Mod p Hecke algebras and dual equivariant cohomology I: the case of GL_2

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Abstract

Let F be a p-adic local field and $\mathbf{G} = GL_2$ over F. Let $\mathcal{H}^{(1)}$ be the pro-p Iwahori-Hecke algebra of the group $\mathbf{G}(F)$ with coefficients in the algebraic closure $\overline{\mathbb{F}}_p$. We show that the supersingular irreducible $\mathcal{H}^{(1)}$-modules of dimension 2 can be realized through the equivariant cohomology of the flag variety of the Langlands dual group $\widehat{\mathbf{G}}$ over $\overline{\mathbb{F}}_p$.

Contents

1 Introduction

Let F be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and let $\mathbf{G}$ be a connected split reductive group over F. Let $\mathcal{H} = R[I \setminus \mathbf{G}(F)/I]$ be the Iwahori-Hecke algebra associated to an Iwahori subgroup $I \subset \mathbf{G}(F)$, with coefficients in an algebraically closed field R. On the other hand, let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{G}$ over R, and $\widehat{\mathcal{B}}$ the flag variety of Borel subgroups of $\widehat{\mathbf{G}}$ over R.

When $R = \mathbb{C}$, the irreducible $\mathcal{H}$-modules appear as subquotients of the Grothendieck group $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{C}}$ of $\widehat{\mathbf{G}}$-equivariant coherent sheaves on $\widehat{\mathcal{B}}$. As such they can be parametrized by the isomorphism classes of irreducible tame $\widehat{\mathbf{G}}(\mathbb{C})$-representations of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F, thereby realizing the tame local Langlands correspondence (in this setting also called the Deligne-Lusztig conjecture for Hecke modules): Kazhdan-Lusztig [KL87], Ginzburg [CG97]. The idea of studying various cohomological invariants of the flag variety by means of Hecke operators (nowadays called Demazure operators) goes back to earlier work of Demazure [D73, D74]. The approach to the Deligne-Lusztig conjecture is based on the construction of a natural $\mathcal{H}$-action on the whole K-group $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{C}}$ which identifies the center of $\mathcal{H}$ with the K-group of the base point $K^{\widehat{\mathbf{G}}}(\mathrm{pt})_{\mathbb{C}}$. The finite part of $\mathcal{H}$ acts thereby via appropriate $\mathbf{q}$-deformations of Demazure operators.

When $R = \overline{\mathbb{F}}_q$ any irreducible $\widehat{\mathbf{G}}(\overline{\mathbb{F}}_q)$-representation of $\mathrm{Gal}(\overline{F}/F)$ is tame and the Iwahori-Hecke algebra needs to be replaced by the bigger pro-p-Iwahori-Hecke algebra

$$\mathcal{H}^{(1)} = \overline{\mathbb{F}}_q[I^{(1)} \setminus \mathbf{G}(F)/I^{(1)}].$$

Here, $I^{(1)} \subset I$ is the unique pro-p Sylow subgroup of I. The algebra $\mathcal{H}^{(1)}$ was introduced by Vignéras and its structure theory developed in a series of papers [V04, V05, V06, V14, V15, V16, V17]. The class of so-called *supersingular* irreducible $\mathcal{H}^{(1)}$-modules figures prominently among all irreducible $\mathcal{H}^{(1)}$-modules, since it is expected to be related to the arithmetic over the field F. For $\mathbf{G} = GL_n$, there is a distinguished correspondence between supersingular irreducible $\mathcal{H}^{(1)}$-modules of dimension n and irreducible $GL_n(\overline{\mathbb{F}}_q)$-representations of $\mathrm{Gal}(\overline{F}/F)$: Breuil [Br03], Vignéras [V04], [V05], Colmez [C10], Grosse-Klönne [GK16], [GK18].

Our aim is to show that the supersingular irreducible $\mathcal{H}^{(1)}$- modules of dimension n can again be realized as subquotients of some $\widehat{\mathbf{G}}$-equivariant cohomology theory of the flag variety $\widehat{\mathcal{B}}$ over $\overline{\mathbb{F}}_q$, although in a way different from the $\mathbb{C}$-coefficient case. Here we discuss the case $n = 2$, and we will treat the case of general n in a subsequent article [PS2].

From now on, let $R = \overline{\mathbb{F}}_q$ and $\mathbf{G} = GL_2$. The algebra $\mathcal{H}^{(1)}$ splits as a direct product of subalgebras $\mathcal{H}^{\gamma}$ indexed by the orbits γ of $\mathfrak{S}_2$ in the set of characters of $(\mathbb{F}_q^{\times})^2$, namely the Iwahori components corresponding to trivial orbits, and the regular components. Accordingly, the category of $\mathcal{H}^{(1)}$-modules decomposes as the product of the module categories for the component algebras. In each component sits a unique supersingular module of dimension 2 with given central character. On the dual side, we have the projective line $\widehat{\mathcal{B}} = \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ over $\overline{\mathbb{F}}_q$ with its natural action by fractional transformations of the algebraic group $\widehat{\mathbf{G}} = GL_2(\overline{\mathbb{F}}_q)$.

For a non-regular orbit γ, the component algebra $\mathcal{H}^{\gamma}$ is isomorphic to the mod p Iwahori-Hecke algebra $\mathcal{H} = \overline{\mathbb{F}}_q[I\backslash \mathbf{G}(F)/I]$ and the quadratic relations in $\mathcal{H}$ are idempotent of type $T_s^2 = -T_s$. The $\widehat{\mathbf{G}}$-equivariant K-theory $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q}$ of $\widehat{\mathcal{B}}$ comes with an action of the classical Demazure operator at $\mathbf{q} = 0$. Our first result is that this action extends uniquely to an action of the full algebra $\mathcal{H}$ on $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_p}$, which is faithful and which identifies the center $Z(\mathcal{H})$ of $\mathcal{H}$ with the base ring $K^{\widehat{\mathbf{G}}}(\mathrm{pt})_{\overline{\mathbb{F}}_q}$. It is constructed from natural presentations of the algebras $\mathcal{H}$ and $Z(\mathcal{H})$ [V04] and through the characteristic homomorphism

$$\mathbb{Z}[\Lambda] \xrightarrow{\simeq} K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$$

which identfies the equivariant K-ring with the group ring of characters Λ of a maximal torus in $\widehat{\mathbf{G}}$. In particular, everything is explicit. We finally show that, given a supersingular central character $\theta : Z(\mathcal{H}) \to \overline{\mathbb{F}}_q$, the central reduction $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\theta}$ is isomorphic to the unique supersingular $\mathcal{H}$-module of dimension 2 with central character θ.

For a regular orbit γ, the component algebra $\mathcal{H}^{\gamma}$ is isomorphic to Vignéras second Iwahori-Hecke algebra $\mathcal{H}_2$ [V04]. It can be viewed as a certain twisted version of two copies of the mod p nil Hecke ring $\mathcal{H}^{\mathrm{nil}}$ (introduced over the complex numbers by Kostant-Kumar [KK86]). In particular, the quadratic relations are nilpotent of type $T_s^2 = 0$. The $\widehat{\mathbf{G}}$-equivariant intersection theory $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q}$ of $\widehat{\mathcal{B}}$ comes with an action of the classical Demazure operator at $\mathbf{q} = 0$. We show that this action extends to a faithful action of $\mathcal{H}^{\mathrm{nil}}$ on $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q}$. To incorporate the twisting, we then pass to the square $\widehat{\mathcal{B}}^2$ of $\widehat{\mathcal{B}}$ and extend the action to a faithful action of $\mathcal{H}_2$ on $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}$. The action identifies a large part $Z^\circ(\mathcal{H}_2)$ of the center $Z(\mathcal{H}_2)$ with the base ring $CH^{\widehat{\mathbf{G}}}(\mathrm{pt})_{\overline{\mathbb{F}}_q}$. As a technical point, one actually has to pass to a certain localization of the Chow groups to realize these actions, but we do not go into this in the introduction. As in the non-regular case, the action is constructed from natural presentations of the algebras $\mathcal{H}_2$ and $Z(\mathcal{H}_2)$ [V04] and through the characteristic homomorphism

$$\mathrm{Sym}(\Lambda) \xrightarrow{\simeq} CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$$

which identifies the equivariant Chow ring with the symmetric algebra on the character group Λ. So again, everything is explicit. We finally show that, given a supersingular central character $\theta : Z(\mathcal{H}_2) \to \overline{\mathbb{F}}_q$, the semisimplification of the $Z^\circ(\mathcal{H}_2)$-reduction of (the localization of) $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}$ equals a direct sum of four copies of the unique supersingular $\mathcal{H}_2$-module of dimension 2 with central character θ.

In a final section we discuss the aforementioned bijection between supersingular irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules of dimension 2 and irreducible smooth $GL_2(\overline{\mathbb{F}}_q)$-representations of $\mathrm{Gal}(\overline{F}/F)$ in the light of our geometric language.

Notation: In general, the letter F denotes a locally compact complete non-archimedean field with ring of integers o_F. Let $\mathbb{F}_q$ be its residue field, of characteristic p and cardinality q. We denote by $\mathbf{G}$ the algebraic group GL_2 over F and by $G := \mathbf{G}(F)$ its group of F-rational points. Let $\mathbf{T} \subset \mathbf{G}$ be the torus of diagonal matrices. Finally, $I \subset G$ denotes the upper triangular standard Iwahori subgroup and $I^{(1)} \subset I$ denotes the unique pro-p Sylow subgroup of I. Without further mentioning, all modules will be left modules.

2 The pro-p-Iwahori-Hecke algebra

Let R be any commutative ring. The *pro-p Iwahori Hecke algebra of G with coefficients in R* is defined to be the convolution algebra

$$\mathcal{H}^{(1)}_R(q) := (R[I^{(1)}\backslash G/I^{(1)}], \star)$$

generated by the $I^{(1)}$-double cosets in G. In the sequel, *we will assume that R is an algebra over the ring*

$$\mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}].$$

The first examples we have in mind are $R = \mathbb{F}_q$ or its algebraic closure $R = \overline{\mathbb{F}}_q$.

2.1 Weyl groups and cocharacters

2.1.1. We denote by

$$\Lambda = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T}) = \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2 \simeq \mathbb{Z} \oplus \mathbb{Z}$$

the lattice of cocharacters of $\mathbf{T}$ with standard basis $\eta_1(x) = \mathrm{diag}(x, 1)$ and $\eta_2(x) = \mathrm{diag}(1, x)$. Then $\alpha = (1, -1) \in \Lambda$ is a root and

$$\begin{array}{rcl} s = s_\alpha = s_{(1,-1)} : \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ (n_1, n_2) & \longmapsto & (n_2, n_1) \end{array}$$

is the associated reflection generating the Weyl group $W_0 = \{1, s\}$. The element s acts on Λ and hence also on the group ring $\mathbb{Z}[\Lambda]$. The two invariant elements

$$\xi_1 := e^{(1,0)} + e^{(0,1)} \quad \text{and} \quad \xi_2 := e^{(1,1)}$$

in $\mathbb{Z}[\Lambda]^s$ define a ring isomorphism

$$\begin{array}{rcl} \xi^+ : \mathbb{Z}[\Lambda^+] = \mathbb{Z}\big[e^{(1,0)}, (e^{(1,1)})^{\pm 1}\big] & \xrightarrow{\simeq} & \mathbb{Z}[\Lambda]^s \\ e^{(1,0)} & \longmapsto & \xi_1 \\ e^{(1,1)} & \longmapsto & \xi_2 \end{array}$$

where $\Lambda^+ := \mathbb{Z}_{\geq 0}(1,0) \oplus \mathbb{Z}(1,1)$ is the monoid of dominant cocharacters.

2.1.2. We introduce the affine Weyl group W_{aff} and the Iwahori-Weyl group W of G:

$$W_{\mathrm{aff}} := e^{\mathbb{Z}(1,-1)} \rtimes W_0 \subset W := e^{\Lambda} \rtimes W_0.$$

With

$$u := e^{(1,0)}s = se^{(0,1)}$$

one has $W = W_{\mathrm{aff}} \rtimes \Omega$ where $\Omega = u^{\mathbb{Z}} \simeq \mathbb{Z}$. Let $s_0 = e^{(1,-1)}s = se^{(-1,1)} = usu^{-1}$. Recall that the pair $(W_{\mathrm{aff}}, \{s_0, s\})$ is a Coxeter group and its length function ℓ can be inflated to W via $\ell|_{\Omega} = 0$.

2.2 Idempotents and component algebras

2.2.1. We have the finite diagonal torus

$$\mathbb{T} := \mathbf{T}(\mathbb{F}_q)$$

and its group ring $R[\mathbb{T}]$. As $q-1$ is invertible in R, so is $|\mathbb{T}| = (q-1)^2$ and hence $R[\mathbb{T}]$ is a semisimple ring. The canonical isomorphism $\mathbb{T} \simeq I/I^{(1)}$ induces an inclusion

$$R[\mathbb{T}] \subset \mathcal{H}_R^{(1)}(q).$$

We denote by $\mathbb{T}^\vee$ the set of characters

$$\lambda : \mathbb{T} \to \mathbb{F}_q^\times$$

of $\mathbb{T}$, with its natural W_0-action given by

$${}^s\lambda(t_1, t_2) = \lambda(t_2, t_1)$$

for $(t_1, t_2) \in \mathbb{T}$. The number of W_0-orbits in $\mathbb{T}^\vee$ equals $\frac{q^2-q}{2}$. Also W acts on $\mathbb{T}^\vee$ through the canonical quotient map $W \to W_0$.

2.2.2. Definition. *For all $\lambda \in \mathbb{T}^\vee$, define*

$$\varepsilon_\lambda := |\mathbb{T}|^{-1} \sum_{t \in \mathbb{T}} \lambda^{-1}(t) T_t \quad \in R[\mathbb{T}]$$

and for all $\gamma \in \mathbb{T}^\vee / W_0$,

$$\varepsilon_\gamma := \sum_{\lambda \in \gamma} \varepsilon_\lambda \quad \in R[\mathbb{T}].$$

Following the terminology of [V04], we call $|\gamma| = 1$ the *Iwahori case* or *non-regular case* and $|\gamma| = 2$ the *regular case.*

2.2.3. Proposition. *For all $\lambda \in \mathbb{T}^\vee$, the element ε_λ is an idempotent. For all $\gamma \in \mathbb{T}^\vee/W_0$, the element ε_γ is a central idempotent in $\mathcal{H}_R^{(1)}(q)$. The R-algebra $\mathcal{H}_R^{(1)}(q)$ is the direct product of its sub-R-algebras $\mathcal{H}_R^{(1)}(q)\varepsilon_\gamma$, i.e.*

$$\mathcal{H}_R^{(1)}(q) = \prod_{\gamma \in \mathbb{T}^\vee / W_0} \mathcal{H}_R^{(1)}(q)\varepsilon_\gamma.$$

Proof. This follows from [V04, Prop. 3.1] and its proof. □

The proposition implies that the category of $\mathcal{H}_R^{(1)}(q)$-modules decomposes into a finite product of the module categories for the individual component rings $\mathcal{H}_R^{(1)}(q)\varepsilon_\gamma$.

2.3 The Iwahori-Hecke algebra

Our reference for the following is [V04, 1.1/2].

2.3.1. Definition. *Let* $\mathbf{q}$ *be an indeterminate. The* generic Iwahori-Hecke algebra *is the* $\mathbb{Z}[\mathbf{q}]$-*algebra* $\mathcal{H}(\mathbf{q})$ *defined by generators*

$$\mathcal{H}(\mathbf{q}) := \bigoplus_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{Z}[\mathbf{q}]T_{e^{(n_1,n_2)}} \oplus \mathbb{Z}[\mathbf{q}]T_{e^{(n_1,n_2)}s}$$

and relations:

- *braid relations*

$$T_w T_{w'} = T_{ww'} \quad \textit{for } w, w' \in W \textit{ if } \ell(w) + \ell(w') = \ell(ww')$$

- *quadratic relations*

$$\begin{cases} T_s^2 = (\mathbf{q}-1)T_s + \mathbf{q} \\ T_{s_0}^2 = (\mathbf{q}-1)T_{s_0} + \mathbf{q}. \end{cases}$$

2.3.2. Setting $S := T_s$ and $U := T_u$, one can check that

$$\mathcal{H}(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = (\mathbf{q}-1)S + \mathbf{q}, \quad U^2 S = SU^2$$

is a presentation of $\mathcal{H}(\mathbf{q})$. For example, $S_0 := T_{s_0} = USU^{-1}$. We also have the generic finite and affine Hecke algebras

$$\mathcal{H}_0(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S] \subset \mathcal{H}_{\mathrm{aff}}(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S_0, S].$$

The algebra $\mathcal{H}_0(\mathbf{q})$ has two characters corresponding to $S \mapsto 0$ and $S \mapsto -1$. Similarly, $\mathcal{H}_{\mathrm{aff}}(\mathbf{q})$ has four characters. The two characters different from the trivial character $S_0, S \mapsto 0$ and the sign character $S_0, S \mapsto -1$ are called *supersingular*.

2.3.3. The center $Z(\mathcal{H}(\mathbf{q}))$ of the algebra $\mathcal{H}(\mathbf{q})$ admits the explicit description via the algebra isomomorphism

$$\begin{array}{rcl} \mathscr{Z}(\mathbf{q}) : \mathbb{Z}[\mathbf{q}]\left[\Lambda^+\right] = \mathbb{Z}[\mathbf{q}]\left[e^{(1,0)}, (e^{(1,1)})^{\pm 1}\right] & \xrightarrow{\simeq} & Z(\mathcal{H}(\mathbf{q})) \\ e^{(1,0)} & \longmapsto & \zeta_1 := U(S - (\mathbf{q}-1)) + SU \\ e^{(1,1)} & \longmapsto & \zeta_2 := U^2. \end{array}$$

In particular,

$$Z(\mathcal{H}(\mathbf{q})) = \mathbb{Z}[\mathbf{q}][US + (1-\mathbf{q})U + SU, U^{\pm 2}] \subset \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}] = \mathcal{H}(\mathbf{q}).$$

2.3.4. Now let $\gamma \in \mathbb{T}^\vee / W_0$ such that $|\gamma| = 1$, say $\gamma = \{\lambda\}$. The ring homomorphism $\mathbb{Z}[\mathbf{q}] \to R$, $\mathbf{q} \mapsto q$, induces an isomorphism of R-algebras

$$\mathcal{H}(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} R \xrightarrow{\simeq} \mathcal{H}_R^{(1)}(q)\varepsilon_\gamma, \quad T_w \mapsto \varepsilon_\lambda T_w.$$

2.4 The second Iwahori-Hecke algebra

Our reference for the following is [V04, 2.2], as well as [KK86] for the basic theory of the nil Hecke algebra. We keep the notation introduced above.

2.4.1. Definition. *The* generic nil Hecke algebra *is the* $\mathbb{Z}[\mathbf{q}]$-*algebra* $\mathcal{H}^{\mathrm{nil}}(\mathbf{q})$ *defined by generators*

$$\mathcal{H}^{\mathrm{nil}}(\mathbf{q}) := \bigoplus_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{Z}[\mathbf{q}]T_{e^{(n_1,n_2)}} \oplus \mathbb{Z}[\mathbf{q}]T_{e^{(n_1,n_2)}s}$$

and relations:

- *braid relations*

$$T_w T_{w'} = T_{ww'} \quad \textit{for } w, w' \in W \textit{ if } \ell(w) + \ell(w') = \ell(ww')$$

- *quadratic relations*

$$\begin{cases} T_s^2 = \mathbf{q} \\ T_{s_0}^2 = \mathbf{q}. \end{cases}$$

2.4.2. Setting $S := T_s$ and $U := T_u$, one can check that

$$\mathcal{H}^{\mathrm{nil}}(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = \mathbf{q}, \quad U^2 S = S U^2$$

is a presentation of $\mathcal{H}^{\mathrm{nil}}(\mathbf{q})$. Again, $S_0 := T_{s_0} = USU^{-1}$. The center $Z(\mathcal{H}^{\mathrm{nil}}(\mathbf{q}))$ admits the explicit description via the algebra isomorphism

$$\begin{array}{rcl} \mathscr{Z}^{\mathrm{nil}}(\mathbf{q}) : \mathbb{Z}[\mathbf{q}]\big[\Lambda^+\big] = \mathbb{Z}[\mathbf{q}]\big[e^{(1,0)}, (e^{(1,1)})^{\pm 1}\big] & \xrightarrow{\simeq} & Z(\mathcal{H}^{\mathrm{nil}}(\mathbf{q})) \\ e^{(1,0)} & \longmapsto & \zeta_1 := US + SU \\ e^{(1,1)} & \longmapsto & \zeta_2 := U^2. \end{array}$$

In particular,

$$Z(\mathcal{H}^{\mathrm{nil}}(\mathbf{q})) = \mathbb{Z}[\mathbf{q}][US + SU, U^{\pm 2}] \subset \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}] = \mathcal{H}^{\mathrm{nil}}(\mathbf{q}).$$

2.4.3. Form the twisted tensor product algebra

$$\mathcal{H}_2(\mathbf{q}) := (\mathbb{Z}[\mathbf{q}] \times \mathbb{Z}[\mathbf{q}]) \otimes'_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{nil}}(\mathbf{q}).$$

With the formal symbols $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$, the ring multiplication is given by

$$(\varepsilon_i \otimes T_w) \cdot (\varepsilon_{i'} \otimes T_{w'}) = (\varepsilon_i \varepsilon_{{}^w i'} \otimes T_w T_{w'})$$

for all $1 \leq i, i' \leq 2$. Here, W acts through its quotient W_0 and $s \in W_0$ acts on the set $\{1,2\}$ by interchanging the two elements. The multiplicative unit element in the ring $\mathbb{Z}[\mathbf{q}] \times \mathbb{Z}[\mathbf{q}]$ is $(1,1) = \varepsilon_1 + \varepsilon_2$ and the multiplicative unit element in the ring $\mathcal{H}_2(\mathbf{q})$ is $(1,1) \otimes 1$. We identify the rings $\mathbb{Z}[\mathbf{q}] \times \mathbb{Z}[\mathbf{q}]$ and $\mathcal{H}^{\mathrm{nil}}(\mathbf{q})$ with subrings of $\mathcal{H}_2(\mathbf{q})$ via the maps $(a,b) \mapsto (a,b) \otimes 1$ and $a \mapsto (1,1) \otimes a$ respectively. In particular, we will write $\varepsilon_1, \varepsilon_2, S_0, S, U \in \mathcal{H}_2(\mathbf{q})$ etc.

We also introduce the generic affine Hecke algebra

$$\mathcal{H}_{2,\mathrm{aff}}(\mathbf{q}) = (\mathbb{Z}[\mathbf{q}] \times \mathbb{Z}[\mathbf{q}]) \otimes'_{\mathbb{Z}[\mathbf{q}]} \mathbb{Z}[\mathbf{q}][S_0, S].$$

It is a subalgebra of $\mathcal{H}_2(\mathbf{q})$ and has two *supersingular* characters χ_1 and χ_2, namely $\chi_1(\varepsilon_1) = 1$ and $\chi_1(\varepsilon_2) = 0$ and $\chi_1(S_0) = \chi_1(S) = 0$. Similarly for χ_2.

2.4.4. The structure of $\mathcal{H}_2(\mathbf{q})$ as an algebra over its center can be made explicit. In fact, there is an algebra isomorphism with an algebra of 2×2-matrices

$$\mathcal{H}_2(\mathbf{q}) \simeq M(2, \mathcal{Z}(\mathbf{q})), \quad \mathcal{Z}(\mathbf{q}) := \mathbb{Z}[\mathbf{q}][X, Y, z_2^{\pm 1}]/(XY)$$

which maps the center $Z(\mathcal{H}_2(\mathbf{q}))$ to the scalar matrices $\mathcal{Z}(\mathbf{q})$. Under this isomorphism, we have

$$S \mapsto \begin{pmatrix} 0 & Y \\ z_2^{-1} X & 0 \end{pmatrix}, \quad U \mapsto \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix},$$

$$\varepsilon_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The induced map $Z(\mathcal{H}_2(\mathbf{q})) \to \mathcal{Z}(\mathbf{q})$ satisfies

$$\zeta_1 \mapsto \begin{pmatrix} X+Y & 0 \\ 0 & X+Y \end{pmatrix}, \quad \zeta_2 \mapsto \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix}.$$

In particular, the subring

$$Z^{\circ}(\mathcal{H}_2(\mathbf{q})) := \mathbb{Z}[\mathbf{q}][\zeta_1, \zeta_2^{\pm 1}] = Z(\mathcal{H}^{\mathrm{nil}}(\mathbf{q})) \subset \mathcal{H}^{\mathrm{nil}}(\mathbf{q}) \subset \mathcal{H}_2(\mathbf{q})$$

lies in fact in the center $Z(\mathcal{H}_2(\mathbf{q}))$ of $\mathcal{H}_2(\mathbf{q})$.

2.4.5. Now let $\gamma \in \mathbb{T}^\vee/W_0$ such that $|\gamma| = 2$, say $\gamma = \{\lambda, {}^s\lambda\}$. The ring homomorphism $\mathbb{Z}[\mathbf{q}] \to R$, $\mathbf{q} \mapsto q$, induces an isomorphism of R-algebras

$$\mathcal{H}_2(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} R \xrightarrow{\simeq} \mathcal{H}_R^{(1)}(q)\varepsilon_\gamma, \quad \varepsilon_1 \otimes T_w \mapsto \varepsilon_\lambda T_w, \quad \varepsilon_2 \otimes T_w \mapsto \varepsilon_{{}^s\lambda} T_w.$$

2.4.6. Remark. We have used the same letters $S_0, S, U, \zeta_1, \zeta_2$ for the corresponding Hecke operators in the Iwahori Hecke algebra and in the second Iwahori Hecke algebra. This should not lead to confusion, as we will always treat non-regular components and regular components separately in our discussion.

3 The non-regular case and dual equivariant K-theory

3.1 Recollections from algebraic $K^{\widehat{\mathbf{G}}}$-theory

For basic notions from equivariant algebraic K-theory we refer to [Th87]. A useful introduction may also be found in [CG97, chap. 5].

3.1.1. We let

$$\widehat{\mathbf{G}} := \mathrm{GL}_{2/\overline{\mathbb{F}}_q}$$

be the Langlands dual group of $\mathbf{G}$ over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. The dual torus

$$\widehat{\mathbf{T}} := \operatorname{Spec} \overline{\mathbb{F}}_q[\Lambda] \subset \widehat{\mathbf{G}}$$

identifies with the torus of diagonal matrices in $\widehat{\mathbf{G}}$. A basic object is

$$R(\widehat{\mathbf{G}}) := \text{the representation ring of } \widehat{\mathbf{G}},$$

i.e. the Grothendieck ring of the abelian tensor category of all finite dimensional $\widehat{\mathbf{G}}$-representations. It can be viewed as the equivariant K-theory $K^{\widehat{\mathbf{G}}}(\mathrm{pt})$ of the base point $\mathrm{pt} = \operatorname{Spec} \overline{\mathbb{F}}_q$. To compute it, we introduce the representation ring $R(\widehat{\mathbf{T}})$ of $\widehat{\mathbf{T}}$ which identifies canonically, as a ring with W_0-action, with the group ring of Λ, i.e.

$$R(\widehat{\mathbf{T}}) = \mathbb{Z}[\Lambda].$$

The formal character $\chi_V \in \mathbb{Z}[\Lambda]^s$ of a representation V is an invariant function and is defined by

$$\chi_V(e^\lambda) = \dim_{\overline{\mathbb{F}}_q} V_\lambda$$

for all $\lambda \in \Lambda$ where V_λ is the λ-weight space of V. The map $V \mapsto \chi_V$ induces a ring isomorphism

$$\chi_\bullet : R(\widehat{\mathbf{G}}) \xrightarrow{\simeq} \mathbb{Z}[\Lambda]^s.$$

The $\mathbb{Z}[\Lambda]^s$-module $\mathbb{Z}[\Lambda]$ is free of rank 2, with basis $\{1, e^{(-1,0)}\}$,

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[\Lambda]^s \oplus \mathbb{Z}[\Lambda]^s e^{(-1,0)}.$$

3.1.2. We let

$$\widehat{\mathcal{B}} := \mathbb{P}^1_{\overline{\mathbb{F}}_q}$$

be the projective line over $\overline{\mathbb{F}}_q$ endowed with its left $\widehat{\mathbf{G}}$-action by fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d}.$$

Here, x is a local coordinate on $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$. The stabilizer of the point $x = \infty$ is the Borel subgroup $\widehat{\mathbf{B}}$ of upper triangular matrices and we may thus write $\widehat{\mathcal{B}} = \widehat{\mathbf{G}}/\widehat{\mathbf{B}}$. We denote by

$$K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) := \text{the Grothendieck group of all } \widehat{\mathbf{G}}\text{-equivariant coherent } \mathcal{O}_{\widehat{\mathcal{B}}}\text{-modules}.$$

Given a representation V and an equivariant coherent sheaf $\mathcal{F}$, the diagonal action of $\widehat{\mathbf{G}}$ makes $\mathcal{F} \otimes_{\overline{\mathbb{F}}_q} V$ an equivariant coherent sheaf. In this way, $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$ becomes a module over the ring $R(\widehat{\mathbf{G}})$.

The *characteristic homomorphism* in algebraic $K^{\widehat{\mathbf{G}}}$-theory is a ring isomorphism

$$c^{\widehat{\mathbf{G}}} : \mathbb{Z}[\Lambda] \xrightarrow{\simeq} K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}).$$

It maps e^{λ} with $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ to the class of the $\widehat{\mathbf{G}}$-equivariant line bundle $\mathcal{O}_{\mathbb{P}^1}(\lambda_1 - \lambda_2) \otimes \det^{\lambda_2}$ where det is the determinant character of $\widehat{\mathbf{G}}$. The characteristic homomorphism is compatible with the character morphism $\chi_\bullet$, i.e. $c^{\widehat{\mathbf{G}}}$ is $\mathbb{Z}[\Lambda]^s \simeq R(\widehat{\mathbf{G}})$-linear.

3.1.3. For the definition of the classical Demazure operators on algebraic K-theory we refer to [D73, D74]. The Demazure operators

$$D_s, D'_s \in \mathrm{End}_{R(\widehat{\mathbf{T}})^s}(R(\widehat{\mathbf{T}}))$$

are defined by:

$$D_s(a) = \frac{a - s(a)}{1 - e^{(1,-1)}} \qquad \text{and} \qquad D'_s(a) = \frac{a - s(a)e^{(1,-1)}}{1 - e^{(1,-1)}}$$

for $a \in R(\widehat{\mathbf{T}})$. They are the projectors on $R(\widehat{\mathbf{T}})^s e^{(-1,0)}$ along $R(\widehat{\mathbf{T}})^s$, and on $R(\widehat{\mathbf{T}})^s$ along $R(\widehat{\mathbf{T}})^s e^{(1,0)}$, respectively. In particular $D_s^2 = D_s$ and $D_s'^2 = D'_s$. One sets

$$D_s(\mathbf{q}) := D_s - \mathbf{q}D'_s \in \mathrm{End}_{R(\widehat{\mathbf{T}})^s[\mathbf{q}]}(R(\widehat{\mathbf{T}})[\mathbf{q}])$$

and checks by direct calculation that

$$D_s(\mathbf{q})^2 = \mathbf{q} - (\mathbf{q} - 1)D_s(\mathbf{q}).$$

In particular, we obtain a well-defined $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\mathscr{A}_0(\mathbf{q}) : \mathcal{H}_0(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S] \longrightarrow \mathrm{End}_{R(\widehat{\mathbf{T}})^s[\mathbf{q}]}(R(\widehat{\mathbf{T}})[\mathbf{q}]), \quad S \longmapsto -D_s(\mathbf{q})$$

which we call the *Demazure representation*.

3.2 The morphism from $R(\widehat{\mathbf{G}})[\mathbf{q}]$ to the center of $\mathcal{H}(\mathbf{q})$

In the following we identify the rings

$$R(\widehat{\mathbf{G}})[\mathbf{q}] \simeq \mathbb{Z}[\mathbf{q}][\Lambda]^s = \mathbb{Z}[\mathbf{q}][\xi_1, \xi_2^{\pm 1}]$$

via the character isomorphism $\chi_\bullet$. We have the $\mathbb{Z}[\mathbf{q}]$-algebra isomorphism coming via base change from the isomorphism ξ^+, cf. 2.1.1:

$$\begin{array}{rcl} \xi^+ : \mathbb{Z}[\mathbf{q}]\left[e^{(1,0)}, (e^{(1,1)})^{\pm 1}\right] & \xrightarrow{\simeq} & \mathbb{Z}[\mathbf{q}][\xi_1, \xi_2^{\pm 1}] \\ e^{(1,0)} & \longmapsto & \xi_1 \\ e^{(1,1)} & \longmapsto & \xi_2. \end{array}$$

On the other hand, the source of ξ^+ is isomorphic to the center $Z(\mathcal{H}(\mathbf{q}))$ of $\mathcal{H}(\mathbf{q})$ via the isomorphism $\mathscr{Z}(\mathbf{q})$, cf. 2.3.2. The composition

$$\begin{array}{rcl} \mathscr{Z}(\mathbf{q}) \circ (\xi^+)^{-1} : R(\widehat{\mathbf{G}})[\mathbf{q}] & \xrightarrow{\simeq} & Z(\mathcal{H}(\mathbf{q})) \\ \xi_1 & \longmapsto & \zeta_1 = U(S - (\mathbf{q} - 1)) + SU \\ \xi_2 & \longmapsto & \zeta_2 = U^2 \end{array}$$

is then a ring isomorphism.

3.3 The extended Demazure representation $\mathscr{A}(\mathbf{q})$

Recall the Demazure representation $\mathscr{A}_0(\mathbf{q})$ of the finite algebra $\mathcal{H}_0(\mathbf{q})$ by $R(\widehat{\mathbf{G}})[\mathbf{q}]$-linear operators on the K-theory $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$, cf. 3.1.3. We have the following first main result.

3.3.1. Theorem. *There is a unique ring homomorphism*

$$\mathscr{A}(\mathbf{q}) : \mathcal{H}(\mathbf{q}) \longrightarrow \mathrm{End}_{R(\widehat{\mathbf{G}})[\mathbf{q}]}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}])$$

which extends the ring homomorphism $\mathscr{A}_0(\mathbf{q})$ and coincides on $Z(\mathcal{H}(\mathbf{q}))$ with the isomorphism

$$\begin{array}{ccc} Z(\mathcal{H}(\mathbf{q})) & \xrightarrow{\simeq} & R(\widehat{\mathbf{G}})[\mathbf{q}] \\ \zeta_1 & \longmapsto & \xi_1 \\ \zeta_2 & \longmapsto & \xi_2. \end{array}$$

The homomorphism $\mathscr{A}(\mathbf{q})$ is injective.

Proof : Such an extension exists if and only if there exists

$$\mathscr{A}(\mathbf{q})(U) \in \mathrm{End}_{R(\widehat{\mathbf{G}})[\mathbf{q}]}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}])$$

satisfying

1. $\mathscr{A}(\mathbf{q})(U)$ is invertible ;
2. $\mathscr{A}(\mathbf{q})(U)^2 = \mathscr{A}(\mathbf{q})(U^2) = \mathscr{A}(\mathbf{q})(\zeta_2) = \xi_2 \,\mathrm{Id}$;
3.

$$\begin{aligned} \mathscr{A}(\mathbf{q})(U)\mathscr{A}_0(\mathbf{q})(S) + (1-\mathbf{q})\mathscr{A}(\mathbf{q})(U) + \mathscr{A}_0(\mathbf{q})(S)\mathscr{A}(\mathbf{q})(U) &= \mathscr{A}(\mathbf{q})(US + (1-\mathbf{q})U + SU) \\ &= \mathscr{A}(\mathbf{q})(\zeta_1) \\ &= \xi_1 \,\mathrm{Id}\,. \end{aligned}$$

To find such an operator $\mathscr{A}(\mathbf{q})(U)$, we write

$$K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}] = R(\widehat{\mathbf{T}})[\mathbf{q}] = R(\widehat{\mathbf{T}})^s[\mathbf{q}] \oplus R(\widehat{\mathbf{T}})^s[\mathbf{q}]e^{(-1,0)},$$

and use the $R(\widehat{\mathbf{T}})^s[\mathbf{q}]$-basis $\{1, e^{(-1,0)}\}$ to identify $\mathrm{End}_{R(\widehat{\mathbf{G}})[\mathbf{q}]}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}])$ with the algebra of 2×2-matrices over the ring $R(\widehat{\mathbf{T}})^s[\mathbf{q}]$. Then, by definition,

$$\mathscr{A}_0(\mathbf{q})(S) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \mathbf{q}\begin{pmatrix} 1 & \xi_1 e^{(-1,-1)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{q} & \mathbf{q}\xi_1 e^{(-1,-1)} \\ 0 & -1 \end{pmatrix}.$$

Hence, if we set

$$\mathscr{A}(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

we get

$$\mathscr{A}(\mathbf{q})(U)^2 = e^{(1,1)}\,\mathrm{Id} \Longleftrightarrow \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} e^{(1,1)} & 0 \\ 0 & e^{(1,1)} \end{pmatrix}$$

and

$$\mathscr{A}(\mathbf{q})(U)\mathscr{A}_0(\mathbf{q})(S) + (1-\mathbf{q})\mathscr{A}(\mathbf{q})(U) + \mathscr{A}_0(\mathbf{q})(S)\mathscr{A}(\mathbf{q})(U) = \xi_1\,\mathrm{Id}$$
$$\Longleftrightarrow \begin{pmatrix} (\mathbf{q}+1)a + \mathbf{q}\xi_1 e^{(-1,-1)}b & \mathbf{q}\xi_1 e^{(-1,-1)}(a+d) \\ 0 & -(\mathbf{q}+1)d + \mathbf{q}\xi_1 e^{(-1,-1)}b \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1 \end{pmatrix}.$$

These two conditions together are in turn equivalent to

$$\left\{ \begin{array}{lcl} a & = & -d \\ bc & = & e^{(1,1)} - a^2 \\ (\mathbf{q}+1)a & = & \xi_1 - \mathbf{q}\xi_1 e^{(-1,-1)}b. \end{array} \right.$$

Moreover, in this case, the determinant

$$ad - bc = -a^2 - (e^{(1,1)} - a^2) = -e^{(1,1)}$$

is invertible. Specialising to $\mathbf{q} = 0$, we find that there is *exactly one* $R(\widehat{\mathbf{G}})[\mathbf{q}]$-algebra homomorphism

$$\mathscr{A}(\mathbf{q}) : \mathcal{H}(\mathbf{q}) \longrightarrow \mathrm{End}_{R(\widehat{\mathbf{G}})[\mathbf{q}]}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}]),$$

extending the ring homomorphism $\mathscr{A}_0(\mathbf{q})$, corresponding to the matrix

$$\mathscr{A}(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} := \begin{pmatrix} \xi_1 & e^{(-1,-1)}\xi_1^2 - 1 \\ -e^{(1,1)} & -\xi_1 \end{pmatrix}.$$

Note that $a, b, c, d \in R(\widehat{\mathbf{T}})^s \subset R(\widehat{\mathbf{T}})^s[\mathbf{q}]$. The injectivity of the map $\mathscr{A}(\mathbf{q})$ will be proved in the next subsection. □

3.4 Faithfulness of $\mathscr{A}(\mathbf{q})$

Let us show that the map $\mathscr{A}(\mathbf{q})$ is injective. It follows from 2.3.2 that the ring $\mathcal{H}(\mathbf{q})$ is generated by the elements

$$1,\ S,\ U,\ SU$$

over its center $Z(\mathcal{H}(\mathbf{q})) = \mathbb{Z}[\zeta_1, \zeta_2^{\pm 1}][\mathbf{q}]$. As the latter is mapped isomorphically to the center $R(\widehat{\mathbf{G}})[\mathbf{q}] = \mathbb{Z}[\xi_1, \xi_2^{\pm 1}][\mathbf{q}]$ of the matrix algebra $\mathrm{End}_{R(\widehat{\mathbf{G}})[\mathbf{q}]}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\mathbf{q}])$ by $\mathscr{A}(\mathbf{q})$, it suffices to check that the images

$$1,\ \mathscr{A}_0(\mathbf{q})(S),\ \mathscr{A}(\mathbf{q})(U),\ \mathscr{A}_0(\mathbf{q})(S)\mathscr{A}(\mathbf{q})(U)$$

of $1, S, U, SU$ by $\mathscr{A}(\mathbf{q})$ are free over $R(\widehat{\mathbf{G}})[\mathbf{q}]$. To ease notation, we will write ξ instead of ξ_1 in the following calculation. So let $\alpha, \beta, \gamma, \delta \in R(\widehat{\mathbf{T}})^s[\mathbf{q}]$ (which is an integral domain) be such that

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} \mathbf{q} & \mathbf{q}\xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} \mathbf{q} & \mathbf{q}\xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = 0.$$

This is equivalent to the expression

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} \beta\mathbf{q} & \beta\mathbf{q}\xi e^{(-1,-1)} \\ 0 & -\beta \end{pmatrix} + \begin{pmatrix} \gamma a & \gamma c \\ \gamma b & -\gamma a \end{pmatrix} + \begin{pmatrix} \delta(\mathbf{q}a + \xi e^{(-1,-1)} b\mathbf{q}) & \delta\mathbf{q}(c - a\xi e^{(-1,-1)}) \\ -\delta b & \delta a \end{pmatrix}$$

being zero, i.e. to the identity

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} \beta\mathbf{q} & \beta\mathbf{q}\xi e^{(-1,-1)} \\ 0 & -\beta \end{pmatrix} + \begin{pmatrix} \gamma a & \gamma c \\ \gamma b & -\gamma a \end{pmatrix} + \begin{pmatrix} \delta(\xi - a) & \delta\mathbf{q}(c - a\xi e^{(-1,-1)}) \\ -\delta b & \delta a \end{pmatrix} = 0.$$

Then

$$\begin{cases} \alpha + \beta\mathbf{q} + \gamma a + \delta(\xi - a) & = & 0 \\ (\gamma - \delta)b & = & 0 \\ \beta\mathbf{q}\xi e^{(-1,-1)} + \gamma c + \delta\mathbf{q}(c - a\xi e^{(-1,-1)}) & = & 0 \\ \alpha - \beta + (\delta - \gamma)a & = & 0. \end{cases}$$

As $b \neq 0$, we obtain $\delta = \gamma$ and

$$\begin{cases} \alpha + \beta\mathbf{q} + \gamma\xi & = & 0 \\ \beta\mathbf{q}\xi e^{(-1,-1)} + \gamma((\mathbf{q}+1)c - \mathbf{q}\xi e^{(-1,-1)}a) & = & 0 \\ \alpha - \beta & = & 0. \end{cases}$$

Hence $\alpha = \beta$ and

$$\begin{cases} \alpha(\mathbf{q}+1) + \gamma\xi & = & 0 \\ \alpha\mathbf{q}\xi e^{(-1,-1)} + \gamma((\mathbf{q}+1)c - \mathbf{q}\xi e^{(-1,-1)}a) & = & 0. \end{cases}$$

The latter system has determinant

$$(\mathbf{q}+1)((\mathbf{q}+1)c - \mathbf{q}\xi e^{(-1,-1)}a) - \mathbf{q}\xi^2 e^{(-1,-1)},$$

which is nonzero (its specialization at $\mathbf{q}=0$ is equal to $c \neq 0$), whence $\alpha=\gamma=0=\beta=\delta$. This concludes the proof and shows that the map $\mathscr{A}(\mathbf{q})$ is injective. We record the following two corollaries of the proof.

3.4.1. Corollary. *The ring $\mathcal{H}(\mathbf{q})$ is a free $Z(\mathcal{H}(\mathbf{q}))$-module on the basis $1, S, U, SU$.*

3.4.2. Corollary. *The representation $\mathscr{A}(0)$ is injective.*

3.5 Supersingular modules

In this section we work at $\mathbf{q}=0$ and over the algebraic closure $\overline{\mathbb{F}}_q$ of the field $\mathbb{F}_q$.

3.5.1. Consider the ring homomorphism $\mathbb{Z}[\mathbf{q}] \to \overline{\mathbb{F}}_q$, $\mathbf{q} \mapsto q = 0$, and let

$$\mathcal{H}_{\overline{\mathbb{F}}_q} = \mathcal{H}(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} \overline{\mathbb{F}}_q = \overline{\mathbb{F}}_q[S, U^{\pm 1}].$$

The characters of $\mathcal{H}_{\overline{\mathbb{F}}_q}$ are parametrised by the set $\{0,-1\} \times \overline{\mathbb{F}}_q^{\times}$ via evaluation on the elements S and U. Let $(\tau_1, \tau_2) \in \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q^{\times}$. A *standard module* over $\mathcal{H}_{\overline{\mathbb{F}}_q}$ of dimension 2 is defined to be a module of type

$$M_2(\tau_1,\tau_2) := \overline{\mathbb{F}}_q m \oplus \overline{\mathbb{F}}_q Um, \qquad Sm = -m, \qquad SUm = \tau_1 m, \qquad U^2 m = \tau_2 m.$$

The center $Z(\mathcal{H}_{\overline{\mathbb{F}}_q}) = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2^{\pm 1}]$ acts on the module $M_2(\tau_1,\tau_2)$ via the character $\zeta_1 \mapsto \tau_1, \zeta_2 \mapsto \tau_2$. The module $M_2(\tau_1,\tau_2)$ is reducible if and only if $\tau_1^2 = \tau_2$. It is called *supersingular* if $\tau_1 = 0$. A supersingular module is thus irreducible. Any simple finite dimensional $\mathcal{H}_{\overline{\mathbb{F}}_q}$-module is either a character or a standard module [V04, 1.4].

3.5.2. Now consider the base change of the representation $\mathscr{A} := \mathscr{A}(0)$ to $\overline{\mathbb{F}}_q$

$$\mathscr{A}_{\overline{\mathbb{F}}_q} : \mathcal{H}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{End}_{R(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q}) = \mathrm{End}_{\overline{\mathbb{F}}_q[\xi_1,\xi_2^{\pm 1}]}(\overline{\mathbb{F}}_q[e^{\pm\eta_1}, e^{\pm\eta_2}]).$$

Recall that the image of $Z(\mathcal{H}_{\overline{\mathbb{F}}_q}) = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2^{\pm 1}]$ is $R(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}]$.

Let us fix a character $\theta : Z(\mathcal{H}_{\overline{\mathbb{F}}_q}) \to \overline{\mathbb{F}}_q$. Following [V04], we call θ *supersingular* if $\theta(\zeta_1) = 0$. Consider the base change of $\mathscr{A}_{\overline{\mathbb{F}}_q}$ along θ

$$\mathcal{H}_\theta := \mathcal{H}_{\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{\overline{\mathbb{F}}_q})} \overline{\mathbb{F}}_q, \quad K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta := K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{\overline{\mathbb{F}}_q})} \overline{\mathbb{F}}_q = K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q} \otimes_{R(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}} \left(R(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{\overline{\mathbb{F}}_q})} \overline{\mathbb{F}}_q \right),$$

$$\mathscr{A}_\theta : \mathcal{H}_\theta \longrightarrow \mathrm{End}_{\overline{\mathbb{F}}_q}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta).$$

3.5.3. Proposition. *The representation $\mathscr{A}_\theta$ is faithful if and only if $\theta(\zeta_1)^2 \neq \theta(\zeta_2)$. In this case, $\mathscr{A}_\theta$ is an algebra isomorphism*

$$\mathscr{A}_\theta : \mathcal{H}_\theta \xrightarrow{\simeq} \mathrm{End}_{\overline{\mathbb{F}}_q}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta).$$

Proof : The discussion in the preceding section 3.4 shows that $\mathcal{H}_\theta$ has $\overline{\mathbb{F}}_q$-basis given by $1, S, U, SU$. Moreover, their images

$$1,\ \mathscr{A}_\theta(S),\ \mathscr{A}_\theta(U),\ \mathscr{A}_\theta(S)\mathscr{A}_\theta(U)$$

by $\mathscr{A}_\theta$ are linearly independent over $\overline{\mathbb{F}}_q$ if and only if the scalar $c = e^{(-1,-1)}\xi_1^2 - 1 \in R(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}$ does not reduce to zero via θ, i.e. if and only if $\zeta_2^{-1}\zeta_1^2 - 1 \notin \ker\theta$. In this case, the map $\mathscr{A}_\theta$ is injective and then bijective since $\dim_{\overline{\mathbb{F}}_q} K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta = 2$. $\square$

3.5.4. Corollary. *The* $\mathcal{H}_{\overline{\mathbb{F}}_q}$*-module* $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta$ *is isomorphic to the standard module* $M_2(\tau_1, \tau_2)$ *where* $\tau_1 = \theta(\zeta_1)$ *and* $\tau_2 = \theta(\zeta_2)$*. In particular, if* θ *is supersingular, then* $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta$ *is isomorphic to the unique supersingular* $\mathcal{H}_{\overline{\mathbb{F}}_q}$*-module with central character* θ*.*

Proof : In the case $\tau_1^2 \neq \tau_2$, the module $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta$ is irreducible by the preceding proposition and hence is standard. In general, it suffices to find $m \in K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta$ with $Sm = -m$ and to verify that $\{m, Um\}$ are linearly independent. For example, $m = e^{\eta_2}$ is a possible choice, cf. below. $\square$

A "standard basis" for the module $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_\theta$ comes from the so-called *Pittie-Steinberg basis* [St75] of $\overline{\mathbb{F}}_q[e^{\pm\eta_1}, e^{\pm\eta_2}]$ over $\overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}]$. It is given by

$$\begin{aligned} z_e &= 1 \\ z_s &= e^{\eta_2}. \end{aligned}$$

It induces a basis of $\overline{\mathbb{F}}_q[e^{\pm\eta_1}, e^{\pm\eta_2}] \otimes_{\overline{\mathbb{F}}_q[\xi_1,\xi_2^{\pm1}],\theta} \overline{\mathbb{F}}_q$ over $\overline{\mathbb{F}}_q$ for any character θ of $\overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}]$. Let $\tau_2 = \theta(\xi_2)$. The matrices of S, U and $S_0 = USU^{-1}$ in the latter basis are

$$S = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & -\tau_2 \\ -1 & 0 \end{pmatrix}, \qquad S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The two characters of $\mathcal{H}_{0,\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[S]$ corresponding to $S \mapsto 0$ and $S \mapsto -1$ are realized by z_e and z_s. From the matrix of S_0, we see in fact that the whole affine algebra $\mathcal{H}_{\mathrm{aff},\overline{\mathbb{F}}_q} := \overline{\mathbb{F}}_q[S_0, S]$ acts on z_e and z_s via the two supersingular characters of $\mathcal{H}_{\mathrm{aff},\overline{\mathbb{F}}_q}$, cf. 2.3.2.

3.5.5. We extend this discussion of the component $\gamma = 1$ to any other non-regular component as follows. Consider the quotient map

$$\mathbb{T}^\vee \longrightarrow \mathbb{T}^\vee / W_0.$$

For any $\gamma \in \mathbb{T}^\vee/W_0$ define the $\overline{\mathbb{F}}_q$-variety

$$\widehat{\mathcal{B}}^\gamma := \widehat{\mathcal{B}} \times \pi^{-1}(\gamma).$$

Suppose $|\gamma| = 1$. We have the algebra isomorphism $\mathcal{H}_{\overline{\mathbb{F}}_q} \overset{\simeq}{\to} \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$ from 2.3.4. It identifies the center $Z(\mathcal{H}_{\overline{\mathbb{F}}_q})$ with the center of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$. In this way, we let the component algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$ act on $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\overline{\mathbb{F}}_q}$ and we denote this representation by $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)_{\overline{\mathbb{F}}_q}$. We may then state, in obvious terminology, that any supersingular character θ of the center of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$ gives rise to the supersingular irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$-module $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)_\theta$.

4 The regular case and dual equivariant intersection theory

4.1 Recollections from algebraic $CH^{\widehat{\mathbf{G}}}$-theory

For basic notions from equivariant algebraic intersection theory we refer to [EG96] and [Bri97]. As in the case of equivariant K-theory, the characteristic homomorphism will make everything explicit.

4.1.1. We denote by $\mathrm{Sym}(\Lambda)$ the symmetric algebra of the lattice Λ endowed with its natural action of the reflection s. The equivariant intersection theory of the base point pt = Spec $\overline{\mathbb{F}}_q$ canonically identifies with the ring of invariants

$$\mathrm{Sym}(\Lambda)^s \simeq CH^{\widehat{\mathbf{G}}}(\mathrm{pt}),$$

cf. [EG96, sec. 3.2]. Recall our basis elements $\eta_1 := (1, 0)$ and $\eta_2 := (0, 1)$ of Λ, so that $\mathrm{Sym}(\Lambda) = \mathbb{Z}[\eta_1, \eta_2]$. We define the invariant elements

$$\xi_1' := \eta_1 + \eta_2 \quad \text{and} \quad \xi_2' := \eta_1\eta_2$$

in $\mathrm{Sym}(\Lambda)^s$. Then

$$\mathrm{Sym}(\Lambda)^s = \mathbb{Z}[\xi_1', \xi_2']$$

and, after inverting the prime 2, the $\mathrm{Sym}(\Lambda)^s$-module $\mathrm{Sym}(\Lambda)$ is free of rank 2, on the basis $\{1, \frac{\eta_1-\eta_2}{2}\}$.

4.1.2. The *equivariant Chern class of line bundles* in the algebraic $CH^{\widehat{\mathbf{G}}}$-theory of $\widehat{\mathcal{B}}$ is a map

$$c_1^{\widehat{\mathbf{G}}} : \mathrm{Pic}^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) \longrightarrow CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$$

which is a group homomorphism. Then, the corresponding *characteristic homomorphism* is a ring isomorphism

$$c^{\widehat{\mathbf{G}}} : \mathrm{Sym}(\Lambda) \xrightarrow{\simeq} CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}),$$

which maps $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ to the equivariant Chern class of the line bundle $\mathcal{O}_{\mathbb{P}^1}(\lambda_1 - \lambda_2) \otimes \det^{\lambda_2}$ on $\widehat{\mathcal{B}} = \mathbb{P}^1_{\overline{\mathbb{F}}_q}$, i.e.

$$c^{\widehat{\mathbf{G}}}(\lambda) = c_1^{\widehat{\mathbf{G}}}(\mathcal{O}_{\mathbb{P}^1}(\lambda_1 - \lambda_2) \otimes \det{}^{\lambda_2}).$$

Note here that the algebraic group $\widehat{\mathbf{G}} = \mathrm{GL}_{2/\overline{\mathbb{F}}_q}$ is *special* (in the sense of [EG96, 6.3]) and the map $c^{\widehat{\mathbf{G}}}$ is therefore already bijective at the integral level [Bri97, sec. 6.6]. The homomorphism $c^{\widehat{\mathbf{G}}}$ is $\mathrm{Sym}(\Lambda)^s \simeq CH^{\widehat{\mathbf{G}}}(\mathrm{pt})$-linear.

To emphasize the duality and the analogy with the case of K-theory (and to ease notation), we abbreviate from now on

$$S(\widehat{\mathbf{T}}) := \mathrm{Sym}(\Lambda) \quad \text{and} \quad S(\widehat{\mathbf{G}}) := \mathrm{Sym}(\Lambda)^s.$$

4.1.3. For the definition of the classical Demazure operators on algebraic intersection theory, we refer to [D73]. The Demazure operators

$$D_s, D_s' \in \mathrm{End}_{S(\widehat{\mathbf{T}})^s}(S(\widehat{\mathbf{T}}))$$

are defined by:

$$D_s(a) = \frac{a - s(a)}{\eta_1 - \eta_2} \quad \text{and} \quad D_s'(a) = \frac{a - s(a)(1 - (\eta_1 - \eta_2))}{\eta_1 - \eta_2}$$

for $a \in S(\widehat{\mathbf{T}})$. Then D_s is the projector on $S(\widehat{\mathbf{T}})^s \frac{\eta_1-\eta_2}{2}$ along $S(\widehat{\mathbf{T}})^s$, and $(-D_s) + D_s' = s$. In particular, $D_s^2 = 0$ and $D_s'^2 = \mathrm{id}$. One sets

$$D_s(\mathbf{q}) := D_s - \mathbf{q}D_s' \in \mathrm{End}_{S(\widehat{\mathbf{T}})^s[\mathbf{q}]}(S(\widehat{\mathbf{T}})[\mathbf{q}])$$

and checks by direct calculation that $D_s(\mathbf{q})^2 = \mathbf{q}^2$. We obtain thus a well-defined $\mathbb{Z}$-algebra homomorphism

$$\mathscr{A}_0^{\mathrm{nil}}(\mathbf{q}) : \mathcal{H}_0^{\mathrm{nil}}(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S] \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{T}})^s[\mathbf{q}]}(S(\widehat{\mathbf{T}})[\mathbf{q}]), \quad \mathbf{q} \longmapsto \mathbf{q}^2, \quad S \longmapsto -D_s(\mathbf{q})$$

which we call the *Demazure representation.*

4.2 The morphism from $S(\widehat{\mathbf{G}})[\mathbf{q}]$ to the center of $\mathcal{H}^{\mathrm{nil}}(\mathbf{q})$

The version of the homomorphism $(\xi^+)^{-1}$ in the regular case is the $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\begin{array}{rcl} S(\widehat{\mathbf{G}})[\mathbf{q}] = \mathbb{Z}[\mathbf{q}][\xi_1', \xi_2'] & \longrightarrow & \mathbb{Z}[\mathbf{q}]\big[e^{(1,0)}, (e^{(1,1)})^{\pm 1}\big] \\ \xi_1' & \longmapsto & e^{(1,0)} \\ \xi_2' & \longmapsto & e^{(1,1)} \end{array}$$

which becomes an isomorphism after inverting ξ_2'. Its composition with $\mathscr{Z}^{\mathrm{nil}}(\mathbf{q})$, cf. 2.4.2, therefore gives a ring isomorphism

$$\begin{array}{rcl} S(\widehat{\mathbf{G}})[\mathbf{q}][\xi_2'^{-1}] & \xrightarrow{\simeq} & Z(\mathcal{H}^{\mathrm{nil}}(\mathbf{q})) \\ \xi_1' & \longmapsto & \zeta_1 = US + SU \\ \xi_2' & \longmapsto & \zeta_2 = U^2. \end{array}$$

4.3 The extended Demazure representation $\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(\mathbf{q})$ at $\mathbf{q}=0$

Recall the Demazure representation $\mathscr{A}^{\rm nil}_0(\mathbf{q})$ of the finite algebra $\mathcal{H}^{\rm nil}_0(\mathbf{q})$ by $S(\widehat{\mathbf{G}})[\mathbf{q}]$-linear operators on the intersection theory $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})$, cf. 4.1.3. In this section we work at $\mathbf{q}=0$. We write $\mathscr{A}^{\rm nil}_0$ for the specialization of $\mathscr{A}^{\rm nil}_0(\mathbf{q})$ at $\mathbf{q}=0$.

For better readibility we make a slight *abuse of notation* and denote the elements ξ'_i by ξ_i in this and the following sections. Moreover, p will always be an *odd* prime.

4.3.1. A ring homomorphism

$$\mathscr{A}^{\rm nil}:\mathcal{H}^{\rm nil}\longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}))$$

which extends $\mathscr{A}^{\rm nil}_0$ and which is linear with respect to the above ring homomorphism $S(\widehat{\mathbf{G}})\to Z(\mathcal{H}^{\rm nil})$ does *not* exist, even after inverting ξ_2. However, there exists a natural good approximation (after inverting the prime 2). We will explain these points in the following.

4.3.2. An extension of $\mathscr{A}^{\rm nil}_0$, linear with respect to $S(\widehat{\mathbf{G}})\to Z(\mathcal{H}^{\rm nil})$, exists if and only if there is an operator

$$\mathscr{A}^{\rm nil}(U)\in \mathrm{End}_{S(\widehat{\mathbf{G}})[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\xi_2^{-1}])$$

satisfying

1. $\mathscr{A}^{\rm nil}(U)$ is invertible ;
2. $\mathscr{A}^{\rm nil}(U)^2=\mathscr{A}^{\rm nil}(U^2)=\xi_2\,\mathrm{Id}$, i.e. $\mathscr{A}^{\rm nil}(U)^2=\xi_2\,\mathrm{Id}$;
3. $\mathscr{A}^{\rm nil}(U)\mathscr{A}^{\rm nil}_0(S)+\mathscr{A}^{\rm nil}_0(S)\mathscr{A}^{\rm nil}(U)=\mathscr{A}^{\rm nil}(US+SU)=\xi_1\,\mathrm{Id}$.

Tensoring by $\mathbb{F}_p$, we may write

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{F}_p}=S(\widehat{\mathbf{G}})_{\mathbb{F}_p}\oplus S(\widehat{\mathbf{G}})_{\mathbb{F}_p}\frac{\eta_1-\eta_2}{2},$$

and identify $\mathrm{End}_{S(\widehat{\mathbf{G}})_{\mathbb{F}_p}}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{F}_p})$ with the algebra of 2×2-matrices over the ring $S(\widehat{\mathbf{G}})_{\mathbb{F}_p}$. The analogous statements hold after inverting ξ_2.

Then, by definition,

$$\mathscr{A}^{\rm nil}_{0,\mathbb{F}_p}(S)=-D_s=\begin{pmatrix}0&-1\\0&0\end{pmatrix}.$$

Hence, if we set

$$\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(U)=\begin{pmatrix}a&c\\b&d\end{pmatrix},$$

we obtain

$$\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(U)^2=\xi_2\,\mathrm{Id}\Longleftrightarrow\begin{pmatrix}a^2+bc&c(a+d)\\b(a+d)&d^2+bc\end{pmatrix}=\begin{pmatrix}\xi_2&0\\0&\xi_2\end{pmatrix}$$

and

$$\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(U)(-D_s)+(-D_s)\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(U)=\xi_1\,\mathrm{Id}$$
$$\Longleftrightarrow\left\{\begin{array}{lcl}a&=&-d\\b&=&-\xi_1,\end{array}\right.$$

and then the first system becomes equivalent to the equation

$$a^2-\xi_1c=\xi_2\in\mathbb{F}_p[\xi_1,\xi_2^{\pm1}].$$

However, since ξ_2 has no square root in the ring $\mathbb{F}_p[\xi_2^{\pm1}]$, this latter equation has no solution (take $\xi_1=0$!). Consequently, there does not exist any matrix $\mathscr{A}^{\rm nil}_{\mathbb{F}_p}(U)$ with coefficients in $S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]$ satisfying conditions 1, 2, 3, above.

As a best approximation, we keep condition 1 and also condition 3 (up to sign), but, because of the square root obstruction above, we modify condition 2 to $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(\zeta_2) = \xi_2^2$. We can then state our second main result.

4.3.3. Theorem. *There is a distinguished ring homomorphism*

$$\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p} : \mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p} \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{F}_p}[\xi_2^{-1}])$$

which extends the ring homomorphism $\mathscr{A}^{\mathrm{nil}}_0$ and coincides on $Z(\mathcal{H}^{\mathrm{nil}})_{\mathbb{F}_p}$ with the homomorphism

$$\begin{array}{ccl} Z(\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p}) & \xrightarrow{\simeq} & \mathbb{F}_p[\xi_1, \xi_2^{\pm 2}] \subset S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}] \\ \zeta_1 & \longmapsto & -\xi_1 \\ \zeta_2 & \longmapsto & \xi_2^2. \end{array}$$

The homomorphism $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ is injective.

Proof. The discussion preceding the theorem shows that the matrix

$$\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U) := \begin{pmatrix} (\frac{\xi_1^2}{2} - \xi_2) & -\xi_1(\frac{\xi_1^2}{4} - \xi_2) \\ \xi_1 & -(\frac{\xi_1^2}{2} - \xi_2) \end{pmatrix}$$

does satisfy the three conditions

1. $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U)$ is invertible ;
2. $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U)^2 = (\xi_2)^2 \,\mathrm{Id}$;
3. $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(US + SU) = -\xi_1 \,\mathrm{Id}$.

The injectivity part of the theorem will be shown in the next subsection. ☐

4.3.4. Remark. The minus sign before ξ_1 appearing in the value of $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ on $\zeta_1 = US + SU$ could be avoided by setting $\mathscr{A}^{\mathrm{nil}}_0(S) := D_s$ instead of $-D_s$ in the Demazure representation. But we will not do this.

4.3.5. Remark. In the Iwahori case, one can check that the action of U coincides with the action of the Weyl element $e^{\eta_1}s$. In the regular case, the action of the element $\eta_1 s$ does not satisfy the conditions 1-3 appearing in the above proof. However, the action of $\eta_1^2 s$ does and, in fact, its matrix is given by matrix $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U)$. So the choice of the matrix $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U)$ is in close analogy with the Iwahori case. Our chosen extension $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ of $\mathscr{A}^{\mathrm{nil}}_{0,\mathbb{F}_p}$ seems to be distinguished for at least this reason. This observation also shows that the action of U can actually be defined integrally, i.e. before inverting the prime 2.

4.4 Faithfulness of $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$

Let us show that the map $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ is injective. It follows from 2.4.2 that the ring $\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p}$ is generated by the elements

$$1,\ S,\ U,\ SU$$

over its center $Z(\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}}) = \mathbb{F}_p[\zeta_1, \zeta_2^{\pm 1}]$. The latter is mapped isomorphically to the subring

$$\mathbb{F}_p[\xi_1, \xi_2^{\pm 2}] \subset S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]$$

of the matrix algebra $\mathrm{End}_{S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{F}_p})$ by $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$. For injectivity, it therefore suffices to show that the images

$$1,\ \mathscr{A}^{\mathrm{nil}}_{0,\mathbb{F}_p}(S),\ \mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U),\ \mathscr{A}^{\mathrm{nil}}_{0,\mathbb{F}_p}(S)\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(U)$$

of $1, S, U, SU$ under $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ are free over $S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]$. To this end, let $\alpha, \beta, \gamma, \delta \in S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]$ (which is an integral domain) be such that

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = 0,$$

i.e.

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & -\beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \gamma a & \gamma c \\ \gamma b & -\gamma a \end{pmatrix} + \begin{pmatrix} -\delta b & \delta a \\ 0 & 0 \end{pmatrix} = 0.$$

Then

$$\begin{cases} \alpha + \gamma a - \delta b & = 0 \\ \gamma b & = 0 \\ -\beta + \gamma c + \delta a & = 0 \\ \alpha - \gamma a & = 0, \end{cases}$$

with $\alpha, \beta, \gamma, \delta \in S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]$. Now recall our choice

$$\mathscr{A}^{\mathrm{nil}}_k(U) = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} := \begin{pmatrix} (\frac{\xi_1^2}{2} - \xi_2) & -\xi_1(\frac{\xi_1^2}{4} - \xi_2) \\ \xi_1 & -(\frac{\xi_1^2}{2} - \xi_2) \end{pmatrix}.$$

In particular, $b = \xi_1$ implies $\gamma = 0$, and then $\alpha = 0$, $\delta = 0$ and $\beta = 0$. This shows that the map $\mathscr{A}^{\mathrm{nil}}$ is injective and concludes the proof. We record the following corollary of the proof.

4.4.1. Corollary. *The ring $\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p}$ is a free $Z(\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p})$-module on the basis $1, S, U, SU$.*

4.5 The twisted representation $\mathscr{A}_{2,\mathbb{F}_p}$

4.5.1. In the algebra

$$\mathcal{H}_2 := \mathcal{H}_2(0) = (\mathbb{Z} \times \mathbb{Z}) \otimes'_{\mathbb{Z}} \mathcal{H}^{\mathrm{nil}}$$

we have the two subrings $\mathcal{H}^{\mathrm{nil}}$ and $\mathbb{Z} \times \mathbb{Z}$. The aim of this section is to extend the representation $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ from $\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p}$ to the whole algebra $\mathcal{H}_{2,\mathbb{F}_p} := \mathcal{H}_2 \otimes_{\mathbb{Z}} \mathbb{F}_p$. To this end, we consider the $\overline{\mathbb{F}}_q$-variety

$$\widehat{\mathcal{B}}^2 := \widehat{\mathcal{B}}_1 \coprod \widehat{\mathcal{B}}_2,$$

where $\widehat{\mathcal{B}}_1$ and $\widehat{\mathcal{B}}_2$ are two copies of $\widehat{\mathcal{B}}$. We have

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2) = CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_1) \times CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_2).$$

After base change to $\mathbb{F}_p$, the ring $\mathcal{H}^{\mathrm{nil}}$ acts $S(\widehat{\mathbf{G}})[\xi_2^{-1}]$-linearly on $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\xi_2^{-1}]$ through the map $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$. We extend this action diagonally to $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]$, thus defining a ring homomorphism

$$\mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}) : \mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p} \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\mathbb{F}_p}[\xi_2^{-1}]).$$

Because of the *twisted* multiplication in the algebra $\mathcal{H}_2$, we need to introduce the permutation action of W

$$\mathrm{perm} : W \twoheadrightarrow W_0 \longrightarrow \mathrm{Aut}_{S(\widehat{\mathbf{G}})}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2))$$

which permutes the two factors of $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)$.

On the other hand, we can consider the projection p_i from $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)$ to $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_i)$ as an $S(\widehat{\mathbf{G}})$-linear endomorphism of $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)$, for $i = 1, 2$. The rule $\varepsilon_i \mapsto p_i$ defines a ring homomorphism

$$\mathrm{proj} : \mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2 \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)).$$

4.5.2. Proposition. *There exists a unique ring homomorphism*

$$\mathscr{A}_{2,\mathbb{F}_p} : \mathcal{H}_{2,\mathbb{F}_p} \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})_{\mathbb{F}_p}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\mathbb{F}_p}[\xi_2^{-1}])$$

such that

- $\mathscr{A}_{2,\mathbb{F}_p}|_{\mathcal{H}^{\mathrm{nil}}_{\mathbb{F}_p}}(T_w) = \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_w) \circ \mathrm{perm}(w)$ *for all* $w \in W$,
- $\mathscr{A}_{2,\mathbb{F}_p}|_{\mathbb{F}_p\varepsilon_1 \times \mathbb{F}_p\varepsilon_2} = \mathrm{proj}\,.$

The homomorphism $\mathscr{A}_{2,\mathbb{F}_p}$ *is injective.*

Proof. Recall that W_0 acts on the set $\{1,2\}$ by interchanging the two elements and then W acts via its projection to W_0. As $\{\varepsilon_i T_w, (i,w) \in \{1,2\} \times W\}$ is a $\mathbb{F}_p$-basis of $\mathcal{H}_{2,\mathbb{F}_p}$, such a ring homomorphism is uniquely determined by the formula

$$\mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w) = p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_w) \circ \mathrm{perm}(w).$$

Conversely, taking this formula as a definition of $\mathscr{A}_{2,\mathbb{F}_p}$, we need to check that the resulting $\mathbb{F}_p$-linear map is a ring homomorphism, i.e.

$$\mathscr{A}_{2,\mathbb{F}_p}((1,1)) = \mathrm{Id}$$

and

$$\mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w \cdot \varepsilon_{i'} T_{w'}) = \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w) \circ \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_{i'} T_{w'}).$$

The first condition is clear because $(1,1) = \varepsilon_1 + \varepsilon_2$ and $p_i + p_{{}^s i} = \mathrm{Id}$. Let us check the second condition. If $i' \neq {}^{w^{-1}}i$, i.e. $i \neq {}^w i'$, then both sides of the claimed equality vanish. Now assume that $i = {}^w i'$. On the left hand side we find

$$\mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w \cdot \varepsilon_{i'} T_{w'}) = \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w T_{w'}),$$

while on the right hand side, we find

$$\begin{aligned} & \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w) \circ \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_{{}^{w^{-1}}i} T_{w'}) \\ = \quad & p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_w) \circ \mathrm{perm}(w) \circ p_{{}^{w^{-1}}i} \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_{w'}) \circ \mathrm{perm}(w') \\ = \quad & p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_w) \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_{w'}) \circ p_{{}^{(w')^{-1}}({}^{w^{-1}}i)} \\ = \quad & p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_w T_{w'}) \circ p_{{}^{(ww')^{-1}}i}. \end{aligned}$$

If $\ell(ww') \neq \ell(w) + \ell(w')$, then $T_w T'_w = 0$ and both sides vanish. Otherwise $T_w T'_w = T_{ww'}$, so that the left hand side becomes

$$\mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_{ww'}) = p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_{ww'}) \circ \mathrm{perm}(ww'),$$

and the right hand side

$$p_i \circ \mathrm{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p})(T_{ww'}) \circ p_{{}^{(ww')^{-1}}i}.$$

These two operators are equal. This proves the existence and the uniqueness of the extension $\mathscr{A}_{2,\mathbb{F}_p}$. Its injectivity will be shown in the next subsection. □

4.6 Faithfulness of $\mathscr{A}_{2,\mathbb{F}_p}$

Let us show that the map $\mathscr{A}_{2,\mathbb{F}_p}$ is injective. This is equivalent to show that the family

$$\{\mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w), (i,w) \in \{1,2\} \times W\}$$

is free over $\mathbb{F}_p$. So let $\{n_{i,w}\} \in \mathbb{F}_p^{(\{1,2\} \times W)}$ such that

$$\sum_{i,w} n_{i,w} \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_i T_w) = 0.$$

Let us fix $i_0 \in \{1,2\}$. Composing by p_{i_0} on the left, we get

$$\sum_w n_{i_0,w} \mathscr{A}_{2,\mathbb{F}_p}(\varepsilon_{i_0} T_w) = 0.$$

The left hand side can be rewritten as

$$\sum_w n_{i_0,w} p_{i_0} \circ \operatorname{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w)) \circ \operatorname{perm}(w) = \sum_w n_{i_0,w} p_{i_0} \circ \operatorname{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w)) \circ p_{w^{-1} i_0}.$$

Now let us fix $w_0 \in W_0$. Composing by $p_{w_0^{-1} i_0}$ on the right, we get

$$\sum_{w \in \Lambda w_0} n_{i_0,w} p_{i_0} \circ \operatorname{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w)) \circ p_{w_0^{-1} i_0} = 0.$$

Then, for each $w \in \Lambda w_0$, remark that

$$p_{i_0} \circ \operatorname{diag}(\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w)) \circ p_{w_0^{-1} i_0} = \iota_{i_0, w_0^{-1} i_0} \circ \mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w) \circ p_{w_0^{-1} i_0}$$

in $\operatorname{End}\big(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_1)[\xi_2^{-1}] \times CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_2)[\xi_2^{-1}]\big)$, where $\iota_{i_0, w_0^{-1} i_0}$ is the canonical map

$$\iota_{i_0, w_0^{-1} i_0} : CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_{w_0^{-1} i_0})[\xi_2^{-1}] = CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_{i_0})[\xi_2^{-1}] \hookrightarrow CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_1)[\xi_2^{-1}] \times CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_2)[\xi_2^{-1}].$$

As the latter is injective, we get

$$0 = \sum_{w \in \Lambda w_0} n_{i_0,w} \mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}(T_w) \circ p_{w_0^{-1} i_0} = \mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}\Big(\sum_{w \in \Lambda w_0} n_{i_0,w} T_w\Big) \circ p_{w_0^{-1} i_0}.$$

Finally, as $p_{w_0^{-1} i_0} : CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}] \to CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_{w_0^{-1} i_0})[\xi_2^{-1}]$ is surjective, and as $\mathscr{A}^{\mathrm{nil}}_{\mathbb{F}_p}$ is injective, cf. 4.4, we get $n_{i_0,w} = 0$ for all $w \in \Lambda w_0$. This concludes the proof that $\mathscr{A}_{2,\mathbb{F}_p}$ is injective.

4.7 Supersingular modules

In this section we work over the algebraic closure $\overline{\mathbb{F}}_q$ of the field $\mathbb{F}_q$.

4.7.1. Recall from 2.4.4 that

$$\mathcal{H}_{2,\overline{\mathbb{F}}_q} = \mathcal{H}_{2,\mathbb{F}_p} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_q = (\overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q) \otimes'_{\overline{\mathbb{F}}_q} \overline{\mathbb{F}}_q[S, U^{\pm 1}]$$

has the structure of a 2×2-matrix algebra over its center $Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q})$. Since $\overline{\mathbb{F}}_q$ is algebraically closed, $Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q})$ acts on any finite-dimensional irreducible $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module by a character (Schur's lemma). Let θ be a character of $Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q})$. Then

$$\mathcal{H}_{2,\theta} := \mathcal{H}_{2,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q}),\theta} \overline{\mathbb{F}}_q$$

is isomorphic to the matrix algebra $M(2, \overline{\mathbb{F}}_q)$. In particular, it is a semisimple (even simple) ring.

4.7.2. The unique irreducible $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module with central character θ is called the *standard module* with character θ. Its $\overline{\mathbb{F}}_q$-dimension is 2 and it is isomorphic to the standard representation $\overline{\mathbb{F}}_q^{\oplus 2}$ of the matrix algebra $M(2, \overline{\mathbb{F}}_q)$. The image of the basis $\{(1,0),(0,1)\}$ of $\overline{\mathbb{F}}_q^{\oplus 2}$ is called a *standard basis.* A central character θ is called *supersingular* if $\theta(X) = \theta(Y) = 0$ (or, equivalently, if $\theta(\zeta_1) = 0$). If θ is supersingular, then the affine algebra $\mathcal{H}_{2,\mathrm{aff},\overline{\mathbb{F}}_q}$ acts on the standard basis of the module via the characters χ_1 respectively χ_2 and the action of U interchanges the two, cf. 2.4.3 and 2.4.4.

For more details we refer to [V04, 2.3].

4.7.3. Now consider the base change of the representation $\mathscr{A}_{2,\mathbb{F}_p}$ to $\overline{\mathbb{F}}_q$

$$\mathscr{A}_{2,\overline{\mathbb{F}}_q} : \mathcal{H}_{2,\overline{\mathbb{F}}_q} \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]) = \mathrm{End}_{\overline{\mathbb{F}}_q[\xi_1,\xi_2^{\pm 1}]}(\overline{\mathbb{F}}_q[\eta_1^{\pm 1}, \eta_2^{\pm 1}]^{\oplus 2}).$$

Recall that the image under the map $\mathscr{A}_{2,\overline{\mathbb{F}}_q}$ of the central subring

$$Z^{\circ}(\mathcal{H}_{2,\overline{\mathbb{F}}_q}) = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2^{\pm 1}] \subset Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q})$$

is the subring of scalars

$$\overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 2}] \subset \overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}] = S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}].$$

4.7.4. Let us fix a supersingular central character θ and denote its restriction to $Z^{\circ} := Z^{\circ}(\mathcal{H}_{2,\overline{\mathbb{F}}_q})$ by θ, too. Then consider the $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-action on the base change

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta} := CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}[\xi_2^{-1}] \otimes_{Z^{\circ}} \overline{\mathbb{F}}_q = CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}[\xi_2^{-1}] \otimes_{S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]} \Big(S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}] \otimes_{Z^{\circ}} \overline{\mathbb{F}}_q \Big).$$

For the base ring, we have

$$S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}] \otimes_{Z^{\circ},\theta} \overline{\mathbb{F}}_q = \overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}] \otimes_{\mathscr{A}_{2,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_q[\zeta_1,\zeta_2^{\pm 1}],\theta} \overline{\mathbb{F}}_q$$

where $\mathscr{A}_{2,\overline{\mathbb{F}}_q}(\zeta_1) = -\xi_1$ and $\mathscr{A}_{2,\overline{\mathbb{F}}_q}(\zeta_2) = \xi_2^2$. Now put $\theta(\zeta_2) =: b \in \overline{\mathbb{F}}_q^{\times}$. Then

$$S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}] \otimes_{Z^{\circ},\theta} \overline{\mathbb{F}}_q = \overline{\mathbb{F}}_q[\xi_1, \xi_2^{\pm 1}]/(\xi_1, \xi_2^2 - b) = \overline{\mathbb{F}}_q[\xi_2]/(\xi_2^2 - b) =: A$$

and so

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta} = \overline{\mathbb{F}}_q[\eta_1^{\pm 1}, \eta_2^{\pm 1}]^{\oplus 2} \otimes_{\overline{\mathbb{F}}_q[\xi_1,\xi_2^{\pm 1}]} \overline{\mathbb{F}}_q[\xi_2]/(\xi_2^2 - b) = \overline{\mathbb{F}}_q[\eta_1^{\pm 1}, \eta_2^{\pm 1}]^{\oplus 2} \otimes_{\overline{\mathbb{F}}_q[\xi_1,\xi_2^{\pm 1}]} A.$$

Note that the $\overline{\mathbb{F}}_q$-algebra A is isomorphic to the direct product $\overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q$ (the isomorphism depending on the choice of a square root of b in $\overline{\mathbb{F}}_q$). An A-basis of $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta}$ is given by the four elements $\{1_i, \frac{\eta_1 - \eta_2}{2} 1_i\}_{i=1,2}$ where

$$1_i \in CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_i) \subset CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_1) \times CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_2) = CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)$$

is the equivariant Chern class of the structure sheaf on $\widehat{\mathcal{B}}_i$, for $i = 1, 2$. The $\overline{\mathbb{F}}_q$-dimension of $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta}$ is therefore 8 and $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$ acts A-linearly. The length of the $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta}$ is 4 and the central character of any irreducible subquotient is necessarily equal to θ, since this is true by construction after restriction to Z°. In the following, we compute explicitly a composition series.

4.7.5. Proposition. *The algebra $\mathcal{H}_{2,\mathrm{aff},\overline{\mathbb{F}}_q}$ acts on $1_i \in CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_{\theta}$ by the supersingular character χ_i, for $i = 1, 2$.*

Proof : The action of $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$ on $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]$ is defined by the map $\mathscr{A}_{2,\overline{\mathbb{F}}_q}$. Hence, by definition,

$$\varepsilon_{i'} \cdot 1_i = \begin{cases} 1_i & \text{if } i' = i \\ 0 & \text{otherwise.} \end{cases}$$

We calculate

$$S \cdot 1_i = \mathrm{diag}(-D_s) \circ \mathrm{perm}(s)(1_i) = \mathrm{diag}(-D_s) 1_{^s i} = 0.$$

Moreover,

$$U^{-1} \cdot 1_i = \mathrm{diag}(U^{-1}) \circ \mathrm{perm}(u^{-1})(1_i) = \mathrm{diag}(U^{-1}) 1_{^s i} = s(\eta_1^{-2}) 1_{^s i} = \eta_2^{-2} 1_{^s i}$$

and

$$D_s(\eta_2^{-2}) = \frac{\eta_2^{-2} - \eta_1^{-2}}{\eta_1 - \eta_2} = (\eta_1\eta_2)^{-2} \frac{\eta_1^2 - \eta_2^2}{\eta_1 - \eta_2} = (\eta_1\eta_2)^{-2}(\eta_1 + \eta_2) = \frac{\xi_1}{\xi_2^2}.$$

Therefore,

$$SU^{-1}\cdot 1_i = \mathrm{diag}(-D_s)\circ \mathrm{perm}(s)(\eta_2^{-2}1_{s_i}) = -D_s(\eta_2^{-2})1_i = -\frac{\xi_1}{\xi_2^2}1_i = 0$$

since $\xi_1 = 0$ in $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}_i)[\xi_2^{-1}]_\theta$. It follows that $S_0\cdot 1_i = USU^{-1}\cdot 1_i = 0$. □

4.7.6. Proposition. *A composition series with simple subquotients of the $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module*

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_\theta$$

is given by

$$\begin{gathered}\{0\}\\ \subset \overline{\mathbb{F}}_q 1_i \oplus \overline{\mathbb{F}}_q(U\cdot 1_i)\\ \subset A1_i \oplus A(U\cdot 1_i) = A1_i\oplus A1_{s_i}\\ \subset A1_i\oplus A1_{s_i}\oplus \overline{\mathbb{F}}_q(\frac{\eta_1-\eta_2}{2}1_i)\oplus \overline{\mathbb{F}}_q(U\cdot\frac{\eta_1-\eta_2}{2}1_i)\\ \subset CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_\theta.\end{gathered}$$

Here the direct sums $\oplus$ are taken in the sense of $\overline{\mathbb{F}}_q$-vector spaces.

Proof. First of all,

$$U\cdot 1_i := \mathrm{diag}(U)\circ\mathrm{perm}(u)(1_i) = \mathrm{diag}(U)1_{s_i} = \eta_1^2 1_{s_i} = -\xi_2 1_{s_i}\in A^\times 1_{s_i}$$

because $0=\xi_1=\eta_1+\eta_2$ and $0=\xi_1^2=\eta_1^2+\eta_2^2+2\xi_2$ in $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\xi_2^{-1}]_\theta$. Hence the three first $\oplus$ appearing in the statement of the proposition are indeed *direct* sums. These three sums are U-stable by construction. Moreover, by the preceding proposition, $\mathcal{H}_{2,\mathrm{aff},\overline{\mathbb{F}}_q}$ acts by the character χ_i on 1_i, hence by the character χ_{s_i} on $U\cdot 1_i$. It follows that $\overline{\mathbb{F}}_q 1_i\oplus\overline{\mathbb{F}}_q(U\cdot 1_i)$ realizes the standard $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module with central character θ, and that $A1_i\oplus A(U\cdot 1_i)$ is an $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-submodule of $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_\theta$ of dimension 4 over $\overline{\mathbb{F}}_q$. In fact, if $L\subset A$ is any $\overline{\mathbb{F}}_q$-line, the same arguments show that $L1_i\oplus L(U\cdot 1_i)$ realizes the standard $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module with central character θ. In particular, the module $A1_i\oplus A(U\cdot 1_i)$ is semisimple.

Now let us compute the action of $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$ on the element $\frac{\eta_1-\eta_2}{2}1_i$, for $i=1,2$. We have

$$\varepsilon_{i'}\cdot\frac{\eta_1-\eta_2}{2}1_i = \begin{cases}\frac{\eta_1-\eta_2}{2}1_i & \text{if } i'=i\\ 0 & \text{otherwise.}\end{cases}$$

Next

$$S\cdot\frac{\eta_1-\eta_2}{2}1_i := \mathrm{diag}(S)\circ\mathrm{perm}(s)(\frac{\eta_1-\eta_2}{2}1_i) = \mathrm{diag}(S)(\frac{\eta_1-\eta_2}{2}1_{s_i}) = -1_{s_i},$$

$$U^{-1}\cdot\frac{\eta_1-\eta_2}{2}1_i := \mathrm{diag}(U^{-1})\circ\mathrm{perm}(u^{-1})(\frac{\eta_1-\eta_2}{2}1_i) = \mathrm{diag}(U^{-1})(\frac{\eta_1-\eta_2}{2}1_{s_i}) = \eta_2^{-2}\frac{\eta_2-\eta_1}{2}1_{s_i},$$

$$D_s(\eta_2^{-2}\frac{\eta_2-\eta_1}{2}) = \frac{1}{\eta_1-\eta_2}(\eta_2^{-2}\frac{\eta_2-\eta_1}{2}-\eta_1^{-2}\frac{\eta_1-\eta_2}{2}) = -\frac{\xi_1^2-2\xi_2}{2\xi_2^2},$$

$$SU^{-1}\cdot\frac{\eta_1-\eta_2}{2}1_i = \mathrm{diag}(S)\circ\mathrm{perm}(s)(\eta_2^{-2}\frac{\eta_2-\eta_1}{2}1_{s_i}) = \frac{\xi_1^2-2\xi_2}{2\xi_2^2}1_i,$$

$$S_0\cdot\frac{\eta_1-\eta_2}{2}1_i := USU^{-1}\cdot\frac{\eta_1-\eta_2}{2}1_i = \mathrm{diag}(U)\circ\mathrm{perm}(u)(\frac{\xi_1^2-2\xi_2}{2\xi_2^2}1_i) = \eta_1^2\frac{\xi_1^2-2\xi_2}{2\xi_2^2}1_{s_i} = 1_{s_i}$$

because $\xi_1=0$ and (hence) $\eta_1^2=-\xi_2$ in $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})[\xi_2^{-1}]_\theta$, and finally

$$U\cdot\frac{\eta_1-\eta_2}{2}1_i = \mathrm{diag}(U)\circ\mathrm{perm}(u)(\frac{\eta_1-\eta_2}{2}1_i) = \mathrm{diag}(U)(\frac{\eta_1-\eta_2}{2}1_{s_i}) = \xi_2\frac{\eta_1-\eta_2}{2}1_{s_i}$$

which lies in $A^{\times}(\frac{\eta_1-\eta_2}{2}1_{{}^si})$. Neither of the two elements $\frac{\eta_1-\eta_2}{2}1_i$ and $U\cdot\frac{\eta_1-\eta_2}{2}1_i$ lies in the (semisimple) module $A1_i\oplus A(U\cdot 1_i)$. Hence the three last $\oplus$ appearing in the statement are indeed direct and they form a sub-$\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module of dimension 6 over $\overline{\mathbb{F}}_q$. So the series appearing in the statement is indeed a composition series with irreducible subquotients. □

4.7.7. Remark. We see from the proof of the preceding proposition that the characters of $\mathcal{H}_{2,\mathrm{aff},\overline{\mathbb{F}}_q}$ in the sub-$\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module

$$A1_i\oplus A1_{{}^si}\oplus\overline{\mathbb{F}}_q(\frac{\eta_1-\eta_2}{2}1_i)\oplus\overline{\mathbb{F}}_q(U\cdot\frac{\eta_1-\eta_2}{2}1_i)$$

are contained in $A1_i\oplus A1_{{}^si}$. Hence this submodule is *not* semi-simple. *A fortiori* the whole module $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^2)[\xi_2^{-1}]_\theta$ is not semisimple and, hence, has no central character.

4.7.8. Now we transfer this discussion to any regular component of the algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ as follows. Let $\gamma=\{\lambda,{}^s\lambda\}\in\mathbb{T}^\vee/W_0$ be a regular orbit and form the $\overline{\mathbb{F}}_q$-variety

$$\widehat{\mathcal{B}}^\gamma=\widehat{\mathcal{B}}\times\pi^{-1}(\gamma)=\widehat{\mathcal{B}}_\lambda\coprod\widehat{\mathcal{B}}_{{}^s\lambda},$$

where $\widehat{\mathcal{B}}_\lambda$ and $\widehat{\mathcal{B}}_{{}^s\lambda}$ are two copies of $\widehat{\mathcal{B}}$. We have the algebra isomorphism $\mathcal{H}_{2,\overline{\mathbb{F}}_q}\xrightarrow{\simeq}\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$ from 2.4.5. In this way, the representation $\mathscr{A}_{2,\overline{\mathbb{F}}_q}$ induces a representation

$$\mathscr{A}^\gamma_{\overline{\mathbb{F}}_q}:\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma\longrightarrow\mathrm{End}_{S(\widehat{\mathbf{G}})_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)_{\overline{\mathbb{F}}_q}[\xi_2^{-1}]).$$

We may then state, in obvious terminology, that any supersingular character θ of the center of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$ gives rise to the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$-module $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)[\xi_2^{-1}]_\theta$ and that the semisimplification of the latter module equals a direct sum of four copies of the unique supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_\gamma$-module with central character θ.

5 Tame Galois representations and supersingular modules

Our reference for basic results on tame Galois representations is [V94].

5.1. Let $\varpi\in o_F$ be a uniformizer and let f be the degree of the residue field extension $\mathbb{F}_q/\mathbb{F}_p$, i.e. $q=p^f$. Let $\mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F. Let $\mathcal{I}\subset\mathrm{Gal}(\overline{F}/F)$ be its inertia subgroup. We fix an element $\varphi\in\mathrm{Gal}(\overline{F}/F)$ lifting the Frobenius $x\mapsto x^q$ on $\mathrm{Gal}(\overline{F}/F)/\mathcal{I}$. The unique pro-$p$-Sylow subgroup of $\mathcal{I}$ is denoted by $\mathcal{P}$ (the wild inertia subgroup) and the quotient $\mathcal{I}/\mathcal{P}$ is pro-cyclic with pro-order prime to p. We choose a lift $v\in\mathcal{I}$ of a topological generator for $\mathcal{I}/\mathcal{P}$. Let $\mathcal{W}\subset\mathrm{Gal}(\overline{F}/F)$ denote the Weil group of F. The quotient group $\mathcal{W}/\mathcal{P}$ is topologically generated by (the images of) φ and v and the only relation between these two generators is $\varphi v\varphi^{-1}=v^q$. There is a topological isomorphism

$$\mathcal{W}/\mathcal{P}\simeq\varprojlim\mathbb{F}^\times_{p^n}$$

where the projective limit is taken with respect to the norm maps $\mathbb{F}^\times_{p^{nm}}\to\mathbb{F}^\times_{p^n}$. We denote by ω_n the projection map $\mathcal{W}/\mathcal{P}\to\mathbb{F}^\times_{p^n}$ followed by the inclusion $\mathbb{F}^\times_{p^n}\subseteq\overline{\mathbb{F}}_q^\times$. We shall only be concerned with the characters ω_f and ω_{2f}. The character ω_f extends from $\mathcal{W}$ to $\mathrm{Gal}(\overline{F}/F)$ by choosing a root $\sqrt[q-1]{-\varpi}$ and letting $\mathrm{Gal}(\overline{F}/F)$ act as

$$g\mapsto\frac{g\sqrt[q-1]{-\varpi}}{\sqrt[q-1]{-\varpi}}\in\mu_{q-1}(F)$$

followed by reduction mod ϖ. The character

$$\omega_f:\mathrm{Gal}(\overline{F}/F)\longrightarrow\mathbb{F}_q^\times$$

depends on the choice of ϖ (but not on the choice of $\sqrt[q-1]{-\varpi}$) and equals the reduction mod ϖ_F of the Lubin-Tate character $\chi_L : \mathrm{Gal}(\overline{F}/F) \to o_F^\times$ associated to the uniformizer ϖ. By changing φ by an element of $\mathcal{I}$, if necessary, we may assume $\omega_f(\varphi) = 1$. We normalize local class field theory $\mathcal{W}^{\mathrm{ab}} \simeq F^\times$ by sending the geometric Frobenius φ^{-1} to ϖ. We view the restriction of ω_f to $\mathcal{W}$ as a character of $F^\times$.

5.2. The set of isomorphism classes of irreducible smooth Galois representations

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \widehat{\mathbf{G}} = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$$

is in bijection with the set of equivalence classes of pairs $(s,t) \in \widehat{\mathbf{G}}^2$ such that

$$s = \begin{pmatrix} 0 & 1 \\ -b & 0 \end{pmatrix} \qquad \text{and} \qquad t = \begin{pmatrix} y & 0 \\ 0 & y^q \end{pmatrix}$$

with $b \in \overline{\mathbb{F}}_q^\times$ and $y \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Here, two pairs (s,t) and (s',t') are equivalent if $s = s'$ and t, t' are $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-conjugate. Note that $\det(s) = b$ and that $sts^{-1} = t^q$. The bijection is induced by the map $\rho \mapsto (\rho(\varphi), \rho(v))$. The number of equivalence classes of such pairs (s,t) equals $\frac{q^2-q}{2}$ and hence coincides with the number of W_0-orbits in $\mathbb{T}^\vee$.

5.3. By the above numerical coincidence (the "miracle" from [V04]), there exist (many) bijections between the isomorphism classes of irreducible smooth two-dimensional Galois representations and the isomorphism classes of supersingular two-dimensional $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules. In the following we discuss a a certain example of such a bijection in our geometric language.

Let ρ be a two-dimensional irreducible smooth Galois representation with parameters (s,t). Since the element $\omega_{2f}(v)$ generates $\mathbb{F}_{q^2}^\times$, the element t uniquely determines an exponent $1 \le h \le q^2 - 1$, such that

$$\omega_{2f}(v)^h = y.$$

Replacing ρ by an isomorphic representation ρ' which replaces y by its Galois conjugate y^q replaces h by the rest of the euclidian division of qh by $q^2 - 1$. We call either of the two numbers an *exponent* of ρ.

5.4. Lemma. *There is $0 \le i \le q-2$ such that $\rho \otimes \omega_f^{-i}$ has an exponent $\le q-1$.*

Proof : This is implicit in the discussion in [V94]. Let $\omega_{2f}(v)^h = y$. Then $h \le q^2 - 2$ since $y \ne 1$. Moreover, $q^2 - 2 - (q-2)(q+1) = q$. Since $\omega_{2f}^{q+1} = \omega_f$, twisting with ω_f reduces to the case $h \le q$. Replacing y by its Galois conjugate y^q, if necessary, reduces then further to $h \le q-1$. $\square$

By the lemma, we may associate two numbers $1 \le h \le q-1$ and $0 \le i \le q-2$ to the representation ρ. We form the character

$$\omega_f^{h-1+i} \otimes \omega_f^i : (F^\times)^2 \longrightarrow \mathbb{F}_q^\times,\ (t_1, t_2) \mapsto \omega_f^{h-1+i}(t_1)\omega_f^i(t_2)$$

and restrict to $\mu_{q-1}(F)^2$. This gives rise to an element $\lambda(\rho)$ of $\mathbb{T}^\vee$ and we take its W_0-orbit γ_ρ.

5.5. Lemma. *The orbit γ_ρ depends only on the isomorphism class of ρ.*

Proof : Suppose $\rho' \simeq \rho$ with

$$\rho'(v) = t' = \begin{pmatrix} y^q & 0 \\ 0 & y \end{pmatrix}.$$

By the preceding lemma, there is $0 \le i \le q-2$ and an exponent $1 \le h \le q-1$ of $\rho \otimes \omega_f^{-i}$. If $1 < h$, then by definition $\omega_{2f}^h(v) = y\omega_f^{-i}(v)$, so that $\omega_{2f}^{qh}(v) = y^q\omega_f^{-i}(v)$, and hence $\omega_{2f}^{q-(h-1)}(v) = y^q\omega_f^{-(h-1+i)}(v)$, using $qh = q - (h-1) + (h-1)(q+1)$. Then $1 \le h' := q - (h-1) \le q-1$ and taking $0 \le i' \le q-2$ congruent to $h-1+i$ mod $q-1$, we obtain that h' is an exponent for $\rho' \otimes \omega_f^{-i'}$. In particular, $\lambda(\rho') := \omega_f^{h'-1+i'} \otimes \omega_f^{i'}$, which is s-conjugate to $\lambda(\rho)$. If $h = 1$, then by definition $\omega_{2f}(v) = y^q\omega_f^{-i}(v)$, which implies $\lambda(\rho') = \lambda(\rho)$ in this case. $\square$

We call ρ *(non-)regular* if the orbit γ_ρ is (non-)regular. On the other hand, we view the element $s = \rho(\varphi)$ as a supersingular character θ_ρ of the center $Z(\mathcal{H}^{\gamma_\rho}_{\overline{\mathbb{F}}_q})$, i.e. $\theta_\rho(\zeta_1) = 0$ and $\theta_\rho(\zeta_2) = b$. Finally, we have the $\overline{\mathbb{F}}_q$-variety

$$\widehat{\mathcal{B}}^\gamma = \widehat{\mathcal{B}} \times \pi^{-1}(\gamma)$$

coming from the quotient map $\mathbb{T}^\vee \to \mathbb{T}^\vee/W_0$. These data give rise to the supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module

$$\mathcal{M}(\rho) := \begin{cases} K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)_{\theta_\rho} & \text{if } \rho \text{ non-regular} \\ CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)[\xi_2^{-1}]_{\theta_\rho} & \text{if } \rho \text{ regular.} \end{cases}$$

Recall that $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\mathcal{M}(\rho)$ via the projection onto $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\varepsilon_{\gamma_\rho}$ followed by the extended Demazure representation $\mathscr{A}^{\gamma_\rho}_{\overline{\mathbb{F}}_q}$. Recall also that the semisimplification of $\mathcal{M}(\rho)$ is a direct sum of four copies of the supersingular standard module, if ρ is regular. By abuse of notation, we denote a simple subquotient of $\mathcal{M}(\rho)$ again by $\mathcal{M}(\rho)$.

5.6. Proposition. *The map $\rho \mapsto \mathcal{M}(\rho)$ gives a bijection between the isomorphism classes of two-dimensional irreducible smooth $\overline{\mathbb{F}}_q$-representations of* $\mathrm{Gal}(\overline{F}/F)$ *and the isomorphism classes of two-dimensional supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules.*

Proof : By construction, the restriction of ω_f^{h-1} to $\mu_{q-1}(F) \simeq \mathbb{F}_q^\times$ is given by the exponentiation $x \mapsto x^{h-1}$. Given $0 \le i \le q-2$ and $1 \le h \le q-1$, and $b \in \overline{\mathbb{F}}_q^\times$, the parameter $y := \omega_{2f}(v)^h$ lies in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and the pair (s,t) determines a Galois representation ρ having h comme exponent. Hence, $\rho \otimes \omega_f^i$ gives rise to the character $\omega_f^{h-1+i} \otimes \omega_f^i$. The elements of type γ_ρ exhaust therefore all orbits in $\mathbb{T}^\vee/W_0$. Since a two-dimensional supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module is determined by its γ-component and its central character, the map $\rho \mapsto \mathcal{M}(\rho)$ is seen to be surjective. It is then bijective, since source and target have the same cardinality. $\square$

5.7. Let F be a finite extension of $\mathbb{Q}_p$. A distinguished natural bijection between irreducible two-dimensional $\mathrm{Gal}(\overline{F}/F)$-representations and supersingular two-dimensional $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules is established by Breuil [Br03] for $F = \mathbb{Q}_p$ (see [Be11] for its relation to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$) and by Grosse-Klönne [GK18] for general $F/\mathbb{Q}_p$. In this final paragraph we will show that the bijection $\rho \mapsto \mathcal{M}(\rho)$ from 5.6 coincides in this case with the bijections [Br03] and [GK18].

The case $F = \mathbb{Q}_p$ follows directly from the explicit formulae given in [Be11, 1.3]. For the general case, we briefly recall the main construction from [GK18] in the case of standard supersingular modules of dimension 2. Let F_ϕ be the special Lubin-Tate group with Frobenius power series $\phi(t) = \varpi t + t^q$. Let F_∞/F be the extension generated by all torsion points of F_ϕ and let $\Gamma = \mathrm{Gal}(F_\infty/F)$. We identify in the following $\Gamma \simeq o_F^\times$ via the character χ_L.

Let $k/\mathbb{F}_q$ be a finite extension and let $\mathcal{H}^{(1)}_k := \mathcal{H}^{(1)}(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} k$ via $\mathbf{q} \mapsto q = 0$. Let M be a two-dimensional standard supersingular $\mathcal{H}^{(1)}_k$-module, arising from a supersingular character $\chi : \mathcal{H}^{(1)}_{\mathrm{aff},k} \to k$. Let $e_0 \in M$ such that $\mathcal{H}^{(1)}_{\mathrm{aff},k}$ acts on e_0 via χ and put $e_1 = T_\omega^{-1} e_0$ (where $\omega = u^{-1}$ in our notation).[1] The character χ determines two numbers $0 \le k_0, k_1 \le q-1$ with $(k_0,k_1) \ne (0,0), (q-1,q-1)$. One considers M a $k[[t]]$-module with $t = 0$ on M. Let $\Gamma = o_F^\times$ act on M via

$$\gamma(m) = T^{-1}_{\eta_1(\overline{\gamma})}(m)$$

for $\gamma \in o_F^\times$ with reduction $\overline{\gamma} \in \mathbb{F}_q^\times$ and $\eta_1(\overline{\gamma})^{-1} = \mathrm{diag}(\overline{\gamma}^{-1}, 1) \in \mathbb{T}$. The $k[[t]][\varphi]$-submodule $\nabla(M)$ of

$$k[[t]][\varphi,\Gamma] \otimes_{k[[t]][\Gamma]} M \simeq k[[t]][\varphi] \otimes_{k[[t]]} M$$

[1] For example, if M is an $\mathcal{H}_\theta$-module on which $U^2 = \zeta_2$ acts via the scalar $\theta(\zeta_2) = \tau_2$, then $U = U^{-1} \cdot \tau_2$ on M and $m := \tau_2^{-1} e_1$ satisfies $Um = T_\omega e_1 = e_0$, i.e. $\{m, Um\}$ is a standard basis for M in the sense of 3.5.1.

is then generated by the two elements $h(e_j) = t^{k_j}\varphi \otimes T_\omega^{-1}(e_j) + 1 \otimes e_j$ thereby defining the relation between the Frobenius φ and the Hecke action of T_ω. Note that in the case of GL_2, the cocharacter e^* of [GK18, 2.1] is equal to η_1.

The module $\nabla(M)$ is stable under the Γ-action and thus the quotient

$$\Delta(M) = \big(k[[t]][\varphi] \otimes_{k[[t]]} M\big)/\nabla(M)$$

defines a $k[[t]][\varphi, \Gamma]$-module. It is torsion standard cyclic with weights (k_0, k_1) in the sense of [GK18, 1.3], according to [GK18, Lemma 5.1]. Let $\Delta(M)^* = \mathrm{Hom}_k(\Delta(M), k)$. By a general construction (which goes back to Colmez and Emerton in the case $F = \mathbb{Q}_p$ and $\phi(t) = (1+t)^p - 1$, as recalled in [Br15, 2.6]) the $k((t))$-vector space

$$\Delta(M)^* \otimes_{k[[t]]} k((t))$$

is in a natural way an étale Lubin-Tate (φ, Γ)-module of dimension 2. The correspondence $M \mapsto \Delta(M)^* \otimes_{k[[t]]} k((t))$ extends in fact to a fully faithful functor from a suitable category of supersingular $\mathcal{H}_k^{(1)}$-modules to the category of étale (φ, Γ)-modules over $k((t))$. The composite functor to the category of continuous $\mathrm{Gal}(\overline{F}/F)$-representations over k is denoted by $M \mapsto V(M)$. It induces the aforementioned bijection between irreducible two-dimensional $\mathrm{Gal}(\overline{F}/F)$-representations and supersingular two-dimensional $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-modules.

5.8. Proposition. *The inverse map to the bijection $M \mapsto V(M)$ is given by the map $\rho \mapsto \mathcal{M}(\rho)$.*

Proof : The correspondence $M \mapsto V(M)$ is compatible with the twist by a character of $F^\times$ and local class field theory, such that the determinant corresponds to the central character restricted to $F^\times$. By its very construction, the same is true for the correspondence $\rho \mapsto \mathcal{M}(\rho)$. It therefore suffices to compare them on irreducible Galois representations having parameters $b = 1$ and $i = 0$. Let $k = \mathbb{F}_{q^2}$ in the following. Let $\mathrm{ind}(\omega_{2f}^h)$ be the Galois representation with exponent $1 \leq h \leq q-1$ and $b = 1$ and $i = 0$. Let D be the (φ, Γ)-module associated to $\rho := \mathrm{ind}(\omega_{2f}^h)$ and let M be a supersingular $\mathcal{H}_k^{(1)}$-module such that $\Delta(M)^* \otimes_{k[[t]]} k((t)) \simeq D$. According to the main result of [PS3] for $n = 2$, the module D has a basis $\{g_0, g_1\}$ such that

$$\gamma(g_j) = \overline{f}_\gamma(t)^{hq^j/(q+1)} g_j$$

for all $\gamma \in \Gamma$ and $\varphi(g_0) = g_1$ and $\varphi(g_1) = -t^{-h(q-1)} g_0$. Here, $\overline{f}_\gamma(t) = \omega_f(t)t/\gamma(t) \in k[[t]]^\times$. Define the triple $(k_0, k_1, k_2) = (h-1, q-h, h-1)$ and let $i_j := q - 1 - k_{2-j}$, so that $i_0 = i_2 = q - h$ and $i_1 = 2q - h - 1$. Define the triple $(h_0, h_1, h_2) = (0, i_1, i_0 + i_1 q)$. Note that $h_2 = h(q-1)$. Put $f_j = t^{h_j} g_j$ for $j = 0, 1$ and let $D^\sharp \subset D$ be the $k[[t]]$-submodule generated by $\{f_0, f_1\}$. Let $(D^\sharp)^*$ be the k-linear dual. Define $e'_i \in (D^\sharp)^*$ via $e'_i(f_j) = \delta_{ij}$ and $e'_i = 0$ on $tD^\sharp$. Using the explicit formulae for the ψ-operator on $k((t))$ as described in [GK18, Lemma 1.1] one may follow the argument of [GK16, Lemma 6.4] and show that $D^\sharp$ is a ψ-stable lattice in D and that $\{e'_0, e'_1\}$ is a k-basis of the t-torsion part of $(D^\sharp)^*$ satisfying

$$t^{k_1}\varphi(e'_0) = e'_1 \ \text{ and } \ t^{k_0}\varphi(e'_1) = -e'_0.$$

But according to [GK18, 1.15] there is only one ψ-stable lattice in $\Delta(M)^* \otimes_{k[[t]]} k((t))$, namely $\Delta(M)^*$. It follows that $\Delta(M) \simeq (D^\sharp)^*$ and so the weights of the torsion standard cyclic $k[[t]][\varphi, \Gamma]$-module $\Delta(M)$ are (k_0, k_1). Since $k_0 = h - 1$, one deduces from [GK18, Lemma 4.1/5.1] that $\epsilon_1 \equiv h - 1 \mod (q-1)$. This means $\lambda \circ \alpha^\vee(x)^{-1} = x^{h-1}$ for the character $\lambda \in \mathbb{T}^\vee$ of M. Since $i = 0$ and hence $a = 0$ (in the notation of [GK16, 2.2]), we arrive therefore at

$$\lambda(\mathrm{diag}(x_1, x_2)) = \lambda(e^*(x_1x_2)\alpha^\vee(x_2)^{-1}) = e^*(x_1x_2)^a x_2^{h-1} = x_2^{h-1}.$$

Hence the image of λ in $\mathbb{T}^\vee/W_0$ coincides with γ_ρ. This implies $M \simeq \mathcal{M}(\rho)$, as claimed. □

References

[Be10] L. Berger, *On some modular representations of the Borel subgroup of* $\mathrm{GL}_2(\mathbb{Q}_p)$, Comp. Math. **146**(1) (2010), 58-80.

[Be11] L. Berger, *La correspondance de Langlands locale p-adique pour* $\mathrm{GL}_2(\mathbb{Q}_p)$, Astérisque **339** (2011), 157-180.

[Br03] C. Breuil, *Sur quelques représentations modulaires et p-adiques de* $GL_2(\mathbb{Q}_p)$*. I*, Compositio Math. **138** (2003), 165-188.

[Br15] C. Breuil, *Induction parabolique et* (φ, Γ)*-modules.*, Algebra and Number Theory. **9** (2015), 2241-2291.

[Bri97] M. Brion, *Equivariant Chow groups for torus actions*, Transformations Groups, Vol. 2, Nr. 3, 1997, pages 225-267.

[CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston,1997.

[C10] P. Colmez, *Représentations de* $GL_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, Astérisque **330** (2010), 281-509.

[D73] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. 21, pages 287-301, 1973.

[D74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4), tome 7 (1974), 53-88. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[EG96] D. Edidin and W. Graham, *Equivariant Intersection Theory*, Invent. Math. 131, pages 595-634, 1996.

[GK16] E. Grosse-Klönne, *From pro-p-Iwahori-Hecke modules to* (φ, Γ)*-modules, I*, Duke Math. Journal 165 No. 8 (2016), 1529-1595.

[GK18] E. Grosse-Klönne, *Supersingular Hecke modules as Galois representations*, Preprint (2018) arXiv:1803.02616.

[Hu05] J.E. Humphreys, *Modular Representations of Finite Groups of Lie Type*, London Math. Soc. Lecture Notes Series 326, Cambridge University Press, 2005.

[KK86] B. Kostant, S. Kumar, *The Nil Hecke ring and cohomology of* G/P *for a Kac-Moody group* G, Advances. Math., **62** (1), 187-237, 1986.

[KL87] D. Kazhdan, G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., **87** (1), 153-215, 1987.

[KR09] M. Kisin, W. Ren, *Galois representations and Lubin-Tate groups*, Doc. Math., **14**, 441-461, 2009.

[O14] R. Ollivier, *Compatibility between Satake and Bernstein isomorphisms in characteristic* p, Algebra and Number Theory **8**(5) (2014), 1071-1111.

[PS2] C. Pépin, T. Schmidt, *Mod p Hecke algebras and dual equivariant cohomology II: the case of* GL_n, in preparation.

[PS3] C. Pépin, T. Schmidt, *On a certain class of Lubin-Tate* (φ, Γ)*-modules*, Preprint 2019.

[Sch17] P. Schneider, *Galois representations and* (φ, Γ)*-modules*, Cambridge Studies in Advanced Mathematics **164** (Cambridge University Press, 2017).

[Se72] J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math., **15**(4) (1972), 259-331.

[St75] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173-117.

[Th87] R.W. Thomason, *Algebraic K-theory of group scheme actions*, Ann. of Math. Stud. **113** (1987), 539-563.

[V94] M.-F. Vigneras, *A propos d'une conjecture de Langlands modulaire*, in: *Finite reductive groups*, Ed. M. Cabanes, Prog. Math., vol. 141 (Birkhäuser, Basel, 1997).

[V96] M.-F. Vigneras, *Représentations ℓ-modulaires d'un groupe réductif p-adique*, Prog. Math., vol. 137 (Birkhäuser, Basel, 1996).

[V04] M.-F. Vigneras *Representations modulo p of the p-adic group $GL(2, F)$*, Compositio Math. 140 (2004) 333-358.

[V05] M.-F. Vigneras *Pro-p-Iwahori Hecke ring and supersingular $\overline{\mathbb{F}}_p$-representations*, Math. Ann. **331** (2005), 523-556. + Erratum

[V06] M.-F. Vigneras *Algèbres de Hecke affines génériques*, Representation Theory **10** (2006), 1-20.

[V14] M.-F. Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group II*, Compositio Math. Muenster J. Math. **7** (2014), 363-379. + Erratum

[V15] M.-F. Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group V (Parabolic induction)*, Pacific J. of Math. **279** (2015), Issue 1-2, 499-529.

[V16] M.-F. Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*, Compositio Math. **152** (2016), 693-753.

[V17] M.-F. Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group III (Spherical Hecke algebras and supersingular modules)*, Journal of the Institue of Mathematics of Jussieu **16** (2017), Issue 3, 571-608. + Erratum

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