Markov Switching Multivariate Space Time model for weather variables

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Abstract. *In this talk, we propose a Markov Switching model which is based on non-separable cross-covariance functions for multivariate space-time data. This model is built as a stochastic weather generator and it is applied to simulate weather data on a large West part of France.*

Keywords. *Multivariate spatio-temporal process, Markov Swtiching Autoregressive model, Matérn covariance.*

1 Introduction

Stochastic weather generators (SWGs) are statistical models that aim at quickly simulating realistic random sequences of atmospheric variables such as temperature, precipitation and wind. Ideally, spatio-temporal dynamics and correlation structures among the variables of interest, as well as weather persistence and natural variability, have to be reproduced accurately in a distributional sense by SWGs [4].
Weather type SWGs include a discrete variable and multivariate statistical distributions modeling the climatic variables conditional on this discrete variable (referred to as *regime* or *weather type*). In many multivariate spatio-temporal SWGs, the discrete variable is inferred a priori (using mixture models for instance) and the spatio-temporal model of the weather variables is fitted conditionally to the clustering [1].
In this talk, we propose a Markov Switching model which allows to perform a global estimation of all the parameters. It has been demonstrated in [2] that it usually leads to more performant SWGs. The main originality of the model is that the matrices of the autoregressive models are defined from a non-separable cross-covariance function for multivariate space-time data.

2 Spatio-temporal Markov Switching model

A MSVAR model is defined as a discrete time stochastic process with two components $(X_k, \mathbf{Y}_k)$ with values in $\{1, \cdots, M\} \times \mathbb{R}^d$ and satisfying the following conditions:
1. The first component is hidden and models a first order Markov chain $\{X_k\}_{k\in\mathbb{Z}}$ taking its values in the set of states $\{1, \cdots, M\}$. The conditional distribution of X_k given $\{X_{k'}, \mathbf{Y}_{k'}\}_{k'<k}$ depends only on X_{k-1} and $\mathbf{Y}_{k-1}$. The transition probabilities are denoted $p(x_k|x_{k-1}, \mathbf{y}_{k-1}) = P(X_k = x_k|X_{k-1} = x_{k-1}, \mathbf{Y}_{k-1} = \mathbf{y}_{k-1})$.

This process is often called regime. In meteorological applications, it usually describes the weather type (e.g. cyclonic, anticyclonic).
2. The second component $\mathbf{Y}_k$ describes the evolution of the observed variables. In first order models, the conditional distribution of $\mathbf{Y}_k$ given $\{\mathbf{Y}_{k'}\}_{k'<k}$ and $\{X_{k'}\}_{k'\leq k}$ only depends on X_k and $\mathbf{Y}_{k-1}$.

$$\mathbf{Y}_k = A_0^{(X_k)} + A_1^{(X_k)} \mathbf{Y}_{k-1} + \left(\Sigma^{(X_k)}\right)^{1/2} \boldsymbol{\epsilon}_k \quad (1)$$

where the unknown parameters $A_0^{(x)}$ are vectors in $\mathbb{R}^d$, $A_1^{(x)}$ are matrices in $\mathbb{R}^{d,d}$ and $\Sigma^{(x)}$ are positive symetric matrices in $\mathbb{R}^{d,d}$. $\{\boldsymbol{\epsilon}_k\}_{k\in\mathbb{Z}}$ is a multivariate sequence of independent and identically distributed Gaussian variables, with zero mean and unit variance, independent of the Markov chain $\{X_k\}$.

One of the main drawbacks in the previous definition of MSVAR, is that it is usually difficult to infer parametric shape for the autoregressive matrices. And without any parametric model, it is not possible to extrapolate the simulation to locations where no observation is available. The aim of this work is to bridge this gap, by proposing a parametric approach for modeling the multivariate autoregressive matrices.

Let us assume that, for each location $s \in \mathbb{R}^2$ and each time $k \in \mathbb{Z}$, $\mathbf{Y}_k(s) \in \mathbb{R}^d$. Eq. (1), can be rewritten by

$$\mathbf{Y}_k(s)|\mathbf{Y}_{k-1}(s') = \mathbf{y}, X_k = x \sim \mathcal{N}\left(\mathbf{A}_0^{(x)}(s) + \mathbf{A}_1^{(x)}(s,s')\mathbf{y}, \boldsymbol{\Sigma}^{(x)}(s,s')\right), \quad (2)$$

thereby defining the conditional distribution of a multivariate Gaussian vector. The intercept vector $\mathbf{A}_0^{(x)}(s) \in \mathbb{R}^d$, the autoregressive matrix $\mathbf{A}_1^{(x)}(s,s') \in \mathbb{R}^{d\times d}$ and the innovation covariance $\boldsymbol{\Sigma}^{(x)}(s,s') \in \mathbb{R}^{d\times d}$ for all regime $x \in \{1,\cdots,M\}$. This model can be generalized to higher autoregressive order.
Let us now write the joint distribution of $(\mathbf{Y}_{k-u}, \mathbf{Y}_k)^T$ given $X_k = x$ as

$$\left(\mathbf{Y}_{k-u}(s'), \mathbf{Y}_k(s)\right)^T |X_k = x \sim \mathcal{N}\left(\left(\begin{array}{c} \mu^{(x)}(s') \\ \mu^{(x)}(s) \end{array}\right), \mathbf{C}^{(x)}(|s-s'|,u)\right) \quad (3)$$

where ^{T} holds for transpose operator. And $\mathbf{C}^{(x)}(h,u)$ can be decomposed as

$$\mathbf{C}^{(x)}(h,u) = \left(\begin{array}{cc} C^{(x)}(0,0) & C^{(x)}(h,u) \\ C^{(x)}(h,u)^T & C^{(x)}(0,0) \end{array}\right)$$

From the properties of Gaussian vectors,

$$\begin{aligned}
&\mathbf{Y}_k(s)|\mathbf{Y}_{k-1}(s') = \mathbf{y}, X_k = x \sim \mathcal{N}\left(\mathbf{m}_x, \mathbf{C}'_x\right) \\
&\mathbf{m}_x = \mu^{(x)}(s) + C^{(x)}(|s-s'|,1)\left(C^{(x)}(0,0)\right)^{-1}(\mathbf{y} - \mu^{(x)}(s')) \\
&\mathbf{C}'_x = C^{(x)}(0,0) - C^{(x)}(|s-s'|,1)\left(C^{(x)}(0,0)\right)^{-1} C^{(x)}(|s-s'|,1)
\end{aligned}$$

By identification of the means in eq. (2) and (3), for each regime x, relations between $\left(\mathbf{A}_0^{(x)}, \mathbf{A}_1^{(x)}\right)$ and $\left(\boldsymbol{\mu}^{(x)}, \mathbf{C}^{(x)}\right)$ are given by the definition of the conditional distribution of $Y_t|Y_{t-1} = y_{t-1}$.

$$\begin{aligned}
&\mathbf{A}_0^{(x)}(s) = \mu^{(x)}(s) - C^{(x)}(|s-s'|,1)\left(C^{(x)}(0,0)\right)^{-1}\mu^{(x)}(s),\ \mathbf{A}_0(s) \in \mathbb{R}^d \\
&\mathbf{A}_1^{(x)}(s,s') = C^{(x)}(|s-s'|,1)\left(C^{(x)}(0,0)\right)^{-1},\ \mathbf{A}_1(s',s) \in \mathbb{R}^{d\times d}
\end{aligned}$$

From Yule-Walker equations or the properties of Gaussian vector above, for all x,

$$\boldsymbol{\Sigma}^{(x)}(s,s') = C^{(x)}(0,0) - \mathbf{A}_1^{(x)}(s,s')C^{(x)}(|s-s'|,1)$$

Now, a covariance model has to be defined.

3 Covariance model

Covariance function $C^{(x)}$ in the regime x has to describe the covariance between the components of $\mathbf{Y}$, the spatial covariances and the first order temporal covariance.
In (Bourotte, 2016), a Matern non separable space time model is proposed. It is defined as follows

$$C_{ij}(\mathbf{h},u) = \frac{\sigma_i\sigma_j}{\psi(u^2)}\rho_{ij}\mathcal{M}\left(\frac{\mathbf{h}}{\psi(u^2)^{1/2}};r_{ij},\nu_{ij}\right) \tag{4}$$

where $\mathcal{M}$ is the Matern covariance, r is such that $1/r_{ij}$ models the cross-range of variables i and j and ν models the smoothness of the underlying process. When ν tends to infinity it leads to the Gaussian covariance and when $\nu = \frac{1}{2}$ to the exponential one.
Constraints are needed to get a well defined covariance model

$$r_{ij} = \left(\frac{r_i^2 + r_j^2}{2}\right)^{1/2}, \quad \nu_{ij} = \frac{\nu_i + \nu_j}{2}, \quad \rho_{ij} = \beta_{ij}\frac{\Gamma(\nu_{ij})}{\Gamma(\nu_i)^{1/2}\Gamma(\nu_j)^{1/2}}\frac{r_i^{\nu_i}r_j^{\nu_j}}{r_{ij}^{2\nu_{ij}}} \tag{5}$$

where ν_i, r_i are positive for all $i = 1,\cdots,d$ and $\beta = (\beta_{ij})_{i,j=1}^d$ is a correlation matrix. The time function ψ may be defined as

$$\psi(x) = (\alpha x^a + 1)^b$$

However, if we focus on autoregressive models of order 1, it is sufficient to know $0 \leq \psi(1) = 1 + \psi_1$. FInally, parameters to be estimated are

$$\sigma \in \mathbf{R}_+{}^d,\ r \in \mathbf{R}_+{}^d,\ \nu \in \mathbf{R}_+{}^d,\ \beta \in [-1,1]^{d\times d} \text{ and positive},\ \psi_1 \in [0,+\infty)^d$$

Esitmation is made by maximizing the likelihood. In practice an Expectation-Maximisation (EM) algorithm is run. In the M step a numerical optimization is performed. The constrains on the parameters are tacken into account by a reparametrization.

4 Application

This section shows some preliminary results of an on-going work.
We consider 3 variables (daily mean wind intensity, daily mean temperature and solar radiation) at 20 locations in the West part of France (see Fig. 1) during the month of June, July and August (11 years). A model with 2 regimes is fitted. In each regime a mean is modeled by a linear trend with respect to the longitude and latitude.
In order to validate the model, we simulate 20 times during 11 years of summer months for the variables. Figure 1 shows that the model correctly captures the trend of the temperature and solar radiation mean. But the wind is spatially less regular and its mean is not well reproduced. Multivariate spatial covariances are plotted in Figure 2. The spatial covariances of temperature and solar radiation are pretty well captured. But for the wind, the model assumption of a common variance for all the locations leads to a srong over estimation of the variances of the wind in almost all the locations. Finally Figure **??** shows some temporal correlations. Here again the model is rather poor for the wind. Its correlation is clearly over estimated.

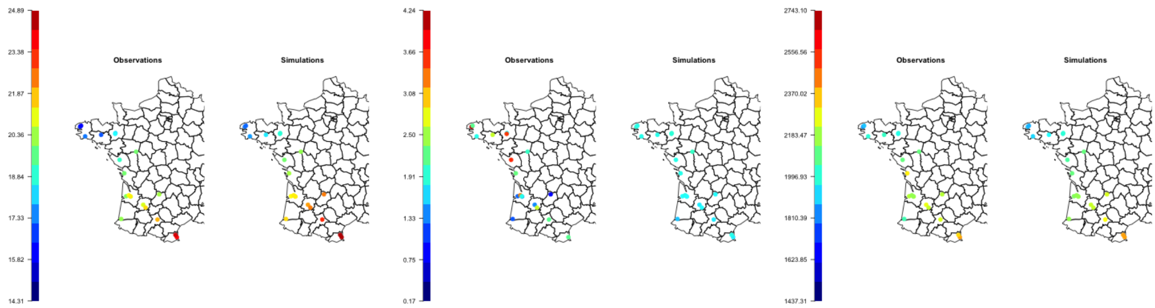

Figure 1: Observed mean (left) and simulated mean (right) for temperature (left panel), wind (middle panel) and solar radiation (right panel)

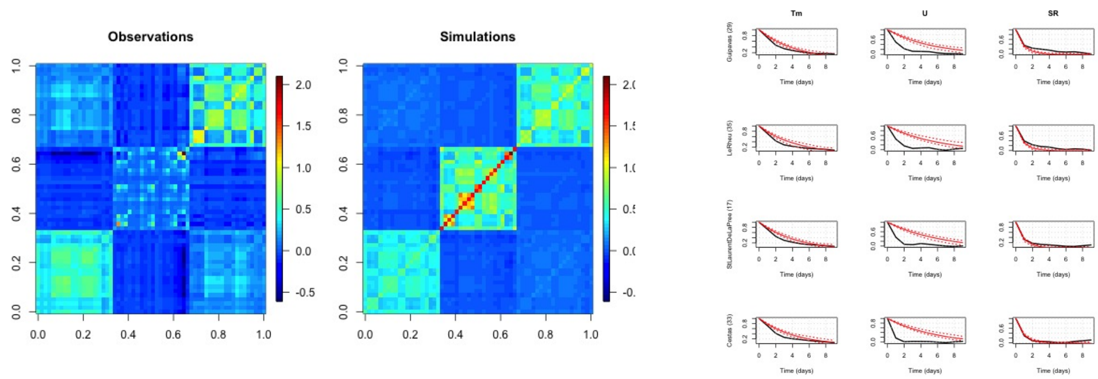

Figure 2: Left panel: observed spatial covariance (left) and simulated spatial covariance (right). In the covariances, the bottom block corresponds to the temperature, the middle one to the wind and the upper on to the solar radiation. In each diagonal block the spatial covariance is plotted. The extra diagonal blocks correspond to the cross covariances. Right panel: temporal correlations for 4 different locations (from North to South) and for temperature (left column), wind (middle column) and solar radiation (right column). Observation is plotted in black and simulation in red with a fluctuation interval.

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