Large-time asymptotic stability of Riemann shocks of scalar balance laws

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Abstract

We prove the large-time asymptotic orbital stability of strictly entropic Riemann shock solutions of first order scalar hyperbolic balance laws, under piecewise regular perturbations provided that the source term is dissipative about endstates of the shock. Moreover the convergence towards a shifted reference state is exponential with a rate predicted by the linearized equations about constant endstates.

Keywords: Riemann shocks; asymptotic stability; scalar balance laws.

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Introduction

In the present contribution, we study the large-time asymptotic behavior of solutions to first order scalar hyperbolic balance laws, that is, of the form

$$\partial_t u + \partial_x\big(f(u)\big) = g(u), \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\,, \tag{1}$$

in a neighborhood of strictly entropy admissible Riemann shocks, that is, about strictly admissible traveling waves with profiles piecewise constant and exhibiting a single discontinuity.

Equations such as (1) are prototypes for dynamics where only convective and reaction effects are relevant, and, as such, are ubiquitous in applications, at least as first-order approximations in some particular regimes. In particular, when $f(u) = c(u)\,u$ and $g(u) = r(u)\,u$, it describes the evolution of a density u of point particles moving with speed c and reacting at rate r.

In comparison with the purely conservative case encoded by homogeneous conservation laws (*i.e.* with $g \equiv 0$), the (local) well-posedness of the standard initial-value (Cauchy) problem for (1) is not significantly altered by the addition of sufficiently smooth (say locally Lipschitz) reaction terms g. In particular the theory of Kružkov [Kru70] applies and there exists a unique bounded local-in-time entropy weak solution for any bounded initial data. However in contrast the large-time asymptotic behavior of the solutions is expected to be deeply impacted by the presence of the source term, even when those do not lead to reaction blow-up, for instance when $g \in W^{1,\infty}(\mathbb{R})$ or g is dissipative at infinity. This expectation is consistent with the simple observations that the purely reactive case (with $f \equiv 0$) assigns a distinguished role to stable zeros of g — that is, those $u_\star$ such that $g(u_\star) = 0$ and $g'(u_\star) < 0$ — and that related growth and decay mechanisms are

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generically exponentially fast, hence much stronger than the algebraic decay involved in the purely conservative large-time dynamics. That the reaction term plays a dominant role — at least near equilibria — is also supported by the fact that the linearized operator about zeros of g, that is

$$L = -f'(u_\star)\partial_x + g'(u_\star),$$

on, say, $BUC^0(\mathbb{R})$ with domain $BUC^1(\mathbb{R})$ when $f'(u_\star) \neq 0$, $BUC^0(\mathbb{R})$ otherwise, is closed densely defined with spectrum $g'(u_\star) + i\mathbb{R}$ if $f'(u_\star) \neq 0$, $\{g'(u_\star)\}$ otherwise. In other words, the spectral stability of zeros of g as equilibria of (1) agrees with their stability as equilibria of the purely reactive equation. Incidentally note that we have used notation BUC^k to denote the set of $\mathcal{C}^k$ functions whose derivatives up to order k are bounded and uniformly continuous.

Another good grasp at new large-time phenomena (compared to the conservative case) is already obtained from the analysis of the structure of relative equilibria, namely in the present case of traveling waves. Since the presence of a source term discards self-similarity, those are the most natural candidates to serve as asymptotic profiles or building blocks of a large-time description. Traveling waves of (1) are given as $u(t,x) = \underline{U}(x - \sigma t)$ with wavespeed σ and waveprofile $\underline{U}$ solving

$$(f(\underline{U}) - \sigma\,\underline{U})' = g(\underline{U})\,.$$

One striking novelty in the non-homogeneous setting is the existence of traveling wave solutions with non-trivial profiles, whereas in the conservative case only piecewise constant profiles are available and the only spatially periodic admissible profiles are constant. The most obvious ones are obtained by picking two consecutive non degenerate zeros u_- and u_+ of g, a speed $\sigma \notin f'([u_-, u_+])$ and solving $\underline{U}' = g(\underline{U})/(f'(\underline{U}) - \sigma)$ between these two zeros. Yet, in this configuration one of the two endstates is spectrally unstable and the corresponding front inherits this instability. More interesting waves are obtained if one allows the presence of a sonic, or characteristic, point in the profile, that is, a point where $f'(\underline{U}) - \sigma$ vanishes. Necessarily then the wavespeed σ must be equal to the sound speed $f'(u_\star)$ at a zero $u_\star$ of g. In the non degenerate bistable case when u_-, $u_\star$ and u_+ are three consecutive zeros of g with $g'(u_-) < 0$, $g'(u_+) < 0$ and $g'(u_\star) > 0$ and $\sigma = f'(u_\star)$, $f''(u_\star) \neq 0$, one indeed derives spectrally stable waves in this way, that are fronts connecting u_- and u_+ through $u_\star$. As a consequence of the foregoing discussion, note that the presence of a nondegenerate source term selects a discrete set of constant solutions, but also a discrete set of wavespeeds for stable fronts. Beyond (discontinuous or smooth) fronts and constant solutions, the equation may also support spatially periodic traveling waves. Those are however necessarily discontinuous and, as a consequence of admissibility, each of their smooth part must also contain a sonic point (see [JNR+18] for details, on a closely related system case).

Under rather natural assumptions on f and g — including the strict convexity of f and the dissipativity at infinity of g —, it has been proved that starting from an L^∞ initial datum that is either spatially periodic or is constant near $-\infty$ and near ∞, the large-time dynamics is indeed well captured in L^∞ topology by piecing together traveling waves (constants, fronts or periodic waves). In the periodic setting [FH93, Lyb94, Sin95, Sin97a], every solution approaches asymptotically either a periodic (necessarily discontinuous) traveling wave, or a constant equilibrium. Moreover, periodic traveling waves are actually unstable and the rate of convergence is exponential in the latter case whereas it may be arbitrarily slow in the former case. Starting from data with essentially compact support [Sin96, MS97], the large-time asymptotics may a priori involve several blocks of different kinds (constants, fronts or periodics). Yet the scenario generating periodic blocks is also non generic and unstable. Note that at the level of regularity considered there the strict convexity assumption on f plays a key role as it impacts the structure of possible discontinuities. The few contributions relaxing the convexity assumption add severe restrictions on g or on the initial data, for instance linearity of g in [Lyb92], Riemann initial data in [Sin97b, Mas00] and monotonicity of the initial data in [Mas98].

At a technical level, one key ingredient in the proofs of the aforementioned series of investigations are generalized characteristics of Dafermos [Daf77]. They provide a formulation of the equation that is well suited to comparison principles thus to asymptotics in L^∞ topology. Our goal here is in a neighborhood of one stable traveling wave (of a specific kind) to complete the picture with a description in stronger topologies assuming more regularity but less localization on initial data. By doing so we expect to contribute to put on a par the stability theory for (1) with the one successfully derived over the years for parabolic systems (see for instance [KP13] for the stability of constants, fronts and solitary waves, and [JNRZ14] for periodic waves). In particular, we derive our asymptotics under spectral stability assumptions that are sharp up to the exclusion of limit cases. Among the difficulties to overcome in carrying out such a general program are the absence of regularization effects sufficiently strong to rely on a Duhamel formula based on a linearization about the reference wave and the presence of discontinuities and/or of sonic points in the profiles themselves that alter even the nature of the underlying spectral problems.

Whereas in a companion paper [DR] we do study waves exhibiting sonic points, we restrict here as announced to the stability of Riemann shocks, that is, to waves given by $\underline{u}(t,x)=\underline{U}(x-(\psi_0+\sigma t))$ with initial shock position $\psi_0\in\mathbb{R}$, speed $\sigma\in\mathbb{R}$ and wave profile $\underline{U}$ such that

$$\underline{U}(x)=\begin{cases}\underline{u}_- & \text{if } x<0\\ \underline{u}_+ & \text{if } x>0\end{cases}$$

where $(\underline{u}_-,\underline{u}_+)\in\mathbb{R}^2$, $\underline{u}_+\neq\underline{u}_-$. $\underline{u}$ is indeed an admissible entropy solution provided that

$$g(\underline{u}_+)=0\,,\qquad g(\underline{u}_-)=0\,,\qquad f(\underline{u}_+)-f(\underline{u}_-)=\sigma(\underline{u}_+-\underline{u}_-)\,,$$

and

$$\begin{cases}\sigma\ \geq\ f'(\underline{u}_+)\,,\\ \frac{f(\tau\,\underline{u}_-+(1-\tau)\,\underline{u}_+)-f(\underline{u}_-)}{\tau\,\underline{u}_-+(1-\tau)\,\underline{u}_+-\underline{u}_-}\geq\frac{f(\tau\,\underline{u}_-+(1-\tau)\,\underline{u}_+)-f(\underline{u}_+)}{\tau\,\underline{u}_-+(1-\tau)\,\underline{u}_+-\underline{u}_+} & \text{for any}\quad\tau\in(0,1)\,,\\ f'(\underline{u}_-)\ \geq\ \sigma\,.\end{cases}\tag{2}$$

One may readily check that

$$g'(\underline{u}_+)\leq 0\qquad\text{and}\qquad g'(\underline{u}_-)\leq 0\tag{3}$$

are necessary to exclude spectral instability of $\underline{u}$. We prove asymptotic orbital stability in $W^{1,\infty}$ topology, with sharp exponential decay rates and asymptotic phase, under BUC^1 perturbations possibly jointly with perturbations on the position and the strength of the discontinuity jump when (2) and (3) hold with strict inequalities. Likewise, we also provide stability results under BUC^k perturbations for any $k\geq 1$. We stress that at this stage no convexity assumption is needed. Yet, our approach may also be extended to cases when perturbations are only piecewise BUC^1 with a finite number[1] of discontinuities of shock-type, and then we do assume that $f''(\underline{u}_-)\neq 0$ and $f''(\underline{u}_+)\neq 0$ (or only half of it if shock-type discontinuities are only introduced on one side of the reference discontinuity).

One important point contrasting with the purely conservative case is that near $\underline{u}$ the positions of discontinuities arising from piecewise smooth perturbations with smooth parts sufficiently small in BUC^1 may be predicted at leading order from the linearized dynamics. This may be intuited by analogy from the consideration of solutions near $\underline{u}\equiv 0$ to

$$\partial_t u+\partial_x\left(\alpha\frac{u^2}{2}\right)=-\beta\,u$$

with $\alpha\in\mathbb{R}$, $\beta\geq 0$. On the latter basic explicit example, by studying $\partial_x u$ along characteristics, one readily checks that the existence of a classical solution and the persistence of regularity holds globally forward-in-time if and only if $\alpha\,\partial_x(u(0,\cdot))\geq-\beta$. Hence when $\beta>0$ and $\alpha\neq 0$, shock formation may be prevented by assuming asymmetric initial smallness on the derivative of the

[1] Yet for exposition purposes, we only provide details about the case where this number is at most one.

initial data. Incidentally note that this asymmetry is fundamental in [Mas98]. This also hints at a classification of discontinuities in initial data between shock-like discontinuities across which f' decreases and rarefaction-like discontinuities across which f' increases. The latter are removable by a density argument in the sense that the generated dynamics may be approximated by the one arising from a family of initial data where the discontinuity is absent. In particular, provided results are proved under sharp asymmetric smallness conditions, there is no loss in generality in assuming that any discontinuity is of shock-type.

Though we hope that similar analyses could be carried out in some system[2] cases, we use here crucially the scalar structure to analyze the evolution of the piecewise regularity in the following way. First we extend each smooth part of the initial datum to a function on $\mathbb{R}$, that is either close to $\underline{u}_-$ or close to $\underline{u}_+$. Then we propagate each of the extended initial data and achieve suitable estimates on the corresponding dynamics near stable constant states. At last we use the evolved extensions to determine the evolution of shock locations by solving the corresponding Rankine-Hugoniot conditions and glue them along the shock curves to obtain the solution for the original discontinuous initial datum. In particular along the way in order to carry out the second step we prove a BUC^1 asymptotic stability result for constant solutions $\underline{u}$, that is, constant functions with value a zero $\underline{u}$ of g, such that $g'(\underline{u}) < 0$. Though in principle the foregoing result could be proved —yet much less readily than L^∞ asymptotics— with classical characteristics and comparison principles[3] (along the lines in [Li94, Chapter 4]), we choose to use tools as close as possible to those in the classical stability theory [KP13, JNRZ14], relying on resolvent estimates and semigroup theory. However, as mentioned hereinabove, since regularization effects are too weak, it is not sufficient to consider the linearized dynamics. Instead we prove that spectral assumptions yield decay estimates for all nearby —time and space dependent— linear dynamics, hence actually use the evolution system (see [Paz83, Chapter 5]) rather than semigroup framework.

In the rest of the present paper, we first study the asymptotic stability of constant states under regular perturbations in $BUC^1(\mathbb{R})$, as stated in Section 1.1 and proved in Sections 1.2 and 1.3. In Section 1.4 we extend our analysis to the case where a constant state is perturbed by a (small) shock. Then we turn to our main concern, the asymptotic stability of (large) Riemann shocks, under perturbations that are either regular (Section 2.1) or piecewise regular with a small shock (Section 2.2).

1 Asymptotic stability of constant states

1.1 Asymptotic stability under shockless perturbations

In this section, first we show the asymptotic stability of constant states with respect to regular perturbations under the natural spectral condition.

Proposition 1.1. *Let $g \in \mathcal{C}^2(\mathbb{R})$ and $\underline{u} \in \mathbb{R}$ be such that*

$$g(\underline{u}) = 0 \qquad \text{and} \qquad g'(\underline{u}) < 0\,. \tag{4}$$

Then for any $C_0 > 1$, there exists $\epsilon > 0$ such that for any $f \in \mathcal{C}^2(\mathbb{R})$, for any $v_0 \in BUC^1(\mathbb{R})$ satisfying

$$\|v_0\|_{L^\infty(\mathbb{R})} \leq \epsilon\,,$$

the unique maximal classical solution to (1), $u \in \mathcal{C}^1([0, T_*(v_0)) \times \mathbb{R}) \cap \mathcal{C}^0([0, T_*(v_0)); BUC^1(\mathbb{R})) \cap \mathcal{C}^1([0, T_*(v_0)); BUC^0(\mathbb{R}))$ with $T_*(v_0) \in (0, \infty]$, generated by the the initial datum $u\,|_{t=0} = \underline{u} + v_0$

[2]During the finalization of the present contribution we have been informed that a system case has been analyzed in [YZ18] with distinct but not disjoint techniques.

[3]Similar results could also be obtained by energy estimates provided one relaxes the essentially sharp BUC^1 framework to the L^2-based H^2 space.

satisfies for any $0 \leq t < T_*(v_0)$

$$\left\|u - \underline{u}\right\|_{L^\infty(\mathbb{R})} \leq \left\|v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}$$

and if moreover $\partial_x v_0 \in L^1(\mathbb{R})$

$$\left\|\partial_x u(t,\cdot)\right\|_{L^1(\mathbb{R})} \leq \left\|\partial_x v_0\right\|_{L^1(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,.$$

The foregoing proposition is a *conditional* asymptotic stability result. Proximity is guaranteed only as long as the solution persists as a classical solution. A strong sign that the result tells nothing about persistence of regularity is that the required smallness is independent of f and does not involve derivatives of v_0. This should be contrasted with the explicit example discussed in the introduction.

In a framework involving a smallness condition with more regularity, one may prove

Proposition 1.2. *Let* f, $g \in \mathcal{C}^2(\mathbb{R})$ *and* $\underline{u} \in \mathbb{R}$ *be such that*

$$g(\underline{u}) = 0 \qquad \text{and} \qquad g'(\underline{u}) < 0\,.$$

Then for any $C_0 > 1$*, there exists* $\epsilon > 0$ *such that for any* $v_0 \in BUC^1(\mathbb{R})$ *satisfying*

$$\left\|v_0\right\|_{W^{1,\infty}(\mathbb{R})} \leq \epsilon\,,$$

the initial datum $u\,|_{t=0} = \underline{u} + v_0$ *generates a global unique classical solution to* (1)
$u \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}^+; BUC^1(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; BUC^0(\mathbb{R}))$ *and it satisfies for any* $t \geq 0$

$$\left\|u(t,\cdot) - \underline{u}\right\|_{L^\infty(\mathbb{R})} \leq \left\|v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\;;$$
$$\left\|u(t,\cdot) - \underline{u}\right\|_{W^{1,\infty}(\mathbb{R})} \leq \left\|v_0\right\|_{W^{1,\infty}(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,.$$

Assuming local convexity/concavity, one may relax part of the foregoing smallness condition

Proposition 1.3. *Let* f, $g \in \mathcal{C}^2(\mathbb{R})$ *and* $\underline{u} \in \mathbb{R}$ *be such that*

$$g(\underline{u}) = 0\,, \qquad g'(\underline{u}) < 0 \qquad \text{and} \qquad f''(\underline{u}) \neq 0\,.$$

Then for any $C_0 > 1$*, there exists* $\epsilon > 0$ *such that for any* $v_0 \in BUC^1(\mathbb{R})$ *satisfying*

$$\left\|v_0\right\|_{L^\infty(\mathbb{R})} \leq \epsilon \qquad \text{and} \qquad \left\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\right\|_{L^\infty(\mathbb{R})} \leq \epsilon\,,$$

the initial datum $u\,|_{t=0} = \underline{u} + v_0$ *generates a global unique classical solution to* (1)
$u \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}^+; BUC^1(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; BUC^0(\mathbb{R}))$ *and it satisfies for any* $t \geq 0$

$$\left\|u(t,\cdot) - \underline{u}\right\|_{L^\infty(\mathbb{R})} \leq \left\|v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,$$
$$\left\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x u(t,\cdot))_-\right\|_{L^\infty(\mathbb{R})} \leq \left\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,$$
$$\left\|\partial_x u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \left\|\partial_x v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,.$$

By a classical approximation/compactness argument one then deduces

Corollary 1.4. *Let* f, $g \in \mathcal{C}^2(\mathbb{R})$ *and* $\underline{u} \in \mathbb{R}$ *be such that*

$$g(\underline{u}) = 0\,, \qquad g'(\underline{u}) < 0 \qquad \text{and} \qquad f''(\underline{u}) \neq 0\,.$$

Then for any $C_0 > 1$, there exists $\epsilon > 0$ such that for any $v_0 \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $(\mathrm{sgn}(f''(\underline{u}))\,\partial_x v_0)_- \in L^\infty(\mathbb{R})$ and

$$\left\|v_0\right\|_{L^\infty(\mathbb{R})} \leq \epsilon \qquad \text{and} \qquad \left\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\right\|_{L^\infty(\mathbb{R})} \leq \epsilon\,,$$

the initial datum $u\big|_{t=0} = \underline{u} + v_0$ generates a global unique entropy solution to (1) and it satisfies for a.e. $t \geq 0$

$$\begin{aligned}\left\|u(t,\cdot) - \underline{u}\right\|_{L^\infty(\mathbb{R})} &\leq \left\|v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,\\ \left\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x u(t,\cdot))_-\right\|_{L^\infty(\mathbb{R})} &\leq \left\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,\end{aligned}$$

and if moreover $v_0 \in BV(\mathbb{R})$

$$\left\|u(t,\cdot)\right\|_{TV(\mathbb{R})} \;\leq\; \left\|v_0\right\|_{TV(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,$$

while if $\partial_x v_0 \in L^\infty(\mathbb{R})$

$$\left\|\partial_x u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \;\leq\; \left\|\partial_x v_0\right\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,.$$

To enlighten the content of Corollary 1.4, we stress that it allows initial data generating small rarefaction waves but not shocks. This does not mean that a similar result cannot hold when small shocks are present but simply that in general, as the explicit example of the introduction shows, they cannot be obtained by a limiting process building on global classical solutions. This is consistent with expectations drawn from general theory, see for instance [Bre00, Chapter 9, Problem 6].

Remark 1.5. *An examination of proofs shows that one may relax everywhere the assumption that $g \in \mathcal{C}^2$. It is sufficient that $g \in \mathcal{C}^1$ and that the modulus of continuity*

$$\omega(r) \;=\; \max_{|u-\underline{u}|\leq r} \|g'(u) - g'(\underline{u})\|$$

is such that $r \mapsto \omega(r)/r$ is locally integrable. This includes the case when $g \in \mathcal{C}^\alpha$, $\alpha > 1$. Indeed the key property is that for any positive C and θ

$$\int_0^\infty \omega(C\,\varepsilon\, e^{-\theta\,t})\,\mathrm{d}\,t \;=\; \frac{1}{\theta}\int_0^{C\,\varepsilon} \frac{\omega(r)}{r}\,\mathrm{d}\,r \;\stackrel{\varepsilon\to 0^+}{\longrightarrow}\; 0\,.$$

The exponential decay in time also holds for higher order derivatives without further restriction on sizes of perturbations.

Proposition 1.6. *Under the assumptions of either Proposition 1.2 or Proposition 1.3, if one assumes additionally that $f \in \mathcal{C}^{k+1}(\mathbb{R})$, $g \in \mathcal{C}^k(\mathbb{R})$ with $k \in \mathbb{N}$, $k \geq 2$ then there exists $C_k > 0$, depending on f, g and k but not on the initial data v_0, such that if $v_0 \in BUC^k(\mathbb{R})$ additionally to constraints in either Proposition 1.2 or Proposition 1.3, then the global unique classical solution to (1) emerging from the initial data $\underline{u}+v_0$ satisfies $u \in \mathcal{C}^k(\mathbb{R}^+\times\mathbb{R}) \cap \bigcap_{0\leq \ell \leq k} W^{\ell,\infty}(\mathbb{R}^+; BUC^{k-\ell}(\mathbb{R}))$ and for any $t \geq 0$*

$$\left\|\partial_x^k u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \left\|\partial_x^k v_0\right\|_{L^\infty(\mathbb{R})} e^{C_k\,\|v_0\|_{W^{1,\infty}}\,(1+\|v_0\|_{W^{1,\infty}}^{k-1})}\, e^{g'(\underline{u})\,t}\,.$$

The local well-posedness theory for (1) at the various levels of regularity considered here is standard. Note in particular that in the foregoing statements without any further constraint uniqueness holds also on any finite time interval. Though we shall not repeat it henceforth this remark applies equally well to all our uniqueness claims. Thus the main upshots of Propositions 1.2 and 1.3, Corollary 1.4 and Proposition 1.6 are global existence of classical solutions and exponential decay in time. For the classical well-posedness theory for scalar balance laws, due to Kružkov, the reader is referred to [Kru70] and[4] [Bre00, Chapter 6].

For our purposes it is expedient to introduce $v \stackrel{\text{def}}{=} u - \underline{u}$ and as long as classical solutions are concerned work with the following quasilinear form of (1)

$$\partial_t v + f'(\underline{u} + v)\partial_x v - g'(\underline{u})v = g(\underline{u} + v) - g(\underline{u}) - g'(\underline{u})v. \tag{5}$$

Note in particular that in the above formulation one cannot allow any "regularity loss" due to a linearization of the transport term. Bearing this in mind, prior to the consideration of a mild formulation of (5) we analyze linear equations of the form

$$\partial_t v + a\partial_x v - bv = r \tag{6}$$

where a is close to $f'(\underline{u})$ and b close to $g'(\underline{u})$ in a suitable sense. Let us anticipate that to deal with the mild formulation of (5) and prove Propositions 1.1 and 1.2 we could stick to the case where $b = g'(\underline{u})$. We shall use the extra flexibility in the choice of b only when tracking asymmetric regularity involved in Proposition 1.3 and Corollary 1.4.

As a preliminary let us discuss the linearized equation

$$\partial_t v + f'(\underline{u})\partial_x v - g'(\underline{u})v = 0\,.$$

A notion of solution may be obtained through the classical semigroup formalism. For instance one may consider $L = -f'(\underline{u})\partial_x + g'(\underline{u})$ on either $L^p(\mathbb{R})$, $1 \leq p < \infty$, or $BUC^0(\mathbb{R})$ with domain $W^{1,p}(\mathbb{R})$ or $BUC^1(\mathbb{R})$, if $f'(\underline{u}) \neq 0$ and $L^p(\mathbb{R})$ or $BUC^0(\mathbb{R})$, otherwise. L is then closed densely-defined with spectrum $g'(\underline{u}) + i\,\mathbb{R}$ if $f'(\underline{u}) \neq 0$, $\{g'(\underline{u})\}$ otherwise. In particular, $g'(\underline{u}) > 0$ would yield spectral instability whereas as follows from the analysis below suitable resolvent estimates show that $g'(\underline{u}) < 0$ provides linear asymptotic stability with exponential rates. We refer the reader to [Paz83, N96] for background on semigroups and their large-time behaviors.

It is already apparent here that though this does not alter significantly the stability properties, the vanishing of transport term impacts dramatically the regularity structure of the spectral problem. As long as we restrict to classical solutions near a constant steady state this is immaterial since going to a uniformly moving frame may remove possible vanishings. This would however not be possible near the traveling wave solutions described in the introduction. In general the presence of an essential characteristic point is a serious cause of trouble, and the reader is referred to [JNR+18, DR] for an example of its impact on spectral problems.

As a consequence it is convenient to change coordinate frame. Explicitly for any $\sigma \in \mathbb{R}$, by introducing $\widetilde{v}$ through $\widetilde{v}(t,x) = v(t, x + \sigma t)$ one replaces (6) with

$$\partial_t \widetilde{v} + (\widetilde{a} - \sigma)\partial_x \widetilde{v} - \widetilde{b}\widetilde{v} = \widetilde{r}\,,$$

with $(\widetilde{a}, \widetilde{b}, \widetilde{r})$ defined by $(\widetilde{a}, \widetilde{b}, \widetilde{r})(t,x) = (a,b,r)(t, x + \sigma t)$. Implicitly some of our assumptions on a will build on the fact that one may choose σ so that $\widetilde{a} - \sigma$ is bounded away from zero.

1.2 Linear equations

To consider (6) with time-dependent a and b, we may either rely on or mimic the available abstract theory for evolution systems, as described in [Paz83, Chapter 5]. In any case the needed elementary block is the solution of problems where a and b are independent of time.

As a consequence we first consider this case. With this restriction we are back to the semigroup framework that may be analyzed directly by resolvent estimates. In the present section we always

[4]Unfortunately, as most of textbooks, for expository reasons [Bre00, Chapter 6] restricts to conservation laws. Yet for local-in-time issues, such as well-posedness, changes needed to extend from conservation laws to balance laws are immaterial.

assume that $a\,,\ b\in BUC^0(\mathbb{R})$ with a bounded away from zero. For such a, b, $L_{a,\,b}=-a\partial_x+b$ is elliptic[5], and is a closed, densely-defined operator on either $L^p(\mathbb{R})$ with domain $W^{1,p}(\mathbb{R})$, $1\leq p<\infty$, or on $BUC^0(\mathbb{R})$ with domain $BUC^1(\mathbb{R})$. The key basic estimate is

Lemma 1.7. *Assume (4), a, $b\in BUC^0(\mathbb{R})$ with a bounded away from zero.*
(i). *Then for any $\lambda\in\mathbb{C}$ such that*

$$\Re(\lambda)>\sup b\,,$$

for any $F\in BUC^0(\mathbb{R})$, there exists a unique $\check{v}(\,\cdot\,;\lambda)\in BUC^1(\mathbb{R})$ such that

$$(\lambda-L_{a,\,b})\,\check{v}(\,\cdot\,;\lambda)\;=\;F$$

and moreover

$$\big\|\check{v}(\,\cdot\,;\lambda)\big\|_{L^\infty}\leq\frac{1}{\Re\lambda-\sup b}\big\|F(\,\cdot\,;\lambda)\big\|_{L^\infty}\,.$$

If b is constant and $F\in W^{1,1}(\mathbb{R})$, then $\check{v}(\,\cdot\,;\lambda)\in W^{1,1}(\mathbb{R})$ and

$$\big\|\partial_x\check{v}(\,\cdot\,;\lambda)\big\|_{L^1}\leq\frac{1}{\Re\lambda-b}\big\|\partial_xF(\,\cdot\,;\lambda)\big\|_{L^1}\,.$$

Moreover if $\lambda\in\mathbb{R}$, $\lambda\in(\sup b,\infty)$ and $F\geq0$ then $\check{v}(\,\cdot\,;\lambda)\geq0$.
(ii). *Assume moreover that*

$$a\in BUC^1(\mathbb{R})\,,\qquad b\text{ is constant}\qquad\text{and}\qquad\Re(\lambda)>\sup(b-a')$$

then for any $F\in BUC^1(\mathbb{R})$, $\check{v}(\,\cdot\,;\lambda)\in BUC^1(\mathbb{R})$ and

$$\big\|\partial_x\check{v}(\,\cdot\,;\lambda)\big\|_{L^\infty}\leq\frac{1}{\Re\lambda-b+\inf a'}\big\|\partial_xF(\,\cdot\,;\lambda)\big\|_{L^\infty}\,.$$

Proof. Let us begin with the uniqueness part. If $(\lambda-L_{a,\,b})\,\check{v}(\,\cdot\,;\lambda)=0$ then actually

$$\check{v}(\,x\,;\lambda)\;=\;e^{\int_0^x\frac{b(z)-\lambda}{a(z)}\,\mathrm{d}\,z}\,\check{v}_0$$

for some constant $\check{v}_0\in\mathbb{C}$. Then if a is positive and bounded away from zero and $\Re(\lambda)>\sup b$, the boundedness near $x=-\infty$ implies $\check{v}_0=0$ since $|e^{\int_0^x\frac{b(z)-\lambda}{a(z)}\,\mathrm{d}\,z}|\geq e^{|x|\,\frac{\Re(\lambda)-\sup b}{\|a\|_{L^\infty}}}$ when $x<0$. Likewise if a is negative and bounded away from zero and $\Re(\lambda)>\sup b$, boundedness near $x=\infty$ yields $\check{v}_0=0$.

From now on for definiteness we assume that a is positive and bounded away from zero. Note that there is no loss of generality since one may go from this case to the opposite one by reversing x into $-x$.

Let $F\in BUC^0(\mathbb{R})$. One readily checks when $\Re(\lambda)>\sup b$ that

$$\check{v}(x;\lambda)\stackrel{\text{def}}{=}\int_{-\infty}^x e^{\int_y^x\frac{b(z)-\lambda}{a(z)}\,\mathrm{d}\,z}\frac{F(y;\lambda)}{a(y)}\,\mathrm{d}\,y$$

defines $\check{v}(\,\cdot\,;\lambda)\in BUC^1(\mathbb{R})$ and that

$$|\check{v}(x;\lambda)|\leq\frac{\big\|F\big\|_{L^\infty}}{\Re\lambda-\sup b}\int_{-\infty}^x e^{\int_y^x\frac{b(z)-\Re\lambda}{a(z)}\,\mathrm{d}\,z}\frac{\Re\lambda-b(y)}{a(y)}\,\mathrm{d}\,y=\frac{\big\|F\big\|_{L^\infty}}{\Re\lambda-\sup b}\,,$$

$$\partial_x\check{v}(x;\lambda)=\frac{F(x;\lambda)}{a(x)}+\int_{-\infty}^x\frac{b(x)-\lambda}{a(x)}e^{\int_y^x\frac{b(z)-\lambda}{a(z)}\,\mathrm{d}\,z}\frac{F(y;\lambda)}{a(y)}\,\mathrm{d}\,y\,.$$

[5]Or, in a more standard terminology, $i\,L_{a,\,b}$ is elliptic.

It is also straightforward to check that if moreover $\lambda \in \mathbb{R}$ and $F \geq 0$ then $\check{v}(\cdot;\lambda) \geq 0$. When moreover b is constant and $\partial_x F \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, the latter expression may be integrated by parts into

$$\begin{aligned}\partial_x \check{v}(x;\lambda) &= \int_{-\infty}^{x} e^{\int_y^x \frac{b-\lambda}{a(z)}\,\mathrm{d}\,z} \frac{\partial_y F(y;\lambda)}{a(x)}\,\mathrm{d}\,y\\ &= \int_{-\infty}^{x} e^{\int_y^x \frac{b-\lambda-a'(z)}{a(z)}\,\mathrm{d}\,z} \frac{\partial_y F(y;\lambda)}{a(y)}\,\mathrm{d}\,y\,.\end{aligned}$$

The latter expression may be used to obtain the $\dot{W}^{1,\infty} \to \dot{W}^{1,\infty}$ bound as we have derived the $L^\infty \to L^\infty$ bound. Concerning the former expression it may be integrated in x to deduce the $\dot{W}^{1,1} \to \dot{W}^{1,1}$ bound since when b is constant for any y

$$\int_{y}^{\infty} e^{\int_y^x \frac{b-\Re(\lambda)}{a(z)}\,\mathrm{d}\,z} \frac{\mathrm{d}\,x}{a(x)} = \frac{1}{\Re(\lambda)-b}\,.$$

□

With the above frozen-time resolvent estimates, for general coefficients a, b one may first change frame to ensure that a is bounded away from zero then apply general theorems on evolution systems. See for instance [Paz83, Chapter 5, Theorem 3.1] with $X = BUC^0(\mathbb{R})$ and $Y = BUC^1(\mathbb{R})$, and apply [Paz83, Chapter 5, Theorem 2.3] to reduce the verification of assumption (H_2) there to the case where b is constant.

Proposition 1.8. *Let $T \in (0,\infty]$, $a \in \mathcal{C}^0([0,T), BUC^1(\mathbb{R}))$, $b \in BUC^0(\mathbb{R})$). Then the family of operators $\mathcal{L}_t = L_{a(t,\cdot),\,b(t,\cdot)}$ generates an evolution system $\mathcal{S}_{a,b}$ on $BUC^0(\mathbb{R})$ such that for any $v_0 \in BUC^0(\mathbb{R})$, any $0 \leq s \leq t < T$*

$$\left\|\mathcal{S}_{a,b}(s,t)\,v_0\right\|_{L^\infty} \leq e^{\int_s^t \sup_{\mathbb{R}} b(\tau,\cdot)\,\mathrm{d}\,\tau}\left\|v_0\right\|_{L^\infty}\,,$$

$v_0 \geq 0$ implies $\mathcal{S}_{a,b}(s,t)\,v_0 \geq 0$, and if b is constant

$$\begin{aligned}\left\|\partial_x \mathcal{S}_{a,b}(s,t)\,v_0\right\|_{L^1} &\leq e^{(t-s)\,b}\left\|\partial_x v_0\right\|_{L^1}\,,\\ \left\|\partial_x \mathcal{S}_{a,b}(s,t)\,v_0\right\|_{L^\infty} &\leq e^{(t-s)\,b-\int_s^t \inf_{\mathbb{R}} \partial_x a(\tau,\cdot)\,\mathrm{d}\,\tau}\left\|\partial_x v_0\right\|_{L^\infty}\,.\end{aligned}$$

1.3 The shockless nonlinear problem

In this section we complete the proofs of results from Section 1.1.

Proof of Proposition 1.1. Let $\epsilon \in (0,1]$. Pick a classical solution $u = \underline{u} + v$ starting from $\underline{u} + v_0$ such that $\left\|v_0\right\|_{L^\infty} \leq \epsilon$. Then if u exists (as a classical solution) on $[0,t_0)$, for any $0 \leq t < t_0$

$$v(t,\cdot) = \mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u})} v_0 + \int_0^t \mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u})}(s,t)\big(g(\underline{u}+v) - g(\underline{u}) - g'(\underline{u})v\big)(s,\cdot)\,\mathrm{d}\,s\,.$$

Therefore if moreover for any $t \in [0,t_0)$, $\left\|v(t,\cdot)\right\|_{L^\infty} \leq 2\epsilon\, e^{g'(\underline{u})\,t}$, then for any $t \in [0,t_0)$

$$e^{-g'(\underline{u})\,t}\left\|v(t,\cdot)\right\|_{L^\infty} \leq \left\|v_0\right\|_{L^\infty} + 2\epsilon\, C_g \int_0^t e^{g'(\underline{u})\,s}\,\left(e^{-g'(\underline{u})\,s}\left\|v(s,\cdot)\right\|_{L^\infty}\right)\mathrm{d}\,s$$

where $C_g = \frac{1}{2}\left\|g''\right\|_{L^\infty([\underline{u}-2\epsilon,\underline{u}+2\epsilon])}$, so that for any $t \in [0,t_0)$,

$$\left\|v(t,\cdot)\right\|_{L^\infty} \leq \left\|v_0\right\|_{L^\infty} e^{g'(\underline{u})\,t}\, e^{2\epsilon\, C_g \int_0^t e^{g'(\underline{u})\,s}\,\mathrm{d}\,s} \leq \left\|v_0\right\|_{L^\infty} e^{g'(\underline{u})\,t}\, e^{\epsilon\,\frac{2C_g}{|g'(\underline{u})|}}\,. \qquad (7)$$

If ϵ is small enough to ensure that $\exp(\epsilon\,\frac{2C_g}{|g'(\underline{u})|}) < 2$ then a continuity argument yields that estimate (7) holds as long as u persists as a classical solution. Since $\exp(\epsilon\,\frac{2C_g}{|g'(\underline{u})|})$ may be brought arbitrarily close to 1 by choosing ϵ small, this proves the L^∞ part of Proposition 1.1. With this bound in hands we deduce even more directly that if moreover $\partial_x v_0 \in L^1$ then

$$\big\|\partial_x v(t,\cdot)\big\|_{L^1} \leq \big\|\partial_x v_0\big\|_{L^1} e^{g'(\underline{u})\,t}\, e^{\epsilon\,\frac{2C_0\,C_g}{|g'(\underline{u})|}}\,.$$

This achieves the proof by taking ϵ even smaller if needed. □

The proof of Proposition 1.2 being completely similar, we omit it.

Proof of Proposition 1.3. First we fix $\epsilon \in (0,1]$ sufficient small to satisfy conclusions of Proposition 1.1 and to ensure that on $[\underline{u}-C_0\epsilon, \underline{u}+C_0\epsilon]$, f'' is of the sign of $f''(\underline{u})$. To proceed we use that if $u = \underline{u} + v$ persists as a classical solution on $[0,t_0)$ then for $t \in [0,t_0)$

$$\partial_x v(t,\cdot) = \mathcal{S}_{f'(\underline{u}+v),\, g'(\underline{u}+v) - f''(\underline{u}+v)\partial_x v}(0,t)\,\partial_x v_0$$

thus by linearity and preservation of non negativity

$$(\mathrm{sgn}(f''(\underline{u}))\partial_x v(t,\cdot))_- \leq \mathcal{S}_{f'(\underline{u}+v),\, g'(\underline{u}+v) - f''(\underline{u}+v)\partial_x v}(0,t)\,(\mathrm{sgn}(f''(\underline{u}))\partial_x v_0)_-\,.$$

Therefore if moreover for any $t \in [0,t_0)$, $\big\|(\mathrm{sgn}(f''(\underline{u}))\partial_x v(t,\cdot))_-\big\|_{L^\infty} \leq 2\epsilon\, e^{g'(\underline{u})\,t}$, then for any $t \in [0,t_0)$

$$\big\|(\mathrm{sgn}(f''(\underline{u}))\partial_x v(t,\cdot))_-\big\|_{L^\infty} \leq \big\|(\mathrm{sgn}(f''(\underline{u}))\partial_x v_0)_-\big\|_{L^\infty} e^{g'(\underline{u})\,t} e^{\epsilon\frac{2C_f + C_0\,C_g}{|g'(\underline{u})|}}$$

with $C_f = \big\|f''\big\|_{L^\infty([\underline{u}-C_0\epsilon, \underline{u}+C_0\epsilon])}$ and $C_g = \big\|g''\big\|_{L^\infty([\underline{u}-C_0\epsilon, \underline{u}+C_0\epsilon])}$. By choosing ϵ sufficiently small so that $e^{\epsilon\frac{2C_f + C_0\,C_g}{|g'(\underline{u})|}} \leq \min(\{2, C_0\})$, one deduces that if $u = \underline{u} + v$ persists as a classical solution on $[0,t_0)$ then for $t \in [0,t_0)$

$$\big\|(\mathrm{sgn}(f''(\underline{u}))\partial_x v(t,\cdot))_-\big\|_{L^\infty} \leq \big\|(\mathrm{sgn}(f''(\underline{u}))\partial_x v_0)_-\big\|_{L^\infty} C_0\, e^{g'(\underline{u})\,t}\,.$$

One concludes by noticing that this implies that if $u = \underline{u}+v$ persists as a classical solution on $[0,t_0)$ then for $t \in [0,t_0)$

$$\big\|\partial_x v(t,\cdot)\big\|_{L^\infty} \leq \big\|\partial_x v_0\big\|_{L^\infty} C_0\, e^{g'(\underline{u})\,t}\,,$$

which rules out finite-time blow-up. □

Proof of Proposition 1.6. Propagation of regularity being classical, we focus on the decay estimate. Note that since we already know that v is small in L^∞

$$\begin{aligned}\partial_x^k v(t,\cdot) &= \mathcal{S}_{f'(\underline{u}+v),\, g'(\underline{u})}(0,t)\,\partial_x^k v_0 \\ &\quad + \int_0^t \mathcal{S}_{f'(\underline{u}+v),\, g'(\underline{u})}(s,t)\Big(c_0(v(s,\cdot))\,v(s,\cdot)\,\partial_x^k v(s,\cdot) + \sum_{\substack{2\leq m\leq|\alpha| \\ \alpha\in(\mathbb{N}^*)^m,\ |\alpha|\in\{k,k+1\}}} c_\alpha(v(s,\cdot))\prod_{i=1}^m \partial_x^{\alpha_i} v(s,\cdot)\Big)\,\mathrm{d}\,s\end{aligned}$$

with c_0, c_α bounded. Note moreover that for any $1 \leq \ell \leq k$, for some $C \geq 0$ and any function w

$$\|\partial_x^\ell w\|_{L^\infty(\mathbb{R})} \leq C\,\min(\{\|w\|_{L^\infty(\mathbb{R})}^{\frac{k-\ell}{k}}\|\partial_x^k w\|_{L^\infty(\mathbb{R})}^{\frac{\ell}{k}},\ \|\partial_x w\|_{L^\infty(\mathbb{R})}^{\frac{k-\ell}{k-1}}\|\partial_x^k w\|_{L^\infty(\mathbb{R})}^{\frac{\ell-1}{k-1}}\})$$

so that for any $2 \leq m \leq k+1$, $\alpha \in (\mathbb{N}^*)^m$, $|\alpha| \in \{k, k+1\}$, there exists C' and C'' such that for any w

$$\begin{aligned}\|\prod_{i=1}^{m} \partial_x^{\alpha_i} w\|_{L^\infty(\mathbb{R})} &\leq C' \min(\{\|w\|_{L^\infty(\mathbb{R})}^{m-\frac{|\alpha|}{k}} \|\partial_x^k w\|_{L^\infty(\mathbb{R})}^{\frac{|\alpha|}{k}},\ \|\partial_x w\|_{L^\infty(\mathbb{R})}^{m-\frac{|\alpha|-m}{k-1}} \|\partial_x^k w\|_{L^\infty(\mathbb{R})}^{\frac{|\alpha|-m}{k-1}}\}) \\ &\leq C'' \|w\|_{W^{1,\infty}}^{m-1} \|\partial_x^k w\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

The proof is then concluded by first invoking the bounds of either Proposition 1.2 or Proposition 1.3 jointly with those of Proposition 1.8 then applying the Grönwall lemma. ☐

Proof of Corollary 1.4. An initial datum as in Corollary 1.4 may be approximated through cut-off with sufficiently slow cut-off functions and convolution with positive kernels by initial data satisfying constraints of Proposition 1.3. Bounds of Propositions 1.1 and 1.3, jointly with equation (1), are then sufficient to extract a subsequence converging pointwise and uniformly bounded. With the latter one may take limits in weak formulations encoding the notion of entropy solution, hence proving the existence of an entropy solution starting from the prescribed initial datum and satisfying claimed bounds. We refer the reader to [Bre00, Section 6.2] for details on the latter compactness arguments. ☐

1.4 Perturbation by small shocks

In this section we extend Proposition 1.3 to the case where the perturbation contains a shock.

We provide a description of the solution u as regular on

$$\Omega^\psi \stackrel{\text{def}}{=} \mathbb{R}_+ \times \mathbb{R} \setminus \{\, (t, \psi(t)) \,|\, t \geq 0 \,\}$$

where ψ follows the position of the shock.

Remark 1.9. *It may be convenient to think of u as being of the form*

$$u : (t, x) \mapsto \widetilde{u}(t, x - \psi(t))$$

with smooth unknowns $\psi : \mathbb{R}_+ \to \mathbb{R}$ and $\widetilde{u} : \mathbb{R}_+ \times \mathbb{R}^\star \to \mathbb{R}$. Though we shall not use this form explicitly here (partly because it is not convenient when two shocks are present), it underlies our strategy and statements. In particular, henceforth $\partial_x u$ will not denote the distributional derivative of $u \in \mathcal{D}'(\mathbb{R})$ but its smooth part

$$\partial_x u : (t, x) \mapsto \partial_x \widetilde{u}(t, x - \psi(t))\,.$$

For such a u to satisfy the equation in distributional sense we require u to satisfy it in a classical sense on Ω^ψ and that also holds the Rankine-Hugoniot condition, for any $t \geq 0$

$$\big(f(u_r(t)) - f(u_l(t))\big) = \psi'(t)(u_r(t) - u_l(t))$$

where $u_l(t) = \lim_{\delta \searrow 0} u(t, \psi'(t) - \delta)$ and $u_r(t) = \lim_{\delta \searrow 0} u(t, \psi'(t) + \delta)$. Moreover when $f''(\underline{u}) \neq 0$ then if u is sufficiently close to $\underline{u}$, its admissibility as an entropy solution is equivalent to Lax admissibility criterion [Bre00, Section 4.5]

$$f'(u_r(t)) < f'(u_l(t)), \qquad t \geq 0\,.$$

Of course this requires initially $f'(u_r(0)) < f'(u_l(0))$. Recall however that discontinuities with $f'(u_r(0)) > f'(u_l(0))$ are already covered by Corollary 1.4.

Proposition 1.10. *Let f, $g \in \mathcal{C}^2(\mathbb{R})$ and $\underline{u} \in \mathbb{R}$ be such that*

$$g(\underline{u}) = 0\,, \qquad g'(\underline{u}) < 0 \qquad \text{and} \qquad f''(\underline{u}) \neq 0\,.$$

For any $C_0 > 1$, there exist positive ϵ and C such that for any $\psi_0 \in \mathbb{R}$ and $\tilde{v}_0 \in BUC^1(\mathbb{R}^\star)$ satisfying

$$\big\|\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} \leq \epsilon \quad \text{and} \quad \big\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x\widetilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^\star)} \leq \epsilon\,, \tag{8}$$

and

$$\lim_{\delta \searrow 0} f'(\underline{u} + \tilde{v}_0(\delta)) < \lim_{\delta \searrow 0} f'(\underline{u} + \tilde{v}_0(-\delta)),$$

there exists $\psi \in \mathcal{C}^2(\mathbb{R}^+)$ and $u \in BUC^1(\Omega^\psi)$ with initial data $\psi(0) = \psi_0$ and $u(0,\cdot) = (\underline{u}+\widetilde{v}_0)(\cdot+\psi_0)$ such that u is an entropy solution to (1) and satisfies for any $t \geq 0$

$$\begin{aligned}
\big\|u(t,\cdot-\psi(t)) - \underline{u}\big\|_{L^\infty(\mathbb{R}\setminus\{\psi(t)\})} &\leq \big\|\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,,\\
\big\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x u(t,\cdot-\psi(t)))_-\big\|_{L^\infty(\mathbb{R}\setminus\{\psi(t)\})} &\leq \big\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x\tilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,,\\
\big\|\partial_x u(t,\cdot-\psi(t))\big\|_{L^\infty(\mathbb{R}\setminus\{\psi(t)\})} &\leq \big\|\partial_x\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,,\\
|\psi'(t) - f'(\underline{u})| &\leq \big\|\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C\, e^{g'(\underline{u})\,t}\,,
\end{aligned}$$

and moreover there exists ψ_∞ such that

$$|\psi_\infty - \psi_0| \,\leq \big\|\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C\,,$$

and for any $t \geq 0$

$$|\psi(t) - \psi_\infty - t\,f'(\underline{u})| \,\leq \big\|\tilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C\, e^{g'(\underline{u})\,t}\,.$$

Remark 1.11. *The consideration of perturbation by small shocks is partly motivated by the fact that smooth perturbations, small in L^∞ but not in $W^{1,\infty}$, may indeed form shocks in finite time. Note however that whereas Proposition 1.1 does follow smooth solutions until shock formation, Proposition 1.10 cannot be used right after shock formation since it requires (asymmetric) smallness of the smooth part of the gradient. Indeed Proposition 1.10 is a counterpart to Proposition 1.3 whereas an analog to Proposition 1.1 would be more appropriate near a shock formation. Note however that then the "smooth" part of solutions would then be controlled only in $W^{1,1}$.*

Remark 1.12. *Since the problem is known to be globally well-posed in the class of L^∞ entropy solutions, one may rightfully wonder whether the result could be extended to such a general class. Such an extension would lead us a way beyond the scope of the present contribution, focused on piece-wise smooth solutions, and very close to front-tracking algorithms. Without going that far, let us now give some hints about first steps required to extend our strategy in this direction. Note first that it is straightforward to extend Proposition 1.10 to cases when the initial datum contains discontinuities leading to rarefaction waves and an arbitrary number of well-separated shocks. Going beyond the latter case to allow for interacting shocks seems a more tedious task but seemingly still achievable with arguments in the spirit of those expounded in the present contribution. In particular even in the latter case one expects that no new discontinuity arises and that paths of discontinuities could be predicted by linearized dynamics. However to relax constraints on derivatives, one would need to follow the path sketched in Remark 1.11 or to approximate L^∞ initial data by piece-wise smooth initial data containing only flat or almost flat smooth parts but an arbitrary large number of shocks. In both cases the prediction of the regularity structure would be a much harder task.*

Proof of Proposition 1.10. To spare notational complexity, we assume henceforth that $\psi_0 = 0$ and accordingly drop tildes on $\widetilde{v}_0$. The general case may be dealt with either by using translation invariance or by propagating notational changes.

We recall that the proof strategy is the following. Given an initial data v_0 satisfying (8), we define two extensions $v_{0,\pm}$, defined on $\mathbb{R}$, satisfying $v_{0,\pm} = v_0$ on $\mathbb{R}^{\pm}$, and fulfilling the hypotheses of Proposition 1.3 near $\underline{u}$. Consider $u_{\pm}$ the two global unique classical solutions to (1) emerging from the initial data $u_{\pm}\,|_{t=0} = \underline{u} + v_{0,\pm}$. The solution u is constructed by patching together u_+ and u_- along the curve $\psi(t)$ defined through the Rankine-Hugoniot condition.

The first step is carried out thanks to the following Lemma.

Lemma 1.13. *For any $C_0^{(0)} > 1$ and any $v_0 \in BUC^1(\mathbb{R}^\star)$ there exist $v_{0,\pm} \in BUC^1(\mathbb{R})$ satisfying*

$$v_0(x) = \begin{cases} v_{0,+}(x) & \text{if } x > 0, \\ v_{0,-}(x) & \text{if } x < 0, \end{cases}$$

and

$$\begin{aligned}
\big\|v_{0,\pm}\big\|_{L^\infty(\mathbb{R})} &\leq \big\|v_0\big\|_{L^\infty(\mathbb{R}^\pm)} C_0^{(0)}\,, \\
\big\|(\operatorname{sgn}(f''(\underline{u}_\pm))\,\partial_x v_{0,\pm})_-\big\|_{L^\infty(\mathbb{R})} &\leq \big\|(\operatorname{sgn}(f''(\underline{u}_\pm))\,\partial_x v_0)_-\big\|_{L^\infty(\mathbb{R}^\pm)}\,, \\
\big\|\partial_x v_{0,\pm}\big\|_{L^\infty(\mathbb{R})} &\leq \big\|\partial_x v_0\big\|_{L^\infty(\mathbb{R}^\pm)}\,.
\end{aligned}$$

Proof. Since the situation is symmetric we only show how to extend the right part of v_0. To do so let us introduce

$$v_0(0^+) \stackrel{\text{def}}{=} \lim_{x\searrow 0} v_0(x) \qquad \text{and} \qquad \partial_x v_0(0^+) \stackrel{\text{def}}{=} \lim_{x\searrow 0} \partial_x v_0(x)$$

whose existence is guaranteed by uniform continuity.

We set

$$\delta = \frac{2(C_0^{(0)} - 1)\big\|v_0\big\|_{L^\infty(\mathbb{R}^+)}}{\max(\{1, |\partial_x v_0(0^+)|\})}$$

and define

$$v_{0,+}(x) = \begin{cases} v_0(0^+) - \frac{\delta}{2}\partial_x v_0(0^+) & \text{if } x \in (-\infty, \delta]\,, \\ v_0(0^+) + \big(x + \frac{1}{2}\delta^{-1}x^2\big)\partial_x v_0(0^+) & \text{if } x \in (-\delta, 0]\,, \\ v_0(x) & \text{if } x > 0\,. \end{cases}$$

One readily checks that $v_{0,+}$ satisfies all prescribed constraints. □

We can now proceed with the proof of Proposition 1.10. We denote C_0 the prescribed amplifying constant and ϵ the smallness parameter as in the statement. First we apply Lemma 1.13 with amplification constant $C_0^{(0)} = \sqrt{C_0}$ to receive extensions $v_{0,\pm}$. Then we apply twice Proposition 1.3, with initial perturbations $v_{0,\pm}$ near $\underline{u}$ and prescribed amplification factors $C_0^{\pm} = \sqrt{C_0}$. This is licit provided we constrain ϵ by

$$\sqrt{C_0}\epsilon \leq \epsilon_0$$

where ϵ_0 encodes the smallness constraint arising from Proposition 1.3. Hence the existence of $u_\pm \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}^+; BUC^1(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; BUC^0(\mathbb{R}))$ global unique classical solutions to (1) with initial data $u_\pm\,|_{t=0} = \underline{u} + v_{0,\pm}$ satisfying for any $t \geq 0$,

$$\begin{aligned}
\big\|u_\pm(t,\cdot) - \underline{u}\big\|_{L^\infty(\mathbb{R})} &\leq \big\|v_0\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,, \\
\big\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x u_\pm(t,\cdot))_-\big\|_{L^\infty(\mathbb{R})} &\leq \big\|(\operatorname{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,, \\
\big\|\partial_x u_\pm(t,\cdot)\big\|_{L^\infty(\mathbb{R})} &\leq \big\|\partial_x v_0\big\|_{L^\infty(\mathbb{R}^\star)} C_0\, e^{g'(\underline{u})\,t}\,.
\end{aligned}$$

We shall construct our solution, u, through the following formula:

$$u(t,x) = \begin{cases} u_-(t,x) & \text{if } x < \psi(t), \\ u_+(t,x) & \text{if } x > \psi(t), \end{cases} \tag{9}$$

where the discontinuity curve, ψ, is defined through the Rankine-Hugoniot condition

$$\big(u_+(t,\psi(t)) - u_-(t,\psi(t))\big)\psi'(t) \;=\; f(u_+(t,\psi(t))) - f(u_-(t,\psi(t)))\,.$$

To this aim, we introduce the slope function associated with f:

$$s_f\,:\,\mathbb{R}\times\mathbb{R}\to\mathbb{R},\quad (a,b)\mapsto\int_0^1 f'\big(a+\tau(b-a)\big)\,\mathrm{d}\,\tau.$$

We have $s_f\in\mathcal{C}^1(\mathbb{R}\times\mathbb{R})$. In particular, $(t,x)\mapsto s_f(u_-(t,x),u_+(t,x))\in BUC^1(\mathbb{R}_+\times\mathbb{R})$, hence there exists a unique $\psi\in\mathcal{C}^2(\mathbb{R}_+)$ satisfying $\psi(0)=0$ and for any $t\geq 0$,

$$\psi'(t)\;=\;s_f(u_-(t,\psi(t)),u_+(t,\psi(t)))\,.$$

It follows that ψ satisfies the Rankine-Hugoniot condition as well as the claimed estimates. Indeed, we have for any $t\geq 0$

$$\psi'(t)-f'(\underline{u})\;=\;s_f(u_-(t,\psi(t)),u_+(t,\psi(t)))-s_f(\underline{u},\underline{u})$$

and since s_f is a locally Lipschitz function, the bound on $\psi'(t)-f'(\underline{u})$ stems directly from the known bounds on $\big\|u_\pm(t,\cdot)-\underline{u}\big\|_{L^\infty}$. Now the bound on $\psi'-f'(\underline{u})$ may be integrated to conclude the desired estimate with

$$\psi_\infty=\int_0^\infty(\psi'(t)-f'(\underline{u}))\,\mathrm{d}\,t\,.$$

To achieve the proof, we need to ensure that lessening ϵ if necessary, the constructed weak solution is an entropy solution. Since $f''(\underline{u})\neq 0$, we can restrict ϵ so that f is either strictly concave or strictly convex on $[\underline{u}-C_0\epsilon,\underline{u}+C_0\epsilon]$ and hence u is an entropy solution if and only if Lax admissibility condition holds, *i.e.*

$$f'(u_+(t,\psi(t)))<f'(u_-(t,\psi(t))),\qquad t\geq 0\,.$$

Since the corresponding inequality holds at time $t=0$ and f' is one-to-one on $[\underline{u}-C_0\epsilon,\underline{u}+C_0\epsilon]$, it is sufficient to prove that

$$w(t):=u_+(t,\psi(t))-u_-(t,\psi(t))\neq 0,\qquad t>0\,.$$

Notice

$$\begin{aligned}w'(t)&=\left(\partial_t u_+ +\psi'(t)\partial_x u_+ -\partial_t u_- -\psi'(t)\partial_x u_-\right)(t,\psi(t))\\&=\Big(g(u_+)-g(u_-)+\big(\psi'(t)-f'(u_+)\big)\partial_x u_+ -\big(\psi'(t)-f'(u_-)\big)\partial_x u_-\Big)(t,\psi(t))\\&=\Phi(t,w(t))\end{aligned}$$

with

$$\begin{aligned}\Phi\,:\,(t,z)\mapsto\;&s_g(u_+(t,\psi(t)),u_-(t,\psi(t)))\,z\\&+\Big(s_f(u_+(t,\psi(t)),u_+(t,\psi(t))-z)-s_f(u_+(t,\psi(t)),u_+(t,\psi(t)))\Big)\partial_x u_+(t,\psi(t))\\&-\Big(s_f(u_-(t,\psi(t))+z,u_-(t,\psi(t)))-s_f(u_-(t,\psi(t)),u_-(t,\psi(t)))\Big)\partial_x u_-(t,\psi(t)).\end{aligned}$$

Since Φ is $\mathcal{C}^1$ and $(\forall t\geq 0,\ \Phi(t,0)=0)$, an application of the Cauchy-Lipschitz theorem concludes the proof. □

2 Asymptotic stability of shocks

2.1 Asymptotic stability under shockless perturbations

In this section under natural spectral assumptions we show the asymptotic stability under regular perturbations of admissible Riemann shocks of (1). More precisely, as described in the introduction we consider a uniformly traveling wave $\underline{u}$,

$$\underline{u}(t,x) = \underline{U}(x - (\psi_0 + \sigma t))\,,$$

with shock position $\psi_0 \in \mathbb{R}$, speed $\sigma \in \mathbb{R}$ and wave profile $\underline{U}$

$$\underline{U}(x) = \begin{cases} \underline{u}_- & \text{if } x < 0 \\ \underline{u}_+ & \text{if } x > 0 \end{cases}$$

where $(\underline{u}_-, \underline{u}_+) \in \mathbb{R}^2$, $\underline{u}_+ \neq \underline{u}_-$. The problem is invariant by translation and ψ_0 is arbitrary, whereas speed and profile are assumed to satisfy conditions enforcing that $\underline{u}$ is a stable entropy solution. To ensure that $\underline{u}$ is a weak solution, we require that $(\sigma, \underline{u}_-, \underline{u}_+)$ satisfies the equilibrium condition

$$g(\underline{u}_+) = 0 \quad \text{and} \quad g(\underline{u}_-) = 0\,; \tag{10}$$

and the Rankine-Hugoniot condition

$$f(\underline{u}_+) - f(\underline{u}_-) = \sigma(\underline{u}_+ - \underline{u}_-)\,. \tag{11}$$

(Strict) entropy admissibility then amounts to the following Oleinik condition

$$\begin{cases} \sigma \,>\, f'(\underline{u}_+)\,, \\ \frac{f(\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+) - f(\underline{u}_-)}{\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+ - \underline{u}_-} > \frac{f(\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+) - f(\underline{u}_+)}{\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+ - \underline{u}_+} \qquad \text{for any} \quad \tau \in (0,1)\,, \\ f'(\underline{u}_-) \,>\, \sigma\,, \end{cases} \tag{12}$$

and the spectral stability is encoded in

$$g'(\underline{u}_+) < 0 \quad \text{and} \quad g'(\underline{u}_-) < 0\,. \tag{13}$$

Note that the entropy condition (12) also contributes — in a more subtle way — to the stability properties of the shock. Yet for the kind of perturbation under consideration here only the Lax condition

$$f'(\underline{u}_+) < \sigma < f'(\underline{u}_-)$$

really contributes to the stabilization. However if we were allowing perturbations breaking the large shock into a “sum" of smaller subshocks the full condition would be involved. See [Bre00, Remark 4.7] for a more detailed discussion and more generally [Bre00, Chapters 4 and 6] for classical background on entropy solutions.

As in Section 1.4 we shall solve (1) in the class of piecewise regular functions and adopt conventions introduced there. The main difference is that now we require as entropy condition, for any $t \geq 0$

$$\begin{cases} \psi'(t) \,>\, f'(u_r(t))\,, \\ \frac{f(\tau\,u_l(t) + (1-\tau)\,u_r(t)) - f(u_l(t))}{\tau\,u_l(t) + (1-\tau)\,u_r(t) - u_l(t)} > \frac{f(\tau\,u_l(t) + (1-\tau)\,u_r(t)) - f(u_r(t))}{\tau\,u_l(t) + (1-\tau)\,u_r(t) - u_r(t)} \qquad \text{for any} \quad \tau \in (0,1)\,, \\ f'(u_l(t)) \,>\, \psi'(t)\,. \end{cases} \tag{14}$$

Theorem 2.1. *Let $f, g \in \mathcal{C}^2(\mathbb{R})$ and $(\sigma, \underline{u}_-, \underline{u}_+) \in \mathbb{R}^3$ satisfying* (10)-(11)-(12)-(13) *and*

$$f''(\underline{u}_+) \neq 0 \quad \text{and} \quad f''(\underline{u}_-) \neq 0 . \tag{15}$$

For any $C_0 > 1$, there exists $\epsilon > 0$ and $C > 0$ such that for any $\psi_0 \in \mathbb{R}$ and $\widetilde{v}_0 \in BUC^1(\mathbb{R}^\star)$ satisfying

$$\begin{aligned} \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} &\leq \epsilon , \\ \big\|(\text{sgn}(f''(\underline{u}_+))\, \partial_x \widetilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^+)} &\leq \epsilon , \\ \big\|(\text{sgn}(f''(\underline{u}_-))\, \partial_x \widetilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^-)} &\leq \epsilon , \end{aligned} \tag{16}$$

there exists $\psi \in \mathcal{C}^2(\mathbb{R}^+)$ and $u \in BUC^1(\Omega^\psi)$ with initial data $\psi(0) = \psi_0$ and $u(0, \cdot) = (\underline{U} + \widetilde{v}_0)(\cdot + \psi_0)$ such that u is an entropy solution to (1) *and satisfies for any $t \geq 0$*

$$\begin{aligned} \big\|u(t, \cdot - \psi(t)) - \underline{u}_\pm\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\, t} , \\ \big\|(\text{sgn}(f''(\underline{u}_\pm))\, \partial_x u(t, \cdot - \psi(t)))_-\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|(\text{sgn}(f''(\underline{u}_\pm))\, \partial_x \widetilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\, t} , \\ \big\|\partial_x u(t, \cdot - \psi(t))\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|\partial_x \widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\, t} , \\ |\psi'(t) - \sigma| &\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)}\, C\, e^{\max(\{g'(\underline{u}_+), g'(\underline{u}_-)\})\, t} , \end{aligned}$$

and moreover there exists ψ_∞ such that

$$|\psi_\infty - \psi_0| \leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C ,$$

and for any $t \geq 0$

$$|\psi(t) - \psi_\infty - t\,\sigma| \leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star)} C\, e^{\max(\{g'(\underline{u}_+), g'(\underline{u}_-)\})\, t} .$$

Remark 2.2. *Theorem 2.1 is a direct counterpart to Proposition 1.3. We could also derive from it an analogous to Corollary 1.4. Likewise as in Proposition 1.2 we could relax totally or partly hypothesis* (15) *if* (16) *is strengthened. This would lead to four different versions of Theorem 2.1. We could also provide a counterpart to Proposition 1.1.*

Modifications required to prove the foregoing claims are straightforward and we have chosen to omit them so as to avoid redundancy.

Remark 2.3. *Note that expressed in classical stability terminology (see for instance [Hen81]) we have proved orbital stability with asymptotic phase. We stress however that the role of phase shifts is here deeper than in the classical stability analysis of smooth waves since it is not only required to provide decay of suitable norms in large-time but also to ensure that those norms are finite locally in time. In particular here there is no freedom, even in finite time, in the definition of phase shifts. See [JNR⁺18, Section 4.1] and [DR] for related (more elaborate) discussions.*

Our proof provides an admissible solution, but does not guarantee its uniqueness. However again, our set of initial data is covered by the theory due to Kružkov [Kru70].

Proof of Theorem 2.1. The proof of Theorem 2.1 follows closely the construction given in the proof of Proposition 1.10. We also assume henceforth that $\psi_0 = 0$, without loss of generality. Using Lemma 1.13 and Proposition 1.3, we find that for $\epsilon > 0$ sufficiently small and for any $v_0 \in BUC^1(\mathbb{R}^\star)$ satisfying (16), there exist $u_\pm \in BUC^1(\mathbb{R}^+ \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}^+; BUC^1(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; BUC^0(\mathbb{R}))$ global classical solutions to (1) with initial data $u_\pm\,|_{t=0} = \underline{u}_\pm + v_{0,\pm}$ and satisfying the desired estimates. We can now construct the solution, u, through (9) where ψ is defined by the differential equation

$$\psi'(t) = s_f(u_-(t, \psi(t)), u_+(t, \psi(t))) ,$$

so that the Rankine-Hugoniot condition as well as the desired bounds on ψ hold since for any $t\geq 0$,

$$\psi'(t)-\sigma\;=\;s_f(u_-(t,\psi(t)),u_+(t,\psi(t)))-s_f(\underline{u}_-,\underline{u}_+)\,.$$

Then the last estimates on ψ are obtained by integration with

$$\psi_\infty=\int_0^\infty(\psi'(t)-\sigma)\,\mathrm{d}\,t\,.$$

To achieve the proof of Theorem 2.1 we only need to ensure that by lessening ϵ further if necessary formula (9) ensures (14). For this purpose we consider

$$S_f:\mathbb{R}\times\mathbb{R}\times[0,1]\to\mathbb{R},\qquad(a,b,\tau)\mapsto s_f(a,\tau\,a+(1-\tau)\,b)-s_f(b,\tau\,a+(1-\tau)\,b)$$

and observe that it is continuous. Since $\{\underline{u}_-\}\times\{\underline{u}_+\}\times[0,1]$ is compact and for any $\tau\in[0,1]$, $S_f(\underline{u}_-,\underline{u}_+,\tau)>0$, one may ensure that provided ϵ is small enough, for any (a,b) such that $|a-\underline{u}_-|\leq C_0\epsilon$ and $|b-\underline{u}_+|\leq C_0\epsilon$, and any $\tau\in[0,1]$, $S_f(a,b,\tau)>0$. From this stems (14) for u built from (9), and the proof is complete. □

We now prove that the exponential decay of higher derivatives holds provided we assume the stronger (symmetric) smallness condition on the first derivative.

Proposition 2.4. *Let $k\in\mathbb{N}$, $k\geq 2$, $f\in\mathcal{C}^{k+1}(\mathbb{R})$, $g\in\mathcal{C}^k(\mathbb{R})$ and $(\sigma,\underline{u}_-,\underline{u}_+)\in\mathbb{R}^3$ satisfying* (10)-(11)-(12)-(13). *There exists $\epsilon>0$ and C_k such that for any $\psi_0\in\mathbb{R}$ and $\widetilde{v}_0\in BUC^k(\mathbb{R}^\star)$ satisfying*

$$\left\|\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\star)}\leq\epsilon\quad\text{ and }\quad\left\|\partial_x\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\star)}\leq\epsilon\,,\tag{17}$$

there exist $\psi\in\mathcal{C}^{k+1}(\mathbb{R}^+)$ and $u\in BUC^k(\Omega^\psi)$ with initial data $\psi(0)=\psi_0$ and $u(0,\cdot)=(\underline{U}+\widetilde{v}_0)(\cdot+\psi_0)$ such that u is an entropy solution to (1) *and satisfies for any $t\geq 0$ and any $j\in\{0,\dots,k\}$*

$$\begin{aligned}\left\|u(t,\cdot-\psi(t))-\underline{u}_\pm\right\|_{W^{j,\infty}(\mathbb{R}^\pm)}\leq\left\|\widetilde{v}_0\right\|_{W^{j,\infty}(\mathbb{R}^\pm)}C_k\,e^{g'(\underline{u}_\pm)\,t}\,,&\\ |\psi^{(j+1)}(t)|\leq\left\|\widetilde{v}_0\right\|_{W^{j,\infty}(\mathbb{R}^\star)}C_k\,e^{\max(\{g'(\underline{u}_+),g'(\underline{u}_-)\})\,t}\,.&\end{aligned}$$

Proof. The result does not follow directly from Proposition 1.6 applied to $u_\pm$ defined in the proof of Theorem 2.1, because the initial data provided by Lemma 1.13 is not sufficiently regular. Here we rather rely on the following Lemma deduced from a standard extension theorem [Ada75, Theorem 4.26].

Lemma 2.5. *Let $k\in\mathbb{N}$, $k\geq 2$. There exists $C_k>0$ such that for any $v_0\in BUC^k(\mathbb{R}^\star)$, there exist $v_{0,\pm}\in\mathcal{C}^k(\mathbb{R})$ satisfying*

$$v_0(x)=\begin{cases}v_{0,+}(x)&\text{if }x>0,\\ v_{0,-}(x)&\text{if }x<0,\end{cases}$$

and for any $j\in\mathbb{N}$, $0\leq j\leq k$,

$$\left\|\partial_x^j v_{0,\pm}\right\|_{L^\infty(\mathbb{R})}\leq\left\|\partial_x^j v_0\right\|_{L^\infty(\mathbb{R}^\pm)}C_k\,.\tag{18}$$

Replacing Lemma 1.13 and Proposition 1.3 with Lemma 2.5 and Proposition 1.6, the proof of Proposition 2.4 is then almost identical to the proof of Theorem 2.1. □

We can also obtain a counterpart to Proposition 2.4 with the asymmetric smallness assumption on first-order derivatives.

Proposition 2.6. *Let $k \in \mathbb{N}$, $k \geq 2$, $f \in \mathcal{C}^{k+1}(\mathbb{R})$, $g \in \mathcal{C}^k(\mathbb{R})$ and $(\sigma, \underline{u}_-, \underline{u}_+) \in \mathbb{R}^3$ satisfying* (10)-(11)-(12)-(13) *and* (15)*. There exist $\epsilon > 0$ and $C_k > 0$ such that for any $\psi_0 \in \mathbb{R}$ and $\widetilde{v}_0 \in BUC^k(\mathbb{R}^\star)$ satisfying* (16)*, the entropy solution defined in Theorem 2.1 satisfies $u \in BUC^k(\Omega^\psi)$, $\psi \in \mathcal{C}^{k+1}(\mathbb{R}^+)$ and for any $t \geq 0$ and any $j \in \{1, \dots, k\}$*

$$\begin{aligned}
&\left\|\partial_x^j u(t, \cdot - \psi(t))\right\|_{L^\infty(\mathbb{R}^\pm)} \\
&\qquad \leq \left\|\widetilde{v}_0\right\|_{W^{j,\infty}(\mathbb{R}^\pm)} C_k (1 + \left\|\partial_x^2 \widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\pm)}^{j-1}) e^{C_k \|\widetilde{v}_0\|_{W^{1,\infty}(\mathbb{R}^\pm)} (1 + \|\widetilde{v}_0\|_{W^{1,\infty}(\mathbb{R}^\pm)}^{j-1})} \; e^{g'(\underline{u}_\pm)\, t}, \\
&|\psi^{(j+1)}(t)| \leq \left\|\widetilde{v}_0\right\|_{W^{j,\infty}} C_k (1 + \left\|\partial_x^2 \widetilde{v}_0\right\|_{L^\infty}^{j-1}) e^{C_k \|\widetilde{v}_0\|_{W^{1,\infty}} (1 + \|\widetilde{v}_0\|_{W^{1,\infty}}^{j-1})} \; e^{\max(\{g'(\underline{u}_+), g'(\underline{u}_-)\})\, t}.
\end{aligned}$$

Proof. Although we follow the same strategy as in the earlier results, we need to ensure that the regular extensions $\widetilde{v}_{0,\pm} \in BUC^k(\mathbb{R})$ preserve the asymmetric smallness hypothesis (16). To this aim, we introduce a smooth cut-off function, χ, such that $\chi(x) = 0$ for $|x| \geq 2/3$, $\chi(x) = 1$ for $|x| \leq 1/3$ and $\chi(x) \in [0, 1]$ for $x \in \mathbb{R}$, and define

$$\widetilde{v}_{0,+}(x) = \begin{cases} \widetilde{v}_0(0^+) + \int_0^{-\delta} w_{0,+}(y) \chi(\delta^{-1} y) \,\mathrm{d}\, y & \text{if } x \in (-\infty, -\delta], \\ \widetilde{v}_0(0^+) + \int_0^{x} w_{0,+}(y) \chi(\delta^{-1} y) \,\mathrm{d}\, y & \text{if } x \in (-\delta, 0], \\ \widetilde{v}_0(x) & \text{if } x > 0. \end{cases}$$

where $w_{0,+}$ is the extension associated with $\partial_x \widetilde{v}_0$ provided by Lemma 2.5. When choosing

$$\delta = \min\left(\left\{ \frac{c_0\, \epsilon}{1 + \left\|\partial_x \widetilde{v}_0\right\|_{W^{1,\infty}(\mathbb{R}^+)}}, 1 \right\} \right),$$

with $c_0 > 0$ sufficiently small and defining symmetrically $\widetilde{v}_{0,-}$, we derive the following Lemma.

Lemma 2.7. *Let $k \in \mathbb{N}$, $k \geq 2$, $C_0 > 1$ and $\epsilon > 0$. There exists $C_k > 0$ such that for any $\widetilde{v}_0 \in BUC^k(\mathbb{R}^\star)$ satisfying* (16)*, there exist $\widetilde{v}_{0,\pm} \in BUC^k(\mathbb{R})$ satisfying*

$$\widetilde{v}_0(x) = \begin{cases} \widetilde{v}_{0,+}(x) & \textit{if } x > 0, \\ \widetilde{v}_{0,-}(x) & \textit{if } x < 0, \end{cases}$$

and the estimates

$$\begin{aligned}
\left\|\widetilde{v}_{0,\pm}\right\|_{L^\infty(\mathbb{R}^\star)} &\leq \min(\{C_0 \epsilon, \left\|\widetilde{v}_0\right\|_{W^{1,\infty}(\mathbb{R}^\star)} C_k\}), \\
\left\|(\mathrm{sgn}(f''(\underline{u}_+))\, \partial_x \widetilde{v}_{0,+})_-\right\|_{L^\infty(\mathbb{R}^\star)} &\leq C_0 \epsilon, \\
\left\|(\mathrm{sgn}(f''(\underline{u}_-))\, \partial_x \widetilde{v}_{0,-})_-\right\|_{L^\infty(\mathbb{R}^\star)} &\leq C_0 \epsilon,
\end{aligned}$$

and for any $j \in \mathbb{N}$, $1 \leq j \leq k$,

$$\left\|\widetilde{v}_{0,\pm}\right\|_{W^{j,\infty}(\mathbb{R})} \leq \left\|\widetilde{v}_0\right\|_{W^{j,\infty}(\mathbb{R}^\pm)} (1 + \left\|\partial_x \widetilde{v}_0\right\|_{W^{1,\infty}(\mathbb{R}^\pm)}^{j-1}) C_k.$$

We can now follow the proof of Theorem 2.1, replacing Lemma 1.13 and Proposition 1.3 with Lemma 2.7 and Proposition 1.6. . □

Remark 2.8. *The non-uniqueness of the intermediate stage of our proofs is particularly striking here, since even for the same initial datum, depending on the level of regularity we aim at, we build distinct extended solutions. Yet in the end the parts actually used in the final gluing process are indeed independent of choices in the extension.*

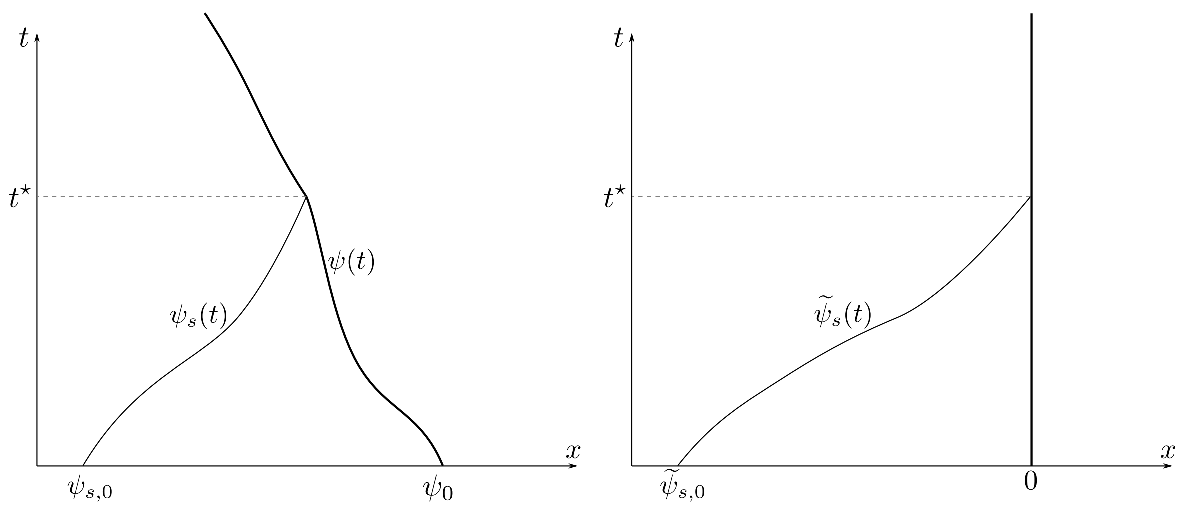
2.2 Perturbation by small shocks

We now elaborate on Proposition 1.10 and Theorem 2.1 and perturb a spectrally stable strictly admissible Riemann shock of (1) with a perturbation containing one shock. For concreteness and concision we assume that the small shock is located on the left of the large shock, the opposite situation being deduced by symmetry considerations. Since the Riemann shock is strictly admissible, sufficiently small perturbations with a small shock will produce two paths of discontinuity eventually merging in a single one, the small shock being essentially absorbed by the large one.

We follow the position of the large shock with $\psi : \mathbb{R} \to \mathbb{R}$ and the position of the small shock, as long as it persists, with $\psi_s : [0, t^\star] \to \mathbb{R}$ where $t^\star > 0$, $\psi_s(t^\star) = \psi(t^\star)$ and, for any $t \in [0, t^\star)$, $\psi_s(t) < \psi(t)$. In particular we seek for a solution that is a classical solution on the domain

$$\Omega_{\psi,\psi_s} \stackrel{\text{def}}{=} \mathbb{R}_+ \times \mathbb{R} \setminus \Big(\{\, (t, \psi_s(t)) \,|\, t \in [0, t^\star] \,\} \cup \{\, (t, \psi(t)) \,|\, t \geq 0 \} \Big)$$

(see Figure 1).



(a) Original domain (b) Straightened domain

Figure 1: Sketch of the shock paths.

Theorem 2.9. *Let $f, g \in \mathcal{C}^2(\mathbb{R})$ and $(\sigma, \underline{u}_-, \underline{u}_+) \in \mathbb{R}^3$ satisfying* (10)-(11)-(12)-(13) *and* (15).

For any $C_0 > 1$, there exists $\epsilon > 0$ and $C > 0$ such that for any $\widetilde{psi}_{s,0} < 0$ and $\psi_0 \in \mathbb{R}$ and any $\widetilde{v}_0 \in BUC^1(\mathbb{R}^\star \setminus \{\widetilde{psi}_{s,0}\})$ satisfying

$$\begin{aligned} \left\|\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\star \setminus \{\widetilde{psi}_{s,0}\})} &\leq \epsilon\,, \\ \left\|(\operatorname{sgn}(f''(\underline{u}_+))\,\partial_x \widetilde{v}_0)_-\right\|_{L^\infty(\mathbb{R}^+)} &\leq \epsilon\,, \\ \left\|(\operatorname{sgn}(f''(\underline{u}_-))\,\partial_x \widetilde{v}_0)_-\right\|_{L^\infty(\mathbb{R}^- \setminus \{\widetilde{psi}_{s,0}\})} &\leq \epsilon\,, \end{aligned} \tag{19}$$

there exist a time $t^\star \in (0, +\infty)$

- *a $\mathcal{C}^0$ function $\psi : \mathbb{R}_+ \to \mathbb{R}$ that is $\mathcal{C}^2$ on $\mathbb{R}_+ \setminus \{t^\star\}$ and such that $\psi(0) = \psi_0$*
- *a $\mathcal{C}^2$ function $\widetilde{psi}_s : [0, t^\star] \to \mathbb{R}_-$ such that $\widetilde{psi}_s$ is negative on $[0, t^\star)$, $\widetilde{psi}_s(0) = \widetilde{psi}_{s,0}$ and $\widetilde{psi}_s(t^\star) = 0$,*

so that, with $\psi_s = \psi_{|[0,t^\star]} + \widetilde{p}si_s$*, the entropy solution to* (1)*,* u*, generated by the initial data* $(\underline{U} + \widetilde{v}_0)(\cdot + \psi_0)$ *belongs to* $BUC^1(\Omega_{\psi,\psi_s})$ *and satisfies*[6] *for any* $t \geq 0$

$$\begin{aligned}
\big\|u(t,\cdot - \psi(t)) - \underline{u}_\pm\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\,t}\,,\\
\big\|(\mathrm{sgn}(f''(\underline{u}_\pm))\,\partial_x u(t,\cdot - \psi(t)))_-\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|(\mathrm{sgn}(f''(\underline{u}_\pm))\,\partial_x \widetilde{v}_0)_-\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\,t}\,,\\
\big\|\partial_x u(t,\cdot - \psi(t))\big\|_{L^\infty(\mathbb{R}^\pm)} &\leq \big\|\partial_x \widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\,t}\,,\\
|\psi_s'(t) - f'(\underline{u}_-)| &\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^-\setminus\{\widetilde{p}si_{s,0}\})}\, C\, e^{g'(\underline{u}_-)\,t}\,, \qquad t \leq t^\star\,,\\
|\psi'(t) - \sigma| &\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star\setminus\{\widetilde{p}si_{s,0}\})}\, C\, e^{\max(\{g'(\underline{u}_+),g'(\underline{u}_-)\})\,t}\,,
\end{aligned}$$

and moreover there exists ψ_∞ *such that*

$$|\psi_\infty - \psi_0| \;\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star\setminus\{\widetilde{p}si_{s,0}\})} C\,,$$

and for any $t \geq 0$

$$|\psi(t) - \psi_\infty - t\,\sigma| \;\leq \big\|\widetilde{v}_0\big\|_{L^\infty(\mathbb{R}^\star\setminus\{\widetilde{p}si_{s,0}\})} C\, e^{\max(\{g'(\underline{u}_+),g'(\underline{u}_-)\})\,t}\,.$$

Proof. Here again we follow the extension/patching strategy used for the previous results, assume without loss of generalty that $\psi_0 = 0$, and correspondingly drop some tildes. With a straightforward adaptation of Lemma 1.13 and using Proposition 1.3, we find that for $\epsilon > 0$ sufficiently small and for any $v_0 \in BUC^1(\mathbb{R}^\star\setminus\{\psi_{s,0}\})$ satisfying (19), there exists $u_l, u_c, u_r \in \mathcal{C}^1(\mathbb{R}^+\times\mathbb{R})\cap\mathcal{C}^0(\mathbb{R}^+; BUC^1(\mathbb{R}))\cap \mathcal{C}^1(\mathbb{R}^+; BUC^0(\mathbb{R}))$ global classical solutions to (1) with initial data such that

$$\begin{cases}
u_l(0,x) = \underline{u}_- + v_0 & \text{if } x < \psi_{s,0}\\
u_c(0,x) = \underline{u}_- + v_0 & \text{if } x \in (\psi_{s,0}, 0)\\
u_r(0,x) = \underline{u}_+ + v_0 & \text{if } x > 0
\end{cases}$$

and satisfying the desired estimates.

We may now identify shock locations. Let ψ_l and ψ_r be defined by the differential equations

$$\psi_l'(t) \;=\; s_f(u_l(t,\psi_l(t)), u_c(t,\psi_l(t))) \text{ and } \psi_r'(t) \;=\; s_f(u_c(t,\psi_r(t)), u_r(t,\psi_r(t)))$$

with initial data $\psi_l(0) = \psi_{l,0}$ and $\psi_r(0) = 0$. Then we observe that $\psi_l, \psi_r \in \mathcal{C}^2(\mathbb{R}^+)$ and

$$\begin{aligned}
|\psi_l'(t) - f'(\underline{u}_-)| &\leq \big\|v_0\big\|_{L^\infty(\mathbb{R}^-\setminus\{\psi_{s,0}\})}\, C\, e^{g'(\underline{u}_-)\,t}\,,\\
|\psi_r'(t) - \sigma| &\leq \big\|v_0\big\|_{L^\infty(\mathbb{R}^\star\setminus\{\psi_{s,0}\})}\, C\, e^{\max(\{g'(\underline{u}_+),g'(\underline{u}_-)\})\,t}\,.
\end{aligned}$$

Since $f'(\underline{u}_-) > \sigma$ and $\psi_{s,0} < 0$ this implies that the time

$$t^\star = \arg\min\,\{t \in \mathbb{R}^+ \mid \psi_l(t) = \psi_r(t)\}$$

is postive and finite. At last ψ_f is defined by the differential equation

$$\psi_f'(t) \;=\; s_f(u_l(t,\psi_f(t)), u_r(t,\psi_f(t)))$$

with "initial" data $\psi(t^\star) = \psi_r(t^\star)$. Note that $\psi_f \in \mathcal{C}^2(\mathbb{R}^+)$ and that $|\psi_f'(t) - \sigma|$ also decays exponentially with the same estimate as $|\psi_r'(t) - \sigma|$. Then we set $\psi_s = (\psi_l)_{|[0,t^\star]}$ and

$$\psi\,:\,\mathbb{R}\to\mathbb{R}\,, \qquad t \mapsto \begin{cases}\psi_r(t) & \text{if } 0 \leq t \leq t^\star\\ \psi_f(t) & \text{if } t > t^\star\end{cases}\,.$$

[6] In the first three inequalities we sacrifice consistency to the sake of concision and readilibility and write $\mathbb{R}^-$ even when $0 \leq t < t^\star$ and notational conventions used elsewhere would require $\mathbb{R}^- \setminus \{\widetilde{p}si_s(t)\}$ or $\mathbb{R}^- \setminus \{\widetilde{p}si_{s,0}\}$.

Again the last estimates on ψ are obtained by integration with

$$\psi_\infty = \int_0^\infty (\psi'(t) - \sigma)\, \mathrm{d}\, t\,.$$

We can now construct the solution u. For any $t \in [0, t^\star]$, we define

$$u(t,x) = \begin{cases} u_l(t,x) & \text{if } x < \psi_s(t) \\ u_c(t,x) & \text{if } \psi_s(t) < x < \psi(t) \\ u_r(t,x) & \text{if } x > \psi(t) \end{cases}.$$

For subsequent times $t \in [t^\star, +\infty)$, we set

$$u(t,x) = \begin{cases} u_l(t,x) & \text{if } x < \psi(t) \\ u_r(t,x) & \text{if } x > \psi(t) \end{cases}.$$

One easily checks that the function u is an entropy solution as soon as ϵ is sufficiently small, following the proof of Proposition 1.10 (along the path $\{(t, \psi_s(t)) \,|\, 0 \leq t \leq t^\star\}$) and Proposition 2.1 (along the path $\{(t, \psi(t)) \,|\, t \geq 0\}$). □

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