INVISCID LIMIT FOR THE TWO-DIMENSIONAL NAVIER-STOKES SYSTEM IN A CRITICAL BESOV SPACE

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ABSTRACT. In a recent paper [12], Vishik proved the global well-posedness of the two-dimensional Euler equation in the critical Besov space $B^2_{2,1}$. In the present paper we prove that the Navier-Stokes system is globally well-posed in $B^2_{2,1}$, with uniform estimates on the viscosity. We prove also a global result of inviscid limit. The convergence rate in L^2 is of order ν.

1. INTRODUCTION

The equations of motion governing an incompressible viscous fluid with viscosity $\nu > 0$ are given by the so-called Navier-Stokes equations,

$$(NS_\nu)\begin{cases} \partial_t v_\nu + v_\nu \cdot \nabla v_\nu - \nu \Delta v_\nu = -\nabla \pi_\nu, \\ \operatorname{div} v_\nu = 0 \\ v_{\nu|t=0} = v^0, \end{cases}$$

the vector field $v_\nu(t,x) = (v^1_\nu, ..., v^d_\nu)(t,x)$ stands for the velocity of the fluid, the quantity π_ν denotes the scalar pressure, and div $v = 0$ means that the fluid is incompressible. When we neglect the diffusion term, then we obtain the Euler equations,

$$(E)\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla \pi \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0. \end{cases}$$

The mathematical study of the Navier-Stokes system was initiated by J. Leray in his pioneering work [10]. In fact, by using a compactness method, this author proved that for any divergence-free initial data v^0 in the energy space L^2, there exits a global solution to (NS_ν). In the case of *two* space dimension that weak solution was proved to be unique. However, for higher dimension ($d \geq 3$) the problem of uniqueness is still a widely open problem. In 60', Fujita-Kato [6] exhibited when the initial data lies in the critical homogeneous Sobolev space $\dot{H}^{\frac{d}{2}-1}$ a class of unique local solutions called strong solutions. These local solutions are obtained, using some integrability properties of the solutions to the heat equation, by solving the

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equivalent integral equation by a standard Picard iteration method. We emphasize that the same result holds true when the initial data belongs to the inhomogeneous Sobolev space H^s, with $s \geq \frac{d}{2} - 1$. In general, the globality in time of these solutions is also an open problem. However a positive answer is given at least in both following cases: either when the initial data is small in the critical space $\dot{H}^{\frac{d}{2}-1}$ which is invariant under the scaling of the the Navier-Stokes equations, or in the space dimension two. Concerning Euler equations, the hyperbolic system theory ensures us the local in time existence and uniqueness in H^s, with $s > \frac{d}{2} + 1$, (see [9]). In the critical case, we mention that we have the same result when we replace the space $H^{1+\frac{d}{2}}$ by the Besov space $B_{p,1}^{1+\frac{d}{p}}, p \in (1, \infty)$ which is contained in the class of Lipschitz functions (see [2, 5], for instance). More recently, M. I. Vishik has proved in [12] that the solution is globally well-posed in these critical Besov spaces when $d = 2$. He used for the proof a subtle logarithmic estimate based on the explicit form of the vorticity in dimension two.
The problem of the convergence of smooth viscous solutions $(v_\nu)_{\nu>0}$ to the Eulerian one as ν goes to zero is well understood. In [11], Majda showed that under the assumption $v^0 \in H^s$ with $s > \frac{d}{2} + 2$, the solutions $(v_\nu)_{\nu>0}$ converge in L^2 norm as $\nu \to 0$ to the unique solution v of (E). The convergence rate is of order νt. We note that in dimension two the previous results are global in time and the proof is based heavily on Brezis-Gallouët logarithmic estimate and the boundedness of the vorticity.
In this paper, we prove that the two-dimensional Navier-Stokes is globally well-posed in Besov space $B^2_{2,1}$, with uniform bounds on the viscosity. We prove also that the convergence rate of the inviscid limit is of order νt for vanishing viscosity. The basic ingredient of the proof is a regularization effect of the vorticity equation which allows us to bound, uniformly on ν, the Lipschitz norm of the viscous velocity. Let us recall the equation of the vorticity in dimension two;

$$(TD_\nu): \quad \partial_t \omega_\nu + v_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0; \quad \omega_\nu = \partial_1 v_\nu^2 - \partial_2 v_\nu^1.$$

We state now our main theorem.

Theorem 1.1. *Let v^0 a divergence-free vector field belonging to $B^2_{2,1}(\mathbb{R}^2)$. Then, the Navier-Stokes system (NS_ν) has a unique global solution in $\mathcal{C}(\mathbb{R}_+; B^2_{2,1})$, satisfying in addition the following uniform estimate*

$$\|v_\nu(t)\|_{B^2_{2,1}} \leq C_0 e^{\exp C_0 t}.$$

Moreover, v_ν goes to the Euler solution v as ν goes to zero. More precisely, we have for all $\nu \in (0, 1]$

$$\|v_\nu(t) - v(t)\|_{L^2} \leq C_0 e^{\exp C_0 t}(\nu t), \tag{1.1}$$

where C_0 is a constant depending on the initial data.

The rest of this paper is structured as follows. In the second section we recall the Littlewood-Paley theory and we give some lemmas needed for the proof of our result. The third one is dedicated to the description of a smoothing effect of the vorticity equation. However, we give in the last section the proof of our main theorem. For the convenience of the reader, we close our paper by an appendix in which we present the proof of a commutator lemma.

2. Preliminaries

In this paragraph we recall the definition of frequency localization operator and the Besov spaces $B^s_{p_1,p_2}$. Some useful lemmas are then given.

Proposition 2.1. *There exist two radially symmetric functions $\chi\in\mathcal{D}(\mathbb{R}^d)$ and $\varphi\in\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ such that*

i) $\chi(\xi)+\sum_{q\geq 0}\varphi(2^{-q}\xi)=1,\quad \frac{1}{3}\leq\chi^2(\xi)+\sum_{q\geq 0}\varphi^2(2^{-q}\xi)\leq 1,$

ii) $\text{supp }\varphi(2^{-p}\cdot)\cap\text{supp }\varphi(2^{-q}\cdot)=\varnothing,$ *if* $|p-q|\geq 2,$

iii) $q\geq 1\Rightarrow \text{supp}\chi\cap\text{supp }\varphi(2^{-q})=\varnothing.$

For every $v\in\mathcal{S}'$ we set

$$\Delta_{-1}v=\chi(\mathrm{D})v\ ;\forall q\in\mathbb{N},\ \Delta_q v=\varphi(2^{-q}\mathrm{D})v\quad\text{ and }\ S_q=\sum_{-1\leq p\leq q-1}\Delta_p.$$

The paradifferential calculous introduced by J.-M. Bony [1] is based on the decomposition (called Bony's decomposition) which split the product uv into three parts:

$$uv=T_uv+T_vu+R(u,v),$$

with

$$T_uv=\sum_q S_{q-1}u\Delta_q v\quad\text{and}\quad R(u,v)=\sum_{|q'-q|\leq 1}\Delta_q u\Delta_{q'}v.$$

Let us now recall what Besov spaces are, through their characterizations via frequency localization. Let $(p_1,p_2)\in[1,+\infty]^2$ and $s\in\mathbb{R}$ then the space $B^s_{p_1,p_2}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p_1,p_2}}:=\Big(2^{qs}\|\Delta_q u\|_{L^{p_1}}\Big)_{\ell^{p_2}}<+\infty.$$

We point out that the Hölderian space C^s with $s\in\mathbb{R}-\mathbb{Z}$ is nothing but the Besov space $B^s_{\infty,\infty}$. When $s\in\mathbb{Z}$, the last space is denoted by C^s_* called Zygmund space. Let $T>0$ and $r\geq 1$, we denote by $L^r_TB^s_{p_1,p_2}$ the space of all function u satisfying

$$\|u\|_{L^r_TB^s_{p_1,p_2}}:=\Big\|\Big(2^{qs}\|\Delta_q u\|_{L^{p_1}}\Big)_{\ell^{p_2}}\Big\|_{L^r_T}<\infty.$$

We will also make continuous use of Bernstein lemma (see for example [3]).

Lemma 2.2. (Bernstein) *Let (r_1, r_2) a pair of strictly positive numbers such that $r_1 < r_2$. There exists a constant C such that for every nonnegatif integer k, for every $1 \leq a \leq b$ and for all function $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp } \widehat{u} \in B(0, \lambda r_1) \quad \Rightarrow \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

$$\text{supp } \widehat{u} \in \mathcal{C}(0, \lambda r_1, \lambda r_2) \quad \Rightarrow \quad C^{-k}\lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

As a consequence, we have the following embeddings:

$$B^{\frac{d}{p}}_{p,1} \hookrightarrow B^{\frac{d}{q}}_{q,1} \hookrightarrow L^\infty; \; \forall 1 \leq p \leq q.$$

The next lemma plays a crucial role in the proof of the regularization effect for the transport-diffusion equation (see [12]).

Lemma 2.3. *Assume $d \geq 2$. There exists a positive constant C such that, for any function f taken in the Schwartz class, and for any diffeomorphism ψ of $\mathbb{R}^d$ preserving the Lebesgue measure, we have for all $p \in [1, +\infty]$ and for all $j, q \geq -1$,*

$$\|\Delta_j(\Delta_q f \circ \psi)\|_{L^p} \leq C 2^{-|j-q|} \|\nabla \psi^{\eta(j,q)}\|_{L^\infty} \|\Delta_q f\|_{L^p},$$

with

$$\eta(j, q) = sign(j - q).$$

Now, we recall the following commutator lemma.

Lemma 2.4. *Let v a divergence-free vector field of $\mathbb{R}^d$ which is lipschitzian, and $a \in L^p$, with $p \in [1, +\infty]$. Then, there exists a constant C depending only on d, such that*

$$\big\|[\Delta_q, v \cdot \nabla]a\big\|_{L^p} \leq C \|\nabla v\|_{L^\infty} \|a\|_{L^p}.$$

We describe by the following lemma the regularization effect of the heat operator (for more details, see [4]).

Lemma 2.5. *Let $\mathcal{C}$ be a given ring. There exist two constants c and C, such that for any pair of positive numbers (t, λ), for every $p \in [1, +\infty]$ and for all function $a \in L^p$, we have :*

$$\text{Supp } \widehat{a} \subset \lambda \mathcal{C} \Rightarrow \|e^{t\Delta} a\|_{L^p} \leq C e^{-ct\lambda^2} \|a\|_{L^p}.$$

3. Smoothing effect

We intend to prove that the integral in time of the vorticity allows us to gain two derivatives with a linear growth on the Lipschitz norm of the velocity. In other words, we have

Proposition 3.1. *Let $\omega^0 \in L^p, p \in [1, +\infty]$ and v_ν a Lipschitz solution for the two-dimensional Navier-Stokes system with initial vorticity ω^0. Then, for all $q \in \mathbb{N}$ we have*

$$\nu 2^{2q} \int_0^t \|\Delta_q \omega_\nu(\tau)\|_{L^p} d\tau \leq C \|\omega^0\|_{L^p} \left(1 + \int_0^t \|\nabla v_\nu(\tau)\|_{L^\infty} d\tau \right).$$

Proof. For the sake of simplicity we will omit the index ν from the vorticity. The function $\omega_q := \Delta_q\omega$ is solution of

$$\partial_t\omega_q + S_{q-1}v\cdot\nabla\omega_q - \nu\Delta\omega_q = (S_{q-1} - \mathrm{Id})v\cdot\nabla\omega_q - [\Delta_q, v\cdot\nabla]\omega := f_q.$$

We set

$$\bar\omega_q(t,x) = \omega_q(t,\,\psi_q(t,\,x)) \quad \text{et} \quad \bar f_q(t,x) = f_q(t,\psi_q(t,x)),$$

where ψ_q is the flow of the regularized velocity $S_{q-1}v$ defined by

$$\psi_q(t,x) = x + \int_0^t S_{q-1}v\big(\tau,\psi_q(\tau,x)\big)d\tau.$$

A simple computation leads to

(3.1)

$$\Delta\omega_q(t)\circ\psi_q(t,x) = \sum_{i=1}^{2}\Big\langle\nabla^2\bar\omega_q(t,x)\cdot(\partial^i\psi_q^{-1})(t,\psi_q(t,x)),(\partial^i\psi_q^{-1})(t,\psi_q(t,x))\Big\rangle + \nabla\bar\omega_q(t,x)\cdot(\Delta\psi_q^{-1})(t,\psi_q(t,x)).$$

Thus, the function $\bar\omega_q$ satisfies

$$\begin{aligned}(\partial_t - \nu\Delta)\bar\omega_q(t) &= \nu\sum_{i=1}^{2}\langle\nabla^2\bar\omega_q(t)g_q^i(t),g_q^i(t)\rangle + 2\nu\sum_{i=1}^{2}\langle\nabla^2\bar\omega_q(t)e_i,g_q^i(t)\rangle\\ &+ \nu\nabla\bar\omega_q(t)\cdot(\Delta\psi_q^{-1})(t,\psi_q(t)) + \bar f_q(t)\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} := \mathcal{R}_q,\end{aligned} \tag{3.2}$$

where the function g_q^i is defined, for all $1\le i\le 2$, by

$$(\partial^i\psi_q^{-1})(t,\psi_q(t,x)) = e_i + g_q^i(t,x),$$

and $(e_i)_{i=1}^2$ is the canonical basis of $\mathbb{R}^2$. By an obvious computation, we get

$$\|g_q(t)\|_{L^\infty} \le CV(t)e^{CV(t)}.$$

Now, we will again apply the operator Δ_j to the new equation (3.2). So by the Duhamel formulation, we obtain

$$\Delta_j\bar\omega_q(t,x) = e^{\nu t\Delta}\Delta_j\omega_q(0) + \int_0^t e^{\nu(t-\tau)\Delta}\Delta_j\mathcal{R}_q(\tau,x)d\tau.$$

On the one hand, Lemma 2.5 and the fact that the operator Δ_j maps L^p to itself uniformly in j yield

$$\begin{aligned}\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{I}+\mathrm{II})(\tau)\|_{L^p} &\le C\nu e^{-c(t-\tau)2^{2j}}\|\mathrm{I}+\mathrm{II}\|_{L^p}\\ &\le CV(\tau)e^{CV(\tau)}\nu e^{-c(t-\tau)2^{2j}}\|\nabla^2\bar\omega_q(\tau)\|_{L^p}.\end{aligned}$$

In the other hand,using Leibniz rule and Bernstein lemma, we get

$$\|\nabla^2\bar\omega_q(\tau)\|_{L^p} \le C2^{2q}e^{CV(t)}\|\omega_q(\tau)\|_{L^p}. \tag{3.3}$$

Thus we infer

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{I}+\mathrm{II})(\tau)\|_{L^p} \le CV(\tau)e^{CV(\tau)}\nu 2^{2q}e^{-c\nu(t-\tau)2^{2j}}\|\omega_q(\tau)\|_{L^p}. \tag{3.4}$$

For the third term we shall use these estimations

$$\|\nabla^2\psi_q^{\mp1}(t)\|_{L^\infty} \leq C2^q V(t)e^{CV(t)};\ \|\nabla\bar{\omega}_q(\tau)\|_{L^p} \leq C2^q e^{CV(t)}\|\omega_q(\tau)\|_{L^p}.$$

The first one is proved, in a classical manner, by using the equation of the flow combined with Gronwall and Bernstein lemmas. Nevertheless, the second one can be proved in the same way as (3.3). Hence, we find

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{III})(\tau)\|_{L^p} \leq CV(\tau)e^{CV(\tau)}\nu 2^{2q}e^{-c\nu(t-\tau)2^{2j}}\|\omega_q(\tau)\|_{L^p}. \tag{3.5}$$

For the fourth term, we deduce from Lemma 2.5

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{IV})(\tau)\|_{L^p} \leq Ce^{-c\nu(t-\tau)2^{2j}}\|\mathrm{IV}(\tau)\|_{L^p}.$$

Thus, Lemma 2.4 yields

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{IV})(\tau)\|_{L^p} \leq Ce^{-c\nu(t-\tau)2^{2j}}\|v(\tau)\|_{\mathrm{Lip}}\|\omega(\tau)\|_{L^p}. \tag{3.6}$$

Combining (3.2), (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}\|e^{\nu(t-\tau)\Delta}\Delta_j\mathcal{R}_q\|_{L^p} &\leq CV(t)e^{V(t)}e^{-c\nu(t-\tau)2^{2j}}\nu 2^{2q}\|\omega_q(\tau)\|_{L^p} + \\ &+ Ce^{-c\nu(t-\tau)2^{2j}}\|v(\tau)\|_{\mathrm{Lip}}\|\omega(\tau)\|_{L^p}.\end{aligned}$$

This yields to

$$\begin{aligned}\|\Delta_j\bar{\omega}_q(t)\|_{L^p} &\leq Ce^{-\nu t2^{2j}}\|\Delta_j\omega_q(0)\|_{L^p} + \\ &+ CV(t)e^{CV(t)}\nu 2^{2q}\int_0^t e^{-c\nu(t-\tau)2^{2j}}\|\omega_q(\tau)\|_{L^p}d\tau \\ &+ C\int_0^t e^{-c\nu(t-\tau)2^{2j}}\|v(\tau)\|_{\mathrm{Lip}}\|\omega(\tau)\|_{L^p}d\tau. \end{aligned}\tag{3.7}$$

Thus, the convolution lemma implies, for all $j \in \mathbb{N}$

$$\begin{aligned}\|\Delta_j\bar{\omega}_q\|_{L^1_tL^p} &\leq C(\nu 2^{2j})^{-1}\|\Delta_j\omega_q(0)\|_{L^p} + CV(t)e^{CV(t)}2^{2(q-j)}\|\omega_q\|_{L^1_tL^p} \\ &+ C(\nu 2^{2j})^{-1}\|\omega\|_{L^\infty_tL^p}\int_0^t\|v(\tau)\|_{\mathrm{Lip}}d\tau.\end{aligned}\tag{3.8}$$

But, we have the classical estimate: $\forall p \in [1,+\infty]$

$$\|\omega\|_{L^\infty_tL^p} \leq \|\omega^0\|_{L^p}.$$

So, the estimate (3.8) becomes

$$\begin{aligned}\|\Delta_j\bar{\omega}_q\|_{L^1_tL^p} &\leq C(\nu 2^{2j})^{-1}\|\Delta_j\omega_q(0)\|_{L^p} + CV(t)e^{CV(t)}2^{2(q-j)}\|\omega_q\|_{L^1_tL^p} \\ &+ C(\nu 2^{2j})^{-1}\|\omega^0\|_{L^p}\int_0^t\|v(\tau)\|_{\mathrm{Lip}}d\tau.\end{aligned}\tag{3.9}$$

Taking an integer N, which will be fixed later. Since the flow preserves the Lebesgue measure, we infer

$$\begin{aligned}\nu 2^{2q}\|\omega_q\|_{L^1_tL^p} &= \nu 2^{2q}\|\bar{\omega}_q\|_{L^1_tL^p} \\ &\leq \nu 2^{2q}\Big(\sum_{|j-q|\leq N}\|\Delta_j\bar{\omega}_q\|_{L^1_tL^p} + \sum_{|j-q|>N}\|\Delta_j\bar{\omega}_q\|_{L^1_tL^p}\Big).\end{aligned}$$

Using (3.9) and the above inequality, we get for all $q \geq N$

$$\begin{aligned} \nu 2^{2q}\|\omega_q\|_{L^1_tL^p} &\leq C\|\omega_q(0)\|_{L^p} + CV(t)e^{CV(t)}2^{2N}\nu 2^{2q}\|\omega_q\|_{L^1_tL^p} \\ &+ CV(t)2^{2N}\|\omega^0\|_{L^p} + \nu 2^{2q}\sum_{|j-q|>N}\|\Delta_j\bar{\omega}_q\|_{L^1_tL^p}. \end{aligned}$$

In the other hand we have in view of Lemma 2.3.

$$\|\Delta_j\bar{\omega}_q(t)\|_{L^p} \leq C2^{-|q-j|}e^{CV(t)}\|\omega_q(t)\|_{L^p}.$$

Thus, for $q > N$, it holds that

$$\begin{aligned} \nu 2^{2q}\|\omega_q\|_{L^1_tL^p} &\leq C\|\omega_q(0)\|_{L^p} + CV(t)2^{2N}\|\omega^0\|_{L^p} + \\ &+ \Big(CV(t)e^{CV(t)}2^{2N} + C2^{-N}e^{CV(t)}\Big)\nu 2^{2q}\|\omega_q\|_{L^1_tL^p}. \end{aligned}$$

For low frequencies, $q \leq N$, we write by Hölder inequality

$$\nu 2^{2q}\|\omega_q\|_{L^1_tL^p} \leq (\nu t)2^{2N}\|\omega^0\|_{L^p}.$$

Hence, we obtain for all $q \geq -1$

$$\begin{aligned} \nu 2^{2q}\|\omega_q\|_{L^1_tL^p} &\leq C\|\omega_q(0)\|_{L^p} + \Big(CV(t)2^{2N} + (\nu t)2^{2N}\Big)\|\omega^0\|_{L^p} + \\ &+ \Big(CV(t)e^{CV(t)}2^{2N} + C2^{-N}e^{CV(t)}\Big)\nu 2^{2q}\|\omega_q\|_{L^1_tL^p}. \end{aligned}$$

Choosing N and t such that

$$CV(t)e^{CV(t)}2^{2N} + Ce^{CV(t)}2^{-N} \leq \frac{1}{2}.$$

This is possible for small time t such that

$$V(t) \leq C_1,$$

where C_1 is a small absolute constant. With this restriction, one obtains

$$\nu 2^{2q}\|\omega_q\|_{L^1_tL^p} \leq C\|\omega^0\|_{L^p}\big(1+\nu t\big)$$

To globalize this estimate we take an arbitrary time T and a partition $(T_i)_0^n$ of the interval $[0,T]$ as follows

$$T_0 = 0 < T_1 < ... < T_n = T; \quad \int_{T_i}^{T_{i+1}}\|v(\tau)\|_{\mathrm{Lip}}d\tau \simeq C_1.$$

Then we repeat the same method and we get

$$\nu 2^{2q}\|\omega_q\|_{L^1(T_i,T_{i+1};L^p)} \leq C\|\omega(T_i)\|_{L^p}\big(1+\nu t\big).$$

However, we know that the L^p norm of the vorticity is bounded for all positive time by its initial value, so

$$\nu 2^{2q}\|\omega_q\|_{L^1(T_i,T_i;L^p)} \leq C\|\omega^0\|_{L^p}\big(1+\nu t\big).$$

Summing these inequalities gives

$$\nu 2^{2q}\|\omega_q\|_{L^1_T;L^p} \leq Cn\|\omega^0\|_{L^p}\big(1+\nu t\big).$$

A simple computation show that

$$C_1 n \simeq V(t).$$

Putting this expression into the above estimate we get the required smoothing effect. □

4. Proof of the Theorem 1.1

The proof is organized as follows: we start by establishing the inviscid limit. We combine for this aim a simple energy estimate together with the global well-posedness result for Euler system as proved in [12]. We show that the convergence rate in L^2 is of order νt. For the uniform estimate on the viscosity of the Navier-Stokes solution in Besov space $B^2_{2,1}$ we need to bound the Lipschitz norm of the velocity. To do so we use in a crucial way the smoothing effect of the vorticity equation and the convergence rate of Navier-Stokes solution to the Euler one.

4.1. **Inviscid limit.** Let us recall the following global estimates for two-dimensional Euler system, due to Vishik [12],

$$\|v(t)\|_{B^2_{2,1}} \leq C_0 e^{\exp C_0 t} \quad \text{and} \quad \|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t}, \tag{4.1}$$

where C_0 denotes a constant depending only on the initial data. Let

$$\bar{v}_\nu = v_\nu - v.$$

Then, combining (NS_ν) and (E) systems gives

$$\begin{cases} \partial_t \bar{v}_\nu + v_\nu \cdot \nabla \bar{v}_\nu - \nu \Delta \bar{v}_\nu + \nabla p_\nu = \nu \Delta v - \bar{v}_\nu \cdot \nabla v \\ \bar{v}_\nu(0) = 0. \end{cases}$$

Taking the L^2 inner product of this equation with $\bar{v}_\nu$, using the incompressibility of the flows

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}_\nu(t)\|^2_{L^2} + \nu\|\nabla \bar{v}_\nu(t)\|^2_{L^2} \leq \nu\|\Delta v\|_{L^2}\|\bar{v}_\nu\|_{L^2} + \|\nabla v\|_{L^\infty}\|\bar{v}_\nu\|^2_{L^2}.$$

Applying Gronwall lemma leads to,

$$\|\bar{v}_\nu(t)\|_{L^2} \leq \nu e^{\int_0^t \|v(\tau)\|_{\mathrm{Lip}} d\tau} \int_0^t \|\Delta v(\tau)\|_{L^2} d\tau.$$

To complete the proof we use the estimates (4.1) combined with this one

$$\|\Delta v\|_{L^2} \leq \|v\|_{B^2_{2,1}}.$$

4.2. **Lipschitz estimate of the velocity.** The object of this paragraph is to bound the quantity $\int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau$. Let N a positive integer, then we have by the triangular inequality

$$\int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau \le \int_0^t \|S_N v(\tau)\|_{\rm Lip} d\tau + \int_0^t \|S_N(v_\nu - v)(\tau)\|_{\rm Lip} d\tau + \int_0^t \|(\mathrm{Id} - S_N)v_\nu(\tau)\|_{\rm Lip} d\tau. \tag{4.2}$$

We stress that the convolution lemma gives

$$\int_0^t \|S_N v(\tau)\|_{\rm Lip} d\tau \le C \int_0^t \|v(\tau)\|_{\rm Lip} d\tau. \tag{4.3}$$

In the other hand, we have according to Bernstein lemma and the estimate (1.1)

$$\begin{aligned}\int_0^t \|S_N(v_\nu - v)(\tau)\|_{\rm Lip} d\tau &\le C 2^{2N} \int_0^t \|(v_\nu - v)(\tau)\|_{L^2} d\tau \\ &\le C_0(\nu\, t)\, 2^{2N}\, e^{\exp C_0 t}.\end{aligned} \tag{4.4}$$

Applying the Proposition 3.1, then the last term of (4.2) is bounded as follows,

$$\begin{aligned}\int_0^t \|(\mathrm{Id} - S_N)v_\nu(\tau)\|_{\rm Lip} d\tau &\le C \sum_{q\ge N} \int_0^t \|\Delta_q \omega_\nu(\tau)\|_{L^\infty} d\tau \\ &\le C \frac{\|\omega^0\|_{L^\infty}}{\nu 2^{2N}} \Big(1 + \int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau\Big).\end{aligned} \tag{4.5}$$

Plugging (4.3), (4.4) and (4.5) into (4.2), leads to

$$\int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau \le C_0\Big(1+\nu\, t\, 2^{2N}\Big)\, e^{\exp C_0 t} + C\frac{\|\omega^0\|_{L^\infty}}{\nu 2^{2N}}\Big(1+\int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau\Big).$$

Choosing N such that,

$$C\frac{\|\omega^0\|_{L^\infty}}{\nu 2^{2N}} \simeq \frac{1}{2},$$

then, we obtain

$$\begin{aligned}\int_0^t \|v_\nu(\tau)\|_{\rm Lip} d\tau &\le C_0(1 + t\|\omega^0\|_{L^\infty}) e^{\exp C_0 t} \\ &\le C_0 e^{\exp C_0 t}.\end{aligned} \tag{4.6}$$

4.3. **Uniform persistence regularity.** We shall prove in this section the uniform control with respect to the viscosity of the velocity in Besov space $B^2_{2,1}$. To begin with, we localize the equation of the viscous vorticity via the operator Δ_q. Let $\omega_q \overset{def}{=} \Delta_q \omega$. Then it satisfies

$$\partial_t \omega_q + v\cdot\nabla\omega_q - \nu\Delta\omega_q = -[\Delta_q, v\cdot\nabla]\omega.$$

Taking the L^2 inner product with ω_q and using the condition $\text{div}v = 0$, then we obtain according to Proposition 5.1, given in the Appendix,

$$\|\omega_q(t)\|_{L^2}^2 \le \|\omega_q(0)\|_{L^2}^2 + C\int_0^t \|\omega_q(\tau)\|_{L^2}\|\nabla v(\tau)\|_{L^\infty}\sum_{j\ge q-N_0} 2^{q-j}\|\Delta_j\omega(\tau)\|_{L^2}d\tau.$$

By Gronwall's lemma, one writes

$$\|\omega_q(t)\|_{L^2} \le \|\omega_q(0)\|_{L^2} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\sum_{j\ge q-N_0} 2^{q-j}\|\Delta_j\omega(\tau)\|_{L^2}d\tau.$$

Multiplying by 2^q and taking the sum over q leads by virtue of Young inequality

$$\|\omega(t)\|_{B^1_{2,1}} \le \|\omega^0\|_{B^1_{2,1}} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{B^1_{2,1}}d\tau.$$

Thus, we obtain according to Gronwall's lemma

$$\|\omega(t)\|_{B^1_{2,1}} \le \|\omega^0\|_{B^1_{2,1}} e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau}. \tag{4.7}$$

Let us now move to the estimate of the velocity in $B^2_{2,1}$. Separating the low and the high frequencies

$$\begin{aligned}\|v(t)\|_{B^2_{2,1}} &\le \|\Delta_{-1}v(t)\|_{L^2} + \sum_{q\in\mathbb{N}} 2^{2q}\|\Delta_q v(t)\|_{L^2}\\ &\le \|v(t)\|_{L^2} + \|\omega(t)\|_{B^1_{2,1}}. \qquad (4.8)\end{aligned}$$

The energy estimate tells us that for all positive number t

$$\|v(t)\|_{L^2} \le \|v^0\|_{L^2} \le \|v^0\|_{B^2_{2,1}}.$$

Plugging this estimate and (4.7) into (4.8),

$$\|v(t)\|_{B^2_{2,1}} \le C\|v^0\|_{B^2_{2,1}} e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

To achieve the proof we use the Lipschitz control (4.6).

5. Appendix

We shall give in this section a precise description of the commutator term $[\Delta_q, v\cdot\nabla]\omega$ in the special case $\omega = \text{rot } v$.

Proposition 5.1. *Let v a regular divergence-free vector field of $\mathbb{R}^2$ and we denote by ω its vorticity. Then, we have for all $q \ge -1$ and for all $p \in [1,+\infty]$,*

$$\|[\Delta_q, v\cdot\nabla]\omega\|_{L^p} \le C\|\nabla v\|_{L^\infty}\sum_{j\ge q-N_0} 2^{q-j}\|\Delta_j\omega\|_{L^p}.$$

where C and N_0 are absolute constants.

Proof. We will make use of Bony's decomposition [1]:

$$[\Delta_q, v\cdot\nabla]\omega = [\Delta_q, T_v\cdot\nabla]\omega + [\Delta_q, T_{\nabla\cdot}\cdot v]\omega + [\Delta_q, R(v,\nabla\cdot)]\omega. \tag{5.1}$$

For the first term, we have by definition of the paraproduct

$$\|[\Delta_q, T_{\nabla\cdot}\cdot v]\omega\|_{L^p} \leq C\sum_{|j-q|\leq N_0}\|S_{j-1}\nabla\omega\|_{L^\infty}\|\Delta_j v\|_{L^p}. \tag{5.2}$$

According to Bernstein lemma and the definition of the vorticity, we have

$$\begin{aligned}\|S_{j-1}\nabla\omega\|_{L^\infty} &\leq C2^j\|\omega\|_{L^\infty}\\ &\leq C2^j\|\nabla v\|_{L^\infty}.\end{aligned}$$

We emphasize that in (5.2) the integers j must be positive , otherwise positive the operator S_{j-1} is null. As a consequence , we get again by Bernstein lemma

$$\|\Delta_j v\|_{L^p} \leq C2^{-j}\|\Delta_j\omega\|_{L^p}.$$

Hence, we obtain

$$\|[\Delta_q, T_{\nabla\cdot}\cdot v]\omega\|_{L^p} \leq C\|\nabla v\|_{L^\infty}\sum_{|j-q|\leq N_0}\|\Delta_j\omega\|_{L^p} \tag{5.3}$$

Concerning the second term in the right side of 5.1, we have by definition of the paraproduct and the commutation property of the operators Δ_q

$$\begin{aligned}[\Delta_q, T_v\cdot\nabla]\omega &= \sum_{k=1}^{2}\sum_j[S_{j-1}v^k\partial_k\Delta_j,\, \Delta_q]\omega,\\ &= \sum_{k=1}^{2}\sum_j[S_{j-1}v^k\partial_k,\, \Delta_q]\Delta_j\omega\end{aligned}$$

The sum is finite and concerns only the integers j satisfying $|j-q|\leq N_0$. This is related to the localization of the Fourier transform of $S_{j-1}v^k\Delta_j\partial_k a$ in a ring of radius 2^j and the fact that $\Delta_q\Delta_j\equiv 0$, if $|j-q|\geq 2$.
To estimate each commutator, we write Δ_q as a convolution

$$[S_{j-1}v^k\partial_k,\, \Delta_q]\Delta_j\omega(\cdot) = 2^{qd}\int h(2^q(\cdot-y))\big(S_{j-1}v^k(\cdot)-S_{j-1}v^k(y)\big)\Delta_j\partial_k\omega(y)dy.$$

Thus, Young inequality yields to

$$\begin{aligned}\big\|[S_{j-1}v^k\partial_k,\, \Delta_q]\Delta_j\omega(\cdot)\big\|_{L^p} &\leq C2^{-q}\|\nabla S_{j-1}v\|_{L^\infty}\|\Delta_j\partial_k\omega\|_{L^p},\\ &\leq C\|\nabla v\|_{L^\infty}2^{j-q}\|\Delta_j\omega\|_{L^p}.\end{aligned} \tag{5.4}$$

Therefore, we obtain

$$\|[\Delta_q, T_v\cdot\nabla]\omega\|_{L^p} \leq C\|\nabla v\|_{L^\infty}\sum_{|j-q|\leq N_0}2^{-|j-q|}\|\Delta_j\omega\|_{L^p}. \tag{5.5}$$

Let us move to the remainder term. It can be written, in view of the definition, as

$$[\Delta_q, R(v,\nabla\cdot)]\omega = \sum_{\substack{j\geq -1\\ i\in\{\mp 1,0\}}} [\Delta_q, \Delta_j v\cdot\nabla]\Delta_{j+i}\omega = \sum_{\substack{j\geq q-N_0\\ i\in\{\mp 1,0\}}} [\Delta_q, \Delta_j v\cdot\nabla]\Delta_{j+i}\omega.$$

We decompose it as follows

$$[\Delta_q, R(v,\nabla\cdot)]\omega = R_q^1(v,a) + R_q^2(v,a), \text{ with}$$

$$\begin{aligned} R_q^1(v,\omega) &= \sum_{\substack{q-N_0\leq j\leq q\\ i\in\{\mp 1,0\}}} [\Delta_q, \Delta_j v\cdot\nabla]\Delta_{j+i}\omega \quad \text{and}\\ R_q^2(v,\omega) &= \sum_{\substack{q+1\leq j\\ i\in\{\mp 1,0\}}} [\Delta_q, \Delta_j v\cdot\nabla]\Delta_{j+i}\omega. \end{aligned}$$

For the first term $R_q^1(v,\omega)$, we imitate the inequality (5.4) and we find

$$\begin{aligned} \|R_q^1(v,\omega)\|_{L^p} &\leq C \sum_{\substack{q-N_0\leq j\leq q\\ i\in\{\mp 1,0\}}} 2^{-q}\|\nabla\Delta_j v\|_{L^\infty}\|\nabla\Delta_{j+1}\omega\|_{L^p}\\ &\leq C\|\nabla v\|_{L^\infty} \sum_{\substack{q-N_0\leq j\leq q\\ i\in\{\mp 1,0\}}} 2^{j-q}\|\Delta_{j+i}\omega\|_{L^p}\\ &\leq C\|\nabla v\|_{L^\infty} \sum_{\substack{q-N_0\leq j\\ i\in\{\mp 1,0\}}} 2^{q-j}\|\Delta_j\omega\|_{L^p}. \end{aligned}$$

Nevertheless, for the second term $R_q^2(v,\omega)$ we use the incompressibility of the flow which implies

$$R_q^2(v,\omega) = \sum_{k=1}^{2} \sum_{\substack{q+1\leq j\\ i\in\{\mp 1,0\}}} \partial_k[\Delta_q, \Delta_j v^k]\Delta_{j+i}\omega.$$

We observe that the Fourier transform of the commutator inside the sum is supported in a ball of radius 2^q. Then, the Bernstein lemma gives for all $(j\in\mathbb{N})$

$$\begin{aligned} \|\partial_k[\Delta_q, \Delta_j v^k]\Delta_{j+i}\omega\|_{L^p} &\leq C2^q\|\Delta_j v^k\|_{L^\infty}\|\Delta_{j+i}\omega\|_{L^p}\\ &\leq C\|\nabla v\|_{L^\infty}2^{q-j}\|\Delta_{j+i}\omega\|_{L^p}. \end{aligned}$$

Hence, we get

$$\|R_q^2(v,\omega)\|_{L^p} \leq C\|\nabla v\|_{L^\infty}\sum_{q+1\leq j} 2^{q-j}\|\Delta_j\omega\|_{L^p}.$$

This completes the proof of the Proposition 5.1. □

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