_{\mathit{eff}} \in L^\infty_\beta$, by the expression in (6.1). Then, there exists $\epsilon_0 > 0$ such that the following bound holds uniformly for $(\epsilon, k) \in [0, \epsilon_0) \times K$:*

$$I^{[q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}, q_{\rm av}+q_\epsilon]}(k) \ \le\ \epsilon^3\ C\left(\,|\!|\!|\, V \,|\!|\!|\,, \sup_{k\in K} |k| \right)\ \max\left(1, \sup_{k\in K} |t^{q_{\rm av}}(k)|\right) \tag{6.4}$$

Theorem 3.3 is then a consequence of the bound (6.4), applied to the right hand side of (6.2). We now turn to derivation of the *effective potential well* $\sigma^\epsilon_{\rm eff}$, and the proof of Proposition 6.1.

6.1 The heart of the matter; derivation of the effective potential well, $\sigma^\epsilon_{\rm eff}(x)$, and the proof of Proposition 6.1

To prove Proposition 6.1 we need to bound $I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff}, q_{\rm av}+q_\epsilon]}$, given by the integral expression in (6.3). We seek a decomposition of the integrand into oscillatory and non-oscillatory terms. Oscillatory terms can be integrated by parts to obtain bounds of high order in ϵ. Non-oscillatory terms are removed by appropriate choice of $\sigma(x)$.

We begin with $f_+^{q_{\rm av}+q_\epsilon}$. Using the Volterra equation (2.16) with $V = q_{\rm av} + q_\epsilon$ and $W = q_{\rm av}$, one has

$$f_+^{q_{\rm av}+q_\epsilon}(x;k) \ =\ f_+^{q_{\rm av}}(x;k) \ +\ J[q_{\rm av}, q_\epsilon](x;k)\ , \tag{6.5}$$

where

$$J[q_{\rm av}, q_\epsilon](\zeta;k) \ \equiv\ \int_\zeta^\infty q_\epsilon(y)\frac{f_+^{q_{\rm av}}(\zeta;k) f_-^{q_{\rm av}}(y;k) - f_-^{q_{\rm av}}(\zeta;k) f_+^{q_{\rm av}}(y;k)}{\mathcal{W}[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]} f_+^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y, \tag{6.6}$$

Therefore,

$$(\ q_\epsilon(\zeta) - \sigma(\zeta)\ )\ f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) \ =\ q_\epsilon(\zeta) f_+^{q_{\rm av}}(\zeta;k) \ -\ \sigma(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) + q_\epsilon(\zeta) J[q_{\rm av}, q_\epsilon](\zeta;k),$$

implying that $I^{[q_{\rm av}+\sigma, q_{\rm av}+q_\epsilon]}$, given by (6.3), can be written as

$$I^{[q_{\rm av}+\sigma, q_{\rm av}+q_\epsilon]} \ =\ \int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma}(\zeta;k)\Big(q_\epsilon(\zeta) f_+^{q_{\rm av}}(\zeta;k) \ -\ \sigma(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) + q_\epsilon(\zeta) J[q_{\rm av}, q_\epsilon](\zeta;k)\Big)\ {\rm d}\zeta. \tag{6.7}$$

We next show that there exists a natural choice, $\sigma = \sigma^\epsilon_{\rm eff}(x) = \mathcal{O}(\epsilon^2)$ such that the contribution of

$$-\sigma(\zeta) f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) + q_\epsilon(\zeta) J[q_{\rm av}, q_\epsilon](\zeta;k)$$

to the integral (6.7) is of order $\mathcal{O}(\epsilon^3)$, for ϵ sufficiently small.

Lemma 6.2 (Cancellation Lemma)**.** *Let $V(x,y)$ satisfy Hypotheses* **(V)***, and $k \in K$ satisfy Hypotheses* **(K)***. Define*

$$\sigma^{\epsilon}_{\mathit{eff}}(x) \;=\; -\frac{\epsilon^2}{(2\pi)^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2} \;=\; -\epsilon^2\Lambda_{\rm eff}(x). \tag{6.8}$$

Then, there exists $\epsilon_0>0$ and $C(V,K)=C(\,\lvert\!\lvert V \rvert\!\rvert\,,\sup_{k\in K}|k|)$ such that

$$\begin{aligned}&-\sigma^{\epsilon}_{\mathit{eff}}(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k)\\ &\qquad=\;\epsilon^2\sum_{j\neq0}\tilde q_j(\zeta)e^{2i\pi\lambda_j\zeta/\epsilon}\;+\;\epsilon^2\sum_{\substack{j,l\neq0\\ j+l\neq0}}\tilde q_{j,l}(\zeta)e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}\;+\;\epsilon^3q_\epsilon(\zeta)R^\epsilon(\zeta;k),\end{aligned}$$

where the following estimate holds for any $(\epsilon,k)\in[0,\epsilon_0)\times K$:

$$\begin{aligned}&\sum_{\substack{j,l\neq0\\ j+l\neq0}}\big(|\tilde q_{j,l}(\zeta)e^{\beta|\zeta|}|+|\tilde q_{j,l}'(\zeta)e^{\beta|\zeta|}|+|\tilde q_{j,l}''(\zeta)e^{\beta|\zeta|}|\big)\le C(V,K),\\ &|R^\epsilon(\zeta;k)|\;+\;\sum_{j\neq0}\big(|\tilde q_j(\zeta)e^{\beta|\zeta|}|+|\tilde q_j'(\zeta)e^{\beta|\zeta|}|+|\tilde q_j''(\zeta)e^{\beta|\zeta|}|\big)\le C(V,K)M_K(1+|\zeta|^2)e^{\alpha|\zeta|},\end{aligned}$$

for $\beta>2\alpha$. Therefore, one has

$$\begin{aligned}I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff},q_{\rm av}+q_\epsilon]}(k)\;=\;&\int_{-\infty}^{\infty}f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\zeta;k)\Big(q_\epsilon(\zeta)f_+^{q_{\rm av}}+\epsilon^2\sum_{j\neq0}\tilde q_j(\zeta)e^{2i\pi\lambda_j\zeta/\epsilon}\\ &+\epsilon^2\sum_{\substack{j,l\neq0\\ j+l\neq0}}\tilde q_{j,l}(\zeta)e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}+\epsilon^3q_\epsilon(\zeta)R^\epsilon(\zeta;k)\Big)\,{\rm d}y.\end{aligned}\tag{6.9}$$

Lemma 6.2 is proved in the next section. We first apply it to conclude the proof of Theorem 3.3. In succession, each term in (6.9) is controlled, for $k\in K$, by the bounds of the following:

Lemma 6.3. *Let $V(x,y)$ satisfy Hypotheses* **(V)***, and $k\in K$ satisfy Hypotheses* **(K)***, then one has*

$$\begin{aligned}\Big|\int_{-\infty}^{\infty}f_-^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(\zeta;k)\;q_\epsilon(\zeta)\;f_+^{q_{\rm av}}(\zeta;k)\;d\zeta\Big|&\;\le\;\epsilon^3\;C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \sum_{j\neq0}\Big|\int_{-\infty}^{\infty}f_-^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(\zeta;k)\;\tilde q_j(\zeta)\;e^{2i\pi\lambda_j\zeta/\epsilon}\;d\zeta\Big|&\;\le\;\epsilon^2\;M_K\;C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \sum_{\substack{j,l\neq0\\ j+l\neq0}}\Big|\int_{-\infty}^{\infty}f_-^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(\zeta;k)\;\tilde q_{j,l}(\zeta)\;e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}\;d\zeta\Big|&\;\le\;\epsilon^2\;C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|),\\ \Big|\int_{-\infty}^{\infty}f_-^{q_{\rm av}+\sigma^\epsilon_{\mathit{eff}}}(\zeta;k)\;q_\epsilon(\zeta)\;R^\epsilon(\zeta;k)\;d\zeta\Big|&\;\le\;M_K\;C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|)\;,\end{aligned}$$

where $C(\,\lvert\!\lvert V\rvert\!\rvert\,,\sup_{k\in K}|k|)$ and $M_K=\max(1,\sup_{k\in K}|t^{q_{\rm av}}(k)|)$ are independent of $\epsilon\in[0,\epsilon_0)$.

Applying Lemma 6.3 to (6.9) yields the desired $\mathcal{O}(\epsilon^3)$ bound on $I^{[q_{\rm av}+\sigma^\epsilon_{\rm eff},q_{\rm av}+q_\epsilon]}(k)$. Proposition 6.1 and therefore Theorem 3.3 follow. We now turn to the proofs of Lemmata 6.2 and 6.3, in Sections 6.2 and 6.3.

6.2 Proof of Lemma 6.2

For ease of presentation, we will use the simplified notation for the expression in (6.6):

$$J[q_{\rm av}, q_\epsilon](\zeta; k) \equiv \sum_{j\neq 0} \int_\zeta^\infty \mathfrak{m}(\zeta, y; k)\ q_j(y)\ e^{c\lambda_j y/\epsilon}\ f(y)\ \ {\rm d}z, \tag{6.10}$$

where $c = 2\pi i,\ f(y) = f_+^{q_{\rm av}+q_\epsilon}(y; k)$ and

$$\mathfrak{m}(\zeta, y; k)\ =\ \frac{f_+^{q_{\rm av}}(\zeta; k) f_-^{q_{\rm av}}(y; k) - f_-^{q_{\rm av}}(\zeta; k) f_+^{q_{\rm av}}(y; k)}{\mathcal{W}[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]}.$$

To make explicit the smallness of certain terms due to cancellations, we shall integrate by parts, keeping in mind that we do not control more than two derivatives of $f \equiv f_+^{q_{\rm av}+q_\epsilon}$. To evaluate boundary terms which arise, we shall use that

$$\{\mathfrak{m}(\zeta, y; k), \partial_y \mathfrak{m}(\zeta, y; k), \partial_y^2 \mathfrak{m}(\zeta, y; k)\}\big|_{y=\zeta} = \{0, 1, 0\}.$$

We now embark on the detailed expansion. From (6.10), using integration by parts, one has

$$J[q_{\rm av}, q_\epsilon](\zeta; k)\ \equiv\ \sum_j \Big(\frac{\epsilon}{c\lambda_j}\Big)^2 \Big[q_j\ f\ e^{c\lambda_j\zeta/\epsilon}\ +\ \int_\zeta^\infty \partial_y^2(\mathfrak{m}\ q_j\ f)\ e^{c\lambda_j y/\epsilon}\ {\rm d}y\Big].$$

Decompose the integrand by using: $\partial_y^2(\mathfrak{m}\ q_j\ f) = \partial_y^2(\mathfrak{m}\ q_j)\ f + 2\partial_y(\mathfrak{m}\ q_j)\ \partial_y f + \mathfrak{m}\ q_j\ \partial_y^2 f$. The first two terms can be integrated by parts once more. This gives for $j \neq 0$:

$$\int_\zeta^\infty \partial_y^2(\mathfrak{m}\ q_j)\ f e^{c\lambda_j y/\epsilon}\ {\rm d}y\ =\ -\frac{\epsilon}{c\lambda_j} \int_\zeta^\infty \partial_y\big(\partial_y^2(\mathfrak{m}\ q_j)\ f\big)\ e^{c\lambda_j y/\epsilon}\ {\rm d}y\ -\ 2\frac{\epsilon}{c\lambda_j} q_j'(\zeta) f(\zeta) e^{c\lambda_j\zeta/\epsilon},$$

$$\int_\zeta^\infty \partial_y(\mathfrak{m}\ q_j)\ \partial_y f e^{c\lambda_j y/\epsilon}\ {\rm d}y\ =\ -\frac{\epsilon}{c\lambda_j} \int_\zeta^\infty \partial_y\big(\partial_y(\mathfrak{m}\ q_j)\ \partial_y f\big)\ e^{c\lambda_j y/\epsilon}\ {\rm d}y\ -\ \frac{\epsilon}{c\lambda_j} q_j(\zeta)\ f'(\zeta) e^{c\lambda_j\zeta/\epsilon}.$$

As for the last term, we use the equation for the Jost solution, f, to express $\partial_y^2 f$ in terms of f: $\partial_y^2 f = \partial_y^2 f_+^{q_{\rm av}+q_\epsilon} = (q_{\rm av} + q_\epsilon - k^2) f_+^{q_{\rm av}+q_\epsilon}$. Thus we eventually obtain:

$$\begin{aligned} J[q_{\rm av}, q_\epsilon](\zeta; k)\ &=\ \sum_{j\neq 0} \big(\frac{\epsilon}{c\lambda_j}\big)^2 \Big[q_j\ f\ e^{c\lambda_j\zeta/\epsilon}\ +\ \int_\zeta^\infty \mathfrak{m} q_j (q_{\rm av} + q_\epsilon - k^2) f\ e^{c\lambda_j y/\epsilon}\ {\rm d}y \\ &\quad +\ \frac{\epsilon}{c\lambda_j}\big\{ \sum_{l,m,n} c_{lmn} \int_\zeta^\infty \big(\ \partial^l \mathfrak{m}\ \partial^m q_j \partial^n f\ \big) e^{c\lambda_j y/\epsilon}\ {\rm d}y\ \ - 2\ (q_j f)'\ e^{c\lambda_j\zeta/\epsilon}\big\}\Big], \end{aligned} \tag{6.11}$$

with $0 \le l, m \le 3,\ 0 \le n \le 2$, and $c_{lmn} \in \mathbb{N}$.

We now study each of the terms of (6.11) separately, beginning with an $\mathcal{O}(\epsilon^3)$ bound on the curly bracket terms in (6.11). Using the estimates of Lemmata A.2 and A.3, one has for any $0 \le l, m \le 3,\ 0 \le n \le 2$,

$$\begin{aligned} \big|\partial_y^l \mathfrak{m}(\zeta, y; k) \partial_y^m q_j(y) \partial_y^n f_+^{q_{\rm av}+q_\epsilon}(y; k)\big|\ &\le\ M_K\ C(1 + |k|^l)\big(1 + |y - \zeta|(1 + |y|)(1 + |\zeta|) e^{\alpha|\zeta|} e^{\alpha|y|}\big) \\ &\qquad \times (1 + |k|^n)(1 + |y|) e^{\alpha|y|} \big|\partial_y^m q_j(y)\big|. \end{aligned}$$

Therefore, the contribution to $J[q_{\rm av}, q_\epsilon]$ of the sum over all integrals in curly brackets in (6.11) is bounded by $\epsilon^3 M_K C\,\big(\,\big|\!\big|\!\big|V\big|\!\big|\!\big|\,, \sup_{k\in K}|k|\big)\,(1 + |\zeta|)^2 e^{\alpha|\zeta|}$, uniformly for $k \in K$. The boundary term in curly brackets satisfy a similar bound. Its contribution is bounded by $\epsilon^3 M_K C\,\big(\,\big|\!\big|\!\big|V\big|\!\big|\!\big|\,, \sup_{k\in K}|k|\big)$.

We now turn to the first two terms, in square brackets, of (6.11). Using the Fourier decomposition of $q_\epsilon(x)$, (1.5), one sees that there are two types of terms: (a) terms where $\lambda_l = -\lambda_j\ (l = -j)$,

$q_{-j}e^{-2i\pi\lambda_j y/\epsilon}$, where no oscillations remain due to phase-cancellation, and (b) contributions from terms where $\lambda_l+\lambda_j \neq 0$, which are highly oscillatory for ϵ small. In these latter terms, an additional factor of ϵ is gained via one more integration by parts. Precisely, one has

$$\int_\zeta^\infty \mathfrak{m} q_j(q_{\rm av}+q_\epsilon-k^2)f\ e^{c\lambda_j y/\epsilon}\ {\rm d}y \ = \ \int_\zeta^\infty \mathfrak{m} q_j q_{-j} f \ \ {\rm d}y \\ + \int_\zeta^\infty \mathfrak{m} q_j f\ \big((q_{\rm av}-k^2)e^{c\lambda_j y/\epsilon} + \sum_{l\notin\{0,-j\}} q_i e^{c(\lambda_l+\lambda_j)y/\epsilon}\big)\ {\rm d}y.$$

The last terms can be integrated by parts; the resulting integral and boundary terms are estimated as above. Finally, recalling that $f=f^{q_{\rm av}+q_\epsilon}$, we obtain

$$J[q_{\rm av},q_\epsilon](\zeta;k) \ = \ \sum_{j\neq 0}\big(\frac{\epsilon}{c\lambda_j}\big)^2\Big[q_j\ f^{q_{\rm av}+q_\epsilon}(\zeta;k)\ e^{c\lambda_j\zeta/\epsilon} \\ + \int_\zeta^\infty \mathfrak{m}(\zeta,y;k)\ q_j(y)q_{-j}(y)f^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y\Big] \ + \ \epsilon^3 R^\epsilon(\zeta;k), \tag{6.12}$$

with $\big|R^\epsilon(\zeta;k)\big| \ \leq \ M_K\ C\Big(|q|_{W^{3,\infty}_\beta}, \sup_{k\in K}|k|\Big)\ (1+|\zeta|^2)\ e^{\alpha|\zeta|}$.

Now multiply (6.12) by $q_\epsilon(\zeta)=\sum_{l\neq0}q_l(\zeta)\exp(2\pi i\lambda_l\zeta/\epsilon)$ and then add the result to $-\sigma f_+^{q_{\rm av}+q_\epsilon}$ to obtain (decomposing again into non-oscillatory and highly oscillatory terms and using the notation $c=2\pi i$):

$$\begin{aligned} &-\sigma(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)\ J[q_{\rm av},q_\epsilon](\zeta;k) \\ &= \left(-\sigma(\zeta)\ +\ \sum_{j\neq0}\left(\frac{\epsilon}{c\lambda_j}\right)^2 q_j(\zeta)q_{-j}(\zeta)\right)\ f_+^{q_{\rm av}+q_\epsilon}(\zeta;k) \\ &\quad+\ \sum_{l\notin\{0,-j\}}\sum_{j\neq0}\left(\frac{\epsilon}{c\lambda_j}\right)^2\left[q_lq_je^{c(\lambda_l+\lambda_j)\zeta/\epsilon}f_+^{q_{\rm av}+q_\epsilon}\right] \\ &\quad+\ \sum_{l\neq0}\sum_{j\neq0}\left(\frac{\epsilon}{c\lambda_j}\right)^2\left[q_le^{c\lambda_l\zeta/\epsilon}\int_\zeta^\infty \mathfrak{m}(\zeta,y)\ q_j(y)q_{-j}(y)f^{q_{\rm av}+q_\epsilon}(y;k)\ {\rm d}y\right] \\ &\quad+\ \epsilon^3 q_\epsilon(\zeta)R^\epsilon(\zeta;k). \end{aligned} \tag{6.13}$$

The first term on the right hand side of (6.13) is non-oscillatory in ζ for small ϵ. We remove it by choosing

$$\sigma(\zeta) \ = \ \sigma^\epsilon_{\rm eff}(\zeta) \ \equiv \ \sum_{j\neq0}\left(\frac{\epsilon}{2i\pi\lambda_j}\right)^2 q_{-j}(\zeta)q_j(\zeta) \ = \ -\frac{\epsilon^2}{4\pi^2}\sum_{j\neq0}\frac{|q_j(\zeta)|^2}{{\lambda_j}^2}\ . \tag{6.14}$$

Then

$$-\sigma^\epsilon_{\rm eff}(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)+q_\epsilon(\zeta)J[q_{\rm av},q_\epsilon](\zeta;k) \\ = \ \epsilon^2\ \sum_{l\neq0}\tilde q_l(\zeta)e^{2i\pi\lambda_l\zeta/\epsilon}\ +\ \epsilon^2\sum_{\substack{j,l\neq0\\ j+l\neq0}}\tilde q_{j,l}(\zeta)e^{2i\pi(\lambda_j+\lambda_l)\zeta/\epsilon}\ +\ \epsilon^3 q_\epsilon(\zeta)R^\epsilon(\zeta;k),$$

which we've written in the form of the statement of Lemma 6.2. Here, $\tilde q_j(\zeta)$ and $\tilde q_{j,l}(\zeta)$ are given by

$$\tilde{q}_l(\zeta) \;\equiv\; q_l(\zeta)\sum_{j\neq 0}\Big(\frac{1}{2i\pi\lambda_j}\Big)^2\int_\zeta^\infty \mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y, \tag{6.15}$$

$$\tilde{q}_{j,l}(\zeta) \;\equiv\; \Big(\frac{1}{2i\pi\lambda_j}\Big)^2 q_l(\zeta)q_j(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k). \tag{6.16}$$

To conclude, we verify the necessary estimates on $\tilde{q}_j$ and $\tilde{q}_{j,l}(\zeta)$, and their first and second derivatives.

As for (6.15), we use Lemmata A.2 and A.3, and obtain

$$\left|\int_\zeta^\infty \mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y\right| \;\leq\; M_K C(\,\|\!| V \|\!|\,,\sup_{k\in K}|k|)(1+|\zeta|^2)e^{\alpha|\zeta|}.$$

For the derivatives, we use

$$\begin{aligned}\partial_\zeta\int_\zeta^\infty \mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y &= \int_\zeta^\infty \partial_\zeta^2\mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y,\\ \partial_\zeta^2\int_\zeta^\infty \mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y &= \int_\zeta^\infty \partial_\zeta^2\mathfrak{m}(\zeta,y;k)q_jq_{-j}(y)f_+^{q_{\rm av}+q_\epsilon}(y;k)\;{\rm d}y\\ &\quad - q_jq_{-j}(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k),\end{aligned}$$

so that the integrals are uniformly bounded in the same way. As these objects are multiplied by q_l, q_l' or q_l'', and since $q_l\in W^{2,\infty}_\beta$, it follows

$$|\tilde{q}_l(\zeta)e^{\beta|\zeta|}|+|\tilde{q}_l'(\zeta)e^{\beta|\zeta|}|+|\tilde{q}_l''(\zeta)e^{\beta|\zeta|}| \;\leq\; M_KC\big(|q_l|_{W^{2,\infty}_\beta},\sup_{k\in K}|k|\big)(1+|\zeta|^2)e^{\alpha|\zeta|},$$

uniformly for $k\in K$.

As for (6.16), one has

$$\begin{aligned}\big|q_l(\zeta)q_j(\zeta)f_+^{q_{\rm av}+q_\epsilon}(\zeta;k)\big| &\leq |q_l(\zeta)||q_jf_+^{q_{\rm av}+q_\epsilon}(\zeta;k)| \leq e^{-\beta|\zeta|}|q_l|_{L^\infty_\beta}|q_jf_+^{q_{\rm av}+q_\epsilon}(\cdot;k)|_{L^\infty}\\ &\leq C(\,\|\!| V \|\!|\,,\sup_{k\in K}|k|)|q_j|_{L^\infty_\beta}|q_l|_{L^\infty_\beta}e^{-\beta|\zeta|},\end{aligned}$$

where we used Lemma A.2 to estimate $f_+^{q_{\rm av}+q_\epsilon}$. The first and second derivatives are bounded in the same way, and the double series converge.

This concludes the proof of the Cancellation Lemma 6.2.

6.3 Proof of Lemma 6.3

The last estimate of Lemma 6.3 follows from bounds on R^ϵ (see Lemma 6.2) and $f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)$ (see Lemma A.2), and the decay Hypotheses **(V)** on q_ϵ. One has

$$\begin{aligned}&\Big|\int_{-\infty}^\infty f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(y;k)q_\epsilon(y)R^\epsilon(y;k)\;\;{\rm d}y\Big|\\ &\leq\; M_K\;C(\,\|\!| V \|\!|\,,\sup_{k\in K}|k|)\;\int_{-\infty}^\infty(1+|y|)^3e^{2\alpha|y|}|q_\epsilon(y)|\;\;{\rm d}y\;\;\leq\; M_K\;C(\,\|\!| V \|\!|\,,\sup_{k\in K}|k|\big).\end{aligned}$$

To prove the ϵ^2-smallness of the second estimate of Lemma 6.3, we integrate by parts:

$$\int_{-\infty}^\infty f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(y;k)\tilde{q}_je^{2i\pi\lambda_j/\epsilon}\;{\rm d}y \;=\; \left(\frac{\epsilon}{2i\pi\lambda_j}\right)^2\int_{-\infty}^\infty(f_-^{q_{\rm av}+\sigma^\epsilon_{\rm eff}}(\cdot;k)\tilde{q}_j)''(y)e^{2i\pi\lambda_jy/\epsilon}\;{\rm d}y.$$

The estimate follows as previously from the bounds on $\tilde{q}_j$ (Lemma 6.2) and the ones on $f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)$ (Lemma A.2), as well as the hypotheses on λ_j: (3.3) in Hypotheses **(V)**.

The third estimate follows as previously, as

$$\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)\tilde{q}_{j,l}e^{2i\pi(\lambda_j+\lambda_l)/\epsilon}\ {\rm d}y = \left(\frac{\epsilon}{2i\pi(\lambda_j+\lambda_l)}\right)^2\int_{-\infty}^{\infty}(f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\cdot;k)\tilde{q}_{j,l})''(y)e^{2i\pi\lambda_j y/\epsilon}\ {\rm d}y.$$

The estimate follows, using now the bounds on $\tilde{q}_{j,l}$ (Lemma 6.2). Finally, we use three integration by parts for the first estimate of Lemma 6.3:

$$\int_{-\infty}^{\infty} f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(y;k)q_j(y)f_+^{q_{\rm av}}(\cdot;k)e^{\frac{2i\pi\lambda_j}{\epsilon}}\ {\rm d}y = \left(\frac{i\epsilon}{2\pi\lambda_j}\right)^3\int_{-\infty}^{\infty}(f_-^{q_{\rm av}+\sigma_{\rm eff}^\epsilon}(\cdot;k)q_jf_+^{q_{\rm av}}(\cdot;k))'''(y)e^{\frac{2i\pi\lambda_j y}{\epsilon}}\ {\rm d}y,$$

which is estimated using the third item of Lemma A.2, and Hypotheses **(V)**.

A Some useful estimates used throughout the paper

We recall that the Jost solution is defined through the Volterra equation

$$f_+^V(x;k)-e^{ikx} = \int_x^{\infty}\frac{\sin(k(y-x))}{2ik}V(y)f_+^V(y;k)\ {\rm d}y. \tag{A.1}$$

A detailed discussion of Jost solutions, $f_\pm(x;k)$, applying to $\Im(k)\geq 0$ can be found in [3], where it is assumed that $V\in\mathcal{L}_2^1$. We present in the following Lemma the results holding when $k\in\mathbb{R}$, and deal with the analytic continuation in a complex strip around the real axis afterwards.

Lemma A.1. *If $k\in\mathbb{R}$ and $V\in\mathcal{L}_2^1$, then one has*

$$|f_\pm^V(x;k)| \leq C(1+|k|)^{-1}(1+|x|), \tag{A.2}$$

$$|\partial_x f_\pm^V(x;k)| \leq C\frac{1+|k|(1+|x|)}{1+|k|} \leq C(1+|x|), \tag{A.3}$$

$$|\partial_x^2 f_\pm^V(x;k)| \leq |V(x)-k^2||f_+^V(x;k)| \leq C(1+|k|)(1+|x|), \tag{A.4}$$

where $C=C(\big|V\big|_{\mathcal{L}_2^1})$. Moreover, if $\partial_x V\in\mathcal{L}_2^1$, then

$$\big|\partial_x^3 f_\pm^V(x;k)\big| \leq C(1+|k|^2)(1+|x|), \quad \text{with } C=C(\big|V\big|_{\mathcal{W}_2^{1,1}}).$$

Proof. As for the first two estimates, equivalent bounds are given in [3], Lemma 1, for the function $m_\pm(x;k) \equiv f_\pm(x;k)e^{\pm ikx}$. The results for $f_\pm(x;k)$ follow straightforwardly. The last two estimates are a direct consequence of (A.1). □

If $e^{2\alpha|x|}V\in L^1$, then $f_\pm(x;k)$ has an analytic continuation to $\Im(k)>-\alpha$. Some results are presented in [11]. In this section we review and obtain the required extensions of these results. In order to simplify the results, we also restrict k to the complex strip $|\Im(k)|<\alpha$.

Lemma A.2. *If $|\Im(k)| < \alpha$ and $V \in L^\infty_\beta$, with $\beta > 2\alpha \geq 0$, then one has*

$$|f^V_\pm(x;k)| \leq C(1+|x|)e^{\alpha|x|}, \tag{A.5}$$

$$|\partial_x f^V_\pm(x;k)| \leq C(1+|k|)(1+|x|)e^{\alpha|x|}, \tag{A.6}$$

$$|\partial_x^2 f^V_\pm(x;k)| \leq |V(x)-k^2||f^V_+(x;k)| \leq C(1+|k|^2)(1+|x|)e^{\alpha|x|}, \tag{A.7}$$

where $C = C\big(|V|_{L^\infty_\beta}\big)$. Moreover, if $V \in W^{1,\infty}_\beta$, then

$$\big|\partial_x^3 f^V_\pm(x;k)\big| \leq C(1+|k|^3)(1+|x|)e^{\alpha|x|}, \quad \text{with } C = C\big(|V|_{W^{1,\infty}_\beta}\big) \ .$$

Proof. We prove bounds for f^V_+. Analogous bounds $f^V_-(x;k)$ are similarly proved and are obtained from the above by replacing x by $-x$, and $x \geq 0$ by $-x \geq 0$ *etc.*

The estimates follow from the Volterra equation (A.1) satisfied by the Jost solutions, and make use of the following bounds: for $k \in \mathbb{C}$, and for $y \geq x$, one has

$$|\cos(k(y-x))| + |\sin(k(y-x))| \leq Ce^{|\Im(k)|(y-x)} \leq Ce^{\alpha|x|}e^{\alpha|y|} \ , \tag{A.8}$$

$$\frac{|\sin(k(y-x))|}{|k|} \leq C\frac{y-x}{1+|k|(y-x)}e^{|\Im(k)|(y-x)} \leq C(y-x)e^{\alpha|x|}e^{\alpha|y|} \ . \tag{A.9}$$

By Theorem XI.57 of [11], one deduces from a careful study of the iterates of the Volterra equation (A.1), that for $x \geq 0$, one has

$$|f^V_+(x;k) - e^{ikx}| \leq e^{\alpha|x|}|e^{Q_k(x)} - 1| \leq Ce^{\alpha|x|}, \tag{A.10}$$

with $Q_k(x) \equiv \int_x^\infty \frac{4y}{1+|k|y}\ |V(y)|\ e^{2\alpha|y|}\ \mathrm{d}y$. Equation (A.5) follows for $x \geq 0$.

As for the case $x \leq 0$, (A.1) yields

$$\begin{aligned}
|f^V_+(x;k)| &= \left|e^{ikx} + \int_x^\infty \frac{\sin(k(y-x))}{k}V(y)f^V_+(y;k)\ \mathrm{d}y\right| \\
&\leq e^{\alpha|x|} + \int_x^\infty (y-x)e^{\alpha|x|}e^{\alpha|y|}|V(y)||f^V_+(y;k)|\ \mathrm{d}y \\
&\leq e^{\alpha|x|}\big[1 + \int_0^\infty ye^{\alpha|y|}|V(y)||f^V_+(y;k)|\ \mathrm{d}y + (-x)\int_x^\infty e^{\alpha|y|}|V(y)||f^V_+(y;k)|\ \mathrm{d}y\big] \\
&\leq e^{\alpha|x|}\big[C_0 + (-x)\int_x^\infty e^{\alpha|y|}|V(y)||f^V_+(y;k)|\ \mathrm{d}y\big].
\end{aligned}$$

We used (A.9) for the first inequality; the last inequality follows from (A.10), with $x = 0$. Therefore, one has with $g(x) \equiv \frac{|f^V_+(x;k)|}{(C_0+(-x))e^{\alpha|x|}}$,

$$|g(x)| \leq 1 + \int_x^\infty e^{\alpha|y|}|V(y)||g(y;k)|(C_0+(-y))e^{\alpha|y|}\ \mathrm{d}y \ .$$

By Gronwall's inequality

$$g(x) \leq \exp\big(\int_x^\infty (C_0+(-y))e^{2\alpha|y|}|V(y)|\ \mathrm{d}y\big) \leq C\big(|V|_{L^\infty_\beta}|\big).$$

Finally, one has

$$f(x;k) \leq C\big(|V|_{L^\infty_\beta}|\big)(C_0+(-x))e^{\alpha|x|} \leq C(1+|x|)e^{\alpha|x|},$$

with $C = C\big(|V|_{L^\infty_\beta}|\big)$. This completes the proof of (A.5).

The proof of (A.6) is similar, and obtained by differentiation and estimation of the Volterra integral equation (A.1). The bound (A.7) is a direct consequence of $\partial_x^2 f^V_+ = (V-k^2)f^V_+$ and the above bounds. □

Lemma A.3. *Let $q_{\rm av} \in W^{1,\infty}_\beta$ and $k \in K$, satisfy Hypotheses* **(K)**. *Define*

$$\mathfrak{m}(x,y;k) \ \equiv\ \frac{f_+^{q_{\rm av}}(x;k)f_-^{q_{\rm av}}(y;k)-f_-^{q_{\rm av}}(x;k)f_+^{q_{\rm av}}(y;k)}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}.$$

Then one has, for $0\le l\le 3$,

$$|\partial_y^l\mathfrak{m}(x,y;k)|+|\partial_x^l\mathfrak{m}(x,y;k)|\le C\ M_K\ (1+|k|)^l\left(1+|y-x|(1+|y|)(1+|x|)e^{\alpha|x|}e^{\alpha|y|}\right), \tag{A.11}$$

where $C=C\big(|q_{\rm av}|_{W^{1,\infty}_\beta}\big)$, and $M_K=\max(1,\sup_{k\in K}|t^{q_{\rm av}}(k)|)<\infty$.

Restricting to $k\in\mathbb{R}$, and assuming only $q_{\rm av}\in\mathcal{W}^{1,1}_2$, one has for $0\le l\le 3$

$$|\partial_y^l\mathfrak{m}(x,y;k)|\ +\ |\partial_x^l\mathfrak{m}(x,y;k)|\ \le\ C(1+|k|)^{l-2}\Big(1+|y-x|(1+|y|)(1+|x|)\Big),$$

where $C=C\big(|q_{\rm av}|_{\mathcal{W}^{1,1}_2}\big)$.

Proof. Let us start with the estimate (A.11) when $l=0$. One can always assume that $y>x$, since $\mathfrak{m}(x,y;k)=-\mathfrak{m}(y,x;k)$. Using Taylor's theorem with remainder in the integral form, one has

$$f_\pm^{q_{\rm av}}(y;k)\ =\ f_\pm^{q_{\rm av}}(x;k)\ +\ (y-x)\big(\partial_y f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=x}\ +\frac12\int_x^y\big(\partial_y^2 f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=t}(y-t)\ {\rm d}t.$$

It follows that

$$\begin{aligned}\mathfrak{m}(x,y;k)\ &=\ (y-x)+\frac12\int_x^y\frac{f_+^{q_{\rm av}}(x;k)f_-^{q_{\rm av}}(t;k)-f_-^{q_{\rm av}}(x;k)f_+^{q_{\rm av}}(t;k)}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}(q_{\rm av}(t)-k^2)(y-t)\ {\rm d}t\\ &=\ (y-x)+\frac12\int_x^y\mathfrak{m}(x,t;k)(q_{\rm av}(t)-k^2)(y-t)\ {\rm d}t.\end{aligned}$$

Therefore, one has with $g_x(y)\ \equiv\ \frac{|\mathfrak{m}(x,y;k)|}{|x-y|}$,

$$g_x(y)\ \le\ 1\ +\ \frac{1}{2|x-y|}\int_x^y g_x(t)|x-t||q_{\rm av}(t)-k^2||y-t|\ {\rm d}t\ \le\ 1\ +\ \frac12\int_x^y g_x(t)|x-t||q_{\rm av}(t)-k^2|\ {\rm d}t,$$

since $|y-t|\le|y-x|$ for $t\in[x,y]$. By Gronwall's inequality, one has

$$g_x(y)\ \le\ \exp\big(\frac12\int_x^y|x-t||q_{\rm av}(t)-k^2|\big)\ {\rm d}t\ \le\ C\big(|q_{\rm av}|_{L^\infty_\beta}\big)e^{\frac14 k^2(y-x)^2}.$$

Therefore, we have an estimate on $|\mathfrak{m}(x,y;k)|$, uniformly for k such that $|k||x-y|\le1$.

When $|k||x-y|\ge1$, one has from Lemma A.2

$$\begin{aligned}|\mathfrak{m}(x,y;k)|&\le C\frac{(1+|x|)e^{\alpha|x|}(1+|y|)e^{\alpha|y|}}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}]}\\ &\le CM_K(1+|x|)(1+|y|)\frac{e^{\alpha|x|}e^{\alpha|y|}}{|k|}\le CM_K(1+|x|)(1+|y|)|x-y|e^{\alpha|x|}e^{\alpha|y|},\end{aligned}$$

where we used that $\frac{1}{W[f_+^{q_{\rm av}},f_-^{q_{\rm av}}](k)}\ =\ \frac{t^{q_{\rm av}}(k)}{-2ik}$ from (2.6), and $|t^{q_{\rm av}}(k)|\ \le M_K$, from Hypotheses **(K)**. The estimate (A.11), when $l=0$, is now straightforward.

Let us now look at $\partial_y\mathfrak{m}(x,y;k)$. Using

$$\partial_y f_\pm^{q_{\rm av}}(y;k)\ =\ \big(\partial_y f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=x}\ +\ \int_x^y\big(\partial_y^2 f_\pm^{q_{\rm av}}(y;k)\big)\big|_{y=t}\ {\rm d}t,$$

one has the identity

$$\partial_y \mathfrak{m}(x,y;k) \ = \ 1 + \int_x^y \mathfrak{m}(x,t;k)(q_{\rm av}(t) - k^2)\ {\rm d}t.$$

If $|k||x-y| \le 1$, we use that $\mathfrak{m}(x,y;k)$ is uniformly bounded, and obtain

$$\big|\partial_y \mathfrak{m}(x,y;k)\big| \le 1+\int_x^y |\mathfrak{m}(x,t;k)||q_{\rm av}(t)-k^2|\ {\rm d}t \le C(1+|x-y|+|k|^2|x-y|) \le C(1+|x-y|)(1+|k|).$$

When $|k||x-y| \ge 1$, one uses the definition of $\mathfrak{m}$ with Lemma A.2, and one obtains as previously

$$|\partial_y \mathfrak{m}(x,y;k)| \le C M_K (1+|k|)(1+|x|)(1+|y|)|x-y|e^{\alpha|x|}e^{\alpha|y|}.$$

Estimate (A.11) follows for $l=1$, using the symmetry $\mathfrak{m}(x,y;k) = -\mathfrak{m}(y,x;k)$.

Estimate (A.11) for $l=2$ is straightforward when remarking that

$$\partial_y^2 \mathfrak{m}(x,y;k) \ = \ (q_{\rm av}(y) - k^2)\mathfrak{m}(x,y;k),$$

and the case $l=3$ follows in the same way.

The proof when $k \in \mathbb{R}$ and $q_{\rm av},\ \partial_x q_{\rm av} \in \mathcal{L}^1_2$ is identical, using the estimates of Lemma A.1 instead of Lemma A.2. Note that $M_K = 1$ for $k \in \mathbb{R}$, using (2.7). □

B Transmission coefficient of $\sigma(x) \ \equiv \ -\epsilon^2\Lambda(x)$

In this section, we study the transmission coefficient of potentials of the form $\sigma(x) \ \equiv \ -\epsilon^2\Lambda(x)$, where $\Lambda \in L^\infty_\beta$, is independent of ϵ. We are particularly interested in the special case where $\sigma(x)$ is the effective potential

$$\sigma^\epsilon_{\rm eff}(x) \ \equiv \ -\frac{\epsilon^2}{4\pi^2}\sum_{j\neq 0}\frac{|q_j(x)|^2}{{\lambda_j}^2},$$

derived earlier.

Lemma B.1 (Transmission coefficient $t^{q_{\rm av}-\epsilon^2\Lambda}(k)$)**.** *Let $q_{\rm av}$ and Λ be any functions in L^∞_β. Then, for $k \in K$ satisfying Hypotheses* **(K)***, one has*

$$\text{(B.1)} \qquad \frac{k}{t^{q_{\rm av}-\epsilon^2\Lambda}(k)} \ = \ \left(\frac{k}{t^{q_{\rm av}}(k)} \ - \ \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y)f_+^{q_{\rm av}}(y;k)\ dy \right) \ + \ \mathcal{O}\big(\epsilon^4\big).$$

Proof. We recall the identity (2.19), satisfied by the transmission coefficient related to *any potential* $V, W \in L^\infty_\beta$:

$$\frac{k}{t^V(k)} \ = \ \frac{k}{t^W(k)} - \frac{I^{[V,W]}(k)}{2i}, \quad \text{with} \quad I^{[V,W]}(k) \equiv \int_{-\infty}^{\infty} f_-^W(y;k)(V-W)(y)f_+^V(y;k)\ {\rm d}y.$$

Now, in the case where $W \equiv q_{\rm av}$ and $V \ \equiv \ q_{\rm av} - \epsilon^2\Lambda(x)$, one has

$$\frac{k}{t^{q_{\rm av}-\epsilon^2\Lambda}(k)} \ - \ \frac{k}{t^{q_{\rm av}}(k)} \ = \ -\frac{i\epsilon^2}{2}I^\epsilon(k), \quad I^\epsilon(k) \ \equiv \ \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y)f_+^{q_{\rm av}-\epsilon^2\Lambda}(y;k)\ {\rm d}y.$$

Then, the Volterra equation (2.16) with $V = q_{\rm av} - \epsilon^2\Lambda$ and $W = q_{\rm av}$, leads to

$$f_+^{q_{\rm av}-\epsilon^2\Lambda}(x;k) = f_+^{q_{\rm av}}(x;k) - \epsilon^2 \int_x^\infty \Lambda(y)\frac{f_+^{q_{\rm av}}(x;k)f_-^{q_{\rm av}}(y;k) - f_-^{q_{\rm av}}(x;k)f_+^{q_{\rm av}}(y;k)}{W[f_+^{q_{\rm av}}, f_-^{q_{\rm av}}]} f_+^{q_{\rm av}-\epsilon^2\Lambda}(y;k)\ {\rm d}y.$$

We can then use the estimates of Lemmata A.2 and A.3, so that

$$\begin{aligned}&\left| I^\epsilon(k) - \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y) f_+^{q_{\rm av}}(y;k)\ {\rm d}y \right| \\ &\leq\ C\epsilon^2 \int_{-\infty}^{\infty} f_-^{q_{\rm av}}(y;k)\Lambda(y) \int_y^\infty \Lambda(z)\mathfrak{m}(y,z;k) f_+^{q_{\rm av}-\epsilon^2\Lambda}(z;k)\ {\rm d}z\ {\rm d}y \\ &\leq \epsilon^2 M_K C, \qquad \text{uniformly for } k\in K.\end{aligned}$$

This concludes the proof. □

A simple consequence is the following

Corollary B.2. *Let* $q_{\rm av}$ *and* Λ *be functions in* L^∞_β*. Then,*

(1) If $q_{\rm av}$ *is generic, in the sense of Definition 2.1, then* $q_{\rm av} - \epsilon^2\Lambda$ *is generic for* ϵ *sufficiently small.*

(2) If $q_{\rm av}$ *is non-generic, and* $\int_{-\infty}^{\infty}\Lambda(y)(f_+^{q_{\rm av}}(y;0))^2\ dy \neq 0$*, then* $q_{\rm av} - \epsilon^2\Lambda$ *is generic for* ϵ *sufficiently small.*

(3) If $q_{\rm av}\equiv 0$*, and* $k\in K$ *satisfy Hypotheses* **(K)***. Then,*

$$\frac{k}{t^{-\epsilon^2\Lambda}(k)}\ =\ k\ -\ \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda(y)\ \ dy\ +\ \mathcal{O}\big(\epsilon^4\big), \tag{B.2}$$

uniformly in $k\in K$*. It follows that if*

$$\left|k - \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda\right|\ \geq\ C\ \max(\epsilon^\tau, |k|), \qquad for \quad \tau<4, \quad k\in K,$$

then one has

$$\left| t^{-\epsilon^2\Lambda}(k)\ -\ \frac{k}{k - \frac{i\epsilon^2}{2}\int_{-\infty}^{\infty}\Lambda}\right|\ =\ \mathcal{O}\big(\epsilon^{4-\tau}\big). \tag{B.3}$$

Proof. As discussed in section 2.2, a potential, V, is generic, if and only if its transmission coefficient satisfies $t^V(0) = 0$ or, equivalently, if $\lim_{k\to 0}\frac{k}{t^V(k)} \neq 0$. Items (1) and (2) are therefore a straightforward consequence of (B.1). As for item (3), since $q_{\rm av}(x)\equiv 0$, we have $t^{q_{\rm av}}\equiv 1$ and $f_\pm^{q_{\rm av}}(x;k) = e^{\pm ikx}$. The result follows by substitution into (B.1), and straightforward computations. □

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