Asymptotic limits for the multilayer shallow water system

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Mathocéan
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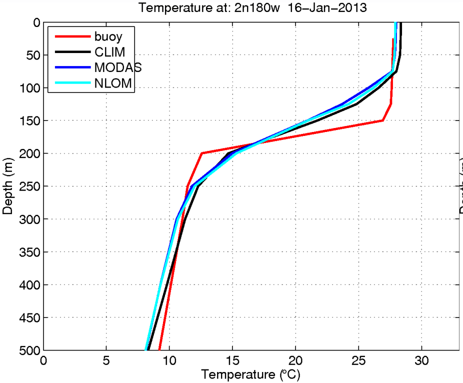
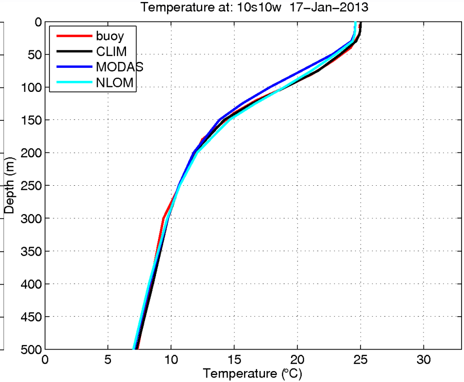
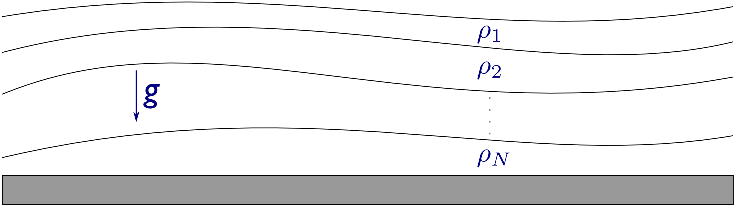
The multilayer Saint-Venant model





Credits: Naval Research Laboratory `www7320.nrlssc.navy.mil/global_nlom`



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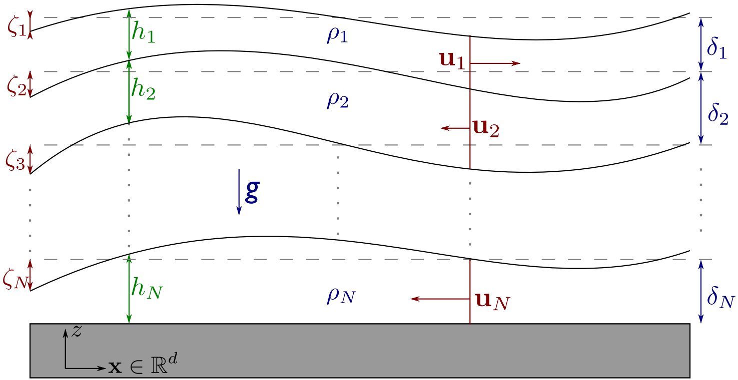

$$\begin{cases} \partial_t \zeta_n + \sum_{i=n}^{N} \nabla \cdot (h_i \, \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \sum_{i=1}^{n} \frac{g(\rho_i - \rho_{i-1})}{\rho_n} \nabla \zeta_n + (\mathbf{u}_n \cdot \nabla) \mathbf{u}_n = \mathbf{0} \end{cases} \qquad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$$

with $h_n = \delta_n + \zeta_n - \zeta_{n+1}$ and conventions $\zeta_{N+1} \equiv 0$ and $\rho_0 = 0$.

A quasilinear system

One can rewrite the system as

$$\partial_t \mathsf{U} + \mathsf{A}^x[\mathsf{U}]\partial_x \mathsf{U} + \mathsf{A}^y[\mathsf{U}]\partial_y \mathsf{U} = \mathbf{0},$$

with $U = (\zeta_1, \dots, \zeta_N, u_1^x, \dots, u_n^x, u_1^y, \dots, u_N^y)^\top$ and

$$\mathsf{A}^x[\mathsf{U}] \stackrel{\text{def}}{=} \begin{pmatrix} \mathsf{M}(u^x) & \mathsf{H}(\zeta) & 0 \\ \mathsf{R} & \mathsf{D}(u^x) & 0 \\ 0 & 0 & \mathsf{D}(u^x) \end{pmatrix}$$

where $\mathsf{D}(u) = \mathrm{diag}(u_1, \dots, u_N)$ and

$$\mathsf{M} = \begin{pmatrix} \ddots & (u_i^x - u_{i-1}^x) \\ (0) & \ddots \end{pmatrix}, \quad \mathsf{H} = \begin{pmatrix} \ddots & (h_i) \\ (0) & \ddots \end{pmatrix}, \quad \mathsf{R} = \begin{pmatrix} \ddots & (0) \\ \left(\frac{g(\rho_i - \rho_{i-1})}{\rho_n}\right) & \ddots \end{pmatrix}.$$

Rotational invariance [Monjarret '14]

$$\mathsf{A}[\mathsf{U}, \xi] \stackrel{\text{def}}{=} \xi^x \mathsf{A}^x[\mathsf{U}] + \xi^y \mathsf{A}^y[\mathsf{U}] = \mathsf{Q}(\xi)^{-1}\mathsf{A}^x[\mathsf{Q}(\xi)\mathsf{U}]\mathsf{Q}(\xi)|\xi|,$$

with $\mathsf{Q}(\xi) = \frac{1}{|\xi|}\begin{pmatrix} |\xi|\mathsf{I}_N & 0 & 0 \\ 0 & \xi^x \mathsf{I}_N & \xi^y \mathsf{I}_N \\ 0 & -\xi^y \mathsf{I}_N & \xi^x \mathsf{I}_N \end{pmatrix}$, homogeneous of degree 0.

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Hyperbolicity

Sufficient conditions for hyperbolicity [Ripa '90, VD '14, Monjarret '14]

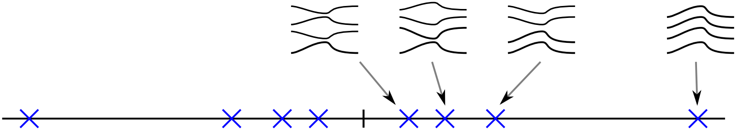
Given $0 < \rho_1 < \rho_2 < \cdots < \rho_N$ and $h_1, \dots, h_N > 0$, there exists $\nu > 0$ s.t.

$$\left|\mathbf{u}_i - \mathbf{u}_{i-1}\right| < \nu,$$

one can construct positive definite (symbolic) symmetrizers of the system.

Proof. By rotational invariance, it suffices to give a symmetrizer of A^x, or $d = 1$.

1. Explicit symmetrizer from the Hessian of the energy + momentum [Ripa '90]
2. if $u_1 = \cdots = u_n = \overline{u}$ then $A[u]$ has $2N$ distinct eigenvalues, $\lambda_{\pm n} = \pm \mu_n^{-1/2}$ with μ_n eigenvalue of a N-by-N symmetric, tridiagonal matrix [Benton '53].



By perturbation $A[u]$ has $2N$ simple, distinct eigenvalues; and $S = \sum P_n^\top P_n$.

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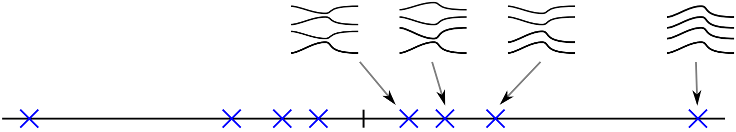

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Well-posedness of the Cauchy problem

Given $\zeta_1^0, \dots, \zeta_N^0, \mathbf{u}_1^0, \dots, \mathbf{u}_N^0 \in H^s(\mathbb{R}^d)$, $s > d/2 + 1$, satisfying the above, there exists $T > 0$ and a unique strong solution to our system in $C([0, T); H^s(\mathbb{R}^d)^{N(d+1)})$ with such initial data.

More detailed result in [Monjarret '14]

Large time well-posedness

$$\mathrm{U} \leftarrow \epsilon \mathrm{U} \qquad \partial_t \leftarrow \epsilon \partial_t \quad \Longrightarrow \quad \partial_t \mathrm{U} + \frac{1}{\epsilon}\mathrm{A}^x[\epsilon \mathrm{U}]\partial_x \mathrm{U} + \frac{1}{\epsilon}\mathrm{A}^y[\epsilon \mathrm{U}]\partial_y \mathrm{U} = \mathbf{0}$$

is well-posed in $H^s, s > \frac{d}{2} + 1$ on $t \in [0, T)$ provided ϵ is sufficiently small.

Uniform energy estimates

If $\mathrm{U} \in L^\infty([0, T); H^s)$ $(s > d/2 + 1)$ then there exists C_0, C_1 such that

$$\forall t \in [0, T], \quad \big\|\mathrm{U}\big\|_{H^s}(t) \leq C_0 \big\|\mathrm{U}\big\|_{H^s}(0) \exp(C_1 t).$$

$$\Big(\mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\Big)\Lambda^s \partial_t \mathrm{U} \; + \; \Big(\mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\Big)\Lambda^s \Big(\frac{1}{\epsilon}\mathrm{A}^{(0)} + \mathrm{A}^{(1)}[\mathrm{U}]\Big)\partial_x \mathrm{U} \; = \; 0.$$

with $\Lambda^s = (\mathrm{Id} - \partial_x^2)^{s/2}$ and $d = 1$. The L^2 inner product with $\Lambda^s \mathrm{U}$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\big(\mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\big)\Lambda^s \mathrm{U}, \Lambda^s \mathrm{U}\Big) &= \frac{1}{2}\Big(\big[\partial_t, \mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\big]\Lambda^s \mathrm{U}, \Lambda^s \mathrm{U}\Big) \\ &+ \Big(\big(\mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\big)\big[\Lambda^s, \frac{1}{\epsilon}\mathrm{A}^{(0)} + \mathrm{A}^{(1)}[\mathrm{U}]\big]\partial_x \mathrm{U}, \Lambda^s \mathrm{U}\Big) \\ &+ \frac{1}{2}\Big(\big[\partial_x, \big(\mathrm{S}^{(0)} + \mathrm{S}^{(1)}[\mathrm{U}]\big)\big(\frac{1}{\epsilon}\mathrm{A}^{(0)} + \mathrm{A}^{(1)}[\mathrm{U}]\big)\big]\Lambda^s \mathrm{U}, \Lambda^s \mathrm{U}\Big)\end{aligned}$$

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$$+ \Big(\big(\mathrm{S}^{(0)} + \epsilon \mathrm{S}^{(1)}[\mathrm{U}]\big)\big[\Lambda^s, \frac{1}{\epsilon}\mathrm{A}^{(0)} + \mathrm{A}^{(1)}[\mathrm{U}]\big]\partial_x \mathrm{U}, \Lambda^s \mathrm{U}\Big)$$
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Asymptotic behavior ($d=2$)

$$\begin{cases} \partial_t \zeta_n + \frac{1}{\epsilon}\sum_{i=n}^N \nabla\cdot(h_i^\epsilon\, \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \frac{1}{\epsilon}\sum_{i=1}^n \frac{g(\rho_i-\rho_{i-1})}{\rho_n}\nabla\zeta_n + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n = \mathbf{0} \end{cases} \qquad (h_n^\epsilon = \delta_n + \epsilon\zeta_n - \epsilon\zeta_{n+1})$$

1. As $\epsilon \to 0$, $\zeta_n \rightharpoonup 0$ and $\mathbf{u}_n \rightharpoonup \mathbf{u}_n^{\mathrm{E}}$ solution to

$$\nabla\cdot\mathbf{u}_n^{\mathrm{E}} = 0, \qquad \partial_t \mathbf{u}_n^{\mathrm{E}} + (\mathbf{u}_n^{\mathrm{E}}\cdot\nabla)\mathbf{u}_n^{\mathrm{E}} = -\nabla p_n \qquad \text{(E)}$$

2. We recover strong convergence on $[0,T)$ by adding $\zeta_n^{\mathrm{ac}}, \mathbf{u}_n^{\mathrm{ac}}$ solution to

$$\partial_t \zeta_n^{\mathrm{ac}} + \frac{1}{\epsilon}\sum_{i=n}^N \nabla\cdot(\delta_i\, \mathbf{u}_i^{\mathrm{ac}}) = 0 \qquad \partial_t \mathbf{u}_n^{\mathrm{ac}} + \frac{1}{\epsilon}\sum_{i=1}^n \frac{g(\rho_i-\rho_{i-1})}{\rho_n}\nabla\zeta_n^{\mathrm{ac}} = \mathbf{0} \quad \text{(AW)}$$

Rk 1 : See [Parisot, Vila '15] for robust numerical schemes.
Rk 2 : One could be more clever and obtain arbitrarily large time results.
Rk 3 : Euler's incompressible limit [Klainerman, Majda, Ukai, Asano *etc.* '80s]
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Mode decomposition

After non-dimensionalization, the multi-layer Saint-Venant system reads

$$\begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \varrho^{-1} \sum_{i=1}^{N} \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \zeta_n + \sum_{i=n}^{N} \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \varrho^{-1} \gamma_n^{-1} \nabla \frac{\zeta_1}{\varrho} + \gamma_n^{-1} \sum_{i=2}^{n} r_i \nabla \zeta_i + (\mathbf{u}_n \cdot \nabla) \mathbf{u}_n = \mathbf{0} \end{cases} \tag{FS}$$

with $h_n = \delta_n + \zeta_n - \zeta_{n+1}$, where $\delta_n, r_n \subset (0, \infty)$, $\gamma_n \approx 1$ and $\boxed{\varrho \ll 1}$.

Claim : As $\varrho \to 0$, one can approach the solution as the superposition of a "*fast mode*" and a "*slow mode*".

Mode decomposition

Slow mode : ~~Free Surface~~ Rigid Lid system with Boussinesq approx.

$$\begin{cases} \cancel{\partial_t \frac{\zeta_1}{\varrho}} + \varrho^{-1} \sum_{i=1}^{N} \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \zeta_n + \sum_{i=n}^{N} \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \cancel{\varrho^{-1} \gamma_n^{-1} \nabla \frac{\zeta_1}{\varrho}} + \nabla p + \cancel{\gamma_n^{-1}}\, 1 \sum_{i=2}^{n} r_i \nabla \zeta_i + (\mathbf{u}_n \cdot \nabla) \mathbf{u}_n = \mathbf{0} \end{cases} \quad \text{(RL)}$$

with $h_1 = \delta_1 + \cancel{\varrho \frac{\zeta_1}{\varrho}} - \zeta_2$, $h_n = \delta_n + \zeta_n - \zeta_{n+1}$

Fast mode : Acoustic wave system

$$\begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \varrho^{-1} \nabla \cdot \left(\sum_{i=1}^{N} h_i \mathbf{u}_i \right) = 0 \\ \partial_t \left(\sum_{i=1}^{N} h_i \mathbf{u}_i \right) + \varrho^{-1} \left(\sum_{i=1}^{N} \delta_i \right) \nabla \frac{\zeta_1}{\varrho} = \mathbf{0} \end{cases} \quad \text{(AW)}$$

Mode decomposition

Slow mode : ~~Free Surface~~ Rigid Lid system with Boussinesq approx.

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Main result

$$\begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \varrho^{-1} \sum_{i=1}^N \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \zeta_n + \sum_{i=n}^N \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \varrho^{-1} \gamma_n^{-1} \nabla \frac{\zeta_1}{\varrho} + \gamma_n^{-1} \sum_{i=2}^n r_i \nabla \zeta_i + (\mathbf{u}_n \cdot \nabla) \mathbf{u}_n = \mathbf{0} \end{cases} \tag{FS}$$

with $h_n = \delta_n + \zeta_n - \zeta_{n+1}$, where $\delta_n, r_n \subset (0, \infty)$, $\gamma_n \approx 1$ and $\boxed{\varrho \ll 1}$.

Main results

Let $\zeta_n^0, \mathbf{u}_n^0 \in H^s$ $(s > 1 + \frac{d}{2})$ such that $h_n \geq h_0 > 0$ and $\left| \frac{\zeta_1^0}{\varrho}, \zeta_n^0, \mathbf{u}_n^0 \right|_{H^s} \leq M$.
There exists $\nu > 0$ such that if $\left| \mathbf{u}_n - \mathbf{u}_{n-1} \right|_{L^\infty} < \nu$, then

1. There exists $T(M, h_0^{-1}) > 0$ and a unique strong solution $U_\varrho \in C([0, T]; H^s)$.
2. As $\varrho \to 0$, $(\zeta_{n,\varrho}, \mathbf{u}_{n,\varrho})$ converges weakly towards a solution to (RL).
3. If $\left| \nabla \frac{\zeta_1}{\varrho} \right|_{H^s} + \left| \sum_{i=1}^N \nabla \cdot (h_i \mathbf{u}_i) \right|_{H^s} \leq \varrho M'$ initially, then the CV is strong.
4. We can construct $U_{\rm app}$=(RL)+(AW), such that $U_\varrho - U_{\rm app} \to 0$ strongly.

Step 1 : Change of variables

$$\begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \varrho^{-1} \sum_{i=1}^N \nabla\cdot(h_i \mathbf{u}_i) = 0 \\ \partial_t \zeta_n + \sum_{i=n}^N \nabla\cdot(h_i\mathbf{u}_i) = 0 \\ \\ \partial_t \mathbf{u}_n + \varrho^{-1}\gamma_n^{-1}\nabla\frac{\zeta_1}{\varrho} + \gamma_n^{-1}\sum_{i=2}^n r_i \nabla\zeta_i \\ \qquad + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n = \mathbf{0} \end{cases} \rightsquigarrow \begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \frac{1}{\varrho}\nabla\cdot\mathbf{w} = 0 \\ \partial_t\zeta_n + \sum_{i=n}^N \nabla\cdot(h_i\mathbf{u}_i) = 0 \\ \partial_t \mathbf{v}_n + r_n\nabla\zeta_n + [\gamma_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i]_{i=n-1}^{i=n} = \mathbf{0} \\ \partial_t \mathbf{w} + \frac{\sum\gamma_i^{-1}h_i}{\varrho}\nabla\frac{\zeta_1}{\varrho} + \sum f_i(\zeta)\nabla\zeta_i \\ \qquad + \sum_{i=1}^N \nabla\cdot(h_i\mathbf{u}_i\otimes\mathbf{u}_i) = \mathbf{0} \end{cases}$$

with $h_1 = \delta_1 + \varrho\frac{\zeta_1}{\varrho} - \zeta_2$, $h_n = \delta_n + \zeta_n - \zeta_{n+1}$. $\qquad \sum_{i=1}^N \gamma_i^{-1} h_i = \sum_{i=1}^N \delta_i + \mathcal{O}(\varrho)$

Define $V \stackrel{\text{def}}{=} (\frac{\zeta_1}{\varrho}, \zeta_n, \mathbf{v}_n, \mathbf{w})$ with $(n = 2, \dots, N)$

$$\mathbf{v}_n \stackrel{\text{def}}{=} \gamma_n\mathbf{u}_n - \gamma_{n-1}\mathbf{u}_{n-1} \quad \text{and} \quad \mathbf{w} \stackrel{\text{def}}{=} \sum_{n=1}^N h_n\mathbf{u}_n.$$

This the Saint-Venant system reads

$$\partial_t V + \frac{1}{\varrho}B_x^{(0)}\partial_x V + B_x^{(1)}[V]\partial_x V + \frac{1}{\varrho}B_y^{(0)}\partial_y V + B_y^{(1)}[V]\partial_y V = 0.$$

and satisfies rotational invariance:

$$\big(B_x^{(0)} + B_x^{(1)}[V]\big)\xi_x + \big(B_y^{(0)} + B_y^{(1)}[V]\big)\xi_y = Q(\xi)^{-1}\big(B_x^{(0)} + B_x^{(1)}[Q(\xi)V]\big)Q(\xi)|\xi|$$

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Step 2 : Symmetrization

The Saint-Venant system reads

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where

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Define $\Pi^{\rm slow}$ such that

$$\Pi^{\rm slow} B_x^{(0)} = \mathcal{O}(\varrho).$$

A "good" symmetrizer

There exists $S_x[V]$ such that

- $S_x[V]$ and $S_x[V]\big(\frac{1}{\varrho} B_x^{(0)} + B_x^{(1)}[V]\big)$ are symmetric;
- $S_x[V] \geq c\, \mathrm{Id}$, provided

$$h_n \geq h_0 > 0 \quad \text{and} \quad |\mathbf{v}_n| \leq \nu. \tag{H}$$

- $S_x[V] = S_x^{(0)} + S_x^{(1)}[V]$ with

$$S_x^{(1)}[V](\mathrm{Id} - \Pi^{\rm slow}) = \mathcal{O}(\varrho) \quad \text{and} \quad S_x^{(1)}[V] - S_x^{(1)}[\Pi^{\rm slow} V] = \mathcal{O}(\varrho).$$

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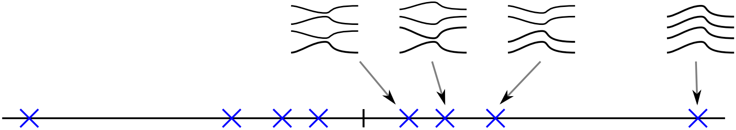
- $\frac{1}{\varrho} B_x^{(0)}$ has $2N$ distinct non-zero eigenvalues
$$-\lambda_1 < \cdots < -\lambda_N < 0 < \lambda_N < \cdots < \lambda_1,$$
and an orthogonal null-space of dimension N, range(Π_y).



The spectral projections $P_{\pm n}$ are smooth and bounded as $\varrho \to 0$.

- Perturbation :
$$\frac{1}{\varrho} B_x[V] = \frac{1}{\varrho} B_x^{(0)} + \mathbf{u}\mathrm{Id} + \delta B_x^{\mathrm{f}}[V](\mathrm{Id} - \Pi^{\mathrm{slow}}) + \delta B_x^{\mathrm{s}}[\Pi^{\mathrm{slow}} V] + \mathcal{O}(\varrho)$$
If $\Pi^{\mathrm{slow}} V$ is sufficiently small, then the eigenvalues remain separated. Moreover, $P_{\pm 1}[V]$ and $\sum_{n=2}^{N} P_n[V] + P_{-n}[V]$ deviate only by $\mathcal{O}(\varrho)$.

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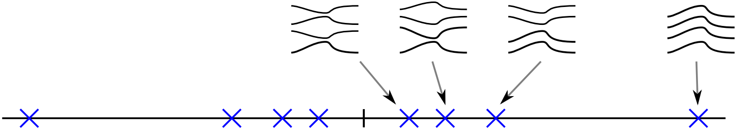

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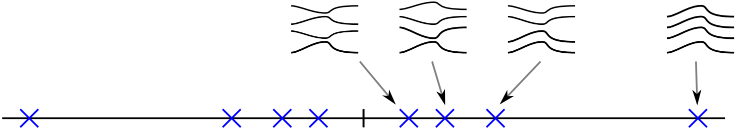

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Step 3 : Energy estimates $(d=1)$

$$\Big(S^{(0)}+S^{(1)}[V]\Big)\Lambda^s\partial_t V \;+\; \Big(S^{(0)}+S^{(1)}[V]\Big)\Lambda^s\Big(\frac{1}{\varrho}B^{(0)}+B^{(1)}[V]\Big)\partial_x V \;=\; 0.$$

with $\Lambda^s=(\mathrm{Id}-\partial_x^2)^{s/2}$ and

$$\Pi^{\rm slow}B^{(0)}=\mathcal{O}(\varrho)\quad;\quad S^{(1)}[V](\mathrm{Id}-\Pi^{\rm slow})=\mathcal{O}(\varrho)\quad;\quad S^{(1)}[V]-S^{(1)}[\Pi^{\rm slow}V]=\mathcal{O}(\varrho).$$

L^2 inner product with $\Lambda^s V$ yields

$$\frac{1}{2}\frac{d}{dt}\Big(\big(S^{(0)}+S^{(1)}[V]\big)\Lambda^s V,\Lambda^s V\Big)=$$

$$\Big(\big(S^{(0)}+S^{(1)}[V]\big)\big[\Lambda^s,\frac{1}{\varrho}B^{(0)}+B^{(1)}[V]\big]\partial_x V,\Lambda^s V\Big)$$

$$+\frac{1}{2}\Big(\big[\partial_x,\big(S^{(0)}+S^{(1)}[V]\big)\big(\frac{1}{\varrho}B^{(0)}+B^{(1)}[V]\big)\big]\Lambda^s V,\Lambda^s V\Big)$$

$$+\frac{1}{2}\Big(\big[\partial_t,S^{(0)}+S^{(1)}[V]\big]\Lambda^s V,\Lambda^s V\Big).$$

$\Longrightarrow$ if $V(t=0)\in H^s$, $s>1+d/2$ and satisfies (H), then

$$\forall t\in[0,T],\quad \big\|V\big\|_{H^s}(t)\le C_0\big\|V\big\|_{H^s}(0),\qquad\text{and}\qquad T^{-1}\le C_0\big\|V\big\|_{H^s}(0).$$

Step 3 : Energy estimates $(d = 1)$

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with $\Lambda^s = (\mathrm{Id} - \partial_x^2)^{s/2}$ and

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&\Big(\big(S^{(0)} + S^{(1)}[V]\big)\Big[\Lambda^s, \frac{1}{\varrho} B^{(0)} + B^{(1)}[V]\Big]\partial_x V, \Lambda^s V\Big)\\
&+ \frac{1}{2}\Big(\Big[\partial_x, \big(S^{(0)} + S^{(1)}[V]\big)\big(\frac{1}{\varrho} B^{(0)} + B^{(1)}[V]\big)\Big]\Lambda^s V, \Lambda^s V\Big)\\
&+ \frac{1}{2}\Big(\big[\partial_t, S^{(0)} + S^{(1)}[V]\big]\Lambda^s V, \Lambda^s V\Big).
\end{aligned}$$

$\Longrightarrow$ if $V(t=0) \in H^s$, $s > 1 + d/2$ and satisfies (H), then

$$\forall t \in [0, T], \quad \big\|V\big\|_{H^s}(t) \le C_0 \big\|V\big\|_{H^s}(0), \qquad \text{and} \qquad T^{-1} \le C_0 \big\|V\big\|_{H^s}(0).$$

Step 3 : Energy estimates $(d=1)$

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$$\forall t\in[0,T],\quad \|V\|_{H^s}(t)\le C_0\|V\|_{H^s}(0),\qquad \text{and}\qquad T^{-1}\le C_0\|V\|_{H^s}(0).$$

Completion of the proof

Proposition (A priori estimate)

Let $s > 1 + d/2$ and $V, W \in C^0([0,T); H^s)$ be such that V satisfies (H) with $h_0, \nu > 0$ and $\partial_t V \in L^\infty((0,T)\times\mathbb{R})$, and

$$\partial_t W \;+\; \frac{1}{\varrho} B_x^{(0)}[V]\partial_x W \;+\; B_x^{(1)}[V]\partial_x W \;+\; \frac{1}{\varrho} B_y^{(0)}[V]\partial_y W \;+\; B_y^{(1)}[V]\partial_y W \;=\; R,$$

with $R \in L^1(0,T;H^s)$. Then one has for any $t \in [0,T]$,

$$\big\|W\big\|_{H^s}(t) \leq C_0 e^{C_1 t}\big\|W\big\|_{H^s}(0) + C_0\int_0^t e^{C_1(t-t')}\big\|R\big\|_{H^s}(t')\mathrm{d}t'$$

with $C_0, C_1 = C(h_0^{-1}, \big\|V\big\|_{L^\infty(0,T;H^s)}, \varrho\big\|\partial_t V\big\|_{L^\infty([0,T]\times\mathbb{R}^d)}, \big\|\Pi^{\rm slow}\partial_t V\big\|_{L^\infty([0,T]\times\mathbb{R}^d)})$.

1. Apply Proposition to $W = V$ solution to (FS) $\Rightarrow$ uniform energy estimate
   Standard blow-up criteria $\Rightarrow$ large time existence.
2. Banach-Alaoglu $\Rightarrow$ $V_\varrho \rightharpoonup V_0$. Passing to the limit in the equation (thanks to Aubin-Lions Lemma for nonlinear terms) $\Rightarrow$ V_0 satisfies (RL).
3. Apply Proposition to $W = \partial_t V \rightsquigarrow$ well-prepared initial data propagates.
   In particular, if V_ϱ is well-prepared, then $V_\varrho \longrightarrow V_0$.
4. Apply Proposition to $V - V_{\rm app}$, where we construct $V_{\rm app} \approx V^{RL} + V^{AW}$
   Strichartz estimate on $V^{AW} \rightsquigarrow$ coupling effects are weak $\rightsquigarrow$ $V - V_{\rm app} \to 0$.

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Completion of the proof

Proposition (A priori estimate)

Let $s > 1 + d/2$ and $V, W \in C^0([0,T); H^s)$ be such that V satisfies (H) with $h_0, \nu > 0$ and $\partial_t V \in L^\infty((0,T) \times \mathbb{R})$, and

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Main result

$$\begin{cases} \partial_t \frac{\zeta_1}{\varrho} + \varrho^{-1} \sum_{i=1}^N \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \zeta_n + \sum_{i=n}^N \nabla \cdot (h_i \mathbf{u}_i) = 0 \\ \partial_t \mathbf{u}_n + \varrho^{-1} \gamma_n^{-1} \nabla \frac{\zeta_1}{\varrho} + \gamma_n^{-1} \sum_{i=2}^n r_i \nabla \zeta_i + (\mathbf{u}_n \cdot \nabla) \mathbf{u}_n = \mathbf{0} \end{cases} \quad \text{(FS)}$$

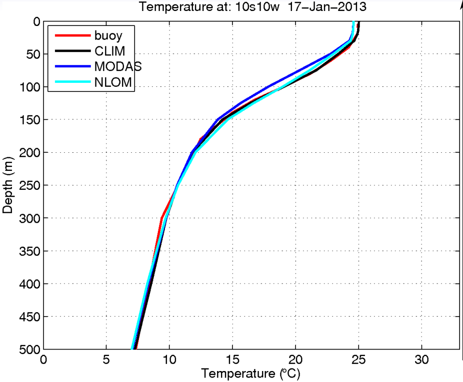
with $h_n = \delta_n + \zeta_n - \zeta_{n+1}$, where $\delta_n, r_n \subset (0, \infty)$, $\gamma_n \approx 1$ and $\boxed{\varrho \ll 1}$.

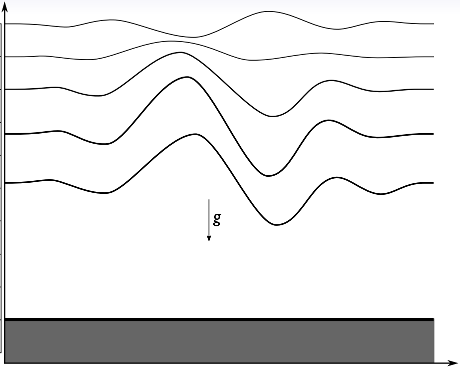
Main results

Let $\zeta_n^0, \mathbf{u}_n^0 \in H^s$ $(s > 1 + \frac{d}{2})$ such that $h_n \geq h_0 > 0$ and $\left| \frac{\zeta_1^0}{\varrho}, \zeta_n^0, \mathbf{u}_n^0 \right|_{H^s} \leq M$.
There exists $\nu > 0$ such that if $\left| \mathbf{u}_n - \mathbf{u}_{n-1} \right|_{L^\infty} < \nu$, then

1. There exists $T(M, h_0^{-1}) > 0$ and a unique strong solution $U_\varrho \in C([0, T]; H^s)$.
2. As $\varrho \to 0$, $(\zeta_{n,\varrho}, \mathbf{u}_{n,\varrho})$ converges weakly towards a solution to (RL).
3. If $\left| \nabla \frac{\zeta_1}{\varrho} \right|_{H^s} + \left| \sum_{i=1}^N \nabla \cdot (h_i \mathbf{u}_i) \right|_{H^s} \leq \varrho M'$ initially, then the CV is strong.
4. We can construct $U_{\rm app}$=(RL)+(AW), such that $U_\varrho - U_{\rm app} \to 0$ strongly.

Temperature vs depth





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In other words,

$$\begin{cases} \partial_t h_n + \nabla\cdot\big(h_n \mathbf{u}_n\big) = 0 \\ \partial_t \mathbf{u}_n + \gamma_n^{-1} \sum_{i=1}^n r_i \sum_{j=i}^N \nabla h_j + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n = \mathbf{0} \end{cases}$$

Formally, one obtains in the limit $N \to \infty$ (continuous stratification)

$$\begin{cases} \partial_t h_z + \nabla\cdot\big(h_z \mathbf{u}_z\big) = 0 \\ \partial_t \mathbf{u}_z + \gamma(z)^{-1} \mathcal{M}\nabla h_z + (\mathbf{u}_z\cdot\nabla)\mathbf{u}_z = \mathbf{0} \end{cases}$$

with $z \in [0,1]$ and

$$(\mathcal{M}\eta)(z) \stackrel{\text{def}}{=} \int_z^1 \frac{-\gamma'(z)}{\gamma(0)-\gamma(1)} \int_0^{z'} \eta(z'')\mathrm{d}z''\mathrm{d}z' + \frac{\gamma(1)}{\gamma(0)-\gamma(1)} \int_0^1 \eta(z')\mathrm{d}z'.$$

Related models

Our system is resembles the Benney system [Benney '73]

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Following [Grenier '96], notice that

$$f(t,x,v) \stackrel{\text{def}}{=} \int_0^1 h(t,x,z)\delta_{u(t,x,z)}(v)\,\mathrm{d}z$$

satisfies the "Vlasov-Dirac-Benney equations" [Bardos-Besse '13]

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \rho) \cdot \nabla_v f = 0$$

with $\rho(t,x) = \mathcal{M}h = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$.

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Well-posedness of the Vlasov-Dirac-Benney equations

WP for analytic data [Grenier '96] . Ill-posed in any Sobolev space.
WP for Sobolev data with the shape of a bump ($d = 1$) [Bardos, Besse '13]
WP for Penrose stable, Sobolev initial data [Han-Kwan, Rousset '15].

Hyperbolicity

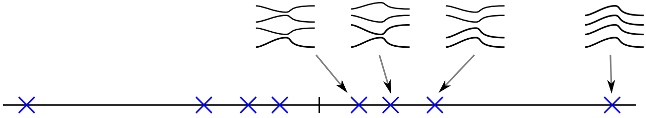
None of the previous approaches pass to the limit as $N \to \infty$.

1. The energy (Hamilton functional) is

$$E = \frac{1}{2}\int_{\mathbb{R}^d} C + \frac{\gamma(1)|\int_0^1 h_z|^2}{\gamma(0)-\gamma(1)} + \int_0^1 \frac{-\gamma'(z)}{\gamma(0)-\gamma(1)}\Big|\int_0^z h_{z'}\Big|^2 + \gamma(z)h_z\big|\mathbf{u}_z\big|^2,$$

but the Hessian D^2E is not positive definite [Abarbanel et al '86, Holm&Long '89, Ripa '90].

2. In absence of shear velocities, eigenvalues accumulate around $\overline{u}$.



However, by [Miles '61, Howard '61], the spectrum is real if

$$\frac{1}{4}\big|\partial_z \mathbf{u}_z\big|^2 < \frac{-\gamma'(z)}{\gamma(0)-\gamma(1)}.$$

⇝ Analytic (Gevrey?) well-posedness.

⇝ Use mass-exchange / capillarity / dispersion / artificial high-frequency cut-off.

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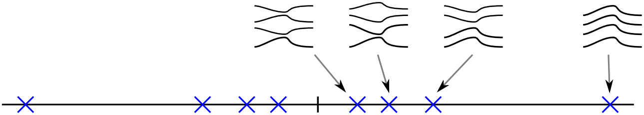

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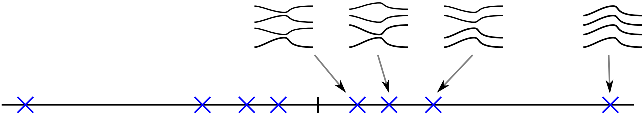

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Thank you for your attention !

Questions ? Ideas ?