

HECKE ALGEBRAS AND AFFINE FLAG VARIETIES IN CHARACTERISTIC p

TOBIAS SCHMIDT

ABSTRACT. Let G be a split semi-simple p -adic group and let \mathcal{H} be its Iwahori-Hecke algebra with coefficients in the algebraic closure $\overline{\mathbb{F}}_p$ of \mathbb{F}_p . Let \mathcal{F} be the affine flag variety associated with G . We show, in the simply connected simple case, that the K' -theory of \mathcal{F} with coefficients in $\overline{\mathbb{F}}_p$ admits an action of \mathcal{H} by Demazure operators and that this provides a model for the regular representation of \mathcal{H} . A similar result holds for the affine Grassmannian and the mod p Satake-Hecke algebra of G .

CONTENTS

1.	Introduction	1
2.	Algebraic loop groups and their affine flag varieties	3
3.	Equivariant K -theory	8
3.1.	Schubert classes	8
3.2.	Demazure operators	10
3.3.	mod p Hecke algebras	14
	References	16

1. INTRODUCTION

Let F be a p -adic local field with ring of integers \mathcal{o}_F and let $\overline{\mathbb{F}}_p$ be the algebraic closure of its residue field. Let G be a connected split reductive linear algebraic group over \mathcal{o}_F and denote by $G(F)$ the group of its F -rational points. For a choice of Iwahori subgroup $I \subset G(F)$, the corresponding Iwahori-Hecke algebra $\mathcal{H}_{\overline{\mathbb{F}}_p}$ over $\overline{\mathbb{F}}_p$ equals the convolution algebra $\overline{\mathbb{F}}_p[I \backslash G(F) / I]$ of $\overline{\mathbb{F}}_p$ -valued functions with finite support on the double cosets of $G(F) \bmod I$. The algebra $\mathcal{H}_{\overline{\mathbb{F}}_p}$ and a certain extension $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$ of it associated to the maximal pro- p subgroup of I - the pro- p Iwahori-Hecke algebra - were systematically studied by Vignéras [24], [25], [26]. Among their first applications is the mod p local Langlands program for the group $G(F)$ [7].

The author would like to acknowledge support of the Heisenberg programme of Deutsche Forschungsgemeinschaft (SCHM 3062/1-1).

In this note we only deal with $\mathcal{H}_{\overline{\mathbb{F}}_p}$ and realize it as an equivariant algebraic K -theory with mod p coefficients of the affine flag variety \mathcal{F} of G . We view this result as a first step to analyze the geometry of the algebras $\mathcal{H}_{\overline{\mathbb{F}}_p}$ and $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$. In particular, we hope that the geometry of affine Springer fibres in \mathcal{F} [6] will eventually shed a light on the representation theory of these algebras, in analogy to the classical case of the finite complex flag variety [13]. We also note that the occurrence of affine geometry in the mod p setting can be viewed as a consequence of the failure of the Bernstein presentation for the algebras $\mathcal{H}_{\overline{\mathbb{F}}_p}$ and $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$ [25].

To give more details about the present article, let us assume that G is in fact simply connected semisimple and simple. Let k denote either the algebraic closure of F or the algebraic closure of the residue field of \mathfrak{o}_F . We denote the base change of G to k by the same letter. Let $T \subset G$ be a split maximal torus defined over \mathfrak{o}_F . We choose an Iwahori subgroup $\mathcal{B} \subset G(k[[t]])$ whose projection to $G(k)$ contains T and let

$$\mathcal{G} = G(k((t)))/G(k[[t]]) \quad \text{resp.} \quad \mathcal{F} = G(k((t)))/\mathcal{B}$$

be the affine Grassmannian resp. the affine flag variety of G over k . They are ind-schemes whose structure can be defined through the increasing union of Schubert varieties S_w indexed by elements w in the algebraic cocharacters Λ of T resp. in the affine Weyl group W of G . By definition, S_w equals the closure of the \mathcal{B} -orbit through w^* and has a natural structure as finite dimensional projective variety over k . Let \tilde{T} be the extension of T by the 'turning torus' \mathbf{G}_m (which corresponds to the derivation element d in the Kac-Moody setting). Then \tilde{T} acts on \mathcal{G} and \mathcal{F} and preserves the stratification into Schubert varieties. Denote by $K'_{\tilde{T}}(S_w)$ the Grothendieck group of \tilde{T} -equivariant coherent module sheaves on S_w and put $\mathbb{K}'_{\tilde{T}}(\cdot) := K'_{\tilde{T}}(\cdot) \otimes \overline{\mathbb{F}}_p$. We define

$$\mathbb{K}'_{\tilde{T}}(\mathcal{F}) := \varinjlim_w \mathbb{K}'_{\tilde{T}}(S_w)$$

and show that the Schubert classes $[\mathcal{O}_{S_w}]$ provide an $\overline{\mathbb{F}}_p[[\tilde{T}]]$ -basis for $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$. Here, $\overline{\mathbb{F}}_p[[\tilde{T}]]$ denotes the ring of algebraic functions on \tilde{T} over $\overline{\mathbb{F}}_p$. The analogous result holds for \mathcal{G} . Letting $\pi : \mathcal{F} \rightarrow \mathcal{G}$ denote the projection, the pull-back

$$\pi^* : \mathbb{K}'_{\tilde{T}}(\mathcal{G}) \hookrightarrow \mathbb{K}'_{\tilde{T}}(\mathcal{F})$$

is a $\overline{\mathbb{F}}_p[[\tilde{T}]]$ -linear injection whose image can be described explicitly. To each simple affine reflection s corresponds a smooth Bruhat-Tits group scheme P_s over $k[[t]]$ with connected fibres and with corresponding partial affine flag variety \mathcal{F}_s . The projection $\pi_s : \mathcal{F} \rightarrow \mathcal{F}_s$ is a \mathbf{P}^1 -bundle. One obtains a Demazure operator $D_s := \pi_s^* \pi_{s*}$ on $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ in analogy to the classical case of the finite flag variety treated in [3]. The operators D_s satisfy braid relations and quadratic relations which are exactly the ones appearing in the Iwahori-Matsumoto presentation of $\mathcal{H}_{\overline{\mathbb{F}}_p}$. The latter is described in terms of the $\overline{\mathbb{F}}_p$ -basis given by the characteristic functions τ_w of the double cosets IwI in the classical way [11], however in

our situation, the parameters satisfy $q_s = 0$ for all s , since $\overline{\mathbb{F}}_p$ has characteristic p . Letting τ_s act through D_s therefore turns $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ into a right module over $\mathcal{H}_{\overline{\mathbb{F}}_p}[\tilde{T}] = \mathcal{H}_{\overline{\mathbb{F}}_p} \otimes \overline{\mathbb{F}}_p[\tilde{T}]$. In this situation, our main result says that mapping the function $(-1)^{\ell(w)}\tau_w$ to the Schubert class $[\mathcal{O}_{S_w}]$ for each w induces an isomorphism

$$\mathcal{H}_{\overline{\mathbb{F}}_p}[\tilde{T}] \xrightarrow{\cong} \mathbb{K}'_{\tilde{T}}(\mathcal{F})$$

as right $\mathcal{H}_{\overline{\mathbb{F}}_p}[\tilde{T}]$ -modules. Here, ℓ denotes the length function on W . Finally, specialising the isomorphism at the unit element $\tilde{T} = 1$ yields an isomorphism between $\mathcal{H}_{\overline{\mathbb{F}}_p}$ and the absolute K' -theory $\mathbb{K}'(\mathcal{F})$ as right $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -modules. In particular, any irreducible right $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -module appears as a quotient of $\mathbb{K}'(\mathcal{F})$. Similarly, we show that $\mathbb{K}'(\mathcal{G})$ realizes the regular representation for the mod p spherical Hecke algebra $\overline{\mathbb{F}}_p[G(o_F)\backslash G(F)/G(o_F)]$ viewed as the monoid algebra on antidominant cocharacters via the mod p Satake isomorphism [8]. The representations are compatible via the map π^* . We remark that all our results hold in fact with the coefficient field $\overline{\mathbb{F}}_p$ replaced by any subfield.

We expect all results to extend to arbitrary connected and split reductive groups. Moreover, replacing the Iwahori-subgroup $\mathcal{B} \subset LG$ by the inverse image of the unipotent radical of the Borel subgroup of G in question should extend all results from $\mathcal{H}_{\overline{\mathbb{F}}_p}$ to $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$. We leave these issues for future work.

To make a brief comparison with the complex case, we recall that over the complex numbers the link between representations of the Iwahori-Hecke algebra and equivariant K -theory was discovered by Lusztig [16] and occupies a central position in applications to the local Langlands program. We only mention the proof of the Deligne-Langlands conjecture by Kazhdan-Lusztig [13] where it is shown that the equivariant K -homology of the complex Steinberg variety of triples affords a natural Hecke action through q -analogues of Demazure operators and that this provides a model for the regular representation of the Hecke algebra. We also point out that the theory of Demazure operators on equivariant K -theory of affine flag manifolds is well-established in the complex Kac-Moody setting, due to the work of Kostant-Kumar [14] and Kashiwara-Shimozono [12].

Acknowledgements: The author would like to thank Elmar Große-Klönne and Marc Levine for some interesting conversations related to this topic, as well as the anonymous referee whose comments and questions led to a substantial improvement of the text.

2. ALGEBRAIC LOOP GROUPS AND THEIR AFFINE FLAG VARIETIES

We recall some definitions and concepts from [4], [18]. Let k be a field and let $K = k((t))$ be the field of formal Laurent series over k . Let G be a simply connected semi-simple and simple group over k which is split over k . Let LG be its associated algebraic *loop group*, i.e. the functor

$$R \mapsto LG(R) = G(R((t)))$$

on the category of k -algebras. It is representable by an ind-scheme over k . Let L^+G be its associated *positive loop group* which is an affine group scheme over k representing the functor

$$R \mapsto L^+G(R) = G(R[[t]]).$$

Let \mathcal{B} be the inverse image under the morphism $\text{pr} : L^+G \rightarrow G, t \mapsto 0$ of a Borel subgroup in G . Thus, \mathcal{B} equals a closed subscheme of an infinite dimensional affine k -space. The *affine Grassmannian* \mathcal{G} resp. the *affine flag variety* \mathcal{F} of LG is the ind-scheme over k

$$\mathcal{G} = LG/L^+G \quad \text{resp.} \quad \mathcal{F} = LG/\mathcal{B}$$

representing the fpqc-sheaf associated with the functor $R \mapsto LG(R)/L^+G(R)$ resp. $R \mapsto LG(R)/\mathcal{B}(R)$. The natural projection map $\pi : \mathcal{F} \rightarrow \mathcal{G}$ is a proper and Zariski locally trivial fibration with fibers $L^+G/\mathcal{B} \simeq G/\text{pr}(\mathcal{B})$ (the split case in [18, 8.e.1]). The group LG acts on \mathcal{G} and \mathcal{F} by left translations. Evaluating G on the structure map $R \mapsto R((t))$ gives an inclusion $G \subset LG$. In this way, G and all its subgroups act on \mathcal{G} and \mathcal{F} .

For example, let $G = \text{SL}_d$, the special linear group. Then LG equals the union of subfunctors L_nG where $L_nG(R)$ is the set of matrices $M = \sum_{i \geq -n} M_i t^i$ of determinant 1 with M_i a $d \times d$ -matrix with entries in R . Each L_nG is representable as a closed subscheme of the affine scheme $\prod_{i \geq -n} \mathbf{A}_k^{d^2}$ and the inclusion $L_nG \subset L_{n+1}G$ is a closed embedding. This defines the ind-structure on LG . To make the ind-structure of \mathcal{G} and \mathcal{F} explicit, recall that a *lattice* $\mathcal{L} \subset R((t))^d$ is a $R[[t]]$ -submodule of $R((t))^d$ such that there is $n \geq 0$ with

$$t^n R[[t]]^d \subseteq \mathcal{L} \subseteq t^{-n} R[[t]]^d$$

and such that the quotient $t^{-n} R[[t]]^d / \mathcal{L}$ is a locally free R -module of finite rank. If $\bigwedge^d \mathcal{L} = R[[t]]$, the lattice is called *special*. We denote the set of special lattices over R by $\mathcal{L}(R)$ and those for fixed n by $\mathcal{L}_n(R)$. Then \mathcal{L}_n is representable by a projective k -scheme. Indeed, the morphism

$$\mathcal{L}_n(R) \rightarrow \text{Grass}_{nd}(t^{-n} k[[t]]^d / t^n k[[t]]^d)(R), \quad \mathcal{L} \mapsto \mathcal{L} / t^n R[[t]]^d$$

defines a closed embedding into the Grassmannian of nd -dimensional subspaces of the $2nd$ -dimensional k -vector space $t^{-n} k[[t]]^d / t^n k[[t]]^d$. Note that multiplication by $1 + t$ induces a (unipotent) automorphism on the latter vector space which induces an automorphism of the Grassmannian. The image of \mathcal{L}_n equals its fixed point variety and hence, is indeed closed. Each element of $LG(R)$ gives rise to a special lattice in $R((t))^d$ by applying it to the standard lattice $R[[t]]^d$. Since $G(R[[t]])$ equals the stabilizer of the standard lattice, we obtain an isomorphism $\mathcal{G}(R) \simeq \mathcal{L}(R)$. This defines the ind-structure of \mathcal{G} in the case of SL_d . Now a *lattice chain* in $R((t))^d$ is a chain

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_{d-1} \supset t\mathcal{L}_0$$

with lattices \mathcal{L}_i in $R((t))^d$ such that each quotient $\mathcal{L}_i / \mathcal{L}_{i+1}$ is a locally free R -module of rank 1. Each element of $LG(R)$ gives rise to a lattice chain with \mathcal{L}_0 special by applying

it to the standard lattice chain

$$\mathcal{L}_i := \bigoplus_{j=1, \dots, d-i} R[[t]]e_j \oplus \bigoplus_{j=d-i+1, \dots, d} tR[[t]]e_j$$

where e_1, \dots, e_d denotes the standard basis of $R((t))^d$. Since $\mathcal{B}(R)$ equals the stabilizer of the standard lattice chain, we obtain an isomorphism between $\mathcal{F}(R)$ and the space of lattice chains \mathcal{L}_\bullet in $R((t))^d$ such that \mathcal{L}_0 is special. In this optic, the projection π is given by sending a chain \mathcal{L}_\bullet to \mathcal{L}_0 and, hence, is a fiber bundle whose fibres are all isomorphic to the usual flag variety $G/\mathrm{pr}(\mathcal{B})$ of G . This defines the ind-structure of \mathcal{F} in the case of SL_d . For more details we refer to [6].

In the following, we are forced to work with a less explicit realization of the ind-structure on \mathcal{F} . To define it requires more notation. Our assumptions on G imply that $\mathcal{B}(k)$ is an Iwahori-subgroup of $G(K)$ and hence, defines a fundamental chamber \mathcal{C} in the Bruhat-Tits building of $G(K)$. We denote by B the associated smooth group scheme with connected fibres over $k[[t]]$. Then $\mathcal{B} = L^+B$ in the notation of [18, 1.b]. Similarly, to any wall F_i (i.e. facet of codimension 1) of this chamber, we let P_i be the smooth group scheme with connected fibres over $k[[t]]$ such that $P_i(k[[t]])$ is equal to the parahoric subgroup of $G(K)$ associated to F_i . To P_i corresponds an infinite-dimensional affine group scheme L^+P_i over k with $L^+P_i(R) = P_i(R[[t]])$. The fpqc-quotient $\mathcal{F}_i := LG/L^+P_i$ is representable by an ind-scheme over k , the *partial affine flag variety* associated to F_i . The projection morphism

$$(2.1) \quad \pi_i : \mathcal{F} \rightarrow \mathcal{F}_i$$

is a principal L^+P_i/L^+B -bundle for the Zariski topology (the split case in [18, Thm. 1.4]). Moreover, $L^+P_i/L^+B \simeq \mathbf{P}_k^1$, the projective line over k , according to [18, Prop. 8.7]. We denote the reflection in the wall F_i by s_i .

Let T be a maximal torus in G contained in the Borel subgroup $\mathrm{pr}(\mathcal{B})$ and let $N_G(T)$ be its normalizer. Let $W_0 = N_G(T)/T$ be the finite Weyl group of G with longest element w_0 . It may be viewed as the group generated by reflections in $\mathcal{S}_0 := \{s_1, \dots, s_l\}$. Let $\Phi = \Phi(G, T)$ be the root system and let Φ^+ be the set of positive roots distinguished by $\mathrm{pr}(\mathcal{B})$. Let $\Lambda := X_*(T)$ denote the group of cocharacters of T . We view Λ as a subgroup of $T(k[[t]])$ via the mapping $\lambda \mapsto \lambda(t)$. In particular, Λ acts on \mathcal{G} and \mathcal{F} .

Let W be the affine Weyl group of G . The group W is an affine Coxeter group with set of generators $\mathcal{S} := \mathcal{S}_0 \cup \{s_0\}$. We denote by \leq the Bruhat-Chevalley (partial) order on W and by $\ell(\cdot)$ the length function. Since G is simply connected, Λ coincides with the coroot lattice whence W may be written as the semi-direct product $W = \Lambda \rtimes W_0$. When considering Λ as a subgroup of W (whose group law is written multiplicatively), we will use exponential notation.

Let P be either B or P_i or G and write for a moment $\mathcal{F}_P = LG/L^+P$. Denote by $* = L^+P \in \mathcal{F}_P$ the base point. Let W^P be the set of representatives in W of minimal

length for the right cosets W/W_P where $W_P \subset W$ equals the subgroup generated by 1 or s_i or \mathcal{S}_0 . The (B, P) -Schubert cell $C_w = C_w(B, P)$ is the reduced subscheme

$$L^+ Bw* \subset \mathcal{F}_P.$$

The (B, P) -Schubert variety $S_w = S_w(B, P)$ is the reduced subscheme with underlying set the Zariski closure of C_w in LG/L^+P . It is a projective variety over k . The varieties S_w can in fact be enumerated by elements in the double quotient $W_B \backslash W/W_P$ [18, Prop.8.1]. Given $u \leq w$ there is a closed immersion $S_u \hookrightarrow S_w$ and

$$LG/L^+P = \bigcup_{w \in W^P} S_w$$

defines the ind-structure on \mathcal{F}_P . If we let \mathcal{F}_P^n be the (finite) union over the S_w with $\ell(w) \leq n$, then $\mathcal{F}_P = \cup_n \mathcal{F}_P^n$ defines the ind-structure on \mathcal{F}_P , too. In case of $P = P_i$, we shall write $\mathcal{F}_i = \cup_n \mathcal{F}_i^n$. In the following, we sometimes drop the base point $*$ from the notation.

Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots in Φ associated with \mathcal{S}_0 and let θ be the highest root. Denote by Φ^{aff} the collection of affine roots (α, m) with $\alpha \in \Phi$ and $m \in \mathbb{Z}$. Then $\alpha_0 := (-\theta, 1)$ is the remaining affine simple root which induces the reflection s_0 . The fundamental chamber \mathcal{C} may then be described as the set of all $x \in \Lambda \otimes \mathbb{R}$ such that $\alpha(x) + m > 0$ for all simple roots (α, m) . For each affine root (α, m) , there is an associated inclusion

$$\phi_{(\alpha, m)} : SL_2 \hookrightarrow LG$$

defined as follows [18, (9.8)]. The vector part α defines a k -morphism $SL_2 \hookrightarrow G$ in the usual way which induces by functoriality $L(SL_2) \hookrightarrow LG$. The map $\phi_{(\alpha, m)}$ equals the composition with the morphism

$$SL_2 \longrightarrow L(SL_2), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bt^m \\ ct^{-m} & d \end{pmatrix}.$$

The image of ϕ_{α_i} is contained in a Levi subgroup of the parabolic $L^+P_i \subset LG$ for $i = 0, \dots, l$. There is an irreducible k -linear representation V_i of the group scheme L^+P_i whose pull-back via ϕ_{α_i} equals the unique irreducible representation of SL_2 of dimension 2. Let $LG \times^{L^+P_i} \mathbf{P}(V_i)$ be the associated projective bundle over \mathcal{F}_i . Let v_i be a highest weight vector in V_i and for $g \in LG$ denote by $[g, v_i]$ the class of $(g, kv_i) \in LG \times^{L^+P_i} \mathbf{P}(V_i)$. We then have a morphism of \mathbf{P}^1 -bundles $\mathcal{F} \rightarrow LG \times^{L^+P_i} \mathbf{P}(V_i)$ given by $gL^+B \mapsto [g, v_i]$.

Lemma 2.2. *The morphism $\mathcal{F} \xrightarrow{\cong} LG \times^{L^+P_i} \mathbf{P}(V_i)$ is an isomorphism.*

Proof. It suffices to check that the morphism is bijective on fibres [15, IV.4.2.E(3)]. By LG -equivariance we may check this over the base point. However, the stabilizer of v_i in L^+P_i equals L^+B and since V_i is irreducible, this implies $L^+P_i/L^+B \simeq \mathbf{P}(V_i)$. \square

In [4, p. 54] Faltings constructs a central extension

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \bar{L}G \xrightarrow{\psi} LG \longrightarrow 1$$

which acts on all line bundles on \mathcal{F} (denoted $\tilde{L}G$ in loc.cit.). The copy of \mathbf{G}_m corresponds to the central element c in the Kac-Moody setting. We consider T as a subgroup of LG and consider its inverse image \bar{T} in $\bar{L}G$, i.e. $\bar{T} := \psi^{-1}(T)$. The central extension has a unique splitting over L^+B and we regard T as being contained in \bar{T} . There is a fundamental character $\rho_i \in X^*(\bar{T})$ whose restriction to T gives the character of V_i .

On the other hand, there is a natural identification

$$LG \times^{L^+P_i} \mathbf{P}(V_i) \simeq \mathbf{P}(LG \times^{L^+P_i} V_i)$$

as projective bundles over \mathcal{F}_i and we have the family of line bundles $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ on this space. Each character $\lambda \in X^*(\bar{T})$ yields an associated line bundle $\mathcal{L}_\lambda = LG \times^{L^+B} k_\lambda$ on \mathcal{F} [18, 10.a.]. Here, k_λ denotes the one-dimensional \bar{T} -representation induced by λ .

Lemma 2.3. *Let $0 \leq i \leq l$. The pull-back of $\mathcal{O}(-1)$ to \mathcal{F} equals \mathcal{L}_{ρ_i} . The \mathcal{L}_{ρ_i} form a basis of the Picard group $\text{Pic}(\mathcal{F})$ of \mathcal{F} .*

Proof. Consider the two-dimensional L^+P_i -representation V_i of weight ρ_i and highest weight vector $v_i \in V_i$ as above. Consider the line bundle $\mathcal{L}_{\rho_i} = G \times^{L^+B} kv_i$ on \mathcal{F} and fix a point $gL^+B \in \mathcal{F}$. The fibre of \mathcal{L}_{ρ_i} in gL^+B equals $[g, v_i]$. Since the isomorphism $\mathcal{F} \simeq \mathbf{P}(G \times^{L^+P_i} V_i)$ takes gL^+B to $[g, v_i]$, we see that \mathcal{L}_{ρ_i} equals the pull-back of the tautological line bundle $\mathcal{O}(-1)$. The second assertion follows from this together with [18, Prop. 10.1]. \square

We need to introduce yet another torus. Let P be either B or P_i or G and abbreviate $\mathcal{F}_P = LG/L^+P$. For a given k -algebra R , its group of units $a \in R^\times$ acts on the rings $R[[t]]$ and $R((t))$ via $t^m \mapsto a^m t^m$ ('turning the loop'). This induces an action of \mathbf{G}_m on LG preserving the subgroup L^+P . We obtain an action on \mathcal{F}_P which commutes with the action of T . We write

$$\tilde{T} := \mathbf{G}_m \times T$$

for the extended torus. This copy of \mathbf{G}_m corresponds to the derivation element d in the Kac-Moody setting. The action of the extended torus preserves the Schubert cell C_w . Indeed, we may write $w = w'e^\lambda$ with $w' \in W_0$ and $\lambda \in \Lambda$. Given $a \in R^\times$ one has

$$a(w*) = a(w')a(\lambda(t))* = w'\lambda(at)* = w'\lambda(a)\lambda(t)* = \lambda(a)w* = w*$$

since $\lambda(a) \in T(R)$. Therefore \tilde{T} indeed stabilizes C_w . It induces an action of \tilde{T} on the Schubert variety S_w . Even more, $w*$ is the only fixpoint in C_w for the action of T and \tilde{T} . Namely, the cell C_w is isomorphic to a finite direct sum over copies of 'root spaces' \mathbf{A}_k^1 for \tilde{T} indexed by certain affine roots (α, m) and with 'root vectors' $t^m X_\alpha$. The isomorphism is \tilde{T} -equivariant and maps $w*$ to the origin in $\bigoplus \mathbf{A}_k^1$. Here, X_α denotes a root vector for the root group $U_\alpha \subset G$ and the torus \tilde{T} acts on $t^m X_\alpha$ through the character $(a, s) \mapsto a^m \alpha(s)$.

In fact, L^+B acts transitively on C_w and the stabilizer of the point $w*$ is $L^+B \cap wL^+Pw^{-1}$. The Lie algebra of the latter has a basis $t^m X_\alpha$ such that $\alpha(x) + m > 0$ for all $x \in \mathcal{C}$ and $\text{Ad}(w)(t^m X_\alpha) \in \text{Lie}(L^+P)$. It follows from this that an affine root (α, m) contributes to the cell C_w if and only if $\alpha(x) + m > 0$ for all $x \in \mathcal{C}$ and $t^{m+v_K(\alpha^{w'}(\lambda))} X_{\alpha^{w'}} \notin \text{Lie}(P)$. Here, $w = e^\lambda w'$ with notation as above and $v_K : K^\times \rightarrow \mathbb{Z}$ denotes the valuation on K . For example in the case of $P = G$ (and where $w = \lambda$) we have $\text{Lie}(L^+G) = \text{Lie}(G) \otimes k[[t]]$ and so the last condition simply means $m + v_K(\alpha(\lambda)) < 0$, compare [5, Lem. 5.7].

3. EQUIVARIANT K -THEORY

3.1. Schubert classes. Let P be either B or P_i or G and let $\mathcal{F}_P = LG/L^+P$. Suppose for a moment that $S \subset \mathcal{F}_P$ is a \tilde{T} -stable ind-subvariety which is, in fact, a finite-dimensional algebraic k -variety. We then have the abelian category of \tilde{T} -equivariant coherent modules on S . We denote the corresponding Grothendieck group by $K'_{\tilde{T}}(S)$, cf. [21]. Let $f : Y \rightarrow S$ be an equivariant morphism to S from another \tilde{T} -variety Y . If f is flat, then pull-back of modules induces a homomorphism $f^* : K'_{\tilde{T}}(S) \rightarrow K'_{\tilde{T}}(Y)$. If f is proper, there is a push-forward homomorphism $f_* : K'_{\tilde{T}}(Y) \rightarrow K'_{\tilde{T}}(S)$ induced by $\sum_j (-1)^j R^j f_*$ [20, 1.10/11]. Note that if f is actually a closed embedding, then of course $R^j f_* = 0$ for $j > 0$. Applying this to the structure map of S , the tensor product endows the group $K'_{\tilde{T}}(S)$ with a module structure for the group ring

$$\mathbb{Z}[\tilde{T}] := \mathbb{Z}[X^*(\tilde{T})] = K'_{\tilde{T}}(\text{Spec } k).$$

Let $H \subseteq \tilde{T}$ be a closed subgroup. Restricting the \tilde{T} -action to H induces a linear homomorphism

$$(3.1) \quad \mathbb{Z}[H] \otimes_{\mathbb{Z}[\tilde{T}]} K'_{\tilde{T}}(\mathcal{F}) \longrightarrow K'_H(\mathcal{F}).$$

It is often bijective [17].

We define the Grothendieck group of \mathcal{F}_P to be

$$K'_{\tilde{T}}(\mathcal{F}_P) := \varinjlim_n K'_{\tilde{T}}(\mathcal{F}_P^n) = \varinjlim_{w \in W^P} K'_{\tilde{T}}(S_w)$$

in analogy with the case of the complex Kashiwara affine flag manifold ($k = \mathbb{C}$) treated in [23, 2.2.1]. The transition maps here are induced by the push-forward along the \tilde{T} -equivariant closed embeddings $\mathcal{F}_P^n \hookrightarrow \mathcal{F}_P^{n+1}$ resp. $S_u \hookrightarrow S_w$ for $u \leq w$.

Let $U_P^n := \mathcal{F}_P^n \setminus \mathcal{F}_P^{n-1}$ with $U_P^0 = \text{Spec } k$. We let $j_n : U_P^n \hookrightarrow \mathcal{F}_P^n$ denote the corresponding open embedding.

Lemma 3.2. *There is a linear isomorphism $K'_{\tilde{T}}(U_P^n) \simeq \bigoplus_{\ell(w)=n} \mathbb{Z}[\tilde{T}]$.*

Proof. We have $U_P^n = \coprod_{\ell(w)=n} C_w$ with a linear \tilde{T} -action on $C_w \simeq \mathbf{A}_k^{\ell(w)}$, cf. discussion at the end of previous section. So the claim follows from homotopy invariance of $K'_{\tilde{T}}$ [21, 4.2]. \square

We have the element $[\mathcal{O}_{S_w}]$ in $K'_{\tilde{T}}(\mathcal{F}_P)$ for $w \in W^P$. In the complex Kac-Moody setting these Schubert classes give rise to ' \mathbb{Z} -bases' in the corresponding K -theories of the affine flag manifold [12], [14]. In the following, we would like to allow the case where the field k has positive characteristic $p > 0$ and where we are not aware of corresponding facts in the literature. The following results have straightforward proofs and are sufficient for our purposes. We work over the coefficient field $\overline{\mathbb{F}}_p$, the algebraic closure of the finite field \mathbb{F}_p with p elements (but any other field of characteristic $p > 0$ would do as well). We let

$$\mathbb{K}'_H(\cdot) := K'_H(\cdot) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p \quad \text{and} \quad \overline{\mathbb{F}}_p[\cdot] := \mathbb{Z}[\cdot] \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$$

for any closed subgroup $H \subseteq \tilde{T}$. If $H = 1$, we write \mathbb{K}' instead of $\mathbb{K}'_{\{1\}}$. Consider the following hypothesis on our field k :

(HYP) the field k is closed under taking p -th roots.

The examples we have in mind are $k = \overline{F}$, the algebraic closure of our p -adic field F , or a perfect field of characteristic $p > 0$. For the rest of this subsection we assume (HYP).

Proposition 3.3. *The $\overline{\mathbb{F}}_p[\tilde{T}]$ -module $\mathbb{K}'_{\tilde{T}}(\mathcal{F}_P^n)$ is free on the basis given by the Schubert classes $[\mathcal{O}_{S_w}]$ with $\ell(w) = n$. If $H \subseteq \tilde{T}$ is closed subgroup, then (3.1) is an isomorphism after tensoring with $\overline{\mathbb{F}}_p$.*

Proof. We have $\mathbb{Z}[H] \otimes_{\mathbb{Z}} K'^{\bullet} = K'_H{}^{\bullet}$ for those H -varieties which have trivial H -action and where K'^{\bullet} denotes ordinary K' -theory. Applying this to $\bullet = 1$ and using (HYP) to see

$$\mathbb{K}^1(\text{Spec } k) = k^{\times} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p = 0,$$

the first equivariant K' -group of $U_P^n = \coprod_{\ell(w)=n} C_w$ thus vanishes upon tensoring with $\overline{\mathbb{F}}_p$. The localization sequence, tensored with $\overline{\mathbb{F}}_p$,

$$0 \longrightarrow \mathbb{K}'_H(\mathcal{F}_P^{n-1}) \longrightarrow \mathbb{K}'_H(\mathcal{F}_P^n) \xrightarrow{j_n^*} \mathbb{K}'_H(U_P^n) \longrightarrow 0$$

is therefore exact [21, 2.7]. Since a preimage under j_n^* of the unit vector at the w -th position in $\oplus_{\ell(w)=n} \mathbb{Z}[H]$ is given by $[\mathcal{O}_{S_w}]$, the first assertion follows now by induction on n with $H = \tilde{T}$. The bijectivity of (3.1) follows now easily from this. \square

Corollary 3.4. *The $\overline{\mathbb{F}}_p[\tilde{T}]$ -module $\mathbb{K}'_{\tilde{T}}(\mathcal{F}_P)$ is free on the basis given by the $[\mathcal{O}_{S_w}]$ with $w \in W^P$.*

Remark: The ind-structure on \mathcal{F}_P may also be defined via the (P, P) -Schubert varieties equal to the closure of the L^+P -orbits in \mathcal{F}_P [18]. Since an L^+P -orbit is not a topological cell anymore, we chose (B, P) -Schubert varieties to calculate the equivariant K' -theory of \mathcal{F}_P .

For each n the set of fixpoints $(\mathcal{F}_P^n)^{\tilde{T}}$ is a non-empty closed subvariety of \mathcal{F}_P^n . We let $\iota_n : (\mathcal{F}_P^n)^{\tilde{T}} \hookrightarrow \mathcal{F}_P^n$ be the corresponding closed embedding. We have the following simple version of the Thomason concentration theorem for fixed points varieties [22, Thm. 2.1].

Proposition 3.5. *The variety $(\mathcal{F}_P^n)^{\tilde{T}}$ equals the union of the finitely many points $w^* \in \mathcal{F}_P^n$ for $\ell(w) \leq n$ and $w \in W^P$. One has a linear isomorphism*

$$\iota_{n*} : \mathbb{K}'_{\tilde{T}}((\mathcal{F}_P^n)^{\tilde{T}}) \xrightarrow{\simeq} \mathbb{K}'_{\tilde{T}}(\mathcal{F}_P^n)$$

which maps the class of the point w^* to $[\mathcal{O}_{S_w}]$.

Proof. One has

$$(\mathcal{F}_P^n)^{\tilde{T}} = \coprod_{\ell(w) \leq n} (C_w)^{\tilde{T}} = \coprod_{\ell(w) \leq n} \{w^*\}$$

by the discussion at the end of the previous section. For the second assertion we consider the cartesian diagramm

$$\begin{array}{ccc} (U_P^n)^{\tilde{T}} & \xrightarrow{j_n} & (\mathcal{F}_P^n)^{\tilde{T}} \\ \downarrow \tilde{\iota}_n & & \downarrow \iota_n \\ U_P^n & \xrightarrow{j_n} & \mathcal{F}_P^n \end{array}$$

Flat equivariant base change for equivariant K' -theory yields $j_n^* \iota_{n*} = \tilde{\iota}_{n*} j_n^*$ [20, 1.11]. We obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{K}'_{\tilde{T}}((\mathcal{F}_P^{n-1})^{\tilde{T}}) & \longrightarrow & \mathbb{K}'_{\tilde{T}}((\mathcal{F}_P^n)^{\tilde{T}}) & \longrightarrow & \mathbb{K}'_{\tilde{T}}((U_P^n)^{\tilde{T}}) \longrightarrow 0 \\ & & \downarrow \iota_{n-1*} & & \downarrow \iota_{n*} & & \simeq \downarrow \tilde{\iota}_{n*} \\ 0 & \longrightarrow & \mathbb{K}'_{\tilde{T}}(\mathcal{F}_P^{n-1}) & \longrightarrow & \mathbb{K}'_{\tilde{T}}(\mathcal{F}_P^n) & \longrightarrow & \mathbb{K}'_{\tilde{T}}(U_P^n) \longrightarrow 0. \end{array}$$

where the left-hand square commutes by functoriality for proper maps [20, 1.11] and the right-hand isomorphism comes from homotopy invariance. The lower horizontal sequence is exact by the proof of the preceding proposition and this implies the exactness of the upper horizontal sequence. The result follows now by induction on n . \square

3.2. Demazure operators. We fix a number n and consider the preimage of \mathcal{F}_i^n under the projection $\pi_i : \mathcal{F} \rightarrow \mathcal{F}_i$ from 2.1. Since π_i is flat and proper we have the homomorphisms

$$K'_{\tilde{T}}(\pi_i^{-1} \mathcal{F}_i^n) \begin{array}{c} \xrightarrow{\pi_{i*}} \\ \xleftarrow{\pi_i^*} \end{array} K'_{\tilde{T}}(\mathcal{F}_i^n).$$

In analogy to the case of the finite flag variety [3] and the complex Kashiwara affine flag manifold [12], we define a linear operator on $K'_{\tilde{T}}(\pi_i^{-1} \mathcal{F}_i^n)$ by

$$D_i := \pi_i^* \circ \pi_{i*}.$$

Lemma 3.6. *The operator D_i extends to an operator on $K'_T(\mathcal{F})$.*

Proof. Since π_i is a principal \mathbf{P}^1 -bundle, each $\pi_i^{-1}\mathcal{F}_i^n$ is a closed algebraic subvariety in \mathcal{F} and these varieties define, for varying n , the ind-structure on \mathcal{F} . It suffices therefore to show that D_i is compatible with the transition map $K'_T(\pi_i^{-1}\mathcal{F}_i^{n-1}) \rightarrow K'_T(\pi_i^{-1}\mathcal{F}_i^n)$. We consider the cartesian diagram

$$\begin{array}{ccc} \pi_i^{-1}\mathcal{F}_i^{n-1} & \xrightarrow{\pi_i} & \mathcal{F}_i^{n-1} \\ \downarrow \tilde{\iota} & & \downarrow \iota \\ \pi_i^{-1}\mathcal{F}_i^n & \xrightarrow{\pi_i} & \mathcal{F}_i^n \end{array}$$

where the map ι is the natural inclusion and the map $\tilde{\iota}$ is induced by it. Flat equivariant base change for equivariant K' -theory yields $\pi_i^*\iota_* = \tilde{\iota}_*\pi_i^*$ and functoriality for proper maps yields $\iota_*\pi_{i*} = \pi_{i*}\tilde{\iota}_*$. Hence

$$\tilde{\iota}_*\pi_i^*\pi_{i*} = \pi_i^*\iota_*\pi_{i*} = \pi_i^*\pi_{i*}\tilde{\iota}_*.$$

□

We denote the resulting operator on $K'_T(\mathcal{F})$ by the same symbol D_i . As in [3, Prop. 2.6], the projective bundle theorem in K -theory identifies the operator D_i as an idempotent projector as follows. We note that under the isomorphism $\mathcal{F} \simeq \mathbf{P}(LG \times^{L^+P_i} V_i)$ of 2.2 the morphism $\pi_i : \mathcal{F} \rightarrow \mathcal{F}_i$ becomes the projective space bundle associated to the rank 2 vector bundle $LG \times^{L^+P_i} V_i \rightarrow \mathcal{F}_i$, $[g, v] \mapsto gL^+P_i$.

Lemma 3.7. *Let $0 \leq i \leq l$. The correspondence*

$$(a_0, a_1) \mapsto 1 \otimes \pi_i^*a_0 + \mathcal{L}_{\rho_i} \otimes \pi_i^*a_1$$

induces a $\mathbb{Z}[\tilde{T}]$ -linear isomorphism

$$K'_T(\mathcal{F}_i^n)^{\oplus 2} \xrightarrow{\simeq} K'_T(\pi_i^{-1}\mathcal{F}_i^n)$$

for any n .

Proof. Since $\mathcal{L}_{\rho_i} = \mathcal{O}(-1)$ by 2.3, we apply the (equivariant) projective bundle theorem [21, Thm. 3.1] to $\pi_i^{-1}\mathcal{F}_i^n \rightarrow \mathcal{F}_i^n$. □

Proposition 3.8. *In the situation of the lemma, the operator D_i on $K'_T(\pi_i^{-1}\mathcal{F}_i^n)$ has the property*

$$D_i(1 \otimes \pi_i^*a_0 + \mathcal{L}_{\rho_i} \otimes \pi_i^*a_1) = 1 \otimes \pi_i^*a_0.$$

Proof. According to [19, Lemma 8.1.1(c)] the modules $M_0 := \pi_i^*a_0$ and $M_1(-1) := \mathcal{O}(-1) \otimes M_1$ with $M_1 := \pi_i^*a_1$ satisfy $\pi_{i*}M_0 = a_0$ and $\pi_{i*}M_1(-1) = 0$ and, moreover, they are regular modules in the sense of Mumford, cf. loc.cit. Then [19, Lemma 8.1.3] implies $R^j\pi_{i*}M_0 = R^j\pi_{i*}M_1(-1) = 0$ for $j > 0$. □

Corollary 3.9. *The operator D_i on $K'_T(\mathcal{F})$ satisfies $D_i^2 = D_i$.*

Remark: According to the preceding proposition, D_i may be also be viewed as an algebraic characteristic p analogue of the operator D_i defined by Kostant-Kumar on the topological equivariant K -theory of the complex analytic Kac-Moody flag manifold [14, Def. (3.6)].

We compute the action of D_i on the elements $[\mathcal{O}_{S_w}]$ of $K'_{\tilde{T}}(\mathcal{F})$.

Lemma 3.10. *Let $Z \subset \mathcal{F}_i^n$ be a closed \tilde{T} -subvariety and let $[\mathcal{O}_Z]$ be its class in $K'_{\tilde{T}}(\mathcal{F}_i^n)$. Then $\pi^*[\mathcal{O}_Z] = [\mathcal{O}_{\pi_i^{-1}Z}]$ in $K'_{\tilde{T}}(\pi_i^{-1}\mathcal{F}_i^n)$.*

Proof. This follows directly from the flatness of π_i . □

For $w \in W^{P_i}$ let S_w^i be a Schubert variety in \mathcal{F}_i . We have $\pi_i^{-1}S_w^i = S_{w'}$ where $w' \in W$ has the property: $w' = w$ if $ws_i < w$ or else $w' = ws_i$. This implies that for any $w \in W$ one has

$$(3.11) \quad \pi_i^{-1}\pi_i(S_w) = S_w \quad \text{if } ws_i < w \quad \text{or else} \quad \pi_i^{-1}\pi_i(S_w) = S_{ws_i}.$$

Remark: Similarly, the inverse image of $S_w \subset \mathcal{G}$ under π equals $S_{ww_0} \subset \mathcal{F}$ where w_0 denotes the longest element in W_0 . Indeed, the inverse image $\pi^{-1}S_w$ is a Schubert variety in \mathcal{F} and ww_0 is the representative of maximal length in its coset modulo W_0 .

Lemma 3.12. *Let $w \in W$ and suppose $S_w \subset \pi_i^{-1}\mathcal{F}_i^n$ for some n . Then*

$$\pi_{i*}[\mathcal{O}_{S_w}] = [\mathcal{O}_{\pi_i(S_w)}]$$

in $K'_{\tilde{T}}(\mathcal{F}_i^n)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} S_w & \xrightarrow{\tilde{\iota}} & \pi_i^{-1}\mathcal{F}_i^n \\ \downarrow \pi_i & & \downarrow \pi_i \\ \pi_i(S_w) & \xrightarrow{\iota} & \mathcal{F}_i^n. \end{array}$$

where ι and $\tilde{\iota}$ are the corresponding closed embeddings. Functoriality for proper maps implies $\pi_{i*}\tilde{\iota}_* = \iota_*\pi_{i*}$ on $K'_{\tilde{T}}(S_w)$. However, ι_* and $\tilde{\iota}_*$ are exact functors on coherent modules, so that we are reduced to show:

$$R^j\pi_{i*}(\mathcal{O}_{S_w}) = 0 \quad \text{for } j > 0 \quad \text{and} \quad \pi_{i*}(\mathcal{O}_{S_w}) = \mathcal{O}_{\pi_i(S_w)}.$$

We distinguish two cases. Assume first that $ws_i < w$. Then $\pi_i^{-1}\pi_i(S_w) = S_w$ by 3.11 so that our commutative diagram is even cartesian. Since π_i is a projective space bundle, our claims follow from [19, 8.1.1/8.1.3]. Assume now $ws_i > w$. Then $\pi_i^{-1}\pi_i(S_w) = S_{ws_i}$ by 3.11 and the map $\pi_i : S_{ws_i} \rightarrow \pi_i(S_w)$ is a proper birational morphism [2, Remark 3.2.3]. Our claims follow since S_{ws_i} and $\pi_i(S_w)$ are normal with at most rational singularities [18, Prop. 9.6]. □

Proposition 3.13. *Let $w \in W$. One has*

$$D_i[\mathcal{O}_{S_w}] = \begin{cases} [\mathcal{O}_{S_w}] & \text{if } ws_i < w \\ [\mathcal{O}_{S_{ws_i}}] & \text{if } ws_i > w. \end{cases}$$

Proof. According to the two preceding lemmas we have $\pi_i^* \pi_{i*}[\mathcal{O}_{S_w}] = [\mathcal{O}_{\pi_i^{-1}\pi_i(S_w)}]$ so that 3.11 implies the assertion. \square

For the rest of this subsection we assume that our field k satisfies (HYP). We consider the endomorphism ring

$$\text{End}_{\overline{\mathbb{F}}_p[\tilde{T}]}(\mathbb{K}'_{\tilde{T}}(\mathcal{F}))^{\text{op}}$$

of $\overline{\mathbb{F}}_p[\tilde{T}]$ -linear operators on $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ with multiplication $(\phi\varphi)(c) = \varphi(\phi(c))$ for $c \in \mathbb{K}'_{\tilde{T}}(\mathcal{F})$. In particular, $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ is naturally a *right* module over this ring. For an element $w \in W$ with reduced decomposition $w = s_{i_1} \cdots s_{i_n}$ we define a linear operator D_w on $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ by

$$D_w := D_{i_1} \cdots D_{i_n}$$

and $D_1 := \text{id}$.

Lemma 3.14. *The operator D_w does not depend on the choice of reduced decomposition for w .*

Proof. According to [1, IV.§1.Prop. 5], it suffices to check for $i \neq j$ and $m_{ij} = \text{ord}(s_i s_j) < \infty$ the relations $D_i D_j D_i \cdots = D_j D_i D_j \cdots$ with m_{ij} factors on both sides. According to 3.4 we can check this on Schubert classes. Since W is crystallographic, we have $m_{ij} \in \{2, 3, 4, 6\}$, so that the claim can be checked case-by-case. We treat the case $m_{ij} = 2$, the other cases follow along the same lines, but with more notation. Since $m_{ij} = 2$, we must show that D_i and D_j commute. So let $w \in W$ and let us argue case-by-case. First, if $ws_i, ws_j < w$, then $[\mathcal{O}_{S_w}]$ is a fixpoint of $D_i D_j$ and $D_j D_i$. Secondly, if $ws_i, ws_j > w$, then $ws_i s_j > ws_j$ and $ws_j s_i > ws_i$ according to case (b) in the proof of [10, Prop. 5.9] which yields

$$[\mathcal{O}_{S_w}](D_i D_j) = [\mathcal{O}_{S_{ws_i}}] D_j = [\mathcal{O}_{S_{ws_i s_j}}] = [\mathcal{O}_{S_{ws_j s_i}}] = [\mathcal{O}_{S_w}](D_j D_i).$$

By symmetry in i and j , it now remains to consider the case $ws_j > w$ and $ws_i < w$. Then $[\mathcal{O}_{S_w}](D_i D_j) = [\mathcal{O}_{S_{ws_j}}]$ and $[\mathcal{O}_{S_w}](D_j D_i) = [\mathcal{O}_{S_{ws_j}}] D_i$. We have either $ws_i s_j < ws_j$ or $ws_i s_j \leq w$ according to [10, Prop. 5.9]. In the first case, we are done. The second case leads via $ws_i s_j \leq w < ws_j$ to the first case. \square

We summarize the properties of the D_w . Let \mathcal{S} be the set of generators s_i for the affine Coxeter group W .

Corollary 3.15. *Let $w, w' \in W$. The linear operators D_w on $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ satisfy the relations*

$$D_w D_{w'} = D_{ww'} \quad \text{if } \ell(w) + \ell(w') = \ell(ww'),$$

$$D_s^2 = D_s \quad \text{for any } s \in \mathcal{S}.$$

Remark: Since D_w acts linearly on $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$, there is an induced action of D_w on

$$\mathbb{K}'_H(\mathcal{F}) \simeq \overline{\mathbb{F}}_p[H] \otimes_{\overline{\mathbb{F}}_p[\tilde{T}]} \mathbb{K}'_{\tilde{T}}(\mathcal{F})$$

for any closed subgroup $H \subseteq \tilde{T}$ according to 3.3.

Remark: The projection $\pi : \mathcal{F} \rightarrow \mathcal{G}$ is a locally trivial $G/\mathrm{pr}(\mathcal{B})$ -fibration and we have the linear homomorphism

$$\pi^* : K'_{\tilde{T}}(S_w) \rightarrow K'_{\tilde{T}}(\pi^{-1}S_w).$$

Flat equivariant base change, as in the proof of 3.6, shows that these homomorphisms commute with transition maps and give in the limit a linear homomorphism $\pi^* : K'_{\tilde{T}}(\mathcal{G}) \rightarrow K'_{\tilde{T}}(\mathcal{F})$. It follows from 3.10 and the subsequent remark that π^* maps the class of $S_w \subset \mathcal{G}$ to the class of $S_{ww_0} \subset \mathcal{F}$. According to 3.4, the induced morphism

$$(3.16) \quad \pi^* : \mathbb{K}'_{\tilde{T}}(\mathcal{G}) \hookrightarrow \mathbb{K}'_{\tilde{T}}(\mathcal{F})$$

is therefore injective.

3.3. mod p Hecke algebras. We assume in the following that k satisfies (HYP). The integral *Coxeter-Hecke algebra* of (W, S) with weights $q_s = 0$ for all $s \in S$ is given as

$$\mathcal{H} = \bigoplus_{w \in W} \mathbb{Z}\tilde{Y}_w$$

with relations

$$\tilde{Y}_w \tilde{Y}_{w'} = \tilde{Y}_{ww'} \quad \text{if } \ell(w) + \ell(w') = \ell(ww'),$$

$$\tilde{Y}_s^2 = -\tilde{Y}_s \quad \text{for any } s \in S,$$

cf. [1, Ex. IV.§2.(24)]. We let $Y_w := (-1)^{\ell(w)} \tilde{Y}_w$ for any w and $\mathcal{H}_R := \mathcal{H} \otimes_{\mathbb{Z}} R$ for a commutative ring R . We have the monoid

$$\Lambda^- = \{\lambda \in \Lambda : \langle \lambda, \alpha \rangle \leq 0 \text{ for all } \alpha \in \Phi^+\}$$

of antidominant cocharacters in Λ and its corresponding monoid ring $\mathbb{Z}[\Lambda^-]$. One has $\ell(e^\lambda) = \langle \lambda, 2\rho \rangle$ for any $\lambda \in \Lambda^-$ where 2ρ equals the sum over elements from Φ^+ . Hence

$$\Theta : \mathbb{Z}[\Lambda^-] \longrightarrow \mathcal{H}, \lambda \mapsto Y_{e^\lambda}$$

is an injective ring homomorphism. In this way, we view $\mathbb{Z}[\Lambda^-]$ as a subring of \mathcal{H} .

Proposition 3.17. *Letting Y_w act through D_w makes $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ a right $\mathcal{H}_{\overline{\mathbb{F}}_p[\tilde{T}]}$ -module.*

Proof. The elements Y_w are a \mathbb{Z} -basis of \mathcal{H} with defining relations $Y_w Y_{w'} = Y_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$ and $Y_s^2 = -Y_s$ for any $s \in S$. According to 3.15 the map $Y_w \mapsto D_w$ extends therefore to a ring homomorphism between $\mathcal{H}_{\overline{\mathbb{F}}_p[\tilde{T}]}$ and $\mathrm{End}_{\overline{\mathbb{F}}_p[\tilde{T}]}(\mathbb{K}'_{\tilde{T}}(\mathcal{F}))^{\mathrm{op}}$. \square

We view the algebra $\mathcal{H}_{\overline{\mathbb{F}}_p[\tilde{T}]}$ via its multiplication as a right module over itself.

Theorem 3.18. *There is an isomorphism of right $\mathcal{H}_{\overline{\mathbb{F}}_p[\tilde{T}]}$ -modules*

$$\Xi : \mathcal{H}_{\overline{\mathbb{F}}_p[\tilde{T}]} \xrightarrow{\cong} \mathbb{K}'_{\tilde{T}}(\mathcal{F}), \quad Y_w \mapsto [\mathcal{O}_{S_w}].$$

Specialising at $\tilde{T} = 1$ yields an isomorphism of right $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -modules $\mathcal{H}_{\overline{\mathbb{F}}_p} \simeq \mathbb{K}'(\mathcal{F})$.

Proof. The first map is $\overline{\mathbb{F}}_p[\tilde{T}]$ -linear and bijective 3.4. It is equivariant for the action of any Y_s by 3.13 and therefore for the action of any Y_w . The second statement follows from the remark after 3.15. \square

We show that the Grassmannian \mathcal{G} gives rise to a Hecke submodule in $\mathbb{K}'_{\tilde{T}}(\mathcal{F})$ for the image of Θ . One has $\ell(e^{-\lambda}w) = \ell(w^{-1}e^\lambda) = \ell(w) + \ell(e^{-\lambda})$ for any $\lambda \in \Lambda^-$ and $w \in W_0$. This means that the elements from $\Lambda^+ := -\Lambda^- = w_0(\Lambda^-)$ are the minimal representatives in their corresponding cosets modulo W_0 . Hence, the set Λ^+ is in bijection with the set of double classes $W_0 \backslash W / W_0$ and so we have $\mathbb{K}'_{\tilde{T}}(\mathcal{G}) = \bigoplus_{\lambda \in \Lambda^+} \overline{\mathbb{F}}_p[\tilde{T}][\mathcal{O}_{S_\lambda}]$ by 3.4. We denote the specialisation of π^* at $\tilde{T} = 1$ by π_1^* and consider the image $\text{Im } \pi_1^* \subset \mathbb{K}'(\mathcal{F})$. Note that the ring $\overline{\mathbb{F}}_p[\Lambda^-]$ acts on $\mathbb{K}'(\mathcal{F})$ by restricting the $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -action via Θ .

Corollary 3.19. *The isomorphism Ξ induces an isomorphism*

$$Y_{w_0} \overline{\mathbb{F}}_p[\Lambda^-] \xrightarrow{\cong} \text{Im } \pi_1^*$$

as right $\overline{\mathbb{F}}_p[\Lambda^-]$ -modules. In particular, $\text{Im } \pi_1^$ is a free $\overline{\mathbb{F}}_p[\Lambda^-]$ -module of rank 1.*

Proof. If $\lambda \in \Lambda^+$, then $e^\lambda w_0 = w_0 e^{w_0(\lambda)}$ with $w_0(\lambda) \in \Lambda^-$. The $\overline{\mathbb{F}}_p[\tilde{T}]$ -module $\text{Im } \pi_1^*$ is therefore freely generated by the classes of the $S_{w_0 e^\lambda}$ where $\lambda \in \Lambda^-$. Given another $\mu \in \Lambda^-$, one has $\ell(w_0 e^\lambda e^\mu) = \ell(w_0 e^\lambda) + \ell(e^\mu)$ and therefore $D_{e^\mu}[\mathcal{O}_{S_{w_0 e^\lambda}}] = [\mathcal{O}_{S_{w_0 e^{\lambda+\mu}}}] \in \text{Im } \pi_1^*$. This implies all assertions. \square

We now turn back to the p -adic setting of the introduction. Let F be a p -adic local field with ring of integers \mathfrak{o}_F . Let G be a simply-connected semi-simple and simple \mathfrak{o}_F -split group scheme. We let $k = \overline{F}$, the algebraic closure of F or any perfect field extension of the residue field of \mathfrak{o}_F . We may then apply all previous results to the k -group $G \otimes_{\mathfrak{o}_F} k$ and its associated affine Grassmannian \mathcal{G} resp. affine flag variety \mathcal{F} over k . On the other hand, let $I \subset G(F)$ be an Iwahori-subgroup of the p -adic group $G(F)$. The *mod p Iwahori-Hecke algebra* of $G(F)$ equals the convolution algebra $\overline{\mathbb{F}}_p[I \backslash G(F) / I]$ of $\overline{\mathbb{F}}_p$ -valued functions with finite support on the double cosets of $G(F) \bmod I$. A $\overline{\mathbb{F}}_p$ -basis is given by the characteristic functions of the double cosets $I \backslash G(F) / I$ which, in turn, are in bijection with the group W . Denote by τ_w the characteristic function of $IwI, w \in W$. By the Iwahori-Matsumoto presentation [11] there is an algebra isomorphism

$$\iota_{\text{IM}} : \overline{\mathbb{F}}_p[I \backslash G(F) / I] \xrightarrow{\cong} \mathcal{H}_{\overline{\mathbb{F}}_p}, \quad \tau_w \mapsto \tilde{Y}_w.$$

Let $G_0 = G(\mathfrak{o}_F)$ and consider the *mod p spherical Hecke algebra* $\overline{\mathbb{F}}_p[G_0 \backslash G(F) / G_0]$. A $\overline{\mathbb{F}}_p$ -basis is given by the characteristic functions of the double cosets $G_0 \backslash G(F) / G_0$. Choosing a uniformizer $\varpi \in \mathfrak{o}_F$, the refined Cartan decomposition gives a bijection between Λ^-

and $G_0 \backslash G(F)/G_0$ by mapping λ to the double class of $\lambda(\varpi)$. Denote by $\tilde{\tau}_\lambda$ the characteristic function of this latter double class. The mod p Satake map from [8] is an algebra isomorphism

$$\iota_S : \overline{\mathbb{F}}_p[G_0 \backslash G(F)/G_0] \xrightarrow{\cong} \overline{\mathbb{F}}_p[\Lambda^-]$$

which allows us to consider the homomorphism

$$\overline{\mathbb{F}}_p[G_0 \backslash G(F)/G_0] \rightarrow \overline{\mathbb{F}}_p[I \backslash G(F)/I]$$

given by the composite $\iota_{\text{IM}}^{-1} \circ \Theta \circ \iota_S$. The value of $\iota_{\text{IM}}^{-1} \circ \Theta \circ \iota_S$ on $\tilde{\tau}_\lambda$ is given by

$$\sum_{\mu \in \Lambda^-, \mu \geq_{\mathbb{R}} \lambda} c_{\lambda, \mu} (-1)^{\ell(e^\mu)} \tau_{e^\mu}.$$

Here, the matrix $c_{\lambda, \mu}$ is upper triangular with entries in $\overline{\mathbb{F}}_p$ and $c_{\lambda, \lambda} = 1$ whose inverse matrix has all nonzero entries equal to 1 [9, Prop. 5.1]. Moreover, $\mu \geq_{\mathbb{R}} \lambda$ if and only if $\mu - \lambda$ is a non-negative real linear combination of positive coroots. We finish by summarizing the discussion.

Theorem 3.20. *The right $\overline{\mathbb{F}}_p[I \backslash G(F)/I]$ -module $\mathbb{K}'(\mathcal{F})$ realizes the regular representation. The right $\overline{\mathbb{F}}_p[G_0 \backslash G(F)/G_0]$ -module $\mathbb{K}'(\mathcal{G})$ realizes the regular representation. The homomorphism $\pi_1^* : \mathbb{K}'(\mathcal{G}) \rightarrow \mathbb{K}'(\mathcal{F})$ is equivariant with respect to the map $\iota_{\text{IM}}^{-1} \circ \Theta \circ \iota_S$.*

REFERENCES

- [1] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.
- [2] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005.
- [3] Michel Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.
- [4] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc. (JEMS)*, 5(1):41–68, 2003.
- [5] Mark Goresky, Robert Kottwitz, and Robert Macpherson. Homology of affine Springer fibers in the unramified case. *Duke Math. J.*, 121(3):509–561, 2004.
- [6] Ulrich Görtz. Affine Springer fibers and affine Deligne-Lusztig varieties. In *Affine flag manifolds and principal bundles*, Trends Math., pages 1–50. Birkhäuser/Springer Basel AG, Basel, 2010.
- [7] Elmar Grosse-Klönne. From pro- p Iwahori-Hecke modules to (ϕ, Γ) -modules I. *Duke Math. J.* (to appear).
- [8] Florian Herzig. A Satake isomorphism in characteristic p . *Compos. Math.*, 147(1):263–283, 2011.
- [9] Florian Herzig. The classification of irreducible admissible mod p representations of a p -adic GL_n . *Invent. Math.*, 186(2):373–434, 2011.
- [10] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [11] Nagayoshi Iwahori and Hideya Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):5–48, 1965.
- [12] Masaki Kashiwara and Mark Shimozono. Equivariant K -theory of affine flag manifolds and affine Grothendieck polynomials. *Duke Math. J.*, 148(3):501–538, 2009.

- [13] David Kazhdan and George Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87(1):153–215, 1987.
- [14] Bertram Kostant and Shrawan Kumar. T -equivariant K -theory of generalized flag varieties. *J. Differential Geom.*, 32(2):549–603, 1990.
- [15] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.
- [16] George Lusztig. Equivariant K -theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985.
- [17] Alexander S. Merkurjev. Equivariant K -theory. In *Handbook of K -theory. Vol. 1, 2*, pages 925–954. Springer, Berlin, 2005.
- [18] Georgios Pappas and Michael Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by T. Haines and Rapoport.
- [19] Daniel Quillen. Higher algebraic K -theory. I. In *Algebraic K -theory, I: Higher K -theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
- [20] Robert W. Thomason. Lefschetz-Riemann-Roch theorem and coherent trace formula. *Invent. Math.*, 85(3):515–543, 1986.
- [21] Robert W. Thomason. Algebraic K -theory of group scheme actions. In *Algebraic topology and algebraic K -theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 539–563. Princeton Univ. Press, Princeton, NJ, 1987.
- [22] Robert W. Thomason. Une formule de Lefschetz en K -théorie équivariante algébrique. *Duke Math. J.*, 68(3):447–462, 1992.
- [23] Michela Varagnolo and Eric Vasserot. Double affine Hecke algebras and affine flag manifolds, I. In *Affine flag manifolds and principal bundles*, Trends Math., pages 233–289. Birkhäuser/Springer Basel AG, Basel, 2010.
- [24] Marie-France Vignéras. Pro- p -Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$ -representations. *Math. Ann.*, 331(3):523–556, 2005.
- [25] Marie-France Vignéras. Algèbres de Hecke affines génériques. *Represent. Theory*, 10:1–20 (electronic), 2006.
- [26] Marie-France Vignéras. The pro- p -Iwahori-Hecke algebra of a reductive p -adic group, II. *Münster J. Math.*, 7:363–379, 2014.

INSTITUTE DE RECHERCHE MATHÉMATIQUE DE RENNES, UNIVERSITÉ DE RENNES 1, CAMPUS DE
BEAULIEU, 35042 RENNES, FRANCE

E-mail address: Tobias.Schmidt@univ-rennes1.fr