Kelvin-Helmholtz instabilities in shallow water
Propagation of large amplitude, long wavelength, internal waves

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Internal gravity waves

Stratification, due to variation of salinity and temperature.

Figure: Temperature vs depth

1. Credits: Naval Research Laboratory
   http://www7320.nrlssc.navy.mil/global_nlom/
Internal gravity waves

Stratification, due to variation of salinity and temperature.

Figure: St. Lawrence Estuary

1. Credits: St. Lawrence Estuary Internal Wave Experiment (SLEIWEX)
http://myweb.dal.ca/kelley/SLEIWEX/index.php
Internal gravity waves

Stratification, due to variation of salinity and temperature.

Figure: Sulu Sea. April 8, 2003

1. Credits: NASA’s Earth Observatory (Picture of the Day July 1, 2003)
   http://earthobservatory.nasa.gov/IOTD/view.php?id=3586
1 Motivation
   - full Euler system
   - Kelvin-Helmholtz instabilities

2 Asymptotic models
   - Asymptotic models
   - Drawbacks
   - New systems

3 Numerical simulations

4 Well-posedness
The full Euler system

- Horizontal dimension $d = 1$, flat bottom, rigid lid.
- Irrotational, incompressible, inviscid, immiscible fluids.
- Fluids at rest at infinity, (small) surface tension.
The full Euler system

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- Irrotational, incompressible, inviscid, immiscible fluids.
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The full Euler system

\[ \mathbf{v}_1 = \nabla_{x,z} \phi_1 \quad \text{div} \, \mathbf{v}_1 = \Delta \phi_1 = 0 \]

\[ \partial_t \phi_1 + \frac{1}{2} |\nabla_{x,z} \phi_1|^2 = -\frac{P}{\rho_1} - gz \]

\[ [P] = -\sigma \, k(\zeta) \]

\[ \partial_t \zeta = \sqrt{1 + |\partial_x \zeta|^2} \partial_n \phi_1 = \sqrt{1 + |\partial_x \zeta|^2} \partial_n \phi_2 \]

\[ \mathbf{v}_2 = \nabla_{x,z} \phi_2 \quad \text{div} \, \mathbf{v}_2 = \Delta \phi_2 = 0 \]

\[ \partial_t \phi_2 + \frac{1}{2} |\nabla_{x,z} \phi_2|^2 = -\frac{P}{\rho_2} - gz \]

\[ \partial_z \phi_2 = 0 \]

\[ \zeta(t, x) \]

\[ d_1 \]

\[ d_2 \]

\[ x \]

The system can be rewritten as two coupled evolution equations in \( \zeta \) and \( \psi \equiv \phi_1|_{\text{interface}} \).

using Dirichlet-Neumann operators.
The full Euler system

The system can be rewritten as two coupled evolution equations in

\[ \zeta \quad \text{and} \quad \psi \equiv \phi_{1|\text{interface}}. \]

using Dirichlet-Neumann operators.

Full Euler system (Zakharov’s formulation)

\[
\begin{align*}
\frac{\partial_t \zeta}{\partial_t \nu} &= -\frac{\partial_x \delta H}{\partial \nu}, \\
\frac{\partial_t \nu}{\partial \zeta} &= -\frac{\partial_x \delta H}{\partial \zeta},
\end{align*}
\]

with \( H = \frac{1}{2} \int_{\mathbb{R}} g(\rho_2 - \rho_1) \zeta^2 \, dx + \frac{\rho_2}{2} \int_{\mathbb{R}} \int_{-d_2}^{\zeta} |\nabla \phi_2|^2 \, dz \, dx + \frac{\rho_1}{2} \int_{\mathbb{R}} \int_{\zeta}^{d_1} |\nabla \phi_1|^2 \, dz \, dx + \sigma \int_{\mathbb{R}} (\sqrt{1 + |\partial_x \zeta|^2} - 1). \)

This system is ill-posed without surface tension.

[Ebin ’88; Iguchi, Tanaka & Tani ’97; Kamotski & Lebeau ’05]
The full Euler system

The system can be rewritten as two coupled evolution equations in
\[
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Full Euler system (Zakharov’s formulation)

\[
\begin{align*}
\partial_t \zeta &= -\partial_x \frac{\delta \mathcal{H}}{\delta \nu}, \\
\partial_t \nu &= -\partial_x \frac{\delta \mathcal{H}}{\delta \zeta},
\end{align*}
\]

with \( \mathcal{H} = \frac{1}{2} \int_{\mathbb{R}} g(\rho_2 - \rho_1) \zeta^2 dx + \frac{\rho_2}{2} \int_{\mathbb{R}} \int_{-d_2}^{\zeta} |\nabla \phi_2|^2 dz dx + \frac{\rho_1}{2} \int_{\mathbb{R}} \int_{\zeta}^{d_1} |\nabla \phi_1|^2 dz dx \\
&\quad + \sigma \int_{\mathbb{R}} (\sqrt{1 + |\partial_x \zeta|^2} - 1).
\]

This system is ill-posed without surface tension.

[Ebin '88; Iguchi, Tanaka&Tani '97; Kamotski&Lebeau '05]
Kelvin-Helmholtz instabilities

Linearize the system around $\zeta = 0, v = v_0$, constant.

Linearized system [Lannes&Ming]

\[
\begin{align*}
\partial_t \zeta + c v_0(D) \partial_x \zeta + b(D) \partial_x v &= 0, \\
\partial_t v + a v_0(D) \partial_x \zeta + c v_0(D) \partial_x v &= 0,
\end{align*}
\]

(L)

with $b > 0$ and

\[
a v_0(D) = g(\rho_2 - \rho_1) - |v_0|^2 \frac{\rho_1 \rho_2}{\rho_2 \tanh(d_1 |D|) + \rho_1 \tanh(d_2 |D|)} |D| - \sigma \partial_x^2
\]

There are growing modes, $e^{i(kx-\omega(k)t)}$ with $\Im(\omega(k)) \neq 0$ iff $a v_0(k) < 0$, i.e.

\[
|v_0|^2 < \left( \frac{\tanh(d_1 |k|)}{\rho_1 |k|} + \frac{\tanh(d_2 |k|)}{\rho_2 |k|} \right) \left( g(\rho_2 - \rho_1) + \sigma |k|^2 \right).
\]

- Modes are stable for $|k|$ small iff $|v_0|^2 < \frac{\rho_2 d_1 + \rho_1 d_2}{\rho_1 \rho_2} g(\rho_2 - \rho_1)$.
- There are always unstable modes if $\sigma = 0$ and $\rho_1, |v_0| \neq 0$.
- All modes are stable if $\sigma \neq 0$ and $\rho_1 \rho_2 |v_0|^2$ is sufficiently small.
Kelvin-Helmholtz instabilities

Linearize the system around $\zeta = 0, v = v_0$, constant.

Linearized system [Lannes&Ming]

$$\begin{cases} 
\frac{\partial}{\partial t} \zeta + c v_0(D) \frac{\partial}{\partial x} \zeta + b(D) \frac{\partial}{\partial x} v = 0, \\
\frac{\partial}{\partial t} v + a v_0(D) \frac{\partial}{\partial x} \zeta + c v_0(D) \frac{\partial}{\partial x} v = 0,
\end{cases} \quad (L)$$

with $b > 0$ and

$$a v_0(D) = g(\rho_2 - \rho_1) - |v_0|^2 \frac{\rho_1 \rho_2}{\rho_2 \tanh(d_1|D|) + \rho_1 \tanh(d_2|D|)} |D| - \sigma \frac{\partial^2}{\partial x^2}$$

There are growing modes, $e^{ikx - \omega(k)t}$ with $\Re(\omega(k)) \neq 0$ iff $a v_0(k) < 0$, i.e.

$$|v_0|^2 < \left( \frac{\tanh(d_1|k|)}{\rho_1|k|} + \frac{\tanh(d_2|k|)}{\rho_2|k|} \right) \left( g(\rho_2 - \rho_1) + \sigma |k|^2 \right).$$

- Modes are stable for $|k|$ small iff $|v_0|^2 < \frac{\rho_2 d_1 + \rho_1 d_2}{\rho_1 \rho_2} g(\rho_2 - \rho_1)$.
- There are always unstable modes if $\sigma = 0$ and $\rho_1, |v_0| \neq 0$.
- All modes are stable if $\sigma \neq 0$ and $\rho_1 \rho_2 |v_0|^2$ is sufficiently small.
Kelvin-Helmholtz instabilities

When surface tension is present, all the modes are stable provided

$$|v_0|^2 < |v_{\text{min}}|^2 \approx 2\frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sqrt{\sigma g (\rho_2 - \rho_1)}.$$
When surface tension is present, all the modes are stable provided

\[ |v_0|^2 < |v_{\text{min}}|^2 \approx 2 \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sqrt{\sigma g (\rho_2 - \rho_1)}. \]

**Nonlinear generalization of this criterion:** [Lannes ’13]
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Asymptotic models
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Numerical simulations

Well-posedness
Asymptotic models may be constructed from asymptotic expansions of the Dirichlet-Neumann operators, w. r. t. given dimensionless parameters.

\[ \epsilon \overset{\text{def}}{=} \frac{a}{d_1}, \quad \mu \overset{\text{def}}{=} \frac{d_1^2}{\lambda^2}, \quad \gamma \overset{\text{def}}{=} \frac{\rho_1}{\rho_2}, \quad \delta \overset{\text{def}}{=} \frac{d_1}{d_2}, \quad \text{Bo}^{-1} \overset{\text{def}}{=} \frac{\sigma}{g(\rho_2 - \rho_1)\lambda^2}. \]
Asymptotic models : examples

Precision $O(\mu)$

Shallow water (a.k.a Saint-Venant) system (+ surface tension)

$$\begin{cases}
\partial_t \zeta + \partial_x w &= 0, \\
\partial_t \left( \frac{h_1 + \gamma h_2}{h_1 h_2} w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) &= \frac{\gamma + \delta}{Bo} \partial_x^2 \left( \frac{\partial_x \zeta}{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \right),
\end{cases}$$

with $h_1 = 1 - \epsilon \zeta$ and $h_2 = \frac{1}{\delta} + \epsilon \zeta$ and

$$\frac{h_1 + \gamma h_2}{h_1 h_2} w \overset{\text{def}}{=} \frac{1}{h_2(t, x)} \int_{-\frac{1}{\delta}}^{\epsilon \zeta(t, x)} \partial_x \phi_2(t, x, z) \, dz - \frac{\gamma}{h_1(t, x)} \int_{\epsilon \zeta(t, x)}^{1} \partial_x \phi_1(t, x, z) \, dz.$$
Asymptotic models : examples

**Precision $O(\mu^2)$**

Green-Naghdi system [Miyata’85; Mal’tseva’89; Choi&Camassa’99]

\[
\begin{cases}
    \partial_t \zeta + \partial_x w = 0, \\
    \partial_t \left( \frac{h_1+\gamma h_2}{h_1 h_2} w + \mu Q[\zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\varepsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) = \mu \varepsilon \partial_x (R[\zeta, w])
\end{cases}
\]

with $h_1 = 1 - \varepsilon \zeta$ and $h_2 = \frac{1}{\delta} + \varepsilon \zeta$ and

\[
\frac{h_1 + \gamma h_2}{h_1 h_2} w \overset{\text{def}}{=} \frac{1}{h_2(t, x)} \int_{-\frac{1}{\delta}}^{\varepsilon(t,x)} \partial_x \phi_2(t, x, z) \, dz - \frac{\gamma}{h_1(t, x)} \int_{\varepsilon(t,x)}^{1} \partial_x \phi_1(t, x, z) \, dz.
\]

\[
Q[\zeta] w \overset{\text{def}}{=} -\frac{1}{3} \left( h_2^{-1} \partial_x \left( h_2^3 \partial_x (h_2^{-1} w) \right) + \gamma h_1^{-1} \partial_x \left( h_1^3 \partial_x (h_1^{-1} w) \right) \right),
\]

\[
R[\zeta, w] \overset{\text{def}}{=} \frac{1}{2} \left( \left( h_2 \partial_x (h_2^{-1} w) \right)^2 - \gamma \left( h_1 \partial_x (h_1^{-1} w) \right)^2 \right) + \frac{1}{3} \left( h_2^{-2} \partial_x \left( h_2^3 \partial_x (h_2^{-1} w) \right) - \gamma h_1^{-2} \partial_x \left( h_1^3 \partial_x (h_1^{-1} w) \right) \right).
\]
**Shallow water models and KH instabilities**

**Qn:** How well do shallow water models predict KH instabilities?
Shallow water models and KH instabilities

Qn : How well do shallow water models predict KH instabilities?

Figure : Stability domains for full Euler (green), Saint-Venant (purple) and Green-Naghdi (blue)

$$
\varepsilon^2 |w|_2^2 \propto \frac{1}{\sqrt{\mu |k|}} \propto \frac{|k|^2}{\mu Bo} \
$$
Shallow water models and KH instabilities

Qn : How well do shallow water models predict KH instabilities?
A : The classical GN model follows the same behavior [Choi&Camassa ’99]
However, whereas Saint-Venant underestimates, Green-Naghdi overestimates KH instabilities [Jo&Choi ’02; Lannes&Ming]

Qn : Can we do better?
A : [Nguyen&Dias ’08; Choi,Barros&Jo ’09; Cotter,Holm&Percival ’10; Boonkasame&Milewski’14; Lannes&Ming]

Strategy :
1. Change of unknowns
2. BBM trick
3. .../...
Shallow water models and KH instabilities

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Strategy :
1. Change of unknowns
2. BBM trick
3. .../...
Construction of our model

The original GN system

\[
\begin{aligned}
\partial_t \zeta + \partial_x w &= 0, \\
\partial_t \left( \frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu Q[\zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) &= \mu \epsilon \partial_x (R[\zeta, w]) \\
&+ \frac{\gamma + \delta}{Bo} \partial_x^2 \left( \frac{\partial_x \zeta}{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \right),
\end{aligned}
\]

with \(h_1 = 1 - \epsilon \zeta\) and \(h_2 = \frac{1}{\delta} + \epsilon \zeta\) and

\[
Q[\zeta] w \overset{\text{def}}{=} -\frac{1}{3} \left( h_2^{-1} \partial_x \left( h_2^3 \partial_x (h_2^{-1} w) \right) + \gamma h_1^{-1} \partial_x \left( h_1^3 \partial_x (h_1^{-1} w) \right) \right),
\]

\[
R[\zeta, w] \overset{\text{def}}{=} \frac{1}{2} \left( \left( h_2 \partial_x (h_2^{-1} w) \right)^2 - \gamma \left( h_1 \partial_x (h_1^{-1} w) \right)^2 \right) + \frac{1}{3} w \left( h_2^{-2} \partial_x \left( h_2^3 \partial_x (h_2^{-1} w) \right) - \gamma h_1^{-2} \partial_x \left( h_1^3 \partial_x (h_1^{-1} w) \right) \right).
\]
Construction of our model

The GN system with improved (and nonlocal!) frequency dispersion

\[
\begin{align*}
\partial_t \zeta + \partial_x w &= 0, \\
\partial_t \left( \frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}_F[\zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) &= \mu \epsilon \partial_x (\mathcal{R}_F[\zeta, w]) \\
&+ \frac{\gamma + \delta}{\text{Bo}} \partial_x^2 \left( \frac{\partial_x \zeta}{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \right),
\end{align*}
\]

with \( h_1 = 1 - \epsilon \zeta \) and \( h_2 = \frac{1}{\delta} + \epsilon \zeta \) and

\[
\begin{align*}
\mathcal{Q}_F[\zeta] w &\overset{\text{def}}{=} -\frac{1}{3} \left( h_2^{-1} \partial_x F_2^\mu \left( h_2^3 \partial_x F_2^\mu (h_2^{-1} w) \right) + \gamma h_1^{-1} \partial_x F_1^\mu \left( h_1^3 \partial_x F_1^\mu (h_1^{-1} w) \right) \right), \\
\mathcal{R}_F[\zeta, w] &\overset{\text{def}}{=} \frac{1}{2} \left( \left( h_2 \partial_x F_2^\mu (h_2^{-1} w) \right)^2 - \gamma \left( h_1 \partial_x F_1^\mu (h_1^{-1} w) \right)^2 \right) \\
&+ \frac{1}{3} \omega \left( h_2^{-2} \partial_x F_2^\mu \left( h_2^3 \partial_x F_2^\mu (h_2^{-1} w) \right) - \gamma h_1^{-2} \partial_x F_1^\mu \left( h_1^3 \partial_x F_1^\mu (h_1^{-1} w) \right) \right).
\end{align*}
\]

where \( F_i^\mu = F_i(\sqrt{\mu|D|}) \).

Notation (Fourier multiplier): \( \mathcal{F}_i^\mu u(\xi) = F_i(\sqrt{\mu|\xi|}) \hat{u}(\xi) \). \( \partial_x = iD \).
Construction of our model

The GN system with improved (and nonlocal !) frequency dispersion

\[
\begin{align*}
\partial_t \zeta + \partial_x w &= 0, \\
\partial_t \left( \frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu Q^F[\zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) &= \mu \epsilon \partial_x (R^F[\zeta, w]) \\
&+ \frac{\gamma + \delta}{Bo} \partial_x^2 \left( \frac{\partial_x \zeta}{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \right),
\end{align*}
\]

\[
Q^F[\zeta] w \overset{\text{def}}{=} -\frac{1}{3} \left( h_2^{-1} \partial_x F_i^\mu \left( h_2^3 \partial_x F_i^\mu (h_2^{-1} w) \right) + \gamma h_1^{-1} \partial_x F_1^\mu \left( h_1^3 \partial_x F_1^\mu (h_1^{-1} w) \right) \right),
\]

\[
R^F[\zeta, w] \overset{\text{def}}{=} \frac{1}{2} \left( \left( h_2 \partial_x F_2^\mu (h_2^{-1} w) \right)^2 - \gamma \left( h_1 \partial_x F_1^\mu (h_1^{-1} w) \right)^2 \right)
+ \frac{1}{3} w \left( h_2^{-2} \partial_x F_2^\mu \left( h_2^3 \partial_x F_2^\mu (h_2^{-1} w) \right) - \gamma h_1^{-2} \partial_x F_1^\mu \left( h_1^3 \partial_x F_1^\mu (h_1^{-1} w) \right) \right).
\]

where \( F_i^\mu = F_i(\sqrt{\mu|D|}) \).

Examples: \( F_i^\mu = \frac{1}{\sqrt{1 + \mu \theta_i |D|^2}} \) or \( F_i^\mu = \sqrt{\frac{3}{\delta - i \sqrt{\mu |D|} \tanh(\delta - i \sqrt{\mu |D|}) - \frac{3}{\delta - 2i \mu |D|^2}}. \)
Motivation

Asymptotic models

Numerical simulations

Well-posedness

Promotion…

Our systems

1. can be tuned to reproduce (formally) the formation of the Kelvin-Helmholtz instabilities, or to suppress them;

2. preserves natural quantities:

\[ I \overset{\text{def}}{=} \int_{-\infty}^{\infty} \zeta \, dx, \quad V_i \overset{\text{def}}{=} \int_{-\infty}^{\infty} h_i^{-1} w + \mu Q_i^F[\zeta] w \, dx, \]

\[ E \overset{\text{def}}{=} \int_{-\infty}^{\infty} (\gamma + \delta) \zeta^2 + \frac{2(\gamma + \delta)}{\mu \epsilon^2 \text{Bo}} (\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} - 1) + \frac{h_1 + \gamma h_2}{h_1 h_2} |w|^2 \]

\[ + \mu \frac{\gamma}{3} h_1^3 (\partial_x F_1^\mu \frac{w}{h_1})^2 + \mu \frac{1}{3} h_2^3 (\partial_x F_2^\mu \frac{w}{h_2})^2 \, dx \]

3. possesses symmetry groups

\[ x \mapsto x + \alpha , \quad t \mapsto t + \alpha , \quad (x, u_1, u_2) \mapsto (x + \alpha t, u_1 + \alpha, u_2 + \alpha) \]

4. enjoys a Hamiltonian structure.
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Well-posedness
Numerical scheme

Spatial discretization: spectral method

- \[ u(t, x) \approx \sum_j u(t, x_j) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h} \text{ with } x_j = jh. \]
- \[ F(D)u \approx \sum_j u(t, x_j)F(D) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h} = M_F(u(t, x_j))_j. \]

\[ \Rightarrow \text{exponential accuracy for smooth functions} \]

\[ 2^9 = 512 \text{ points gives machine precision } (10^{-18}) \]

Time evolution: High order Runge-Kutta scheme (Matlab’s ode45)

- Reasonable CFL condition \( \Delta t \leq Ch \)
- High order scheme = high accuracy (here, typically \( 10^{-10} \))
Numerical scheme

Spatial discretization: spectral method

- \( u(t, x) \approx \sum_j u(t, x_j) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h} \) with \( x_j = jh \).
- \( F(D)u \approx \sum_j u(t, x_j)F(D) \frac{\sin(\pi(x-x_j)/h)}{\pi(x-x_j)/h} = M_F(u(t, x_j))_j \).

\[ \Rightarrow \text{exponential accuracy for smooth functions} \]
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Strategy (ideas…) 

The system may be rewritten as

\[
\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ w \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ w \end{pmatrix} = \text{“l.o.t”}
\]

where

\[
\begin{align*}
a \bullet & \overset{\text{def}}{=} \left( a_0(\epsilon \zeta) - \epsilon^2 \tilde{a}_0(\epsilon \zeta)|w|^2 \right) \bullet - \frac{1}{B_0} \partial_x \left( a_1(\epsilon \zeta) \partial_x \bullet \right) + \mu \partial_x F_i^\mu \left( \tilde{a}_1(\epsilon \zeta) w^2 \partial_x F_i^\mu \bullet \right) \\
\end{align*}
\]

\[
\begin{align*}
b \bullet & \overset{\text{def}}{=} b_0(\epsilon \zeta) \bullet - \mu \partial_x F_i^\mu \left( b_1(\epsilon \zeta) \partial_x F_i^\mu \bullet \right) \\
c \bullet & \overset{\text{def}}{=} c_0(\epsilon \zeta) \bullet - \mu \partial_x F_i^\mu \left( c_1(\epsilon \zeta) \partial_x F_i^\mu \bullet \right)
\end{align*}
\]

The symmetrizer defines a natural energy:

\[
\vert(\zeta, w)\vert^2_{X_0} \overset{\text{def}}{=} \left( a \zeta, \zeta \right)_{L^2} + \left( b w, w \right)_{L^2}.
\]

Hyperbolicity conditions

If \( h_1 = 1 - \epsilon \zeta > h_0 \) and \( h_2 = \delta^{-1} + \epsilon \zeta > h_0 \), then \( a_0, a_1, b_0, b_1 > 0 \).
Strategy (ideas…)

The system may be rewritten as

\[
\begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix} \partial_t \begin{pmatrix}
\zeta \\
w
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
a & c
\end{pmatrix} \partial_x \begin{pmatrix}
\zeta \\
w
\end{pmatrix} = \text{“l.o.t”}
\]

where

\[
a \bullet \overset{\text{def}}{=} (a_0(\epsilon \zeta) - \epsilon^2 \tilde{a}_0(\epsilon \zeta)|w|^2) \bullet - \frac{1}{B_0} \partial_x (a_1(\epsilon \zeta) \partial_x \bullet) + \mu \partial_x F_i^\mu (\tilde{a}_1(\epsilon \zeta) w^2 \partial_x F_i^\mu \bullet)
\]

\[
b \bullet \overset{\text{def}}{=} b_0(\epsilon \zeta) \bullet - \mu \partial_x F_i^\mu (b_1(\epsilon \zeta) \partial_x F_i^\mu \bullet)
\]

\[
c \bullet \overset{\text{def}}{=} c_0(\epsilon \zeta) \bullet - \mu \partial_x F_i^\mu (c_1(\epsilon \zeta) \partial_x F_i^\mu \bullet)
\]

The symmetrizer defines a natural energy:

\[
\left| (\zeta, w) \right|^2_{X^0} \overset{\text{def}}{=} \left( a \zeta, \zeta \right)_{L^2} + \left( b w, w \right)_{L^2}
\]

Hyperbolicity conditions

If \( h_1 = 1 - \epsilon \zeta > h_0 \) and \( h_2 = \delta^{-1} + \epsilon \zeta > h_0 \), then \( a_0, a_1, b_0, b_1 > 0 \).
Strategy (ideas...)

\[
\begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix}
\partial_t \begin{pmatrix}
\zeta \\
w
\end{pmatrix}
+ \begin{pmatrix}
0 & 1 \\
a & c
\end{pmatrix}
\partial_x \begin{pmatrix}
\zeta \\
w
\end{pmatrix}
= \text{“l.o.t”}
\]

\[
\begin{align*}
\mathbf{a} & \overset{\text{def}}{=} (a_0(\epsilon \zeta) - \epsilon^2 \tilde{a}_0(\epsilon \zeta) |w|^2) \cdot - \frac{1}{B_0} \partial_x (a_1(\epsilon \zeta) \partial_x \cdot) + \mu \partial_x F_i^\mu (\tilde{a}_1(\epsilon \zeta) w^2 \partial_x F_i^\mu \cdot)
\end{align*}
\]

\[
\begin{align*}
\mathbf{b} & \overset{\text{def}}{=} b_0(\epsilon \zeta) \cdot - \mu \partial_x F_i^\mu (b_1(\epsilon \zeta) \partial_x F_i^\mu \cdot)
\end{align*}
\]

The symmetrizer defines a natural energy:

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Hyperbolicity conditions

If \( h_1 = 1 - \epsilon \zeta > h_0 \) and \( h_2 = \delta^{-1} + \epsilon \zeta > h_0 \), then \( a_0, a_1, b_0, b_1 > 0 \).

If \( F_i(\xi) \lesssim |\xi|^{-\sigma} \) and \( \epsilon^2 |w|^2 (1 + (\mu B_0)^{1-\sigma}) \) sufficiently small, then

\[
\left| (\zeta, w) \right|^2_{X_0} \approx \left| \zeta \right|^2_{L^2} + \frac{1}{B_0} \left| \partial_x \zeta \right|^2_{L^2} + |w|^2_{L^2} + \mu \left| \partial_x F_i^\mu w \right|^2_{L^2}.
\]
A priori estimates

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} \partial_t \begin{pmatrix}
zeta \\
\omega
\end{pmatrix} + \begin{pmatrix}
0 & a \\
a & c
\end{pmatrix} \partial_x \begin{pmatrix}
zeta \\
\omega
\end{pmatrix} = \text{"l.o.t"}
\]

Usual strategy: multiply the system with \( \Lambda^s = (1 + |D|^2)^{s/2} \) and use commutator estimates to control \( X^s \) norm, \( s > 1/2 + 1 \):

\[
\| (\zeta, \omega) \|_{X^s}^2 \approx |zeta|_{H^s}^2 + \frac{1}{Bo} |\partial_x zeta|_{H^s}^2 + |\omega|_{H^s}^2 + \mu |\partial_x F_i \partial^\alpha \omega|_{H^s}^2.
\]

Here, we cannot estimate

\[
\sim ([\Lambda^s, a] \partial_x \omega, \Lambda^s \zeta)_{L^2} \quad \text{and/or} \quad ([\Lambda^s, a] \partial_t \zeta, \Lambda^s \zeta)_{L^2}.
\]

Instead, work with

\[
\| (\zeta, \omega) \|_{EN}^2 \approx \sum |\partial^\alpha zeta|_{L^2}^2 + \frac{1}{Bo} |\partial_x \partial^\alpha zeta|_{L^2}^2 + |\partial^\alpha \omega|_{L^2}^2 + \mu |\partial_x F_i \partial^\alpha \omega|_{L^2}^2.
\]

where the sum is over \( |\alpha| \leq N \), \( \alpha \) multi-indices in space \textit{and time}.

Differentiate \( \alpha \) times, extract leading order system for \((\partial^\alpha zeta, \partial^\alpha \omega)\)

\[
\implies \text{control of } \| (\partial^\alpha zeta, \partial^\alpha \omega) \|_{X^0}.
\]
A priori estimates

\[ \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \partial_t \begin{pmatrix} \zeta \\ \omega \end{pmatrix} + \left( \begin{array}{cc} 0 & a \\ a & c \end{array} \right) \partial_x \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = \text{"l.o.t"} \]

Usual strategy: multiply the system with \( \Lambda^s = (1 + |D|^2)^{s/2} \) and use commutator estimates to control \( X^s \) norm, \( s > 1/2 + 1 \):

\[ |(\zeta, \omega)|^2_{X^s} \approx |\zeta|_{H^s}^2 + \frac{1}{Bo} |\partial_x \zeta|_{H^s}^2 + |\omega|_{H^s}^2 + \mu |\partial_x F^\mu_i \omega|_{H^s}^2. \]

Here, we cannot estimate

\[ \rightsquigarrow \left( [\Lambda^s, a] \partial_x \omega, \Lambda^s \zeta \right)_{L^2} \text{ and/or } \left( [\Lambda^s, a] \partial_t \zeta, \Lambda^s \zeta \right)_{L^2}. \]

Instead, work with

\[ |(\zeta, \omega)|_{\mathcal{E}N} \approx \sum |\partial^\alpha \zeta|_{L^2}^2 + \frac{1}{Bo} |\partial_x \partial^\alpha \zeta|_{L^2}^2 + |\partial^\alpha \omega|_{L^2}^2 + \mu |\partial_x F^\mu_i \partial^\alpha \omega|_{L^2}^2. \]

where the sum is over \( |\alpha| \leq N \), \( \alpha \) multi-indices in space \( \text{and time} \).

Differentiate \( \alpha \) times, extract leading order system for \( (\partial^\alpha \zeta, \partial^\alpha \omega) \)

\[ \implies \text{control of } |(\partial^\alpha \zeta, \partial^\alpha \omega)|_{X^0}. \]
Well-posedness

Assume that \((\zeta^0, w^0) \in E^N\) with \(N\) large enough (time-derivatives given by the system);

\[
|\epsilon \zeta^0|_{L^\infty} < \min \left\{ 1, \frac{1}{\delta} \right\}, \quad \epsilon^2 \sum_{|\alpha| \leq 1} \left( |\partial^\alpha w^0|_{L^\infty}^2 + (\mu \text{ Bo})^{1-\sigma} \sum_{|j| \leq M} \left| (\sqrt{\mu \partial_x F}_i^\mu)^j \partial^\alpha w^0 \right|_{L^\infty}^2 \right) < C(\epsilon \zeta^0).
\]

Then there exists \(C, T\) and a unique strong solution \((\zeta, w)\) to the Green-Naghdi system for \(t \in [0, T)\), and

\[
\forall t < T, \quad |(\zeta, w)|_{EN}(t) \leq C |(\zeta^0, w^0)|_{EN} \exp(Ct),
\]

and the hyperbolicity conditions remain satisfied.

Remarks:

- This result allows to fully justify the models w.r.t. the full Euler system [Lannes ‘13]
- \(C, T^{-1}\) are uniformly bounded with respect to \(\gamma \in [0, 1], \epsilon \in [0, 1], \mu \in [0, \mu_{\text{max}}]\), and also \(\sigma \in \mathbb{R}^+\).
- The time domain is \([0, T/\epsilon)\) if a stronger hyperbolicity condition is satisfied.
- If \(\sigma \geq 1\) or \(\mu = 0\), the result is also valid without surface tension (\(\text{Bo}^{-1} = 0\)), and

\[
|(\zeta, w)|_{EN} = |\zeta|_{H^N}^2 + |w|_{H^N}^2.
\]
Well-posedness

Well-posedness of the Green-Naghdi models

Assume that \((\zeta^0, w^0) \in E^N\) with \(N\) large enough (time-derivatives given by the system);

\[
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- The time domain is \([0, T/\epsilon]\) if a stronger hyperbolicity condition is satisfied.
- If \(\sigma \geq 1\) or \(\mu = 0\), the result is also valid without surface tension \((\Bo^{-1} = 0)\), and

\[
\left| (\zeta, w) \right|_{EN} = \zeta^2_{HN} + w^2_{HN}.
\]
Thank you for your attention!