INCOMPRESSIBLE VISCOUS FLOWS IN BORDERLINE BESOV SPACES

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ABSTRACT. This paper is dedicated to the global persistence of the Besov regularity $B_{p,1}^{\frac{2}{p}+1}$ for the two-dimensional Navier-Stokes equations. We prove some uniform estimates, with respect to the viscosity, on the velocity in these spaces and we use it to a obtain a global result of inviscid limit.

1. INTRODUCTION

The equations of motion governing an incompressible viscous fluid with viscosity $\nu > 0$ are given by the so-called Navier-Stokes equations,

$$(NS_\nu)\begin{cases} \partial_t v_\nu + v_\nu \cdot \nabla v_\nu - \nu\Delta v_\nu = -\nabla \pi_\nu, \\ \operatorname{div} v_\nu = 0 \\ v_{\nu|t=0} = v^0, \end{cases}$$

the vector field $v_\nu(t,x) = (v_\nu^1, ..., v_\nu^d)(t,x)$ stands for the velocity of the fluid, the quantity π_ν denotes the scalar pressure, and div $v = 0$ means that the fluid is incompressible. When we neglect the diffusion term, then we obtain the Euler equations,

$$(E)\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla \pi \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0. \end{cases}$$

The mathematical study of the Navier-Stokes system was initiated by J. Leray in his pioneering work [10]. In fact, by using a compactness method, this author proved that for any divergence-free initial data v^0 in the energy space L^2, there exits a global solution to (NS_ν). In the case of *two* space dimension that weak solution was proved to be unique. However, for higher dimension $(d \geq 3)$ the problem of uniqueness is still a widely open problem. In 60', Fujita-Kato [6] exhibited when the initial data lies in the critical homogeneous Sobolev space $\dot{H}^{\frac{d}{2}-1}$ a class of unique local solutions called strong solutions. These local solutions are obtained, using some integrability properties of the solutions to the heat equation, by solving the equivalent integral equation by a standard Picard iteration method. We emphasize that the same result holds true when the initial data belongs to the inhomogeneous Sobolev space H^s, with $s \geq \frac{d}{2} - 1$. In general, the globality in time of these solutions is also an open problem. However a positive answer is given at least in both following cases: either when the initial data is small in the critical space $\dot{H}^{\frac{d}{2}-1}$ which is invariant

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under the scaling of the the Navier-Stokes equations, or in the space dimension two. Concerning Euler equations, the hyperbolic system theory ensures us the local in time existence and uniqueness in H^s, with $s > \frac{d}{2}+1$, (see [9]). In the critical case, we mention that we have the same result when we replace the space $H^{1+\frac{d}{2}}$ by the Besov space $B_{p,1}^{1+\frac{d}{p}}, p \in (1,\infty)$ which is contained in the class of Lipschitz functions (see [2, 5], for instance). More recently, M. I. Vishik has proved in [12] that the solution is globally well-posed in these critical Besov spaces when $d = 2$. He used for the proof a subtle logarithmic estimate based on the explicit form of the vorticity in dimension two.
The problem of the convergence of smooth viscous solutions $(v_\nu)_{\nu>0}$ to the Eulerian one as ν goes to zero is well understood. In [11], Majda showed that under the assumption $v^0 \in H^s$ with $s > \frac{d}{2}+2$, the solutions $(v_\nu)_{\nu>0}$ converge in L^2 norm as $\nu \to 0$ to the unique solution v of (E). The convergence rate is of order νt. We note that in dimension two the previous results are global in time and the proof is based heavily on Brezis-Gallouët logarithmic estimate and the boundedness of the vorticity.
This paper is the sequel to [8] in which we continue to study essentially two problems. The first one is related to the uniform persistence on vanishing viscosity of critical Besov regularity ($v^0 \in B_{p,1}^{2/p+1}$). However we deal in the second one with the inviscid limit for the two-dimensional Navier-Stokes system. In [8] we have given a positive answer to these problems when $p \in]1,2]$. The basic ingredient of the proof is a regularization effect of the vorticity equation

$$\partial_t \omega_\nu + v_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0; \quad \omega_\nu = \partial_1 v_\nu^2 - \partial_2 v_\nu^1.$$

We show essentially the following " linear estimate on the Lipschitz norm"

$$\nu \Big\| \int_0^t \omega_\nu(\tau) d\tau \Big\|_{C_*^2} \leq C \|\omega^0\|_{L^\infty} \Big(1 + \int_0^t \|\nabla v_\nu(\tau)\|_{L^\infty} d\tau \Big),$$

where C_*^2 stands for the Calderón-Zygmund space. We extend here these results for all $p > 2$ following an other approach. For this purpose, we establish a logarithmic estimate for Navier-Stokes system, similar to the one used by Vishik in [12] for the Eulerian case. The basic tool is Littlewood-Paley theory and some results from harmonic analysis. Let us note that the logarithmic estimate allows us to bound uniformly on ν the Lipschitz norm of the velocity, which is crucial for the persistence of the regularity. To give a better rate of convergence for the inviscid limit, we develop some new estimates related to the viscous vorticity. We state now our main theorem:

Theorem 1.1. *Let $p \in (2,\infty)$ and v^0 a divergence-free vector field belonging to the space $B_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)$. Then, the Navier-Stokes system (NS_ν) has a unique global solution in $\mathcal{C}(\mathbb{R}_+; B_{p,1}^{\frac{2}{p}+1})$, satisfying in addition the following uniform estimate*

$$\|v_\nu(t)\|_{B_{p,1}^{\frac{2}{p}+1}} \leq C_0 e^{\exp C_0 t}.$$

Moreover, v_ν goes to the Euler solution v as ν goes to zero. More precisely, we have for all $\nu \in (0,1]$

$$\|v_\nu(t) - v(t)\|_{L^p} \le C_0 e^{\exp C_0 t} (\nu t)^{\frac12 + \frac1p},$$

where C_0 is a constant independent of the viscosity and only depending on the initial data and p.

The rest of this paper is organized as follows. In § 2, we introduce the main notations related to the Littlewood-Paley theory and we give some useful lemmas. In § 3, we study some aspects for a transport-diffusion equation as the smoothing effect. The proof of the main result is given in § 4.

2. Preliminaries

In this paragraph we recall the definition of frequency localization operator and the Besov spaces $B^s_{p_1,p_2}$. Some useful lemmas are then given.

Proposition 2.1. *There exist two radially symmetric functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that*

i) $\chi(\xi) + \sum_{q\ge 0} \varphi(2^{-q}\xi) = 1, \quad \frac13 \le \chi^2(\xi) + \sum_{q\ge0} \varphi^2(2^{-q}\xi) \le 1,$

ii) $\mathrm{supp}\ \varphi(2^{-p}\cdot) \cap \mathrm{supp}\ \varphi(2^{-q}\cdot) = \varnothing,$ *if* $|p-q| \ge 2,$

iii) $q \ge 1 \Rightarrow \mathrm{supp}\chi \cap \mathrm{supp}\ \varphi(2^{-q}) = \varnothing.$

For every $v \in \mathcal{S}'$ we set

$$\Delta_{-1} v = \chi(\mathrm{D}) v \ ; \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D}) v \quad \text{and} \quad S_q = \sum_{-1\le p \le q-1} \Delta_p$$

The paradifferential calculus introduced by L J.-M. Bony [1] is based on the decomposition (called Bony's decomposition) which split the product uv into three parts:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q'-q|\le 1} \Delta_q u \Delta_{q'} v.$$

Let us now recall what Besov spaces are, through their characterizations via frequency localization. Let $(p_1,p_2) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ then the space $B^s_{p_1,p_2}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p_1,p_2}} := \Big(2^{qs} \|\Delta_q u\|_{L^{p_1}} \Big)_{\ell^{p_2}} < +\infty.$$

We point out that the Hölderian space C^s with $s \in \mathbb{R} - \mathbb{Z}$ is nothing but the Besov space $B^s_{\infty,\infty}$. When $s \in \mathbb{Z}$, the last space is denoted by C^s_* called Zygmund space. Let $T > 0$ and $r \ge 1$, we denote by $L^r_T B^s_{p_1,p_2}$ the space of all function a satisfying

$$\|u\|_{L^r_T B^s_{p_1,p_2}} := \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^{p_1}} \Big)_{\ell^{p_2}} \Big\|_{L^r_T} < \infty.$$

We say that a function u is an element of the space $\widetilde{L^r_T}B^s_{p_1,p_2}$ if

$$\|u\|_{\widetilde{L^r_T}B^s_{p_1,p_2}} := \Big(2^{qs}\|\Delta_q u\|_{L^r_T L^{p_1}}\Big)_{\ell^{p_2}} < +\infty.$$

The relation between these spaces is detailed by the following lemma, which is a direct consequence of the Minkowski inequalities.

Lemma 2.2. *Let $s\in\mathbb{R}$, $r\geq 1$ and $\big(p_1,p_2\big)\in[1,\infty]^2$. Then we have the following inclusions*

$$L^r_T B^s_{p_1,p_2} \hookrightarrow \widetilde{L^r_T}B^s_{p_1,p_2}, \text{ if } \quad r\leq p_2,$$

$$\widetilde{L^r_T}B^s_{p_1,p_2} \hookrightarrow L^r_T B^s_{p_1,p_2}, \text{ if } \quad r\geq p_2.$$

In the proof of Theorem 1.1, the following result of real interpolation is needed:

Lemma 2.3. *Let $T>0, 1\leq p_1,p_2,r\leq+\infty$ and $s_1<s<s_2$. Let $\theta\in(0,1)$ satisfy $s=\theta s_1+(1-\theta)s_2$. The following interpolation inequality holds true*

$$\|u\|_{\widetilde{L^r_T}B^s_{p_1,p_2}} \leq C\|u\|^{\theta}_{\widetilde{L^r_T}B^{s_1}_{p_1,\infty}}\|u\|^{1-\theta}_{\widetilde{L^r_T}B^{s_2}_{p_1,\infty}},$$

for some constant C depending only on r and θ.

We will also make continuous use of Bernstein lemma (see for example [3]).

Lemma 2.4. (Bernstein) *Let (r_1,r_2) a pair of strictly positive numbers such that $r_1<r_2$. There exists a constant C such that for every nonnegative integer k, for every $1\leq a\leq b$ and for all function $u\in L^a(\mathbb{R}^d)$, we have*

$$\text{supp }\widehat{u}\in B(0,\lambda\, r_1) \quad\Rightarrow\quad \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^k\,\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},$$

$$\text{supp }\widehat{u}\in \mathcal{C}(0,\lambda r_1,\lambda r_2) \quad\Rightarrow\quad C^{-k}\lambda^k\|u\|_{L^a}\leq \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^k\lambda^k\|u\|_{L^a}.$$

As a consequence, we have the following embeddings:

$$B^{\frac{d}{p}}_{p,1}\hookrightarrow B^{\frac{d}{q}}_{q,1}\hookrightarrow L^\infty;\ \forall 1\leq p\leq q.$$

The next lemma plays a crucial role in the proof of the regularization effect for the transport-diffusion equation.

Lemma 2.5. *Assume $d\geq 2$. There exists a positive constant C such that, for any function f taken in the Schwartz class, and for any diffeomorphism ψ of $\mathbb{R}^d$ preserving the Lebesgue measure, we have for all $p\in[1,+\infty]$ and for all $j,q\geq -1$,*

$$\|\Delta_j(\Delta_q f\circ\psi)\|_{L^p}\leq C2^{-|j-q|}\|\nabla\psi^{\eta(j,q)}\|_{L^\infty}\|\Delta_q f\|_{L^p},$$

with

$$\eta(j,q)=sign(j-q).$$

Now, we recall the following commutator lemma.

Lemma 2.6. *Let v a divergence-free vector field of $\mathbb{R}^d$ which is lipschitzian, and $a \in B^s_{p_1,p_2}$, such that $(s, p_1, p_2) \in]-1, 1[\times[1, +\infty]^2$. Then, there exists a constant C depending only on s and d, such that*

$$\big\|[\Delta_q, v\cdot\nabla]a\big\|_{L^{p_1}} \leq C2^{-qs}c_q\|\nabla v\|_{L^\infty}\|a\|_{B^s_{p_1,p_2}},$$

with

$$\sum_{q\geq -1} c_q^{p_2} = 1.$$

We describe by the following lemma the regularization effect of the heat operator (for more details, see [4]).

Lemma 2.7. *Let $\mathcal{C}$ be a given ring. There exist two constants c and C, such that for any pair of positive numbers (t, λ), for every $p \in [1, +\infty]$ and for all function $a \in L^p$, we have :*

$$\text{Supp }\widehat{a} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}a\|_{L^p} \leq Ce^{-ct\lambda^2}\|a\|_{L^p}.$$

We recall now a lemma dealing with the persistence of the Besov regularity in the transport equation, which is proved in [3].

Lemma 2.8. *Let $s \in (-1, 1)$, $(p_1, p_2) \in [1,\infty]^2$, $r \in [1,\infty]$ and v a divergence-free vector field belonging to $L^1_{loc}(\mathbb{R}_+; \text{Lip}(\mathbb{R}^d))$. Let a be a solution of*

$$\partial_t a + v\cdot\nabla a = f.$$

Then the following estimate holds true:

$$\|a(t)\|_{B^s_{p_1,p_2}} \leq Ce^{CV(t)}\Big(\|a(0)\|_{B^s_{p_1,p_2}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p_1,p_2}}d\tau\Big),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau.$$

3. Some a priori estimates for transport-diffusion equations

The object of this section is to derive some Besov estimates related to the transport-diffusion equation. The difficult concerns the Hölderian one which appears as a limit case. This problem has been discussed in [7] and what we will give here is not than a bit generalization. We need for this purpose the following lemma proved in [7].

Lemma 3.1. *Let $p \in [1, +\infty]$ and v a divergence-free vector field belonging to $L^1_{loc}(\mathbb{R}_+; \text{Lip}(\mathbb{R}^d))$. Giving two functions $f \in L^1_{loc}(\mathbb{R}_+; L^p)$ and $a^0 \in L^p$. Then, for any solution a of the problem*

$$\partial_t a + v\cdot\nabla a - \nu\Delta a = f;\ a(0) = a^0,$$

we get for all positive number t:

$$\|a(t)\|_{L^p} \leq \|a^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p}d\tau.$$

We give now the main result of this section.

Proposition 3.2. *Let $s\in(-1,1)$, $(p_1,p_2)\in[1,\infty]^2$, $r\in[1,\infty]$ and v a divergence-free vector field belonging to $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip}(\mathbb{R}^d))$. Let a be a solution of*

$$\partial_t a+v\cdot\nabla a-\nu\Delta a=0.$$

Then the following estimate holds true:

$$\nu^{\frac{1}{r}}\|a\|_{\widetilde{L^r_t}B^{s+\frac{2}{r}}_{p_1,p_2}}\le Ce^{CV(t)}\big(1+(\nu t)^{\frac{1}{r}}\big)\|a(0)\|_{B^s_{p_1,p_2}},$$

with $V(t)=\displaystyle\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau$. The constant C depends only on s and d.

Proof. The proof proceeds in two steps. First, we prove the persistence regularity in Besov spaces (case $r=+\infty$). Second, we use this result to establish the regularization effect when r is finite. The function $a_q:=\Delta_q a$ is solution of

$$\partial_t a_q+S_{q-1}v\cdot\nabla a_q-\nu\Delta a_q=(S_{q-1}-\mathrm{Id})v\cdot\nabla a_q-[\Delta_q,v\cdot\nabla]a:=f_q.$$

By Lemma 3.1, we obtain

$$\|a_q(t)\|_{L^\infty_t L^{p_1}}\le\|a_q(0)\|_{L^{p_1}}+\int_0^t\|f_q(\tau)\|_{L^{p_1}}d\tau.$$

To estimate f_q we write

$$\|f_q\|_{L^{p_1}}\le C\|\nabla v\|_{L^\infty}\|a_q\|_{L^{p_1}}+\big\|[\Delta_q,v\cdot\nabla]a\big\|_{L^{p_1}}.$$

Combined with the commutator estimate in Lemma 2.6 this yields

$$\|a_q\|_{L^\infty_t L^{p_1}}\le\|a_q(0)\|_{L^{p_1}}+C2^{-qs}\int_0^t c_q(\tau)\|\nabla v(\tau)\|_{L^\infty}\|a(\tau)\|_{B^s_{p_1,p_2}}d\tau,$$

with $\sum_q c_q^{p_2}=1$. Multiplying by 2^{qs} and taking the ℓ^{p_2} norm we obtain, via Minkowski inequality,

$$\|a\|_{\widetilde{L^\infty_t}B^s_{p_1,p_2}}\le\|a(0)\|_{B^s_{p_1,p_2}}+C\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|a(\tau)\|_{B^s_{p_1,p_2}}d\tau.\tag{3.1}$$

We can conclude now by the embedding detailed in Lemma 2.2 combined with Gronwall's inequality. Now, let us move to the proof of the case of r finite. We set

$$\bar a_q(t)=a_q(t,\psi_q(t))\quad\text{and}\quad\bar f_q(t)=f_q(t,\psi_q(t)),$$

where ψ_q is the flow of the regularized velocity $S_{q-1}v$

$$\psi_q(t,x)=x+\int_0^t S_{q-1}v\big(\tau,\psi_q(\tau,x)\big)d\tau.$$

A simple computation gives

$$\Delta a_q(t)\circ\psi_q(t)=\sum_{i=1}^d\Big\langle\nabla^2\bar a_q(t)\cdot(\partial^i\psi_q^{-1})(t,\psi_q(t)),(\partial^i\psi_q^{-1})(t,\psi_q(t))\Big\rangle+$$
$$+\nabla\bar a_q(t)\cdot(\Delta\psi_q^{-1})(t,\psi_q(t)).$$

Therefore, the function $\bar{a}_q$ satisfies the equation

$$\begin{aligned}(\partial_t - \nu\Delta)\bar{a}_q(t) &= \nu\sum_{i=1}^{d}\langle\nabla^2\bar{a}_q(t)g_q^i, g_q^i(t)\rangle + 2\nu\sum_{i=1}^{d}\langle\nabla^2\bar{a}_q(t)e_i, g_q^i(t)\rangle \\ &+ \nu\nabla\bar{a}_q(t)\cdot(\Delta\psi_q^{-1})(t,\psi_q(t)) + \bar{f}_q(t) := \mathcal{R}_q(t) \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}, \qquad (3.2)\end{aligned}$$

where the function g_q is defined, for all $1 \leq i \leq d$, by

$$(\partial^i\psi_q^{-1})(t,\psi_q(t,x)) = e_i + g_q^i(t,x),$$

and $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$. By an obvious computation, we get

$$\|g_q(t)\|_{L^\infty} \leq CV(t)e^{CV(t)}.$$

Now, we will again apply the operator Δ_j to the new equation (3.2). So by the Duhamel formulation, we obtain

$$\Delta_j\bar{a}_q(t,x) = e^{\nu t\Delta}\Delta_j a_q(0) + \int_0^t e^{\nu(t-\tau)\Delta}\Delta_j\mathcal{R}_q(\tau,x)d\tau.$$

The Lemma 2.7 combined with the fact that the operator Δ_j maps L^{p_1} to itself uniformly in j, allows us

$$\begin{aligned}\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{I}+\mathrm{II})(\tau)\|_{L^{p_1}} &\leq C\nu e^{-c(t-\tau)2^{2j}}\|\mathrm{I}+\mathrm{II}\|_{L^{p_1}} \\ &\leq CV(\tau)e^{CV(\tau)}\nu e^{-c(t-\tau)2^{2j}}\|\nabla^2\bar{a}_q(\tau)\|_{L^{p_1}}\end{aligned}$$

In the other hand, we obtain by virtue of Leibniz rule and Bernstein lemma,

$$\|\nabla^2\bar{a}_q(\tau)\|_{L^{p_1}} \leq C2^{2q}e^{CV(t)}\|a_q(\tau)\|_{L^{p_1}}. \qquad (3.3)$$

Thus we infer

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{I}+\mathrm{II})(\tau)\|_{L^{p_1}} \leq CV(\tau)e^{CV(\tau)}\nu 2^{2q}e^{-c\nu(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^{p_1}}. \qquad (3.4)$$

We turn to the third term. We shall use these estimates

$$\|\nabla^2\psi_q^{\mp1}(t)\|_{L^\infty} \leq C2^qV(t)e^{CV(t)}; \; \|\nabla\bar{a}_q(\tau)\|_{L^{p_1}} \leq C2^qe^{CV(t)}\|a_q(\tau)\|_{L^{p_1}}.$$

The first one is proved in a classical manner by using the equation of the flow combined with Gronwall and Bernstein lemmas. Nevertheless, the second one can be proved in the same way as (3.3). Hence, we find

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{III})(\tau)\|_{L^{p_1}} \leq CV(\tau)e^{CV(\tau)}\nu 2^{2q}e^{-c\nu(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^{p_1}}. \qquad (3.5)$$

We are going to estimate the fourth term. We deduce from Lemma 2.7

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{IV})(\tau)\|_{L^{p_1}} \leq Ce^{-c\nu(t-\tau)2^{2j}}\|\mathrm{IV}(\tau)\|_{L^{p_1}}.$$

Thus, Lemma 2.6 yields

$$\|e^{\nu(t-\tau)\Delta}\Delta_j(\mathrm{IV})(\tau)\|_{L^{p_1}} \leq C2^{-qs}c_q(\tau)e^{-c\nu(t-\tau)2^{2j}}\|v(\tau)\|_{\mathrm{Lip}}\|a(\tau)\|_{B^s_{p_1,p_2}}, \qquad (3.6)$$

such that $\sum_q c_q^{p_2}=1$. Combining (3.2), (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}\|\Delta_j\bar a_q(t)\|_{L^{p_1}} &\leq Ce^{-\nu t2^{2j}}\|\Delta_j a_q(0)\|_{L^{p_1}}+\\ &+ CV(t)e^{CV(t)}\nu 2^{2q}\int_0^t e^{-c\nu(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^{p_1}}d\tau\\ &+ C2^{-qs}\int_0^t e^{-c\nu(t-\tau)2^{2j}}c_q(\tau)\|v(\tau)\|_{\mathrm{Lip}}\|a(\tau)\|_{B^s_{p_1,p_2}}d\tau. \end{aligned}\tag{3.7}$$

Thus, the Young lemma implies, for all $j\in\mathbb{N}$

$$\begin{aligned}\|\Delta_j\bar a_q\|_{L^r_tL^{p_1}} &\leq C(\nu 2^{2j})^{\frac{-1}{r}}\|\Delta_j a_q(0)\|_{L^{p_1}}+CV(t)e^{CV(t)}2^{2(q-j)}\|a_q\|_{L^r_tL^{p_1}}\\ &+ C2^{-qs}(\nu 2^{2j})^{\frac{-1}{r}}\|a\|_{\widetilde{L^\infty_t}B^s_{p_1,p_2}}\int_0^t c_q(\tau)\|v(\tau)\|_{\mathrm{Lip}}d\tau.\end{aligned}\tag{3.8}$$

Taking an integer N, which will be fixed later. As the flow preserves the Lebesgue measure, then we show

$$\begin{aligned}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}} &= \nu^{\frac1r}2^{q(s+\frac2r)}\|\bar a_q\|_{L^r_tL^{p_1}}\\ &\leq \nu^{\frac1r}2^{q(s+\frac2r)}\Big(\sum_{|j-q|\leq N}\|\Delta_j\bar a_q\|_{L^r_tL^{p_1}}+\sum_{|j-q|>N}\|\Delta_j\bar a_q\|_{L^r_tL^{p_1}}\Big).\end{aligned}$$

Using this inequality together with (3.8), we can easily show that for all $q\geq N$

$$\begin{aligned}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}} &\leq C2^{qs}\|a_q(0)\|_{L^{p_1}}+CV(t)e^{CV(t)}2^{2N}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}}\\ &+ C2^{\frac{2N}{r}}\|a\|_{\widetilde{L^\infty_t}B^s_{p_1,p_2}}\int_0^t c_q(\tau)\|v(\tau)\|_{\mathrm{Lip}}d\tau+\\ &+ \nu^{\frac1r}2^{q(s+\frac2r)}\sum_{|j-q|>N}\|\Delta_j\bar a_q\|_{L^r_tL^{p_1}}.\end{aligned}$$

In the other hand, we have by virtue of Lemma 2.5,

$$\|\Delta_j\bar a_q(t)\|_{L^p}\leq C2^{-|q-j|}e^{CV(t)}\|a_q(t)\|_{L^p}.$$

Thus, we infer

$$\begin{aligned}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}} &\leq C2^{qs}\|a_q(0)\|_{L^{p_1}}+CV(t)e^{CV(t)}2^{2N}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}}\\ &+ C2^{\frac{2N}{r}}\|a\|_{\widetilde{L^\infty_t}B^s_{p_1,p_2}}\int_0^t c_q(\tau)\|v(\tau)\|_{\mathrm{Lip}}d\tau+\\ &+ C2^{-N}e^{CV(t)}\nu^{\frac1r}2^{q(s+\frac2r)}\|a_q\|_{L^r_tL^{p_1}}.\end{aligned}$$

For low frequencies, $q\leq N$, we write by Hölder inequality

$$\nu^{\frac1r}2^{q(s+\frac1r)}\|a_q\|_{L^r_tL^{p_1}}\leq(\nu t)^{\frac1r}2^{\frac Nr}\|a_q\|_{L^\infty_tL^{p_1}}.$$

Taking th ℓ^{p_2} norm an using the Minkowski inequality,

$$\begin{aligned}\nu^{\frac1r}\|a\|_{\widetilde{L^r_t}B^{s+\frac2r}_{p_1,p_2}} &\leq C\|a(0)\|_{B^s_{p_1,p_2}}+CV(t)e^{CV(t)}2^{2N}\nu^{\frac1r}\|a\|_{\widetilde{L^r_t}B^{s+\frac2r}_{p_1,p_2}}\\ &+ C2^{-N}e^{CV(t)}\nu^{\frac1r}\|a\|_{\widetilde{L^r_t}B^{s+\frac2r}_{p_1,p_2}}+C2^{\frac Nr}\big(V(t)+(\nu t)^{\frac1r}\big)\|a\|_{\widetilde{L^\infty_t}B^s_{p_1,p_2}}.\end{aligned}$$

Choosing N and t such that

$$CV(t)e^{CV(t)}2^{2N}+Ce^{CV(t)}2^{-N}\leq\frac{1}{4}.$$

This is possible for small time t such that

$$V(t)\leq C_1,$$

where C_1 is a small absolute constant. Under this restriction, one obtains

$$\nu^{\frac{1}{r}}\|a\|_{\widetilde{L_t^r}B^{s+\frac{2}{r}}_{p_1,p_2}}\leq C\|a(0)\|_{B^s_{p_1,p_2}}+C\big(1+(\nu t)^{\frac{1}{r}}\big)\|a\|_{\widetilde{L_t^\infty}B^s_{p_1,p_2}}.$$

In the other hand, we can deduce from (3.1) that

$$\|a\|_{\widetilde{L_t^\infty}B^s_{p_1,p_2}}\leq Ce^{CV(t)}\|a(0)\|_{B^s_{p_1,p_2}}.$$

Hence, if $V(t)\leq C_1$, then

$$\nu^{\frac{1}{r}}\|a\|_{\widetilde{L_t^r}B^{s+\frac{2}{r}}_{p_1,p_2}}.\leq C\big(1+(\nu t)^{\frac{1}{r}}\big)\|a(0)\|_{B^s_{p_1,p_2}}.$$

We emphasize that this local result is only related to the Lipschitz norm of the velocity and to the growth of the quantity $\|a(t)\|_{B^s_{p_1,p_2}}$. However the first part of the proof tells us that the persistence regularity occurs globally in time. So we can make an iterative procedure leading to the required estimate. □

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 which is divided in two parts. In the first one which is the most difficult we show the persistence of the Besov regularity to the Navier-Stokes solution with a uniform bound on the viscosity. The principal ingredient of the proof is a new logarithmic estimate governing the viscous vorticity. In the second part, we prove the inviscid limit based essentially on the Proposition 3.2.

Let us now state the logarithmic estimate in more general case.

Proposition 4.1. *Let v a divergence-free vector field belonging to the space $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip}(\mathbb{R}^d))$ and a a solution to the problem*

$$\begin{cases}\partial_t a+v\cdot\nabla a-\nu\Delta a=0\\ a_{|t=0}=a^0.\end{cases}$$

If the initial data $a^0\in B^0_{\infty,1}$, then we have for all $t\in\mathbb{R}_+$

$$\|a(t)\|_{B^0_{\infty,1}}\leq C\|a^0\|_{B^0_{\infty,1}}\Big(1+\int_0^t\|v(\tau)\|_{\mathrm{Lip}}d\tau\Big).$$

Proof. Let $\tilde{a}_q$ be the unique global solution of the initial value problem:

$$\begin{cases}\partial_t\tilde{a}_q+v\cdot\nabla\tilde{a}_q-\nu\Delta\tilde{a}_q=0\\ \tilde{a}_q(0)=\Delta_q a^0.\end{cases}$$

By the Proposition 3.2 we have for all $0\leq\epsilon<1$ both estimates:

$$\|\tilde{a}_q(t)\|_{C^{\mp\epsilon}}\leq C\|\tilde{a}_q(0)\|_{C^{\mp\epsilon}}\exp\Big(C_\epsilon\int_0^t\|v(\tau)\|_{\mathrm{Lip}}d\tau\Big).$$

They imply in view of the definition of Hölder space that for all $j \geq -1$

$$\|\Delta_j \tilde{a}_q(t)\|_{L^\infty} \leq C2^{-\epsilon|j-q|}\|\tilde{a}_q(0)\|_{L^\infty} \exp\Big(C_\epsilon \int_0^t \|v(\tau)\|_{\text{Lip}} d\tau\Big). \tag{4.1}$$

Now by linearity one can write

$$a(t,x) = \sum_{q\geq -1} \tilde{a}_q(t,x).$$

Given an integer N, which will be carefully chosen later. So by applying the Littlewood-Paley operator Δ_j for both sides and after a simple computation, we get

$$\begin{aligned} \|a(t)\|_{B^0_{\infty,1}} &\leq \sum_{j,q\geq -1} \|\Delta_j \tilde{a}_q(t)\|_{L^\infty} \\ &\leq \sum_{|j-q|\geq N} \|\Delta_j \tilde{a}_q(t)\|_{L^\infty} + \sum_{|j-q|<N} \|\Delta_j \tilde{a}_q(t)\|_{L^\infty} \end{aligned} \tag{4.2}$$

Thus, by the estimate (4.1) we obtain

$$\sum_{|j-q|\geq N} \|\Delta_j \tilde{a}_q(t)\|_{L^\infty} \leq C2^{-\epsilon N}\|a^0\|_{B^0_{\infty,1}} \exp\Big(C_\epsilon \int_0^t \|v(\tau)\|_{\text{Lip}} d\tau\Big). \tag{4.3}$$

For the second term of the right side of (4.2), we use two facts: the first one is that the operator Δ_j maps uniformly L^∞ into itself while the second is the use of maximum principle. So we get

$$\begin{aligned} \sum_{|j-q|<N} \|\Delta_j \tilde{a}_q(t)\|_{L^\infty} &\leq C \sum_{|j-q|<N} \|\tilde{a}_q(t)\|_{L^\infty} \\ &\leq C \sum_{|j-q|<N} \|\Delta_q a^0\|_{L^\infty} \\ &\leq CN\|a^0\|_{B^0_{\infty,1}}. \end{aligned} \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we have

$$\|a(t)\|_{B^0_{\infty,1}} \leq C2^{-\epsilon N}\|a^0\|_{B^0_{\infty,1}} \exp\Big(C_\epsilon \int_0^t \|v(\tau)\|_{\text{Lip}} d\tau\Big) + CN\|a^0\|_{B^0_{\infty,1}}.$$

Choosing

$$N = \Big[\frac{C_\epsilon V(t)}{\epsilon \log 2} + 1\Big],$$

then, the above inequality completes the proof of the proposition. □

As an application to the two-dimensional Navier-Stokes system, we obtain the following result.

Corollary 4.2. *Let $p > 2$ and v^0 a divergence-free vector field belonging to the space $B^{\frac{2}{p}+1}_{p,1}(\mathbb{R}^2)$. Then the Navier-Stokes solution v_ν corresponding to the initial data v^0 satisfies the following estimate.*

$$\forall t \in \mathbb{R}_+,\ \|\nabla v_\nu(t)\|_{L^\infty} \leq C_0 e^{C_0 t},$$

where C_0 is an absolute constant, independent of the viscosity.

Proof. For the sake of simplicity we will omit the index ν from the velocity. Using the dyadic decomposition and the Bernstein inequality one gets

$$\|\nabla v(t)\|_{L^\infty} \leq \|\Delta_{-1} v(t)\|_{L^\infty} + \sum_{q\geq 0} \|\Delta_q \nabla v(t)\|_{L^\infty}.$$

In the other hand, we have for the high frequencies $q \in \mathbb{N}$,

$$\|\Delta_q \nabla v(t)\|_{L^\infty} \leq C\|\Delta_q \omega(t)\|_{L^\infty}.$$

It follows that

$$\|\nabla v(t)\|_{L^\infty} \leq \|\Delta_{-1} v(t)\|_{L^\infty} + C\|\omega(t)\|_{B^0_{\infty,1}}.$$

From Proposition 4.1, we infer that

$$\|\nabla v(t)\|_{L^\infty} \leq \|\Delta_{-1} v(t)\|_{L^\infty} + C\|\omega^0\|_{B^0_{\infty,1}}\Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big). \tag{4.5}$$

Next, we move to the case of low frequencies. We can write thanks to Bernstein inequality.

$$\|\Delta_{-1} v(t)\|_{L^\infty} \leq C\|v(t)\|_{L^p}.$$

Applying Lemma 3.1 gives

$$\|v(t)\|_{L^p} \leq \|v^0\|_{L^p} + \int_0^t \|\nabla \pi(\tau)\|_{L^p} d\tau. \tag{4.6}$$

On the other hand, we obtain by the Calderón-Zygmund inequality

$$\begin{aligned} \|\nabla \pi\|_{L^p} &\leq \sum_{j=1}^{j=2} \|\nabla \partial_j \Delta^{-1}\big(v \cdot \nabla v_j\big)\|_{L^p} \\ &\leq C\|v\|_{L^\infty}\|\nabla v\|_{L^p} \leq C\|v\|_{L^\infty}\|\omega\|_{L^p}. \end{aligned}$$

We infer from the L^p estimate of the vorticity that

$$\|\nabla \pi(t)\|_{L^p} \leq C\|v(t)\|_{L^\infty}\|\omega^0\|_{L^p}. \tag{4.7}$$

Let us now bound $\|v(t)\|_{L^\infty}$. Since

$$\|v\|_{L^\infty} \leq \|\Delta_{-1} v\|_{L^\infty} + \sum_{j\in\mathbb{N}} \|\Delta_j v\|_{L^\infty},$$

we get in view of Bernstein lemma,

$$\begin{aligned} \|v\|_{L^\infty} &\leq \|\Delta_{-1} v\|_{L^p} + \sum_{j\in\mathbb{N}} 2^{j(-1+\frac{2}{p})}\|\Delta_j \omega\|_{L^p} \\ &\leq C\|v\|_{L^p} + C\|\omega\|_{L^p} \leq C\|v\|_{L^p} + C\|\omega^0\|_{L^p}. \end{aligned}$$

In the last estimate we have used the boundness of the L^p norm of the vorticity by its initial value. Hence, we deduce from (4.7)

$$\|\nabla \pi(t)\|_{L^p} \leq C\|\omega^0\|^2_{L^p} + C\|\omega^0\|_{L^p}\|v\|_{L^p}.$$

Adding this estimate to (4.6), we obtain

$$\|v(t)\|_{L^p} \leq \|v^0\|_{L^p} + C\|\omega^0\|_{L^p}^2\, t + C\|\omega^0\|_{L^p} \int_0^t \|v(\tau)\|_{L^p} d\tau.$$

According to the Gronwall's inequality, we find

$$\|v(t)\|_{L^p} \leq C_0 e^{C_0 t}, \tag{4.8}$$

where C_0 is a constant depending on the initial data.
It follows from (4.5) that

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t} + C\|\omega^0\|_{B^0_{\infty,1}} \Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big).$$

Again, from Gronwall's inequality we obtain the required estimate stated in the corollary. □

To achieve the proof of the first part of the Theorem 1.1, we apply the Proposition 3.2 to the viscous vorticity ω_ν. So, we obtain in view of the Corollary 4.2

$$\begin{aligned} \|\omega_\nu(t)\|_{B^{\frac{2}{p}}_{p,1}} \leq \|\omega_\nu\|_{\widetilde{L^\infty_t} B^{\frac{2}{p}}_{p,1}} &\leq C\|\omega^0\|_{B^{\frac{2}{p}}_{p,1}} e^{C\int_0^t \|\nabla v_\nu(\tau)\|_{L^\infty} d\tau} \\ &\leq C_0 e^{\exp C_0 t}. \end{aligned} \tag{4.9}$$

Finally, to estimate the velocity, we use the classical estimate:

$$\|v_\nu\|_{B^{\frac{2}{p}+1}_{p,1}} \leq C\|v_\nu(t)\|_{L^p} + C\|\omega_\nu(t)\|_{B^{\frac{2}{p}}_{p,1}}.$$

Therefore, we conclude by applying the estimates (4.9) and (4.8).
Let us now move to the proof of the inviscid limit. The following proposition is more precise than the result announced in Theorem 1.1: we can replace the Lebesgue space L^p by the subspace $B^0_{p,1}$.

Proposition 4.3. *Let $p > 2$ and v^0 a divergence-free vector field belonging to $B^{\frac{2}{p}+1}_{p,1}$. Then the Navier-Stokes solution and the Euler one satisfy, globally in time,*

$$\|(v_\nu - v)(t)\|_{B^0_{p,1}} \leq C_0 e^{\exp C_0 t} (\nu t)^{\frac{1}{2}+\frac{1}{p}} (1 + \nu t)^{\frac{1}{2}-\frac{1}{p}}.$$

Proof. Let us define the functions

$$\bar{v}_\nu = v_\nu - v \quad \text{and} \quad \bar{\pi}_\nu = \pi_\nu - \pi.$$

So combining (NS_ν) and Euler systems gives

$$\widetilde{(NS_\nu)} \begin{cases} \partial_t \bar{v}_\nu + v_\nu \cdot \nabla \bar{v}_\nu = \nu \Delta v_\nu - \bar{v}_\nu \cdot \nabla v - \nabla \bar{\pi}_\nu \\ \bar{v}_\nu(0) = 0. \end{cases}$$

Hence, we get, in view of Lemma 2.8,

$$\|\bar{v}_\nu(t)\|_{B^0_{p,1}} \leq C e^{CV_\nu(t)} \int_0^t e^{-CV_\nu(\tau)} \Big(\nu\|\Delta v_\nu\|_{B^0_{p,1}} + \|\bar{v}_\nu \cdot \nabla v\|_{B^0_{p,1}} + \|\nabla \bar{\pi}_\nu\|_{B^0_{p,1}}\Big) d\tau. \tag{4.10}$$

To estimate the first member of the right side we write by the Lemma 2.3

$$\begin{aligned}\|\Delta v_\nu\|_{L^1_t B^0_{p,1}} &\leq \|\omega_\nu\|_{\widetilde{L^1_t} B^1_{p,1}}\\ &\leq \|\omega_\nu\|^{\frac12+\frac1p}_{\widetilde{L^1_t} B^{\frac2p}_{p,\infty}} \|\omega_\nu\|^{\frac12-\frac1p}_{\widetilde{L^1_t} B^{2+\frac2p}_{p,\infty}}.\end{aligned}$$

In one hand, we have by Hölder inequality and the estimate (4.9)

$$\begin{aligned}\|\omega_\nu\|^{\frac12+\frac1p}_{\widetilde{L^1_t} B^{\frac2p}_{p,\infty}} &\leq t^{\frac12+\frac1p}\|\omega_\nu\|^{\frac12+\frac1p}_{\widetilde{L^\infty_t} B^{\frac2p}_{p,1}}\\ &\leq C_0 t^{\frac12+\frac1p} e^{\exp C_0 t}.\end{aligned}$$

In the other hand, the Proposition 3.2 allows us,

$$\begin{aligned}\|\omega_\nu\|_{\widetilde{L^1_t} B^{2+\frac2p}_{p,\infty}} &\leq \nu^{-1} e^{CV_\nu(t)}(1+\nu t)\|\omega^0\|_{B^{\frac2p}_{p,1}}\\ &\leq C_0 e^{\exp C_0 t}\nu^{-1}(1+\nu t).\end{aligned}$$

Thus, we get

$$\nu\|\Delta v_\nu\|_{L^1_t B^0_{p,1}} \leq C_0 e^{\exp C_0 t}(\nu t)^{\frac12+\frac1p}(1+\nu t)^{\frac12-\frac1p}. \tag{4.11}$$

For the second term in the right member of (4.10), we use the following lemma. Its proof will be discussed at the end of the section.

Lemma 4.4. *Let v a divergence-free vector field belonging to $B^0_{p,1}$ and f a function of $B^1_{p,1}$. Then,*

$$\|v\cdot\nabla f\|_{B^0_{p,1}} \leq C\|v\|_{B^0_{p,1}}\|f\|_{B^1_{\infty,1}}.$$

Applying this lemma, gives

$$\|\bar v_\nu\cdot\nabla v(t)\|_{B^0_{p,1}} \leq \|\bar v_\nu(t)\|_{B^0_{p,1}}\|v(t)\|_{B^1_{\infty,1}}.$$

The chain rule in Besov spaces tells us that

$$\|v(t)\|_{B^1_{\infty,1}} \leq C\|v(t)\|_{B^{1+\frac2p}_{p,1}}.$$

So by the uniform estimates developed in the first part of Theorem 1.1, we obtain

$$\|\bar v_\nu\cdot\nabla v(t)\|_{B^0_{p,1}} \leq C_0 e^{\exp C_0 t}\|\bar v_\nu(t)\|_{B^0_{p,1}}. \tag{4.12}$$

For the pressure we use the following identity based on the fact $\mathrm{div}\, v = \mathrm{div}\,\bar v_\nu = 0$

$$\mathrm{div}\big(v\cdot\nabla\bar v_\nu\big) = \mathrm{div}\big(\bar v_\nu\cdot\nabla v\big).$$

Applying the divergence operator to the system $\widetilde{(NS_\nu)}$, gives

$$\Delta\bar\pi_\nu = \mathrm{div}(\bar v_\nu\cdot\nabla v_\nu + v\cdot\nabla\bar v_\nu).$$

Thus, we have

$$-\Delta\bar\pi_\nu = \mathrm{div}\big(\bar v_\nu\cdot\nabla(v_\nu+v)\big).$$

This yields

$$\|\nabla\bar\pi_\nu\|_{B^0_{p,1}} = \|(-\Delta)^{-1}\nabla\mathrm{div}\big(\bar v_\nu\cdot\nabla(v_\nu+v)\big)\|_{B^0_{p,1}}.$$

Hence, we deduce by Calderón-Zygmund inequality that

$$\|\nabla\bar{\pi}_\nu\|_{B^0_{p,1}} \leq \|\bar{v}_\nu\cdot\nabla(v_\nu+v)\|_{B^0_{p,1}}.$$

Then we conclude by Lemma 4.4 that

$$\begin{aligned}\|\nabla\bar{\pi}_\nu(t)\|_{B^0_{p,1}} &\leq C\|(v_\nu+v)(t)\|_{B^1_{\infty,1}}\|\bar{v}_\nu(t)\|_{B^0_{p,1}}\\ &\leq C_0 e^{\exp C_0 t}\|\bar{v}_\nu(t)\|_{B^0_{p,1}}.\end{aligned}\tag{4.13}$$

Plugging (4.13), (4.12) and (4.11) into (4.10), gives

$$\|\bar{v}_\nu(t)\|_{B^0_{p,1}} \leq C_0 e^{\exp C_0 t}\Big((\nu t)^{\frac12+\frac1p}(1+\nu t)^{\frac12-\frac1p}+\int_0^t e^{-\exp C_0\tau}\|\bar{v}_\nu(\tau)\|_{B^0_{p,1}}d\tau\Big).$$

To complete the proof of Proposition 4.1, we use Gronwall's inequality. □

We shall now give the proof of Lemma 4.4 which is based on Bony's decomposition. We write

$$v\cdot\nabla f = T_v\cdot\nabla f + T_{\nabla f}\cdot v + R(v,\nabla f).$$

By definition of the paraproduct, we have

$$\begin{aligned}\|T_v\cdot\nabla f\|_{B^0_{p,1}} &\leq C\sum_{q\in\mathbb{N}}\|S_{q-1}v\|_{L^p}\|\Delta_q\nabla f\|_{L^\infty}\\ &\leq C\|v\|_{B^0_{p,1}}\sum_{q\in\mathbb{N}}2^q\|\Delta_q f\|_{L^\infty}\\ &\leq C\|v\|_{B^0_{p,1}}\|f\|_{B^1_{\infty,1}}.\end{aligned}$$

By the same way, we show that

$$\|T_{\nabla f}\cdot v\|_{B^0_{p,1}} \leq C\|v\|_{B^0_{p,1}}\|f\|_{B^1_{\infty,1}}.$$

To estimate the last term, we use the incompressibility hypothesis

$$R(v,\nabla f) = \mathrm{div}\ R(v,f).$$

So we have by definition of the remainder R

$$\|\mathrm{div}R(v,f)\|_{B^0_{p,1}} \leq \sum_{q\geq -1}\big\|\mathrm{div}\Delta_q\sum_{j\geq -1 i\in\{0,1\}}\Delta_j v\Delta_{j\mp i}f\big\|_{L^p}.$$

Using the fact that the Fourier transform of $\Delta_j v\Delta_{j\mp i}f$ is supported in a ball of radius 2^j, so we get, in view of Bernstein lemma

$$\begin{aligned}\|\mathrm{div}R(v,f)\|_{B^0_{p,1}} &\leq C\sum_{q\geq -1}2^q\sum_{j\geq q-M; i\in\{0,1\}}\|\Delta_j v\|_{L^p}\|\Delta_{j\mp i}f\|_{L^\infty}\\ &\leq C\|f\|_{B^1_{\infty,1}}\sum_{q\geq -1}2^q\sum_{j\geq q-M}2^{-j}\|\Delta_j v\|_{L^p}.\end{aligned}$$

The classical property of convolution between ℓ^1 and itself gives the result.

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