NUMERICAL STOCHASTIC MODELS OF CONDITIONAL METEOROLOGICAL FIELDS

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Outline

- **Numerical stochastic models of space and space-time fields of daily precipitation**
- **Numerical stochastic models of conditional meteorological fields**
- **Stochastic interpolation of indicator heterogeneous fields**

Simulation of a Gaussian vector

$$\vec{\xi}_{(n)} = (\vec{\xi}_1^{\,T}, \vec{\xi}_2^{\,T}, \dots, \vec{\xi}_n^{\,T})^T, \qquad \vec{\xi}_i = (\xi_{i1}, \xi_{i2}, \dots, \xi_{im})^T$$

$$R_{(n)} = \begin{Vmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{Vmatrix}, \qquad R_{ij} = \begin{Vmatrix} r_{11}^{ij} & r_{12}^{ij} & \dots & r_{1m}^{ij} \\ r_{12}^{ij} & r_{22}^{ij} & \dots & r_{1m}^{ij} \\ \dots & \dots & \dots & \dots \\ r_{m1}^{ij} & r_{m2}^{ij} & \dots & r_{mm}^{ij} \end{Vmatrix},$$

$$\vec{\xi}_{(n)} = P_{(n)}\Lambda_{(n)}^{1/2}\vec{\varphi}_{(n)} \qquad P_{(n)}\Lambda P_{(n)}^T = R_{(n)} \qquad P_{(n)} = \begin{Vmatrix} \vec{P}_1 & \cdots & \vec{P}_{mn} \end{Vmatrix} \qquad R_{(n)}\vec{P}_i = \lambda_i \vec{P}_i \qquad \Lambda = \begin{Vmatrix} \lambda_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda_{mn} \end{Vmatrix}$$

$$\vec{\xi}_{(n)} = A_{(n)}\vec{\varphi}_{(n)}, \quad A_{(n)}A_{(n)}^T = R_{(n)}$$

$$\vec{\xi}_1 = C_0\vec{\phi}_1, \qquad R_{ij} = R_{|i-j|}$$

$$\vec{\xi}_2 = \vec{B}^T[1]J_{(1)}\vec{\xi}_{(1)} + C_1\vec{\phi}_2, \qquad C_iC_i^T = Q_i. \quad Q_k = R_0 - \vec{B}^T[k]\tilde{R}_{(k)}\vec{B}[k],$$

$$\cdots$$

$$\vec{\xi}_n = \vec{B}^T[n-1]J_{(n-1)}\vec{\xi}_{(n-1)} + C_{n-1}\vec{\phi}_n, \qquad \tilde{R}_{(k)}\vec{B}[k] = \vec{R}_k, \;\; \tilde{R}_{(k)} = J_{(k)}R_{(k)}J_{(k)}, \quad J_{(k)} = \begin{Vmatrix} 0 & \dots & I \\ \dots & \dots & \dots \\ I & \dots & 0 \end{Vmatrix}$$

$$\vec{\varphi}_{(n)} = (\vec{\varphi}_1^{\,T}, \vec{\varphi}_2^{\,T}, \dots, \vec{\varphi}_n^{\,T})^T \qquad \vec{\varphi}_i = (\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{im})^T$$

$$\vec{\varphi}_1, \vec{\varphi}_2, \dots, \vec{\varphi}_n : E\vec{\varphi}_k\vec{\varphi}_k^T = I, E\vec{\varphi}_k\vec{\varphi}_l^T = 0, k \neq l,$$

Inverse distribution functions method (1966, Pirahashvilly Z.A.)

$$\xi_t = F_t^{-1}\left(\Phi\left(\eta_t\right)\right), \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2} du$$

η_t – Gaussian serie (field) with $\quad M\eta_t = 0, \quad D\eta_t = 1, \quad M\eta_i\eta_j = g_{ij},$

$$r_{ij} = \frac{M\xi_i\xi_j - M\xi_i M\xi_j}{\sqrt{D\xi_i D\xi_j}} = R_{F_iF_j}(g_{ij}),$$

$$R_{F_iF_j}(g_{ij}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i^{-1}(\Phi(x))F_j^{-1}(\Phi(y))\varphi(x,y,g_{ij})dxdy,$$

$$\varphi(x,y,g_{ij}) = \left[2\pi\sqrt{1-g_{ij}^2}\exp\left(\frac{2g_{ij}xy - x^2 - y^2}{2(1-g_{ij}^2)}\right)\right] \quad \text{где} \quad g_{ij} = R_{F_iF_j}^{-1}(r_{ij})$$

Simulation based on the normalization of the real series (1977, Svanigze G.G.)

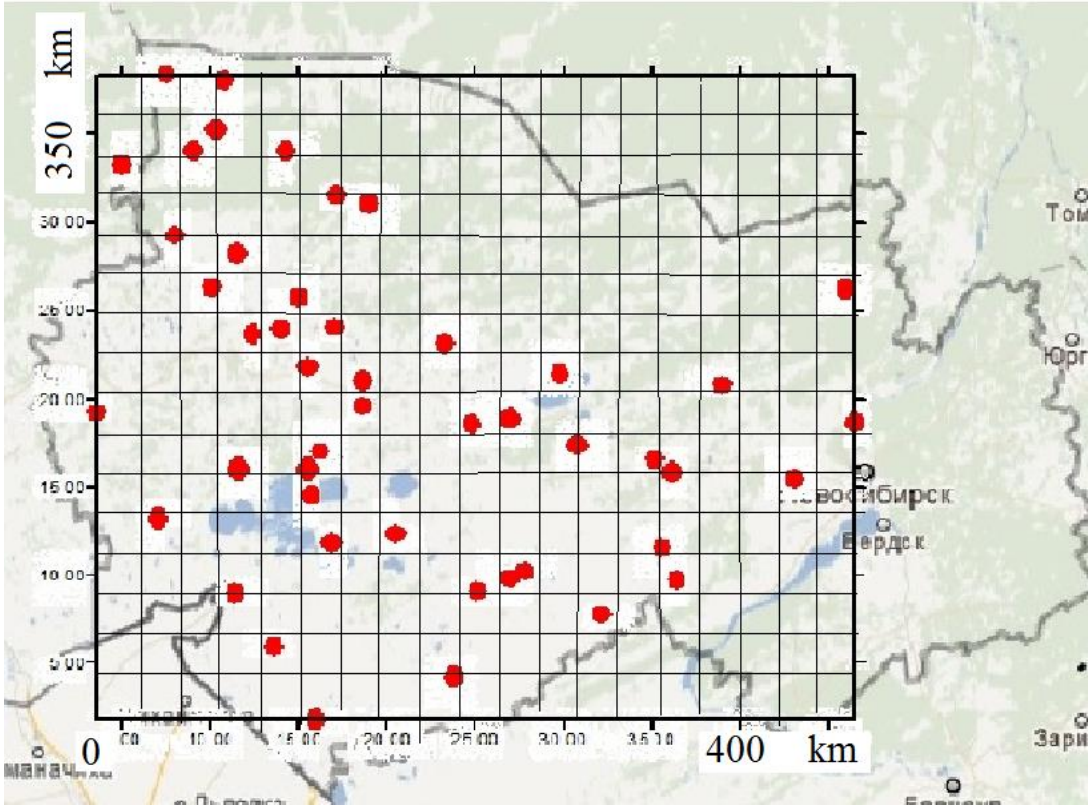
1. Normalization of real series (fields) $\quad \xi_t^* = \Phi^{-1}(F(\eta_t^*)).$

2. Estimation of the normalized series (fields) correlation function $\quad R_{t,t+\tau}^*$

3. Simulation of Gaussian sequence (field) ξ_t with the correlation function $R_{t,t+\tau}^*$

4. The final process $\quad \eta_t = F^{-1}(\Phi(\xi_t)).$

Numerical stochastic model of spatial fields of daily precipitation



Multiplicative representation

$$\eta_{ik} = \omega_{ik} \chi_{ik}$$

Simulation algorithm

1) $\omega_{ik} = \begin{cases} 1, & \xi_{ik} \le c, \\ 0, & \xi_{ik} > c, \end{cases}$ - indicator field with corr. matrix $\{s_{ik,jl}\}$

where $\{\xi_{ik}\}$ - Gaussian field with corr. matrix $\{g_{ik,jl}\}$,

$$p = P(\omega_{ik} = 1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{1}{2}u^2} du,$$

,

$$s_{ik,jl} = 1 - \frac{2}{p(1-p)} T(c, a_{ik,jl}),$$

$$T(c, a_{ik,jl}) = \frac{1}{2\pi} \int_0^{a_{ik,jl}} e^{-\frac{c^2(1+u^2)}{2}} \frac{du}{1+u^2}, \; a_{ik,jl} = \sqrt{\frac{1-g_{ik,jl}}{1+g_{ik,jl}}}.$$

2) $\chi_{ik} = F^{-1}\left(\Phi(\zeta_{ik})\right)$ - field of presipitation with corr. matrix $\{q_{ik,jl}\}$

where $\{\zeta_{ik}\}$ - Gaussian field with corr. matrix $\{h_{ik,jl}\}$,

$$q_{ik,jl} = f(h_{ik,jl}).$$

One-dimensional distribution of homogeneous spatial fields of daily precipitation

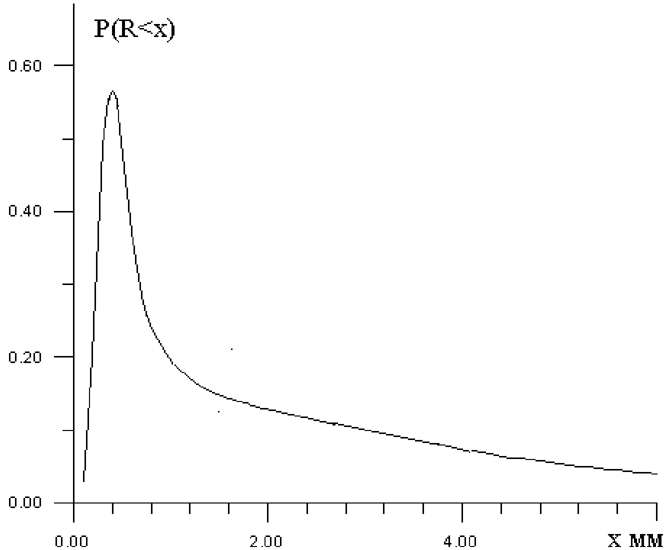
$$0 = x_0 \le x_1 < \ldots < x_{m+1} < \ldots < x_{n+1} < \infty$$

$$0 = F_0^* \le F_1^* \le \ldots \le F_{m+1}^* \le \ldots \le F_{n+1}^* < 1$$

Distribution function (1986, Marchenko A.S.)

$$F(x) = \begin{cases} U(x), & x_0 \le x \le x_1 \\ F_i(x), & x_i \le x \le x_{i+1},\ i = 1, \ldots, m \\ V(x), & x_{m+1} \le x < \infty \end{cases}$$

$$U(x_0) = 0, \quad U(x_1) = F_1, \quad V(x_{m+1}) = F_{m+1}, \quad V(\infty) = 1$$



Probability density

$$F_i(x) = F_i^* + f_i h_i y_i [1 + a_i(1 - y_i) + b_i(1 - y_i^2)],$$

$$h_i = x_{i+1} - x_i, \ h_i y_i = x - x_i, \ f_i h_i = F_{i+1}^* - F_i^*.$$

$$U(x) = cx, \quad 0 \le x < x_1$$

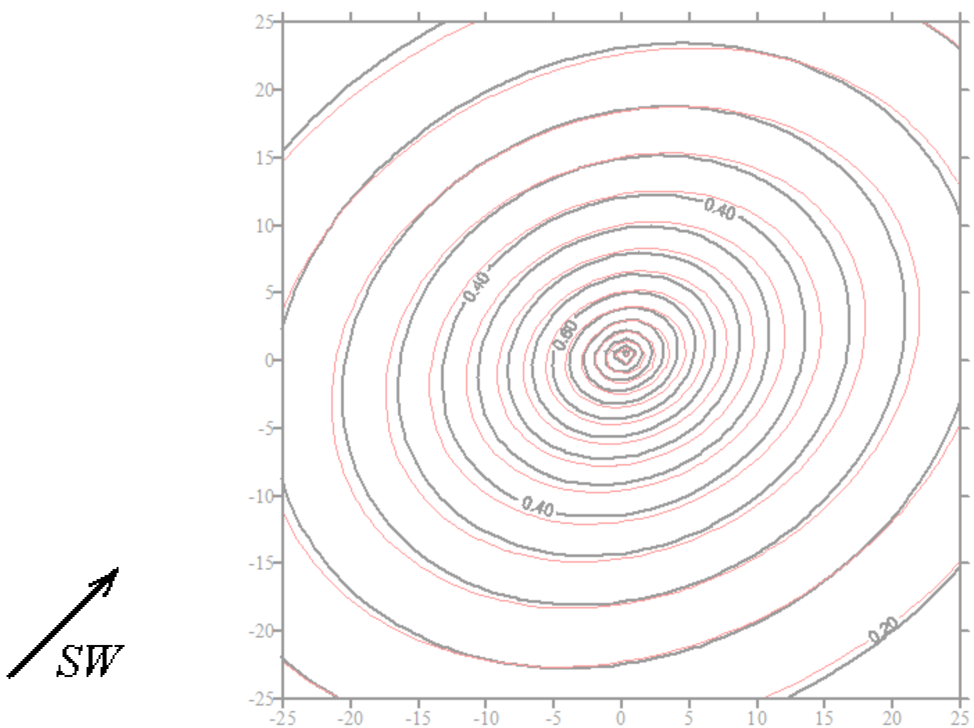
$$V(x) = (1 - W(x))^{\alpha}, W(x) = e^{-ax^b}$$

$$L = \sum_{x_i \ge x_{m+1}} \left(F_i^* - V(x_i) \right)^2 = \min$$

Correlation structure of space fields of daily precipitation

$$r(x,y)=\exp(-\alpha[ax^2+bxy+cy^2]^\theta)$$

Spatial correlation function



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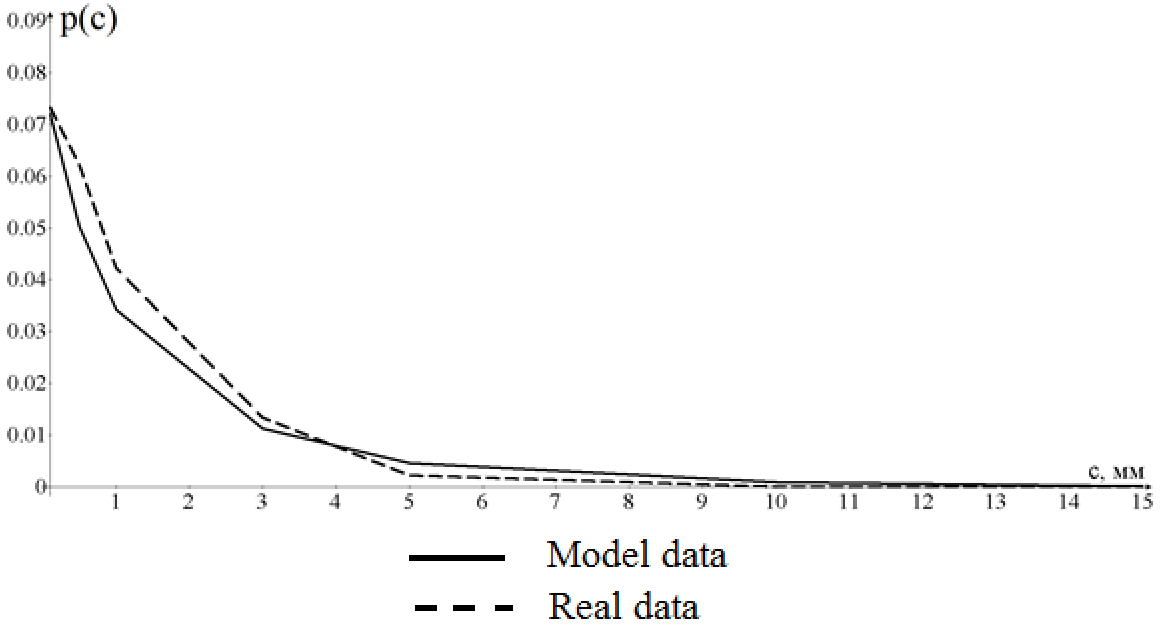
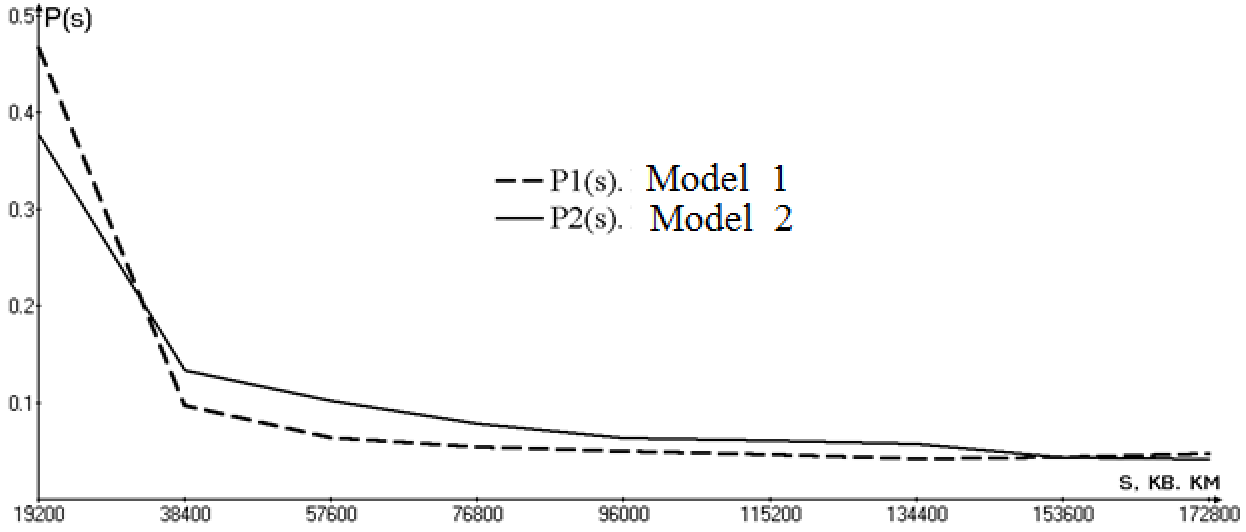
Verification of the model



Probability of event : sum of precipitation on 6 stations is above given level c . Novosibirsk region, June

Probability distributions of the area occupied by precipitation



1. $\eta_{ik\nu} = F^{-1}\left(\Phi(\zeta_{ik\nu})\right)$

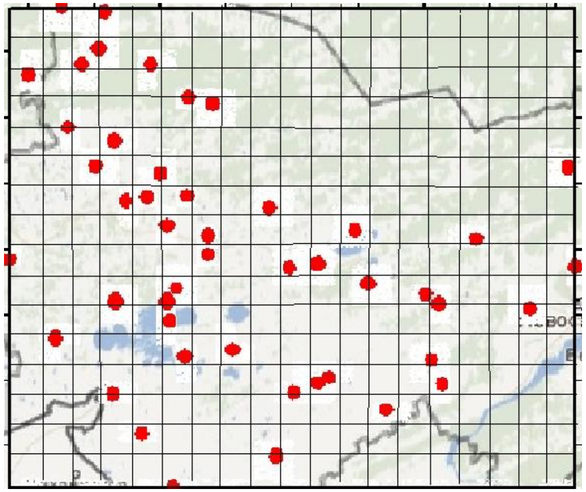
2. $\eta_{ik\nu} = \omega_{ik\nu}\chi_{ik\nu}$

Simulation of conditional Gaussian fields

$$\vec{\zeta} = (\vec{\zeta}_1^{\,T}, \vec{\zeta}_2^{\,T})^T \text{ is } N(0, G)$$

$$\vec{\mu} = E\vec{\zeta} = (\vec{\mu}_1^{\,T}, \vec{\mu}_2^{\,T})^T,$$

$$G = E(\vec{\zeta} - \vec{\mu})(\vec{\zeta} - \vec{\mu})^T = \begin{Vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{Vmatrix},$$



$$E(\vec{\zeta}_1 - G_{12}G_{22}^{-1}\vec{\zeta}_2) = \vec{\mu}_1 - G_{12}G_{22}^{-1}\vec{\mu}_2, \qquad G_{11.2} = G_{11} - G_{12}G_{22}^{-1}G_{21},$$

$$\mu_{1.2} = \vec{\mu}_1 + R_{12}R_{22}^{-1}(\vec{x}_2 - \vec{\mu}_2), \qquad \vec{\xi}_{1.2} = \mu_{1.2} + \vec{\zeta}_{1.2},$$

$$\vec{\zeta}_{1.2} \text{ is } N(0, G_{11.2})$$

Simulation of conditional non-Gaussian fields

Let us consider random field $\eta(x, y)$ with a corr. f. $r(x, y)$ and p.d.f. $F(x)$ on a regular grid $\{x_i, y_j\}$, $i = 1,\ldots, n_1$, $j = 1,\ldots, n_2$ and at a system of weater stations $\{x_l, y_l\}, l = 1,\ldots, n_3$ as a vector $\vec{\eta} = (\vec{\eta}_1^T, \vec{\eta}_2^T)^T$ of dimension $n = n_1 n_2 + n_3$.

It is required to build the vector $\vec{\eta}_1^T$ provided that the vector $\vec{\eta}_2^T$ is given.

$$E\vec{\eta} = (E\vec{\eta}_1^T, E\vec{\eta}_2^T)^T = \vec{\mu},$$

$$E(\vec{\eta} - \vec{\mu})(\vec{\eta} - \vec{\mu})^T = R = (r_{km}) = \begin{Vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{Vmatrix}$$

Simulation of conditional non-Gaussian fields

Algorithm 1.

1. $\zeta_\nu^* = \Phi^{-1}\left(F(\eta_{2\nu}^*)\right),\ \nu = 1,\ldots,n_3,\quad \Phi\left(x\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}x^2}dx$

$\vec{\zeta} = (\vec{\zeta}_1^T, \vec{\zeta}_2^T)^T$ is the Gaussian vector:

$$E(\vec{\zeta} - \vec{v})(\vec{\zeta} - \vec{v})^T = G = (g_{km}) = \begin{Vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{Vmatrix}$$

$$r_{km} = \frac{1}{\sigma^2}\left(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F^{-1}(\Phi(x))F^{-1}(\Phi(y))\,f(x,y;g_{km})dxdy - \mu^2\right)$$

2. $\vec{\zeta}_1 = \mu_{1.2} + \vec{\zeta}_{1.2},\ \mu_{1.2} = G_{12}G_{22}^{-1}\vec{\zeta}_2^*,\ \vec{\zeta}_{1.2}$ is $N(0, G_{11.2} = G_{11} - G_{12}G_{22}^{-1}G_{21})$

3. $\eta_{ik} = F^{-1}\left(\Phi(x_{ik}\,\middle|\,\vec{\zeta}_2 = \vec{\zeta}_2^*)\right),\quad \Phi(x_{ik}\,|\,\vec{\zeta}_2 = \vec{\zeta}_2^*) = \frac{1}{\sqrt{2\pi}\sigma_{ik.2}}\int\limits_{-\infty}^{x_{ik}} \exp\left(-\left(\frac{u - \mu_{ik.2}}{2\sigma_{ik.2}}\right)\right)du.$

Simulation of conditional indicator fields

$$\omega_m = \begin{cases} 1, & \xi_m \le C_m \\ 0, & \xi_m > C_m \end{cases}, \quad m = 1,\ldots,n_1 n_2 + n_3 \qquad (1)$$

$\vec{\xi} = (\vec{\xi}_1^{\,T}, \vec{\xi}_2^{\,T})^T$ is a Gaussian vector with corr. matrix $G = (g_{km})$,

$$p_m = P(\xi_m \le C_m) = P(\omega_m = 1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{C_m} e^{-\frac{1}{2}z^2} dz$$

$$f(x_1,\ldots x_{n_1 n_2} \,|\, \xi_{n_1 n_2+1} > C_{n_1 n_2+1},\ldots,\xi_{n_1 n_2+k} > C_{n_1 n_2+k}, \xi_{n_1 n_2+k+1} < C_{n_1 n_2+k+1},\ldots,\xi_n < C_n)$$

Let $\omega_{n_1 n_2+1} = 1, \ldots, \omega_{n_1 n_2+k} = 1, \omega_{n_1 n_2+k+1} = 0, \ldots, \omega_n = 0$

Algorithm 2.

1. We simulate vector $\vec{\xi} = (\vec{\xi}_1^{\,T}, \vec{\xi}_2^{\,T})^T$

2. If $\xi_{n_1 n_2+1} > C_{n_1 n_2+1},\ldots,\xi_{n_1 n_2+k} > C_{n_1 n_2+k}, \xi_{n_1 n_2+k+1} < C_{n_1 n_2+k+1},\ldots,\xi_n < C_n$ go to 3. or else go to 1.

3. We simulate $\vec{\omega} = (\vec{\omega}_1^{\,T}, \vec{\omega}_2^{\,T})^T$ ($e.g.$, formula (1))

Approximate algorithms for numerical simulation of conditional fields of daily precipitation

$\vec{\eta} = \left(\vec{\eta}_1^T, \vec{\eta}_2^T\right)^T$, $\vec{\eta}_1$ is a field of precipitation on the grid

$\vec{\eta}_2$ is a field of precipitation at weather stations

$\vec{\eta}_{1.2} = (\vec{\eta}_1 \left| \vec{\eta}_2 = \vec{\eta}_2^*\right.)$ is a conditional field (for example $\vec{\eta}_2^* = (0,0,0.4,\ldots,0,1.5,0)^T$)

$\vec{\eta}_2^* = \left(\vec{\eta}_{21}^{*T}, \vec{\eta}_{22}^{*T}\right)^T$, $\vec{\eta}_{21}^{*T} = \vec{0}$, $\vec{\eta}_{22}^{*T} \neq \vec{0}$

Algorithm 3.

1. $\zeta_{22,\nu}^* = \Phi^{-1}\left(F(\eta_{22,\nu}^*)\right)$, $\nu = 1,\ldots,n'$, $\vec{\zeta}_{22}^* = (\zeta_{22,1}^*,\ldots,\zeta_{22,n'}^*)^T$

   $\vec{\zeta}_{1.2} = \vec{\mu}_{1.2} + \vec{\zeta}_{1.2}$, $\mu_{1.2} = R_{22}^{12}(R_{22}^{22})^{-1}\vec{\zeta}_{22}^*$, $\vec{\zeta}_{1.2}$ is $N(0, R_{11.2} = R_{22}^{11} - R_{22}^{12}(R_{22}^{22})^{-1}R_{22}^{21})$

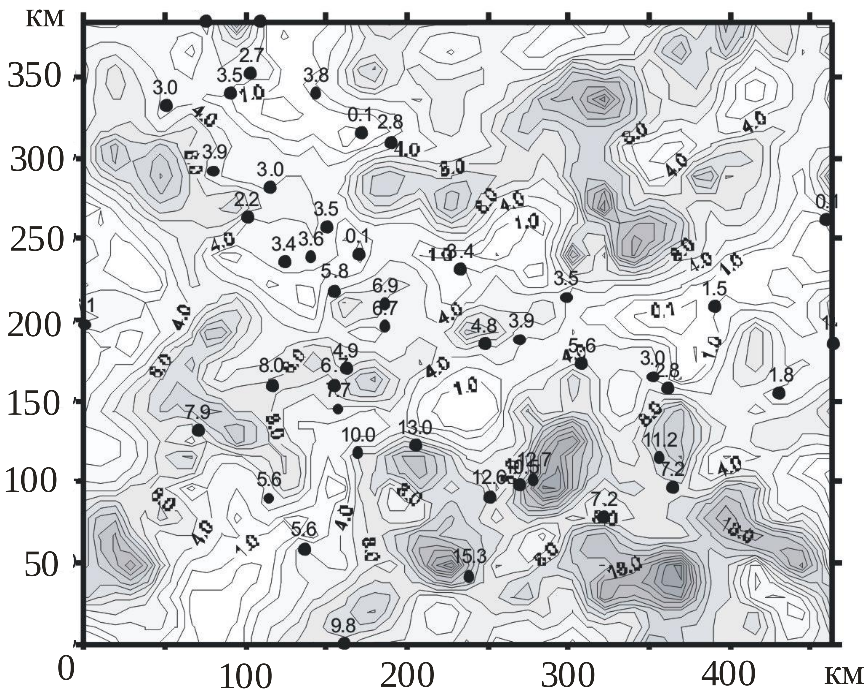
2. We simulate vector $\vec{\zeta}_{1.2}$. If all components $\in \left(-\infty, \Phi^{-1}\left(F(\varepsilon)\right)\right)$

   then $\vec{\zeta}_{12}^* = \vec{\zeta}_{1.2}$.

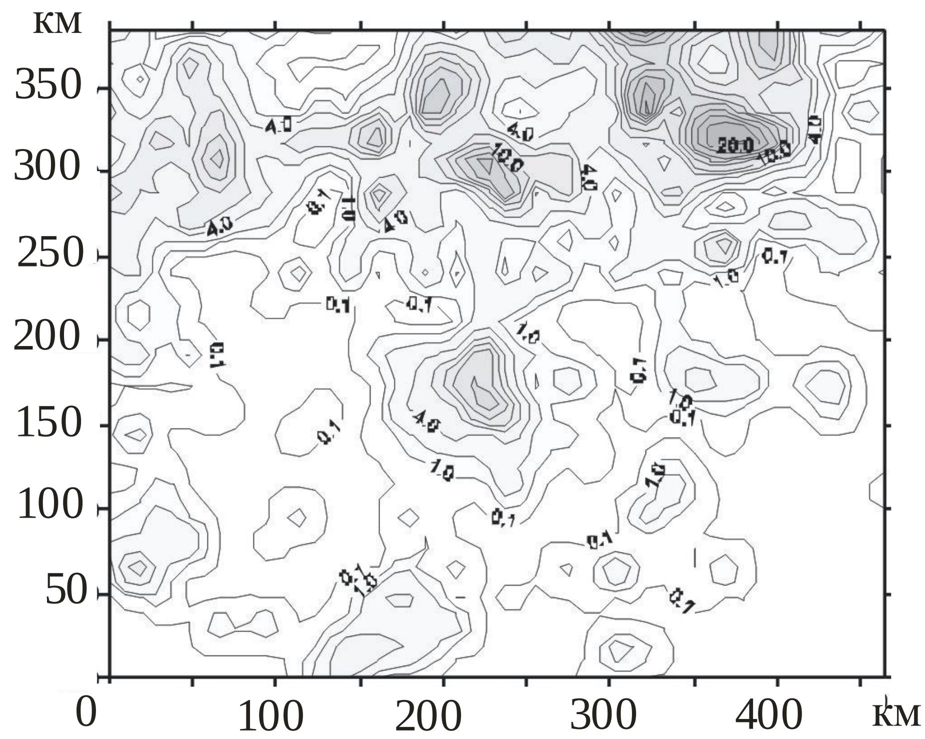
3. We construct the vector $\vec{\zeta}_2^* = (\vec{\zeta}_{12}^{*T}, \vec{\zeta}_{22}^{*T})^T$.

For simulation $\vec{\eta}_{1.2}$ we use Algorithm 1.

Realizations conditional and nonconditional fields of daily precipitation

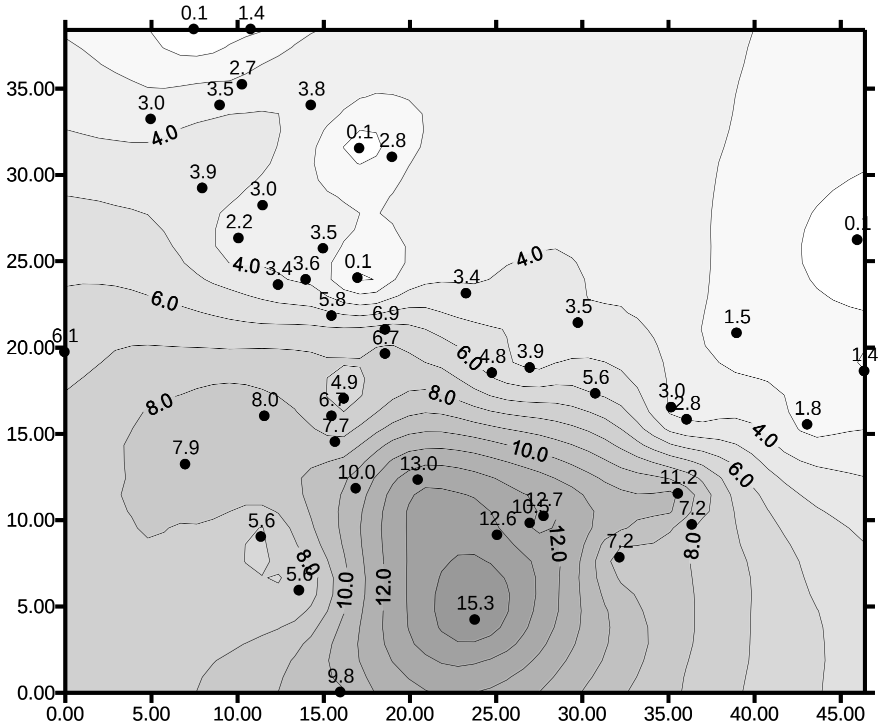


conditional field



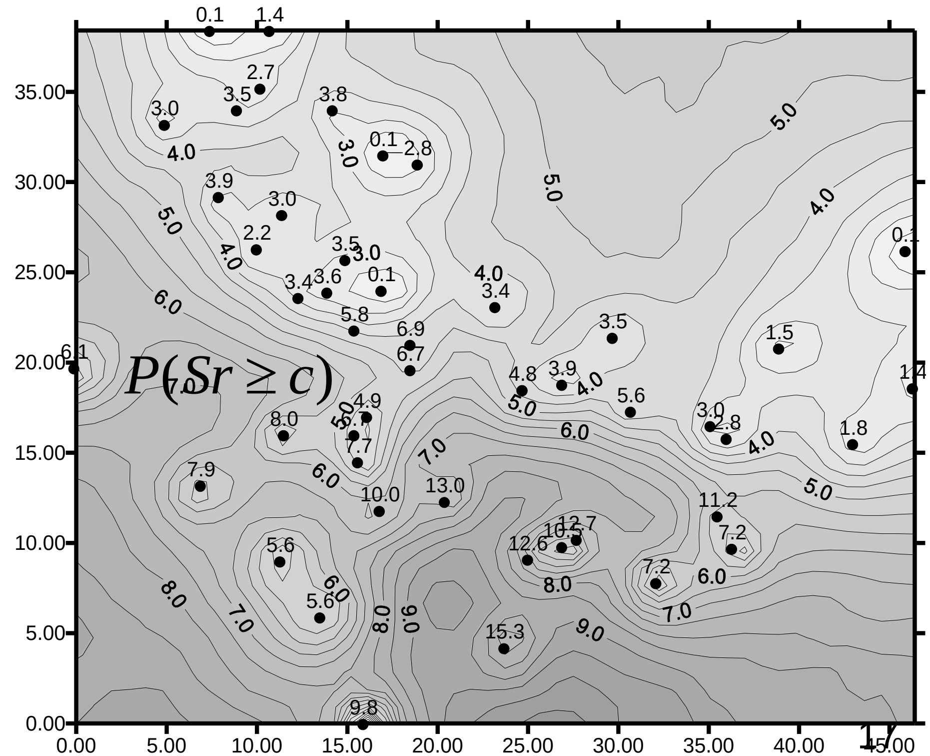
nonconditional field

Conditional average and standard deviation of precipitation field



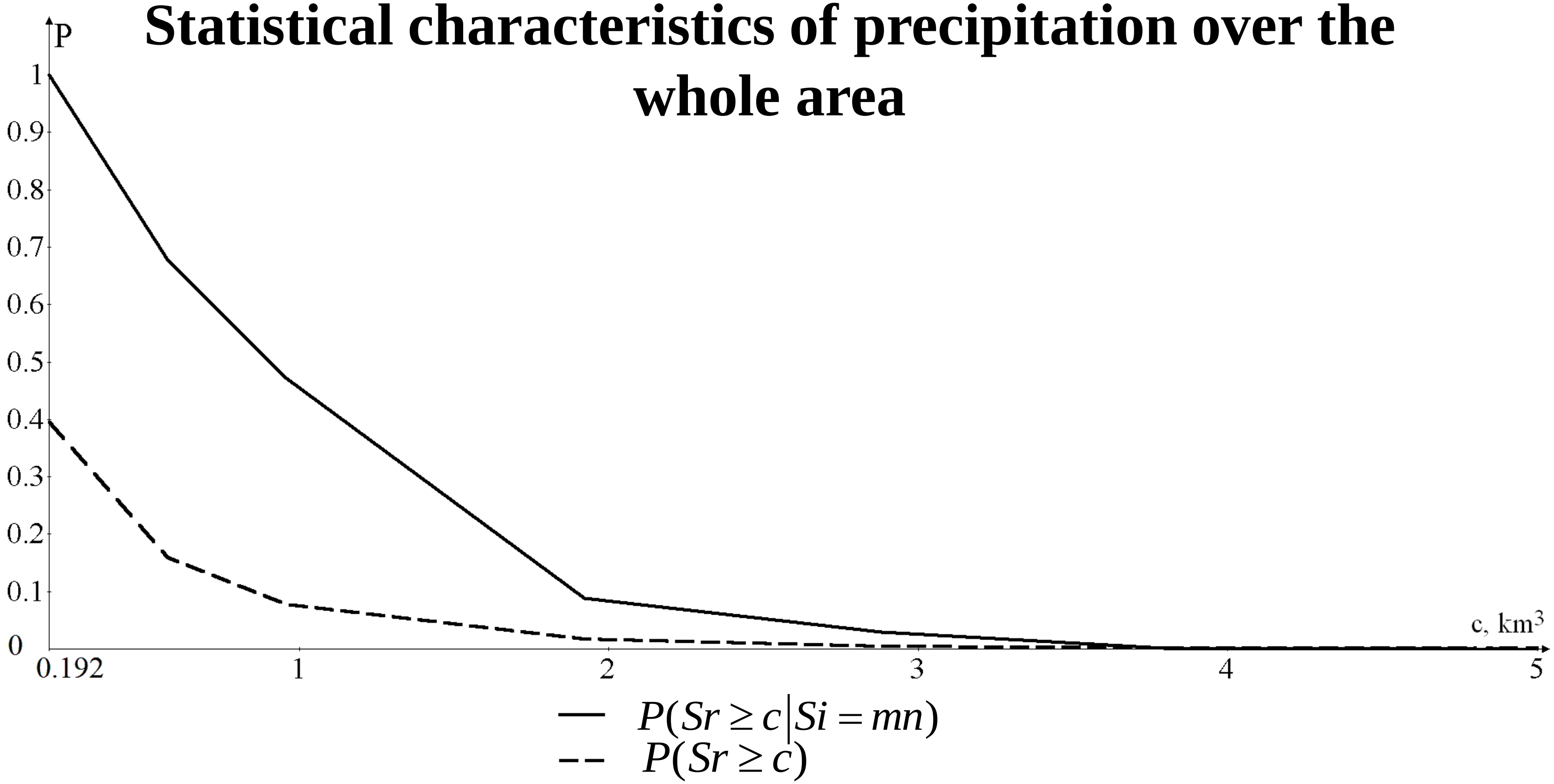
Conditional standard deviation

Conditional average of precipitation field when values at weather stations are given



$P(Sr \geq c)$

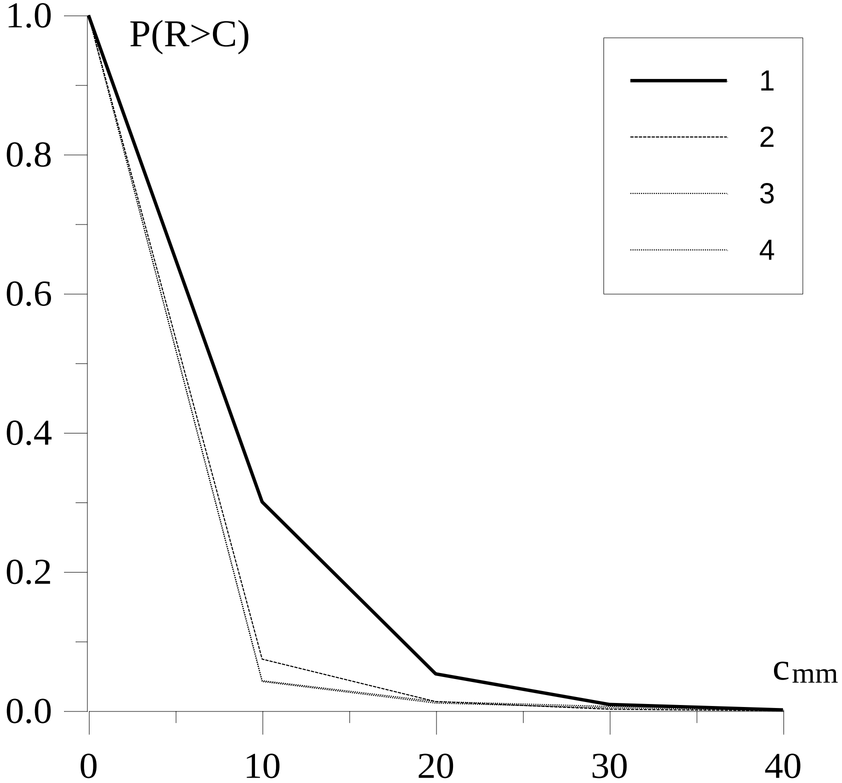
Statistical characteristics of precipitation over the whole area



Probabilities $P(Sr \geq c \mid Si = mn)$ and $P(Sr \geq c)$ calculated by conditional and nonconditional models, where Sr is total rainfall at the area,

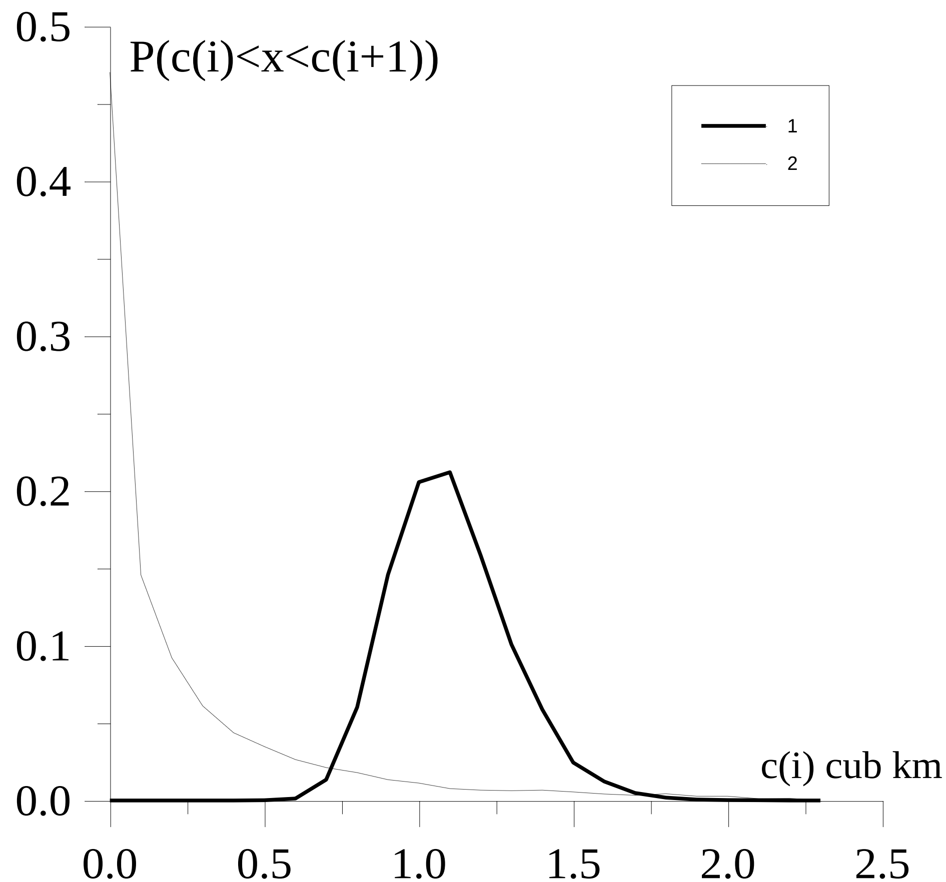
$Si = \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_{ij}$ is sum of precipitation indicators on the grid

Statistical characteristics of precipitation fields

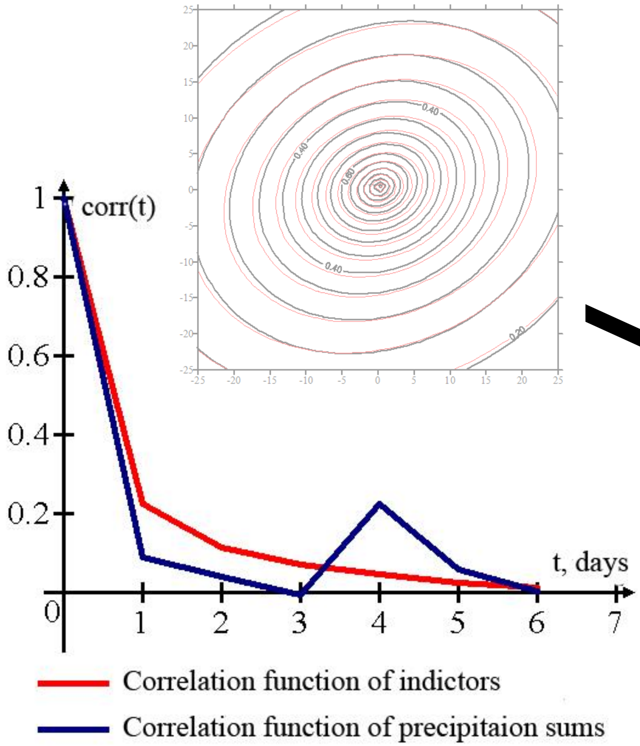


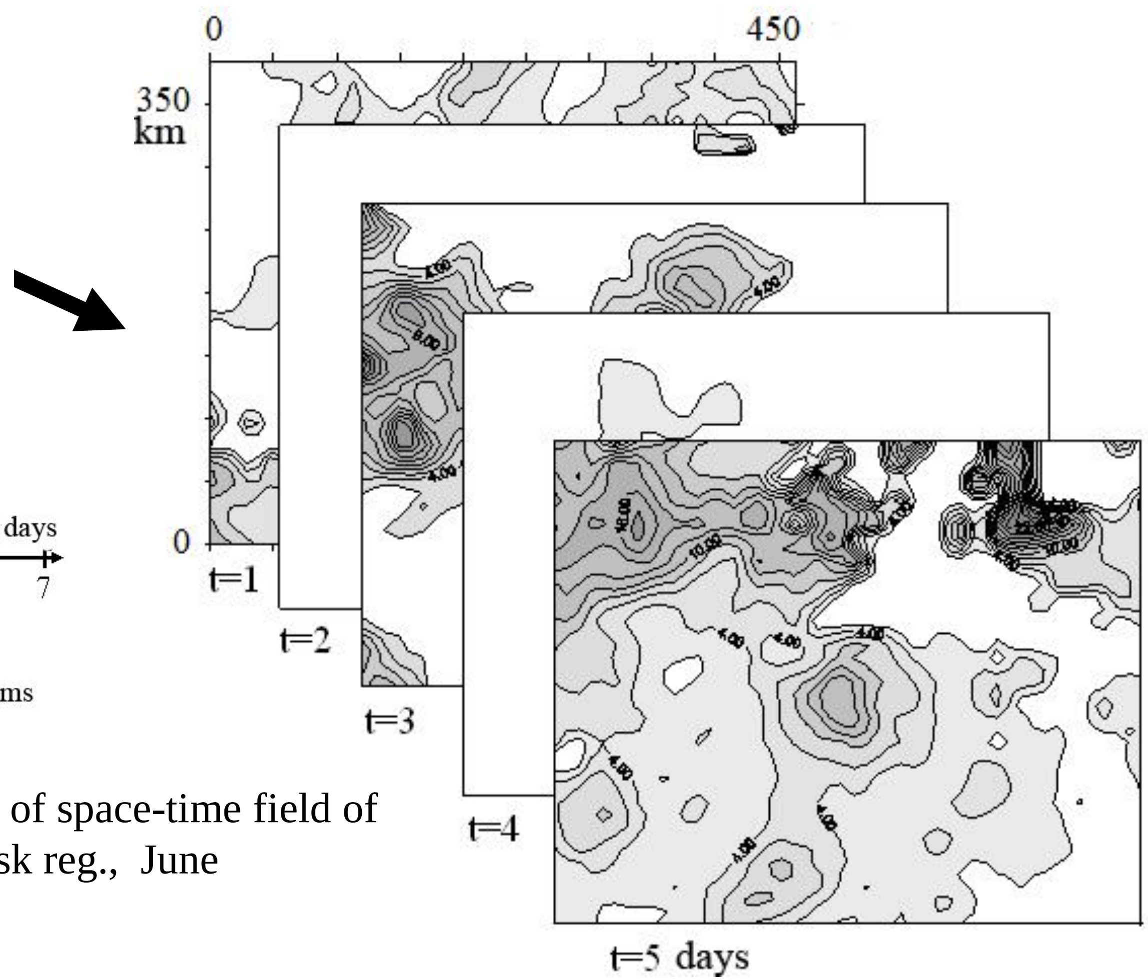
Probabilities $P(\eta > c)$ calculated by conditional and nonconditional models.

Probabilities $P(c(i) < \eta \leq c(i+1))$ of the total precipitation (day) at the area calculated by conditional (1) and nonconditional (2) models



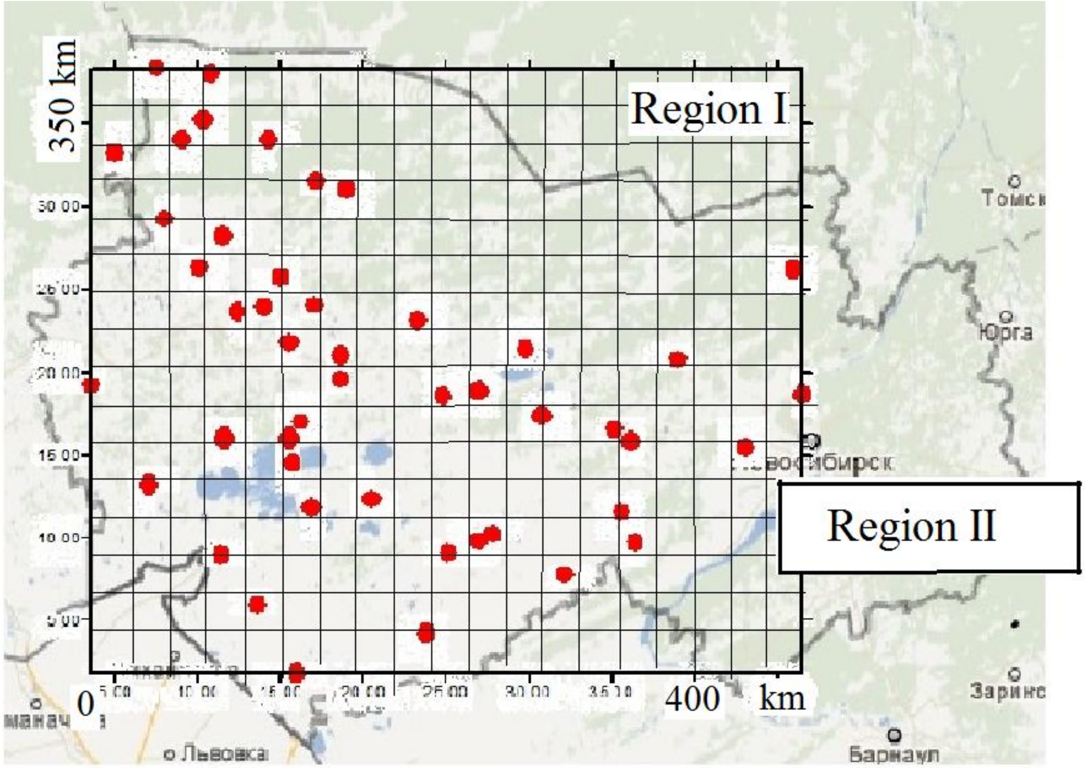
Numerical stochastic models of space-time fields of daily precipitation

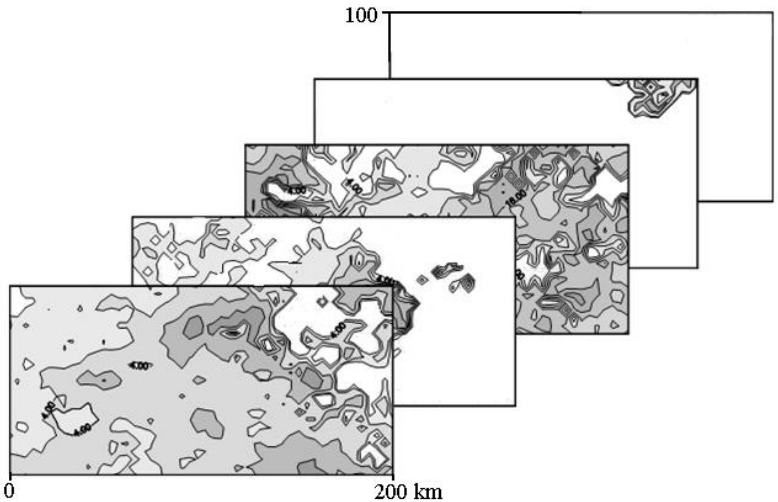


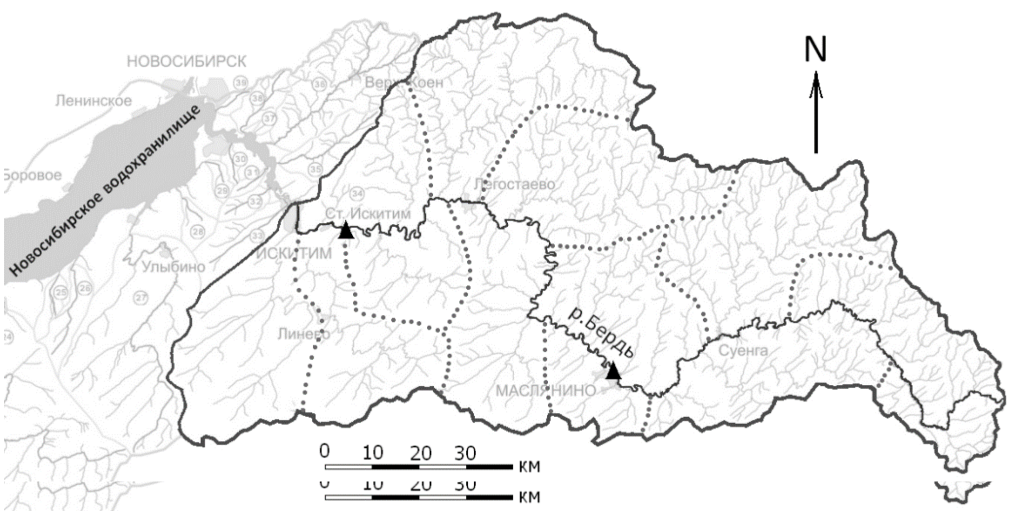


An example of the realization of space-time field of daily precipitation, Novosibirsk reg., June

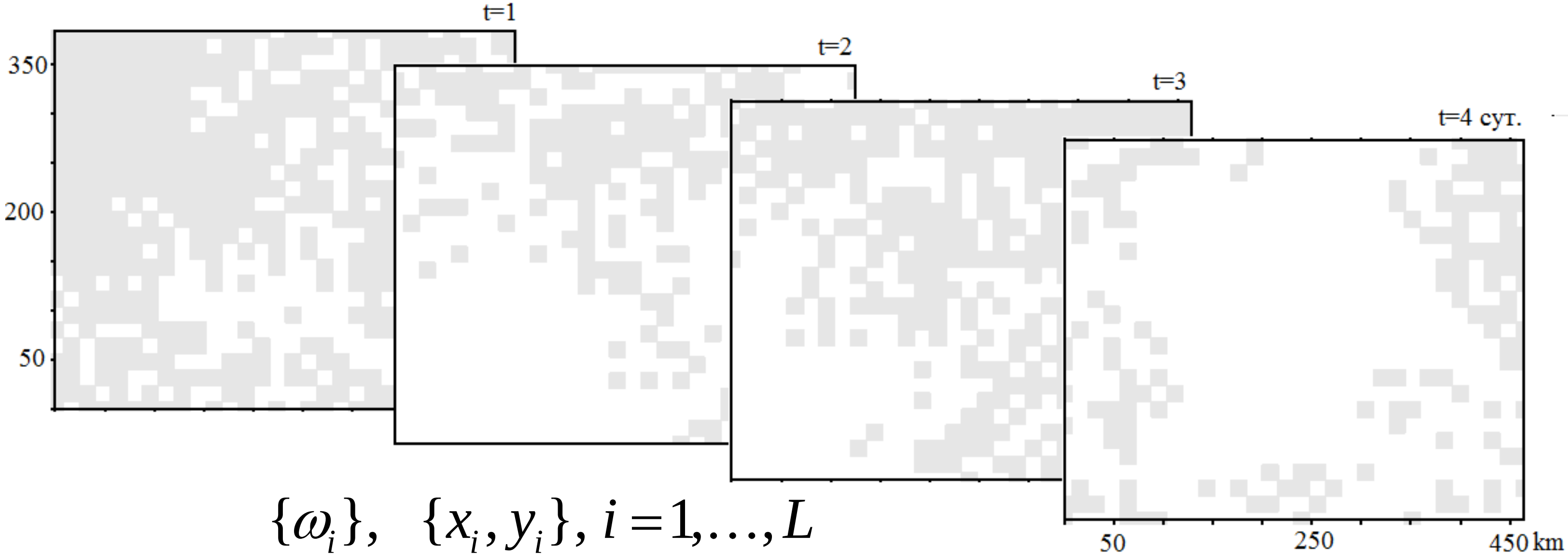
Application models of space-time field of daily precipitation to the problems of Hydrology







Stochastic interpolation from weather stations to grid points for simulation of indicadicator nonhomogeneous fields



$$\{\omega_i\}, \quad \{x_i, y_i\},\ i = 1, \dots, L$$

$$\{\omega_{lk}\}, \quad \{x_{lk}, y_{lk}\}, \quad l = 1, \dots, n, \quad k = 1, \dots, m$$

$$\bar{w}_{lk,i} = \frac{1}{\sqrt{(x_{lk} - x_i)^2 + (y_{lk} - y_i)^2}}, \quad w = \sum_{i=1}^{L} \bar{w}_{lk,i}, \quad w_{lk,i} = \frac{\bar{w}_{lk,i}}{w}, \quad i = 1, \dots, L$$

$$\omega_{lk} = \omega_i$$

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Thank you for your attention