On the global well-posedness for Boussinesq system

H. ABIDI* et T. HMIDI†

Abstract

In this paper, we give a global well-posedness result for the two dimensional Boussinesq system with partial viscosity, when the initial data $v^0 \in B^{-1}_{\infty,1}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $\theta^0 \in B^0_{2,1}(\mathbb{R}^2)$.

1 Introduction

In this paper, we address the problem of the global well-posedness of the two-dimensional Boussinesq system with partial viscosity,

$$(B)\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla\Pi = \theta e_2, \\ \partial_t\theta + v\cdot\nabla\theta = 0, \\ \mathrm{div}\, v = 0, \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

Here, e_2 denotes the vector $(0,1)$, $v = (v_1, v_2)$ is the velocity field, Π the scalar pressure and θ the temperature. The coefficient ν, called kinematic viscosity, is assumed to be strictly positive and will be taken to be 1 throughout this paper.

This system is used as a model describing many geophysical phenomena, see e.g. [16] and has lately received significant attention in mathematical fluid dynamics due to its connection to three-dimensional incompressible flows. Indeed, the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies a transport-diffuson equation with second member $\partial_1\theta$, which, in turn, obeys a transport equation with second member $(\nabla v)\partial_1\theta$. This quantity is a stretching term in the three-dimensional incompressible Euler vorticity equation (for more details about this subject we can refer to [13]).

The existence and uniqueness of local smooth solutions can be done without any difficulty as in the case of Euler or Navier-Stokes systems but the issue of whether a finite-time singularity can be developed is not directly attained by analyzing the vorticity. To be more precise, we recall from [2, 17] that the formation of singularities occurs when there

*Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie. E-mail: abidi@ann.jussieu.fr
†IRMAR, Université Rennes1. E-mail: thmidi@univ-rennes1.fr

is an accumulation, in time, of vorticity, as what happens in incompressible Navier-Stokes or Euler equations. In 2-dimensional space, the global smooth solutions of the two latter equations mentioned are simple since we can easily show that the L^∞-norm is bounded at any time by its initial value.

However the situation for Boussinesq system is more subtle and we cannot directly prove the boundedness of the vorticity without making use of the regularization effect of the heat semi-flow in an adequate manner.

Actually, the problem of finite-time formation of singularities was listed by H. K. Moffatt in [15] among other interesting questions on fluid flows. Some recent progress have been obtained by numerous authors. In [8], D. Cordóba, C. Fefferman and R. De La Llave gave, using a geometrical approach, a partial answer asserting that some special type of singularities called "squirt singularities" cannot be developed in finite time. Afterwards, D. Chae [4] and T. Y. Hou and C. Li [13] proved independently the global-in-time regularity when the initial data v^0 and θ^0 belong to the Sobolev space H^s, with $s > 2$. The proofs rest essentially on two facts. First is the use of the smoothing effects of the vorticity equation in an adequate manner, allowing them to diminish the required regularity for the temperature. The second technique is the use of a sharp Sobolev embedding estimate in two spatial dimensions with a logarithmic correction. Let us note that in this context, the velocity and the temperature are Lipschitz and this assumption is crucial for their analysis.

More recently, in a joint work with S. Keraani, the second author has proved the global existence in the energy space, *i.e.*, $v^0 \in H^s$, with $0 \leq s \leq 2$ and $\theta^0 \in L^2$. However the uniqueness of such solutions are known only under some additional hypotheses, namely, $v^0 \in H^s$, with $s > 0$ and $\theta^0 \in B^0_{2,1} \cap B^0_{p,\infty}$, with $p > 2$ (for the definition of Besov spaces see next paragraph).

The proof of the existence is based on the paradifferential calculus and on some compactness arguments. However, the uniqueness proof seems to be more complicated and laborious. It is divided into two major parts: the first one is obtaining a global estimate of the velocity in $L^1_T \mathrm{Lip}(\mathbb{R}^2)$, in which one uses Vishik's logarithmic estimate type. The second one is choosing a suitable space wherein one can estimate the difference of two solutions leading to a "linear" estimate to which we apply Gronwall lemma.

In this paper, we pursue our investigation of the uniqueness problem for initial data belonging to spaces having the same scaling as the energy space L^2. However, our method does not show uniqueness when v^0, $\theta^0 \in L^2$, so that it remains an open problem .
Let us now state our main result.

Theorem 1.1. *Let v^0 be a divergence-free vector field belonging to the space $L^2 \cap B^{-1}_{\infty,1}$ and $\theta^0 \in B^0_{2,1}$. Then there exists a unique global solution (v, θ, Π) for the system (B), such that*

$$v \in \mathcal{C}(\mathbb{R}_+; L^2 \cap B^{-1}_{\infty,1}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+; B^1_{\infty,1}),$$
$$\theta \in \mathcal{C}_b(\mathbb{R}_+; B^0_{2,1}) \quad \textit{and} \quad \nabla\Pi \in L^1_{\mathrm{loc}}(\mathbb{R}_+; B^0_{2,1}).$$

If one scrutinizes carefully the proof of Theorem 1.1, he will see that one can replace the $L^2 \cap B^{-1}_{\infty,1}$ space by a small one like the Besov space $B^0_{2,1}$. On the contrary, this regularity is globally preserved. More precisely we have the following statement:

Corollary 1. *Let v^0 be a divergence-free vector field belonging to $B^0_{2,1}$ and $\theta^0 \in B^0_{2,1}$. Then there exists a unique global solution (v, θ, Π) for the system (B), such that*

$$v \in \mathcal{C}(\mathbb{R}_+; B^0_{2,1}) \cap L^2_{\rm loc}(\mathbb{R}_+; B^1_{2,1}) \cap L^1_{\rm loc}(\mathbb{R}_+; B^1_{\infty,1}),$$
$$\theta \in \mathcal{C}_b(\mathbb{R}_+; B^0_{2,1}) \quad \textit{and} \quad \nabla\Pi \in L^1_{\rm loc}(\mathbb{R}_+; B^0_{2,1}).$$

Throughout this paper, C stands for some real positive constant which may be different in each occurrence, and C_0 a real positive constant depending on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$.

2 Notation and preliminaries

In this preparatory section, we recall the so-called Littlewood-Paley operators and their elementary properties. It will be also convenient to introduce some function spaces and review some important lemma that will be used constantly in the following pages.
Let us start with the definition of the dyadic decomposition of the full space $\mathbb{R}^d$(see [5]).

Proposition 2.1. *There exist two radially symmetric functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that*

i) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \frac{1}{3} \leq \chi^2(\xi) + \sum_{q\geq 0} \varphi^2(2^{-q}\xi) \leq 1,$

ii) supp $\varphi(2^{-p}\cdot) \cap$ supp $\varphi(2^{-q}\cdot) = \varnothing$, *if* $|p-q| \geq 2$,

iii) $q \geq 1 \Rightarrow$ supp$\chi \cap$ supp $\varphi(2^{-q}) = \varnothing$.

For every $v \in \mathcal{S}'$ we define the non-homogeneous Littlewood-Paley operators:

$$\Delta_{-1} v = \chi(\mathrm{D})v \,;\, \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D})v \qquad \text{and} \ \ S_q = \sum_{-1\leq p\leq q-1} \Delta_p.$$

The homogeneous operators are defined as follows.

$$\dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q = \sum_{j\leq q-1} \dot{\Delta}_q u.$$

From the paradifferential calculus introduced by J.-M. Bony [1] the product uv can be formally divided into three parts as follows:

$$uv = T_u v + T_v u + R(u,v) = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

where

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \qquad\qquad R(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v,$$
$$\dot{T}_u v = \sum_q \dot{S}_{q-1} u \Delta_q v \qquad \text{and} \qquad \dot{R}(u,v) = \sum_q \dot{\Delta}_q u \dot{\widetilde{\Delta}}_q v,$$

$$\text{with} \quad \widetilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i} \quad \text{and} \quad \dot{\widetilde{\Delta}}_q = \sum_{i=-1}^{1} \dot{\Delta}_{q+i}.$$

$T_u v$ is called paraproduct of v by u and $R(u,v)$ the remainder term. Let us now recall what Besov spaces are. Let $(p_1, p_2) \in [1, +\infty]^2$ and $s \in \mathbb{R}$, then the non-homogeneous Besov space $B^s_{p_1,p_2}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p_1,p_2}} := \Big(2^{qs} \|\Delta_q u\|_{L^{p_1}} \Big)_{\ell^{p_2}(\mathbb{N} \cup \{-1\})} < +\infty.$$

The homogeneous Besov spaces $\dot{B}^s_{p_1,p_2}$ is the set of $u \in \mathcal{S}'$ such that,

$$\lim_{q \to -\infty} S_q u = 0 \quad \text{in} \quad \mathcal{S}' \quad \text{and} \quad \|u\|_{\dot{B}^s_{p_1,p_2}} := \Big(2^{qs} \|\dot{\Delta}_q u\|_{L^{p_1}} \Big)_{\ell^{p_2}(\mathbb{Z})} < +\infty.$$

We notice that there is an equivalent norm for the negative-indexed homogeneous Besov space in terms of some integrability properties through the heat semiflow:
Let s be a strictly positive real number and $p_1, p_2 \in [1, +\infty]$. Then the following norm is equivalent to $\| \cdot \|_{\dot{B}^{-s}_{p_1,p_2}}$:

$$\Big\| t^{\frac{s}{2}} \|e^{t\Delta} \cdot \|_{L^{p_1}} \Big\|_{L^{p_2}(\mathbb{R}_+, \frac{dt}{t})}. \tag{2.1}$$

It is convenient to recall some mixed function spaces. Let $T > 0$ and $r \geq 1$, we denote by $L^r_T B^s_{p_1,p_2}$ the space of all function u satisfying

$$\|u\|_{L^r_T B^s_{p_1,p_2}} := \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^{p_1}} \Big)_{\ell^{p_2}} \Big\|_{L^r_T} < \infty.$$

We say that a function u is an element of the space $\widetilde{L}^r_T B^s_{p_1,p_2}$ if

$$\|u\|_{\widetilde{L}^r_T B^s_{p_1,p_2}} := \Big(2^{qs} \|\Delta_q u\|_{L^r_T L^{p_1}} \Big)_{\ell^{p_2}} < +\infty.$$

The relationships between these spaces are detailed in the following lemma, which is a direct consequence of the Minkowski inequalities.

Lemma 2.2. *Let $s \in \mathbb{R}, \epsilon > 0, r \geq 1$ and $\big(p_1, p_2\big) \in [1, \infty]^2$. Then we have the following embeddings*

$$L^r_T B^s_{p_1,p_2} \hookrightarrow \widetilde{L}^r_T B^s_{p_1,p_2} \hookrightarrow L^r_T B^{s-\epsilon}_{p_1,p_2}, \text{ if} \quad r \leq p_2,$$
$$L^r_T B^{s+\epsilon}_{p_1,p_2} \hookrightarrow \widetilde{L}^r_T B^s_{p_1,p_2} \hookrightarrow L^r_T B^s_{p_1,p_2}, \text{ if} \quad r \geq p_2.$$

A further important result that will be constantly used is the so-called Bernstein lemma (see for example [5]). Here the notation $\mathcal{F}$ is reserved for Fourier transform in the whole space.

Lemma 2.3. (BERNSTEIN) *Let (r_1, r_2) a pair of strictly positive numbers such that $r_1 < r_2$. There exists a constant C such that for every nonnegative integer k, and for every $1 \leq a \leq b$ and for all function $u \in L^a(\mathbb{R}^d)$, we have*

$$\begin{aligned}
\text{supp}\,\mathcal{F}u \in B(0, \lambda\, r_1) \quad &\Rightarrow \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k\, \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},\\
\text{supp}\,\mathcal{F}u \in \mathcal{C}(0, \lambda r_1, \lambda r_2) \quad &\Rightarrow \quad C^{-k}\lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.
\end{aligned}$$

As will be seen some refined estimates about the heat semiflow will be very useful when we plan to establish some a priori estimates. For more details about the proof of the following lemma one can refer to [6]).

Lemma 2.4. *Let $\mathcal{C}$ be a given ring. There exist two constants c and C, such that for any pair of positive numbers (t, λ), for every $p \in [1, +\infty]$ and for all function $a \in L^p$, we have :*

$$\text{Supp}\, \widehat{a} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta} a\|_{L^p} \leq Ce^{-ct\lambda^2} \|a\|_{L^p}.$$

To prove the uniqueness part of our main theorem, a logarithmic estimate type will be needed. For the convenience of the reader, we will briefly describe its proof.

Lemma 2.5. *Let $\varepsilon > 0$ and u be a smooth function. Then there exists an absolute constant C such that the following estimate holds true:*

$$\|u\|_{L^1_t L^\infty} \leq C\varepsilon^{-1} \|u\|_{\widetilde{L}^1_t B^1_{2,\infty}} \ln\Big(e + \frac{\|u\|_{\widetilde{L}^1_t B^\varepsilon_{\infty,\infty}}}{\|u\|_{\widetilde{L}^1_t B^1_{2,\infty}}}\Big), \tag{2.2}$$

Proof. We write in view of the continuous embedding $B^0_{\infty,1} \hookrightarrow L^\infty$,

$$\|u\|_{L^1_t L^\infty} \leq \|u\|_{L^1_t B^0_{\infty\, 1}} = \sum_{q \geq -1} \|\Delta_q u\|_{\widetilde{L}^1_t L^\infty}.$$

Let N be an arbitrary integer which will be fixed later. Then, separating the above sum into low and high frequencies gives

$$\begin{aligned}
\|u\|_{L^1_t L^\infty} &\leq \sum_{-1 \leq q \leq N} \|\Delta_q u\|_{L^1_t L^\infty} + \sum_{q \geq N+1} 2^{-q\varepsilon} 2^{q\varepsilon} \|\Delta_q u\|_{L^1_t L^\infty}\\
&\leq CN \|u\|_{\widetilde{L}^1_t B^1_{2,\infty}} + C 2^{-N\varepsilon} \|u\|_{\widetilde{L}^1_t B^\varepsilon_{\infty,\infty}}.
\end{aligned}$$

Choosing N as follows

$$N = 1 + \Big[\frac{1}{\varepsilon} \log_2 \Big(\frac{\|u\|_{\widetilde{L}^1_t B^\varepsilon_{\infty,\infty}}}{\|u\|_{\widetilde{L}^1_t B^1_{2,\infty}}}\Big)\Big]$$

gives the desired estimate. $\square$

How the paraproducts and remainder term act on Besov spaces is partially described by the following proposition (for more details see [9, 5] and the references therein):

Proposition 2.6. *Let u and v be two smooth vector fields. Then we have the following estimates.*

- *For every $s \in \mathbb{R}$*
$$\|T_u v\|_{B^s_{2,\infty}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{2,\infty}}.$$
- *For every $s < r$,*
$$\|T_u v\|_{B^s_{2,\infty}} \lesssim \|u\|_{B^{s-r}_{2,\infty}} \|v\|_{B^r_{\infty,\infty}}.$$
- *In addition, if $\operatorname{div} u = 0$, then the following estimates hold true.*

$$\|R(u^j, \partial_j v)\|_{B^s_{2,\infty}} \lesssim \begin{cases} \|u\|_{B^{s_1}_{2,\infty}} \|v\|_{B^{s_2}_{2,\infty}}, \ \textit{if } s+2>0, \ \textit{with } \ s+2 = s_1 + s_2 \\ \|u\|_{B^1_{\infty,r_1}} \|v\|_{B^s_{2,r_2}}, \ \textit{if } s+1 \geq 0, \ \textit{with } \frac{1}{r_1} + \frac{1}{r_2} = 1. \end{cases}$$

Remark 1. *The summation convention over repeated indices is used here, i.e.,*

$$R(u^j, \partial_j v) = \sum_j R(u^j, \partial_j v).$$

Remark 2. *Similar estimates to Proposition 2.6 can be shown for the $\widetilde{L}^p_T B^s_{2\,\infty}$ spaces. For example, for $s < r$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have*

$$\|T_u v\|_{\widetilde{L}^p_T B^s_{2,\infty}} \lesssim \|u\|_{\widetilde{L}^{p_2} B^{s-r}_{2,\infty}} \|v\|_{\widetilde{L}^{p_1} B^r_{\infty,\infty}}.$$

For the sake of completeness, we will prove a commutator estimate frequently used throughout the succeeding sections.

Lemma 2.7. *Let v be a smooth free-divergence vector field and u a smooth function. Then we have*

$$\sup_{q \geq -1} 2^{-q} \big\| [\Delta_q, v \cdot \nabla] u \big\|_{L^2} \lesssim \|v\|_{B^1_{\infty,1}} \|u\|_{B^{-1}_{2,\infty}}.$$

Proof. By using Bony's decomposition and the zero divergence condition, the commutator may be decomposed as follows (see for example [5])

$$[\Delta_q, v \cdot \nabla] u = -\Delta_q \partial_j R(u, v^j) - \Delta_q T_{\partial_j u} v^j + T'_{\Delta_q \partial_j u} v^j + [T_{v^j}, \Delta_q] \partial_j u = \sum_{i=1}^4 R^i_q,$$

where $T'_u v$ stands for $T_u v + R(u,v)$.
According to Proposition 2.6, one obtains

$$\sup_q 2^{-q}\|R_q^1\|_{L^2} \lesssim \|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}. \tag{2.3}$$

To estimate R_q^2, we write by definition, *i.e.*,

$$R_q^2 = -\Delta_q T_{\partial_j u} v^j = \sum_{|q-k|\leq 4} \Delta_q(S_{k-1}\partial_j u \Delta_k v^j).$$

Again, applying Proposition 2.6 with $s=-1$ and $r=1$ leads to

$$\begin{aligned} 2^{-q}\|R_q^2\|_{L^2} &\lesssim \|T_{\partial_j u} v^j\|_{B^{-1}_{2,\infty}} \lesssim C\|v\|_{B^1_{\infty,\infty}}\|\nabla u\|_{B^{-2}_{2,\infty}} \\ &\lesssim C\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}. \end{aligned} \tag{2.4}$$

It is easy to verify from the definition that R_q^3 can be expressed as

$$R_q^3 = T'_{\Delta_q \partial_j u} v^j = \sum_{k\geq q-2} S_{k+2}\Delta_q \partial_j u \Delta_k v^j$$

Thus applying Hölder and Young inequalities give

$$\begin{aligned} 2^{-q}\|R_q^3\|_{L^2} &\lesssim 2^{-q}\|\Delta_q u\|_{L^2} \sum_{k\geq q-2} 2^{q-k}\, 2^k \|\Delta_k v\|_{L^\infty} \\ &\lesssim \|u\|_{B^{-1}_{2,\infty}}\|v\|_{B^1_{\infty,\infty}}. \end{aligned} \tag{2.5}$$

An obvious computation using the localizing properties of Littlewood-Paley operators shows that

$$R_q^4 = [T_{v^j}, \Delta_q]\partial_j u = \sum_{|k-q|\leq 4} [S_{k-1}v^j, \Delta_q]\Delta_k \partial_j u$$

On the other hand we have the following classical estimate (see for example [5])

$$\|[S_{k-1}v^j, \Delta_q]\Delta_k \partial_j u\|_{L^2} \lesssim \|\nabla v\|_{L^\infty}\|\Delta_k u\|_{L^2}.$$

Therefore we get

$$\sup_q 2^{-q}\|R_q^4\|_{L^2} \lesssim \|\nabla v\|_{L^\infty}\|u\|_{B^{-1}_{2,\infty}}. \tag{2.6}$$

Whence the statement of Lemma 2.7 follows on putting together estimates (2.3), (2.4), (2.5) and (2.6).

$\square$

3 On Stokes problem and transport-diffusion equations

In this section we intend to establish some important results about Stokes and transport-diffusion equations. This is the crucial step for the proof of our global existence and uniqueness result. Consider the following Stokes system:

$$
(\mathrm{S}) \begin{cases} \partial_t u + v\cdot\nabla u - \Delta u + \nabla\Pi = f \\ \mathrm{div}\, u = 0 \\ u_{|t=0} = u^0. \end{cases}
$$

Our first result deals with the persistent regularity of initial data belonging to some critical Besov spaces. This completes some results due to numerous authors and for more general survey see [5, 10] and the references therein.

Proposition 3.1. *Let v be a divergence-free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}_+; B^1_{\infty,1})$ and $f \in \widetilde{L}^1_{\mathrm{loc}}(\mathbb{R}_+; B^{-1}_{2,\infty})$. Then any solution u of the system (S), with $u^0 \in B^{-1}_{2,\infty}$ satisfies*

$$
\|u\|_{L^\infty_T B^{-1}_{2,\infty}} + \|u\|_{\widetilde{L}^1_T B^1_{2,\infty}} + \|\nabla\Pi\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}} \le Ce^{CT + C\|v\|_{L^1_T B^1_{\infty,1}}}\big(\|u^0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}}\big).
$$

The constant C is absolute.

Proof. We start by localizing the equation of u through Littlewood-Paley operator Δ_q. Then by setting $u_q := \Delta_q u$, we get

$$
\partial_t u_q + v\cdot\nabla u_q - \Delta u_q + \nabla\Pi_q = -[\Delta_q, v\cdot\nabla]u + f_q := \mathcal{R}_q.
$$

Taking the L^2 inner product of this equation with u_q leads to

$$
\frac{1}{2}\frac{d}{dt}\|u_q\|^2_{L^2} + \|\nabla u_q\|^2_{L^2} \le \|u_q\|_{L^2}\|\mathcal{R}_q\|_{L^2}. \tag{3.1}
$$

Thus we infer that

$$
\|u_q(t)\|_{L^2} \le \|u_q(0)\|_{L^2} + \|\mathcal{R}_q\|_{L^1_t L^2}.
$$

Multiplying both sides by 2^{-q} and taking the supremum over q give

$$
\begin{aligned}
\|u(t)\|_{B^{-1}_{2,\infty}} &\le \|u^0\|_{B^{-1}_{2,\infty}} + \sup_q 2^{-q}\|\mathcal{R}_q\|_{L^1_t L^2} \\
&\le \|u^0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + \sup_q 2^{-q}\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^1_t L^2} \\
&\le \|u^0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + \int_0^t \sup_q 2^{-q}\big\|[\Delta_q, v\cdot\nabla]u(\tau)\big\|_{L^2} d\tau.
\end{aligned}
$$

Consequently, we obtain from Lemma 2.7

$$
\|u(t)\|_{B^{-1}_{2,\infty}} \le \|u^0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + C\int_0^t \|v(\tau)\|_{B^1_{\infty,1}}\|u(\tau)\|_{B^{-1}_{2,\infty}} d\tau. \tag{3.2}
$$

Let us now turn to the regularization effect for the velocity. We combine Parseval identity with (3.1) obtaining for $q \in \mathbb{N}$

$$\frac{d}{dt}\|u_q\|_{L^2} + 2^{2q}\|u_q\|_{L^2} \leq C\|\mathcal{R}_q\|_{L^2}.$$

Integrating this differential inequality, we get

$$\|u_q(t)\|_{L^2} \leq Ce^{-t2^{2q}}\|u_q(0)\|_{L^2} + C\int_0^t e^{-(t-\tau)2^{2q}}\|\mathcal{R}_q(\tau)\|_{L^2}d\tau.$$

Making use of the convolution inequality, one further obtains

$$2^q\|u_q\|_{L^1_tL^2} \leq C2^{-q}\|u_q(0)\|_{L^2} + C2^{-q}\|\mathcal{R}_q\|_{L^1_tL^2}.$$

Thus it follows from Lemma 2.7 that

$$\sup_{q\in\mathbb{N}} 2^q\|u_q\|_{L^1_tL^2} \leq C\|u^0\|_{B^{-1}_{2,\infty}} + C\|f\|_{\widetilde{L}^1_tB^{-1}_{2,\infty}} + C\int_0^t \|v(\tau)\|_{B^1_{\infty,1}}\|u(\tau)\|_{B^{-1}_{2,\infty}}d\tau.$$

For low frequencies, we have by definition

$$\|u_{-1}\|_{L^1_tL^2} \leq C\int_0^t \|u(\tau)\|_{B^{-1}_{2,\infty}}d\tau.$$

Combining the last two estimates yield

$$\|u\|_{\widetilde{L}^1_tB^1_{2,\infty}} \leq C\|u^0\|_{B^{-1}_{2,\infty}} + C\|f\|_{\widetilde{L}^1_tB^{-1}_{2,\infty}} + C\int_0^t \big(1 + \|v(\tau)\|_{B^1_{\infty,1}}\big)\|u(\tau)\|_{B^{-1}_{2,\infty}}d\tau. \quad (3.3)$$

Putting (3.2) and (3.3) together and using Gronwall's Lemma give

$$\|u(t)\|_{B^{-1}_{2,\infty}} + \|u\|_{\widetilde{L}^1_tB^1_{2,\infty}} \leq Ce^{Ct+C\|v\|_{L^1_tB^1_{\infty,1}}}\big(\|u^0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_tB^{-1}_{2,\infty}}\big). \quad (3.4)$$

Let us now move to the proof of the pressure control. We apply the divergence operator to the linearized equation,

$$\Delta\Pi = \mathrm{div}\, f - \mathrm{div}(v\cdot\nabla u).$$

Then we have

$$\nabla\Pi = \nabla\Delta^{-1}\mathrm{div}\, f - \nabla\Delta^{-1}\mathrm{div}(v\cdot\nabla u).$$

Since $\mathrm{div}\, u = \mathrm{div}\, v = 0,$ then we have

$$\mathrm{div}(v\cdot\nabla u) = \mathrm{div}(u\cdot\nabla v).$$

This yields

$$\nabla\Pi = \nabla\Delta^{-1}\mathrm{div}\, f - \nabla\Delta^{-1}\mathrm{div}(u\cdot\nabla v).$$

Since Riesz transforms map the Besov space $B^{-1}_{2,\infty}$ continuously into itself, then we deduce that

$$\|\nabla\Pi\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} \leq C\|f\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}} + C\|u\cdot\nabla v\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}}.$$

On the other hand we have from Bony's decomposition

$$u\cdot\nabla v = R(u^j, \partial_j v) + T_{u^j}\partial_j v + T_{\partial_j v}u^j.$$

Applying Proposition 2.6 and using simultaneously Lemma 2.2, we get

$$\|u\cdot\nabla v\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} \leq C\int_0^t \|u(\tau)\|_{B^{-1}_{2,\infty}}\|v(\tau)\|_{B^1_{\infty,1}}d\tau.$$

Thus we infer

$$\|\nabla\Pi\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} \leq C\|g\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + C\int_0^t \|u(\tau)\|_{B^{-1}_{2,\infty}}\|v(\tau)\|_{B^1_{\infty,1}}d\tau.$$

Whence the pressure control follows at once by combining the above inequality with (3.4). The proof of Proposition 3.1 is completed. □

As a direct consequence of the proof of Proposition 3.1 we have the following result:

Corollary 2. *Let v be a divergence-free vector field belonging to the space $L^1_{\rm loc}(\mathbb{R}_+; B^1_{\infty,1}), a_0 \in B^{-1}_{2,\infty}$ and $f \in \widetilde{L}^1_{\rm loc}(\mathbb{R}_+; B^{-1}_{2,\infty})$.We assume that a function $a \in L^\infty_{\rm loc}(\mathbb{R}_+; B^{-1}_{2\,\infty})$ is a solution of the transport equation:*

$$\begin{cases} \partial_t a + v\cdot\nabla a = f \\ a_{|t=0} = a_0. \end{cases}$$

Then there exists an absolute constant C such that we have for all $t\in\mathbb{R}_+$,

$$\|a\|_{L^\infty_t B^{-1}_{2,\infty}} \leq Ce^{C\|v\|_{L^1_t B^1_{\infty,1}}}\big(\|a_0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}}\big). \tag{3.5}$$

We recall now a logarithmic estimate for transport-diffusion equation used by the second author in another context (for the proof, see [12]). It generalizes a similar result obtained by Vishik [18] for the two-dimensional Euler vorticity equation.

Proposition 3.2. *Let $p,\, r\in[1,+\infty]$, v be a divergence-free vector field belonging to the space $L^1_{loc}(\mathbb{R}_+;{\rm Lip}(\mathbb{R}^d))$ and let a be a scalar solution to the following problem (with $\nu\geq 0$),*

$$\begin{cases} \partial_t a + v\cdot\nabla a - \nu\Delta a = f \\ a_{|t=0} = a^0. \end{cases}$$

If the initial data $a^0\in B^0_{p,r}$, then we have for all $t\in\mathbb{R}_+$

$$\|a\|_{\widetilde{L}^\infty_t B^0_{p,r}} \leq C\big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde{L}^1_t B^0_{p,r}}\big)\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau\Big),$$

where C depends only on the dimension d and not on vanishing viscosity.

We will conclude this section with the following classical result:

Lemma 3.3. *Let $s \in \mathbb{R}, (p,r,q) \in [1,+\infty]^3$. We assume that u is a solution of the heat equation,*

$$\begin{cases} \partial_t u - \Delta u = f + g \\ u_{|t=0} = u^0, \end{cases}$$

such that $u^0 \in \dot{B}^s_{p,r}$, $f \in L^1_T \dot{B}^s_{p,r}$ and $g \in L^q_T \dot{B}^{s+\frac{2}{q}-2}_{p,r}$. Then we have the following estimate:

$$\|u\|_{\widetilde{L}^\infty_T \dot{B}^s_{p,r}} \leq C\big(\|u^0\|_{\dot{B}^s_{p,r}} + \|f\|_{L^1_T \dot{B}^s_{p,r}} + \|g\|_{L^q_T \dot{B}^{s+\frac{2}{q}-2}_{p,r}}\big).$$

4 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. For conciseness, we shall provide the *a priori* estimates supporting the claims of the theorem and give a complete proof of the uniqueness, while the proof of the existence part will be shortened and briefly described since it is somewhat contained in [12].

4.1 Some a priori estimates

The aim of this subsection is to give some refined estimates for two dimensional Navier-Stokes system with external force and we shall see later how we can apply these results to the Boussinesq system. In particular, we derive an $L^2_t L^\infty$ estimate for Leray's solution. It is worth pointing out that this estimate is recently obtained by Chemin and Gallagher [7] with a better control term. For the convenience of the reader, we will give another concise proof.

Consider the incompressible Navier-Stokes system,

$$(NS) \begin{cases} \partial_t v + \mathbb{P}(v \cdot \nabla v) - \Delta v = \mathbb{P} f \\ \operatorname{div} v = 0, \\ v_{|t=0} = v^0, \end{cases}$$

where $\mathbb{P}$ is the orthogonal projector over solenoidal vectorfields.

The main result of this section is stated in the following theorem:

Theorem 4.1. *Let v be the solution of the two-dimensional Navier-Stokes system such that the initial data $v^0 \in L^2$ and $f \in L^1_{\rm loc}(\mathbb{R}_+; L^2)$. Then for every $T \in \mathbb{R}_+$, we have*

$$\|v\|_{L^2_T L^\infty} \leq C E_{0,T}(1 + E^{\frac{1}{2}}_{0,T}),$$

with $E_{0,T} = \|v^0\|^2_{L^2} + \|f\|^2_{L^1_T L^2}$.
Moreover if $v^0 \in L^2 \cap \dot{B}^{-1}_{\infty,1}$ and $f \in L^1_{\rm loc}(\mathbb{R}_+; L^2 \cap \dot{B}^{-1}_{\infty,1})$, then we have for all $T \in \mathbb{R}_+$

$$\|v\|_{L^1_T \dot{B}^1_{\infty,1}} \leq C\big(\|v^0\|_{\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T \dot{B}^{-1}_{\infty,1}} + \mathcal{E}_{0,T}\big)(1 + \mathcal{E}_{0,T}),$$

with $\mathcal{E}_{0,T} = E_{0,T}(1 + E_{0,T}^{\frac{1}{2}})$.

Proof. We notice that throughout the proof, we will only make use of the homogeneous Littlewood-Paley operators and to avoid any confusion, we will adopt the same notations for the non-homogeneous operators. As in [7], we split v into two functions $v = v^1 + v^2$, such that

$$\begin{cases} \partial_t v^1 - \Delta v^1 = \mathbb{P} f \\ v^1{}_{|t=0} = v^0 \end{cases}$$

and

$$\begin{cases} \partial_t v^2 - \Delta v^2 = -\mathbb{P}(v \cdot \nabla v) \\ v^2{}_{|t=0} = 0. \end{cases}$$

To estimate v_1, we write, using Duhamel formula and the characterization of the Besov space given in (2.1), and have the following:

$$\begin{aligned} \|v^1\|_{L^2_t L^\infty} &\leq \|v^0\|_{\dot{B}^{-1}_{\infty,2}} + \int_0^t \|f(\tau)\|_{\dot{B}^{-1}_{\infty,2}} d\tau \\ &\lesssim \|v^0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2} d\tau. \end{aligned}$$

For the last inequality, we used the continuous embedding property $L^2 \hookrightarrow \dot{B}^{-1}_{\infty,2}$.
Let us now move to the estimate of the term v^2. Applying the operator Δ_q to the equation of v^2 and using Lemma 2.4, we have

$$\|v_q^2(t)\|_{L^\infty} \lesssim \int_0^t e^{-c(t-\tau)2^{2q}} \|\Delta_q(v \cdot \nabla v)\|_{L^\infty} d\tau,$$

By using the convolution inequality, one obtains

$$\|v_q^2\|_{L^2_T L^\infty} \lesssim 2^{-q} \|\Delta_q(v \otimes v)\|_{L^2_T L^\infty}.$$

This leads to

$$\|v^2\|_{\widetilde{L}^2_T \dot{B}^0_{\infty\,1}} \lesssim \|v \otimes v\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,\,1}}. \tag{4.1}$$

From Bony's decomposition, we can write

$$\|v \otimes v\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,\,1}} \leq 2\|T_v \cdot v\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,\,1}} + \|R(v,v)\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,\,1}}.$$

From the paraproduct definition and Bernstein inequality we have

$$\begin{aligned} \|T_v \cdot v\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,\,1}} &\lesssim \sum_{q \in \mathbb{Z}} 2^{-q} \|S_{q-1} v\|_{L^\infty_T L^\infty} \|\Delta_q v\|_{L^2_T L^\infty} \\ &\lesssim \sum_{j \leq q-1} 2^q \|\Delta_j v\|_{L^\infty_T L^\infty} \|\Delta_q v\|_{L^2_T L^\infty} \\ &\lesssim \sum_{j \leq q-1} 2^{j-q} \|\Delta_j v\|_{L^\infty_T L^2} \|\Delta_q \nabla v\|_{L^2_T L^2}. \end{aligned}$$

Thus, we obtain by virtue of the convolution inequality

$$\|T_v \cdot v\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,1}} \leq C\|v\|_{\widetilde{L}^\infty_T L^2} \|\nabla v\|_{L^2_T L^2}. \tag{4.2}$$

For the remainder term, we apply Bernstein and Hölder inequalities and get

$$\begin{aligned} \|R(v,v)\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,1}} &\leq \sum_q 2^{-q} \sum_{j \geq q-3} \|\Delta_q(\Delta_j v\, \widetilde{\Delta}_j v)\|_{L^2_T L^\infty} \\ &\lesssim \sum_q 2^q \sum_{j \geq q-3} \|\Delta_j v\, \widetilde{\Delta}_j v\|_{L^2_T L^1} \\ &\lesssim \sum_{j \geq q-3} 2^q \|\Delta_j v\|_{L^\infty_T L^2} \|\widetilde{\Delta}_j v\|_{L^2_T L^2} \\ &\lesssim \sum_{j \geq q-3} 2^{q-j} \|\Delta_j v\|_{L^\infty_T L^2} \|\widetilde{\Delta}_j \nabla v\|_{L^2_T L^2}. \end{aligned}$$

Applying again the convolution inequality to the above sum, we obtain

$$\|R(v,v)\|_{\widetilde{L}^2_T \dot{B}^{-1}_{\infty,1}} \leq C\|v\|_{\widetilde{L}^\infty_T L^2} \|\nabla v\|_{L^2_T L^2}. \tag{4.3}$$

Putting together the estimates (4.2), (4.3) and (4.1) yield

$$\|v^2\|_{\widetilde{L}^2_T \dot{B}^0_{\infty,1}} \leq C\|v\|_{\widetilde{L}^\infty_T L^2} \|\nabla v\|_{L^2_T L^2}. \tag{4.4}$$

To conclude the $L^2_T L^\infty$ estimate, it suffices to show that $v \in \widetilde{L}^\infty_T L^2$. To do this, we apply Lemma 3.3 to the Navier-Stokes system and get

$$\|v\|_{\widetilde{L}^\infty_T L^2} \lesssim \|v^0\|_{L^2} + \|v \cdot v\|_{L^2_T L^2} + \|f\|_{L^1_T L^2}.$$

Now, using Galgiardo-Nirenberg inequality, we further have

$$\begin{aligned} \|v\|^2_{\widetilde{L}^\infty_T L^2} &\lesssim \|v^0\|^2_{L^2} + \int_0^T \|v(\tau)\|^2_{L^2} \|\nabla v(\tau)\|^2_{L^2} d\tau + \|f\|^2_{L^1_T L^2} \\ &\lesssim \|v^0\|^2_{L^2} + \|v\|^2_{L^\infty_T L^2} \|v\|^2_{L^2_T \dot{H}^1} + \|f\|^2_{L^1_T L^2}. \end{aligned}$$

Thus we get

$$\|v\|^2_{\widetilde{L}^\infty_T L^2} \leq C E_{0,T}(1 + E_{0,T}). \tag{4.5}$$

Inserting (4.5) into (4.4) gives

$$\|v^2\|_{\widetilde{L}^2_T \dot{B}^0_{\infty,1}} \leq C E_{0,T}(1 + E^{\frac{1}{2}}_{0,T}). \tag{4.6}$$

Therefore we get from this estimate and from the standard embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$,

$$\|v^2\|_{L^2_T L^\infty} \leq C E_{0,T}(1 + E^{\frac{1}{2}}_{0,T}).$$

This completes the proof of the $L^2_T L^\infty$ estimate of the velocity.
Let us now move to the second part of Theorem 4.1. To this end, we will first prove that $v \in \widetilde{L}^2_T \dot{B}^0_{\infty,1}$ which will be crucial for the proof of the $L^1_T \dot{B}^1_{\infty,1}$ bound of the velocity. According to the estimate (4.6), it suffices to show that v^1 belongs to the space $\widetilde{L}^2_T \dot{B}^0_{\infty,1}$. One can write from Lemma 2.4

$$\|v^1_q(t)\|_{L^\infty} \lesssim e^{-ct2^{2q}}\|v_q(0)\|_{L^\infty} + \int_0^t e^{-c(t-\tau)2^{2q}}\|f_q(\tau)\|_{L^\infty} d\tau. \tag{4.7}$$

Using the convolution inequality we obtain easily after summing over $q \in \mathbb{Z}$

$$\begin{aligned}\|v^1\|_{\widetilde{L}^2_T \dot{B}^0_{\infty,1}} &\lesssim \sum_{q\in\mathbb{Z}} 2^{-q}\|v^1_q(0)\|_{L^\infty} + \sum_{q\in\mathbb{Z}} 2^{-q}\|f_q\|_{L^1_T L^\infty}\\ &\lesssim \|v^0\|_{\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T \dot{B}^{-1}_{\infty,1}}.\end{aligned}$$

Hence we get

$$\|v\|_{\widetilde{L}^2_T \dot{B}^0_{\infty,1}} \leq C\Big(\|v^0\|_{\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T \dot{B}^{-1}_{\infty,1}} + E_{0,T}(1 + E^{\frac{1}{2}}_{0,T})\Big). \tag{4.8}$$

Our next purpose is to give the proof of the $L^1_T \dot{B}^1_{\infty,1}$ estimate. To start off, we apply convolution inequality to (4.7) leading to

$$\|v^1\|_{L^1_T \dot{B}^1_{\infty,1}} \lesssim \|v^0\|_{\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T \dot{B}^{-1}_{\infty,1}}.$$

For the term v^2, we proceed as in the proof of the $L^2_T L^\infty$ bound, (see [4.1]), and we find

$$\begin{aligned}\|v^2\|_{L^1_T \dot{B}^1_{\infty,1}} &\lesssim \|v \otimes v\|_{L^1_T \dot{B}^0_{\infty,1}}\\ &\lesssim \|T_v \cdot v\|_{L^1_T \dot{B}^0_{\infty,1}} + \|R(v,v)\|_{L^1_T \dot{B}^0_{\infty,1}}.\end{aligned}$$

We estimate the paraproduct term as follows:

$$\begin{aligned}\|T_v \cdot v\|_{L^1_T \dot{B}^0_{\infty,1}} &\lesssim \sum_q \|S_{q-1}v\|_{L^2_T L^\infty}\|v_q\|_{L^2_T L^\infty}\\ &\lesssim \|v\|_{L^2_T L^\infty}\|v\|_{\widetilde{L}^2_T \dot{B}^0_{\infty,1}}.\end{aligned}$$

Thus we get from (4.5) and (4.8)

$$\|T_v \cdot v\|_{L^1_T \dot{B}^0_{\infty,1}} \lesssim \Big(\|v_0\|_{\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T \dot{B}^{-1}_{\infty,1}} + E_{0,T}(1 + E^{\frac{1}{2}}_{0,T})\Big)E_{0,T}(1 + E^{\frac{1}{2}}_{0,T}).$$

Let us now turn to the remainder term. By using Bernstein and Hölder inequalities we find

$$\begin{aligned}
\|R(v,v)\|_{L^1_T\dot B^0_{\infty,1}} &\le \sum_q \sum_{j\ge q-3} \|\Delta_q(\Delta_j v\, \widetilde\Delta_j v)\|_{L^1_T L^\infty}\\
&\lesssim \sum_q 2^{2q} \sum_{j\ge q-3} \|\Delta_j v\, \widetilde\Delta_j v\|_{L^1_T L^1}\\
&\lesssim \sum_{j\ge q-3} 2^{2q} \|\Delta_j v\|_{L^2_T L^2} \|\widetilde\Delta_j v\|_{L^2_T L^2}\\
&\lesssim \sum_{j\ge q-3} 2^{2(q-j)} \|\Delta_j \nabla v\|_{L^2_T L^2} \|\widetilde\Delta_j \nabla v\|_{L^2_T L^2}\\
&\lesssim \|v\|^2_{L^2_T \dot H^1}.
\end{aligned}$$

Combining these estimates leads to

$$\|v\|_{L^1_T\dot B^1_{\infty,1}} \le C\big(\|v^0\|_{\dot B^{-1}_{\infty,1}} + \|f\|_{L^1_T\dot B^{-1}_{\infty,1}} + \mathcal{E}\big)(1+\mathcal{E}), \tag{4.9}$$

with $\mathcal{E} := E_{0,T}(1+E^{\frac12}_{0,T})$. This completes the proof of Theorem 4.1. $\square$

Remark 3. *A further result from the previous analysis that will be used later for the inhomogeneous case is the following inequality*

$$\begin{aligned}
\|v-v_{-1}\|_{L^1_T B^1_{\infty,1}} &\lesssim \|v^1 - v^1_{-1}\|_{L^1_T\dot B^1_{\infty,1}} + \|v^2 - v^2_{-1}\|_{L^1_T\dot B^1_{\infty,1}}\\
&\lesssim \big(\|v^0 - v^0_{-1}\|_{\dot B^{-1}_{\infty,1}} + \|f - f_{-1}\|_{L^1_T\dot B^{-1}_{\infty,1}} + \mathcal{E}\big)(1+\mathcal{E})\\
&\lesssim \big(\|v^0\|_{B^{-1}_{\infty,1}} + \|f\|_{L^1_T B^{-1}_{\infty,1}} + \mathcal{E}\big)(1+\mathcal{E}). \qquad (4.10)
\end{aligned}$$

We shall now apply Theorem 4.1 to the Boussinesq system yielding to the following proposition.

Proposition 4.2. *Let (v,θ) be a smooth global solution for the Boussinesq system such that $v^0 \in L^2\cap B^{-1}_{\infty,1}$ and $\theta^0 \in B^0_{2,1}$. Then we have for every time T,*

$$\begin{aligned}
\|v\|^2_{L^\infty_T L^2} + \|v\|^2_{L^2_T\dot H^1} &\le C_0(1+T^2),\\
\|v\|^2_{\widetilde L^\infty_T L^2} + \|v\|_{\widetilde L^\infty_T B^{-1}_{\infty,1}} &\le C_0(1+T^3) \quad \textit{and}\\
\|v\|_{L^1_T B^1_{\infty,1}} + \|\theta\|_{\widetilde L^\infty_T B^0_{2,1}} + \|\nabla\Pi\|_{L^1_T B^0_{2,1}} &\le C_0 e^{C_0 T^4}.
\end{aligned}$$

with C_0 a constant depending only on the initial data.

Proof. For the energy estimate, we take the L^2 inner product of the momentum equation with the velocity obtaining, due to the zero divergence condition, the following:

$$\begin{aligned}
\frac12\frac{d}{dt}\|v(t)^2_{L^2} + \|\nabla v(t)\|^2_{L^2} &\le \|v(t)\|_{L^2}\|\theta(t)\|_{L^2}\\
&\le \|\theta^0\|_{L^2}\|v(t)\|_{L^2}.
\end{aligned}$$

We have used here the estimate $\|\theta(t)\|_{L^2} = \|\theta^0\|_{L^2}$. So the desired estimate follows readily by integration from the above differential inequality.
The estimate $\widetilde{L}^\infty_T L^2$ can be easily derived from (4.5) and from the energy inequality:

$$E_{0,T} \leq C_0(1+T^2).$$

To show the $\widetilde{L}^\infty_T B^{-1}_{\infty,1}$ we intend to use on an adequate manner Lemma 3.3. We set

$$f = -\mathbb{P}R(v, \nabla v) \quad \text{and}$$

$$g = -\mathbb{P}(T_v \cdot \nabla v + T_{\nabla v} \cdot v) + \mathbb{P}\theta e_2.$$

Then an obvious computation shows that

$$\partial_t v - \Delta v = f + g.$$

Applying Lemma 3.3 one obtains

$$\|v - v_{-1}\|_{\widetilde{L}^\infty_T B^{-1}_{\infty,1}} \lesssim \|v^0\|_{B^1_{\infty,1}} + \|f - f_{-1}\|_{L^1_T B^{-1}_{\infty,1}} + \|g - g_{-1}\|_{\widetilde{L}^\infty_T B^{-3}_{\infty,1}}.$$

Using the definition of the remainder and Bernstein inequality we get easily

$$\|f - f_{-1}\|_{L^1_T B^{-1}_{\infty,1}} \lesssim \|v\|^2_{L^2_T H^1}.$$

By the same way we obtain

$$\|g - g_{-1}\|_{\widetilde{L}^\infty_T B^{-3}_{\infty,1}} \lesssim \|v\|^2_{\widetilde{L}^\infty_T L^2} + \|\theta^0\|_{L^2}.$$

Combining previous estimates yield

$$\|v - v_{-1}\|_{\widetilde{L}^\infty_T B^{-1}_{\infty,1}} \leq C_0(1+T^3).$$

For low frequency we have from Bernstein inequality and energy estimate,

$$\|v_{-1}\|_{\widetilde{L}^\infty_T B^{-1}_{\infty,1}} \lesssim \|v_{-1}\|_{L^\infty_T L^2} \lesssim 1+T.$$

Let us now state the proof of the smoothing effects, that is, $\|v\|_{L^1_T B^1_{\infty,1}}$. We first separate low and high frequencies and write

$$\|v\|_{L^1_T B^1_{\infty,1}} \leq \|v_{-1}\|_{L^1_T L^\infty} + \|v - v_{-1}\|_{L^1_T B^1_{\infty,1}}. \tag{4.11}$$

Using Bernstein inequality and the energy estimate, we find

$$\begin{aligned} \|v_{-1}\|_{L^1_T L^\infty} &\lesssim T\|v\|_{L^\infty_T L^2} \\ &\lesssim T E^{\frac{1}{2}}_{0,T}. \end{aligned}$$

Thus we get

$$\|v_{-1}\|_{L^1_T L^\infty} \le C_0(1+T^2). \tag{4.12}$$

For high frequencies, we use the inequality (4.10) and the embedding $B^0_{2,1} \hookrightarrow B^{-1}_{\infty,1}$

$$\|v - v_{-1}\|_{L^1_T B^1_{\infty,1}} \lesssim \big(\|v^0\|_{B^{-1}_{\infty,1}} + \|\theta\|_{L^1_T B^0_{2,1}} + \mathcal{E}\big)(1+\mathcal{E}).$$

Combining the definition of $\mathcal{E}$ seen in Theorem 4.1 and the energy estimate one obtains

$$\mathcal{E} \le C_0(1+T^3).$$

Therefore we get

$$\|v - v_{-1}\|_{L^1_T B^1_{\infty,1}} \le C_0(1+T^6) + C_0(1+T^3)\int_0^T \|\theta(\tau)\|_{B^0_{2,1}} d\tau.$$

Now Proposition 3.2 ensures, in particular, that

$$\|\theta(t)\|_{B^0_{2,1}} \lesssim \|\theta_0\|_{B^0_{2,1}}\Big(1 + \int_0^t \|v(\tau)\|_{B^1_{\infty,1}} d\tau\Big).$$

We have used here the continuous embedding $B^1_{\infty,1} \hookrightarrow \mathrm{Lip}$. Therefore

$$\|v - v_{-1}\|_{L^1_T B^1_{\infty,1}} \le C_0(1+T^6) + C_0(1+T^3)\int_0^T \|v\|_{L^1_\tau B^1_{\infty,1}} d\tau. \tag{4.13}$$

Putting together (4.11), (4.12) and (4.13)

$$\|v\|_{L^1_T B^1_{\infty,1}} \le C_0(1+T^6) + C_0(1+T^3)\int_0^t \|v\|_{L^1_\tau B^1_{\infty,1}} d\tau$$

It suffices now to use Gronwall's inequality

$$\|v\|_{L^1_T B^1_{\infty,1}} \le C_0 e^{C_0 T^4}.$$

Combining this estimate with Proposition 3.2, we have

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_T B^0_{2,1}} &\lesssim \|\theta^0\|_{B^0_{2,1}}\Big(1 + \int_0^t \|v(\tau)\|_{B^1_{\infty,1}} d\tau\Big)\\ &\le C_0 e^{C_0 T^4}.\end{aligned}$$

What remains to be done now is to estimate the pressure. First, we have the following identity that we can derive easily from the zero divergence condition

$$\nabla\Pi = \nabla\Delta^{-1}\partial_2\theta + \nabla\Delta^{-1}\mathrm{div}\,(v\cdot\nabla v).$$

Thus we get from Riesz inequality

$$
\begin{aligned}
\|\nabla\Pi\|_{L^1_T B^0_{2,1}} &\lesssim \|\theta\|_{L^1_T B^0_{2,1}} + \|v\cdot\nabla v\|_{L^1_T B^0_{2,1}} \\
&\lesssim T\|\theta\|_{L^\infty_T B^0_{2,1}} + \|v\cdot\nabla v\|_{L^1_T B^0_{2,1}}.
\end{aligned}
$$

By a simple computation applying Bony's decomposition and using Hölder inequalities, we obtain

$$
\|v\cdot\nabla v\|_{L^1_T B^0_{2,1}} \lesssim \|v\|_{L^\infty_T L^2}\|v\|_{L^1_T B^1_{\infty,1}}.
$$

Combining this inequality with (4.5) gives the desired result and so the proof of Proposition 4.2 is done. □

4.2 Existence

Let us now outline briefly the proof of the existence of global solutions to the Boussinesq system which is standard and similar to [12]. We smooth out the initial data and so we obtain from the result of [4] the existence of a family of unique global solutions (v^n,θ^n). It follows from the a priori estimates that this family is bounded in our solving spaces, hence it converges weakly to (v,θ), up to the extraction of a subsequence. However, to pass to the limit in the equations we have to establish the local strong convergence. This relies upon compactness properties of the sequence which are obtained by considering the time derivative of the solution. More precisely we use the following uniform estimate, for all $\eta>0$ and $T>0$,

$$
\begin{aligned}
\|\partial_\tau v_n\|_{L^2_T H^{-1-\eta}} &\leq C\|\Delta v_n\|_{L^2_T H^{-1-\eta}} + C_\eta\|v_n\|_{L^\infty_T L^2}\|v_n\|_{L^2_T H^1} + \|\theta_n\|_{L^2_t L^2} \\
&\leq C_{0,\eta}(1+T^2). \qquad (4.14)
\end{aligned}
$$

which is derived from the equation of the velocity, law products in Sobolev spaces and energy estimates. Thus we get the strong convergence according to Ascoli Theorem.
It remains to show the continuity in-time of any solution (v,θ) already constructed. Inasmuch as the velocity satisfies (4.5), we affirm that for every $\epsilon>0$ there exists an integer N such that

$$
\sum_{q\geq N}\|\Delta_q v\|^2_{L^\infty_T L^2} \leq \frac{\epsilon^2}{16}.
$$

Let $t,\ t'\in[0,T]$, then it follows from Taylor formula and Hölder inequality,

$$
\begin{aligned}
\|v(t)-v(t')\|_{L^2} &\leq \|S_N v(t)-S_N v(t')\|_{L^2} + 2\big(\sum_{q\geq N}\|\Delta_q v\|^2_{L^\infty_T L^2}\big)^{\frac12} \\
&\leq |t-t'|^{\frac12}\|\partial_\tau S_N v\|_{L^2_T L^2} + \frac{\epsilon}{2} \\
&\leq C2^{N(1+\eta)}|t-t'|^{\frac12}\|\partial_\tau S_N v\|_{L^2_T H^{-1-\eta}} + \frac{\epsilon}{2} \\
&\leq C_0 2^{N(1+\eta)}|t-t'|^{\frac12}(1+T) + \frac{\epsilon}{2}.
\end{aligned}
$$

This proves the continuity of the function v taking value in L^2. We can show by the same method that v belongs to $\mathcal{C}(\mathbb{R}_+; B^{-1}_{\infty,1})$. It is to be observed that the continuity of the temperature θ may be proven in the same line as the above proof since θ lies in $\widetilde{L}^\infty_{\rm loc} B^0_{2,1}$, according to Proposition 4.2.

4.3 Uniqueness

In this section we shall give the proof of the uniqueness part.

Theorem 4.3. *Let $\{(\theta^i, v^i, \nabla\Pi^i)\}_{i=1}^2$ be two solutions of the system (B) with the same initial data (θ_0, v_0), such that $\theta_0 \in B^0_{2\,1}(\mathbb{R}^2)$ and $v_0 \in B^{-1}_{\infty\,1}(\mathbb{R}^2) \cap L^2$ with $\operatorname{div} v_0 = 0$. We assume that for $i \in \{1, 2\}$,*

$$\begin{aligned} \theta^i &\in L^\infty_{loc}(\mathbb{R}_+;\, B^0_{2,1}),\\ v^i &\in L^\infty_{loc}(\mathbb{R}_+;\, L^2) \cap L^2_{loc}(\mathbb{R}_+;\, H^1) \cap L^1_{Loc}(\mathbb{R}_+;\, B^1_{\infty,1}),\\ \nabla\Pi^i &\in L^1_{Loc}(\mathbb{R}_+;\, B^0_{2,1}). \end{aligned}$$

The we have $(\theta^2, v^2, \nabla\Pi^2) = (\theta^1, v^1, \nabla\Pi^1)$.

Proof. We set $w = v^2 - v^1$, $\eta = \theta^2 - \theta^1$ and $\Pi = \Pi^2 - \Pi^1$. They satisfy

$$\begin{cases} \partial_t w + v^2 \cdot \nabla w - \Delta w + \nabla \Pi = \eta\, e_2 - w \cdot \nabla v^1 \\ \partial_t \eta + v^2 \cdot \nabla \eta = -w \cdot \nabla \theta^1 \\ \operatorname{div} w = 0. \end{cases}$$

We intend to estimate $(w, \nabla\Pi, \eta)$ in the space

$$E_T := L^\infty_T B^{-1}_{2\,\infty} \cap \widetilde{L}^1_T B^1_{2\,\infty} \times \widetilde{L}^1_T B^{-1}_{2\,\infty} \times L^\infty_T B^{-1}_{2\,\infty}.$$

First, we notice that $(w, \eta, \Pi) \in E_T$. From Proposition 3.1 and Minkowski inequality, we have

$$\|w\|_{L^\infty_T B^{-1}_{2,\infty}} + \|w\|_{\widetilde{L}^1_T B^1_{2,\infty}} + \|\nabla\Pi\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}} \le C e^{T + \|v^2\|_{L^1_T B^1_{\infty,1}}} \Big(\|\eta\|_{L^1_T B^{-1}_{2,\infty}} + \\ + \|w \cdot \nabla v^1\|_{L^1_T B^{-1}_{2,\infty}} \Big).$$

One can easily show by using Bony's decomposition and Proposition 2.6 that

$$\|w \cdot \nabla v^1\|_{B^{-1}_{2,\infty}} \le C \|w\|_{B^{-1}_{2,\infty}} \|v^1\|_{B^1_{\infty,1}}.$$

Applying Corollary 2 and Minkowski inequality lead to

$$\|\eta(t)\|_{B^{-1}_{2\,\infty}} \le C e^{\|v^2\|_{L^1_t B^1_{\infty,1}}} \|w \cdot \nabla \theta^1\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}}.$$

We have again from this proposition

$$\begin{aligned}\|w\cdot\nabla\theta^1\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} &\le \|R(w^j,\partial_j\theta^1)\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + \|T_{w^j}\partial_j\theta^1\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} + \|T_{\partial_j\theta^1}w^j\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}}\\ &\lesssim \|w\|_{\widetilde{L}^1_t B^1_{2,\infty}}\|\theta^1\|_{\widetilde{L}^\infty_t B^0_{2,1}} + \|w\|_{L^1_t L^\infty}\|\theta^1\|_{L^\infty_t B^0_{2,\infty}}.\end{aligned}$$

The logarithmic estimate (2.2) yield

$$\|w\|_{L^1_t L^\infty} \lesssim \|w\|_{\widetilde{L}^1_t B^1_{2,\infty}} \ln\Big(e + \frac{\|v^1\|_{L^1_t B^1_{\infty,1}} + \|v^2\|_{L^1_t B^1_{\infty,1}}}{\|w\|_{\widetilde{L}^1_t B^1_{2,\infty}}}\Big).$$

Putting together the previous estimates lead to

$$\begin{aligned}\|w\|_{L^\infty_T B^{-1}_{2,\infty}} + \|w\|_{\widetilde{L}^1_T B^1_{2,\infty}} + \|\nabla\Pi\|_{\widetilde{L}^1_T B^{-1}_{2,\infty}} &\lesssim e^{T+\|v^2\|_{L^1_T B^1_{\infty,1}}}\\ \times\Big\{\int_0^T \|w\|_{\widetilde{L}^1_t B^1_{2,\infty}} \ln\Big(e + &\frac{\|v^1\|_{L^1_t B^1_{\infty,1}} + \|v^2\|_{L^1_t B^1_{\infty,1}}}{\|w\|_{\widetilde{L}^1_t B^1_{2,\infty}}}\Big)\|\theta^1\|_{\widetilde{L}^\infty_t B^0_{2,1}}dt\\ + \int_0^T \|w\|_{B^{-1}_{2,\infty}}\|v^1\|_{B^1_{\infty,1}}dt\Big\}.\end{aligned}$$

We define

$$\gamma(t) = \|w\|_{L^\infty_t B^{-1}_{2,\infty}} + \|w\|_{\widetilde{L}^1_t B^1_{2,\infty}} + \|\nabla\Pi\|_{\widetilde{L}^1_t B^{-1}_{2,\infty}} \quad \text{and} \quad \alpha(t) = \sum_{1\le i\le 2}\|v^i\|_{L^1_t B^1_{\infty,1}}.$$

Now, applying Gronwall's Lemma, we get

$$\gamma(T) \lesssim \|\theta^1\|_{\widetilde{L}^\infty_T B^0_{2,1}} e^{T+\alpha(T)}\int_0^T \gamma(t)\ln\Big(e + \frac{\alpha(T)}{\gamma(t)}\Big)dt.$$

We have used here the fact that the function $x \to x\log(e + \frac{\alpha(T)}{x})$ is increasing over $\mathbb{R}_+$. Thus we conclude from Osgood lemma that $\gamma(T) = 0$ which implies that $\eta = 0$, and this concludes the uniqueness proof. □

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