One-dimensional scattering and localization properties of highly oscillatory potentials

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Dec 01, 2011
1 Introduction
- A motivation for our problem
- Scattering on the line
- The case of a highly oscillatory potential

2 Discontinuities in the potential
- Jump conditions and interface correctors
- A rigorous approach

3 Low energy analysis
- Generic and exceptional potentials
- Main result
- Consequences
Motivation

\[ u(x, z) e^{i \omega t} \]

\[ \partial_x^2 u(x, z) + \partial_z^2 u(x, z) + k^2 n_0^2 u(x, z) = k^2 (n_0^2 - n^2(x)) u(x, z), \]

\[ k = \frac{\omega}{c}, \quad n(x) \text{ the refractive index, and } n_0 \text{ the mean}. \]
Motivation

$u(x, z)$ satisfies the Helmholtz equation

$$\partial_x^2 u(x, z) + \partial_z^2 u(x, z) + k^2 n_0^2 u(x, z) = k^2 (n_0^2 - n^2(x)) u(x, z),$$

$k = \omega / c$, $n(x)$ the refractive index, and $n_0$ the mean.

Define $u(x, z) = F(x, z) e^{ikn_0 z}$ with

**Paraxial approximation**: $|2ikn_0 \partial_z F| \gg |\partial_x^2 F|$

yields the Schrödinger equation:

$$2ikn_0 \partial_z F = (-\partial_x^2 + k^2 (n_0^2 - n(x)^2)) F \equiv H F.$$

with solution

$$F = e^{-iz(2ikn_0)^{-1} H} F(x, 0).$$
Motivation

$u(x, z)$ satisfies the Helmholtz equation

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with solution

$$F = e^{-i z (2ikn_0)^{-1} H} F(x, 0).$$

We study the 1d, time-independent Schrödinger equation

$$(-\partial_x^2 + V(x) - k^2) \psi = 0,$$

(S) with $V$ a localized, highly oscillatory potential.
Scattering on the line

\[( -\partial_x^2 + V(x) - k^2 ) \psi = 0, \quad -\infty < x < \infty. \]

**Qn**: What can we say when the potential is highly oscillatory?

\[ V(x) \equiv q_\varepsilon(x) \equiv q(x, x/\varepsilon), \]

with \( x \mapsto q(x, \cdot) \) localized and \( y \mapsto q(\cdot, y) \) 1-periodic.
Scattering on the line

\[
( -\partial_x^2 + V(x) - k^2 ) \psi = 0, \quad -\infty < x < \infty.
\]

\( t(k) e^{i k x} \quad \text{and} \quad r(k) e^{-i k x} \)

\[ e^q(x; k) \quad \text{and} \quad e^{-q(x; k)} \]

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Homogenization

We seek the distorted plane waves of (S) under the form

\[ e^{q_\varepsilon}(x) \equiv F_\varepsilon(x, x/\varepsilon) \equiv F_0(x, x/\varepsilon) + \varepsilon F_1(x, x/\varepsilon) + \varepsilon^2 F_2(x, x/\varepsilon) + \ldots \]

Plug the Ansatz into equation

\[ \left( -\left( \frac{\partial}{\partial x} + \frac{1}{\varepsilon} \frac{\partial}{\partial y} \right)^2 + q(x, y) - k^2 \right) F_\varepsilon(x, y) = 0, \]

and solve at each order.

One obtains

- \( F_0(x, y) = e^{q_{av}}(x) \), satisfies \((-d^2/dx^2 + q_{av} - k^2)e^{q_{av}} = 0\), with \( q_{av}(x) \equiv \int_0^1 q(x, y) dy \);
- \( F_1 \equiv 0 \);
- \( F_2 \equiv F_2^{(h)}(x) + F_2^{(p)}(x, y) \), with
  - \( F_2^{(p)}(x, y) = -\frac{e^{q_{av}}(x)}{4\pi^2} \sum_{|j| \geq 1} \frac{q_j(x)}{j^2} e^{2i\pi jy} \), when \( q(x, y) = \sum_j q_j(x) e^{2i\pi jy} \),
  - \((-d^2/dx^2 + q_{av}(x) + \int_0^1 q(x, y) F_2^{(p)}(x, y) dy - k^2)F_2^{(h)}(x) = 0.\)
Homogenization

We seek the distorted plane waves of (S) under the form

\[ e_+^{q\varepsilon}(x) \equiv F^\varepsilon(x, x/\varepsilon) \equiv F_0(x, x/\varepsilon) + \varepsilon F_1(x, x/\varepsilon) + \varepsilon^2 F_2(x, x/\varepsilon) + \ldots \]

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Asymptotic expansion of the transmission coefficient.

\[
t^{q\varepsilon}(k) = t_0(k) + \varepsilon^2 t_2(k) + O(\varepsilon^3)
\]

where

- \( t_0 = t^{q_{av}}(k) \), the transmission coefficient of \( q_{av} \);
- \( t_2 \) depends on \( q \) and \( k \), but not on \( \varepsilon \).
The homogenization fails in the two following cases:

1. \( x \mapsto q(x, \cdot) \) is discontinuous.

2. \( k \ll 1 \) and \( q_{av} \equiv 0 \).

   If \( q_{av} \equiv 0 \), then \( t^{q_{av}}(k) = 1 \), for any \( k \) (exceptional!)

   In the generic case, \( t^V(k) \to 0 \) when \( k \to 0 \).

   \[ \implies t^{q_\varepsilon}(k) = t^{q_{av}}(k) + \varepsilon^2 t_2(k) + O(\varepsilon^3) \] is not uniform in \( k \).
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$$\Rightarrow t^{q_{\varepsilon}}(k) = t^{q_{av}}(k) + \varepsilon^2 t_2(k) + \mathcal{O}(\varepsilon^3)$$

is not uniform in $k$. 

\[\begin{array}{c|c|c|c|c|c|c}
\varepsilon & 0.05 & 0.1 & 0.2 \\
\hline
|t^{q_{\varepsilon}}(k)| & 1 & 1 & 1 \\
\end{array}\]
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Discontinuities in the potential

$x \mapsto q(x, x/\varepsilon)$ has discontinuities at

$x_1 < x_2 < \cdots < x_n$.

Jump conditions. Any solution $\psi$ of (S) satisfies

$$\left[ \frac{d}{dx} \psi \right]_x = [\psi]_x = 0, \quad \forall x \in \mathbb{R},$$

where $[\psi]_x \equiv \psi(x^+) - \psi(x^-)$.

Interface correctors. In the homogenization expansion, one can introduce interface correctors, of the form

$$\psi_a(x) \equiv \begin{cases} \alpha \psi_-(x; k) & \text{if } x < a, \\ \beta \psi_+(x; k) & \text{if } x > a, \end{cases}$$

with

$$\begin{cases} \left( -\frac{d^2}{dx^2} + q_{av} - k^2 \right) \psi_\pm = 0, \\ \psi_\pm(x) \sim e^{\pm ikx}, \quad x \to \pm \infty. \end{cases}$$

Application to the transmission coefficient

$$t^\varepsilon(k) = t_0(k) + \varepsilon t_1^\varepsilon(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^\varepsilon(k) + \ldots$$
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t^{\varepsilon}(k) = t_0(k) + \varepsilon t_1^{\varepsilon}(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^{\varepsilon}(k) + \ldots
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**Application to the transmission coefficient**

\[
t_\varepsilon(k) = t_0(k) + \varepsilon t_1^\varepsilon(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^\varepsilon(k) + \ldots
\]

where

- \( t_1^\varepsilon(k) \) comes from discontinuities in \( x \mapsto q(x, \cdot) \);
- \( t_2^\varepsilon(k) \) comes from discontinuities in \( x \mapsto \partial_x q(x, \cdot) \).
A rigorous approach

\[ V = q_{av} + Q, \] with \( Q \) localized at high frequencies.

\[ |||Q||| \equiv \| \langle D \rangle^{-1} \chi^{-1} Q \chi^{-1} \langle D \rangle^{-1} \|_{L^2 \to L^2} \ll 1, \]

where \( \langle D \rangle^{s} \equiv \left(1 - \frac{d^2}{dx^2}\right)^{s/2} \) and \( \chi(x) \equiv (1 + x^2)^{-\sigma}, \sigma > 2. \)

Lippmann-Schwinger equation. \( e_{+}^{V} \), as a solution of (S), satisfies

\[ e_{+}^{V} = \left(I + \left(-\partial_{x}^2 + q_{av} - k^2\right)^{-1} Q\right)^{-1} e_{+}^{q_{av}} = \left(I + R_{V} Q\right)^{-1} e_{+}^{q_{av}} \]

\[ ' = ' e_{+}^{q_{av}} - R_{V} Q e_{+}^{q_{av}} + R_{V} Q R_{V} Q e_{+}^{q_{av}} + \ldots \]

Application to the transmission coefficient.

\[ t^{\varepsilon}(k) = t_0(k) + t_1[Q] + t_2[Q; Q] + \ldots \]
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**Lippmann-Schwinger equation.** \( e_V^+ \), as a solution of (S), satisfies

\[
e^V_+ = \left( I + \left( -\frac{\partial^2}{\partial x^2} + q_{av} - k^2 \right)^{-1} Q \right)^{-1} e^{q_{av}}_+ \equiv \left( I + R_V Q \right)^{-1} e^{q_{av}}_+
\]

\[ ' = ' e^{q_{av}}_+ - R_V Q e^{q_{av}}_+ + R_V Q R_V Q e^{q_{av}}_+ + \ldots \]

**Application to the transmission coefficient.**

\[ t^c(k) = t_0(k) + t_1[Q] + t_2[Q; Q] + \ldots \]
A rigorous approach

\[ V = q_{av} + Q, \text{ with } Q \text{ localized at high frequencies.} \]

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Lippmann-Schwinger equation. \( e^V_+ \), as a solution of \((S)\), satisfies

\[ \langle D \rangle \chi e^V_+ = (I + \langle D \rangle \chi \mathcal{R}_V \chi \langle D \rangle \langle D \rangle^{-1} \chi^{-1}Q\chi^{-1}\langle D \rangle^{-1})^{-1} \langle D \rangle \chi e^{q_{av}}_+ \]

\[ = \langle D \rangle \chi e^{q_{av}}_+ - \mathcal{R}_V Q \langle D \rangle \chi e^{q_{av}}_+ + \mathcal{R}_V Q \mathcal{R}_V Q \langle D \rangle \chi e^{q_{av}}_+ + \ldots \]

Application to the transmission coefficient.

\[ t^\varepsilon(k) = t_0(k) + t_1[Q] + t_2[Q; Q] + \ldots \]
A rigorous approach

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Lippmann-Schwinger equation. \( e^V_+ \), as a solution of (S), satisfies

\[ \langle D \rangle \chi e^V_+ = \left( I + \langle D \rangle \chi R_V \chi \langle D \rangle \langle D \rangle^{-1} \chi^{-1} Q \chi^{-1} \langle D \rangle^{-1} \right)^{-1} \langle D \rangle \chi e^{q_{av}}_+ \]

\[ = \langle D \rangle \chi e^{q_{av}}_+ - R_V Q \langle D \rangle \chi e^{q_{av}}_+ + R_V Q R_V Q \langle D \rangle \chi e^{q_{av}}_+ + \ldots \]

Application to the transmission coefficient.

\[ t^\varepsilon(k) = t_0(k) + t_1[Q] + t_2[Q; Q] + \ldots \]
Back to the periodic case

\[ V = q_{av} + Q, \text{ with } Q(x) \equiv q(x, x/\varepsilon). \]

\[ \implies |||Q||| = \mathcal{O}(\varepsilon). \]

\[ t^\varepsilon(k) = t_0(k) + t_1[Q] + t_2[Q; Q] + \ldots \]

where

- \[ t_0 = t^{q_{av}}(k), \text{ the transmission coefficient of } q_{av}; \]
- \[ t_1[Q] = \int f(x)Q(x)dx = \int f(x)q(x, x/\varepsilon)dx = \sum_j \int f(x)q_j(x)e^{ijx/\varepsilon} \]
  \[ \implies \varepsilon t_1^\varepsilon(k) + \varepsilon^2 t_2^\varepsilon(k) + \ldots \]
- \[ t_2[Q; Q] \approx \int g(x)Q(x)Q(x)dx = \sum_{j,k} \int f(x)q_j(x)q_k(x)e^{i(j+k)x/\varepsilon} \]
  \[ \implies \varepsilon^2 t_2(k) + \ldots \]

We recover

\[ t^\varepsilon(k) = t_0(k) + \varepsilon t_1^\varepsilon(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^\varepsilon(k) + \ldots \]
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Generic and exceptional potentials

\[ t^V(k) = \frac{2ik}{2ik - I^V(k)}, \quad I^V(k) \equiv \int_{-\infty}^{\infty} V(x)e^{-ikx}f_+^V(x; k)dx. \]

**Generic potential**: \( I^V(k) \to \gamma \neq 0 \), and \( t^V(k) \to 0 \).

**Exceptional case**: \( I^V(k) \to 0 \), and \( t^V(k) \to 0 \).

\( V \equiv 0 \) is exceptional!

Thus if \( q_{av} \equiv 0 \) (or more generally exceptional), the expansion

\[ t^{q_\varepsilon}(k) = t^{q_{av}}(k) + \varepsilon^2 t_2(k) + O(\varepsilon^3) \]

is not uniform in \( k \).
Generic and exceptional potentials

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1d scattering and localization properties of highly oscillatory potentials

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Volterra equations

The Jost solutions are uniquely defined as the solution of Volterra equations

\[ f_+^V(x; k) = e^{ikx} + \int_x^\infty \frac{e^{ik(y-x)} - e^{ik(x-y)}}{2ik} V(y)f_+^V(y)dy. \]

This can be generalized to

\[ f_+^V(x; k) = f_+^W(x; k) + \int_x^\infty \frac{f_+^W(x; k)f_-^W(y; k) - f_-^W(x; k)f_+^W(y; k)}{\text{Wron}[f_+^W(x; k), f_-^W(x; k)]} V(y)f_+^V(y)dy. \]

\[ \Rightarrow \quad \frac{k}{t^V(k)} = \frac{k}{t^W(k)} - \frac{1}{2i} l^{[V,W]}(k), \quad l^{[V,W]}(k) \equiv \int f_-^W(\cdot; k)(V-W)f_+^V(\cdot; k). \]

Our analysis uses mostly integration by parts on these identities, with well-chosen potentials.

- Requires \( q_\varepsilon \equiv q(x, x/\varepsilon) \), (almost-)periodic in the fast variable, and some regularity in the slow variable.
- Allows \( k \) to lie in a complex strip \( \Im(k) < \alpha \).
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\[ \Rightarrow \quad \frac{k}{t^V(k)} = \frac{k}{t^W(k)} - \frac{1}{2i} I_{[V,W]}(k), \quad I_{[V,W]}(k) \equiv \int f_-^W(\cdot; k)(V-W)f_+^V(\cdot; k). \]

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Introduction

Discontinuities in the potential

Low energy analysis

Main result

Convergence of the transmission coefficient

Assume \( q_\varepsilon = q(x, x/\varepsilon) = \sum_{j \neq 0} q_j(x) e^{2i\pi j x/\varepsilon} \) is smooth and exponentially decaying at infinity. Then there exists \( \varepsilon_0 > 0 \) and \( K \) a compact subset of \( \mathbb{C} \) such that \( (\varepsilon, k) \in [0, \varepsilon_0) \times K \), one has

\[
\left| \frac{k}{t^{\sigma_{\text{eff}}^\varepsilon}(k)} - \frac{k}{tq_\varepsilon(k)} \right| \leq \varepsilon^3 C(K, |V|),
\]

where \( \sigma_{\text{eff}}^\varepsilon \) is the effective potential well defined by

\[
\sigma_{\text{eff}}^\varepsilon(x) \equiv -\varepsilon^2 \Lambda_{\text{eff}}(x) \equiv -\frac{\varepsilon^2}{(2\pi)^2} \sum_{j \neq 0} |q_j(x)|^2 j^2.
\]
Consequences

\[
\frac{k}{t^{q\varepsilon}(k)} = k + \frac{\varepsilon^2}{2i} \int_{-\infty}^{\infty} \Lambda_{\text{eff}}(x) \, dx + \mathcal{O}(\varepsilon^3),
\]

This allows to expand \( t^{q\varepsilon}(k) \), apart from a shrinking subset around \( k^* \equiv i \frac{\varepsilon^2}{2} \int \Lambda_{\text{eff}}. \)

This is true in particular

- uniformly for \( k \in \mathbb{R} \): \( \sup_{k \in \mathbb{R}} |t^{\sigma_{\text{eff}}}(k) - t^{q\varepsilon}(k)| = \mathcal{O}(\varepsilon). \)
- if \( k = \varepsilon^2 \kappa, \kappa \neq i \frac{\int \Lambda_{\text{eff}}}{2} \):

\[
\lim_{\varepsilon \to 0} t^{q\varepsilon}(\varepsilon^2 \kappa) = \frac{\kappa}{\kappa - i \frac{\int \Lambda_{\text{eff}}}{2}}.
\]

This universal scaled limit is the transmission coefficient for a Dirac-distribution potential: \((-\partial_x^2 - \delta(x) \int \Lambda_{\text{eff}} - \kappa^2)\psi = 0.\)
Consequences
Introduction

Discontinuities in the potential

Low energy analysis

Consequences (continued)

\[
\frac{k}{t^{q_\varepsilon}(k)} = k + \frac{\varepsilon^2}{2i} \int_{-\infty}^{\infty} \Lambda_{\text{eff}}(x) dx + O(\varepsilon^3),
\]

t^{q_\varepsilon} has a pole in the upper half plane

\[
k_\varepsilon \approx i\frac{\varepsilon^2}{2} \int \Lambda_{\text{eff}} + O(\varepsilon^3).
\]

(using Rouché argument).

Edge bifurcation of point spectrum

\[H_{q_\varepsilon} \equiv (-\partial_x^2 + q_\varepsilon)\] has a point eigenvalue at energy

\[E_\varepsilon = k_\varepsilon^2 \approx -\frac{\varepsilon^4}{4} \left(\int \Lambda_{\text{eff}}\right)^2 + O(\varepsilon^5).\]
Consequences (continued)

Edge bifurcation of point spectrum

\( H_{q\varepsilon} \equiv (-\partial_x^2 + q\varepsilon) \) has a point eigenvalue at energy

\[
E_{\varepsilon} = k_{\varepsilon}^2 \approx -\frac{\varepsilon^4}{4} \left( \int \Lambda_{\text{eff}} \right)^2 + \mathcal{O}(\varepsilon^5).
\]

This indicates the existence of a solution \( u(x, z) e^{i\omega t} \), localized in \( x \), for a careful choice of \( k = \omega/c \).