GLOBAL WELL-POSEDNESS FOR EULER-BOUSSINESQ SYSTEM WITH CRITICAL DISSIPATION

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ABSTRACT. In this paper we study a fractional diffusion Boussinesq model which couples the incompressible Euler equation for the velocity and a transport equation with fractional diffusion for the temperature. We prove global well-posedness results.

1. Introduction

Boussinesq systems of the type

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \theta e_2 + \nu \mathcal{D}_v v \\ \partial_t \theta + v \cdot \nabla \theta = \kappa \mathcal{D}_\theta \theta \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0 \end{cases}$$

are simple models widely used in the modelling of oceanic and atmospheric motions. These models also appear in many other physical problems, we refer for instance to [5, 3] for more details. Here, we focus on the two-dimensional case, the space variable $x = (x_1, x_2)$ is in $\mathbb{R}^2$, the velocity field v is given by $v = (v^1, v^2)$ and the pressure p and the temperature θ are scalar functions. The factor θe_2 in the velocity equation, the vector e_2 being given by $(0, 1)$, models the effect of gravity on the fluid motion. The operator $\mathcal{D}_v$ and $\mathcal{D}_\theta$ whose form may vary are used to take into account the possible effects of diffusion and dissipation in the fluid motion, thus the constants $\nu \geq 0$, $\kappa \geq 0$ can be seen as the inverse of Reynolds numbers.
Mathematically, the simplest model to study is the fully viscous model when $\nu > 0$, $\kappa > 0$ and $\mathcal{D}_v = \Delta$, $\mathcal{D}_\theta = \Delta$. The properties of the system are very similar to the one of the two-dimensional Navier-Stokes equation and similar global well-posedness results can be obtained.
The most difficult model for the mathematical study is the inviscid one, i.e. when $\nu = \kappa = 0$. A local existence result of smooth solution can be proven as for symmetric hyperbolic quasilinear systems, nevertheless, it is not known if smooth solutions can develop singularities in finite time. Indeed, the temperature θ is the solution of a transport equation and the vorticity $\omega = \operatorname{curl} v = \partial_1 v^2 - \partial_2 v^1$ solves the equation

$$\partial_t \omega + v \cdot \nabla \omega = \partial_1 \theta. \tag{1.1}$$

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The main difficulty is that to get an L^∞ estimate on ω which is crucial to prove global existence of smooth solutions for Euler type equation, one needs to estimate $\int_0^T ||\partial_1\theta||_{L^\infty}$ and, unfortunately, no *a priori* estimate on $\partial_1\theta$ is known.
In order to understand the coupling between the two equations in Boussinesq type systems, there have been many recent works studying Boussinesq systems with partial viscosity i.e. with a viscous term only in one equation. For $\kappa > 0$, $\nu = 0$ and $\mathcal{D}_\theta = \Delta$, the question of global existence is solved recently in a series of papers. In [7], Chae proved the global existence and uniqueness for initial data $(v^0, \theta^0) \in H^s \times H^s$, with $s > 2$, see also [20]. This result was recently extended in [16] by the two first authors to initial data $v^0 \in B_{p,1}^{\frac{2}{p}+1}$ and $\theta^0 \in B_{p,1}^{-1+\frac{2}{p}} \cap L^r, r > 2$. More recently, the study of global existence of Yudovich solutions for this system has been done in [14]. We also mention that in [15], Danchin and Paicu were able to construct global strong solutions (still for $\kappa > 0$, $\nu = 0$) for a dissipative term of the form $\mathcal{D}_\theta = \partial_{11}\theta$ instead of $\Delta\theta$. Recently the first author and Zerguine [19] proved the global well-posedness for fractional diffusion $\mathcal{D}_\theta = -|\mathrm{D}|^\alpha$ for $\alpha \in]1, 2[$ where the operator $|\mathrm{D}|^\alpha$ is defined by

$$\mathcal{F}(|\mathrm{D}|^\alpha u)(\xi) = |\xi|^\alpha (\mathcal{F}u)(\xi).$$

In these works, the global existence result relies on the fact that the only smoothing effect due to the transport-(fractional) diffusion equation

$$\partial_t\theta + v \cdot \nabla\theta + |\mathrm{D}|^\alpha\theta = 0$$

governing the temperature is sufficient to counterbalance the amplification of the vorticity. However the case $\alpha = 1$ is not reached by their method. The main reason is that this case can be seen as critical in the previous approaches in the sense that the smoothing effect for the temperature equation does not provide the $L^1_T(L^\infty)$ bound for $\partial_1\theta$ which seems needed to control the amplification of the L^∞ norm of the vorticity. Note that such an estimate is nevertheless almost true since $\partial_1\theta$ can be estimated in the space $\tilde{L}^1_T(L^\infty)$ which has the same scaling (see below for the definition). The aim of this paper is the study of the well-posedness for this case, i.e., we focus on the system

$$(1.2) \qquad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \theta e_2 \\ \partial_t\theta + v \cdot \nabla\theta + |\mathrm{D}|\theta = 0 \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

Note that we have taken $\kappa = 1$ which is legitimate since we study global well-posedness issues. Indeed, we can always change the coefficient κ into 1 by a change of scale.
We also point out that at first sight, the system (1.2) contains the mathematical difficulties of the critical quasigeostrophic equation introduced in [10]

$$(1.3) \qquad \partial_t\theta + v \cdot \nabla\theta + |\mathrm{D}|\theta = 0, \quad v = \nabla^\perp |\mathrm{D}|^{-1}\theta$$

which was much studied recently. Indeed, in (1.2) the link between v and θ is not given by the Riesz transform but by a dynamical equation, the first equation of (1.2). Nevertheless, from this velocity equation one gets that v has basically the regularity of θ as in the quasigeostrophic equation. The global well-posedness for (1.3) was obtained recently by Kiselev, Nasarov and Volberg [21]. We also refer to the work [6] by Caffarelli and Vasseur about the regularity of weak solutions. Other discussions can be found in [1, 11, 12].

The main result of this paper is a global well-posedness result for the system (1.2) (see section 2 for the definitions and the basic properties of Besov spaces).

Theorem 1.1. *Let $p \in]2,\infty[$, $v^0 \in B^1_{\infty,1} \cap \dot W^{1,p}$ be a divergence-free vector field of $\mathbb{R}^2$ and $\theta^0 \in B^0_{\infty,1} \cap L^p$. Then there exists a unique global solution (v,θ) to the system* (1.2) *with*

$$v \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; B^1_{\infty,1} \cap \dot W^{1,p}\big), \qquad \theta \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; B^0_{\infty,1} \cap L^p\big) \cap \widetilde L^1_{\mathrm{loc}}(\mathbb{R}_+; B^1_{p,\infty}).$$

A few remarks are in order.

Remark 1.2. If we take $\theta = 0$ then the system (1.2) is reduced to the well-known 2D incompressible Euler system. It is well known that this system is globally well-posed in H^s for $s > 2$. The main argument for globalization is the BKM criterion [2] ensuring that the development of finite time singularities for Kato's solutions is related to the blowup of the L^∞ norm of the vorticity near the maximal time existence. In [26] Vishik has extended the global existence of strong solutions result to initial data lying on the spaces $B^{1+2/p}_{p,1}$. Notice that these spaces have the same scaling as Lipschitz functions (the space which is relevant for the hyperbolic theory) and in this sense they are called critical. We emphasize that the application of the BKM criterion requires a super-lipschitzian regularity (H^s with $s > 2$ for example). For that reason the question of global existence in the critical spaces $B^{1+2/p}_{p,1}$ is hard to deal with because these spaces have only a lipschitzian regularity and the BKM criterion cannot be used.
Since $B^{1+2/p}_{p,1} \hookrightarrow B^1_{\infty,1} \cap \dot W^{1,r}$ for all $p \in [1,+\infty[$ and $r > \max\{p,2\}$, then the space of initial velocity in our theorem contains all the critical spaces $B^{1+2/p}_{p,1}$ except the biggest one, that is $B^1_{\infty,1}$. For the limiting case we have been able to prove the global existence only up to the extra assumption $\nabla v^0 \in L^p$ for some $p \in]2,\infty[$. The reason behind this extra assumption is the fact that to obtain a global L^∞ bound for the vorticity we need before to establish an L^p estimate for some $p \in]2,\infty[$ and it is not clear how to get rid of this condition.

Remark 1.3. Since $\nabla v, \nabla\theta \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty)$ (see Remark 5.6 below for θ) then we can easily propagate all the higher regularities: critical (*i.e.* $v_0 \in B^{1+2/p}_{p,1}$ with p finite) and sub-critical (for example $v_0 \in H^s$, for $s > 2$).

The main idea in the proof of Theorem 1.1 is to really used the structural properties of the system solved by (ω,θ), $\omega = \mathrm{curl}\, v = \partial_1 v^2 - \partial_2 v^1$. Indeed, if we neglect the nonlinear terms for the moment, one gets the system

$$\partial_t \omega = \partial_1 \theta, \quad \partial_t \theta = -|\mathrm{D}|\theta$$

and we notice that its symbol given by

$$\mathcal{A}(\xi) = \begin{pmatrix} 0 & i\xi_1 \\ 0 & -|\xi| \end{pmatrix}$$

is diagonalizable for $\xi \neq 0$ with two real distinct eigenvalues which are 0 and $-|\xi|$. By using the Riesz transform $\mathcal{R} = \partial_1/|\mathrm{D}|$, one gets that the diagonal form of the system is given by

$$\partial_t \mathcal{R}\theta = |\mathrm{D}|\mathcal{R}\theta, \quad \partial_t\big(\omega + \mathcal{R}\theta\big) = 0.$$

This last form of the system is much more convenient in order to perform *a priori* estimates. To prove Theorem 1.1, we shall use the same idea, we shall diagonalize the linear part of the system and then get *a priori* estimates from the study of the new system. The main technical difficulty in this program when one takes the nonlinear terms into account is to evaluate in a sufficiently sharp way the commutator $[\mathcal{R}, v\cdot\nabla]$ between the Riesz transform and the convection operator. Such commutator estimates are stated and proven in section 3 of the paper.
The diagonalization approach used in this paper also allows to prove global well-posedness in different spaces for a "Boussinesq-Navier-Stokes" system i.e. the system which corresponds to $\mathcal{D}_\theta = 0$ and $\mathcal{D}_v = -|\mathrm{D}|$. This is discussed in a companion paper [18].

The remaining of the paper is organized as follows. In section 2 we recall some functional spaces and we give some of their useful properties. Section 3 is devoted to the study of some commutators involving the Riesz transform. In section 4 we study a linear transport-(fractional) diffusion equation. Especially, we establish some smoothing effects and a logarithmic estimate type.
In section 5 we give the proof of Theorem 1.1 which is It is splitted into three parts. We first establish some *a priori* estimates, then we prove the uniqueness and finally we briefly explain how one can easily combine a procedure of smoothing out of the initial data with the *a priori* estimates to get the existence part of the theorem. An appendix is devoted to the proof of a technical commutator lemma.

2. Notations and preliminaries

2.1. **Notations.** Throughout this work we will use the following notations.
- For any positive A and B the notation $A \lesssim B$ means that there exist a positive harmless constant C such that $A \le CB$.
- For any tempered distribution u both $\hat{u}$ and $\mathcal{F}u$ denote the Fourier transform of u.
- Pour every $p \in [1,\infty]$, $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p.
- The norm in the mixed space time Lebesgue space $L^p([0,T], L^r(\mathbb{R}^d)$ is denoted by $\|\cdot\|_{L^p_T L^r}$ (with the obvious generalization to $\|\cdot\|_{L^p_T\mathcal{X}}$ for any normed space $\mathcal{X}$).
- For any pair of operators P and Q on some Banach space $\mathcal{X}$, the commutator $[P,Q]$ is given by $PQ - QP$.
- For $p \in [1,\infty]$, we denote by $\dot{W}^{1,p}$ the space of distributions u such that $\nabla u \in L^p$.

2.2. **Functional spaces.** Let us introduce the so-called Littlewood-Paley decomposition and the corresponding cut-off operators. There exists two radial positive functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d\backslash\{0\})$ such that

i) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall\, q \geq 1, \text{supp}\ \chi \cap \text{supp}\ \varphi(2^{-q}) = \varnothing$

ii) $\text{supp}\ \varphi(2^{-j}\cdot) \cap \text{supp}\ \varphi(2^{-k}\cdot) = \varnothing$, if $|j-k| \geq 2$.

For every $v \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1} v = \chi(\mathrm{D})v\ ;\ \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad S_q = \sum_{j=-1}^{q-1} \Delta_j.$$

The homogeneous operators are defined by

$$\dot{\Delta}_q v = \varphi(2^{-q}\mathrm{D})v, \quad \dot{S}_q v = \sum_{j\leq q-1} \dot{\Delta}_j v, \quad \forall q \in \mathbb{Z}.$$

From [4] we split the product uv into three parts:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u,v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

Let $T>0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \Big\| \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^\rho_T L^p}\Big)_{\ell^r} < +\infty.$$

By a direct application of the Minkowski inequality, we have the following links between these spaces. Let $\varepsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \text{ if } \quad r \geq \rho,$$

$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \text{ if } \quad \rho \geq r.$$

We will make continuous use of Bernstein inequalities (see [8] for instance).

Lemma 2.1. *There exists a constant C such that for $q,k\in\mathbb{N}$, $1\le a\le b$ and for $f\in L^a(\mathbb{R}^d)$,*

$$\begin{aligned}\sup_{|\alpha|=k}\|\partial^\alpha S_q f\|_{L^b} &\le C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q f\|_{L^a},\\ C^{-k}2^{qk}\|\Delta_q f\|_{L^a} &\le \sup_{|\alpha|=k}\|\partial^\alpha\Delta_q f\|_{L^a}\le C^k 2^{qk}\|\Delta_q f\|_{L^a}.\end{aligned}$$

3. Riesz transform and commutators

In the next proposition we gather some properties of the Riez operator $\mathcal{R}=\partial_1/|D|$.

Proposition 3.1. *Let $\mathcal{R}$ be the Riez operator $\mathcal{R}=\partial_1/|D|$. Then the following hold true.*

1. *For every $p\in]1,+\infty[$,*
$$\|\mathcal{R}\|_{\mathcal{L}(L^p)}\lesssim 1.$$
2. *Let $\chi\in\mathcal{D}(\mathbb{R}^d)$. Then, there exists $C>0$ such that*
$$\||\mathrm{D}|^s\chi(2^{-q}|\mathrm{D}|)\mathcal{R}\|_{\mathcal{L}(L^p)}\le C2^{qs},$$
for every $(p,s,q)\in[1,\infty]\times]0,+\infty[\times\mathbb{N}$.
We notice that the previous results hold true if we change $|\mathrm{D}|^s$ by ∇^s with $s\in\mathbb{N}$.
3. *Let $\mathcal{C}$ be a fixed ring. Then, there exists $\psi\in\mathcal{S}$ whose spectum does not meet the origin such that*
$$\mathcal{R}f=2^{qd}\psi(2^q\cdot)\star f$$
for every f with Fourier transform supported in $2^q\mathcal{C}$.

Proof. (1) It is a classical Calderón-Zygmund theorem (see [24] for instance).
(2) Let $K\in\mathcal{S}'$ such that $\mathcal{F}K(\xi)=|\xi|^s\chi(|\xi|)\frac{\xi_i}{|\xi|}$. K is a tempered distribution such that its Fourier transform $\mathcal{F}K$ is $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ and satisfies for every $\alpha\in\mathbb{N}^d$

$$|\partial_\xi^\alpha\mathcal{F}K(\xi)|\le C_\alpha|\xi|^{s-|\alpha|}.$$

According to Mikhlin-Hörmander Theorem (see [24] for instance) we have for every $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d\setminus\{0\}$,

$$|\partial_x^\alpha K(x)|\le C'_\alpha|x|^{-s-d-|\alpha|}.$$

Then, it ensues that

$$|K(x)|\le C|x|^{-d-s}\qquad\forall x\neq 0.$$

Since $\mathcal{F}K$ is obviously in L^1, we also have that $K\in\mathcal{C}_0(\mathbb{R}^d)$. This removes the singularity at the origin and gives

$$|K(x)|\le C(1+|x|)^{-d-s}\qquad\forall x\in\mathbb{R}^d.$$

Therefore we get that the kernel $K\in L^1$. Now for every $u\in L^p$ we have $|\mathrm{D}|^s\mathcal{R}\chi(|\mathrm{D}|)u=K\star u$. We can now conclude the case $q=0$ by using the classical Young inequality for convolution products.
The case $q\ge 1$ can be derived from $q=0$ via an obvious argument of homogeneity.

(3) This is can be done easily by introducing a judiciously chosen cut-off function. □

The following lemma will be useful in the proof of many commutator estimates.

Lemma 3.2. *Given $(p,m)\in[1,\infty]^2$ such that $p\geq m'$ with m' the conjugate exponent of m. Let f,g and h be three functions such that $\nabla f\in L^p, g\in L^m$ and $xh\in L^{m'}$. Then,*

$$\|h\star(fg)-f(h\star g)\|_{L^p}\leq\|xh\|_{L^{m'}}\|\nabla f\|_{L^p}\|g\|_{L^m}.$$

Proof. We have by definition and Taylor formula

$$\begin{aligned}h\star(fg)(x)-f(h\star g)(x)&=\int_{\mathbb{R}^d}h(x-y)g(y)\big(f(y)-f(x)\big)dy\\&=\int_0^1\int_{\mathbb{R}^d}g(y)h(x-y)\Big[(y-x)\cdot\nabla f(x+t(y-x))\Big]dydt.\end{aligned}$$

Using Hölder inequalities and making a change of variables $z=t(x-y)$ we get

$$|h\star(fg)(x)-f(h\star g)(x)|\leq\|g\|_{L^m}\int_0^1\Big(\int_{\mathbb{R}^d}t^{-d}h_1^{m'}(t^{-1}z)|\nabla f|^{m'}(x-z)dz\Big)^{\frac{1}{m'}},$$

where we set $h_1(z)=|z||h(z)|$. Using Convolution inequalities we obtain since $p\geq m'$

$$\begin{aligned}\|h\star(fg)-f(h\star g)\|_{L^p}&\leq\|g\|_{L^m}\|h_1^{m'}\|_{L^1}^{\frac{1}{m'}}\||\nabla f|^{m'}\|_{L^{\frac{p}{m'}}}^{\frac{1}{m'}}\\&\leq\|g\|_{L^m}\|h_1\|_{L^{m'}}\|\nabla f\|_{L^p}.\end{aligned}$$

□

As explained in the introduction, the control of the commutator between $\mathcal{R}$ and the convection operator $v\cdot\nabla$ is a crucial ingredient in the proof of Theorem 1.1.

Theorem 3.3. *Let v be is a smooth divergence-free vector field.*

(1) *For every $(p,r)\in[2,\infty[\times[1,\infty]$ there exists a constant $C=C(p,r)$ such that*

$$\|[\mathcal{R},v\cdot\nabla]\theta\|_{B^0_{p,r}}\leq C\|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,r}}+\|\theta\|_{L^p}\big),$$

for every smooth scalar function θ.

(2) *For every $(r,\rho)\in[1,\infty]\times]1,\infty[$ and $\epsilon>0$ there exists a constant $C=C(r,\rho,\varepsilon)$ such that*

$$\|[\mathcal{R},v\cdot\nabla]\theta\|_{B^0_{\infty,r}}\leq C(\|\omega\|_{L^\infty}+\|\omega\|_{L^\rho})\big(\|\theta\|_{B^\epsilon_{\infty,r}}+\|\theta\|_{L^\rho}\big),$$

for every smooth scalar function θ.

Proof. We split the commutator into three parts, according to Bony's decomposition,

$$\begin{aligned}[\mathcal{R}, v\cdot\nabla]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R}, S_{q-1}v\cdot\nabla]\Delta_q\theta + \sum_{q\in\mathbb{N}}[\mathcal{R}, \Delta_q v\cdot\nabla]S_{q-1}\theta \\ &+ \sum_{q\geq -1}[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta \\ &= \sum_{q\in\mathbb{N}}\mathrm{I}_q + \sum_{q\in\mathbb{N}}\mathrm{II}_q + \sum_{q\geq -1}\mathrm{III}_q \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

We start with the estimate of the first term I. According to the point (3) of Proposition 3.1 there exists $h\in\mathcal{S}$ whose spectum does not meet the origin such that

$$\mathrm{I}_q(x) = h_q\star(S_{q-1}v\cdot\nabla\Delta_q\theta) - S_{q-1}v\cdot(h_q\star\nabla\Delta_q\theta),$$

where $h_q(x) = 2^{dq}h(2^q x)$. Applying Lemma 3.2 with $m=\infty$ we get

$$\begin{aligned}\|\mathrm{I}_q\|_{L^p} &\lesssim \|xh_q\|_{L^1}\|\nabla S_{q-1}v\|_{L^p}\|\Delta_q\nabla\theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p}\|\Delta_q\theta\|_{L^\infty}. \end{aligned}\tag{3.1}$$

In the last line we've used Bernstein inequality and $\|xh_q\|_{L^1} = 2^{-q}\|xh\|_{L^1}$. Combined with the trivial fact

$$\Delta_j\sum_q \mathrm{I}_q = \sum_{|j-q|\leq 4}\mathrm{I}_q$$

this yields

$$\begin{aligned}\|\mathrm{I}\|_{B^0_{p,r}} &\lesssim \Big(\sum_{q\geq -1}\|\mathrm{I}_q\|^r_{L^p}\Big)^{\frac{1}{r}} \\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,r}}.\end{aligned}$$

Let us move to the second term II. As before one writes

$$\mathrm{II}_q(x) = h_q\star(\Delta_q v\cdot\nabla S_{q-1}\theta) - \Delta_q v\cdot(h_q\star\nabla S_{q-1}\theta),$$

and then we obtain the estimate

$$\begin{aligned}\|\mathrm{II}_q\|_{L^p} &\lesssim 2^{-q}\|\Delta_q\nabla v\|_{L^p}\|S_{q-1}\nabla\theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p}\sum_{j\leq q-2}2^{j-q}\|\Delta_j\theta\|_{L^\infty}.\end{aligned}$$

Combined with convolution inequalities this yields

$$\|\mathrm{II}\|_{B^0_{p,r}} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,r}}.$$

Let us now deal with the third term III. Using that the divergence of $\Delta_q v$ vanishes, we rewrite III as

$$\begin{aligned}\mathrm{III} &= \sum_{q\geq 2}\mathcal{R}\mathrm{div}(\Delta_q v\,\widetilde{\Delta}_q\theta) - \sum_{q\geq 2}\mathrm{div}(\Delta_q v\,\mathcal{R}\widetilde{\Delta}_q\theta) + \sum_{q\leq 1}[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta \\ &= J_1 + J_2 + J_3.\end{aligned}$$

Using Proposition 3.1-(2), we get

$$\big\|\Delta_j\mathcal{R}\mathrm{div}(\Delta_q v\,\widetilde{\Delta}_q\theta)\big\|_{L^p}\lesssim 2^j\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}.$$

Also, since $\widetilde{\Delta}_q\theta$ is supported away from zero for $q\geq 2$ then Proposition 3.1 (3) yields

$$\begin{aligned}\big\|\Delta_j\mathrm{div}(\Delta_q v\,\mathcal{R}\widetilde{\Delta}_q\theta)\big\|_{L^p} &\lesssim 2^j\|\Delta_q v\|_{L^p}\|\mathcal{R}\widetilde{\Delta}_q\theta\|_{L^\infty}\\ &\lesssim 2^j\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}.\end{aligned}$$

Therefore we get

$$\begin{aligned}\|\Delta_j(J_1+J_2)\|_{L^p} &\lesssim \sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}} 2^j\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}} 2^{j-q}\|\Delta_q\theta\|_{L^\infty},\end{aligned}$$

where we have again used Bernstein inequality to get the last line. It suffices now to use convolution inequalities to get

$$\|J_1+J_2\|_{B^0_{p,r}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,r}}.$$

For the last term J_3 we can write

$$\sum_{-1\leq q\leq 1}[\mathcal{R},\Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta(x)=\sum_{q\leq 1}[\mathrm{div}\;\widetilde{\chi}(\mathrm{D})\mathcal{R},\Delta_q v]\widetilde{\Delta}_q\theta(x),$$

where $\widetilde{\chi}$ belongs to $\mathcal{D}(\mathbb{R}^d)$. Proposition 3.1 ensures that div $\widetilde{\chi}(\mathrm{D})\mathcal{R}$ is a convolution operator with a kernel $\tilde{h}$ satisfying

$$|\tilde{h}(x)|\lesssim(1+|x|)^{-d-1}.$$

Thus

$$J_3=\sum_{q\leq 1}\tilde{h}\star(\Delta_q v\cdot\tilde{\Delta}_q\theta)-\Delta_q v\cdot(\tilde{h}\star\tilde{\Delta}_q\theta).$$

First of all we point out that $\Delta_j J_3=0$ for $j\geq 6$, thus we just need to estimate the low frequencies of J_3. Noticing that $x\tilde{h}$ belongs to $L^{p'}$ for $p'>1$ then using Lemma 3.2 with $m=p\geq 2$ we obtain

$$\begin{aligned}\|\Delta_j J_3\|_{L^\infty} &\lesssim \sum_{q\leq 1}\|x\tilde{h}\|_{L^{p'}}\|\Delta_q\nabla v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^p}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{-1\leq q\leq 1}\|\Delta_q\theta\|_{L^p}.\end{aligned}$$

This yields finally

$$\|J_3\|_{B^0_{p,r}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{L^p}.$$

This completes the proof of the first part of Theorem 3.3. The second part can be done in the same way so that we will only give here a shorten proof. To estimate the terms I and II we use two facts: the first one is $\|\Delta_q\nabla u\|_{L^\infty}\approx\|\Delta_q\omega\|_{L^\infty}$ for all $q\in\mathbb{N}$. The second one is

$$\begin{aligned}\|\nabla S_{q-1}v\|_{L^\infty} &\lesssim \|\nabla\Delta_{-1}v\|_{L^\infty}+\sum_{j=0}^{q-2}\|\Delta_j\nabla v\|_{L^\infty}\\ &\lesssim \|\omega\|_{L^\rho}+q\|\omega\|_{L^\infty}.\end{aligned}$$

For the remainder term we do strictly the same analysis as before except for J_3: we apply Lemma 3.2 with $p=\infty$ and $m=\rho$ leading to

$$\begin{aligned}\|\Delta_j J_3\|_{L^p} &\lesssim \sum_{q\leq 1}\|x\tilde{h}\|_{L^{\rho'}}\|\Delta_q\nabla v\|_{L^\infty}\|\widetilde{\Delta}_q\theta\|_{L^\rho}\\ &\lesssim \|\nabla v\|_{L^\rho}\sum_{-1\leq q\leq 1}\|\Delta_q\theta\|_{L^\rho}\\ &\lesssim \|\omega\|_{L^\rho}\|\theta\|_{L^\rho}.\end{aligned}$$

This ends the proof of the theorem.

□

4. Transport-Diffusion equation

In this section we will give some useful estimates for any smooth solution of a linear transport-diffusion model given by

$$\text{(TD)}\qquad \begin{cases}\partial_t\theta+v\cdot\nabla\theta+|\mathrm{D}|\theta=f\\ \theta_{|t=0}=\theta^0.\end{cases}$$

We will discuss three kinds of estimates: L^p estimates, smoothing effects and logarithmic estimates.
The proof of the following L^p estimates can be found in [13].

Proposition 4.1. *Let v be a smooth divergence-free vector field of $\mathbb{R}^d$ and θ be a smooth solution of* (TD)*. Then we have for every $p\in[1,\infty]$*

$$\|\theta(t)\|_{L^p}\leq\|\theta^0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}d\tau.$$

We intend to prove the following smoothing effect.

Theorem 4.2. *Let v be a smooth divergence-free vector field of $\mathbb{R}^d$ with vorticity ω. Then, for every $p\in[1,\infty[$ there exists a constant C such that*

$$\sup_{q\in\mathbb{N}}2^q\|\Delta_q\theta\|_{L^1_tL^p}\leq C\|\theta^0\|_{L^p}+C\|\theta^0\|_{L^\infty}\|\omega\|_{L^1_tL^p},$$

for every smooth solution θ of (TD) *with $f\equiv 0$.*

Proof. We start with localizing in frequencies the equation: for $q\geq -1$ we set $\theta_q:=\Delta_q\theta$. Then

$$\partial_t\theta_q+v\cdot\nabla\theta_q+|\mathrm{D}|\theta_q=-[\Delta_q,v\cdot\nabla]\theta.$$

Recall that θ_q is real function since the functions involved in the dyadic partition of the unity are radial. Then multiplying the above equation by $|\theta_q|^{p-2}\theta_q$, integrating by parts and using Hölder inequalities we get

$$\frac{1}{p}\frac{d}{dt}\|\theta_q\|_{L^p}^p+\int_{\mathbb{R}^2}(|\mathrm{D}|\theta_q)|\theta_q|^{p-2}\theta_q dx\leq\|\theta_q\|_{L^p}^{p-1}\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}.$$

Recall from [9] the following generalized Bernstein inequality

$$c2^q\|\theta_q\|_{L^p}^p\leq\int_{\mathbb{R}^2}(|\mathrm{D}|\theta_q)|\theta_q|^{p-2}\theta_q dx,$$

where c depends on p. Inserting this estimate in the previous one we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta_q\|_{L^p}^p+c2^q\|\theta_q\|_{L^p}^p\lesssim\|\theta_q\|_{L^p}^{p-1}\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}.$$

Thus we find

$$\frac{d}{dt}\|\theta_q\|_{L^p}+c2^q\|\theta_q\|_{L^p}\lesssim\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}. \tag{4.1}$$

To estimate the right hand-side, we shall use the following lemma (see the appendix for the proof of this lemma).

Lemma 4.3. *Let v be a smooth divergence-free vector field and θ be a smooth scalar function. Then, for all $p\in[1,\infty]$ and $q\geq-1$,*

$$\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.$$

Combined with (4.1) this lemma yields

$$\begin{aligned}\frac{d}{dt}\big(e^{ct2^q}\|\theta_q(t)\|_{L^p}\big)&\lesssim e^{ct2^q}\|\nabla v(t)\|_{L^p}\|\theta(t)\|_{B^0_{\infty,\infty}}\\&\lesssim e^{ct2^q}\|\omega(t)\|_{L^p}\|\theta^0\|_{L^\infty}.\end{aligned}$$

To get the last line, we have used the conservation of the L^∞ norm of θ and the classical fact

$$\|\nabla v\|_{L^p}\lesssim\|\omega\|_{L^p}\qquad\forall p\in]1,+\infty[.$$

Integrating the differential inequality we get

$$\|\theta_q(t)\|_{L^p}\lesssim\|\theta_q^0\|_{L^p}e^{-ct2^q}+\|\theta^0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^q}\|\omega(\tau)\|_{L^p}d\tau.$$

Integrating in time yields finally

$$\begin{aligned}2^q\|\theta_q\|_{L^1_tL^p}&\lesssim\|\theta_q^0\|_{L^p}+\|\theta^0\|_{L^\infty}\int_0^t\|\omega(\tau)\|_{L^p}d\tau\\&\lesssim\|\theta^0\|_{L^p}+\|\theta^0\|_{L^\infty}\int_0^t\|\omega(\tau)\|_{L^p}d\tau,\end{aligned}$$

which is the desired result. □

Let us now move to the last part of this section which deals with some logarithmic estimates generalizing the results of [26, 16]. First we recall the following result of propagation of Besov regularities.

Proposition 4.4. *Let* $(p,r)\in[1,\infty]^2, s\in]-1,1[$ *and* θ *a smooth solution of* (TD). *Then we have*

$$\|\theta\|_{\widetilde{L}^\infty_t B^s_{p,r}}\lesssim e^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,r}}+\int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

where $V(t)=\|\nabla v\|_{L^1_tL^\infty}$.

The proof of this result is omitted here and it can be done similarly to the inviscid case [8], using especially Proposition 4.1.
Now we will show that for the index regularity $s=0$ we can obtain a better estimate with a linear growth on Lipschitz norm of the velocity.

Theorem 4.5. *There exists* $C>0$ *such that if* $\kappa\geq 0$, $p\in[1,\infty]$ *and* θ *a solution of*

$$(\partial_t+v\cdot\nabla+\kappa|\mathrm{D}|)\theta=f,$$

then we have

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{p,1}}\leq C\Big(\|\theta^0\|_{B^0_{p,1}}+\|f\|_{L^1_tB^0_{p,1}}\Big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big).$$

Proof. We mention that the result is first proved in [26] for the case $\kappa=0$ by using the special structure of the transport equation. In [17] the first two authors generalized Vishik's result for a transport-diffusion equation where the dissipation term has the form $-\kappa\Delta\theta$. The method described in [17] can be easily adapted here for our model.
Let $q\in\mathbb{N}\cup\{-1\}$ and denote by $\overline{\theta}_q$ the unique global solution of the initial value problem

$$\left\{\begin{array}{l}\partial_t\overline{\theta}_q+v\cdot\nabla\overline{\theta}_q+|\mathrm{D}|\overline{\theta}_q=\Delta_q f,\\ \overline{\theta}_{q|t=0}=\Delta_q\theta^0.\end{array}\right.\tag{4.2}$$

Using Proposition 4.4 with $s=\pm\frac12$ we get

$$\|\overline{\theta}_q\|_{\widetilde{L}^\infty_t B^{\pm\frac12}_{p,\infty}}\lesssim\big(\|\Delta_q\theta^0\|_{B^{\pm\frac12}_{p,\infty}}+\|\Delta_q f\|_{L^1_tB^{\pm\frac12}_{p,\infty}}\big)e^{CV(t)},$$

where $V(t)=\|\nabla v\|_{L^1_tL^\infty}$. Combined with the definition of Besov spaces this yields for $j,q\geq -1$

$$\|\Delta_j\overline{\theta}_q\|_{L^\infty_tL^p}\lesssim 2^{-\frac12|j-q|}\big(\|\Delta_q\theta^0\|_{L^p}+\|\Delta_q f\|_{L^1_tL^p}\big)e^{CV(t)}.\tag{4.3}$$

By linearity and again the definition of Besov spaces we have

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{p,1}}\leq\sum_{|j-q|\geq N}\|\Delta_j\overline{\theta}_q\|_{L^\infty_tL^p}+\sum_{|j-q|<N}\|\Delta_j\overline{\theta}_q\|_{L^\infty_tL^p},\tag{4.4}$$

where $N\in\mathbb{N}$ is to be chosen later. To deal with the first sum we use (4.3)

$$\begin{aligned}\sum_{|j-q|\geq N}\|\Delta_j\overline{\theta}_q\|_{L^\infty_tL^p}&\lesssim 2^{-N/2}\sum_{q\geq-1}\big(\|\Delta_q\theta^0\|_{L^p}+\|\Delta_q f\|_{L^1_tL^p}\big)e^{CV(t)}\\&\lesssim 2^{-N/2}\big(\|\theta^0\|_{B^0_{p,1}}+\|f\|_{L^1_tB^0_{p,1}}\big)e^{CV(t)}.\end{aligned}$$

We now turn to the second sum in the right-hand side of (4.4).
It is clear that

$$\sum_{|j-q|<N}\|\Delta_j\overline{\theta}_q\|_{L^\infty_t L^p}\lesssim\sum_{|j-q|<N}\|\overline{\theta}_q\|_{L^\infty_t L^p}.$$

Applying Proposition 4.1 to the system (4.2) yields

$$\|\overline{\theta}_q\|_{L^\infty_t L^p}\leq\|\Delta_q\theta^0\|_{L^p}+\|\Delta_q f\|_{L^1_t L^p}.$$

It follows that

$$\sum_{|j-q|<N}\|\Delta_j\overline{\theta}_q\|_{L^\infty_t L^p}\lesssim N\big(\|\theta^0\|_{B^0_{p,1}}+\|f\|_{L^1_t B^0_{p,1}}\big).$$

The outcome is the following

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{p,1}}\lesssim\big(\|\theta^0\|_{B^0_{p,1}}+\|f\|_{L^1_t B^0_{p,1}}\big)\Big(2^{-N/2}e^{CV(t)}+N\Big).$$

Choosing

$$N=\Big[\frac{2CV(t)}{\log 2}\Big]+1,$$

we get the desired result.

□

5. Proof of Theorem 1.1

Throughout this section we use the notation Φ_k to denote any function of the form

$$\Phi_k(t)=C_0\underbrace{\exp(\ldots\exp}_{k\,times}(C_0t^2)\ldots),$$

where C_0 depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentionning it) of the following trivial facts

$$\int_0^t\Phi_k(\tau)d\tau\leq\Phi_k(t)\qquad\text{and}\qquad\exp\Big(\int_0^t\Phi_k(\tau)d\tau\Big)\leq\Phi_{k+1}(t).$$

The proof of Theorem 1.1 will be done in several steps. The first one deals with some *a priori* estimates for the equations (1.2). In the second one we prove the uniqueness part. Finally, we will discuss the construction of the solutions at the end of this section.

5.1. *a priori estimates.* As we will see the important quantities to bound for all times are $\|\omega(t)\|_{L^\infty}$ and $\|\nabla v(t)\|_{L^\infty}$. It seems that for subcritical regularities like for example $H^s, s>2$ or more generally $B^s_{p,r}$, $s>1+\frac{2}{p}$ we need only to bound the quantity $\|\partial_1\theta\|_{L^1_t L^\infty}$, which in turn controls $\|\omega(t)\|_{L^\infty}$, due to Brezis-Galloüet logarithmic estimate, see for example [23]. Even though these quantities seem to be less regular than $\|\nabla v\|_{L^1_t L^\infty}$, it is not at all clear how to estimate them without involving the latter quantity.
When we deal with critical regularities which is our subject here one needs to bound the Lipschitz norm of the velocity and this will require some refinement analysis,

especially Theorem 4.5 seems to be very crucial. To obtain a Lipschitz bound we will proceed in several steps: one of the main step is to give an L^∞-bound of the vorticity but due to some technical difficulties related to Riesz transforms this will be not done in a straight way. We prove before an L^p estimate with $2<p<\infty$ and this allows us to bound the vorticity in L^∞.
We start with recalling the L^p estimate for the temperature function. It is a direct consequence of Proposition 4.1.

Proposition 5.1. *Let (v,θ) be a smooth solution of (1.2), then for all $p\in[1,+\infty]$*

$$\|\theta(t)\|_{L^p}\le\|\theta^0\|_{L^p}.$$

We intend now to bound the L^p-norm of the vorticity and to describe a smoothing effect for the temperature.

Proposition 5.2. *If $\omega^0\in L^p$ and $\theta^0\in L^p\cap L^\infty$ with $p\in]2,\infty[$, then*

$$\|\omega(t)\|_{L^p}+\|\theta\|_{\widetilde L^1_tB^1_{p,\infty}}\le\Phi_1(t).$$

Proof. Applying Riesz transform $\mathcal{R}$ to the temperature equation we get

$$\partial_t\mathcal{R}\theta+v\cdot\nabla\mathcal{R}\theta+|\mathrm{D}|\mathcal{R}\theta=-[\mathcal{R},v\cdot\nabla]\theta. \tag{5.1}$$

Since $|\mathrm{D}|\mathcal{R}=\partial_1$ then the function $\Gamma:=\omega+\mathcal{R}\theta$ satisfies

$$\partial_t\Gamma+v\cdot\nabla\Gamma=-[\mathcal{R},v\cdot\nabla]\theta. \tag{5.2}$$

According to the first part of Theorem 3.3 applied with $r=2$ we have

$$\big\|[\mathcal{R},v\cdot\nabla]\theta\big\|_{B^0_{p,2}}\lesssim\|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,2}}+\|\theta\|_{L^p}\big).$$

Using the classical embedding $B^0_{p,2}\hookrightarrow L^p$ which is true only for $p\in[2,\infty)$

$$\big\|[\mathcal{R},v\cdot\nabla]\theta\big\|_{L^p}\le\|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,2}}+\|\theta\|_{L^p}\big).$$

Since $\operatorname{div} v=0$ then we get from the transport equation (5.2)

$$\|\Gamma(t)\|_{L^p}\le\|\Gamma^0\|_{L^p}+\int_0^t\|[\mathcal{R},v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.$$

Putting together the last two estimates we get

$$\begin{aligned}\|\Gamma(t)\|_{L^p}&\lesssim\|\Gamma^0\|_{L^p}+\int_0^t\|\nabla v(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^0_{\infty,2}}+\|\theta\|_{L^p}\big)d\tau\\&\lesssim\|\omega^0\|_{L^p}+\|\theta^0\|_{L^p}+\int_0^t\|\omega(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^0_{\infty,2}}+\|\theta^0\|_{L^p}\big)d\tau.\end{aligned}$$

We have used here the Calderón-Zygmund estimates: for $p\in(1,\infty)$

$$\|\nabla v\|_{L^p}\le C\|\omega\|_{L^p}\quad\text{and}\quad\|\mathcal{R}\theta^0\|_{L^p}\le C\|\theta^0\|_{L^p}.$$

On the other hand, from the continuity of the Riesz transform and Proposition 5.1

$$\begin{aligned}\|\omega(t)\|_{L^p}&\le\|\Gamma(t)\|_{L^p}+\|\mathcal{R}\theta\|_{L^p}\\&\lesssim\|\Gamma(t)\|_{L^p}+\|\theta^0\|_{L^p}.\end{aligned}$$

This leads to

$$\|\omega(t)\|_{L^p} \lesssim \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \|\omega(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^0_{\infty,2}} + \|\theta^0\|_{L^p}\big)d\tau.$$

According to Gronwall lemma we get

$$\|\omega(t)\|_{L^p} \leq C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}}. \tag{5.3}$$

Let $N \in \mathbb{N}$, then we have by Bernstein inequalities and Proposition 5.2

$$\begin{aligned}
\|\theta\|_{L^1_t B^0_{\infty,2}} &\leq \|S_N\theta\|_{L^1_t B^0_{\infty,2}} + \|(\mathrm{Id} - S_N)\theta\|_{L^1_t B^0_{\infty,1}} \\
&\lesssim t\|\theta^0\|_{L^\infty}\sqrt{N} + \sum_{q \geq N} \|\Delta_q \theta\|_{L^1_t L^\infty} \\
&\lesssim \sqrt{N}\|\theta^0\|_{L^\infty} t + \sum_{q \geq N} 2^{q\frac{2}{p}} \|\Delta_q \theta\|_{L^1_t L^p}.
\end{aligned}$$

Using Theorem 4.2 and $p > 2$ we obtain

$$\begin{aligned}
\sum_{q \geq N-1} 2^{q\frac{2}{p}} \|\Delta_q \theta\|_{L^1_t L^p} &\lesssim \sum_{q \geq N-1} 2^{q(\frac{2}{p}-1)}\Big(\|\theta^0\|_{L^p} + \|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^p} d\tau\Big) \\
&\lesssim \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})}\|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^p} d\tau.
\end{aligned}$$

Thus, we get

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \lesssim \sqrt{N}\|\theta^0\|_{L^\infty} t + \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})}\|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^p} d\tau.$$

We choose N as follows

$$N = \left[\frac{\log\Big(e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau\Big)}{(1 - 2/p)\log 2}\right] + 1.$$

Then it follows

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t \log^{\frac{1}{2}}\Big(e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau\Big).$$

Combining this estimate with (5.3) we get

$$\begin{aligned}
\|\theta\|_{L^1_t B^0_{\infty,2}} &\lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t\, \log^{\frac{1}{2}}\Big(e + C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}}\Big) \\
&\leq C_0(1 + t^2) + C\|\theta^0\|_{L^\infty} t \|\theta\|^{\frac{1}{2}}_{L^1_t B^0_{\infty,2}}.
\end{aligned}$$

Thus we get for every $t \in \mathbb{R}+$

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \leq C_0(1 + t^2).$$

It follows from (5.3)

$$\|\omega(t)\|_{L^p} \leq \Phi_1(t). \tag{5.4}$$

Applying Theorem 4.2 and (5.4) we get

$$2^q\|\Delta_q\theta\|_{L^1_tL^p}\leq\Phi_1(t),\qquad\forall q\in\mathbb{N}\tag{5.5}$$

and thus

$$\|\theta\|_{\widetilde{L}^1_tB^1_{p,\infty}}\leq\Phi_1(t).$$

This ends the proof of Proposition 5.2.

Remark 5.3. It is not hard to see that from (5.5) one can obtain that for every $s<1$

$$\|\theta\|_{L^1_tB^s_{p,1}}\leq\|\theta\|_{\widetilde{L}^1_tB^1_{p,\infty}}\leq\Phi_1(t).\tag{5.6}$$

Combined with Bernstein inequalities and the fact that $p>2$ this yields

$$\|\theta\|_{L^1_tB^\epsilon_{\infty,1}}\leq\Phi_1(t),\tag{5.7}$$

for every $\epsilon<1-\frac{2}{p}$.

□

We aim now at giving an L^∞-bound of the vorticity.

Proposition 5.4. *Let* (v,θ) *be a smooth solution of* (1.2) *such that* $\omega^0,\theta^0\in L^p\cap L^\infty$ *and* $\mathcal{R}\theta^0\in L^\infty$, *with* $2<p<\infty$. *Then we have*

$$\|\omega(t)\|_{L^\infty}+\|\mathcal{R}\theta(t)\|_{L^\infty}\leq\Phi_2(t)\tag{5.8}$$

and

$$\|v(t)\|_{L^\infty}\leq\Phi_3(t).\tag{5.9}$$

Proof.
• *Proof of* (5.8). By using the maximum principle for the transport equation (5.2), we get

$$\|\Gamma(t)\|_{L^\infty}\leq\|\Gamma^0\|_{L^\infty}+\int_0^t\|[\mathcal{R},v\cdot\nabla]\theta(\tau)\|_{L^\infty}d\tau.$$

Since the function $\mathcal{R}\theta$ satisfies the equation

$$\big(\partial_t+v\cdot\nabla+|\mathrm{D}|\big)\mathcal{R}\theta=-[\mathcal{R},v\cdot\nabla]\theta,\tag{5.10}$$

we get by using Proposition 4.1 for $p=\infty$ that

$$\|\mathcal{R}\theta(t)\|_{L^\infty}\leq\|\mathcal{R}\theta(t)\|_{L^\infty}+\int_0^t\|[\mathcal{R},v\cdot\nabla]\theta(\tau)\|_{L^\infty}d\tau.$$

Combining the last two estimates yields

$$\begin{aligned}\|\Gamma(t)\|_{L^\infty}+\|\mathcal{R}\theta(t)\|_{L^\infty}&\leq\|\Gamma^0\|_{L^\infty}+\|\mathcal{R}\theta^0\|_{L^\infty}+2\int_0^t\|[\mathcal{R},v\cdot\nabla]\theta(\tau)\|_{L^\infty}d\tau\\&\leq C_0+\int_0^t\|[\mathcal{R},v\cdot\nabla]\theta(\tau)\|_{B^0_{\infty,1}}d\tau.\end{aligned}$$

It follows from the second estimate of Theorem 3.3 and Proposition 5.2

$$\begin{aligned}
\|\omega(t)\|_{L^\infty}+\|\mathcal{R}\theta(t)\|_{L^\infty} &\lesssim C_0+\int_0^t\|\omega(\tau)\|_{L^\infty\cap L^p}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta(\tau)\|_{L^p}\big)d\tau\\
&\lesssim C_0+\|\omega\|_{L^\infty_t L^p}\big(\|\theta\|_{L^1_t B^\epsilon_{\infty,1}}+t\|\theta^0\|_{L^p}\big)\\
&+\int_0^t\|\omega(\tau)\|_{L^\infty}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta^0\|_{L^p}\big)d\tau.
\end{aligned}$$

Let $0<\epsilon<1-\frac{2}{p}$ then using (5.7) we get

$$\|\omega(t)\|_{L^\infty}+\|\mathcal{R}\theta(t)\|_{L^\infty}\lesssim\Phi_1(t)+\int_0^t\|\omega(\tau)\|_{L^\infty}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta^0\|_{L^p}\big)d\tau.$$

Therefore we obtain by the Gronwall lemma and a new use of (5.7) that

$$\|\omega(t)\|_{L^\infty}+\|\mathcal{R}\theta(t)\|_{L^\infty}\le\Phi_2(t).$$

• *Proof of* (5.9).

Let $N\in\mathbb{N}$ to be chosen later. Using the fact that $\|\dot\Delta_q v\|_{L^\infty}\approx 2^{-q}\|\dot\Delta_q\omega\|_{L^\infty}$, we then have

$$\begin{aligned}
\|v(t)\|_{L^\infty} &\le \|\chi(2^N|\mathrm{D}|)v(t)\|_{L^\infty}+\sum_{q\ge -N}2^{-q}\|\dot\Delta_q\omega(t)\|_{L^\infty}\\
&\le \|\chi(2^N|\mathrm{D}|)v(t)\|_{L^\infty}+2^N\|\omega(t)\|_{L^\infty}.
\end{aligned}$$

Applying the frequency localizing operator to the velocity equation we get

$$\chi(2^N|\mathrm{D}|)v=\chi(2^N|\mathrm{D}|)v_0+\int_0^t\mathcal{P}\chi(2^N|\mathrm{D}|)\theta(\tau)d\tau+\int_0^t\mathcal{P}\chi(2^N|\mathrm{D}|)\mathrm{div}(v\otimes v)(\tau)d\tau.$$

where $\mathcal{P}$ stands for Leray projector. From Bernstein inequalities, Calderón-Zygmund estimate and the uniform boundness of $\chi(2^{-N}|\mathrm{D}|)$ we get

$$\begin{aligned}
\int_0^t\|\chi(2^N|\mathrm{D}|)\mathcal{P}\theta(\tau)\|_{L^\infty}d\tau &\lesssim 2^{-N\frac{2}{p}}\int_0^t\|\theta(\tau)\|_{L^p}d\tau\\
&\lesssim t\|\theta^0\|_{L^p}.
\end{aligned}$$

Using Proposition 3.1-(2) we find

$$\int_0^t\|\mathcal{P}\chi(2^N|\mathrm{D}|)\mathrm{div}(v\otimes v)(\tau)\|_{L^\infty}d\tau\lesssim 2^{-N}\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau.$$

The outcome is

$$\begin{aligned}
\|v(t)\|_{L^\infty} &\lesssim \|v^0\|_{L^\infty}+t\|\theta_0\|_{L^p}+2^{-N}\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau+2^N\|\omega(t)\|_{L^\infty}\\
&\lesssim 2^{-N}\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau+2^N\Phi_2(t)
\end{aligned}$$

Choosing judiciously N we find

$$\|v(t)\|_{L^\infty} \leq \Phi_2(t)\Big(1+\Big(\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau\Big)^{\frac{1}{2}}\Big).$$

From Gronwall lemma we get

$$\|v(t)\|_{L^\infty} \leq \Phi_3(t).$$

□

Now we will describe the last part of the *a priori* estimates. Following the program exposed in the beginning, the aim is to get estimates on ∇v.

Proposition 5.5. *Let (v,θ) be a smooth solution of (1.2) and $\omega^0,\theta^0 \in B^0_{\infty,1}\cap L^p$ with $p\in]2,\infty[$. Then*

$$\|\theta(t)\|_{B^0_{\infty,1}} + \|\omega(t)\|_{B^0_{\infty,1}} + \|v(t)\|_{B^1_{\infty,1}} \leq \Phi_3(t).$$

Proof. By using the logarithmic estimates of Theorem 4.5 for the equations (5.1) and (5.10), we obtain

$$\|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}\theta(t)\|_{B^0_{\infty,1}} \lesssim \Big(C_0 + \big\|[\mathcal{R}, v\cdot\nabla]\theta\big\|_{L^1_tB^0_{\infty,1}}\Big)\Big(1+\|\nabla v\|_{L^1_tL^\infty}\Big). \tag{5.11}$$

Thanks to Theorem 3.3, Propositions 5.4, 5.2 and (5.7) we get

$$\begin{aligned}\big\|[\mathcal{R}, v\cdot\nabla]\theta\big\|_{L^1_tB^0_{\infty,1}} &\lesssim \int_0^t (\|\omega(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^p})\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}} + \|\theta(\tau)\|_{L^p}\big)d\tau\\ &\lesssim \Phi_2(t).\end{aligned}$$

By easy computations we get

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v\|_{L^\infty} + \sum_{q\in\mathbb{N}}\|\Delta_q\nabla v\|_{L^\infty}\\ &\lesssim \|\omega\|_{L^p} + \sum_{q\in\mathbb{N}}\|\Delta_q\omega\|_{L^\infty}\\ &\lesssim \Phi_1(t) + \|\omega(t)\|_{B^0_{\infty,1}}.\end{aligned} \tag{5.12}$$

Putting together (5.11) and (5.12) leads to

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq \|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}\theta(t)\|_{B^0_{\infty,1}} \leq \Phi_2(t)\Big(1+\int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}}d\tau\Big).$$

Thus we obtain from Gronwall inequality

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq \Phi_3(t). \tag{5.13}$$

Coming back to (5.12) we get

$$\|\nabla v(t)\|_{L^\infty} \leq \Phi_3(t).$$

Let us move to the estimate of v in the space $B^1_{\infty,1}$. By definition we have

$$\|v(t)\|_{B^1_{\infty,1}} \lesssim \|v(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}.$$

Combined with (5.9) and (5.13) this yields

$$\|v(t)\|_{B^1_{\infty,1}} \leq \Phi_3(t)$$

as claimed. To estimate $\|\theta(t)\|_{B^0_{\infty,1}}$ we use Theorem 4.5 and the Lipschitz estimate of the velocity.

$$\begin{aligned}\|\theta(t)\|_{B^0_{\infty,1}} &\lesssim \|\theta^0\|_{B^0_{\infty,1}}\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big)\\ &\lesssim \Phi_3(t).\end{aligned}$$

The proof of Proposition 5.5 is now achieved. □

Remark 5.6. Notice that the *a priori* estimates above imply that $\nabla\theta \in L^1_{\text{loc}}(\mathbb{R}_+; L^\infty)$. Indeed, we can establish the following estimate by combining the smoothing effect of the temperature equation with the logarithmic estimate described by Proposition 4.5,

$$\|\theta\|_{L^1_t B^1_{\infty,1}} \leq C\|\theta^0\|_{B^0_{\infty,1}}(1+\|\nabla v\|^2_{L^1_t L^\infty}).$$

Although we expect to use this estimate for the uniqueness part, it seems that this is not sufficient for our purpose and some technical problems arise due to the fact that Riesz transforms do not map continuously L^∞ to itself. The crucial information that we need for the uniqueness is $\theta \in \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; B^1_{p,\infty})$.

5.2. **Uniqueness.** We now show that the Boussinesq system (1.2) has a unique solution in the class

$$\mathcal{X}_T = (L^\infty_T B^0_{\infty,1} \cap L^1_T B^1_{\infty,1}) \times (L^\infty_T L^p \cap \widetilde{L}^1_T B^1_{p,\infty}), \quad 2 < p < \infty.$$

Let (v^1, θ^1) and (v^2, θ^2) two solutions of (1.2) belonging to the space $\mathcal{X}_T$, and denote

$$v = v^2 - v^1, \quad \theta = \theta^2 - \theta^1.$$

Then we have the equations

$$\begin{cases} \partial_t v + v^2\cdot\nabla v = -\nabla p - v\cdot\nabla v^1 + \theta e_2\\ \partial_t\theta + v^2\cdot\nabla\theta + |\mathrm{D}|\theta = -v\cdot\nabla\theta^1\\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0.\end{cases}$$

According to Theorem 4.5 we have

$$\|v(t)\|_{B^0_{\infty,1}} \lesssim (1+V_1(t))\Big(\|v^0\|_{B^0_{\infty,1}} + \|\nabla p\|_{L^1_t B^0_{\infty,1}} + \|v\cdot\nabla v^1\|_{L^1_t B^0_{\infty,1}} + \|\theta\|_{L^1_t B^0_{\infty,1}}\Big),$$

with $V_1(t) = \|\nabla v^1\|_{L^1_t L^\infty}$. A straightforward calculus using the incompressibility of the flows gives

$$\begin{aligned}\nabla p &= -\nabla\Delta^{-1}\text{div }(v\cdot\nabla(v^1+v^2)) + \nabla\Delta^{-1}\partial_2\theta\\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

To estimate the first term of the RHS we use the definition

$$\|\mathrm{I}\|_{B^0_{\infty,1}} \lesssim \|(\nabla\Delta^{-1}\text{div })\text{div }\Delta_{-1}(v\otimes(v^1+v^2))\|_{L^\infty} + \|v\cdot\nabla(v^1+v^2)\|_{B^1_{\infty,1}}$$

From Proposition 3.1-(2) and Besov embeddings we have

$$\begin{aligned}\|(\nabla\Delta^{-1}\text{div })\text{div }\Delta_{-1}(v\otimes(v^1+v^2))\|_{L^\infty} &\lesssim \|v\otimes(v^1+v^2)\|_{L^\infty}\\ &\lesssim \|v\|_{B^0_{\infty,1}}\|v^1+v^2\|_{B^0_{\infty,1}}.\end{aligned}$$

Using the incompressibility of v and using Bony's decomposition one can easily obtain

$$\|v\cdot\nabla(v^1+v^2)\|_{B^0_{\infty,1}}\lesssim\|v\|_{B^0_{\infty,1}}\|v^1+v^2\|_{B^1_{\infty,1}}.$$

Putting together these estimates yields

$$\|\mathrm{I}\|_{B^0_{\infty,1}}\lesssim\|v\|_{B^0_{\infty,1}}\|v^1+v^2\|_{B^1_{\infty,1}}. \tag{5.14}$$

Let us now show how to estimate the second term II. By using Besov embeddings and Calderón-Zygmund estimate we get

$$\begin{aligned}\|\mathrm{II}\|_{B^0_{\infty,1}} &\lesssim \|\nabla\Delta^{-1}\partial_2\theta\|_{B^{\frac{2}{p}}_{p,1}}\\ &\lesssim \|\theta\|_{B^{\frac{2}{p}}_{p,1}}.\end{aligned}$$

Combining this estimate with (5.14) yields

$$\begin{aligned}\|v(t)\|_{B^0_{\infty,1}}\lesssim(1+V_1(t))\Big(\|v^0\|_{B^0_{\infty,1}} &+ \int_0^t\|v(\tau)\|_{B^0_{\infty,1}}\big[1+\|(v^1,v^2)(\tau)\|_{B^1_{\infty,1}}\big]d\tau\Big)\\ &+ (1+V_1(t))\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}.\end{aligned}\tag{5.15}$$

We have now to estimate $\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}$. By applying Δ_q to the equation of θ and arguing similarly to the proof of Theorem 4.2 we obtain

$$\begin{aligned}\|\theta_q(t)\|_{L^p}\lesssim e^{-ct2^q}\|\theta_q^0\|_{L^p} &+ \int_0^t e^{-c2^q(t-\tau)}\|\Delta_q(v\cdot\nabla\theta^1)(\tau)\|_{L^p}d\tau\\ &+ \int_0^t e^{-c2^q(t-\tau)}\|\big[v^2\cdot\nabla,\Delta_q\big]\theta(\tau)\|_{L^p}d\tau.\end{aligned}$$

Remark, first, that an obvious Hölder inequality yields that for every $\varepsilon\in[0,1]$ there exists an absolute constant C such that

$$\int_0^t e^{-c\tau2^q}d\tau\le Ct^\varepsilon2^{-q(1-\varepsilon)},\qquad\forall\, t\ge0.$$

Using this fact and integrating in time

$$\begin{aligned}2^{q\frac{2}{p}}\|\theta_q\|_{L^1_tL^p} &\lesssim 2^{q(-1+\frac{2}{p})}\|\theta_q^0\|_{L^p}\\ &+ t^\varepsilon2^{q(-1+\varepsilon+\frac{2}{p})}\int_0^t\Big(\|\Delta_q(v\cdot\nabla\theta^1)(\tau)\|_{L^p}+\|[v^2\cdot\nabla,\Delta_q]\theta(\tau)\|_{L^p}\Big)d\tau\\ &= 2^{q(-1+\frac{2}{p})}\|\theta_q^0\|_{L^p}+\mathrm{I}_q(t)+\mathrm{II}_q(t).\end{aligned}\tag{5.16}$$

Using Bony's decomposition we get easily

$$\begin{aligned}\|\Delta_q(v\cdot\nabla\theta^1)(t)\|_{L^p} &\lesssim \|v(t)\|_{L^\infty}\sum_{j\leq q+2}2^j\|\Delta_j\theta^1(t)\|_{L^p}\\ &+ 2^q\|v(t)\|_{L^\infty}\sum_{j\geq q-4}\|\Delta_j\theta^1(t)\|_{L^p}.\end{aligned}$$

Integrating in time we get

$$\begin{aligned}\mathrm{I}_q(t) &\lesssim t^\varepsilon\|v\|_{L^\infty_tL^\infty}2^{q(-1+\frac{2}{p}+\varepsilon)}(|q|+1)\|\theta^1\|_{\widetilde{L}^1_tB^1_{p,\infty}}\\ &+ t^\varepsilon\|v\|_{L^\infty_tL^\infty}2^{q(\frac{2}{p}+\varepsilon)}\sum_{j\geq q-4}\|\Delta_j\theta^1\|_{L^1_tL^p}\\ &\lesssim t^\varepsilon\|v\|_{L^\infty_tL^\infty}2^{q(-1+\frac{2}{p}+\varepsilon)}(|q|+1)\|\theta^1\|_{\widetilde{L}^1_tB^1_{p,\infty}}.\end{aligned}\tag{5.17}$$

To estimate the term II_q we use the following classical commutator $(2/p<1)$

$$\|[v^2\cdot\nabla,\Delta_q]\theta\|_{L^p}\lesssim 2^{-q\frac{2}{p}}\|\nabla v^2\|_{L^\infty}\|\theta\|_{B^{\frac{2}{p}}_{p,1}}.$$

Thus we obtain,

$$\mathrm{II}_q(t)\lesssim t^\varepsilon 2^{q(-1+\varepsilon)}\|\nabla v^2\|_{L^\infty_tL^\infty}\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}\tag{5.18}$$

We choose $\varepsilon>0$ such that $-1+\frac{2}{p}+\varepsilon<0$, which is possible since $p>2$. Then summing (5.16) and using (5.17) and (5.18) we get

$$\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}} \lesssim \|\theta^0\|_{L^p}+t^\varepsilon\|v\|_{L^\infty_tL^\infty}\|\theta^1\|_{\widetilde{L}^1_tB^1_{p,\infty}}+t^\varepsilon\|\nabla v^2\|_{L^\infty_tL^\infty}\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}.$$

For small $t\in[0,\delta]$ one can obtain

$$\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}\lesssim\|\theta^0\|_{L^p}+t^\varepsilon\|v\|_{L^\infty_tL^\infty}\|\theta^1\|_{\widetilde{L}^1_tB^1_{p,\infty}}.$$

Plugging this estimate into (5.15) we find

$$\|v\|_{L^\infty_tB^0_{\infty,1}}\lesssim e^{CV(t)}\Big(\|v^0\|_{B^0_{\infty,1}}+\|\theta^0\|_{L^p}+t\|v\|_{L^\infty_tB^0_{\infty,1}}+t^\varepsilon\|v\|_{L^\infty_tL^\infty}\|\theta^1\|_{\widetilde{L}^1_tB^1_{p,\infty}}\Big),$$

where $V(t):=\|(v^1,v^2)\|_{L^1_tB^1_{\infty,1}}$. If δ is sufficiently small then we get for $t\in[0,\delta]$

$$\|v\|_{L^\infty_tB^0_{\infty,1}}\lesssim\|v^0\|_{B^0_{\infty,1}}+\|\theta^0\|_{L^p}.\tag{5.19}$$

This gives in turn

$$\|\theta\|_{L^1_tB^{\frac{2}{p}}_{p,1}}\lesssim\|v^0\|_{B^0_{\infty,1}}+\|\theta^0\|_{L^p}.\tag{5.20}$$

This gives in particular the uniqueness on $[0,\delta]$. An iteration of this argument yields the uniqueness in $[0,T]$.

5.3. **Existence.** We consider the following system

$$(\mathrm{B}_n)\qquad \begin{cases} \partial_t v_n + v_n\cdot\nabla v_n + \nabla p_n = \theta_n e_2\\ \partial_t\theta_n + v_n\cdot\nabla\theta_n + |\mathrm{D}|\theta_n = 0\\ \mathrm{div} v_n = 0\\ v_{n|t=0} = S_n v^0, \quad \theta_{n|t=0} = S_n\theta^0\end{cases}$$

By using the same method as [16] we can prove that this system has a unique local smooth solution (v_n,θ_n). The global existence of these solutions is governed by the following criterion: we can push the construction beyond the time T if the quantity $\|\nabla v_n\|_{L^1_T L^\infty}$ is finite. Now from the *a priori* estimates the Lipschitz norm can not blow up in finite time and then the solution (v_n,θ_n) is globally defined. Once again from the *a priori* estimates we have for $2<p<\infty$

$$\|v_n\|_{L^\infty_T B^1_{\infty,1}} + \|\omega_n\|_{L^\infty_T L^p} + \|\theta_n\|_{L^\infty_T(B^0_{\infty,1}\cap L^p)} \le \Phi_3(T).$$

Then it follows that up to an extraction the sequence (v_n,θ_n) is weakly convergent to (v,θ) belonging to $L^\infty_T B^1_{\infty,1}\times L^\infty_T(B^0_{\infty,1}\cap L^p)$, with $\omega\in L^\infty_T L^p$. For $(n,m)\in\mathbb{N}^2$ we set $v_{n,m}=v_n-v_m$ and $\theta_{n,m}=\theta_n-\theta_m$ then according to the estimate (5.19) and (5.20) we get for $T=\delta$

$$\|v_{n,m}\|_{L^\infty_T B^0_{\infty,1}} + \|\theta_{n,m}\|_{L^1_T B^{\frac{2}{p}}_{p,1}} \le \Phi_3(t)\big(\|S_n v^0 - S_m v^0\|_{B^0_{\infty,1}} + \|S_n\theta^0 - S_m\theta^0\|_{L^p}\big).$$

This shows that (v_n,θ_n) is a Cauchy sequence in the Banach space $L^\infty_T B^0_{\infty,1}\times L^1_T B^{\frac{2}{p}}_{p,1}$ and then it converges strongly to (v,θ). This allows to pass to the limit in the system (B_n) and then we get that (v,θ) is a solution of the Boussinesq system (1.2).

Appendix: proof of Lemma 4.3

We have from Bony's decomposition

$$\begin{aligned}[\Delta_q, v\cdot\nabla]\theta &= \sum_{|j-q|\le 4}[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j\theta + \sum_{|j-q|\le 4}[\Delta_q,\Delta_j v\cdot\nabla]S_{j-1}\theta\\ &+ \sum_{j\ge q-4}[\Delta_q,\Delta_j v\cdot\nabla]\widetilde{\Delta}_j\theta\\ &:= \mathrm{I}_q + \mathrm{II}_q + \mathrm{III}_q.\end{aligned}$$

Observe first that

$$\mathrm{I}_q = \sum_{|j-q|\le 4} h_q\star(S_{j-1}v\cdot\nabla\Delta_j\theta) - S_{j-1}v\cdot(h_q\star\nabla\Delta_j\theta)$$

where $\hat{h}_q(\xi)=\varphi(2^{-q}\xi)$. Thus, Lemma 3.2 and Bernstein inequalities yield

$$\begin{aligned}\|\mathrm{I}_q\|_{L^p} &\lesssim \sum_{|j-q|\le 4}\|xh_q\|_{L^1}\|\nabla S_{j-1}v\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|h_0\|_{L^1}\sum_{|j-q|\le 4}2^{j-q}\|\Delta_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

To estimate the second term we use once again Lemma 3.2

$$\begin{aligned}\|\mathrm{II}_q\|_{L^p} &\lesssim \sum_{|j-q|\leq 4} 2^{-q}\|\Delta_j\nabla v\|_{L^p}\|\nabla S_{j-1}\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{\substack{|j-q|\leq 4\\ k\leq j-2}} 2^{k-q}\|\Delta_k\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

Let us now move to the remainder term. We separate it into two terms: high frequencies and low frequencies.

$$\begin{aligned}\mathrm{III}_q &= \sum_{\substack{j\geq q-4\\ j\in\mathbb{N}}}[\Delta_q\partial_i,\Delta_j v^i]\widetilde{\Delta}_j\theta+[\Delta_q,\Delta_{-1}v\cdot\nabla]\widetilde{\Delta}_{-1}\theta\\ &:= \mathrm{III}_q^1+\mathrm{III}_q^2.\end{aligned}$$

For the first term we don't need to use the structure of the commutator. We estimate separately each term of the commutator by using Bernstein inequalities.

$$\begin{aligned}\|\mathrm{III}_q^1\|_{L^p} &\lesssim \sum_{\substack{j\geq q-4\\ j\in\mathbb{N}}} 2^q\|\Delta_j v\|_{L^p}\|\widetilde{\Delta}_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{j\geq q-4} 2^{q-j}\|\widetilde{\Delta}_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

For the second term we use Lemma 3.2 combined with Bernstein inequalities.

$$\begin{aligned}\|\mathrm{III}_q^2\|_{L^p} &\lesssim \|\nabla\Delta_{-1}v\|_{L^p}\|\nabla\widetilde{\Delta}_{-1}\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

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