Asymptotic models for internal waves

and the rigid-lid approximation

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Stratified fluids and internal waves

Stratification, due to variations of salinity and temperature.

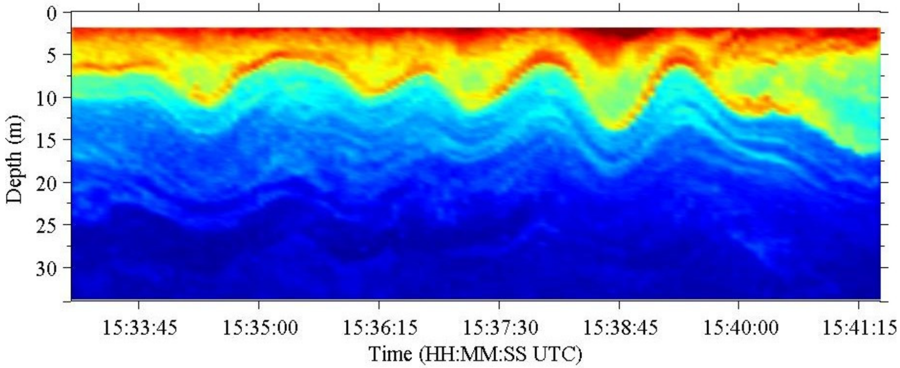
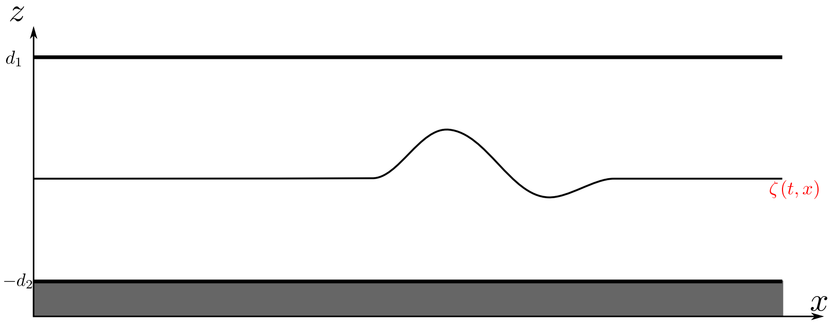


Figure : Internal wave in the St. Lawrence Estuary [1]

1. Credits : St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) http://myweb.dal.ca/kelley/SLEIWEX/index.php

The full Euler system

- Horizontal dimension $d = 1$, flat bottom, **rigid lid**.
- The domains are described by the graph of a function.
- Irrotational, incompressible, inviscid, immiscible fluids.
- Fluids at rest at infinity, no surface tension.



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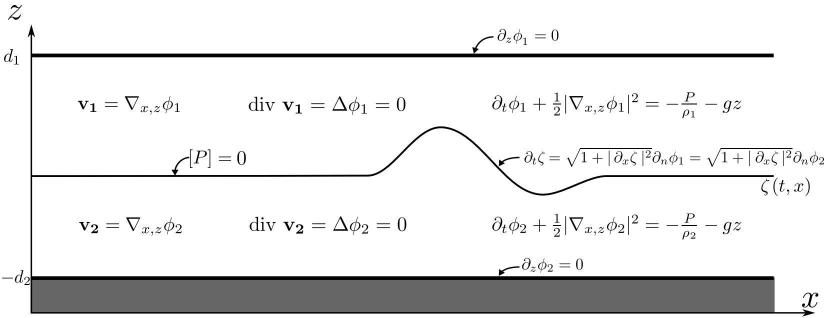

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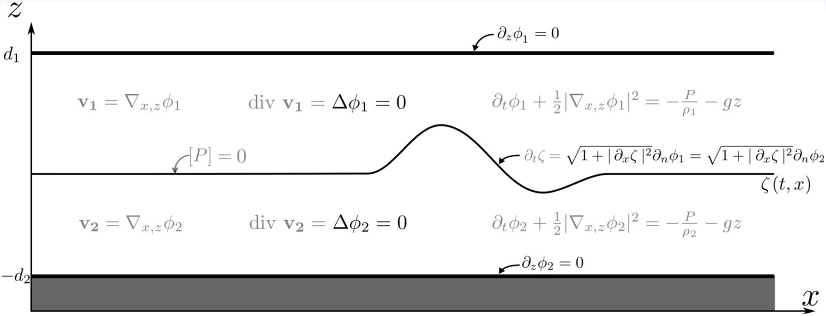


The system can be rewritten as two coupled evolution equations in

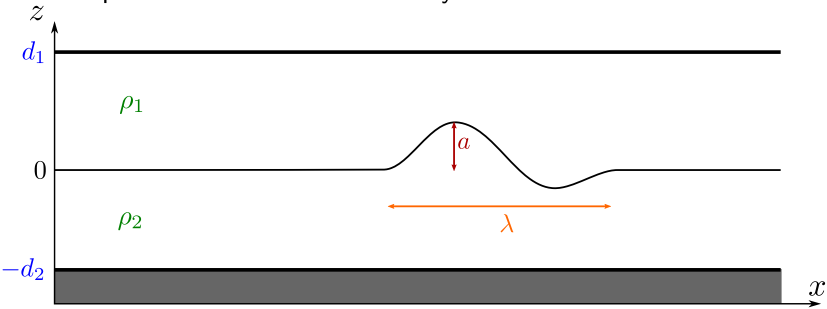
$$\zeta \quad \text{and} \quad \psi \equiv \phi_{2\,|\text{interface}}.$$

using Dirichlet-Neumann operators [Zakharov '68,Craig-Sulem-Sulem '92]

$$\mathcal{G}[\zeta]\psi \equiv \big(\ (\partial_n\phi_2)_{|\text{interface}}\ ,\ \phi_{1\,|\text{interface}}\ \big)$$

Asymptotic models

First step : non-dimensionalize the system



$$\epsilon \equiv \frac{a}{d_1}, \quad \mu \equiv \frac{{d_1}^2}{\lambda^2}, \quad \gamma \equiv \frac{\rho_1}{\rho_2}, \quad \delta \equiv \frac{d_1}{d_2}.$$

The linearized system around equilibrium is exactly solvable
$\rightsquigarrow$ natural scaling.

Full justification

The strategy for fully justifying a model is :

1. **Construction.** Asymptotic expansions of the Dirichlet-Neumann operators in the selected regime. [Bona-Lannes-Saut '08,VD '11, Xu'12...]
   Flattening of the domain, a priori expansion, elliptic estimates
   Ex : $\psi, \zeta \in H^{s+N} \Longrightarrow \big| \mathcal{G}^{\mu}[\epsilon\zeta]\psi - \mu\partial_x\big((1-\epsilon\zeta)\partial_x\psi\big)\big|_{H^s} \leq C\mu^2.$
   $\rightsquigarrow$ **consistency**
2. **Validation.** A priori control of solutions in some energy space
   $\rightsquigarrow$ **well-posedness, stability**

$\Longrightarrow$ Control of the difference between the exact and approximate solution (with corresponding initial data).

Important information include the rate of convergence, level of regularity and/or additional assumptions required, lifespan of solutions, *etc.*

Any approximate solution of the asymptotic model (in the sense of consistency) is close to the corresponding *exact* solution.

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Rigid lid approximation

Figure : Internal wave in the St. Lawrence Estuary [2]

Free Surface vs Rigid Lid.

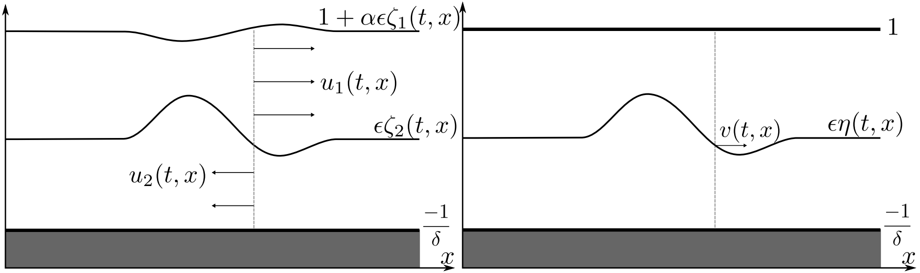
2. Credits : St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) http://myweb.dal.ca/kelley/SLEIWEX/index.php

Shallow water models

Assume $\mu \equiv \frac{{d_1}^2}{\lambda^2} \ll 1$ and neglect terms of size $\mathcal{O}(\mu)$. [3]

(equivalently, horizontal velocity is constant throughout the depth of each layer).

$\rightsquigarrow$ shallow water (Saint Venant, 1871) models



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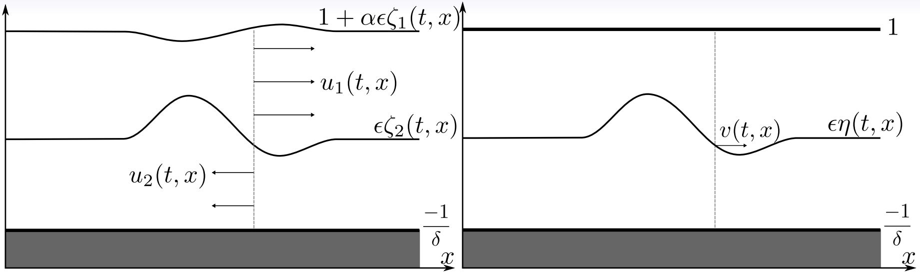


(FS)

$$\begin{aligned}
&\partial_t\zeta_1 + \frac{1}{\varrho}\Big(\partial_x(h_1u_1)+\partial_x(h_2u_2)\Big) = 0,\\
&\partial_t\zeta_2 + \partial_x(h_2u_2) = 0,\\
&\partial_t u_1 + \frac{1}{\varrho}\partial_x\zeta_1 + \frac{\epsilon}{2}\partial_x\left(|u_1|^2\right) = 0,\\
&\partial_t u_2 + (\delta+\gamma)\partial_x\zeta_2 + \frac{\gamma}{\varrho}\partial_x\zeta_1 + \frac{\epsilon}{2}\partial_x\left(|u_2|^2\right) = 0.
\end{aligned}$$

(RL)

$$\begin{aligned}
&\partial_t\eta + \partial_x\left(\frac{h_1h_2}{h_1+\gamma h_2}v\right) = 0,\\
&\partial_t v + (\gamma+\delta)\partial_x\eta + \frac{\epsilon}{2}\partial_x\left(\frac{h_1^2-\gamma h_2^2}{(h_1+\gamma h_2)^2}|v|^2\right) = 0.
\end{aligned}$$

$h_1 = 1-\epsilon\zeta_2(+\epsilon\varrho\zeta_1)$, $h_2 = \delta^{-1}+\epsilon\zeta_2$ are the upper and lower layer depths.

$$\varrho := \sqrt{\frac{1-\gamma}{\delta+\gamma}} \to 0$$

We want to compare the solution of these two models as $\gamma \to 1$ ($\varrho \to 0$).

Well-posedness results

Well-posedness of (RL) [Guyenne-Lannes-Saut'10]

Let $s > 3/2$, and $U^0 = (\eta^0, v^0)^\top \in H^s(\mathbb{R})^2$ s.t. $\exists h_0 > 0$ with

$$h_1^0, h_2^0 \geq h_0 \quad \text{and} \quad (\gamma+\delta)(h_1^0+\gamma h_2^0)^3 - \epsilon^2\gamma(h_1^0+h_2^0)^2|v^0|^2 \geq h_0.$$

Then there exists a unique, $T_{\max} > 0$ and

$$U_{\rm RL} = (\eta, v)^\top \in C([0, T_{\max}); H^s(\mathbb{R})^2) \cap C^1([0, T_{\max}); H^{s-1}(\mathbb{R})^2),$$

solution to (RL), with initial data $U_{\rm RL}(t=0,\cdot) = U^0$. Moreover, one has

$$T_{\max} \gtrsim 1/(\epsilon \big| U^0 \big|_{H^s(\mathbb{R})^2}).$$

Well-posedness of (FS) [VD, sub.]

Same as above, but :

$$h_1^0, h_2^0 \geq h_0 \quad \text{and} \quad (\gamma+\delta)h_2^0 - \epsilon^2|u_2^0 - u_1^0|^2 \geq h_0,$$

and
$$T_{\max} \gtrsim \varrho/(\epsilon \big| U^0 \big|_{H^s(\mathbb{R})^4}).$$

Long time result

Justification of rigid-lid assumption [VD, sub.]

Let $\zeta_1^0, \zeta_2^0, u_1^0, u_2^0 \in H^{s+1}(\mathbb{R})$ $(s > 3/2)$ satisfy additionally

$$\text{(H)} \qquad \begin{aligned} \epsilon\big|\zeta_2^0\big|_{H^{s+1}} \ + \ \epsilon\big|u_2^0 - u_1^0\big|_{H^{s+1}} \ &\leq \ M, \\ \epsilon\big|\zeta_1^0\big|_{H^{s+1}} \ + \ \epsilon\big|h_1 u_1^0 \ + \ h_2 u_2^0\big|_{H^{s+1}} \ &\leq \ M \ \varrho. \end{aligned}$$

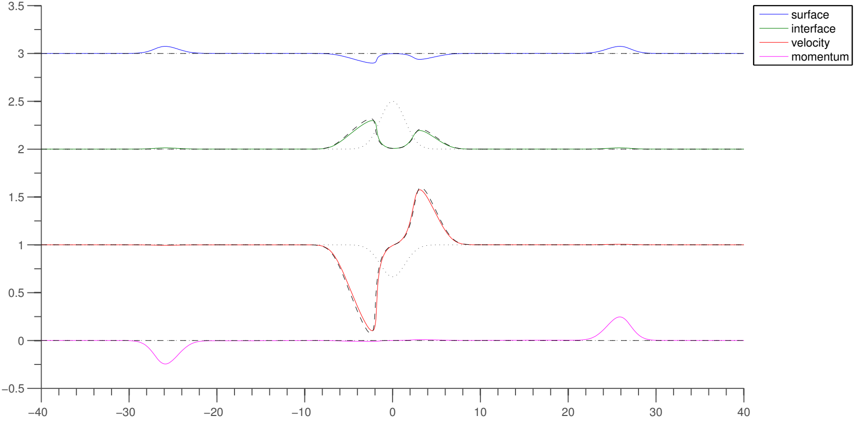
Then there exists $T, C > 0$ such that

1. There exists (η, v) a unique strong solution to (RL) with initial data $(\eta(t=0,\cdot) = \zeta_2^0,\ v(t=0,\cdot) = u_2^0 - \gamma u_1^0)$. Moreover, $T_{\max} \geq T/M$.
2. There exists $(\zeta_1, \zeta_2, u_1, u_2)$ a unique strong solution to (FS), with initial data $(\zeta_1^0, \zeta_2^0, u_1^0, u_2^0)$. Moreover, $T_{\max} \geq T/\max\{M, \varrho\}$.
3. Pour tout $0 \leq t < T/\max\{M, \varrho\}$,

$$\begin{aligned} \epsilon\big\|\eta - \zeta_2\big\|_{L^\infty([0,t];H^s)} + \epsilon\big\|v - (u_2 - \gamma u_1)\big\|_{L^\infty([0,t];H^s)} \ &\leq \ C \ M \ \varrho, \\ \epsilon\big\|\zeta_1\big\|_{L^\infty([0,t];H^s)} \ + \ \epsilon\big\|h_1 u_1 \ + \ h_2 u_2\big\|_{L^\infty([0,t];H^s)} \ &\leq \ C \ M \ \varrho. \end{aligned}$$

Numerical experiment

$$\forall t \in [0, T], \qquad \big\| V - V_{\mathrm{RL}} \big\|_{L^\infty([0,t];X^s)} \;\leq\; C\; M\; \varrho\;.$$



▸ Sketch of the proof
▸ Higher order approximation

Concluding remarks

We proved well-posedness and stability of the flow (for large time) predicted by Saint Venant models.

Can we rigorously justify the Saint Venant models, and the rigid lid assumption for the full Euler system in the shallow water regime ?

The full Euler system is ill-posed in Sobolev spaces !
[Ebin '88, Iguchi-Tanaka-Tani '97, Kamotski-Lebeau '05]
Discontinuity of the tangential velocity at the interface induces **Kelvin-Helmholtz instabilities**

The flow is regularized when

- the effect of surface tension is taken into account [Lannes '13] ;
- we replace the sharp interface with continuous stratification ;
- mixing is allowed [Audusse-Bristeau-Perthame-Sainte Marie '11].

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Thank you for your attention !

Sketch of the proof

1. **Change of variables.** Use $V \equiv (\zeta_1, \zeta_2, u_s = u_2 - \gamma u_1, m = \gamma h_1 u_1 + h_2 u_2)$.

$$\begin{cases} \partial_t \zeta_1 + \frac{1}{\varrho}\partial_x m + \frac{1-\gamma}{\gamma\varrho}\partial_x \left(h_1 \frac{m - h_2 u_s}{h_1 + h_2} \right) = 0, \\ \partial_t \zeta_2 + \partial_x \left(\frac{h_2}{h_1+h_2}(h_1 u_s + m) \right) = 0, \\ \partial_t u_s + (\delta + \gamma)\partial_x \zeta_2 + \dfrac{1}{2}\partial_x \left(\frac{\gamma(m+h_1 u_s)^2 - (m - h_2 u_s)^2}{\gamma(h_1+h_2)^2} \right) = 0, \\ \partial_t m + \gamma \frac{h_1+h_2}{\varrho}\partial_x \zeta_1 + (\gamma+\delta)h_2 \partial_x \zeta_2 + \partial_x \left(\frac{h_1(m-h_2 u_s)^2 + \gamma h_2 (m + h_1 u_s)^2}{\gamma(h_1+h_2)^2} \right) = 0. \end{cases}$$

$$\partial_t V \ + \ \big(\frac{1}{\varrho} L_\varrho + B[V]\big)\partial_x V \ = \ 0.$$

2. **Construction of an approximate solution.** $V = V^{\rm app} + W$.

$$\partial_t W \ + \ \big(\frac{1}{\varrho} L_\varrho + B[V^{\rm app} + W]\big)\partial_x W \ = \ \mathcal{O}(\varrho) \quad \text{and} \quad W\big|_{t=0} = \mathcal{O}(\varrho).$$

3. **Separation between "modes".**

Back

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Show that :

- Control of $V_{\rm app} \approx (0, \eta, v, 0)^\top + \mathcal{O}(\varrho)$ for $t \in [0, T/M]$;
- A priori control of W : bounded of size $\mathcal{O}(\varrho)$ on $[0, T/\max\{M, \varrho\}]$;
- Blow-up criterion $\Rightarrow V = V_{\rm app} + W$ well-defined on $[0, T/\max\{M, \varrho\}]$;
- $V - V^{\rm RL} = W + \mathcal{O}(\varrho) = \mathcal{O}(\varrho)$.

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3. **Separation between "modes".** Roughly speaking, the flow behaves as the superposition of two modes :
   - slow (or baroclinic) mode supported on variables ζ_2, v ;
   - fast (or barotropic) mode supported on variables ζ_1, m ;

   The heart of the matter is to prove that coupling effects between the two modes are small.

Back

Higher order approximate solution

$$\forall t \in [0, T], \qquad \left\| V - V^s_{\rm app} + V^f_{\rm app} \right\|_{L^\infty([0,t];X^s_{\rm ul})} \ \leq \ C \ M \ \varrho^2 \ .$$

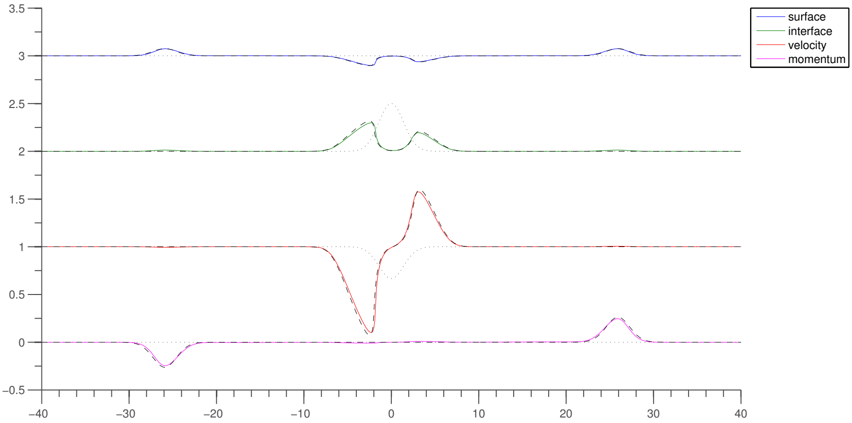

Figure : Comparison with improved approximation ◂ Back

Steps of the strategy

- **Slow mode approximate solution**

$$\exists V^s_{\text{app}} \equiv (\varrho\breve{\zeta}_1, \zeta_2, v, \varrho^2\breve{m}) \text{ satisfying (FS) with precision } \mathcal{O}(\varrho^2).$$

But : $\varrho\breve{\zeta}_1 - \zeta_1^0 = \mathcal{O}(\varrho), \varrho^2\breve{m} - m^0 = \mathcal{O}(\varrho)$.

- **Fast mode approximate solution**
  From initial data $(\zeta_1^0 - \varrho\breve{\zeta}_1, 0, 0, m^0)$,

$$\exists V^f_{\text{app}} \equiv u_+(x - \sqrt{1+\delta^{-1}}/\varrho) + u_-(x - \sqrt{1+\delta^{-1}}/\varrho)$$

  satisfying (FS) with precision $\mathcal{O}(\varrho^2)$.
  But : we have nonlinearities (coupling effects)

- **Control of coupling effects**
  Use (initial) localization in space :

$$(1+|\cdot|^2)^\sigma\zeta_1^0, (1+|\cdot|^2)^\sigma\zeta_2^0, (1+|\cdot|^2)^\sigma u_s^0, (1+|\cdot|^2)^\sigma m^0 \in H^s(\sigma > 1/2).$$

  Spatial localization persists with time.
  $\rightsquigarrow$ The two modes interact strongly only for $t = \mathcal{O}(\varrho)$.

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