One-dimensional scattering and localization properties of highly oscillatory potentials

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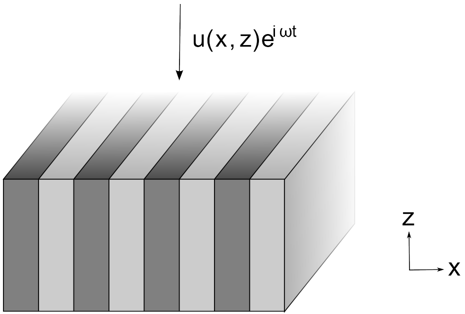
1. Introduction
   - A motivation for our problem
   - Scattering on the line
   - The case of a highly oscillatory potential

2. Discontinuities in the potential
   - Jump conditions and interface correctors
   - A rigorous approach

3. Low energy analysis
   - Generic and exceptional potentials
   - Main result
   - Consequences

Motivation



$u(x, z)$ satisfies the Hemholtz equation

$$\partial_x^2 u(x,z) \;+\; \partial_z^2 u(x,z) \;+\; k^2 n_0^2 u(x,z) \;=\; k^2(n_0^2 - n^2(x))u(x,z),$$

$k = \omega/c$, $n(x)$ the refractive index, and n_0 the mean.

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Define $u(x,z) = F(x,z)e^{ikn_0 z}$ with
Paraxial approximation : $|2ikn_0\partial_z F| \gg |\partial_z^2 F|$
yields the Schrödinger equation :

$$2ikn_0 \ \partial_z F \ = \ \left(-\partial_x^2 + k^2(n_0^2 - n(x)^2)\right) F \ \equiv \ H \ F.$$

with solution

$$F \ = \ e^{-iz(2ikn_0)^{-1}H} F(x,0).$$

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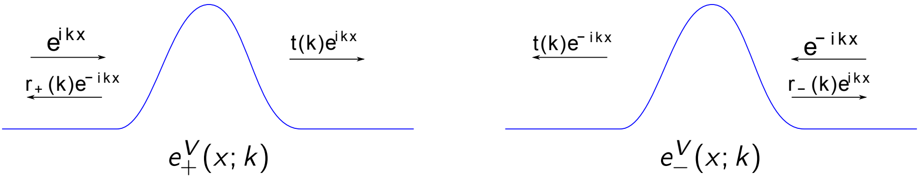
We study the 1d, time-independent Schrödinger equation

$$\left(-\partial_x^2 + V(x) - k^2\right)\psi \;=\; 0, \qquad \text{(S)}$$

with V a localized, highly oscillatory potential.

Scattering on the line

$$\left(-\partial_x^2 + V(x) - k^2 \right) \psi = 0, \qquad -\infty < x < \infty.$$



Qn : What can we say when the potential is highly oscillatory ?

$$V(x) \equiv q_\varepsilon(x) \equiv q(x, x/\varepsilon),$$

with $x \mapsto q(x, \cdot)$ localized and $y \mapsto q(\cdot, y)$ 1-periodic.

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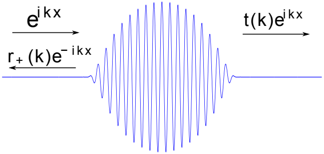
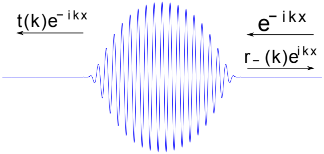


$e_+^{q_\varepsilon}(x;k)$



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Homogenization

We seek the distorted plane waves of (S) under the form

$$e_+^{q_\varepsilon}(x) \equiv F^\varepsilon(x, x/\varepsilon) \equiv F_0(x, x/\varepsilon) + \varepsilon F_1(x, x/\varepsilon) + \varepsilon^2 F_2(x, x/\varepsilon) + \dots$$

Plug the *Ansatz* into equation

$$\left(-\left(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y} \right)^2 + q(x,y) - k^2 \right) F^\varepsilon(x,y) = 0,$$

and solve at each order .

One obtains

- $F_0(x,y) = e_+^{q_{av}}(x)$, satisfies $(-\frac{d^2}{dx^2} + q_{av} - k^2)e_+^{q_{av}} = 0$, with $q_{av}(x) \equiv \int_0^1 q(x,y)dy$;
- $F_1 \equiv 0$;
- $F_2 \equiv F_2^{(h)}(x) + F_2^{(p)}(x,y)$, with
  - $F_2^{(p)}(x,y) = -\frac{e_+^{q_{av}}(x)}{4\pi^2}\sum_{|j|\geq 1}\frac{q_j(x)}{j^2}e^{2i\pi jy}$, when $q(x,y) = \sum_j q_j(x)e^{2i\pi jy}$,
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Asymptotic expansion of the transmission coefficient.

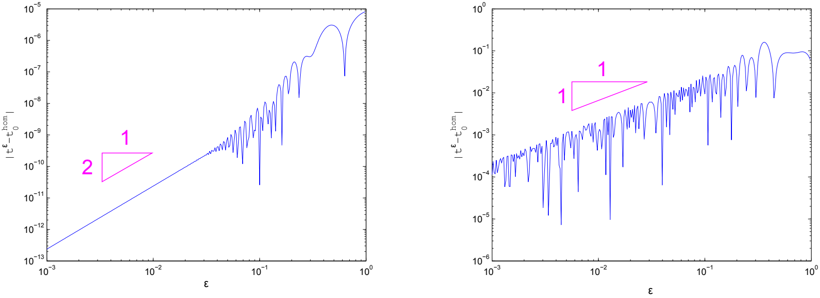
$$t^{q_\varepsilon}(k) \ = \ t_0(k) \ + \ \varepsilon^2 t_2(k) \ + \ \mathcal{O}(\varepsilon^3)$$

where

- $t_0 \ = \ t^{q_{av}}(k)$, the transmission coefficient of q_{av} ;
- t_2 depends on q and k, but not on ε.

The homogenization fails in the two following cases :

1. $x \mapsto q(x,\cdot)$ is discontinuous.



Smooth potential — Discontinuous potential

2. $k \ll 1$ and $q_{av} \equiv 0$.
   If $q_{av} \equiv 0$, then $t^{q_{av}}(k) = 1$, for any k (exceptional !)
   In the generic case, $t^V(k) \to 0$ when $k \to 0$.

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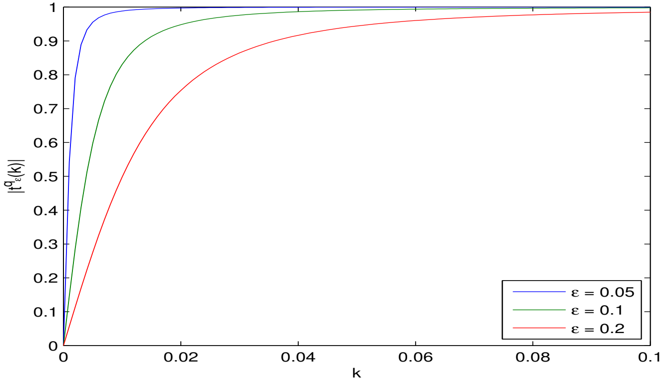

$x \mapsto q(x, x/\varepsilon)$ has discontinuities at $x_1 < x_2 < \cdots < x_n$.

Jump conditions. Any solution ψ of (S) satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\psi\right]_x = [\psi]_x = 0, \qquad \forall x \in \mathbb{R},$$

where $[\psi]_x \equiv \psi(x^+) - \psi(x^-)$.

Interface correctors. In the homogenization expansion, one can introduce *interface correctors*, of the form

$$\psi_a(x) \equiv \begin{cases} \alpha\ \psi_-(x;k) & \text{if } x < a, \\ \beta\ \psi_+(x;k) & \text{if } x > a, \end{cases} \quad \text{with} \quad \begin{cases} \left(-\frac{d^2}{dx^2} + q_{av} - k^2\right)\psi_\pm = 0, \\ \psi_\pm(x) \sim e^{\pm ikx}, \ x \to \pm\infty. \end{cases}$$

Application to the transmission coefficient

$$t^\varepsilon(k) = t_0(k) + \varepsilon t_1^\varepsilon(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^\varepsilon(k) + \ldots$$

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where

- $t_1^\varepsilon(k)$ comes from discontinuities in $x \mapsto q(x,\cdot)$;
- $t_2^\varepsilon(k)$ comes from discontinuities in $x \mapsto \partial_x q(x,\cdot)$.

A rigorous approach

$$V \;=\; q_{av} \;+\; Q, \text{ with } Q \text{ localized at high frequencies.}$$

$$|||Q||| \;\equiv\; \|\langle D\rangle^{-1}\chi^{-1}Q\chi^{-1}\langle D\rangle^{-1}\|_{L^2\to L^2} \;\ll\; 1,$$

where $\langle D\rangle^s \equiv \left(1-\frac{d^2}{dx^2}\right)^{s/2}$ and $\chi(x)\equiv(1+x^2)^{-\sigma},\ \sigma>2.$

Lippmann-Schwinger equation. e_+^V, as a solution of (S), satisfies

$$e_+^V \;=\; \Big(\, I + \big(-\partial_x^2 + q_{av} - k^2\big)^{-1} Q \,\Big)^{-1} e_+^{q_{av}} \;\equiv\; \big(\, I + R_V Q \,\big)^{-1} e_+^{q_{av}}$$

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$$t^\varepsilon(k) \;=\; t_0(k) \;+\; t_1[Q] \;+\; t_2[Q;Q] \;+\; \dots$$

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$$= \;\langle D\rangle\chi e_+^{q_{av}} - \mathcal{R}_V\mathcal{Q}\,\langle D\rangle\chi e_+^{q_{av}} + \mathcal{R}_V\mathcal{Q}\mathcal{R}_V\mathcal{Q}\,\langle D\rangle\chi e_+^{q_{av}} + \dots$$

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$$t^\varepsilon(k) \;=\; t_0(k) \;+\; t_1[Q] \;+\; t_2[Q;Q] \;+\; \dots$$

Back to the periodic case

$$V = q_{av} + Q, \text{ with } Q(x) \equiv q(x, x/\varepsilon).$$

$$\Longrightarrow \quad |||Q||| = \mathcal{O}(\varepsilon).$$

$$t^{\varepsilon}(k) = t_0(k) + t_1[Q] + t_2[Q;Q] + \ldots$$

where

- $t_0 = t^{q_{av}}(k)$, the transmission coefficient of q_{av} ;
- $t_1[Q] = \int f(x)Q(x)\mathrm{d}x = \int f(x)q(x,x/\varepsilon)\mathrm{d}x = \sum_j \int f(x)q_j(x)e^{ij\frac{x}{\varepsilon}}$
  $\hookrightarrow \varepsilon t_1^{\varepsilon}(k) + \varepsilon^2 t_2^{\varepsilon}(k) + \ldots$
- $t_2[Q;Q] \approx \int g(x)Q(x)Q(x)\mathrm{d}x = \sum_{j,k} \int f(x)q_j(x)q_k(x)e^{i(j+k)\frac{x}{\varepsilon}}$
  $\hookrightarrow \varepsilon^2 t_2(k) + \ldots$

We recover

$$t^{\varepsilon}(k) = t_0(k) + \varepsilon t_1^{\varepsilon}(k) + \varepsilon^2 t_2(k) + \varepsilon^2 t_2^{\varepsilon}(k) + \ldots$$

Generic and exceptional potentials

$$t^V(k) = \frac{2ik}{2ik - I^V(k)}, \qquad I^V(k) \equiv \int_{-\infty}^{\infty} V(x) e^{-ikx} f_+^V(x;k) dx.$$

Generic potential : $I^V(k) \to \gamma \neq 0$, and $t^V(k) \to 0$.

Exceptional case : $I^V(k) \to 0$, and $t^V(k) \to 0$.

$V \equiv 0$ is exceptional !

Thus if $q_{av} \equiv 0$ (or more generally exceptional), the expansion

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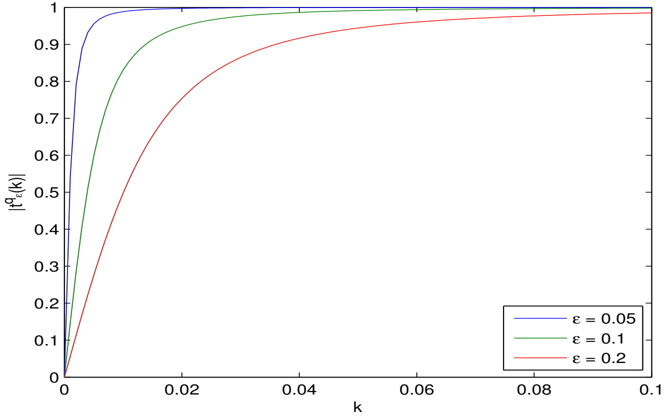

Volterra equations

The Jost solutions are uniquely defined as the solution of Volterra equations

$$f_+^V(x;k) \;=\; e^{ikx} \;+\; \int_x^\infty \frac{e^{ik(y-x)} - e^{ik(x-y)}}{2ik} V(y) f_+^V(y) dy.$$

This can be generalized to

$$f_+^V(x;k) \;=\; f_+^W(x;k) + \int_x^\infty \frac{f_+^W(x;k) f_-^W(y;k) - f_-^W(x;k) f_+^W(y;k)}{Wron[f_+^W(x;k), f_-^W(x;k)]} V(y) f_+^V(y) dy.$$

$$\Longrightarrow \;\; \frac{k}{t^V(k)} \;=\; \frac{k}{t^W(k)} - \frac{1}{2i} I^{[V,W]}(k), \quad I^{[V,W]}(k) \equiv \int f_-^W(\cdot;k)(V-W) f_+^V(\cdot;k).$$

Our analysis uses mostly integration by parts on these identities, with well-chosen potentials.

- Requires $q_\varepsilon \equiv q(x, x/\varepsilon)$, (almost-)periodic in the fast variable, and some regularity in the slow variable.
- Allows k to lie in a complex strip $\Im(k) < \alpha$.

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This can be generalized to

$$f_+^V(x;k) \; = \; f_+^W(x;k) + \int_x^\infty \frac{f_+^W(x;k) f_-^W(y;k) - f_-^W(x;k) f_+^W(y;k)}{Wron[f_+^W(x;k), f_-^W(x;k)]} V(y) f_+^V(y) dy.$$

$$\Longrightarrow \; \frac{k}{t^V(k)} \; = \; \frac{k}{t^W(k)} - \frac{1}{2i} I^{[V,W]}(k), \quad I^{[V,W]}(k) \equiv \int f_-^W(\cdot;k)(V-W) f_+^V(\cdot;k).$$

Our analysis uses mostly integration by parts on these identities, with well-chosen potentials.

- Requires $q_\varepsilon \equiv q(x, x/\varepsilon)$, (almost-)periodic in the fast variable, and some regularity in the slow variable.
- Allows k to lie in a complex strip $\Im(k) < \alpha$.

Main result

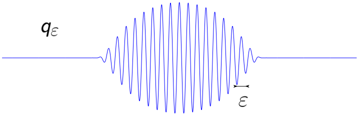
Convergence of the transmission coefficient

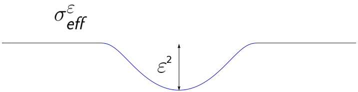
Assume $q_\varepsilon = q(x, x/\varepsilon) = \sum_{j\neq 0} q_j(x) e^{2i\pi j \frac{x}{\varepsilon}}$ is smooth and exponentially decaying at infinity. Then there exists $\varepsilon_0 > 0$ and K a compact subset of $\mathbb{C}$ such that $(\varepsilon, k) \in [0, \varepsilon_0) \times K$, one has

$$\left| \frac{k}{t^{\sigma^\varepsilon_{eff}}(k)} - \frac{k}{t^{q_\varepsilon}(k)} \right| \leq \varepsilon^3 \, C(K, \| V \|),$$

where σ^ε_{eff} is the *effective potential well* defined by

$$\sigma^\varepsilon_{eff}(x) \equiv -\varepsilon^2 \, \Lambda_{\rm eff}(x) \equiv -\frac{\varepsilon^2}{(2\pi)^2} \sum_{j\neq 0} \frac{|q_j(x)|^2}{j^2}.$$





Consequences

$$\frac{k}{t^{q_\varepsilon}(k)} = k + \frac{\varepsilon^2}{2i}\int_{-\infty}^{\infty}\Lambda_{\rm eff}(x)dx + \mathcal{O}(\varepsilon^3),$$

This allows to expand $t^{q_\varepsilon}(k)$, apart from a shrinking subset around $k^\star \equiv i\frac{\varepsilon^2}{2}\int \Lambda_{\rm eff}$.

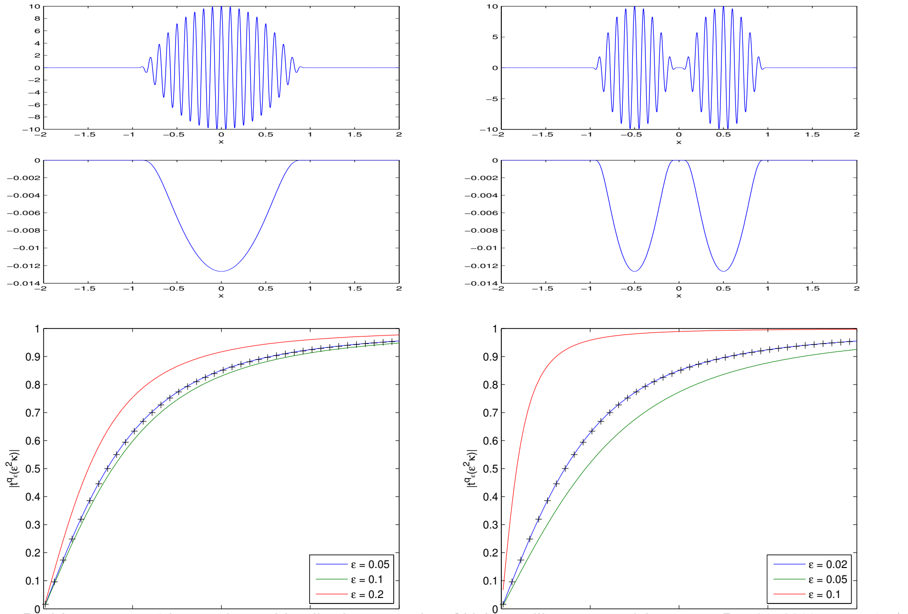
This is true in particular

- uniformly for $k \in \mathbb{R}$: $\sup_{k\in\mathbb{R}} \left| t^{\sigma^\varepsilon_{\rm eff}}(k) - t^{q_\varepsilon}(k)\right| = \mathcal{O}(\varepsilon)$.
- if $k = \varepsilon^2\kappa$, $\kappa \neq i\frac{\int \Lambda_{\rm eff}}{2}$:

$$\lim_{\varepsilon\to 0} t^{q_\varepsilon}(\varepsilon^2\kappa) = \frac{\kappa}{\kappa - i\frac{\int\Lambda_{\rm eff}}{2}}.$$

This universal scaled limit is the transmission coefficient for a Dirac-distribution potential : $(-\partial_x^2 - \delta(x)\int\Lambda_{\rm eff} - \kappa^2)\psi = 0$.

Consequences



Consequences (continued)

$$\frac{k}{t^{q_\varepsilon}(k)} = k + \frac{\varepsilon^2}{2i}\int_{-\infty}^{\infty}\Lambda_{\rm eff}(x)dx + \mathcal{O}(\varepsilon^3),$$

t^{q_ε} has a pole in the upper kalf plane

$$k_\varepsilon \approx i\frac{\varepsilon^2}{2}\int \Lambda_{\rm eff} + \mathcal{O}(\varepsilon^3).$$

(using Rouché argument).

Edge bifurcation of point spectrum

$H_{q_\varepsilon} \equiv (-\partial_x^2 + q_\varepsilon)$ has a point eigenvalue at energy

$$E_\varepsilon = k_\varepsilon^2 \approx -\frac{\varepsilon^4}{4}\left(\int \Lambda_{\rm eff}\right)^2 + \mathcal{O}(\varepsilon^5).$$

Consequences (continued)

Edge bifurcation of point spectrum

$H_{q_\varepsilon} \equiv (-\partial_x^2 + q_\varepsilon)$ has a point eigenvalue at energy

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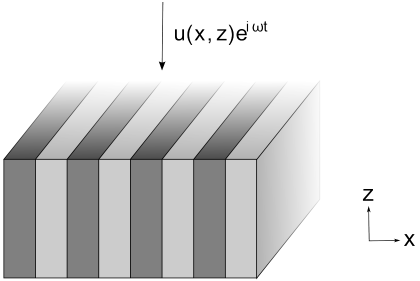

This indicates the existence of a solution $u(x,z)$, localized in x, for a careful choice of $k = \omega/c$.