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Systematic stochastic modeling of atmospheric variability

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Outline

1) Motivation

2) Stochastic Mode Reduction

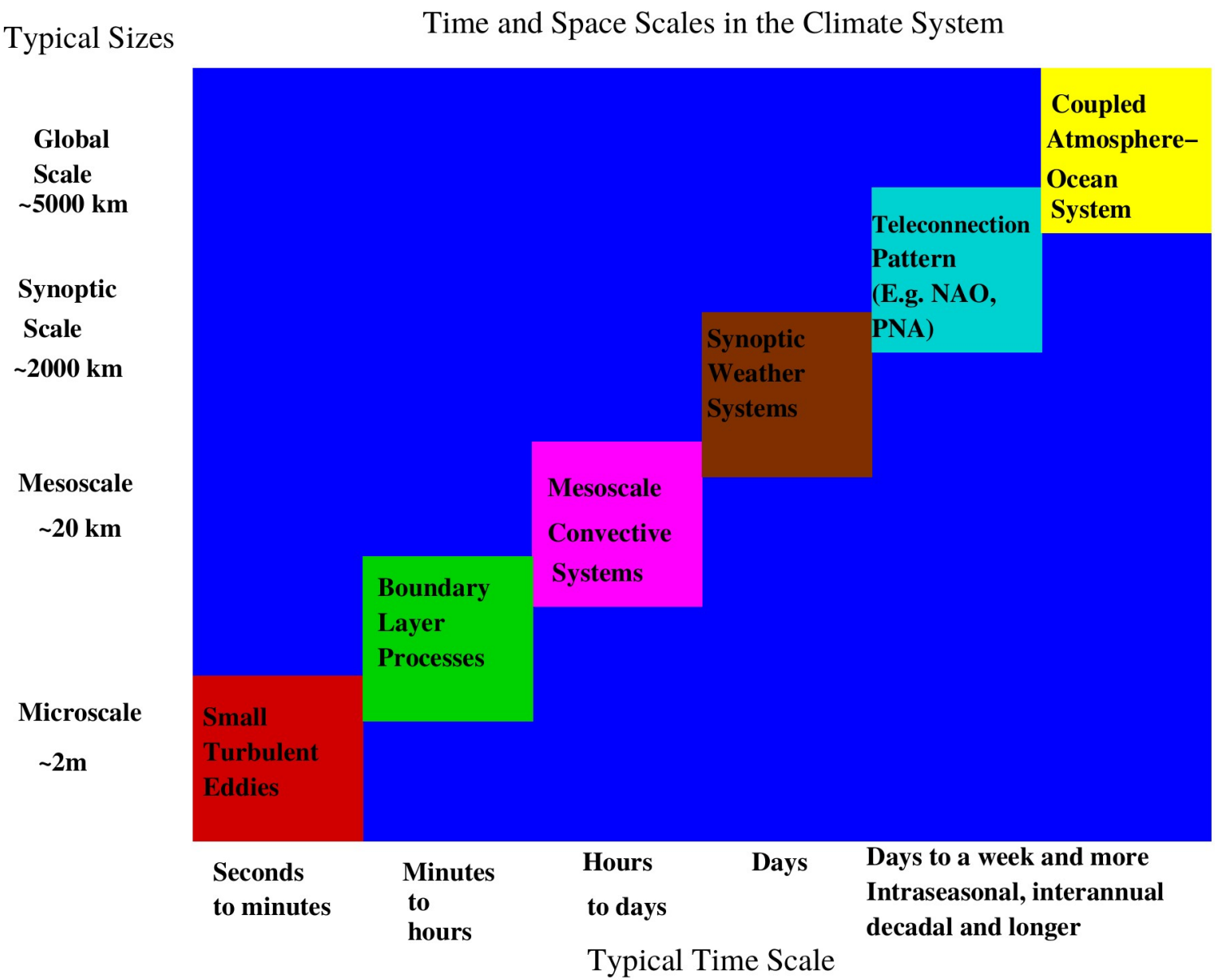
3) Physical Constraints

4) Bayesian Parameter Estimation

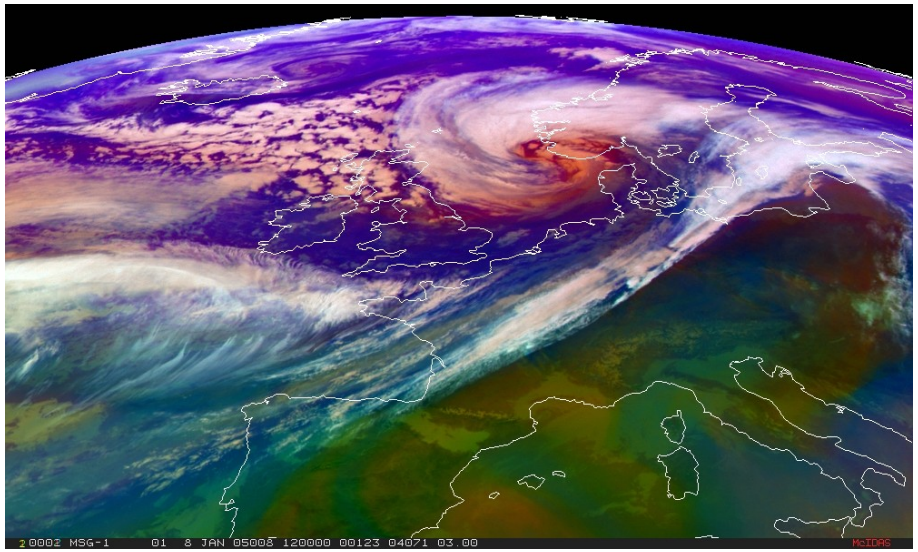
5) Results

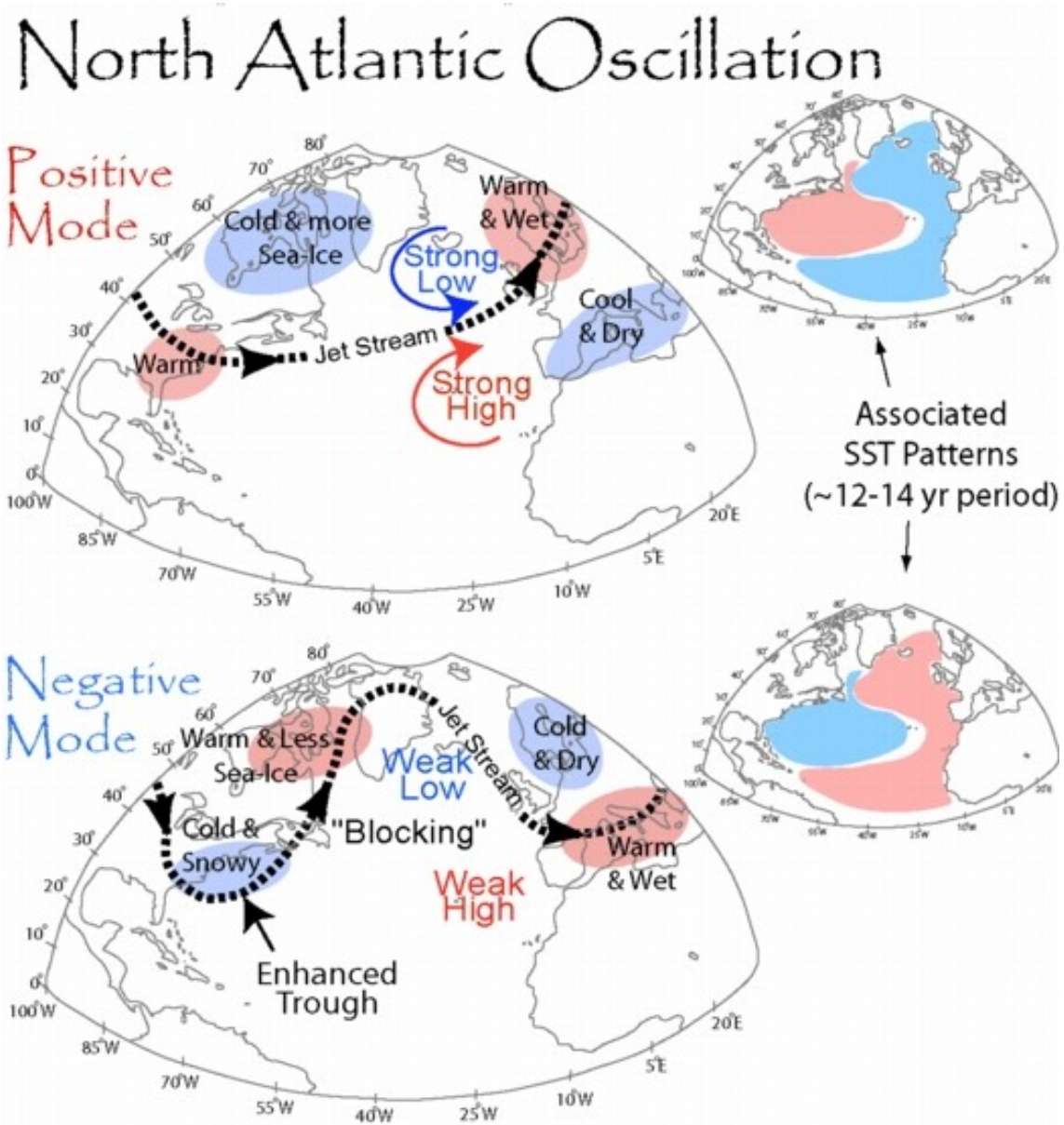
6) Summary

Scales


Typical Sizes
Time and Space Scales in the Climate System
Global Scale ~5000 km
Synoptic Scale ~2000 km
Mesoscale ~20 km
Microscale ~2m
Coupled Atmosphere–Ocean System
Teleconnection Pattern (E.g. NAO, PNA)
Synoptic Weather Systems
Mesoscale Convective Systems
Boundary Layer Processes
Small Turbulent Eddies
Seconds to minutes
Minutes to hours
Hours to days
Days
Days to a week and more Intraseasonal, interannual decadal and longer
Typical Time Scale

Long-Range Forecasts





Reduced Stochastic Climate Models

→ Computationally much cheaper

→ Capture essential dynamics

- Improved extended range forecasting
- Large ensemble forecasting
- Long-term climate studies (e.g. paleo-climate)
- Long control simulations to estimate extremes
- Extreme Event Prediction

Reduced Order Models

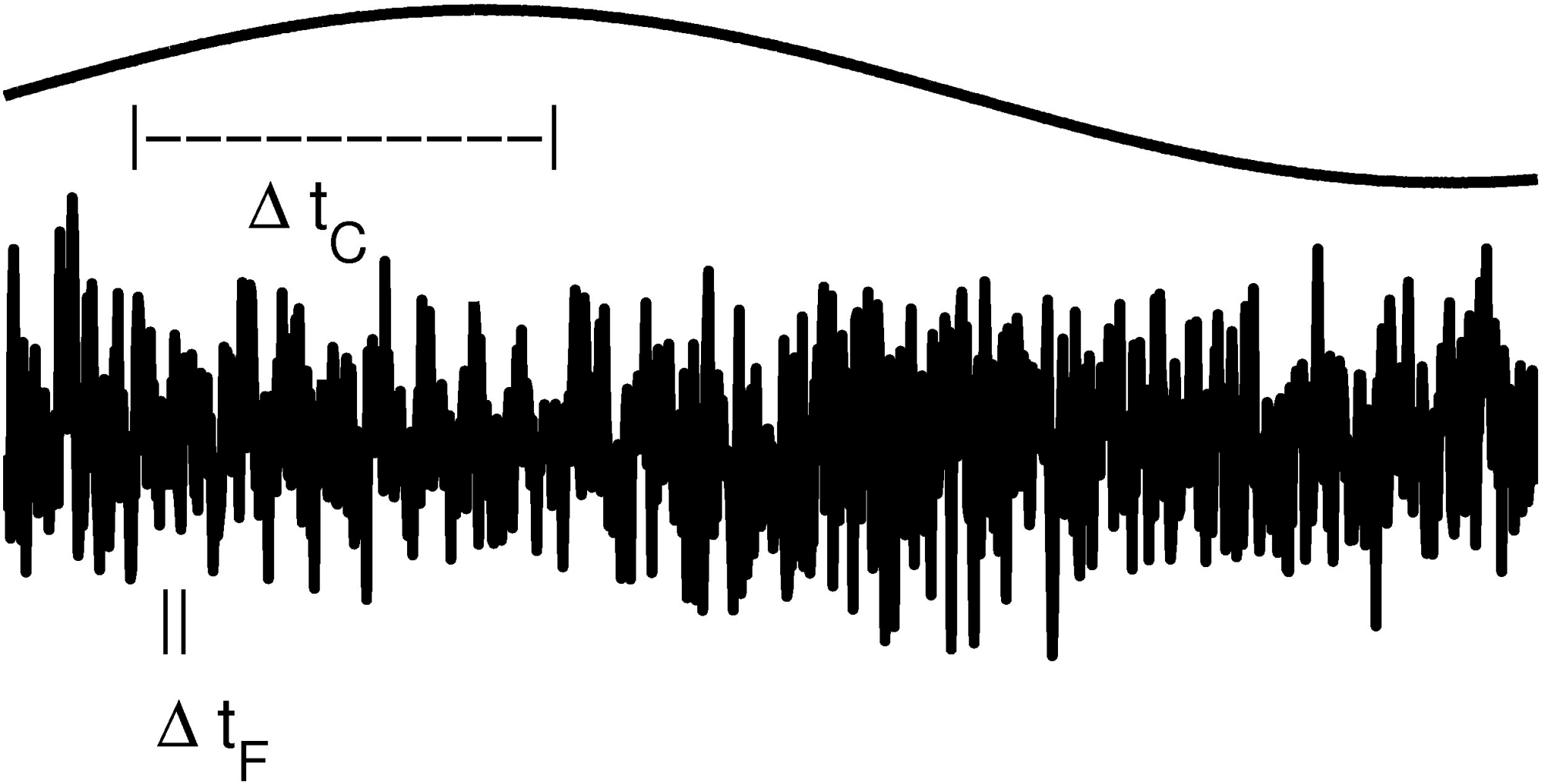
Slow Climate Modes $$\frac{d\alpha(t)}{dt} = G(\alpha(t), \beta(t))$$

Fast Weather Modes $$\frac{d\beta(t)}{dt} = \frac{1}{\varepsilon} F(\alpha(t), \beta(t))$$

Assumption:
Time scale separation

Reduced Model $$d\alpha(t) = G_{\text{eff}}(\alpha(t))dt + D_{\text{eff}}(\alpha(t))dW$$

Reduced Order Models



Stochastic Modeling

Advanced Review

Stochastic climate theory and modeling

Christian L. E. Franzke,[1,*] Terence J. O'Kane,[2] Judith Berner,[3] Paul D. Williams[4] and Valerio Lucarini[1,5]

Stochastic methods are a crucial area in contemporary climate research and are increasingly being used in comprehensive weather and climate prediction models as well as reduced order climate models. Stochastic methods are used as subgrid-scale parameterizations (SSPs) as well as for model error representation, uncertainty quantification, data assimilation, and ensemble prediction. The need to use stochastic approaches in weather and climate models arises because we still cannot resolve all necessary processes and scales in comprehensive numerical weather and climate prediction models. In many practical applications one is mainly interested in the largest and potentially predictable scales and not necessarily in the small and fast scales. For instance, reduced order models can simulate and predict large-scale modes. Statistical mechanics and dynamical systems theory suggest that in reduced order models the impact of unresolved degrees of freedom can be represented by suitable combinations of deterministic and stochastic components and non-Markovian (memory) terms. Stochastic approaches in numerical weather and climate prediction models also lead to the reduction of model biases. Hence, there is a clear need for systematic stochastic approaches in weather and climate modeling. In this review, we present evidence for stochastic effects in laboratory experiments. Then we provide an overview of stochastic climate theory from an applied mathematics perspective. We also survey the current use of stochastic methods in comprehensive weather and climate prediction models and show that stochastic parameterizations have the potential to remedy many of the current biases in these comprehensive models.

How to cite this article:
WIREs Clim Change 2015, 6:63–78. doi: 10.1002/wcc.318

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Stochastic Mode Reduction

Equations of motions

$$\frac{du}{dt} = F + Lu + I(u,u)$$

Energy conservation

$$u * I(u,u) = 0$$

Decompose u into (x,y)

x: slow compoment
y: fast component

Stochastic Mode Reduction

$$\frac{dx_i}{dt} = F_i + \sum_j \left(L_{ij}x_j + I_{ij}x_i x_j\right)$$
$$+ \sum_p \left(L_{ip}y_p + I^M_{ip}y_p x_i + I^A_{ip}y_p y_p + \dots\right)$$
$$\frac{dy_p}{dt} = \sum_j \left(L_{pj}x_j + F_p - I^M_{pj}x_j x_j - I^A_{pj}y_p x_j + \dots\right)$$
$$- \frac{\gamma_p}{\varepsilon}y_p + \frac{\sigma_p}{\sqrt{\varepsilon}}\dot{W}_p$$

e.g. Ohrnstein-Uhlenbeck Process
→ Continous time version of AR(1)

Fast nonlinear interactions: I(y,y)

Majda et al.1999, 2001, 2008; Franzke et al. 2005; Franzke and Majda 2006

Stochastic Mode Reduction

Equations of motions

$$\frac{du}{dt} = F + Lu + I(u,u)$$

Energy conservation

$$u * I(u,u) = 0$$

Mode Reduction

Reduced Model:

$$dx = \left(\widetilde{F} + \widetilde{L}x + \widetilde{I}(x,x) + M(x,x,x)\right)dt + \sigma_A dW_A + \sigma_M(x) dW_M$$

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L} \cdot \vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon} y + \frac{\sigma}{\sqrt{\varepsilon}} \dot{W}(t).$$

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L}\cdot\vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon}y + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}(t).$$

Solve for y:

$$y(t) = e^{-\gamma t/\varepsilon}y(0) + \int_0^t e^{-\gamma(t-s)/\varepsilon}[b_3 x_1(s)x_2(s) - \vec{L}\cdot\vec{x}(s) + F_3(s)]\,ds + g(t)$$

$$g(t) = \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t e^{-\gamma(t-s)/\varepsilon}\,dW(s).$$

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L}\cdot\vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon}y + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}(t).$$

For $\varepsilon \to 0$.

$$\int_0^t e^{-\gamma(t-s)/\varepsilon}[b_3 x_1(s)x_2(s) - \vec{L}\cdot\vec{x}(s) + F_3]\,ds$$

$$\to \frac{\varepsilon}{\gamma}[b_3 x_1(t)x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3].$$

$$g(t)\,dt \to \sqrt{\varepsilon}\frac{\sigma}{\gamma}\,dW(t).$$

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L}\cdot\vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon}y + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}(t).$$

Plug into equation for x:

$$d\vec{x}(t) = \vec{F}_0(\vec{x}; F)\,dt + \frac{\varepsilon}{\gamma}\vec{L}[b_3 x_1 x_2 - \vec{L}\cdot\vec{x} + F_3]\,dt$$
$$+ \frac{\varepsilon}{\gamma}\begin{bmatrix} b_1 x_2(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \\ b_2 x_1(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \end{bmatrix} dt$$
$$+ \sqrt{\varepsilon}\frac{\sigma}{\gamma}\begin{bmatrix} L_1 + b_1 x_2(t) \\ L_2 + b_2 x_1(t) \end{bmatrix} \circ dW(t).$$

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L}\cdot\vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon}y + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}(t).$$

Plug into equation for x:

$$d\vec{x}(t) = \vec{F}_0(\vec{x}; F)\,dt + \frac{\varepsilon}{\gamma}\vec{L}[b_3 x_1 x_2 - \vec{L}\cdot\vec{x} + F_3]\,dt$$
$$+ \frac{\varepsilon}{\gamma}\begin{bmatrix} b_1 x_2(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \\ b_2 x_1(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \end{bmatrix} dt$$
$$+ \sqrt{\varepsilon}\frac{\sigma}{\gamma}\begin{bmatrix} L_1 + b_1 x_2(t) \\ L_2 + b_2 x_1(t) \end{bmatrix} \circ dW(t).$$

CAM Noise

Stochastic Mode Reduction

$$\frac{d\vec{x}}{dt} = D\vec{x} + (\omega + a_1 x_1 + a_2 x_2)\vec{x}^{\perp} + \vec{F} + \vec{L}y + \begin{bmatrix} b_1 x_2 y \\ b_2 x_1 y \end{bmatrix},$$

$$\frac{dy}{dt} = -\vec{L}\cdot\vec{x} + b_3 x_1 x_2 + F_3 - \frac{\gamma}{\varepsilon}y + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}(t).$$

Plug into equation for x:

$$d\vec{x}(t) = \vec{F}_0(\vec{x}; F)\,dt + \frac{\varepsilon}{\gamma}\vec{L}[b_3 x_1 x_2 - \vec{L}\cdot\vec{x} + F_3]\,dt$$

$$+ \frac{\varepsilon}{\gamma}\begin{bmatrix} b_1 x_2(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \\ b_2 x_1(t)\{b_3 x_1(t) x_2(t) - \vec{L}\cdot\vec{x}(t) + F_3\} \end{bmatrix} dt$$

$$+ \sqrt{\varepsilon}\frac{\sigma}{\gamma}\begin{bmatrix} L_1 + b_1 x_2(t) \\ L_2 + b_2 x_1(t) \end{bmatrix} \circ dW(t).$$

Cubic Term

CAM Noise

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Constraints on Stochastic Climate Models

$$\frac{dx_i}{dt} = (\text{Bare truncation})_i + A_i\vec{x}$$

$$+ F_i + \tilde{B}_i(\vec{x}, \vec{x}) + \sum_p \sigma_{ip}^A \dot{W}_{ip}^A$$

$$- \sum_p \frac{{I_{ip}^M}^2}{\gamma_p} x_i^3 + \sum_{j \neq i, p} \frac{I_{ip}^M I_{pj}^M}{\gamma_p} x_i x_j^2$$

$$+ \sum_p \frac{\sigma_p}{\gamma_p} \left(L_{ip} - I_{ip}^M x_i\right) \circ \dot{W}_p$$

Constraints on Stochastic Climate Models

$$\frac{1}{2}\frac{dx_i^2}{dt} = \left(\sum_{j \neq i} \sum_p \frac{A_{ijp} A_{pij}}{\gamma_p} x_j^2 - \sum_p \frac{I_{ip}^2}{\gamma_p} x_i^2 \right) x_i^2$$

$$= \sum_i \left(\sum_{j \neq i} \tilde{A}_{ij} x_j^2 - \tilde{I}_i x_i^2 \right) x_i^2$$

$$E = |\vec{x}|^2, \tilde{A}_{ij} = \sum_p \frac{I_{ip}^M I_{pj}^M}{\gamma_p} \text{ and } \tilde{I}_i = \sum_p \frac{{I_{ip}^M}^2}{\gamma_p}$$

Only considering cubic terms

Majda et al. 2009; Peavoy et al. 2015

Constraints on Stochastic Climate Models

$$\frac{1}{2}\frac{dE}{dt} = (\vec{x}^2)^T \left(\tilde{A} - \tilde{I}\mathbb{E}\right) \vec{x}^2 = (\vec{x}^2)^T Q \vec{x}^2$$

Stability: $\frac{dE}{dt} < 0$

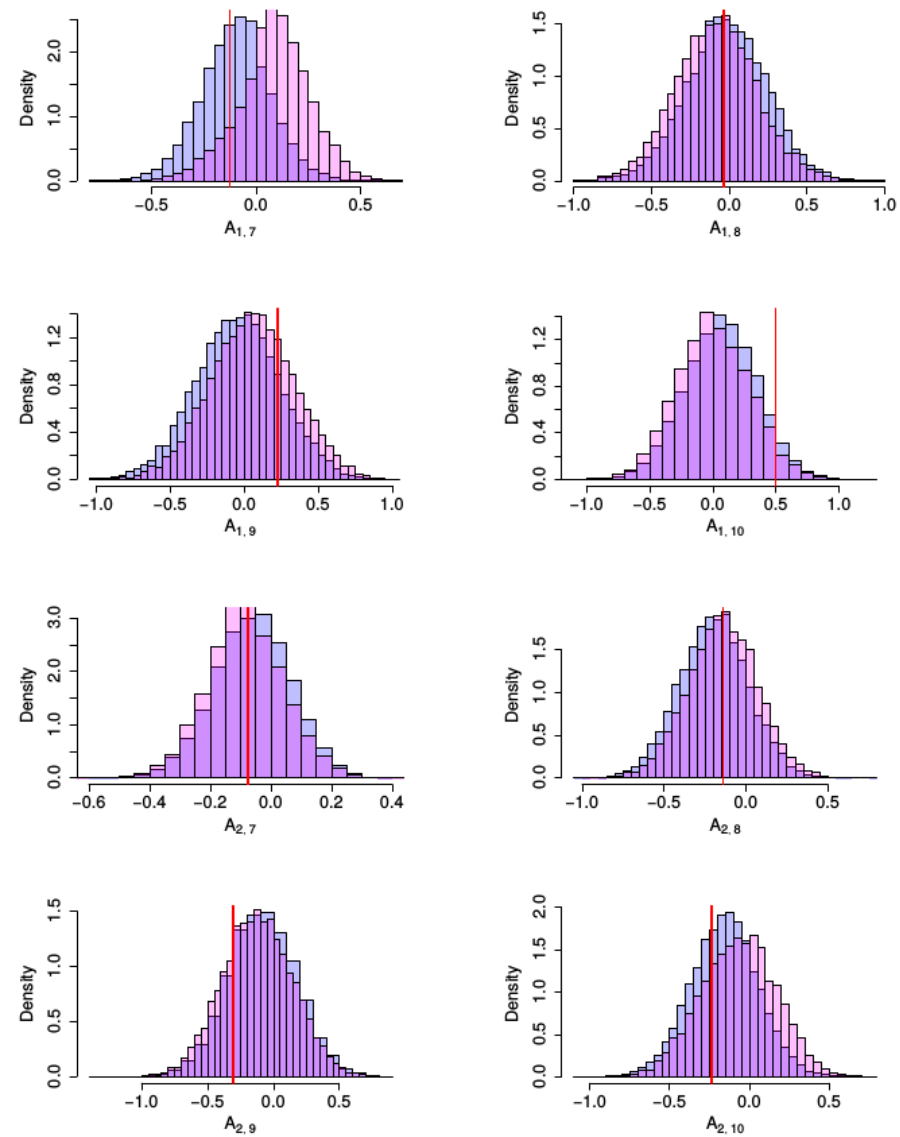
→ Quadratic form Q is negative-definite

→ Allows the system to be linearly unstable

Majda et al. 2009; Peavoy et al. 2015

Physical Constraints

Without constraint about 40% of parameter estimates lead to unstable solutions



Peavoy et al. 2015

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Bayesian Parameter Estimation Procedure

$$d\boldsymbol{X}_t = \boldsymbol{\mu}(\boldsymbol{X}_t, \boldsymbol{\theta})dt + \boldsymbol{a}(\boldsymbol{X}_t, \boldsymbol{\theta})d\boldsymbol{W}_t\,, \;\; \boldsymbol{X}_0 = \boldsymbol{x}_0\,, \;\; t \in [0, T]$$

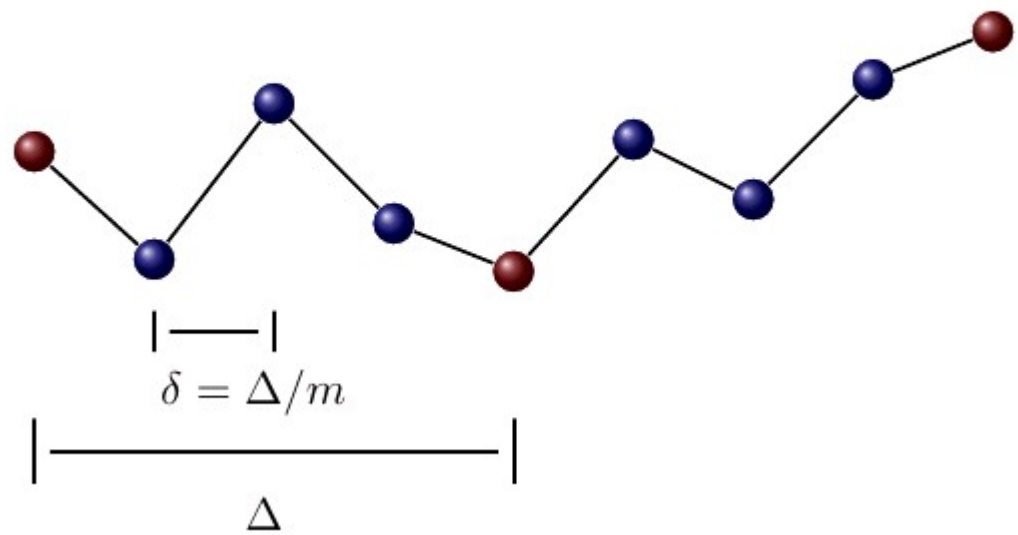
Discretisation: Euler-Maruyama scheme

$$\boldsymbol{X}_{k+1} = \boldsymbol{X}_k + \boldsymbol{\mu}(\boldsymbol{X}_k, \boldsymbol{\theta})\Delta t_k + \boldsymbol{a}(\boldsymbol{X}_k, \boldsymbol{\theta})(\boldsymbol{W}_{k+1} - \boldsymbol{W}_k)$$

Likelihood based parameter estimation (MCMC)

$$l_N^{\text{Euler}}(\boldsymbol{\theta}) = -\frac{1}{2\Delta t_k}\sum_{k=0}^{N-1}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k - \Delta t_k \boldsymbol{\mu}(\boldsymbol{x}_k, \boldsymbol{\theta}))^T \Sigma(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k - \Delta t_k \boldsymbol{\mu}(\boldsymbol{x}_k, \boldsymbol{\theta}))$$

Imputing of Data



Modified Linear Bridge

Peavoy et al. 2015

How to sample negative-definite matrices?

Wishart Distribution

Truncated Normal Algorithm:

A n×n matrix is negative definite if and only if all k≤n leading principal minors obey $|M^k|(-1)^k > 0$.
The k-th principal minor is the determinant of the upper left k×k sub-matrix.

for $i = 1$ to n **do**
  $U_i = 0$
  for $j = i$ to n **do**
    $x = -\left(\sum^{j}_{\substack{k \neq i \\ k=1}} (-1)^{i+k} M_{ik} |M^{(j)}_{\{-i\},\{-k\}}|\right) / |M^{(j)}_{\{-i\},\{-i\}}|$
  end for
  if $x < U_i$ **then**
    $U_i = x$
  end if
  $M_{ii} \sim \mathcal{N}_-(\mu_i, U_i, \sigma_i^2)$
end for

Diagonal Elements

for $i = 1$ to n **do**
  for $j = i+1$ to n **do**
    $u^+ = \infty$
    $u^- = -\infty$
    for $k = j$ to n **do**
      Calculate $a_{ij}^{(k)}$, $b_{ij}^{(k)}$ and $c_{ij}^{(k)}$ and solve $a_{ij}^{(k)}x^2 + b_{ij}^{(k)}x + c_{ij}^{(k)} = 0$.
      Set mn $= \min(x_1, x_2)$ and mx $= \max(x_1, x_2)$
    end for
    if $mx < u^+$ **then**
      $u^+ = mx$
    end if
    if $mn > u^-$ **then**
      $u^- = mn$
    end if
    $M_{ij} \sim \mathcal{N}_-^+(\mu_{ij}, u^-, u^+, \sigma_{ij}^2)$
  end for
end for

Off-Diagonal Elements

Peavoy et al. 2015

Modelling Memory Effects via Latent Variables

$$dx = (a_1 + a_2 x + a_3 x^2 + a_4 x^3)dt + ydt$$

$$dy = -\gamma y dt + \sigma dW$$

Red Noise

Peavoy et al. 2015

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Triad Model Example

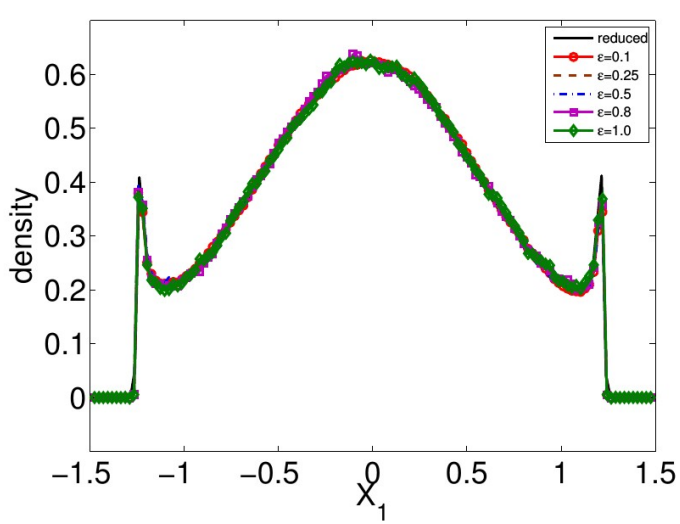
$$
\begin{aligned}
\frac{dx_1}{dt} &= \frac{b_1}{\epsilon} x_2 y_1 \\
\frac{dx_2}{dt} &= \frac{b_2}{\epsilon} x_1 y_1 \\
\frac{dy_k}{dt} &= \frac{b_3}{\epsilon} x_1 x_2 \delta_{1,k} - \mathrm{Re}\frac{ik}{2\epsilon^2} \sum_{p+q+k=0} \hat{u}_p^* \hat{u}_q^* \\
\frac{dz_k}{dt} &= -\mathrm{Im}\frac{ik}{2\epsilon^2} \sum_{p+q+k=0} \hat{u}_p^* \hat{u}_q^*,
\end{aligned}
$$

Model Reduction

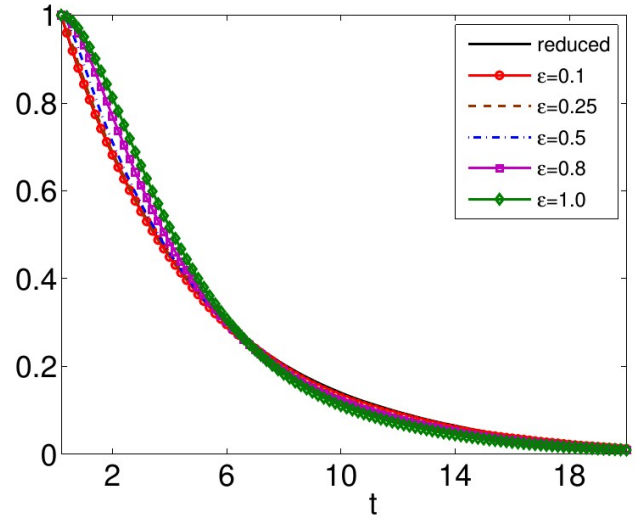
$$
\begin{aligned}
dx_1(t) &= \frac{b_1}{\gamma}(b_3 x_2^2(t) + \frac{\sigma^2}{2\gamma} b_2) x_1(t) dt + \frac{\sigma}{\gamma} b_1 x_2(t) dW_t \\
dx_2(t) &= \frac{b_2}{\gamma}(b_3 x_1^2(t) + \frac{\sigma^2}{2\gamma} b_1) x_2(t) dt + \frac{\sigma}{\gamma} b_2 x_1(t) dW_t
\end{aligned}
$$

Peavoy et al. 2015

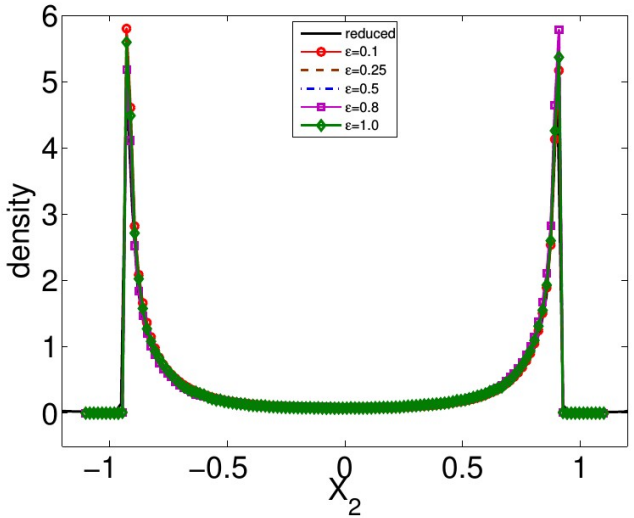
Triad Model Example



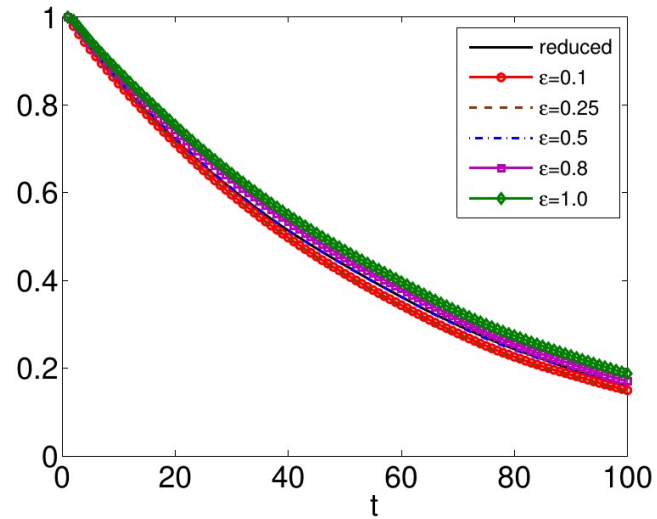
(a) Stationary distribution for x_1



(b) Autocorrelation function for x_1



(c) Stationary distribution for x_2



(d) Autocorrelation function for x_2

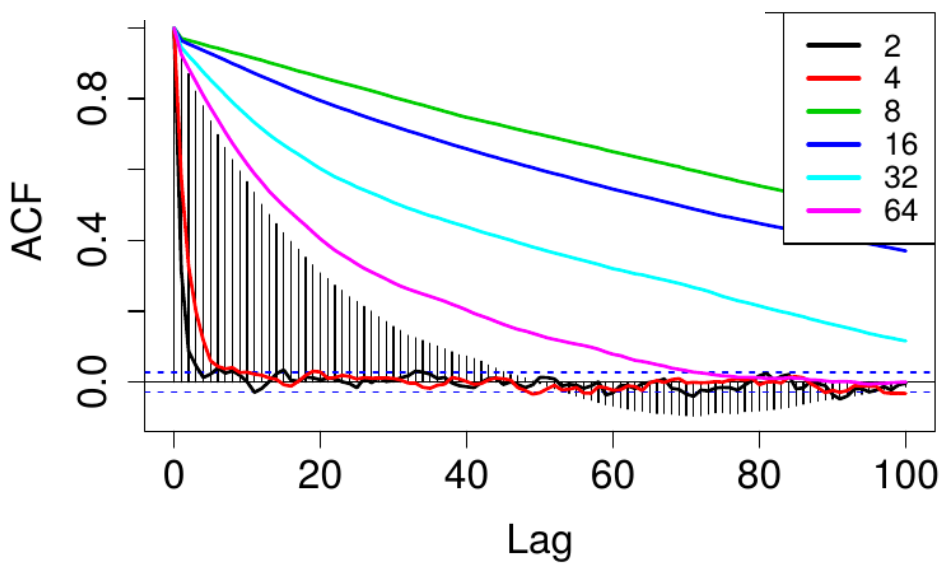
Peavoy et al. 2015

Test: Chaotic Lorenz Model

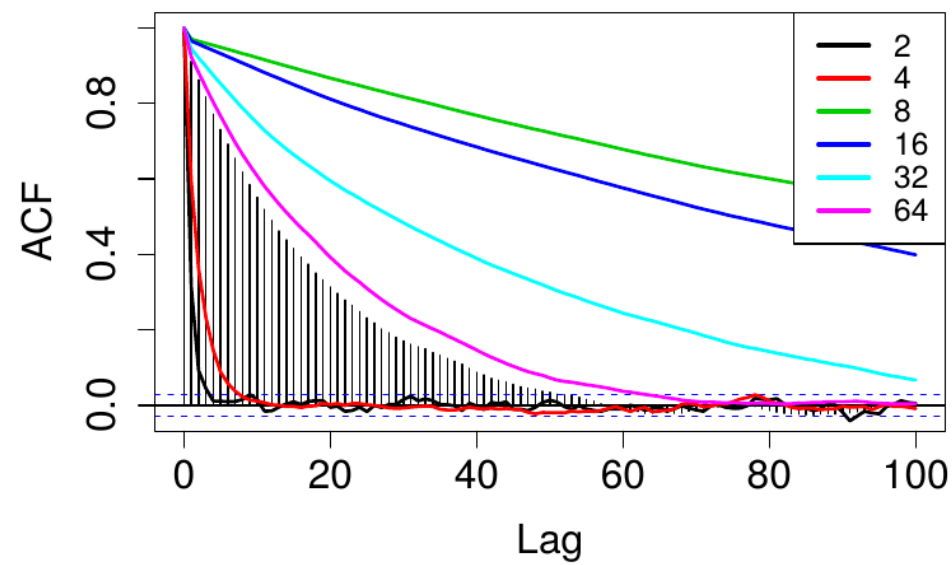
$$\frac{dx}{dt} = x - x^3 + \frac{4}{90\epsilon}y_2$$
$$\frac{dy_1}{dt} = \frac{10}{\epsilon^2}(y_2 - y_1)$$
$$\frac{dy_2}{dt} = \frac{1}{\epsilon^2}(28y_1 - y_2 - y_1y_3)$$
$$\frac{dy_3}{dt} = \frac{1}{\epsilon^2}(y_1y_2 - \frac{8}{3}y_3)\,.$$

$$dX_t = (a_1 + a_2X_t + a_3X_t^2 + a_4X_t^3)dt + \sigma dW_t$$

Reduced order model:



ε = 0.1



ε = 0.01

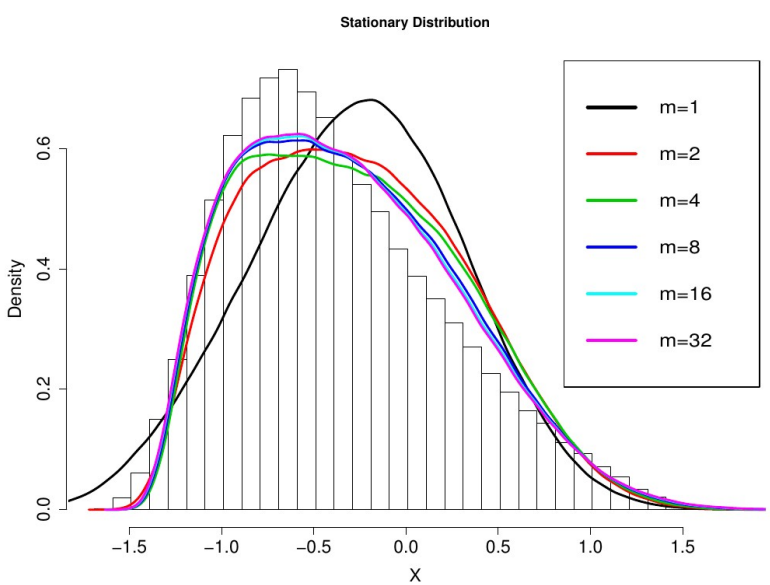
Peavoy et al. 2015

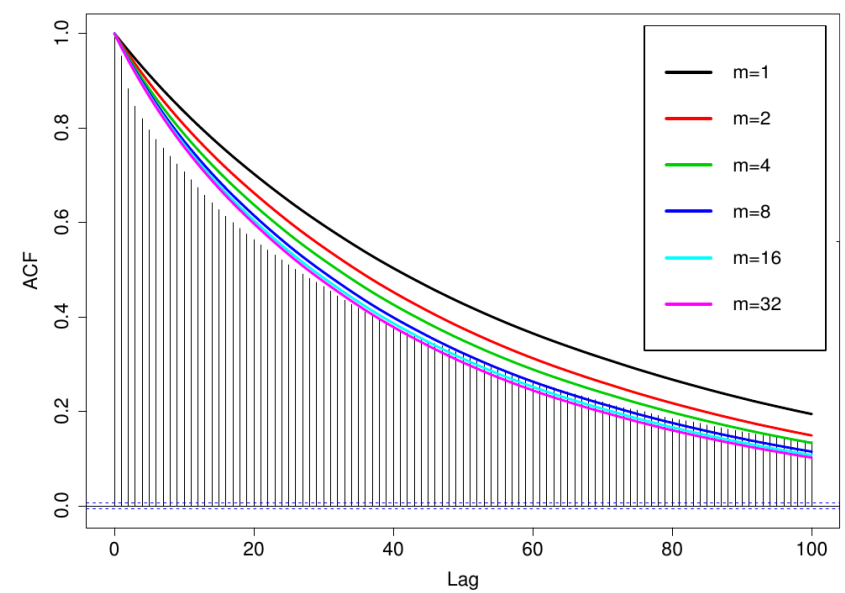
Flow over topography on a ß-plane

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = 0$$

$$q = \Delta\psi + h, \quad \frac{dU}{dt} = \frac{1}{4\pi^2}\int h\frac{\partial \psi}{\partial x}dxdy\,.$$

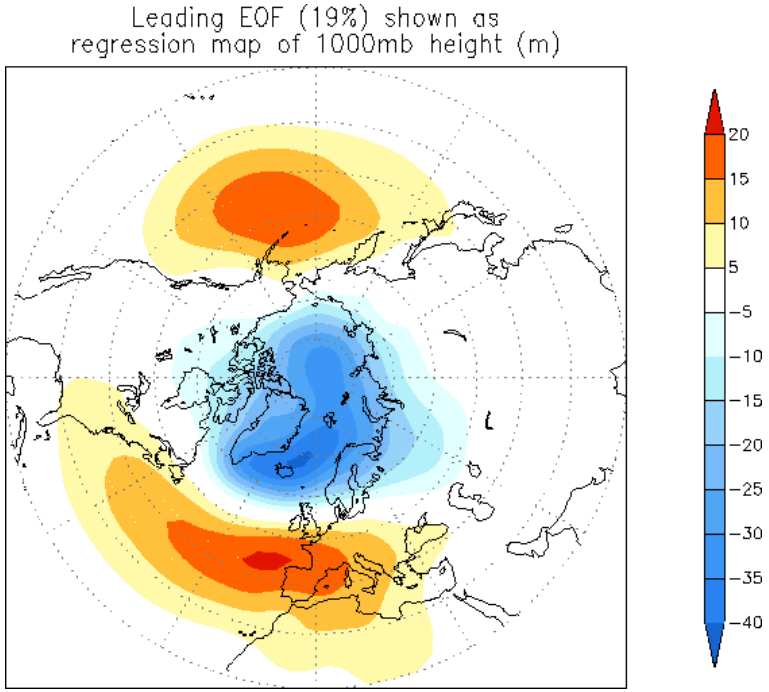
$$dU = (a_1 + a_2U + a_3U^2 + a_4U^3)dt + (\sigma_1 + \sigma_2U)dB_t$$





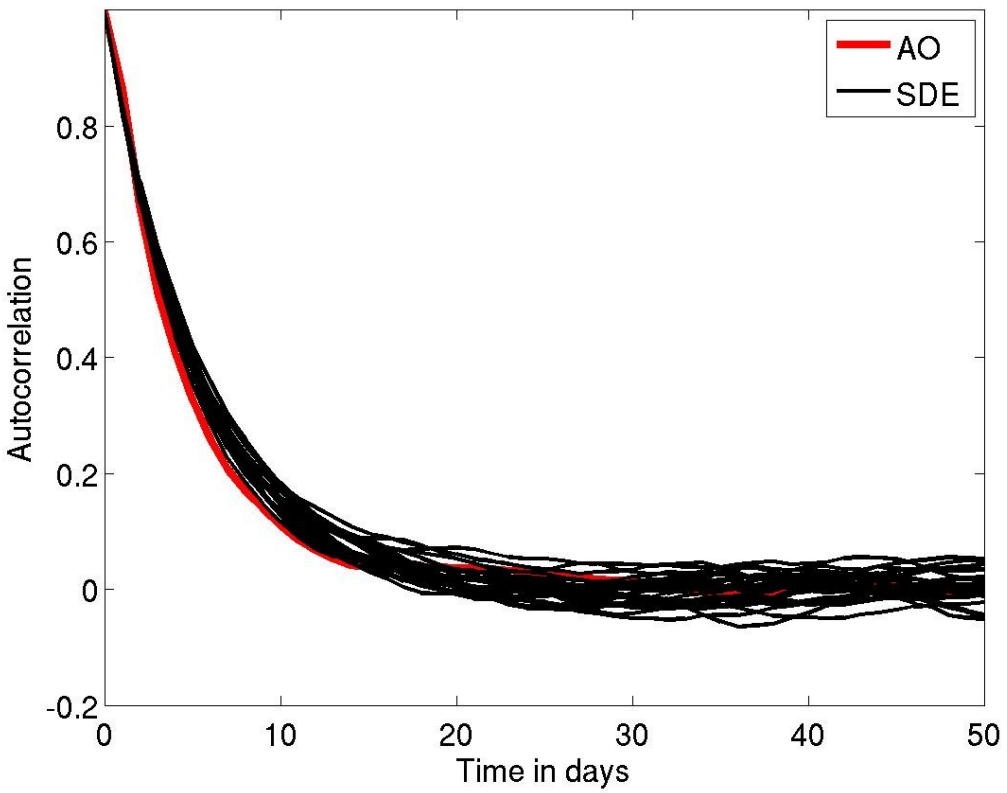
Peavoy et al. 2015

Arctic Oscillation Index



From NCAR-NCEP reanalysis data covering period 1948-2010

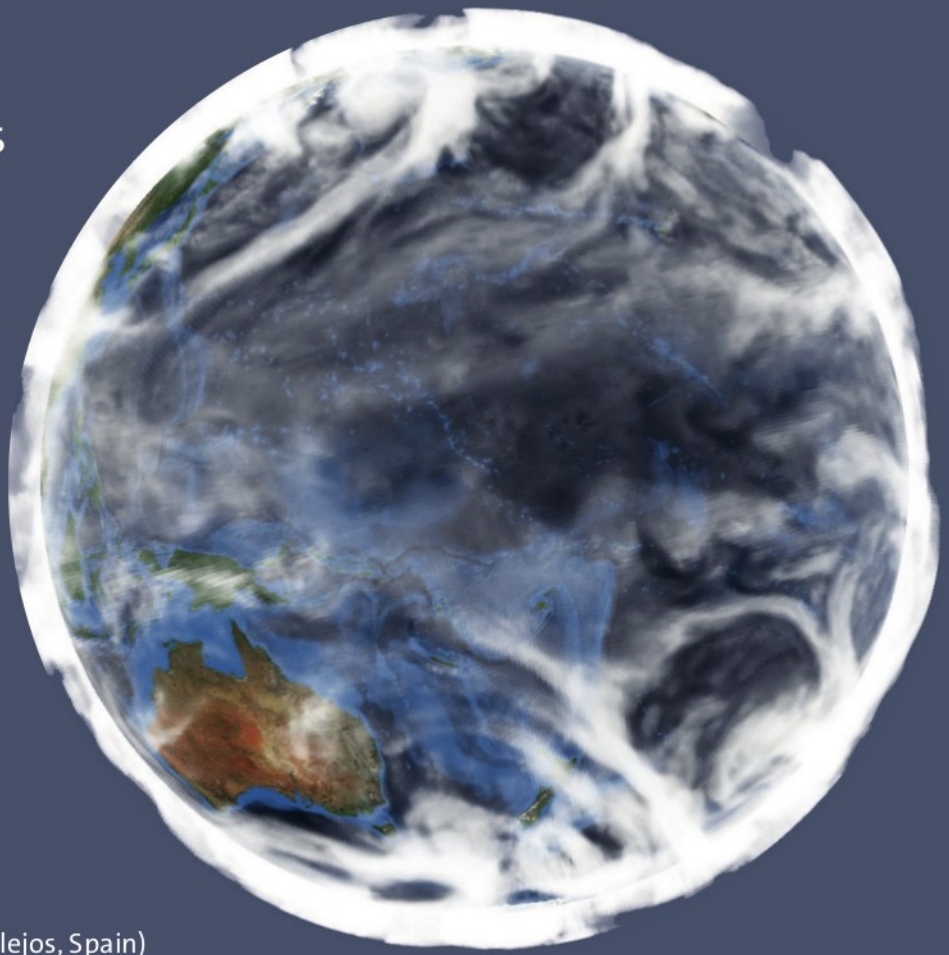
Autocorrelation Function



DAMES Conference 2016

Biannual International Conference Series
Data Analysis and Modeling in Earth Sciences

26-28 September 2016
Hamburg, Germany



Scientific Themes

> Instabilities and Predictability
> Multi-Scale Effects and Stochastic Parameterizations
> Data assimilation and Big Data
> Coupled Data Assimilation
> Complex Network Approaches
> Stochastic and Dynamical Systems Approaches
> Nonlinear Time Series Analysis
> Modeling of Extreme Events
> Environmental Risk Analysis and Sustainability

Invited Speakers

> Sonia I. Seneviratne (ETH Zürich, Switzerland)
> Sandra Chapman (University of Warwick, UK)
> Marc Bocquet (CEREA, Joint laboratory of École des Ponts ParisTech, France)
> Laure Zanna (Oxford University, UK)
> Erik Chavez (Imperial College London, UK)
> Emilio Hernandez-Garcia (Instituto de Fisica Interdisciplinar y Sistemas Complejos, Spain)
> Mickael Chekroun (UCLA, USA)
> Judith Berner (NCAR, USA)
> Francisco Doblas-Reyes (ICREA-BSC, Spain)

Information and registration: http://www.clisap.de/dames

Deadline: 15.06.2016

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Summary

- Normal form for reduced stochastic climate models predict a **cubic nonlinear drift** and a **correlated additive and multiplicative CAM noise**.
- Bayesian Framework for Physics Constrained Parameter Estimation
- Reduced stochastic climate models perform well

References:

- Majda, Franzke and Crommelin, 2009: Normal forms for reduced stochastic climate models. Proc. Natl. Acad. Sci. USA, 106, 3649-3653.
- Peavoy, Franzke and Roberts, 2015: Physics constrained parameter estimation of stochastic differential equations. Comp. Stat. Data Ana., 83, 182-199.
- Franzke, C., T. O'Kane, J. Berner, P. Williams and V. Lucarini, 2015: Stochastic Climate Theory and Modelling. WIREs Climate Change, 6, 63-78.
- Gottwald, G., D. Crommelin and C. Franzke, 2016: Stochastic Climate Theory. To appear in Nonlinear and Stochastic Climate Dynamics, Cambridge University Press.