# WAVE OPERATOR BOUNDS FOR 1-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS AND APPLICATIONS

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Abstract. Boundedness of wave operators for Schrödinger operators in one space dimension for a class of singular potentials, admitting finitely many Dirac delta distributions, is proved. Applications are presented to, for example, dispersive estimates and commutator bounds.

## 1. Introduction

*Wave operators* provide a means for converting operator bounds for a "free" dynamics generated by a constant coefficient Hamiltonian, $H_0 = -\Delta$ to analogous operator bounds about "interacting" dynamics associated with a variable coefficient Hamiltonian, $H = -\Delta + V$, on its continuous spectral subspace. Indeed let $W_\pm$ and $W_\pm^*$ denote wave operators associated with the free and interacting Hamiltonians $H_0$ and $H$ (defined by (2.1) and (2.2)). Then we have

$$W_\pm W_\pm^* = P_c, \quad W_\pm^* W_\pm = Id \tag{1.1}$$

$$f(H)P_c = W_\pm f(H_0) W_\pm^*,\ f(H_0) = W_\pm^* f(H) W_\pm, \quad f \text{ Borel on } \mathbb{R}\ . \tag{1.2}$$

It follows that bounds on $f(H)P_c$ acting between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$ can be derived from bounds on $f(H_0)$ between these spaces if the wave operators $W_\pm$ are bounded between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$ for $k_j \geq 0$ and $p \geq 1$. Here, $W^{k,p}(\mathbb{R}^d)$, $k \geq 1$, $p \geq 1$ denotes the Sobolev space of functions having derivatives up to order $k$ in $L^p(\mathbb{R}^d)$.

Boundedness of wave operators in $W^{k,p}(\mathbb{R}^d)$, under smoothness and decay assumptions on $V(x)$ was proved by Yajima [31] in dimensions $d \geq 2$. Weder [30] proved boundedness in dimension one; see also the article of D'Ancona and Fanelli [3]. In [30] it is assumed that $V \in L^1_\gamma(\mathbb{R})$, the space of all complex-valued measurable functions $\phi$ defined on $\mathbb{R}$ such that

$$\|\phi\|_{L^1_\gamma} = \int |\phi(x)|(1+|x|)^\gamma dx < \infty. \tag{1.3}$$

For $V$ in a class of generic potentials, the assumption is $\gamma > 3/2$, and otherwise it is assumed $\gamma > 5/2$. Wave operator bounds can be used to establish dispersive estimates, namely

$$\left\|e^{-iHt}P_c(H)f\right\|_{L^p(\mathbb{R}^d)} = \left\|W_\pm e^{-iH_0 t} W_\pm^* f\right\|_{L^p(\mathbb{R}^d)} \leq C\ |t|^{-\frac{d}{2}-\frac{d}{p}}\ \|f\|_{L^q(\mathbb{R}^d)}, \quad p^{-1}+q^{-1}=1, \quad p \geq 1.$$

Applications of wave operator bounds for singular potentials appear in [21, 5, 16]. Schrödinger operators with singular potentials arise in several mathematical models, which have recently been extensively investigated. For example, see [9, 16, 12, 13, 14, 8, 19, 7, 21], where Dirac delta function potentials are considered. Boundedness of wave operators in $W^{1,2}(\mathbb{R})$ for singular potentials is used implicitly in references [16] and [9], but this property appears not to have been addressed previously. This gap in the literature is addressed in the present work. Another motivation for the present work is the study of scattering for highly oscillatory potentials, containing local singularities, in the homogenization limit [5]. In this work, bounds on $(m^2 + H)^{-1} P_c(H)(m^2 - \partial_x^2)$, where $H = -\partial_x^2 + V(x)$ is a Schrödinger operator with a singular (distribution) part to the potential $V(x)$, are required; see section 8.

This article is devoted to an extension of the one-dimensional results [30] to the case of singular potentials. Specifically, our results apply to Hamiltonians of the form

$$H \ = \ -\partial_x^2 \ + \ V(x),$$

where $V(x)$ satisfies:

**Hypotheses (V)**

$$V(x) \ = \ V_{sing}(x) \ + \ V_{reg}(x), \tag{1.4}$$

$$V_{sing}(x) \ = \ \sum_{j=0}^{N-1} q_j \ \delta(x - y_j), \quad q_j, y_j \in \mathbb{R}, \quad y_j < y_{j+1}, \quad q_j \neq 0, \tag{1.5}$$

$$\|V_{reg}\|_{L^1_{\frac{3}{2}+}(\mathbb{R})} \ \equiv \ \int_{\mathbb{R}} (1 + |s|)^{\frac{3}{2}+} |V_{reg}(s)| \ ds \ < \ \infty. \tag{1.6}$$

The paper is structured as follows. In section 2 we state our main result, Theorem 1, concerning boundedness of wave operators. In section 3 the strategy of proof is outlined. Section 4 summarizes facts about Jost solutions, distorted plane waves, reflection and transmission coefficients, *etc.* In section 5 we state a general result, Theorem 3, from which Theorem 1 follows. The proof of Theorem 3 is given in section 6, and the completion of Theorem 1 in section 7. Finally, in section 8 we present examples (multi-delta function potentials) and applications to dispersive estimates, commutator bounds and well posedness.

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## 2. Main results

We first define and review properties of the wave operators. For basic results on wave operators see, for example, [1, 23, 25].

Introduce the self-adjoint operators $H_0 = -\Delta$ and $H = -\Delta + V$. Here, $V$ is a real-valued potential, satisfying assumptions given below; see Section 5. Let $P_c = P_c(H)$ denote the continuous spectral projection associated with $H$. The wave operators, $W_\pm$ and their adjoints $W_\pm^*$ are defined by

$$W_\pm \equiv s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{2.1}$$

$$W_\pm^* \equiv s - \lim_{t \to \pm\infty} e^{itH_0} e^{-itH} P_c. \tag{2.2}$$

The wave operators satisfy the properties (1.1) and (1.2). The notion of wave operators is intimately related to the idea of distorted Fourier bases, which are discussed in detail in [1], [15], [22]. In one dimension, this is directly related to the Jost solutions, studied in general self-adjoint Schrödinger operators in [22, 4] and for a certain class of non-self-adjoint operators in [18].

Theorem 3 of section 5, combined with the calculations of section 7, implies the following:

**Theorem 1.** *Consider the Schrödinger operator with a potential, $V(x)$, satisfying* **Hypotheses (V)**. *Then $W_\pm$ and $W_\pm^*$ originally defined on $W^{1,p} \cap L^2$, $1 \le p \le \infty$, have extensions to bounded operators on $W^{1,p}$, $1 < p < \infty$. Moreover, there are constants $C_p$ such that:*

$$\|W_\pm f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})},\ \ \|W_\pm^* f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})},\ \ f \in W^{1,p}(\mathbb{R}),\ \ 1 < p < \infty. \tag{2.3}$$

**Remark 2.1.** In general, the wave operators are not bounded on $L^1$. The constraint $p > 1$ is due to the Hilbert transform, $\mathcal{H}$ not being bounded on $L^1$; see [30].

## 3. Strategy of Proof

We use the approach for wave operators on $\mathbb{R}$ initiated by Weder in [30]. The heart of the matter concerns the detailed low and high frequency behavior of Jost solutions, worked out by Deift and Trubowitz [4], or a consequence of their methods. The idea is to split the wave operators into high and low frequency components:

$$W_\pm = W_{\pm,high} + W_{\pm,low}.$$

For the high frequency component we prove for $\phi \in \mathcal{S}$,

$$W_{\pm,high}\phi = \sum_j S_{A_j}\phi, \quad \text{where} \quad S_A\phi \equiv \int_{-\infty}^{\infty} A(x,y)\phi(y)dy.$$

For each $A = A_j$, we use the criterion (Young's inequality [6]) for $L^p$, $1 \le p \le \infty$ boundedness:

$$\begin{aligned} C_A \ &\equiv\ \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\ dy\ +\ \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\ dx\ <\ \infty \\ &\Longrightarrow\ \|S_A\phi\|_{L^p}\ \le\ C_A\ \|\phi\|_{L^p}. \end{aligned}$$

to prove

$$\|W_{\pm,high}\phi\|_{W^{k,p}}\ \le\ C_p\ \|\phi\|_{W^{k,p}},\quad 1 < p < \infty,\quad k \ge 0. \tag{3.1}$$

For the low frequency components, we have

$$W_{\pm,low} \sim \mathcal{H}\ +\ \sum_j S_{A_j},$$

where $S_{A_j}$ is as above and $\mathcal{H}$ denotes the Hilbert Transform

$$(\mathcal{H}\phi)(x)\ =\ \frac{1}{\pi}\ \text{P.V.} \int \frac{\phi(x-y)}{y}dy\ =\ \int_{-\infty}^{\infty} e^{ikx}\ \ (-i\ \text{sgn}(k))\ \ \hat{\phi}(k)dk \tag{3.2}$$

Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier Transform on $\mathbb{R}$ and its inverse, defined by

$$\hat{\phi}(k)\ \equiv\ \mathcal{F}\phi(k) = \frac{1}{2\pi}\int e^{-ikx}\phi(x)dx,\quad \check{\Phi}(x)\ \equiv\ \mathcal{F}^{-1}\Phi(x) = \int e^{ikx}\Phi(k)dk. \tag{3.3}$$

Thus, for low frequencies, boundedness

$$\|W_{\pm,low}\phi\|_{W^{k,p}}\ \le\ C_p\ \|\phi\|_{W^{k,p}},\quad 1 < p < \infty,\quad k \ge 0 \tag{3.4}$$

reduces to the boundedness properties of the Hilbert transform [27]:

**Theorem 2.** $\mathcal{H} : W^{k,p} \to W^{k,p}$, for $1 < p < \infty$ and $k \ge 0$, *with* $\|\mathcal{H}\phi\|_{W^{k,p}(\mathbb{R})}\ \le\ K_p\ \|\phi\|_{W^{k,p}(\mathbb{R})}$.

Estimates (3.1) and (3.4) then imply the theorem. The proof of (3.1) and (3.4) is given in section 6. We now develop some background for implementing the strategy.

## 4. Background spectral theory of $H = -\partial_x^2 + V$

4.1. **Distorted plane waves,** $e_\pm(x;k)$**.** Consider the operator $H = -\partial_x^2 + V(x)$, defined as a self-adjoint operator on $L^2(\mathbb{R})$. Denote by $P_d$ and $P_c$ the discrete and continuous spectrum projections. $P_d$ and $P_c$ are orthogonal projections with $P_c = Id - P_d$.

Denote by $R_0$ the outgoing "free" resolvent operator $R_0(k) = (-\partial_x^2 - k^2)^{-1}$ with kernel

$$R_0(k)(x,y) = -(2ik)^{-1}\exp(ik|x-y|)$$

and finally introduce the *distorted plane waves*, $e_\pm(x;k)$:

**Definition 4.1.** The functions $u = e_\pm(x;k)$ are the unique solutions to $(H-k^2)u = 0$ satisfying

$$e_\pm(x;k) \;=\; e^{\pm ikx} \;+\; \text{outgoing}(x), \tag{4.1}$$

where a function $U$ is said to be outgoing as $|x| \to \infty$ if

$$(\;\partial_x \;\mp\; ik\;)\,U \to\; 0,\;\; x \to \pm\infty.$$

Thus, $e_\pm(x;k)$ is given by the integral equation:

$$e_\pm(x;k) = e^{\pm ikx} - R_0(k)Ve_\pm(x;k) \tag{4.2}$$

or equivalently

$$e_\pm(x;k) = e^{\pm ikx} - R_V(k)Ve^{\pm ikx}. \tag{4.3}$$

The continuous spectral projection, $P_c$, is given by

$$P_cf(x) = \frac{1}{2\pi}\int\int_0^\infty \left(e_+(x,k)\;\overline{e_+(y,k)} + e_-(x,k)\;\overline{e_-(y,k)}\right) f(y)dkdy. \tag{4.4}$$

see, for example, [29].

We write

$$P_cf \;\equiv\; F_+^*\;F_+\;f, \quad \text{where it follows from (4.4) that}$$

$$F_+f \;\equiv\; \int_{\mathbb{R}} \overline{\Psi_+(y,k)}\;f(y)\;dy, \qquad F_+^*f \;\equiv\; \int_{\mathbb{R}} \Psi_+(y,k)\;f(k)\;dk \;\;\text{and} \tag{4.5}$$

$$\Psi_+(y,k) = \frac{1}{\sqrt{2\pi}}\begin{cases} e_+(x;k) & k \geq 0, \\ e_-(x;-k) & k < 0. \end{cases} \tag{4.6}$$

We also define $\Psi_-(x,k) = \overline{\Psi_+(x,-k)}$.

4.2. **Jost solutions.** To make direct use of the arguments in [30] and [4], we express the results of the preceding subsection in terms of *Jost solutions*, commonly introduced for one-dimensional Schrödinger operators.

Given the Schrödinger equation

$$-\frac{d^2}{dx^2}u + Vu = k^2u,\;\; k \in \mathbb{C}, \tag{4.7}$$

we define the Jost solutions, $f_j(x,k)$, $j=1,2$, $\text{Im}k \geq 0$, to be the unique solutions of (4.7) satisfying the conditions:

$$\begin{aligned} f_1(x,k) \ &- \ e^{ikx} \to 0, \quad x \to \infty, \quad \text{and} \\ f_2(x,k) \ &- \ e^{-ikx} \to 0, \quad x \to -\infty. \end{aligned} \tag{4.8}$$

The Jost solutions are linearly independent solutions of (4.7) for $k \neq 0$. Therefore, there are unique functions $T(k)$, $R_j(k)$, $j=1,2$ such that for $k \in \mathbb{R} \setminus 0$

$$f_2(x,k) \ = \ \frac{R_1(k)}{T(k)} \ f_1(x,k) + \frac{1}{T(k)} \ f_1(x,-k), \tag{4.9}$$

$$f_1(x,k) \ = \ \frac{R_2(k)}{T(k)} \ f_2(x,k) + \frac{1}{T(k)} \ f_2(x,-k) \tag{4.10}$$

For a potential, $V$, with compact support within $(-r,r)$, $R_j(k)$ and $T(k)$ are defined via the solutions:

$$e_+(x;k) = t(k) f_1(x;k) = \begin{cases} e^{ikx} + R_2(k)e^{-ikx}, & x < -r, \\ T(k)e^{ikx}, & x > r \end{cases} \tag{4.11}$$

$$e_-(x;k) = t(k) f_2(x;k) = \begin{cases} T(k)e^{-ikx}, & x < -r, \\ e^{-ikx} + R_1(k)e^{ikx}, & x > r. \end{cases} \tag{4.12}$$

Generically,

$$T(k) = \alpha k + o(k), \qquad 1 + R_j(k) = \alpha_j k + o(k), \qquad j = 1,2, \quad k \to 0. \tag{4.13}$$

$T(k)$ is called the transmission coefficient associated with $H$. $R_1(k)$ is the right to left reflection coefficient, and $R_2(k)$ the left to right reflection coefficient.

Finally, it is convenient to denote by $m_j(x,k)$, $j=1,2$

$$m_1(x,k) \ = \ e^{-ikx} f_1(x,k), \qquad \text{and} \qquad m_2(x,k) \ = \ e^{ikx} f_2(x,k). \tag{4.14}$$

It follows from (4.1), (4.8) and (4.9) that

$$\Psi_+(x,k) \ = \ \frac{1}{\sqrt{2\pi}} \begin{cases} T(k) \ e^{ikx} \ m_1(x,k) & k \geq 0, \\ T(-k) \ e^{ikx} \ m_2(x,-k) & k < 0, \end{cases} \tag{4.15}$$

where $m_1(x,k) - 1 \to \ 0$ as $x \to \infty$ and $m_2(x,k) - 1 \ \to \ 0$ as $x \to -\infty$. The detailed smoothness and decay properties, in $x$ and $k$, of $m_j(x;k) - 1$ are required in estimates. These are given in section 7.

## 5. Statement of the Central Theorem

Our central result, from which Theorem 1 follows, is:

**Theorem 3.** *Let $H = -\partial_x^2 + V(x)$ be self-adjoint on $L^2(\mathbb{R})$ for which the transmission and reflection coefficients (see (4.9)) satisfy the bounds:*

$$|R_1(k)| \ + \ |R_2(k)| \ + \ |T(k) - 1| \quad \leq \quad \frac{C}{\langle k \rangle}, \tag{5.1}$$

$$|\partial_k R_1(k)| \ + \ |\partial_k R_2(k)| \ + \ |\partial_k T(k)| \quad = \quad \mathcal{O}\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \tag{5.2}$$

*Let $S_1$ and $S_2$ be defined by*

$$(S_j\Phi)(x) \equiv \int_{\mathbb{R}} R_j(x,y)\ \Phi(y)\ dy, \quad \text{where} \tag{5.3}$$

$$R_j(x,y) \equiv \int_{\mathbb{R}} e^{ikx}\,(m_j(x,k)-1)\,e^{-iky}\ dk. \tag{5.4}$$

*and assume, for $1<p<\infty$, that $S_1$ is bounded on $W^{1,p}(\mathbb{R}_+)$ and $S_2$ is bounded on $W^{1,p}(\mathbb{R}_-)$ .*

*Then $W_\pm$ and $W_\pm^*$ originally defined on $W^{1,p}\cap L^2$, $1\le p\le\infty$, extend to bounded operators on $W^{1,p}$, $1<p<\infty$. Furthermore, there are constants $C_p$ such that:*

$$\|W_\pm f\|_{W^{1,p}} \le C_p\|f\|_{W^{1,p}},\ \ \|W_\pm^* f\|_{W^{1,p}} \le C_p\|f\|_{W^{1,p}},\ \ f\in W^{1,p}\cap L^2,\ \ 1<p<\infty. \tag{5.5}$$

**Remark 5.1.** Deift and Trubowitz [4] establish the hypotheses of the Theorem for any potential $V(x)$, for which $(1+|x|)^{\frac{3}{2}+}|V(x)|\in L^1(\mathbb{R})$ (see section 7). We show in section 7.2 that their proof also applies to a potential of the type in Hypothesis **(V)**, $V=V_{sing}+V_{reg}$, where $V_{sing}$ has a finite set of Dirac masses within an interval $(-A,A)$, and such that $(1+|x|)^{\frac{3}{2}+}|V_{reg}(x)|\in L^1(\mathbb{R})$.

**Remark 5.2.** In fact, less restrictive bounds on $V_{reg}$ as developed in [3] would suffice. However, for simplicity we will follow the work of [30] as it makes some computations more explicit.

## 6. Proof of Central Theorem 3

We follow the strategy described in section 3. Theorem 1 will follow from 3 by verifying the hypotheses of Theorem 3 for $V=V_{sing}+V_{reg}$. This verification is computed in section 7.

Let $\chi(x\ge 1)\in C^\infty(\mathbb{R})$ denote non-decreasing cut-off functions such that

$$\chi(x\ge 1)=\begin{cases} 0 & x\le \frac{1}{2}, \\ 1 & x\ge 1\ . \end{cases} \tag{6.1}$$

To localize in frequency space, introduce $\psi(|k|\le k_0)\in C_0^\infty(\mathbb{R})$ be a compactly supported cut-off function, depending on a parameter, $k_0$, to be chosen, such that

$$\psi(|k|\le k_0)=\begin{cases} 1 & |k|\le\ k_0, \\ 0 & |k|\ge\ 2k_0\ \ . \end{cases} \tag{6.2}$$

We decompose any $\phi\in L^2(\mathbb{R})$ into its low and high frequency parts:

$$\phi(x)\ =\ \phi_{low}(x)\ +\ \phi_{high}(x), \qquad \text{where using} \quad D\equiv -i\partial_x, \tag{6.3}$$

$$\phi_{low}(x)\ \equiv\ \psi(|D|\le k_0)\phi(x)\ \equiv\ \int_{\mathbb{R}} e^{ikx}\psi(|k|\le k_0)\,\hat\phi(k)\ dk, \tag{6.4}$$

$$\phi_{high}(x)\ \equiv\ (\ 1\ -\ \psi(|D|\le k_0))\ \ \phi(x)\ \equiv\ \int_{\mathbb{R}} e^{ikx}\,(\ 1-\psi(|k|\le k_0)\ )\,\hat\phi(k)\ dk\ . \tag{6.5}$$

### 6.1. Bounds on $W_+\phi_{low}$.

For $x\ge 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_1(x,k)$, and for $x\le 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_2(x,k)$. Since the cases $x\ge 0$ and $x\le 0$ are very similar, we only carry this calculation out in detail for $x \geq 0$. We have, using the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
W_+\phi_{low} &= F_+^*\mathcal{F}\ \psi(|D| \leq k_0)\phi \\
&= \int_0^\infty e^{ikx}\ T(k)\ m_1(x,k)\ \psi\left(|k| \leq k_0\right)\ \hat{\phi}(k)\ dk + \int_{-\infty}^0 e^{ikx}\ T(-k)\ m_2(x,-k)\ \psi\left(|k| \leq k_0\right)\ \hat{\phi}(k)\ dk \\
&= \int_0^\infty e^{ikx}\ T(k)\ m_1(x,k)\ \psi\left(|k| \leq k_0\right)\ \hat{\phi}(k)\ dk \\
&\quad + \int_{-\infty}^0 e^{ikx}\ [R_1(-k)e^{-2ikx}\ m_1(x,-k) + m_1(x,k)]\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)\ dk \\
&= \int_0^\infty e^{ikx}\ m_1(x,k)\ [T(k) + R_1(k)P]\ \psi\left(|k| \leq k_0\right)\ \hat{\phi}(k)\ dk + \int_{-\infty}^0 e^{ikx}\ m_1(x,k)\ \hat{\phi}(k)\ dk, \quad x \geq 0,
\end{aligned}$$

where we have applied (4.5) and (4.15).

We continue by using that $\int_0^\infty\ [\dots]\ dk = \frac{1}{2}\int_{-\infty}^\infty\ (1 + \mathrm{sgn}(k))\ [\dots]\ dk$, we have

$$\begin{aligned}
W_+\phi_{low} &= \frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\ e^{ikx}(m_1(x,k) - 1)T(k)\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk \\
&+\frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\ e^{ikx}(m_1(x,k) - 1)R_1(k)P\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk \\
&+\frac{1}{2}\int_{-\infty}^\infty (1 - \mathrm{sgn}(k))\ e^{ikx}(m_1(x,k) - 1)\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk \\
&+\frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\ e^{ikx}T(k)\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk \\
&+\frac{1}{2}\int_{-\infty}^\infty (1 + \mathrm{sgn}(k))\ e^{ikx}R_1(k)P\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk \\
&+\frac{1}{2}\int_{-\infty}^\infty (1 - \mathrm{sgn}(k))\ e^{ikx}\ \psi\left(|k| \leq k_0\right) \hat{\phi}(k)dk, \quad x \geq 0.
\end{aligned} \tag{6.6}$$

For $x \leq 0$ an analogous representation holds with $m_1(x,k)$ replaced by $m_2(x,k)$.

We now show that $W_{+,1,low}$ is a bounded operator on $W^{k,p}(\mathbb{R}_+)$. Each term in the first three lines of (6.6) is of the form:

$$\phi\ \mapsto\ S_1\ \circ\ (I \pm i\ \mathcal{H})\ \circ\ \Psi(D)\ \phi\ , \tag{6.7}$$

and each term in the last three lines is of the form

$$\phi\ \mapsto\ (I \pm i\ \mathcal{H})\ \circ\ \Psi(D)\ \phi\ , \tag{6.8}$$

where $S_1$, is defined in (5.3)-(5.4), $\mathcal{H}$ denotes the Hilbert transform (3.2), and

$$\begin{aligned}
\Psi(D) &= \mathcal{F}^{-1}\ \hat{\Psi}(k)\ \mathcal{F} \quad \text{and} \\
\hat{\Psi}(k) &= T(k)\ \psi(|k| \leq k_0) \text{ or } R_1(k)\ P\ \psi(|k| \leq k_0) \text{ or } \psi(|k| \leq k_0).
\end{aligned}$$

(For $x \leq 0$, the argument is parallel with $S_1$ replaced by $S_2$.)

By hypotheses on $T(k)$ and $R(k)$, $\hat{\Psi}(k)$ is a multiplier on $W^{k,p}(\mathbb{R})$ for $1 < p < \infty$ [27]. The Hilbert transform is bounded (Theorem 2), so that the boundedness of the operators in (6.7) and (6.8) on $W^{k,p}$ for $1 < p < \infty$ follows from boundedness of $S_j$, that holds by hypothesis. Therefore, one has

$$\|W_+\phi_{low}\|_{W^{1,p}(\mathbb{R})}\ \leq\ C\ \|\phi\|_{W^{1,p}(\mathbb{R})}\ , \tag{6.9}$$

and this completes the low frequency analysis.

6.2. **High Frequencies.** We have, using (4.9) and the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
W_+\phi_{high} &= F_+^*\mathcal{F}\left(1-\psi(|D|\le k_0)\right)\phi \\
&= \int_0^\infty T(k)e^{ikx}m_1(x,k)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&\quad + \int_{-\infty}^0 T(-k)e^{ikx}m_2(x,-k)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&= \int_0^\infty T(k)e^{ikx}m_1(x,k)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+ \int_{-\infty}^0 e^{ikx}[R_1(-k)e^{-2ikx}m_1(x,-k)+m_1(x,k)](1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&= \int_0^\infty e^{ikx}m_1(x,k)[T(k)+R_1(k)P](1-\psi\,(|k|\le k_0))\hat\phi(k)dk + \int_{-\infty}^0 e^{ikx}m_1(x,k)\hat\phi(k)dk.
\end{aligned}$$

For $x \ge 0$ we rewrite this expression as

$$\begin{aligned}
W_+\phi_{high} &= \frac12\int_{-\infty}^\infty e^{ikx}\ (1+\mathrm{sgn}(k))\ (m_1(x,k)-1)T(k)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+\frac12\int_{-\infty}^\infty e^{ikx}\ (1+\mathrm{sgn}(k))(m_1(x,k)-1)R_1(k)P(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+\frac12\int_{-\infty}^\infty e^{ikx}\ (1-\mathrm{sgn}(k))(m_1(x,k)-1)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+\frac12\int_{-\infty}^\infty e^{ikx}\ (1+\mathrm{sgn}(k))T(k)(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+\frac12\int_{-\infty}^\infty e^{ikx}\ (1+\mathrm{sgn}(k))R_1(k)P(1-\psi\,(|k|\le k_0))\hat\phi(k)dk \\
&+\frac12\int_{-\infty}^\infty e^{ikx}\ (1-\mathrm{sgn}(k))(1-\psi\,(|k|\le k_0))\hat\phi(k)dk, \quad x\ge 0.
\end{aligned}$$

An analogous expression, with $m_1(x,k)$ replaced by $m_2(x,k)$, is used for $x \le 0$. We proceed now to show that each term is bounded on $W^{1,p}(\mathbb{R}_+)$, $p \ge 1$.

Each summand in this decomposition of $W_+\phi_{high}$ is of the form:

$$\phi \mapsto S_j \circ \rho(D)\ \phi, \qquad \text{or} \qquad \phi \mapsto \rho(D)\phi. \tag{6.10}$$

where $\rho(D) = \mathcal{F}^{-1}\hat\rho(k)\mathcal{F}$. Here, $S_j$, $j = 1,2$, defined in (5.3) and (5.4), is bounded on $W^{1,p}(\mathbb{R}_+)$ for $1 < p < \infty$ by hypothesis. Moreover, $\rho(k)$ is a multiplier on $W^{1,p}(\mathbb{R})$ for $1 < p < \infty$ due to hypotheses on $R(k), T(k)-1, \partial_k R(k)$ and $\partial_k T(k)$, and the fact that $1-\psi(|k|\le k_0)$ is smooth, asymptotically constant as $k\to\infty$ and vanishing in a neighborhood of 0. It follows that

$$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R}_+)} \le C\|V\|_{L^1_{\frac32+}(\mathbb{R})}\ \|\phi\|_{W^{1,p}(\mathbb{R}_+)}. \tag{6.11}$$

An estimate analogous to (6.11), similarly proved using a representation of $W_+\phi_{high}(x)$ for $x\le 0$, in terms of $S_2$, also holds. Thus,

$$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R})} \le C\|V\|_{L^1_{\frac32+}(\mathbb{R})}\ \|\phi\|_{W^{1,p}(\mathbb{R})}\ . \tag{6.12}$$

The decomposition (6.3) and the bounds (6.9) and (6.12) imply the result. This completes the proof of the central result, Theorem 3.

## 7. Completion of the proof of Theorem 1

This section is devoted to the completion of the proof of Theorem 1, as a consequence of Theorem 3. The hypotheses of Theorem 3 are satisfied for potentials $V \in L^1_{\frac{3}{2}+}(\mathbb{R})$, by results in [4]. We briefly recall the argument below, and then we generalize it to potentials of the form (1.4), $V = V_{sing} + V_{reg}$, in section 7.2.

7.1. **The case of regular potentials.** From the relation $m_1(x,k) = e^{-ikx}\ f_1(x,k), k \in \mathbb{C}$, we have that $m_1(x,k)$ is the unique solution of

$$\frac{d^2}{dx^2}m_1 + 2ik\ \frac{d}{dx}m_1 = V\ m_1, \text{ and } \ m_1(x;k) \to 1, \text{ as } x \to \infty\ . \tag{7.1}$$

Consequently, we have

$$m_1(x,k) = 1 + \int_x^\infty D_k(y-x)V(y)m_1(y,k)dy, \quad \text{where} \ \ D_k(x) \equiv \int_0^x e^{2iky}dy. \tag{7.2}$$

Indeed, for $V \in L^1_{\frac{3}{2}+}(\mathbb{R})$, the iterates of the Volterra integral are bounded by $\frac{\gamma(x)^n}{n!}$, with

$$\gamma(x) \equiv \int_x^\infty (t-x)|V(t)|\ dt,$$

. Summing on $n$, we find that the majoring series converges and $m_1(x,k)$ satisfies the bound

$$|m_1(x,k)|\ \le\ e^{\gamma(x)}\gamma(x).$$

By a careful analysis for $x \to -\infty$, one has the improved estimate

$$|m_1(x,k)|\ \le\ C(1+\max(-x,0))\int_x^\infty (1+|t|)|V(t)|\ dt. \tag{7.3}$$

As a consequence, the function $m_1(x,k)-1$ is in the Hardy space, and therefore there exists $B_1 \in L^2(\mathbb{R}\times\mathbb{R}_+)$ such that

$$m_1(x,k) = 1 + \int_0^\infty B_1(x,y)e^{2iky}dy. \tag{7.4}$$

Now, the function $B_1(x,y)$ is equivalently defined with

$$B_1(x,y)\ \equiv\ \int_{x+y}^\infty V(t)\ dt\ +\ \int_0^y\int_{x+y-z}^\infty V(t)B_1(t,z)\ dt\ dz \tag{7.5}$$

$$= \sum_{n=0}^\infty K_n(x,y), \tag{7.6}$$

where $K_n$ is defined by induction with

$$K_0(x,y) = \int_{x+y}^\infty V(t)dt\ , \qquad K_{n+1}(x,y) = \int_0^y\int_{x+y-z}^\infty V(t)K_n(t,z)dtdz.$$

It is then easy to prove by induction that

$$|K_n(x,y)|\ \le\ \frac{\gamma^n(x)}{n!}\eta(x+y), \quad \text{with } \eta(x)\ \equiv\ \int_x^\infty |V(t)|\ dt.$$

This allows us to confirm that the sum in (7.6) is well-defined and satisfies (7.5), plus the estimates

$$|B_1(x,y)| \le e^{\gamma(x)}\eta(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \le e^{\gamma(x)}\gamma(x). \tag{7.7}$$

From (7.7), taking the $x-$derivative of (7.5), we have

$$|\partial_x B_1(x,y)| \leq C\, e^{\gamma_1(x)} \left( V(x+y) + \int_{x+y}^{\infty} |V(t)|dt \right), \ x \in \mathbb{R},\ y>0. \tag{7.8}$$

The construction above, and (7.4) with the estimates (7.7) and (7.8), are sufficient to prove

**Lemma 7.1.** *$S_1$ is bounded on $W^{1,p}(\mathbb{R}_+)$ and $S_2$ is bounded on $W^{1,p}(\mathbb{R}_-)$ for $1<p<\infty$.*

*Proof.* We focus on the bound for $S_1$ on $W^{k,p}(\mathbb{R}_+)$. The bound for $S_2$ is bounded on $W^{k,p}(\mathbb{R}_-)$ is similar. To prove boundedness of $S_1$ and $\partial S_1$ on $L^p$ of the operator we use that the operator

$$S_R\Phi(x) = \int_{\mathbb{R}} R(x,y)\ \Phi(y)\ dy,$$

is bounded on $L^p$ with estimate

$$\|S_R\Phi\,\|_{L^p} \leq C_R\ \|\Phi\|_{L^p}, \ \ 1\leq p\leq\infty \tag{7.9}$$

if

$$C_R \equiv \sup_{x\geq 0} \int_{\mathbb{R}} |R(x,y)|\ dy + \sup_{y\geq 0}\int_{\mathbb{R}} |R(x,y)|\ dx < \infty\ . \tag{7.10}$$

Using the representation formula (7.4) we have

$$R_j(x,y) \equiv \int_{\mathbb{R}} e^{ik(x-y)} \int_0^\infty e^{2ikz}\ B_1(x,z)\ dk\ dz = B_1\left(x, \frac{y-x}{2}\right).$$

Thus, the operator $S_1$ simplifies to

$$(S_1\Phi)(x) = \int_x^\infty B_1\left(x,\frac{y-x}{2}\right)\Phi(y)dy = \int_0^\infty B_1\left(x,\frac{\zeta}{2}\right)\ \Phi(\zeta-x)\ d\zeta, \quad x\geq 0\ .$$

Since we must estimate $S_1$ on $W^{1,p}$ we also compute

$$\begin{aligned}\partial_x(S_1\Phi)(x) &= \int_0^\infty B_1\left(x,\frac{\zeta}{2}\right)\ (-\partial_\zeta)\Phi(\zeta-x)\ d\zeta + \int_0^\infty \partial_x B_1\left(x,\frac{\zeta}{2}\right)\ \Phi(\zeta-x)\ d\zeta \\ &= \int_x^\infty B_1\left(x,\frac{y-x}{2}\right)\ (-\partial_y)\Phi(y)\ dy + \int_x^\infty \partial_x B_1\left(x,\frac{y-x}{2}\right)\ \Phi(y)\ dy, \quad x\geq 0\ .\end{aligned}$$

Note that by (7.7) and (7.8) we have for large enough $x$ that

$$|B_1(x,z)| \lesssim \int_{x+z}^\infty |V(s)|ds \ \text{ and } \ |\partial_x B_1(x,z)| \lesssim |V(x)| + \int_{x+z}^\infty |V(s)|ds. \tag{7.11}$$

Therefore,

$$\begin{aligned}&\sup_{x\geq 0}\int \mathbf{1}_{y\geq x}\left|B_1\left(x,\frac{y-x}{2}\right)\right|\ dy + \sup_{y\geq 0}\int \mathbf{1}_{y\geq x}\left|B_1\left(x,\frac{y-x}{2}\right)\right|\ dx \\ &\leq 2\sup_{x\geq 0}\int_0^\infty \int_{\frac{x+y}{2}}^\infty |V(s)|ds\ dy \\ &\leq 2\int_0^\infty \left(1+\frac{x+y}{2}\right)^{-\frac{3}{2}-} \int_{\frac{x+y}{2}}^\infty (1+s)^{\frac{3}{2}+}|V(s)|ds \\ &\leq const \times \|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}.\end{aligned}$$

A similar bound applies to the kernel $\mathbf{1}_{x\geq y}\partial_x B_1\left(x, \frac{y-x}{2}\right)$. Thus, we have

$$\|S_1\Phi\|_{W^{1,p}(\mathbb{R}_+)} \equiv \|S_1\Phi\|_{L^p(\mathbb{R}_+)} + \|\partial_x\left(S_1\Phi\right)\|_{L^p(\mathbb{R}_+)} \leq\ C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\ \|\Phi\|_{W^{1,p}(\mathbb{R}_+)}\ .$$

Applying similar arguments with $S_1$ replaced by $S_2$ for $x \leq 0$ yields boundedness of $S_2$ on $W^{1,p}$. □

**Lemma 7.2.**

$$|R_j(k)|,\ |T(k)-1| \leq \frac{C}{\langle k\rangle}\quad \forall k\in\mathbb{R},\qquad |\partial_k T(k)|,\ |\partial_k R_1(k)|,\ |\partial_k R_2(k)| \leq \frac{C}{|k|}\quad \textit{as } |k|\to\infty.$$

*Proof of Lemma 7.2 for $V = V_{reg}$:*

We follow again the method of [4]. From (7.2), one has

$$\begin{aligned} m_1(x,k) &= 1 + \frac{1}{2ik}\int_0^\infty (e^{2ik(y-x)}-1)V(y)m_1(y,k)\ dy \\ &=\ e^{-2ikx}\left(\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_1(y,k)V(y)\ dy\right) \\ &\quad+\ \left(1-\frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\ dy\right) + o(1),\quad x\to-\infty. \end{aligned}$$

Moreover, one has from (4.8) and (4.9)

$$m_1(x,k)\ =\ e^{-2ikx}\frac{R_2(k)}{T(k)}\ +\ \frac{1}{T(k)}\ +\ o(1),\quad x\to-\infty.$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

$$\frac{1}{T(k)}\ =\ 1-\frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\ dy, \tag{7.12}$$

$$\frac{R_2(k)}{T(k)}\ =\ \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_1(y,k)V(y)\ dy, \tag{7.13}$$

$$\frac{R_1(k)}{T(k)}\ =\ \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky}m_2(y,k)V(y)\ dy. \tag{7.14}$$

These integral representations, together with the $L^1_{\frac{3}{2}+}$ decay assumption on the potential and the uniform bounds (7.3), lead immediately to

$$|R_j(k)|,\ |T(k)-1| \leq \frac{C}{\langle k\rangle}.$$

Now, differentiating (7.4) with respect to $k$ leads to the uniform estimate

$$|\ \partial_k m_1(x,k)\ |\ \leq\ C\langle k\rangle\langle x\rangle,$$

so that (7.12) yields

$$|\partial_k T(k)| \leq \frac{C}{|k|},\qquad \text{as}\qquad |k|\to\infty.$$

The equivalent estimates for $R_1(k), R_2(k)$ follow similarly from (7.13) and (7.14), and Lemma 7.2 holds.

7.2. **The case of potentials with a singular component.** In this section we prove that one can generalize the construction above for generalized potentials, satisfying Hypothesis **(V)**, with equivalent estimates, so that Lemma 7.1 and 7.2 hold. As a consequence, the hypotheses of Theorem 3 are satisfied, and Theorem 1 is proved.

*Proof of Lemma 7.1 for $V$ satisfying Hypotheses (**V**):*

We prove the desired estimates for $m_1(x,k)$, $x \geq 0$, and similar results apply to $m_2(x,k)$, $x \leq 0$. Let us define the function $B_1(x,y)$ with

$$B_1(x,y) \equiv \int_{x+y}^{\infty} V_{reg}(t)\ dt \ + \ \sum_{l=0}^{N-1} c_l \mathbf{1}(x_l-(x+y)) \ + \ \int_0^y \int_{x+y-z}^{\infty} V_{reg}(t) B_1(t,z)\ dt\ dz$$
$$+ \int_0^y \sum_{l=0}^{N-1} c_l B_1(x_l,z)\mathbf{1}(x_l-(x+y-z))\ dz \tag{7.15}$$
$$= \sum_{n=0}^{\infty} K_n(x,y), \tag{7.16}$$

with $\mathbf{1}$ the classical symmetric Heaviside function defined such that

$$\mathbf{1}(x) = \begin{cases} 1\ , & x>0, \\ \frac{1}{2}\ , & x=0, \\ 0\ , & x<0, \end{cases}$$

and $K_n$ defined by induction, with

$$K_0(x,y) = \int_{x+y}^{\infty} V_{reg}(t)dt \ + \ \sum_{l=0}^{N-1} c_l \mathbf{1}(x_l-(x+y)),$$
$$K_{n+1}(x,y) = \int_0^y \int_{x+y-z}^{\infty} V_{reg}(t) K_n(t,z) dt dz \ + \ \sum_{l=0}^{N-1} c_l \int_0^y K_n(x_l,z)\mathbf{1}(x_l-(x+y-z))\ dz.$$

Following the proof of Lemma 3, [4], it is easy to show by induction the following pointwise bound

$$|K_n(x,y)| \ \leq \ \frac{\gamma_1^n(x)}{n!}\eta_1(x+y),$$

with $\gamma_1$ and $\eta_1$ defined as

$$\gamma_1(x) \ \equiv \ \int_x^{\infty} (t-x)|V_{reg}(t)|\ dt \ + \ \sum_{l=0}^{N-1} |c_l|(x_l-x)\mathbf{1}(x_l-x),$$
$$\eta_1(x) \ \equiv \ \int_x^{\infty} |V_{reg}(t)|\ dt \ + \ \sum_{l=0}^{N-1} |c_l|\mathbf{1}(x_l-x).$$

This allows us to confirm that the sum in (7.16) is well-defined and satisfies (7.15), plus the estimates

$$|B_1(x,y)| \leq e^{\gamma_1(x)}\eta_1(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \leq e^{\gamma(x)}\gamma(x) \tag{7.17}$$

and, differentiating with respect to $x$,

$$|\partial_x B_1(x,y)| \leq \ C\ e^{\gamma_1(x)} \ \left( \ V(x+y) \ + \ \int_{x+y}^{\infty} |V(t)|dt \ \right),\ x\in\mathbb{R},\ y>0. \tag{7.18}$$

Abusing notation (see justification below), we define

$$m_1(x,k) \equiv 1 + \int_0^\infty B_1(x,y)e^{2iky}\,dy, \tag{7.19}$$

it is easy to deduce from (7.15) that

$$\begin{aligned}
\partial_x m_1(x,k) &= \int_0^\infty \left(\partial_x B_1(x,y) - \partial_y B_1(x,y)\right) e^{2iky}\,dy + \int_0^\infty \partial_y B_1(x,y)e^{2iky}\,dy \\
&= -\int_0^\infty \int_x^\infty V_{reg}(t)B_1(t,y)\,dt\,e^{2iky}\,dy - \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y)\mathbf{1}(x_l-x)e^{2iky}\,dy \\
&\quad - \int_0^\infty 2ikB_1(x,y)e^{2iky}\,dy - B_1(x,0). \qquad (7.20)\\
\partial_x^2 m_1(x,k) &= \int_0^\infty V_{reg}(x)B_1(x,y)e^{2iky}\,dy + \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y)\delta(x_l-x)e^{2iky}\,dy \\
&\quad - \int_0^\infty 2ik\partial_x B_1(x,y)e^{2iky}\,dy + V_{reg}(x) + \sum_{l=0}^{N-1} c_l\delta(x_l-x). \qquad (7.21)
\end{aligned}$$

Therefore, $m_1(x,k)$ is the unique function satisfying

$$\frac{d^2}{dx^2}m_1 + 2ik\,\frac{d}{dx}m_1 = \sum_l c_l\delta(x-x_l) + V_{reg}\;m_1,\ \ k\in\mathbb{C},$$

with $m_1(x;k)\to 1$ as $x\to\infty$. Equation (7.19) is thus justified.

Finally, Lemma 7.1 follows from (7.19), with the estimates (7.17) and (7.18).

*Proof of Lemma 7.2.* We follow again the method of [4]. The generalization of (7.2) to potentials satisfying Hypotheses **(V)** is

$$\begin{aligned}
m_1(x,k) &= 1 + \frac{1}{2ik}\int_0^\infty (e^{2ik(y-x)}-1)V_{reg}(y)m_1(y,k)\,dy + \frac{1}{2ik}\sum_{l=0}^{N-1} c_l(e^{2ik(x_l-x)}-1)m_1(x_l,k)\mathbf{1}(x_l-x) \\
&= \frac{e^{-2ikx}}{2ik}\left(\int_{-\infty}^{+\infty} e^{2iky}m_1(y,k)V_{reg}(y)\,dy + \sum_{l=0}^{N-1} c_l e^{2ikx_l}m_1(x_l,k)\right) \\
&\quad + 1 - \frac{1}{2ik}\left(\int_{-\infty}^{+\infty} m_1(y,k)V_{reg}(y)\,dy + \sum_{l=0}^{N-1} c_l m_1(x_l,k)\right) + o(1)(x\to-\infty).
\end{aligned}$$

Moreover, one has from (4.8) and (4.9)

$$m_1(x,k) = e^{-2ikx}\frac{R_2(k)}{T(k)} + \frac{1}{T(k)} + o(1)(x\to-\infty).$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

$$\frac{1}{T(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V_{reg}(y)\ dy \ - \ \frac{1}{2ik}\sum_{l=0}^{N-1} c_l m_1(x_l,k), \tag{7.22}$$

$$\frac{R_2(k)}{T(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k)V_{reg}(y)\ dy \ + \ \frac{1}{2ik}\sum_{l=0}^{N-1} c_l e^{2ikx_l} m_1(x_l,k), \tag{7.23}$$

$$\frac{R_1(k)}{T(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_2(y,k)V_{reg}(y)\ dy \ + \ \frac{1}{2ik}\sum_{l=0}^{N-1} c_l e^{2ikx_l} m_2(x_l,k). \tag{7.24}$$

The identity (7.19), with the estimates (7.17), guarantees the uniform bounds

$$|m_1(x,k) \le C\langle x\rangle, \qquad |\partial_k m_1(x,k)| \ \le \ C\langle k\rangle\langle x\rangle.$$

Therefore the $L^1_{\frac{3}{2}+}$ decay assumption on the potential $V_{reg}$ leads immediately to

$$|R_j(k)|, \ |T(k)-1| \le \frac{C}{\langle k\rangle}.$$

Now, differentiating (7.22) with respect to $k$ yields

$$|\partial_k T(k)| \le \frac{C}{|k|}, \qquad \text{as} \qquad |k| \to \infty.$$

The equivalent estimates for $R_1(k), R_2(k)$ follow similarly from (7.23) and (7.24), and Lemma 7.2 holds.

## 8. Examples and Applications

8.1. $V(x) =$ **a sum of Dirac delta masses.** In this section we verify directly the hypotheses of Theorem 3 for the case of a potential, which is the sum of Dirac delta functions, thereby establishing the applicability of our main results to this case.

We follow the analysis from [12] and [29], see also [10], [11] for specific examples. Seek solutions of the form

$$\left( H_{\vec{q},\vec{y}} - \frac{1}{2}k^2 \right) e_\pm(x,k) = 0, \tag{8.1}$$

where $H_{\vec{q},\vec{y}} = \sum_{j=0}^{N-1} q_j\delta(x-y_j)$ when $\vec{q} = (q_0,\cdots,q_{N-1})$, $\vec{y} = (y_0,\cdots,y_{N-1})$, and where $e_\pm(x,k)$ represent the distorted Fourier basis functions as defined (4.1). Thus,

$$e_+(x,k) = \begin{cases} e^{ikx} + B_0 e^{-ikx} \text{ for } x < y_0, \\ A_1 e^{ikx} + B_1 e^{-ikx} \text{ for } y_0 < x < y_1, \\ \vdots \\ A_N e^{ikx} \text{ for } x > y_{N-1}, \end{cases} \tag{8.2}$$

where we have taken $A_0 = 1$ and $B_N = 0$. With this choice of notation, we have, referring to (4.11) and (4.12), $A_N = T$ the transmission coefficient and $B_0 = R_1$ the reflection coefficient for the "incoming" plane wave $e^{ikx}$ from $-\infty$. Then, we have the following system of equations implied by continuity and jump conditions at the points $\{y_j\}$ for $j = 0, \ldots, N-1$:

$$\begin{aligned}
e^{iky_0} + B_0 e^{-iky_0} &= A_1 e^{ikx_0} + B_1 e^{-iky_0} \\
ik\left[A_1 e^{iky_0} - B_1 e^{-iky_0} - e^{iky_0} + B_0 e^{-iky_0}\right] &= 2q_0\left[A_1 e^{iky_0} + B_1 e^{-iky_0}\right] \\
&\vdots \\
A_{N-1} e^{iky_{N-1}} + B_{N-1} e^{-iky_{N-1}} &= A_N e^{iky_{N-1}} \\
ik\left[A_N e^{iky_{N-1}} - A_{N-1} e^{iky_0} + B_{N-1} e^{-iky_0}\right] &= 2q_{N-1}\left[A_N e^{iky_{N-1}}\right].
\end{aligned}$$

Note, the above system guarantees unitarity, or that

$$|B_0|^2 + |A_N|^2 = 1. \tag{8.3}$$

We can define similarly

$$e_-(x,k) = \begin{cases} D_0 e^{-ikx} \text{ for } x < y_0, \\ C_1 e^{ikx} + D_1 e^{-ikx} \text{ for } y_0 < x < y_1, \\ \vdots \\ C_N e^{ikx} + e^{-ikx} \text{ for } x > y_{N-1}, \end{cases} \tag{8.4}$$

where now the incoming wave is $e^{-ikx}$ from $\infty$ and the scattering matrix is determined by the transmission coefficients $D_0 = T$ and the reflection coefficient $C_N = R_2$ for the "incoming" plane wave $e^{-ikx}$ from $\infty$.

8.1.1. *Bounds on $m_1$, $m_2$:* In addition, for general singular potentials with compact support, we have

$$m_1(x,k) = e^{-ikx} f_1(x,k) = \begin{cases} e^{-ikx} \frac{e_+(x,k)}{T(k)} \text{ for } x < y_{N-1}, \\ 1, \text{ for } x > y_{N-1}, \end{cases}$$

$$m_2(x,k) = e^{ikx} f_2(x,k) = \begin{cases} e^{ikx} \frac{e_-(x,k)}{T(k)} \text{ for } x > y_0, \\ 1, \text{ for } x < y_0. \end{cases}$$

Hence, there exists constants $C^1_\alpha(y_{N-1})$ and $C^2_\alpha(y_0)$ such that

$$|\partial_k^\alpha m_1(x,k)| \leq C^1_\alpha(y_{N-1}) \text{ for } y_{N-1} > x \geq 0, \tag{8.5}$$

$$|\partial_k^\alpha m_2(x,k)| \leq C^2_\alpha(y_0) \text{ for } y_0 < x \leq 0. \tag{8.6}$$

As a result, we see that an arbitrary collection of $\delta$ functions satisfies the required estimate for the proof of Lemma 7.1.

We conclude this subsection with explicit computations of the transmission and reflection coefficients for single and double $\delta$ well potentials:

8.1.2. **Single $\delta$ potential ($H_q = -q\delta(x)$)***:* Setting up the appropriate equations, we have

$$R_1 = r_q = \frac{q}{ik - q}, \tag{8.7}$$

$$T = t_q = \frac{ik}{ik - q}, \tag{8.8}$$

where $r_q$, $t_q$ are the reflection and transmission coefficients for $H_q$ respectively. We must show the bounds from (5.1) hold, however such bounds follow clearly for (8.8), (8.7).

8.1.3. **Double $\delta$ potential ($H_{q,L} = -q(\delta(x+L)+\delta(x-L))$):** Setting up the appropriate equations, we have

$$R_1 = r_{q,L} = \left( \frac{q(ik-q)e^{2ikL} + q(ik+q)e^{-2ikL}}{q^2e^{2ikL} - (ik+q)^2e^{-2ikL}} \right) e^{-2ikL}, \tag{8.9}$$

$$T = t_{q,L} = \left( \frac{k^2}{q^2e^{2ikL} - (ik+q)^2e^{-2ikL}} \right) e^{-2ikL}, \tag{8.10}$$

where $r_{q,L}$, $t_{q,L}$ are the reflection and transmission coefficients for $H_{q,L}$ respectively.

Again, we must verify Lemma (7.2), hence we must prove for instance

$$|\partial_k t_{q,L}(k)| \le C(1+|k|)^{-1},$$

provided $qL \neq 1/2$. Indeed, we have

$$\partial_k t_{q,L}(k) = \frac{2k(k^2 - 2ikq + q^2(e^{4ikL}-1)) - 2ik^2(2Lq^2e^{4ikL} - (ik+q))}{(k^2 - 2ikq + q^2(e^{4ikL}-1))^2},$$

which satisfies

$$|\partial_k t_{q,L}(k)| \sim \mathcal{O}(|k|^{-1})$$

as $k \to \infty$ and

$$|\partial_k t_{q,L}(k)| \sim \mathcal{O}\left( \frac{1}{4q^2L - 2q} \right)$$

as $k \to 0$. A similar computation holds for $r_{q,L}$.

8.2. **Commutator / Resolvent type bounds.** In [5], where homogenization of high contrast oscillatory structures with defects is studied, bounds on $(H_0+1)^{-1}(H_{\vec{q},\vec{y}}+1)$ are required to estimate a Lipmann Schwinger equation. We have, by our main theorem that

$$(H_0+1)^{-1}(H_{\vec{q},\vec{y}}+1)P_c = (H_0+1)^{-1}W_+(H_0+1)W_+^* : L^2 \to L^2.$$

8.3. **Dispersive and Strichartz estimates in $H^1$ for $\delta$-Schrödinger.** We may represent

$$e^{-itH}P_cf = \frac{1}{2\pi}\int\int_0^\infty e^{-\frac{itk^2}{2}} \left( e_+(x,k)\,\overline{e_+(x,k)} + e_-(x,k)\,\overline{e_-(x,k)} \right) f(y)dkdy. \tag{8.11}$$

From here, we may use direct computations to arrive at Strichartz estimates and apply Weder's results on wave operators since the potentials are all in $L^1$ with compact support.

Using the properties of wave operators, we have

$$\|e^{iHt}P_cf\|_{L^p} = \|W_\pm e^{itH_0}W_\pm^* f\|_{L^p} \tag{8.12}$$

and using standard dispersive estimates for the linear Schrödinger operator (see for instance [28] for a concise overview) arrive at

$$\|e^{iHt}P_cf\|_{L^p} \le C_p t^{-(\frac{1}{2}-\frac{1}{p})}\|f\|_{W^{1,p}}. \tag{8.13}$$

Define a Strichartz pair $(q,r)$ to be admissible if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r} \tag{8.14}$$

with $2 \le r < \infty$. Then, we arrive at the celebrated Strichartz estimates

$$\|e^{iHt}P_cu_0\|_{L^qW^{1,r}} \lesssim \|u_0\|_{W^{1,2}} \tag{8.15}$$

and

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^q W^{1,r}} \lesssim \|f(x,t)\|_{L_t^{\tilde{q}} W_x^{1,\tilde{r}}} \tag{8.16}$$

using duality techniques and once again the boundedness of the wave operators.

As a side note, using positive commutators and well crafted local smoothing spaces, from [20] we have the Strichartz estimate

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^\infty L^2} \lesssim \|f(x,t)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}}. \tag{8.17}$$

Now, by boundedness of wave operators on $W^{1,p}$ spaces for singular potentials as proved in Theorem 3, we have the following useful relation

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^\infty H^1} \lesssim \|f(x,t)\|_{L_t^{\tilde{p}} W_x^{1,\tilde{q}}}, \tag{8.18}$$

where $(\tilde{p}, \tilde{q})$ is a dual Strichartz pair without first going through the dispersive estimates.

8.4. **Local Well-Posedness in $H^1$ for $\delta$-NLS.** Consider the nonlinear Schrödinger / Gross-Pitaevskii, with a potential consisting of a finite set of Dirac delta functions:

$$\begin{cases} i\partial_t u + H_{\vec{q},\vec{y}} u - |u|^{2\sigma} u = 0, \\ u(x,0) = u_0(x) \in H^1, \end{cases}$$

for $0 < \sigma < \infty$. We seek a solution in the following sense:

$$u = \Lambda[u],$$

where

$$\Lambda[u](t) = e^{-iH_{\vec{q},\vec{y}} t} u_0 - i \int_0^t e^{-iH_{\vec{q},\vec{y}}(t-s)} |u|^{2\sigma} u(s) ds. \tag{8.19}$$

We claim that local well-posedness can be established via the contraction mapping principle in the space $C^0([0,T); H^1(\mathbb{R}))$ for $T$ sufficiently small. To prove the necessary boundedness and contraction estimates, it is natural to apply the operator $(I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c$, which commutes with the group $e^{-iH_{\vec{q},\vec{y}} t}$ to (8.19). Then, estimates follow in a straightforward way, using that $H^1(\mathbb{R})$ is an algebra, provided the space

$$\mathcal{H}^1(\mathbb{R}) = \left\{ f : (I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c f \in L^2(\mathbb{R}) \right\} \tag{8.20}$$

is equivalent to the classical Sobolev space $H^1$. This follows from the relations

$$(I+H)^{\frac{1}{2}} P_c = W (I - \partial_x^2)^{\frac{1}{2}} W^*, \quad W^* (I+H)^{\frac{1}{2}} W = (I - \partial_x^2)^{\frac{1}{2}}$$

and our results on the boundedness of wave operators associated with $H_{\vec{q},\vec{y}}$ on $H^1$.

8.5. **Long time dynamics for $NLS$ with a double $\delta$ well potential.** In [21], the long time dynamics of solutions to the nonlinear Schrödinger / Gross-Pitaevskii equation

$$i\partial_t u = (-\Delta + V(x)) u \ + \ gK \left[ |u|^2 \right] u, \tag{8.21}$$

where $V$ is a symmetric, double well potential, are studied. In particular, under appropriate spectral assumptions on the operator $H = -\partial_x^2 + V(x)$, in a neighborhood of a symmetry breaking bifurcation point, there are different classes of oscillating solutions (8.21) which *shadow* periodic orbits of a finite dimensional reduction on very long, but finite, time scales. These solutions correspond to states with mass concentrations oscillating between the two wells of a symmetric potential well. The proof requires dispersive

/ Strichartz type estimates. The results of this paper imply that the results of [21] extend to (8.21) for the case of singular potentials, such as

$$V(x) = -q[\delta(x-L) + \delta(x+L)].$$

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