On the Favrie-Gavrilyuk system

as an approximation to the Serre-Green-Naghdi model

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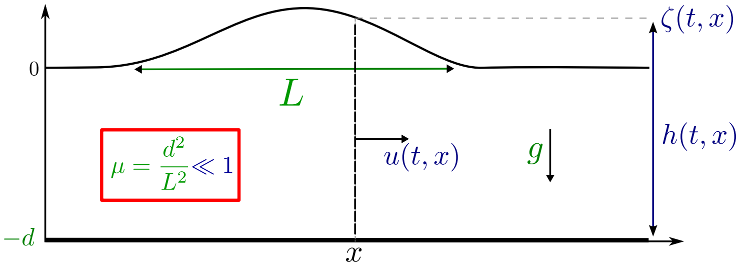
Lund Workshop, June 25–29, 2018

The Green-Naghdi system



The Green-Naghdi system (GN) is

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\mu}{3h} \nabla \left(h^2 \ddot{h} \right) = 0, \end{cases} \tag{GN}$$

where $\dot{h} = \partial_t h + \mathbf{u} \cdot \nabla h$ and $\ddot{h} = \partial_t \dot{h} + \mathbf{u} \cdot \nabla \dot{h}$.

- Derived in [Serre'53][Su,Gardner'69][Green,Naghdi'76][Miles-Salmon'85]...
- Asymptotic model for water-waves with precision $\mathcal{O}(\mu^2)$ [Lannes,Bonneton '09]
- Cauchy problem is locally well-posed on the natural timescale [Li'06][Alvarez-Samaniego,Lannes'08] [Fujiwara,Iguchi'15][VD,Israwi'18]

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Solving the evolutionary equations require to invert

$$\mathfrak{T}[h] : \mathbf{v} \mapsto \mathbf{v} - \frac{\mu}{3h}\nabla\big(h^3\nabla\cdot\mathbf{v}\big).$$

Costly for numerical computations.

- Direct approaches [Le Métayer,Gavrilyuk,Hank '10] [Mitsotakis,Synolakis,McGuinness '17] among many others
- Solve new model with same precision [Lannes,Marche '15][Duran,Marche '15]
- Alternative approach [Favrie,Gavrilyuk '17]

The Favrie-Gavrilyuk system

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The Favrie-Gavrilyuk system (FG) is

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\lambda\mu}{3h}\nabla\big(\frac{\eta}{h}(\eta - h)\big) = 0, \\ \partial_t \eta + \mathbf{u} \cdot \nabla \eta = w, \\ \partial_t w + \mathbf{u} \cdot \nabla w = -\frac{\lambda}{h^2}\big(\eta - h\big). \end{cases} \tag{FG}$$

As $\lambda \to \infty$, we hope

$$\eta = h + \mathcal{O}(\lambda^{-1}) \quad \text{and} \quad \lambda\big(\eta - h\big) = -h^2\ddot{\eta} = -h^2\ddot{h} + \mathcal{O}(\lambda^{-1})$$

This is valid if $U, \partial_t U, \partial_t^2 U, \partial_t^3 U$ are regular and bounded uniformly with respect to $\lambda \gg 1$ and $\mu \ll 1$, which requires to prepare the initial data.

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Outline

Dispersion relation

The linearized system around $h = 1, \mathbf{u} = 0, \eta = 1, w = 0$ has plane wave solutions of the form $e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(|\mathbf{k}|)t}$ if (when $d = 2$)

$$\omega \left((\omega^2 - |\mathbf{k}|^2)(\omega^2 - \lambda) - \frac{\lambda\mu}{3} |\mathbf{k}|^2 \omega^2 \right) = 0.$$

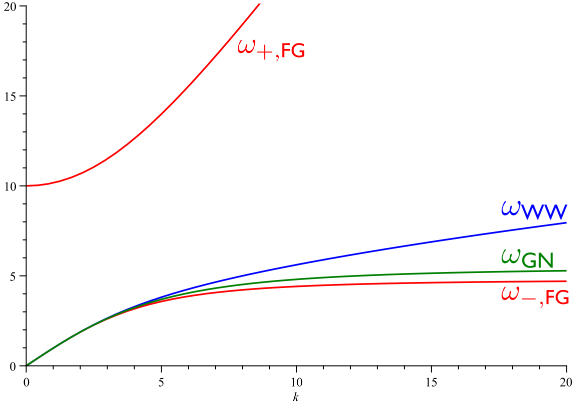

Figure: $\mu = 0.1$, $\lambda = 100$

Hyperbolicity

Recall the Favrie-Gavrilyuk system

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The principal symbol is smoothly diagonalizable as soon as $h > 0$.

- Kernel of dimension 2 ($+1$ for vorticity if $d = 2$).
- Non-trivial eigenvalues $\pm\sqrt{h + \mu\lambda\frac{\eta^2}{3h}} \approx \pm\sqrt{1 + \frac{\mu\lambda}{3}}\sqrt{h}$

Small time well-posedness

Let $s > 1 + d/2$, $\lambda, \mu \in (0, \infty)$ and initial data $h_0 - 1, \mathbf{u}_0, \eta_0 - 1, w \in H^s$ with $h \geq h_\star > 0$. Then there exists $T(\lambda, \mu) > 0$ and a unique maximal strong solution in $C^0([0, T); H^s) \cap C^1([0, T); H^{s-1})$.

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Symmetric structure

Introducing the new unknowns

$$\iota \overset{\text{def}}{=} (\mu\lambda)^{1/2}(\eta - h) \quad ; \quad \kappa \overset{\text{def}}{=} \mu^{1/2} h^{-1} w,$$

the Favrie-Gavrilyuk system (FG) is equivalent to

$$\begin{cases} \partial_t \zeta + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta - \frac{1}{3h}\nabla\Big((\mu\lambda)^{1/2}\iota + \frac{\iota^2}{h}\Big) = 0, \\ \partial_t \iota + \mathbf{u}\cdot\nabla\iota = \lambda^{1/2}(h\kappa + \mu^{1/2} h\nabla\cdot\mathbf{u}), \\ \partial_t(h\kappa) + \mathbf{u}\cdot\nabla(h\kappa) = -\lambda^{1/2} h^{-2}\iota. \end{cases} \tag{FG'}$$

It is easy to rewrite the above under the form

$$\mathcal{S}_t(V)\partial_t V + \mathcal{S}_x(V)\partial_x V + \mathcal{S}_y(V)\partial_y V = \lambda^{1/2}\mathsf{J}^{\mu} V + G(V),$$

where $\mathcal{S}_t, \mathcal{S}_x, \mathcal{S}_y$ are smooth functions with values into symmetric matrices, G is vector-valued and smooth and

$$\mathsf{J}^{\mu} = \begin{pmatrix} 0 & 0_{1,d} & & \\ 0_{d,1} & 0_{d,d} & \mu^{1/2}\nabla & \\ & \mu^{1/2}\nabla^{\top} & 0 & 1 \\ & & -1 & 0 \end{pmatrix}.$$

Aim: Uniform energy estimates for solutions to

$$\mathcal{S}_t(V)\partial_t V + \mathcal{S}_x(V)\partial_x V + \mathcal{S}_y(V)\partial_y V = \lambda^{1/2} \mathsf{J}^{\mu} V + G(V)$$

where $V \in C^j([0,T); H^{s-j})$ with $s > d/2 + 1$ and $0 \leq j \leq s$.

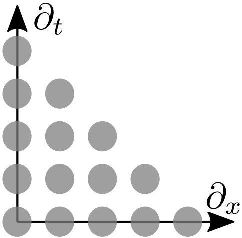
Strategy: [Schochet '86]

We denote (bounded initially)

$$\|V\|_0 \approx \|V\|_{H^s} \quad ; \quad \|V\|_j \approx \lambda^{\frac{1-j}{2}} \|\partial_t^j V\|_{H^{s-j}} \ (j \geq 1).$$

- Test against $(1-\Delta)^s V$ and integrate by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V\|_0^2 \leq C(\|V\|_0, \|V\|_1)\|V\|_0^2.$$



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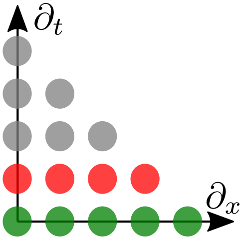

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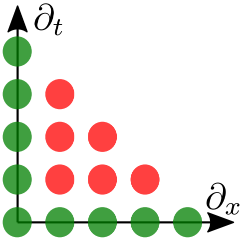
- Apply ∂_t^j to the equation and testing against $\partial_t^j V$. Commutator term $[\partial_t^j, \mathcal{S}_t(V)]\partial_t V$ is controlled. Other terms are not singular. By the above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^j V\|_{L^2}^2 \leq C(\|V\|_0, \dots, \|V\|_j)\|\partial_t^j V\|_{L^2}^2.$$



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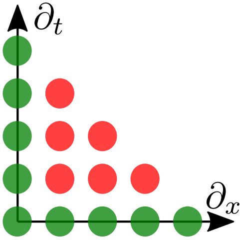
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^j V\|_{L^2}^2 \leq C(\|V\|_0, \dots, \|V\|_j)\|\partial_t^j V\|_{L^2}^2.$$

- Mixed time-space derivatives cannot be estimated in this way due to the contribution of

$$[\partial_x^{s-j}, \mathcal{S}_t(V)]\partial_t^{j+1} V.$$



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with

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Define Π^{reg} and Π^{sing} the (orthogonal) projections onto the kernel and non-zero eigenvalues of J^{μ}, respectively. We have the following

Lemma

Restricted to $\mathsf{J}^{\mu}|_{\Pi^s} : H^n \to H^{n-1}$ is invertible and

$$\left\|(\mathsf{J}^{\mu}|_{\Pi^s})^{-1}\right\|_{H^{n-1}\to H^n} \leq (1+\mu^{-1/2}).$$

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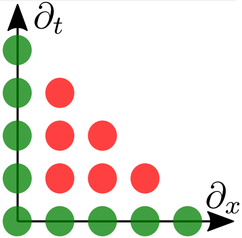
- Applying $\partial_t^j \Pi^{\mathrm{sing}}$ and $\partial_t^{j-1}\Pi^{\mathrm{reg}}$ to the equation and testing against $\partial_t^j V$:

$$\left\|V\right\|_j \leq C(\left\|V\right\|_0, \left\|\partial_t V\right\|_{L^2}, \cdots, \left\|\partial_t^s V\right\|_{L^2})$$

uniformly in λ but not in μ.



Main result

Large time well-posedness

Let $m, s \in \mathbb{N}$ with $s > 1 + d/2$ and $1 \leq m \leq s$, $h_\star, M_0 > 0$. There exists $\nu_\star, \tau, C_0 > 0$ such that for any (λ, μ) satisfying $\lambda \geq 1$, $\mu \leq 1$, $\lambda\mu \geq \nu_\star$, and for any $V \in C^0([0,T); H^s(\mathbb{R}^d))$ maximal strong solution to (FG') such that one has $h\big|_{t=0} \geq h_\star$ and

$$\left(\sum_{j=0}^{m} \big\| \partial_t^j V \big\|_{H^{s-j}} + \sum_{j=m+1}^{s} (\lambda\mu)^{\frac{m-j}{2}} \big\| \partial_t^j V \big\|_{H^{s-j}} \right)(t=0) \leq M_0,$$

one has $T > (M_0\tau)^{-1}$ and for any $t \in [0, (M_0\tau)^{-1}]$,

$$\left(\sum_{j=0}^{m} \big\| \partial_t^j V \big\|_{H^{s-j}} + \sum_{j=m+1}^{s} (\lambda\mu)^{\frac{m-j}{2}} \big\| \partial_t^j V \big\|_{H^{s-j}} \right)(t) \leq C_0 M_0.$$

Rk: If $\mu \approx 1$, then it suffices to control $\left(\sum_{j=0}^{m} \big\|\partial_t^j V\big\|_{H^{s-j}}\right)(t=0)$.

Main result

Large time well-posedness

Let $s \in \mathbb{N}$ with $s > 1 + d/2$, $h_\star, M_0 > 0$ and $\delta_\star \in (0,1)$. There exists $\tau, C_0 > 0$ such that for any (λ, μ) satisfying $\mu \leq 1$, $\lambda \geq \mu^{-1}$, and for any $V \in C^0([0,T); H^s(\mathbb{R}^d))$ maximal strong solution to (FG') satisfying

$$h\big|_{t=0} \geq h_\star, \quad \big|(\kappa h)\big|_{t=0} \leq (1-\delta_\star)(\lambda\mu)^{1/2}, \quad \big|(\iota h^{-1})\big|_{t=0}\big| \leq (1-\delta_\star)(\lambda\mu)^{1/2}$$

and

$$\left(\sum_{j=0}^{s} \big\|\partial_t^j V\big\|_{H^{s-j}}\right)(t=0) \leq M_0,$$

one has $T > (M_0\tau)^{-1}$ and for any $t \in [0, (M_0\tau)^{-1}]$,

$$\left(\sum_{j=0}^{s} \big\|\partial_t^j V\big\|_{H^{s-j}}\right)(t) \leq C_0 M_0.$$

Qn 1: Is it really *necessary* to prepare the initial data?

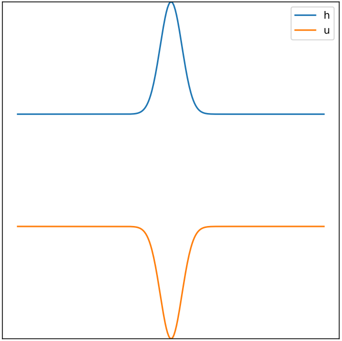
Qn 2: Is it really *possible* to prepare the initial data?

Toy models

Order 1

$$\begin{cases} \partial_t h = 0, \\ \partial_t u + \frac{1}{\epsilon} h \partial_x u = 0 \end{cases}$$

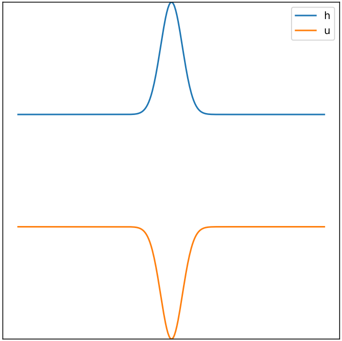
Animation



Order 0

$$\begin{cases} \partial_t h = 0, \\ \partial_t u + \frac{i}{\epsilon} h u = 0 \end{cases}$$

Animation



Well-prepared initial data

Proposition

Let $s, m \in \mathbb{N}$, $s > d/2 + 1$, $s \geq m + 1$ and $h_\star, M_0 > 0$. There exists $C_m > 0$ such that for any $\mu \leq 1$, $\lambda \geq \mu^{-1}$, and $h_0 - 1, \mathbf{u}_0 \in H^s$ with

$$h_0 \geq h_\star \quad \text{and} \quad \big\|\zeta_0\big\|_{H^s} + \big\|\mathbf{u}_0\big\|_{H^s} \leq M_0,$$

the following holds. There exists $c^{(j)} \in H^s(\mathbb{R}^d)$ for $j \in \{1, \cdots, m\}$ such that the unique strong solution to (FG) with initial data $(h_0, \mathbf{u}_0, \eta_0^{(m)}, w_0^{(m)})$

$$\text{where } w_0^{(m)} = \sum_{j \text{ odd}} \lambda^{-(j-1)/2} c^{(j)} \quad \text{and} \quad \eta_0^{(m)} = h_0 + \sum_{j \text{ even}} \lambda^{-j/2} c^{(j)}$$

satisfies

$$\left(\sum_{j=0}^{m+1} \big\|\partial_t^j U^{(m)}\big\|_{H^{s-j}}(0) + \lambda \sum_{j=0}^{m} \big\|\partial_t^j(\eta^{(m)} - h^{(m)})\big\|_{H^{s-j}} \right)(t=0) \leq C_m\, M_0.$$

Proposition (continued)

In the previous proposition, we can choose

$$w_0^{(J)} = c^{(1)} + \dots \quad \text{and} \quad \eta_0^{(J)} = h_0 + \lambda^{-1} c^{(2)} + \dots$$

with $c^{(1)} = -h_0 \nabla \cdot \mathbf{u}_0$ and $c^{(2)}$ the unique solution to

$$\mathfrak{t}[h_0] c^{(2)} = h_0^3 \left(\mathbf{u}_0 \cdot \nabla (\nabla \cdot \mathbf{u}_0) - (\nabla \cdot \mathbf{u}_0)^2 - \Delta \zeta_0 - \nabla \cdot ((\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0) \right)$$

where we define

$$\mathfrak{t}[h]\psi \stackrel{\text{def}}{=} \psi - \frac{\mu}{3} h^3 \nabla \cdot \left(h^{-1} \nabla \psi \right).$$

Strategy: [Browning,Kreiss '82]

Proof

Recall

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\lambda\mu}{3h} \nabla\left(\frac{\eta}{h}(\eta - h)\right) = 0, \\ \partial_t \eta + \mathbf{u} \cdot \nabla \eta = w, \\ \partial_t w + \mathbf{u} \cdot \nabla w = -\frac{\lambda}{h^2}\left(\eta - h\right). \end{cases} \qquad \text{(FG)}$$

We have, denoting $U = (h, \mathbf{u}, \eta, w)$.

$$\partial_t^2(\eta - h) = \mathfrak{r}[U] + \lambda\mu\,\mathfrak{s}[U, \eta - h] - \lambda h^{-2}\mathfrak{t}[h](\eta - h)$$

where $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{t}$ are nonlinear differential operators (in space) of order two. Moreover

- $\mathfrak{r}[h_0, \mathbf{u}_0, -h_0\nabla \cdot \mathbf{u}_0, 0] = h_0\big(\mathbf{u}_0 \cdot \nabla(\nabla \cdot \mathbf{u}_0) - (\nabla \cdot \mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla \cdot ((\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0)\big)$
- $\mathfrak{s}$ is quadratic in $\eta - h$. In particular, $\mathfrak{r}[h_0, \mathbf{u}_0, -h_0\nabla \cdot \mathbf{u}_0, 0] = 0$.
- we collected singular terms in the operator $\mathfrak{t}$:

$$\mathfrak{t}[h]\psi = \psi - \frac{\mu}{3} h^3 \nabla \cdot \left(h^{-1}\nabla\psi\right).$$

From this an induction strategy can be pursued to control $\partial_t^j U$ and $\lambda\partial_t^j(\eta - h)$.

Proof

Recall

$$\begin{cases} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\lambda\mu}{3h}\nabla\big(\frac{\eta}{h}(\eta-h)\big) = 0, \\ \partial_t \eta + \mathbf{u}\cdot\nabla\eta = w, \\ \partial_t w + \mathbf{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta-h\big). \end{cases} \tag{FG}$$

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$$\partial_t^2(\eta-h) = \mathfrak{r}[U] + \lambda\mu\,\mathfrak{s}[U,\eta-h] - \lambda h^{-2}\mathfrak{t}[h](\eta-h)$$

where $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{t}$ are nonlinear differential operators (in space) of order two. Moreover

- $\mathfrak{r}[h_0,\mathbf{u}_0,-h_0\nabla\cdot\mathbf{u}_0,0] = h_0\big(\mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_0) - (\nabla\cdot\mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla\cdot((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0)\big)$
- $\mathfrak{s}$ is quadratic in $\eta - h$. In particular, $\mathfrak{r}[h_0,\mathbf{u}_0,-h_0\nabla\cdot\mathbf{u}_0,0] = 0$.
- we collected singular terms in the operator $\mathfrak{t}$:

$$\mathfrak{t}[h]\psi = \psi - \frac{\mu}{3}h^3\nabla\cdot\big(h^{-1}\nabla\psi\big)\,.$$

From this an induction strategy can be pursued to control $\partial_t^j U$ and $\lambda\partial_t^j(\eta-h)$.

Proof

Recall

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\lambda\mu}{3h} \nabla\left(\frac{\eta}{h}(\eta - h)\right) = 0, \\ \partial_t \eta + \mathbf{u} \cdot \nabla \eta = w, \\ \partial_t w + \mathbf{u} \cdot \nabla w = -\frac{\lambda}{h^2}\left(\eta - h\right). \end{cases} \qquad \text{(FG)}$$

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- $\mathfrak{r}[h_0, \mathbf{u}_0, -h_0\nabla \cdot \mathbf{u}_0, 0] = h_0\big(\mathbf{u}_0 \cdot \nabla(\nabla \cdot \mathbf{u}_0) - (\nabla \cdot \mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla \cdot ((\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0)\big)$
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Conclusion

If you want to numerically simulate shallow-water gravity waves, you may

1. Non-dimensionalize the variables and extract $\mu = d^2/L^2$.
2. Set $\lambda \gtrsim \mu^{-1}$.
3. Set $w_0 = -h_0 \nabla \cdot \mathbf{u}_0$ and $\eta_0 = h_0 + \lambda^{-1} c^{(2)}$.
4. Solve the Favrie-Gavrilyuk system

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla h + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\lambda\mu}{3h} \nabla\left(\frac{\eta}{h}(\eta - h)\right) = 0, \\ \partial_t \eta + \mathbf{u} \cdot \nabla \eta = w, \\ \partial_t w + \mathbf{u} \cdot \nabla w = -\frac{\lambda}{h^2}\left(\eta - h\right). \end{cases} \tag{FG}$$

5. Enjoy precision $\mathcal{O}(\mu\lambda^{-1})$ w.r.t Green-Naghdi and hence $\mathcal{O}(\mu^2)$ w.r.t. water-waves.

▸ Animation

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Animation

Thank you for your attention !