ON THE GLOBAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL BOUSSINESQ SYSTEM WITH A ZERO DIFFUSIVITY

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Abstract. In this paper we prove the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity, for rough initial data.

1. Introduction

We consider the two-dimensional Boussinesq system,

$$(B_{\nu,\kappa})\begin{cases}\partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla\pi = \theta e_2,\\ \partial_t\theta + v\cdot\nabla\theta - \kappa\Delta\theta = 0,\\ \operatorname{div} v = 0,\\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0.\end{cases}$$

Here, e_2 denotes the vector $(0,1)$, $v=(v_1,v_2)$ is the velocity field, π the scalar pressure and θ stands for the temperature. The coefficients ν and κ are assumed to be positive; ν is called the kinematic viscosity and κ the molecular conductivity.
In the case of strictly positive coefficients ν and κ both velocity and temperature have sufficiently smoothing effects leading to the global well-posedness results proven by numerous authors in various function spaces (see [3, 10] and the references therein).
However, in the case $\kappa = 0$ and $\nu > 0$ the issue of whether a finite time singularity can form out of smooth initial data seems to be more difficult since the temperature obeys to a convective equation without any viscous effects. More precisely, let us mention that an analogous B-K-M criterion of blowing-up smooth solutions is established for Boussinesq system (see for example [2, 14]). It asserts in particular that if there is not an accumulation of the L^∞ norm of the vorticity then the solution does not develop a finite time singularity. We note that in space dimension two the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies the following transport-diffusion equation

$$\partial_t\omega + v\cdot\nabla\omega - \nu\Delta\omega = \partial_1\theta. \tag{1.1}$$

An obvious consideration shows that in the case of zero initial temperature the system $(B_{\nu,0})$ is reduced to the incompressible Navier-Stokes one, which is globally well-posed since the vorticity is bounded by using the maximum principle. Nevertheless, the situation for an arbitrary smooth initial temperature is more subtle: to bound the vorticity for every time we need to control the growth of the gradient of the temperature which is not an easy problem because the spatial derivative of the temperature obeys to a transport equation with a stretching term similar to the vorticity equation in space dimension three.

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Actually, the problem of finite-time formation of singularities was listed by H. K. Moffatt in [13] among other interesting questions on the fluid flows. Since some progress toward settling this problem are obtained by numerous authors. In [7], D. Cordóba, C. Fefferman and R. De La Llave gave a partial answer asserting that some special type of singularities called "squirt singularities" cannot be developed in finite time. Their argument is simple and shows that if this type of singularities appears at a finite time T then necessarily we have $\int_0^T \|v(\tau)\|_{L^\infty} d\tau = +\infty$, which cannot occur by using the evolution equations. More recently, D. Chae [4] and T. Y. Hou and C. Li [11] proved independently the global-in-time regularity when the initial data v^0 and θ^0 belong to the Sobolev space H^s, with $s > 2$. Their proofs rest essentially on two facts. The first one is the use on an adequate manner of the smoothing effects of the vorticity equation allowing them to diminish the required regularity for the temperature. The second technique is the use of a a sharp Sobolev embeddings estimate in two spatial dimensions with a logarithmic correction. Let us note that in this context the velocity and the temperature are Lipschitz and this assumption is crucial for their analysis.
The aim of this paper is to improve their results for more rough initial data which are not necessarily Lipschitz. Before stating our main results we recall that H^s denotes the usual Sobolev space (see the next section for more details about these spaces). Our first result deals with the existence of global solutions.

Theorem 1.1. *Let $\theta^0 \in L^2$ and v^0 be a divergence-free vector field belonging to the space H^s with $s \in [0, 2[$. Then there exists a weak global solution (v, θ) for the Boussinesq system $(B_{\nu,0})$, such that $v \in \mathcal{C}(\mathbb{R}_+; H^s) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{\min\{s+1,2\}})$ and $\theta \in \mathcal{C}_b(\mathbb{R}_+; L^2)$.*

The proof uses the paradifferential calculus, giving some *a priori* bounds, combined with the standard compactness arguments, as used for the existence of Leray's solutions to the Navier-Stokes system [12]. We mention that Lemma 3.1 plays an important role for the proof of the continuity in time of θ since the flow exists and is continuous. For the uniqueness of weak solutions we are only able to provide a positive answer under some additional assumptions on the initial data.

Theorem 1.2. *Let $s \in]0, 2]$ and $p \in]2, +\infty]$. Assume that v^0 is a divergence-free vector field belonging to the space H^s and $\theta^0 \in B^0_{2,1} \cap B^0_{p,\infty}$. Then there exists a unique global solution for the system $(B_{\nu,0})$ such that*

$$v \in \mathcal{C}(\mathbb{R}_+; H^s) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{\min\{s+1,2\}}) \cap L^1_{\text{loc}}(\mathbb{R}_+; B^2_{2,1}) \quad \textit{and} \quad \theta \in \mathcal{C}(\mathbb{R}_+; B^0_{2,1} \cap B^0_{p,\infty}).$$

The proof of both existence an uniqueness results is heavily related to Lemma 4.1 which gives a bound of the velocity in $L^1_{\text{loc}}(\mathbb{R}_+; \text{Lip}(\mathbb{R}^2))$. The main idea of the proof is to introduce in the calculus a parameter of frequency cut off N that will be judiciously choosen with respect to the growth in time of some suitable quantities. Another key of the proof is Proposition B.1 generalizing a logarithmic result due to Vishik [15].
The rest of this paper is organized as follows. In section 2, we recall some function spaces and gather several important estimates. Section 3 is devoted to the proof of Theorem 1.1. In section 4 we give the proof of Theorem 1.2. Some auxiliary results are given in the appendix.

2. Notations and preliminaries

We shall denote by C some real positive constants which may be different in each occurrence and by C_0 a real positive constant depending on the initial data.

This is a preparatory section in which we review the characterization of Sobolev and Besov spaces through the frequency localization operators and we give some useful results.

Let us start with the definition of the dyadic decomposition of $\mathbb{R}^2$ and recall the Littlewood-Paley operators (see for example [5]): there exist two radially functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \frac{1}{3} \leq \chi^2(\xi) + \sum_{q\geq 0} \varphi^2(2^{-q}\xi) \leq 1,$

ii) $\operatorname{supp} \varphi(2^{-p}\cdot) \cap \operatorname{supp} \varphi(2^{-q}\cdot) = \varnothing$, if $|p-q| \geq 2$,

iii) $q \geq 1 \Rightarrow \operatorname{supp}\chi \cap \operatorname{supp} \varphi(2^{-q}) = \varnothing$.

For every $v \in \mathcal{S}'$ we set

$$\Delta_{-1} v = \chi(\mathrm{D})v \ ; \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad S_q = \sum_{-1\leq p\leq q-1} \Delta_p.$$

The paradifferential calculus introduced by J.-M. Bony [1] is based on the decomposition (called Bony's decomposition) of the product uv into three parts:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u, v) = \sum_{|q'-q|\leq 1} \Delta_q u \Delta_{q'} v.$$

In the whole space $\mathbb{R}^2$, Sobolev spaces are defined in terms of integrability properties in the frequency space, using the Fourier transform $\mathcal{F}$. For $s \in \mathbb{R}$, the inhomogeneous Sobolev space H^s denotes the set of tempered distribution u such that $\mathcal{F}u \in L^1_{\text{loc}}(\mathbb{R}^2)$ and

$$\|u\|_{H^s} := \Big(\int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi\Big)^{\frac{1}{2}} < \infty.$$

An other equivalent H^s norm is defined through the dyadic decomposition.

$$\|u\|^2_{H^s} \simeq \sum_q 2^{2qs} \|\Delta_q u\|^2_{L^2}.$$

We can generalize the last norm leading to what we call Besov spaces.

Let $(p_1, p_2) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ then the Besov space $B^s_{p_1,p_2}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p_1,p_2}} := \Big(2^{qs} \|\Delta_q u\|_{L^{p_1}}\Big)_{\ell^{p_2}} < +\infty.$$

Let $T > 0$ and $r \geq 1$, we denote by $L^r_T B^s_{p_1,p_2}$ the space of all function u satisfying

$$\|u\|_{L^r_T B^s_{p_1,p_2}} := \Big\|\Big(2^{qs} \|\Delta_q u\|_{L^{p_1}}\Big)_{\ell^{p_2}}\Big\|_{L^r_T} < \infty.$$

We say that a function u is an element of the space $\widetilde{L^r_T} B^s_{p_1,p_2}$ if

$$\|u\|_{\widetilde{L^r_T} B^s_{p_1,p_2}} := \Big(2^{qs} \|\Delta_q u\|_{L^r_T L^{p_1}}\Big)_{\ell^{p_2}} < +\infty. \tag{2.1}$$

The relationships between these spaces are detailed by the following lemma, which is a direct consequence of the Minkowski inequality.

Lemma 2.1. *Let $s\in\mathbb{R},\epsilon>0$, $r\geq 1$ and $\big(p_1,p_2\big)\in[1,\infty]^2$. Then we have the following embeddings*

$$L^r_TB^s_{p_1,p_2}\hookrightarrow \widetilde{L^r_T}B^s_{p_1,p_2}\hookrightarrow L^r_TB^{s-\epsilon}_{p_1,p_2},\ \text{if}\quad r\leq p_2,$$
$$L^r_TB^{s+\epsilon}_{p_1,p_2}\hookrightarrow \widetilde{L^r_T}B^s_{p_1,p_2}\hookrightarrow L^r_TB^{s}_{p_1,p_2},\ \text{if}\quad r\geq p_2.$$

We will also make continuous use of Bernstein lemma (see for example [5]).

Lemma 2.2. (Bernstein) *Let (r_1,r_2) a pair of strictly positive numbers such that $r_1<r_2$. There exists a constant C such that for every nonnegative integer k, for every $1\leq a\leq b$ and for all function $u\in L^a(\mathbb{R}^2)$, we have*

$$\text{supp }\mathcal{F}u\in B(0,\lambda\, r_1)\quad\Rightarrow\quad \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^k\,\lambda^{k+2(\frac1a-\frac1b)}\|u\|_{L^a},$$
$$\text{supp }\mathcal{F}u\in \mathcal{C}(0,\lambda r_1,\lambda r_2)\quad\Rightarrow\quad C^{-k}\lambda^k\|u\|_{L^a}\leq \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^k\lambda^k\|u\|_{L^a}.$$

The proof of the next lemma can be found in [6].

Lemma 2.3. *There exists a constant C such that for every free-divergence vector field v on $\mathbb{R}^2$ and for every $q\in\mathbb{N}$ we have,*

$$|<\Delta_q(v\cdot\nabla v),\Delta_q v>_{L^2}|\leq Cc_q\|\nabla v\|^2_{L^2}\|\Delta_q v\|_{L^2},\quad \textit{with}\quad \|(c_q)_q\|_{\ell^2}=1.$$

We state now a classical result for the linearized momentum equation (for a more general study see Proposition 2.2 [8] and the references therein).

$$(\text{LM})\begin{cases}\partial_t u+v\cdot\nabla u-\nu\Delta u+\nabla\pi=f+g\\ \text{div }u=0\\ u_{|t=0}=u^0.\end{cases}$$

Proposition 2.4. *Let $r\in[1,+\infty]$ and $s\in]-1,1[$. We assume that v is a divergence-free vector field belonging to $L^1_{\text{loc}}(\mathbb{R}_+;\text{Lip}(\mathbb{R}^2))$, $f\in\widetilde{L^1_{\text{loc}}}(\mathbb{R}_+;B^s_{2,r})$ and $g\in\widetilde{L^\infty_{\text{loc}}}(\mathbb{R}_+;B^{s-2}_{2,r})$. Then any solution u of the equation* (LM), *($\nu>0$) with $u^0\in B^s_{2,r}$ satisfies for all $r\in[1,+\infty]$,*

$$\|u\|_{\widetilde{L^\infty_t}B^s_{2,r}}\leq C\big(\|u^0\|_{B^s_{2,r}}+\|f\|_{\widetilde{L^1_t}B^s_{2,r}}+(1+t)\|g\|_{\widetilde{L^\infty_t}B^{s-2}_{2,r}}\big)e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

The constant C depends on ν and s.
When $u=v$, the above estimate holds true if $s>-1$.

The following persistence result will be needed for the uniqueness result (for the proof see Proposition A.1 [9] and the references therein).

Proposition 2.5. *Let $r\in[1,\infty]$, $\ell\in[1,\infty]$, $\bar\ell$ its conjugate exponent and s such that*

$$-1-\frac{2}{\max\{\bar\ell,2\}}<s<1+\frac{2}{\max\{\ell,2\}}.$$

Let a be a solution of the transport-diffusion equation ($\nu\geq0$)

$$\partial_t a+v\cdot\nabla a-\nu\Delta a=f,\ a_{|t=0}=a^0,\ \text{div }v=0,$$

such that $a^0 \in B^s_{\ell,r}(\mathbb{R}^2)$ *and* $f \in L^1_{\text{loc}}(\mathbb{R}_+; B^s_{\ell,r})$. *Then for every* $t \in \mathbb{R}_+$,

$$\|a(t)\|_{B^s_{\ell,r}} \leq Ce^{CV(t)}\Big(\|a^0\|_{B^s_{\ell,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{\ell,r}} d\tau\Big),$$

where $V(t) = \displaystyle\int_0^t \|\nabla v(\tau)\|_{B^1_{2,1}} d\tau$. *The constant* C *is independent on* ν *and* r.

3. Proof of Theorem 1.1

The proof is based upon some *a priori* estimates combined with compactness argument. Without any loss of generality we take $\nu = 1$. The rest of the proof will proceed in three steps.

• *Step 1: A priori estimates.* Let us start with the energy estimates corresponding to the case $s = 0$. We take the inner product in L^2 between the first equation of $(B_{1,0})$ and v, leading after some integration by parts to

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \leq \|\theta(t)\|_{L^2}\|v(t)\|_{L^2}.$$

A simple computation gives

$$\|v(t)\|_{L^2} \leq \|v^0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2} d\tau.$$

Since div $v = 0$, we have $\|\theta(t)\|_{L^2} = \|\theta^0\|_{L^2}$. Thus, it follows that

$$\|v(t)\|_{L^2} \leq \|v^0\|_{L^2} + \|\theta^0\|_{L^2} t.$$

Integrating the differential inequality and using the last estimate,

$$\begin{aligned} \|v(t)\|^2_{L^2} + 2\int_0^t \|\nabla v(\tau)\|^2_{L^2} d\tau &\leq \|v^0\|^2_{L^2} + 2\|\theta^0\|_{L^2}\|v^0\|_{L^2} t + 2\|\theta^0\|^2_{L^2} t^2 \\ &\leq 4(\|\theta^0\|_{L^2} + \|v^0\|_{L^2})^2(1+t^2). \end{aligned} \tag{3.1}$$

The following smoothing lemma will be useful later.

Lemma 3.1. *There exists* $C > 0$*, such that for all* $t \geq 0$

$$\|v\|_{\widetilde{L}^1_t H^2} \leq C(a + a^2)\big(1 + t^2\big), \tag{3.2}$$

with $\quad a = \|v^0\|_{L^2} + \|\theta^0\|_{L^2}$.

Proof. Let $q \in \mathbb{N}$ and set $v_q := \Delta_q v$. It is easy to check that

$$\partial_t v_q - \Delta v_q = -\Delta_q(v \cdot \nabla v) - \nabla \pi_q + \theta_q e_2.$$

Taking L^2 inner product of this equation with v_q and using Lemma 2.3 combined with Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_q(t)\|^2_{L^2} + \|\nabla v_q\|^2_{L^2} \leq Cc_q(t)\|\nabla v(t)\|^2_{L^2}\|v_q(t)\|_{L^2} + \|\theta_q(t)\|_{L^2}\|v_q(t)\|_{L^2},$$

with $\sum |c_q(t)|^2 = 1$. However, in view of Parseval formula, there exits $\alpha > 0$ such that

$$\|\nabla v_q(t)\|^2_{L^2} \geq \alpha 2^{2q}\|v_q(t)\|^2_{L^2}, \quad \forall t \geq 0, \forall q \in \mathbb{N}.$$

So we get the following differential inequality

$$\frac{d}{dt}\|v_q(t)\|_{L^2}+\alpha 2^{2q}\|v_q(t)\|_{L^2}\leq Cc_q(t)\|\nabla v(t)\|_{L^2}^2+\|\theta_q(t)\|_{L^2},$$

which can be rewritten, via Duhamel formula,

$$\begin{aligned}\|v_q(t)\|_{L^2}\leq e^{-\alpha t2^{2q}}\|v_q^0\|_{L^2} \quad &+ \quad C\int_0^t e^{-\alpha(t-\tau)2^{2q}}c_q(\tau)\|\nabla v(\tau)\|_{L^2}^2d\tau+\\ &+\quad \int_0^t e^{-\alpha(t-\tau)2^{2q}}\|\theta_q(\tau)\|_{L^2}d\tau.\end{aligned}$$

Integrating again both sides and using Young inequality lead to

$$2^{2q}\|v_q\|_{L^1_tL^2}\leq C\|v_q^0\|_{L^2}+C\int_0^t c_q(\tau)\|\nabla v(\tau)\|_{L^2}^2d\tau+C\|\theta_q\|_{L^1_tL^2}.$$

Remark that

$$\|v\|_{\widetilde{L}^1_tH^2}\leq \|v_{-1}\|_{L^1_tL^2}+\Big(\sum_{q\in\mathbb{N}}2^{4q}\|v_q\|_{L^1_tL^2}^2\Big)^{1/2}\leq \|v\|_{L^1_tL^2}+\Big(\sum_{q\in\mathbb{N}}2^{4q}\|v_q\|_{L^1_tL^2}^2\Big)^{1/2}.$$

Hence, we deduce that

$$\|v\|_{\widetilde{L}^1_tH^2}\leq \|v\|_{L^1_tL^2}+C\|v^0\|_{L^2}+C\Big(\sum_{q\in\mathbb{N}}\Big(\int_0^t c_q(\tau)\|\nabla v(\tau)\|_{L^2}^2d\tau\Big)^2\Big)^{\frac12}+C\|(\|\theta_q\|_{L^1_tL^2})\|_{\ell^2}.$$

But, by Minkowski inequality, we have

$$\begin{aligned}\Big(\sum_{q\in\mathbb{N}}\Big(\int_0^t c_q(\tau)\|\nabla v(\tau)\|_{L^2}^2d\tau\Big)^2\Big)^{\frac12} \quad &\leq \quad \int_0^t\Big(\sum_{q\in\mathbb{N}}c_q(\tau)^2\|\nabla v(\tau)\|_{L^2}^4\Big)^{\frac12}d\tau\\ &\leq \quad \int_0^t\|\nabla v(\tau)\|_{L^2}^2d\tau.\end{aligned}$$

Besides we have used the identity: $\sum_{q\in\mathbb{N}}c_q(\tau)^2=1$. Similarly, one has

$$\|(\|\theta_q\|_{L^1_tL^2})\|_{\ell^2}\leq\|\theta\|_{L^1_tL^2}.$$

The outcome is the following

$$\|v\|_{\widetilde{L}^1_tH^2}\leq \|v\|_{L^1_tL^2}+C\|v^0\|_{L^2}+C\int_0^t\|\nabla v(\tau)\|_{L^2}^2d\tau+C\|\theta\|_{L^1_tL^2}.$$

Combined with L^2-energy estimate (3.1) this gives the result. □

Let us now examine the case $s\in]0,2[$ and estimate v in Sobolev space H^s. Applying Δ_q to the first equation of the system $(B_{1,0})$, we get

$$\partial_t v_q+v\cdot\nabla v_q-\Delta v_q+\nabla\pi_q=-[\Delta_q,v\cdot\nabla]v+\theta_qe_2:=\mathcal{R}_q+\theta_qe_2.$$

Taking the L^2-scalar product of the above equation with v_q, then after some obvious computations based on integration by parts and Parseval formula, we obtain

$$\frac{d}{dt}\|v_q(t)\|_{L^2}+\alpha 2^{2q}\|v_q(t)\|_{L^2}\leq C\|\mathcal{R}_q(t)\|_{L^2}+\|\theta_q(t)\|_{L^2},$$

which can be rewritten, via Duhamel formula, as

$$\|v_q(t)\|_{L^2} \leq e^{-\alpha t 2^{2q}}\|v_q^0\|_{L^2} + \int_0^t e^{-\alpha(t-\tau)2^{2q}}\Big(C\|\mathcal{R}_q(\tau)\|_{L^2} + \|\theta_q(\tau)\|_{L^2}\Big)d\tau.$$

According to Proposition A.1 (see Appendix A)

$$\|\mathcal{R}_q(\tau)\|_{L^2} \leq C c_q(\tau) 2^{q(1-s)}\|\nabla v(\tau)\|_{L^2}\|v(\tau)\|_{H^s}, \quad \text{with} \quad \sum_{q\in\mathbb{N}} |c_q(\tau)|^2 = 1,$$

which implies

$$\|v_q(t)\|_{L^2} \leq e^{-\alpha t 2^{2q}}\|v_q^0\|_{L^2} + C2^{q(1-s)}\int_0^t e^{-\alpha(t-\tau)2^{2q}} c_q(\tau)\|\nabla v(\tau)\|_{L^2}\|v(\tau)\|_{H^s}d\tau + \\ + \int_0^t e^{-\alpha(t-\tau)2^{2q}}\|\theta_q(\tau)\|_{L^2}d\tau. \tag{3.3}$$

Multiplying by 2^{qs} and using Young inequality give, for every $q \in \mathbb{N}$,

$$2^{2qs}\|v_q\|^2_{L^\infty_t L^2} \leq C2^{2qs}\|v_q^0\|^2_{L^2} + C\int_0^t c_q^2(\tau)\|\nabla v(\tau)\|^2_{L^2}\|v(t)\|^2_{H^s}d\tau + C2^{2q(s-2)}\|\theta_q\|^2_{L^\infty_t L^2}.$$

Note that, from the convolution law and the energy inequality, we have

$$\|\theta_q\|_{L^\infty_t L^2} \leq C\|\theta\|_{L^\infty_t L^2} \leq C\|\theta^0\|_{L^2}, \qquad \forall q \in \mathbb{N}.$$

Therefore, one obtains

$$2^{2qs}\|v_q\|^2_{L^\infty_t L^2} \leq C2^{2qs}\|v_q^0\|^2_{L^2} + C\int_0^t c_q^2(\tau)\|\nabla v(\tau)\|^2_{L^2}\|v(\tau)\|^2_{H^s}d\tau + C2^{2q(s-2)}\|\theta^0\|^2_{L^2}.$$

Summing over q and using the energy estimate (3.1) yields for $s \in [0,2[$,

$$\begin{aligned}
\|v\|^2_{\widetilde{L}^\infty_t H^s} &\leq C\Big(\|\Delta_{-1}v\|^2_{L^\infty_t L^2} + \|v^0\|^2_{H^s} + \int_0^t \|\nabla v(\tau)\|^2_{L^2}\|v\|^2_{\widetilde{L}^\infty_\tau H^s}d\tau + \|\theta^0\|^2_{L^2}\Big)\\
&\leq C\Big(\|v\|^2_{L^\infty_t L^2} + \|v^0\|^2_{H^s} + \int_0^t \|\nabla v(\tau)\|^2_{L^2}\|v\|^2_{\widetilde{L}^\infty_\tau H^s}d\tau + \|\theta^0\|^2_{L^2}\Big)\\
&\leq C_0(1+t^2) + C\int_0^t \|\nabla v(\tau)\|^2_{L^2}\|v\|^2_{\widetilde{L}^\infty_\tau H^s}d\tau.
\end{aligned}$$

Combined with Gronwall inequality and the energy estimate (3.1) the last estimate yields

$$\|v\|_{\widetilde{L}^\infty_t H^s} \leq C_0(1+t)e^{C\|\nabla v\|^2_{L^2_t L^2}} \leq C_0 e^{C_0 t^2}. \tag{3.4}$$

Let us now give a bound of $\|v\|_{L^2_t H^\beta}$, where $\beta = \min\{s+1, 2\}$. Applying convolution inequality to (3.3) gives

$$\begin{aligned}
\|v_q\|^2_{L^2_t L^2} &\leq C2^{-2q}\|v_q^0\|^2_{L^2} + C2^{-2q(1+s)}\int_0^t c_q(\tau)^2\|\nabla v\|^2_{L^2}\|v\|^2_{H^s}d\tau + C2^{-4q}\|\theta_q\|^2_{L^2_t L^2}\\
&\leq C2^{-2q(1+s)}(2^{2qs}\|v_q^0\|^2_{L^2}) + C2^{-2q(1+s)}\int_0^t c_q(\tau)^2\|\nabla v(\tau)\|^2_{L^2}\|v(\tau)\|^2_{H^s}d\tau + \\
&\quad + C2^{-4q}\|\theta_q\|^2_{L^2_t L^2}.
\end{aligned}$$

Multiplying by $2^{2q\beta}$ and summing over q, after separating low frequency, lead to

$$\begin{aligned}\|v\|^2_{L^2_tH^\beta} &\leq \|\Delta_{-1}v\|^2_{L^2_tL^2}+C\|v^0\|^2_{H^s}+C\int_0^t\|\nabla v(\tau)\|^2_{L^2}\|v(\tau)\|^2_{H^s}d\tau+C\|\theta\|^2_{L^2_tL^2}\\ &\leq \|v\|^2_{L^2_tL^2}+C\|v^0\|^2_{H^s}+C\int_0^t\|\nabla v(\tau)\|^2_{L^2}\|v(\tau)\|^2_{H^s}d\tau+C\|\theta\|^2_{L^2_tL^2}.\end{aligned}$$

Therefore we obtain from (3.1) and (3.4)

$$\|v\|_{L^2_tH^\beta}\leq C_0e^{C_0t^2}. \tag{3.5}$$

• *Step 2: Compactness argument.* Let us now sketch the proof of the existence of global solutions to the Boussinesq system which is standard. We smooth out the initial data and so we obtain from the result of [4] the existence of a family of unique global solutions (v^n,θ^n). It follows from the *a priori* estimates that this family is bounded in our solving spaces, hence it converges weakly to (v,θ), up to the extraction of a subsequence. However, to pass to the limit in the equations we have to establish the local strong convergence. This relies upon compactness properties of the sequence which are obtained by considering the time derivative of the solution. More precisely, by using the usual product laws and (3.1) we get, for all $\eta>0$ and $T>0$,

$$\begin{aligned}\|\partial_\tau v\|_{L^2_TH^{-1-\eta}} &\leq C_\eta\|v\|_{L^\infty_TL^2}\|v\|_{L^2_TH^1}+\|\theta\|_{L^2_tL^2}\\ &\leq C_{0,\eta}(1+T^2),\end{aligned} \tag{3.6}$$

which gives the strong convergence in view of Ascoli Theorem.

• *Step 3: Continuity in-time.* First of all, according to (3.4), for every $\epsilon>0$ there exists an integer N such that

$$\sum_{q\geq N}2^{2qs}\|\Delta_qv\|^2_{L^\infty_TL^2}\leq\frac{\epsilon^2}{16}.$$

Let $t,\, t'\in[0,T]$, then it follows from Taylor formula and Hölder inequality,

$$\begin{aligned}\|v(t)-v(t')\|_{H^s} &\leq \|S_Nv(t)-S_Nv(t')\|_{H^s}+2\big(\sum_{q\geq N}2^{2qs}\|\Delta_qv\|^2_{L^\infty_TL^2}\big)^{\frac12}\\ &\leq |t-t'|^{\frac12}\|\partial_\tau S_Nv\|_{L^2_TH^s}+\frac{\epsilon}{2}\\ &\leq C2^{N(s+1+\eta)}|t-t'|^{\frac12}\|\partial_\tau S_Nv\|_{L^2_TH^{-1-\eta}}+\frac{\epsilon}{2}\\ &\leq C_02^{N(s+1+\eta)}|t-t'|^{\frac12}(1+T)+\frac{\epsilon}{2}.\end{aligned}$$

This proves the continuity of the function v.

Let us move to the proof of the continuity of θ. First, since $v\in\widetilde{L}^1_{\text{loc}}H^2$ (see (3.2)), then we deduce from [6] that this velocity has a unique global flow $\psi(t,x)$ which is in $\mathcal{C}(\mathbb{R}_+\times\mathbb{R}^2;\mathbb{R}^2)$. Thus, the unique solution of the transport equation is then given explicitly by

$$\theta(t,x)=\theta^0(\psi^{-1}(t,x)).$$

We suppose that θ^0 is a continuous function. Then we can see easily from the preserving Lebesgue measure by the flow that

$$\forall t \in \mathbb{R}_+,\ \|\theta(t)\|_{L^2} = \|\theta(t_0)\|_{L^2} \quad \text{and} \quad \lim_{t\to t_0} \theta(t,x) = \theta(t_0,x),\ \forall x \in \mathbb{R}^2.$$

We conclude now from Fatou lemma that

$$\lim_{t\to t_0} \|\theta(t) - \theta(t_0)\|_{L^2} = 0.$$

If θ^0 is not continuous then we proceed by approximation. The proof of Theorem 1.1 is now complete.

4. Proof of Theorem 1.2

We divide the proof into two steps. In the first one we prove the global existence and in the second one we show the uniqueness.

• *Step 1: Global existence.* The existence of solutions for initial data $v^0 \in H^s$ and $\theta^0 \in B^0_{2,1}$ has already been proved in Theorem 1.1 since θ^0 belongs to the space L^2. We have in particular $v \in \mathcal{C}(\mathbb{R}_+; H^s) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{\min s+1,2})$. Thus, to finish the proof of the existence part of Theorem 1.2, it remains only to show that $v \in L^1_{\text{loc}}(\mathbb{R}_+; B^2_{2,1})$ and the persistence regularity for the temperature θ. This is the object of the following lemma.

Lemma 4.1. *Let $\theta^0 \in B^0_{2,1} \cap B^0_{p,\infty}$, with $p > 2$ and $v^0 \in H^s$, with $s \in]0,2]$. Then there exists $C_0 > 0$ depending on the initial data, such that for every $t \geq 0$,*

$$\|\nabla v\|_{L^1_t L^\infty} + \|\theta\|_{\widetilde{L}^\infty_t(B^0_{2,1}\cap B^0_{p,\infty})} + \|v\|_{L^1_t B^2_{2,1}} \leq C_0 e^{C_0 t^2}.$$

Proof. First, we take an arbitrary integer N and we denote $v_q := \Delta_q v$. Then separating low and high frequencies one can write

$$\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \leq \int_0^t \|\nabla S_N v(\tau)\|_{L^\infty} d\tau + \sum_{q\geq N} \int_0^t \|\nabla v_q(\tau)\|_{L^\infty} d\tau.$$

Applying successively Bernstein lemma and Hölder inequality yield,

$$\begin{aligned} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau &\leq C2^N \int_0^t \|\nabla v(\tau)\|_{L^2} d\tau + C \sum_{q\geq N} 2^q \int_0^t \|v_q(\tau)\|_{L^\infty} d\tau \\ &\leq C2^N t^{\frac{1}{2}} \|\nabla v\|_{L^2_t L^2} + C \sum_{q\geq N} 2^{2q} \int_0^t \|v_q(\tau)\|_{L^2} d\tau. \end{aligned} \tag{4.1}$$

To give a suitable estimate for the remainder term we will localize in frequency the first equation of the system $(B_{1,0})$ through the operator Δ_q,

$$\partial_t v_q + v \cdot \nabla v_q - \Delta v_q + \nabla \pi_q = -[\Delta_q, v \cdot \nabla] v + \theta_q e_2.$$

As was seen before a simple L^2-energy estimate combined with zero divergence condition and Parseval formula lead to

$$\frac{d}{dt} \|v_q(t)\|_{L^2} + \alpha 2^{2q} \|v_q(t)\|_{L^2} \leq C \|[\Delta_q, v \cdot \nabla] v(t)\|_{L^2} + C \|\theta_q(t)\|_{L^2}.$$

However, from Proposition A.1 (see Appendix A) gives

$$\|[\Delta_q, v\cdot\nabla]v(t)\|_{L^2}\le C2^{-qs}\|\nabla v(t)\|_{L^\infty}\|v(t)\|_{H^s}.$$

Inserting this estimate into the differential inequality and integrating in time

$$\|v_q(t)\|_{L^2}\le e^{-\alpha t2^{2q}}\|v_q^0\|_{L^2}+C2^{-qs}\int_0^t e^{-\alpha(t-\tau)2^{2q}}\|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{H^s}d\tau+ \\ +C\int_0^t e^{-\alpha(t-\tau)2^{2q}}\|\theta_q(\tau)\|_{L^2}d\tau.$$

Integrating again over the time and using the convolution inequality

$$2^{2q}\int_0^t\|v_q(\tau)\|_{L^2}d\tau\le C\|v_q^0\|_{L^2}+C2^{-qs}\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{H^s}d\tau+ \\ +C\int_0^t\|\theta_q(\tau)\|_{L^2}d\tau. \tag{4.2}$$

Substituting this estimate into (4.1) yields

$$\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\le C2^Nt^{\frac12}\|\nabla v\|_{L^2_tL^2}+C2^{-Ns}\|v\|_{L^\infty_tH^s}\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau+ \\ +C\|v^0\|_{B^0_{2,1}}+C\int_0^t\|\theta(\tau)\|_{B^0_{2,1}}d\tau.$$

According to Proposition B.1 (see Appendix B), one writes

$$\|\theta\|_{\widetilde{L^\infty_t}B^0_{2,1}}\le\|\theta^0\|_{B^0_{2,1}}\big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\big). \tag{4.3}$$

Combining both last estimates, we infer

$$V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\le C2^Nt^{\frac12}\|\nabla v\|_{L^2_tL^2}+C\|v^0\|_{B^0_{2,1}}+Ct\|\theta^0\|_{B^0_{2,1}}+ \\ +C2^{-Ns}\|v\|_{L^\infty_tH^s}V(t)+C\|\theta^0\|_{B^0_{2,1}}\int_0^tV(\tau)d\tau.$$

Choosing N such that

$$2^{-Ns}\|v\|_{L^\infty_tH^s}\simeq\frac12.$$

Thus we get by using Gronwall's inequality

$$V(t)\le Ce^{Ct\|\theta^0\|_{B^0_{2,1}}}\Big(\big(1+\|v\|^{\frac1s}_{L^\infty_tH^s}\big)t^{\frac12}\|\nabla v\|_{L^2_tL^2}+C\|v^0\|_{B^0_{2,1}}+1\Big).$$

This gives, by virtue of (3.4) and (3.1),

$$\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\le C_0e^{C_0t}. \tag{4.4}$$

Inserting the estimate (4.4) in (4.3) we get

$$\|\theta(t)\|_{B^0_{2,1}}\le\|\theta\|_{\widetilde{L^\infty_t}B^0_{2,1}}\le C_0e^{C_0t^2}. \tag{4.5}$$

Applying again Proposition B.1 to the temperature equation, we get

$$\|\theta(t)\|_{B^0_{p,\infty}} \leq \|\theta\|_{\widetilde{L^\infty_t} B^0_{p,\infty}} \leq C_0 e^{C_0 t^2}. \tag{4.6}$$

Finally, from inequality (4.2) we obtain

$$\|v\|_{L^1_t B^2_{2,1}} \leq C\|v^0\|_{B^0_{2,1}} + \|v\|_{L^\infty_t H^s} V(t) + \|\theta\|_{L^1_t B^0_{2,1}},$$

and (3.4), (4.4) and (4.5) give together

$$\|v\|_{L^1_t B^2_{2,1}} \leq C_0 e^{C_0 t^2}. \tag{4.7}$$

To achieve the existence part it remains to prove the continuity in-time of θ which is a consequence of (4.5) and (4.6) and the following estimate: for all $1 > \eta > 0$ we have

$$\begin{aligned} \|\partial_t \theta\|_{L^\infty_T H^{-2+\eta}} &\leq \|v\,\theta\|_{L^\infty_T H^{-1+\eta}} \\ &\leq C\|v\|_{L^\infty_T H^\eta} \|\theta\|_{L^\infty_T L^2}, \end{aligned}$$

which is nothing else than the products law in Sobolev spaces in space dimension two. □

• *Step 2: Uniqueness.* Now we focus our attention on the uniqueness result. Let $\{(v^i, \theta^i)\}_{i=1}^2$, two solutions for the system $(B_{1,0})$ such that the initial data $v^{i,0} \in B^0_{2,1}$ and $\theta^{i,0} \in B^0_{2,1} \cap B^0_{p,\infty}$, with $p > 2$. Furthermore, we suppose that for $i = 1, 2$,

$$\theta^i \in C(\mathbb{R}_+; B^0_{2,1}) \quad \text{and} \quad v^i \in C(\mathbb{R}_+; B^0_{2,1}) \cap L^1_{\text{loc}}(\mathbb{R}_+; B^2_{2,1}).$$

We emphasize that from the existence part these hypothesis are satisfied when the initial velocity belongs to H^s, with $s > 0$. We set

$$v = v^1 - v^2, \quad \theta = \theta^1 - \theta^2 \quad \text{and} \quad \pi = \pi^1 - \pi^2.$$

Obviously (v, π, θ) satisfies the following system

$$\partial_t v + v^1 \cdot \nabla v - \Delta v + \nabla \pi = -v \cdot \nabla v^2 + \theta e_2, \tag{4.8}$$

$$\partial_t \theta + v^1 \cdot \nabla \theta = -v \cdot \nabla \theta^2. \tag{4.9}$$

Our first step is to give an adequate estimate for $\|v\|_{B^0_{2,1}}$. To this end we will make use of Proposition 2.4,

$$\begin{aligned} \|v(t)\|_{B^0_{2,1}} \leq e^{\int_0^t \|\nabla v^1(\tau)\|_{L^\infty} d\tau} \Big(\|v^0\|_{B^0_{2,1}} &+ \int_0^t e^{-\int_0^\tau \|\nabla v^1(t')\|_{L^\infty} dt'} \|v \cdot \nabla v^2(\tau)\|_{B^0_{2,1}} d\tau \\ &+ (1+t)\|\theta\|_{\widetilde{L^\infty_t} B^{-2}_{2,1}} \Big). \end{aligned} \tag{4.10}$$

By using the condition div $v = 0$ and the Bony's decomposition we can easily check that

$$\|v \cdot \nabla v^2\|_{B^0_{2,1}} \leq C\|v\|_{B^0_{2,1}} \|v^2\|_{B^1_{\infty,1}} \leq C\|v\|_{B^0_{2,1}} \|v^2\|_{B^2_{2,1}},$$

so we get , thanks to Gronwall's inequality,

$$\|v(t)\|_{B^0_{2,1}} \leq C\, e^{C\|(v^1,v^2)\|_{L^1_t B^2_{2,1}}} \Big(\|v^0\|_{B^0_{2,1}} + (1+t)\|\theta\|_{\widetilde{L^\infty_t} B^{-2}_{2,1}} \Big).$$

At this stage we choose an $\epsilon \in]0,1[$ sufficiently small so that $\epsilon + \frac{2}{p} < 1$. In view of Lemma 2.1, it holds that

$$\|v(t)\|_{B^0_{2,1}} \leq C\, e^{C\|(v^1,v^2)\|_{L^1_t B^2_{2,1}}} \Big(\|v^0\|_{B^0_{2,1}} + (1+t)\|\theta\|_{L^\infty_t B^{-2+\epsilon}_{2,1}} \Big). \tag{4.11}$$

Let us now move to the estimate of θ. We apply Proposition 2.5 to the equation (4.9), with $s = -2+\epsilon$, $\ell = 2$ and $r = 1$

$$\|\theta\|_{L^\infty_t B^{-2+\epsilon}_{2,1}} \leq e^{C\|v^1\|_{L^1_t B^2_{2,1}}} \Big(\|\theta^0\|_{B^{-2+\epsilon}_{2,1}} + \int_0^t e^{-C\|v^1\|_{L^1_\tau B^2_{2,1}}} \|v\cdot\nabla\theta^2(\tau)\|_{B^{-2+\epsilon}_{2,1}} d\tau \Big). \tag{4.12}$$

At this stage we need following lemma.

Lemma 4.2. *For every $p > 2$ and $\epsilon > 0$ satisfying $\epsilon + \frac{2}{p} < 1$, one has*

$$\|v\cdot\nabla\theta^2\|_{B^{-2+\epsilon}_{2,1}} \leq C\|v\|_{B^0_{2,1}} \|\theta^2\|_{B^0_{p,\infty}}.$$

Proof. In view of the zero-divergence condition it is sufficent to prove that

$$\|v\cdot\theta^2\|_{B^{-1+\epsilon}_{2,1}} \leq C\|v\|_{B^0_{2,1}} \|\theta^2\|_{B^0_{p,\infty}}. \tag{4.13}$$

From Bony's decomposition we have

$$\|v\cdot\theta^2\|_{B^{-1+\epsilon}_{2,1}} \leq \|T_v\, \theta^2\|_{B^{-1+\epsilon}_{2,1}} + \|T_{\theta^2}\, v\|_{B^{-1+\epsilon}_{2,1}} + \|R(v,\theta^2)\|_{B^{-1+\epsilon}_{2,1}}.$$

The first term can be estimated from Bernstein lemma as follows:

$$\begin{aligned} \|T_v\, \theta^2\|_{B^{-1+\epsilon}_{2,1}} &\leq C \sum_{q\in\mathbb{N}} 2^{q(-1+\epsilon)} \|S_{q-1}v\|_{L^2} \|\Delta_q\theta^2\|_{L^\infty} \\ &\leq C\|v\|_{L^2} \sum_{q\in\mathbb{N}} 2^{q(-1+\epsilon+\frac{2}{p})} \|\Delta_q\theta^2\|_{L^p} \\ &\leq C\|v\|_{L^2} \|\theta^2\|_{B^0_{p,\infty}}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \|T_{\theta^2}\, v\|_{B^{-1+\epsilon}_{2,1}} &\leq C \sum_{q\in\mathbb{N}} 2^{q(-1+\epsilon)} \|S_{q-1}\theta^2\|_{L^\infty} \|\Delta_q v\|_{L^2} \\ &\leq C\|v\|_{L^2} \|\theta^2\|_{B^0_{p,\infty}} \sum_{q\in\mathbb{N}} 2^{q(-1+\epsilon+\frac{2}{p})}(q+2) \\ &\leq C\|v\|_{L^2} \|\theta^2\|_{B^0_{p,\infty}}. \end{aligned}$$

Let us now turn to the remainder term and setting n such that $\frac{1}{n} = \frac{1}{2} + \frac{1}{p}$. Using again Bernstein lemma one may obtain

$$\|\Delta_q R(v,\theta^2)\|_{L^2} \leq C 2^{q(\frac{2}{n}-1)} \|\Delta_q R(v,\theta^2)\|_{L^n}.$$

On the other hand we have from the definition and Hölder inequality,

$$\begin{aligned} \|\Delta_q R(v,\, \theta^2)\|_{L^n} &\leq C \sum_{\substack{q'\geq q-3 \\ i\in\{\mp1,0\}}} \|\Delta_{q'} v\|_{L^2} \|\Delta_{q'+i}\theta^2\|_{L^p} \\ &\leq C\|v\|_{B^0_{2,1}} \|\theta^2\|_{B^0_{p,\infty}}. \end{aligned}$$

Thus we conclude from these estimates that

$$\begin{aligned}2^{q(-1+\epsilon)}\|\Delta_q R(v,\,\theta^2)\|_{L^2} &\leq C2^{q(-1+\epsilon+\frac{2}{n}-1)}\|v\|_{B^0_{2,1}}\|\theta^2\|_{B^0_{p,\infty}}\\ &\leq C2^{q(-1+\epsilon+\frac{2}{p})}\|v\|_{B^0_{2,1}}\|\theta^2\|_{B^0_{p,\infty}}.\end{aligned}$$

It follows that

$$\|R(v,\,\theta^2)\|_{B^{-1+\epsilon}_{2,1}} \leq C\|v\|_{B^0_{2,1}}\|\theta^2\|_{B^0_{p,\infty}}.$$

This concludes the proof of (4.13). □

Coming back to the proof of Theorem 1.2. Combined with (4.12) this lemma yields

$$\|\theta\|_{L^\infty_t B^{-2+\epsilon}_{2,1}} \leq e^{C\|v^1\|_{L^1_t B^2_{2,1}}}\Big(\|\theta^0\|_{B^{-2+\epsilon}_{2,1}} + \int_0^t e^{-C\|v^1\|_{L^1_\tau B^2_{2,1}}}\|v(\tau)\|_{B^0_{2,1}}\|\theta^2(\tau)\|_{B^0_{p,\infty}}d\tau\Big).$$

However, in view of Proposition B.1 and (4.4), we have

$$\|\theta^2(t)\|_{B^0_{p,\infty}} \leq C\|\theta^{2,0}\|_{B^0_{p,\infty}}\big(1+\int_0^t\|\nabla v^2(\tau)\|_{L^\infty}d\tau\big) := h(t),$$

which implies

$$\|\theta\|_{L^\infty_t B^{-2+\epsilon}_{2,1}} \leq e^{C\|v^1\|_{L^1_t B^2_{2,1}}}\Big(\|\theta^0\|_{B^{-2+\epsilon}_{2,1}} + \int_0^t e^{-C\|v^1\|_{L^1_\tau B^2_{2,1}}}\|v(\tau)\|_{B^0_{2,1}}h(\tau)d\tau\Big). \tag{4.14}$$

Plugging (4.14) into (4.11) and using Gronwall's inequality lead to

$$\|v(t)\|_{B^0_{2,1}} + \|\theta(t)\|_{B^0_{2,1}} \leq C(1+t)\,e^{C_0(1+\|h\|_{L^1_t})\exp C\|(v^1,v^2)\|_{L^1_t B^2_{2,1}}}\Big(\|v^0\|_{B^0_{2,1}} + \|\theta^0\|_{B^{-2+\epsilon}_{2,1}}\Big),$$

and then conclude the proof of Theorem 1.2.

□

Appendix A. Commutator estimates

Our task in this first appendix is to prove the following commutator lemma.

Proposition A.1. *Let v a divergence-free vector field of $\mathbb{R}^2$ which is Lipschitz and $u \in B^s_{p_1,p_2}$, with $p_1, p_2 \in [1,+\infty]$ and $s \in]-1,1[$. Then, there exists a constant C depending only on s, such that for all $q \geq -1$,*

$$\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^{p_1}} \leq Cc_q 2^{-qs}\|\nabla v\|_{L^\infty}\|u\|_{B^s_{p_1,p_2}}, \text{ with } \quad \sum_q c_q^{p_2} = 1.$$

If we take $u = v$, then the commutator result holds true for every $s > -1$.
Moreover we have for $-1 < s < 2$ and $q \geq -1$,

$$\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^2} \leq Cc_q 2^{q(1-s)}\|\nabla u\|_{L^2}\|v\|_{H^s}, \quad \textit{with} \quad \|(c_q)_q\|_{\ell^2} = 1.$$

Proof. The proof of first part of this Proposition can be found in [5]. So we restrict ourselves only to the proof of the second one. The principal tool is Bony's decomposition [1]:

$$[\Delta_q, v\cdot\nabla]u = [\Delta_q, T_v\cdot\nabla]u + [\Delta_q, T_{\nabla\cdot}\cdot v]u + [\Delta_q, R(v\cdot\nabla,.)]u, \tag{A.1}$$

where

$$[\Delta_q, T_v\cdot\nabla]u = \Delta_q(T_v\cdot\nabla u) - T_v\cdot\nabla\Delta_q u$$
$$[\Delta_q, T_{\nabla\cdot}\cdot v]u = \Delta_q(T_{\nabla u}\cdot v) - T_{\nabla\Delta_q u}\cdot v$$
$$[\Delta_q, R(v\cdot\nabla, .)]u = \Delta_q(R(v\cdot\nabla, u)) - R(v\cdot\nabla, \Delta_q u).$$

From the definition of the paraproduct one writes

$$\|[\Delta_q, T_{\nabla\cdot}\cdot v]u\|_{L^2} \le C\sum_{|j-q|\le N_0}\|S_{j-1}\nabla u\|_{L^\infty}\|\Delta_j v\|_{L^2}. \tag{A.2}$$

According to Bernstein lemma, we have

$$\|S_{j-1}\nabla u\|_{L^\infty} \le C2^j\|\nabla u\|_{L^2}.$$

Hence, we obtain

$$\begin{aligned}\|[\Delta_q, T_{\nabla\cdot}\cdot v]u\|_{L^2} &\le C\|\nabla u\|_{L^2}\sum_{|j-q|\le N_0}2^j\|\Delta_j v\|_{L^2}\\ &\le C2^{q(1-s)}c_q\|\nabla u\|_{L^2}\|v\|_{H^s}, \quad \text{with} \quad \|(c_q)_q\|_{\ell^2}=1 \end{aligned}\tag{A.3}$$

Concerning the second term in the right side of A.1, we have by definition of the paraproduct and the commutation property of the operators Δ_q

$$\begin{aligned}[\Delta_q, T_v\cdot\nabla]u &= \sum_{j\ge1}[S_{j-1}v\cdot\nabla\Delta_j,\, \Delta_q]u,\\ &= \sum_{|j-q|\le N_0}[S_{j-1}v\cdot\nabla,\, \Delta_q]\Delta_j u.\end{aligned}$$

To estimate each commutator, we write Δ_q as a convolution

$$[S_{j-1}v\cdot\nabla,\, \Delta_q]\Delta_j u(\cdot) = 2^{qd}\int h(2^q(\cdot-y))\big(S_{j-1}v(\cdot)-S_{j-1}v(y)\big)\cdot\nabla\Delta_j u(y)dy.$$

Thus, Young and Bernstein inequalities yield, for $|j-q|\le N_0$, to

$$\begin{aligned}\big\|[S_{j-1}v\cdot\nabla,\, \Delta_q]\Delta_j u(\cdot)\big\|_{L^2} &\le C2^{-q}\|\nabla S_{j-1}v\|_{L^\infty}\|\Delta_j\nabla u\|_{L^2},\\ &\le C\|\nabla u\|_{L^2}\|v\|_{H^s}c_j2^{-q}\sum_{j'\le j-2}2^{j'(2-s)}c_{j'}.\end{aligned}$$

Therefore, we obtain for $s<2$

$$\|[\Delta_q, T_v\cdot\nabla]u\|_{L^2} \le Cc_q2^{q(1-s)}\|\nabla u\|_{L^2}\|v\|_{H^s}. \tag{A.4}$$

Let us move to the remainder term. It can be written, in view of the definition, as

$$\mathrm{J}_q := [\Delta_q, R(v\cdot\nabla, .)]u = \sum_{\substack{j\ge q-N_0, j\ge0\\ i\in\{\mp1,0\}}}[\Delta_q, \Delta_{j+i}v]\cdot\Delta_j\nabla u + \sum_{i\in\{0,1\}}[\Delta_q, \Delta_{-1+i}v]\cdot\Delta_{-1}\nabla u.$$

It follows from the zero divergence condition that

$$\mathrm{J}_q = \sum_{i\in\{0,1\}}[\Delta_q, \Delta_{-1+i}v]\cdot\Delta_{-1}\nabla u + \sum_{\substack{j\ge q-N_0, j\ge0\\ i\in\{\mp1,0\}}}\operatorname{div}\big([\Delta_q, \Delta_{j+i}v]\otimes\Delta_j u\big) = \mathrm{I}_q + \mathrm{II}_q.$$

If $q \geq N_0$ then the first sum of the second member is null. So we have for every q

$$\|\mathrm{I}_q\|_{L^2} \leq C\mathbf{1}_{]-\infty,N_0]}(q)\|v\|_{H^s}\|\nabla u\|_{L^2}.$$

To estimate the second term we use Bernstein inequality

$$\begin{aligned}\|\mathrm{II}_q\|_{L^2} &\leq C\sum_{\substack{j\geq q-N_0, j\geq 0\\ i\in\{\mp 1,0\}}} 2^{2q}\|\Delta_{j+i}v\|_{L^2}\|\Delta_j u\|_{L^2}\\ &\leq C2^{q(1-s)}\|v\|_{H^s}\|\nabla u\|_{L^2}\sum_{\substack{j\geq q-N_0\\ i\in\{\mp 1,0\}}} 2^{(q-j)(1+s)}\tilde{c}_j^2\\ &\leq Cc_q 2^{q(1-s)}\|v\|_{H^s}\|\nabla u\|_{L^2}.\end{aligned}$$

This completes the proof of Proposition A.1. □

Appendix B. Logarithmic estimate

We give here an improvement of Vishik's logarithmic estimate, see Theorem 4.2 [15]. Our approach is merely different from Vishik's one since we don't use the representation of the solution through the flow but only the structure of the equation.

Proposition B.1. *Let p, $r \in [1,+\infty]$, v be a divergence-free vector field belonging to the space $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip}(\mathbb{R}^2))$ and a be a scalar solution to the following problem,*

$$\begin{cases}\partial_t a + v\cdot\nabla a = 0\\ a_{|t=0} = a^0.\end{cases}$$

If the initial data $a^0 \in B^0_{p,r}$, then we have for all $t\in\mathbb{R}_+$

$$\|a\|_{\widetilde{L^\infty_t}B^0_{p,r}} \leq C\|a^0\|_{B^0_{p,r}}\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big),$$

where C is an absolute constant.

Proof. We denote by $\tilde{a}_q$ the unique global solution of the initial value problem:

$$\begin{cases}\partial_t \tilde{a}_q + v\cdot\nabla \tilde{a}_q = 0\\ \tilde{a}_q(0) = \Delta_q a^0.\end{cases}$$

Using Proposition 2.5 with $r=+\infty$ and $s=\epsilon$, for $0<\epsilon<1$, one obtains

$$\|\tilde{a}_q(t)\|_{B^\epsilon_{p,\infty}} \leq C\|\Delta_q a^0\|_{B^\epsilon_{p,\infty}} e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

Thus we deduce from the definition of Besov spaces that for all $j\geq -1$

$$\|\Delta_j\tilde{a}_q(t)\|_{L^p} \leq C2^{\epsilon(q-j)}\|\Delta_q a^0\|_{L^p} e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

This estimate holds true when we exchange ϵ with $-\epsilon$. Hence we have for every $j\geq -1$,

$$\|\Delta_j\tilde{a}_q(t)\|_{L^p} \leq C2^{-\epsilon|j-q|}\|\Delta_q a^0\|_{L^p} e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

We can easily deduce from the above estimate

$$\|\Delta_j\tilde{a}_q\|_{L^\infty_t L^p} \leq C2^{-\epsilon|j-q|}\|\Delta_q a^0\|_{L^p} e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}. \tag{B.1}$$

Now by linearity one can write

$$a(t,x)=\sum_{q\geq -1}\tilde{a}_q(t,x).$$

Taking $N\in\mathbb{N}$ and which will be carefully chosen later. Then we write by definition

$$\begin{aligned}
\|a\|_{\widetilde{L_t^\infty}B^0_{p,r}} &\leq \Big(\sum_j\Big(\sum_q\|\Delta_j\tilde{a}_q\|_{L_t^\infty L^p}\Big)^r\Big)^{\frac1r}\\
&\leq \Big(\sum_j\Big(\sum_{|q-j|\geq N}\|\Delta_j\tilde{a}_q\|_{L_t^\infty L^p}\Big)^r\Big)^{\frac1r}+\Big(\sum_j\Big(\sum_{|q-j|<N}\|\Delta_j\tilde{a}_q\|_{L_t^\infty L^p}\Big)^r\Big)^{\frac1r}\\
&= \mathrm{I}+\mathrm{II}. \qquad \text{(B.2)}
\end{aligned}$$

To estimate the first term we use (B.1) and the convolution inequality for the series

$$\begin{aligned}
\mathrm{I} &\leq C2^{-\epsilon N}e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}\|\|{a_q}^0\|_{L^p}\|_{\ell^r}\\
&\leq C2^{-\epsilon N}\|a^0\|_{B^0_{p,r}}e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}. \qquad \text{(B.3)}
\end{aligned}$$

To treat the second term of the right side of (B.2), we use two arguments: the first one is that the operator Δ_j maps uniformly L^p into itself while the second is the use of the L^p energy estimate. So we find

$$\begin{aligned}
\mathrm{II} &\leq C\Big(\sum_j\Big(\sum_{|q-j|<N}\|\tilde{a}_q\|_{L_t^\infty L^p}\Big)^r\Big)^{\frac1r}\\
&\leq C\Big(\sum_j\Big(\sum_{|q-j|<N}\|a^0_q\|_{L^p}\Big)^r\Big)^{\frac1r}\\
&\leq CN\|a^0\|_{B^0_{p,r}}. \qquad \text{(B.4)}
\end{aligned}$$

Plugging estimates (B.3) and (B.4) into (B.2), we get

$$\|a\|_{\widetilde{L_t^\infty}B^0_{p,r}}\leq C\|a^0\|_{B^0_{p,r}}\Big(2^{-\epsilon N}e^{C\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}+N\Big).$$

Putting

$$N=\Big[\frac{C_\epsilon V(t)}{\epsilon\log 2}+1\Big],$$

then the above inequality finishes the proof of Proposition B.1.

□

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