BLOWUP THEORY FOR THE CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS REVISITED

TAOUFIK HMIDI AND SAHBI KERAANI

Abstract. In this note we prove a refined version of compactness lemma adapted to the blowup analysis and we use it to give direct and simple proofs to some classical results of blowup theory for critical nonlinear Schrödinger equations (mass concentration, classification of the singular solutions with minimal mass...).

1. Introduction

We consider the L^2-critical nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplace operator on $\mathbb{R}^d$ and $u_0 : \mathbb{R}^d \longrightarrow \mathbb{C}$. It is well-known from the result by Ginibre and Velo (see [3] for a review) that Cauchy problem (1.1) is locally well-posed in H^1: there exists $T \in (0, +\infty]$ and a solution $u \in \mathcal{C}([0,T), H^1)$, with the following blowup alternative: either $T = \infty$ (the solution is global) or $T < +\infty$ (the solution blows up in finite time) and

$$\lim_{t \uparrow T} \|u(t,\cdot)\|_{H^1} = +\infty.$$

The unique solution has the following conservation law:

$$\begin{aligned} \mathcal{N}(t) &= \int_{\mathbb{R}^d} |u(t,x)|^2 dx, \\ E(t) &= \frac{1}{2} \int |\nabla u|^2 dx - \frac{d}{4+2d} \int |u|^{\frac{4}{d}+2} dx \end{aligned} \tag{1.2}$$

Also, if $u_0 \in \Sigma = \{f \in H^1, xf \in L^2\}$, then the solution satisfies the *Virial* identity

$$\frac{d^2}{dt} \int |x|^2 |u(t,x)|^2 dx = 16E(0).$$

If $E(0) < 0$ then $t \mapsto \int |x|^2 |u(t,x)|^2 dx$ is an inverted parabola which becomes negative in finite time. Thus, the solution cannot exist globally and blows up at finite time. This was the starting point of a blowup theory of Schroödinger

2000 *Mathematics Subject Classification.* 35Q55, 35B40, 35B05.

Key words and phrases. Time dependent Schrödinger equation, blowup, mass concentration.

equations which has been developed in the two last decays (see [3], [13], [11] and the references therein). This theory is mainly connected to the notion of ground state: the unique positive radial solution of the elliptic problem

$$\Delta Q - Q + |Q|^{\frac{4}{d}} Q = 0.$$

In [15], M. I. Weinstein exhibited the following refined Gagliardo-Nirenberg inequality

$$\|\psi\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}} \leq C_d \|\psi\|^{\frac{4}{d}}_{L^2} \|\nabla \psi\|^2_{L^2} \qquad \forall \psi \in H^1, \tag{1.3}$$

with $C_d = \frac{d+2}{d} \|Q\|^{-\frac{4}{d}}_{L^2}$. Combined with the conservation of energy, this implies that $\|Q\|_{L^2}$ is the critical mass for the formation of singularities: for every $u_0 \in H^1$ such that

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

the solution of (1.1) with initial data u_0 is global. Also, this bound is optimal. In fact, by using the conformal invariance, one constructs

$$u(t,x) = (T-t)^{-d/2} e^{[(i/(T-t))+(-i|x|^2/T-t)]} Q(\frac{x}{T-t})$$

a sigular solution of with critical mass. In this sens $\|Q\|_{L^2}$ is the minimal mass necessary to ignite a wave collapse. There is an abundant literature devoted to the study of the blow up mechanism (see [3] and [13] for a review). In this note we prove a compactness lemma adapted to the analysis of the blowup phenomenon of the nonlinear Schroödinger equation and use it to give elementary proofs of some classical blowup results. Our main tool is the following.

Theorem 1.1 (Comactness Lemma)**.** *Let $\{v_n\}_{n=1}^{\infty}$ be a bounded family of $H^1(\mathbb{R}^d)$, such that*

$$\limsup_{n\to\infty} \|\nabla v_n\|_{L^2} \leq M \qquad \textit{and} \qquad \limsup_{n\to\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m.$$

Then, there exits $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ such that, up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V$$

with $\|V\|_{L^2} \geq (\frac{d}{d+2})^{d/4} \frac{m^{\frac{2}{d}+1}}{M^{d/2}} \|Q\|_{L^2}$.

Remark 1.2. *The lower-bound on the L^2 norm of V is optimal. In fact, if we take $v_n = Q$ then we get equality.*

The plan of the rest of the note is as follows. In section 2 we restate and prove some well-known blow up results. Section 3 is devoted to the proof of Theorem 1.1.

2. Blow up theory revisited

2.1. **Concentration.** For $d \geq 2$ and spherically symmetric blow up solutions, it has been shown (see [12],[14],[17]) that there is a minimal amount pf concentration of the L^2 norm at the origin. Below we give an easy proof for the general case.

Theorem 2.1. *Let u be a solution of* (1.1) *which blows up at finite time $T > 0$, and $\lambda(t) > 0$ any function, such that $\frac{\sqrt{T-t}}{\lambda(t)} \longrightarrow 0$ as $t \uparrow T$. Then, there exits $x(t) \in \mathbb{R}^d$, such that*

$$\liminf_{t \uparrow T} \int_{|x-x(t)| \leq \lambda(t)} |u(t,x)|^2 dx \geq \int Q^2.$$

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $t_n \uparrow T$. We set

$$\rho_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t_n, \cdot)\|_{L^2}} \qquad \text{and} \qquad v_n = \rho_n^{d/2} u(t_n, \rho_n x).$$

The sequence $\{v_n\}_{n=1}^{\infty}$ satisfies:

$$\|v_n\|_{L^2} = \|u_0\|_{L^2}, \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

Furthermore, by conservation of the energy and blowup criteria, it ensues that

$$E(v_n) = \rho_n^2 E(0) \longrightarrow 0, \qquad \text{as} \quad n \to \infty$$

which yields, in particular,

$$\|v_n\|_{\mathrm{L}^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \longrightarrow \frac{d+2}{d} \|\nabla Q\|_{L^2}^2, \qquad \text{as} \quad n \to \infty.$$

The family $\{v_n\}_{n=1}^{\infty}$ satisfies the conditions of the Theorem 1.1 above with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2}^2 \qquad \text{and} \qquad M^2 = \|\nabla Q\|_{L^2}^2.$$

Thus, *there exists exits $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ such that, up to a subsequence,*

$$v_n(\cdot + x_n) = \rho_n^{d/2} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \in H^1$$

with $\|V\|_{L^2} \geq \|Q\|_{L^2}$.[Note taht this asymptotic is proved in [16] via concentration-compactness Lemma by P.L. Lions [8].] From this, it follows that

$$\liminf_{n \to \infty} \int_{|x| \leq A} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx,$$

for every $A > 0$. Thus,

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq A\rho_n} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx. \tag{2.1}$$

Since $\frac{\sqrt{T-t}}{\lambda(t)} \longrightarrow 0$ as $t \to T$, it ensues that $\frac{\rho_n}{\lambda(t_n)} \longrightarrow 0$ and then

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx$$

for every A, which means that

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{\{|x-y|\leq\lambda(t_n)\}}|u(t_n,x)|^2dx \geq \int_{\mathbb{R}^d}|V|^2dx \geq \int Q^2.$$

Since this is true for every sequence $t_n\to T$, then

$$\liminf_{t\to T}\sup_{y\in\mathbb{R}^d}\int_{\{|x-y|\leq\lambda(t)\}}|u(t,x)|^2dx \geq \int Q^2.$$

as claimed. □

2.2. **Universality of the profile with critical mass.** In the context of the proof of Theorem 2.1 above, if we assume $\|u_0\|_{L^2}=\|Q\|_{L^2}$ then[1]

$$\|V\|_{L^2}=\|Q\|_{L^2}.$$

Thus,

$$v_n(\cdot-x_n)\to V \qquad \text{in} \qquad L^2.$$

Also, since it's bounded in H^1, we have

$$v_n(\cdot-x_n)\to V \qquad \text{in} \qquad L^4.$$

In view of Galiardo-Nirenberg inequality, this leads to

$$\|\nabla V\|_{L^2}\geq\|\nabla Q\|_{L^2}.$$

Since $\|\nabla V\|_{L^2}\leq\limsup\|\nabla v_n\|_{L^2}=\|\nabla Q\|_{L^2}$, then

$$\|\nabla v_n\|_{L^2}\longrightarrow\|\nabla V\|_{L^2}.$$

This means that the strong convergence holds in H^1. This fact implies, in particular,

$$E(V)=0.$$

Let us summarize the properties of the limit V:

$$V\in H^1,\quad \|V\|_{L^2}=\|Q\|_{L^2},\quad \|\nabla V\|_{L^2}=\|\nabla Q\|_{L^2}\quad\text{and}\quad E(V)=0.$$

The variational characterization of the ground state implies that

$$V(x)=e^{i\theta}Q(x+x_0),$$

for some $\theta\in[0,2\pi[$ and $x_0\in\mathbb{R}^d$. This prove the following result[2]: *if u is a singular solution with critical mass then there exist $x(t)$ and $\theta(t)$ such that*

$$(\lambda(t))^{d/2}e^{i\theta(t)}u(t,\lambda(t)x+x(t))\longrightarrow Q,\qquad t\to T. \tag{2.2}$$

---

[1]Since $\|V\|_{L^2}\leq\|u_0\|_{L^2}$.

[2]This result is due to Weinstein [16] and Kwong [7].

2.3. **Characterization of the singular solutions with critical mass.** In the sequel we use the notation:

$$\mathcal{A} = \{\lambda^{d/2} e^{i\theta} R(\lambda x + y), \quad y \in \mathbb{R}^d, \lambda \in \mathbb{R}_*^+, \theta \in [0, 2\pi[\}.$$

Theorem 2.2 ([9]). *Let u be a blowing up solution of (1.1) at finite time $T > 0$ such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Then, there exits $x_0 \in \mathbb{R}^d$ such that $e^{i\frac{|x-x_0|^2}{4T}} u_0 \in \mathcal{A}$.*

Proof. Let $t_n \to T$ any sequence. It is clear that (2.2) implies

$$|u(t_n, x)|^2 dx - \|R\|_{L^2}^2 \delta_{x=x_n} \rightharpoonup 0.$$

Up to extract a subsequence and translation, one assume $x_n \to x_0 \in \{0, \infty\}$. Let ϕ a nonnegative radial $C_0^\infty(\mathbb{R}^d)$ function such that

$$\phi(x) = |x|^2, \quad |x| < 1 \quad and \quad |\nabla\phi(x)|^2 \leq C\phi(x).$$

For every $p \in \mathbb{N}$ one defines

$$\phi_p(x) = p^2 \phi(\frac{x}{p}) \quad and \quad g_p(t) = \int \phi_p(x)|u(t,x)|^2 dx.$$

Using the Cauchy-Schwartz estimates by V. Banica [2], we get (remember $|\nabla\phi_p|^2 \leq C\phi_p$)

$$\begin{aligned} |\dot{g}_p(t)| = |2\Im \int \bar{u}(x)\nabla u(x)\nabla\phi_p(x)dx| &\leq (8E(u_0)\int |u|^2|\nabla\phi_p|^2 dx)^{1/2} \\ &\leq C(u_0)\sqrt{g_p(t)}, \end{aligned}$$

for every $t \in [0, T[$. By integration we obtain, for every $t \in [0, T[$,

$$|\sqrt{g_p(t)} - \sqrt{g_p(t_n)}| \leq C|t_n - t|.$$

We let t_n goes to T; and we get (since $\phi_p(x_0) = 0$ for both finite or infinite case)

$$g_p(t)| \leq C(u_0)(T-t)^2.$$

We fix $t \in [0; T[$ and get p goes to infinity[3] to obtain

$$8t^2 E(e^{i\frac{|x|^2}{4t}} u_0) = \int |x|^2 |u(t,x)|^2 dx \leq C(u_0)(T-t)^2. \tag{2.3}$$

The first identity, which is just another way the write Viriel identity, is easy. We let then t goes to T and get[4]

$$E(e^{i\frac{|x|^2}{4T}} u_0) = 0.$$

Since $\|e^{i\frac{|x|^2}{4T}} u_0\|_{L^2} = \|u_0\|_{L^2} = \|R\|_{L^2}$, then the variational characterization of R implies that $e^{i\frac{|x|^2}{4T}} u_0 \in \mathcal{A}$. This ends the proof of this theorem. □

[3]We can take first the limit at $t = 0$ to get $u_0 \in \Sigma$. This implies that $\int |x|^2 |u(t,x)|^2 dx$ is well defined for every $t \in [0, T[$. We take then the limit in all $t \in [0, T[$.

[4]We may use the uniform bound (2.4) to prove that $\lim x_n \neq \infty$ and then equal, up to a translation, to 0.

Appendix. Proof of Theorem 1.1

In the sequel we put $2 = \infty$ if $d = 1, 2$, and $2 = \frac{2d}{d-2}$ if $d \geq 3$. Theorem 1.1 is a consequence of a profile decomposition of the bounded sequences in H^1 following the work by P. Gérard [4] (see also [1] and [6]). More precisely, we have the following

Proposition 2.3. *Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then there exist a subsequence of $\{v_n\}_{n=1}^{\infty}$ (still denoted $\{v_n\}_{n=1}^{\infty}$), a family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ of sequences in $\mathbb{R}^d$ and a sequence $\{V^j\}_{j=1}^{\infty}$ of H^1 functions, such that*

i) *for every $k \neq j$, $\quad |x_n^k - x_n^j| \underset{n\to\infty}{\longrightarrow} +\infty$;*

ii) *for every $\ell \geq 1$ and every $x \in \mathbb{R}^d$, we have*

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^{\ell}(x),$$

with

$$\limsup_{n\to\infty} \|v_n^{\ell}\|_{L^p(\mathbb{R}^d)} \underset{\ell\to\infty}{\longrightarrow} 0 \tag{2.4}$$

for every $2 < p < 2^*$.

Moreover, we have

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^{\ell}\|_{L^2}^2 + o(1), \tag{2.5}$$

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^{\ell}\|_{L^2}^2 + o(1), \tag{2.6}$$

as $n \to \infty$.

Proof. Let $\mathbf{v} = \{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Let $\mathcal{V}(\mathbf{v})$ be the set of functions obtained as weak limits in H^1 of subsequences of the translated $v_n(. + x_n)$ with $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$. We denote

$$\eta(\mathbf{v}) = \sup\{\|V\|_{H^1}, \quad V \in \mathcal{V}(\mathbf{v})\}.$$

Clearly

$$\eta(\mathbf{v}) \leq \limsup_{n\to\infty} \|v_n\|_{H^1}.$$

We shall prove the existence of a sequence $\{V^j\}_{j=1}^{\infty}$ of $\mathcal{V}(\mathbf{v})$ and a family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ of sequences of $\mathbb{R}^d$, such that

$$k \neq j \Longrightarrow |x_n^k - x_n^j| \underset{n\to\infty}{\longrightarrow} \infty,$$

and, up to extracting a subsequence, the sequence $\{v_n\}_{n=1}^\infty$ can be written as

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x), \quad \eta(\mathbf{v}^\ell) \underset{\ell\to\infty}{\longrightarrow} 0,$$

such that the identities (2.5)-(2.6) hold. Indeed, if $\eta(\mathbf{v}) = 0$, we can take $V^j \equiv 0$ for all j, otherwise we choose $V^1 \in \mathcal{V}(\mathbf{v})$, such that

$$\|V^1\|_{H^1} \geq \frac{1}{2}\eta(\mathbf{v}) > 0.$$

By definition, there exists some sequence $\mathbf{x}^1 = \{x_n^1\}_{n=1}^\infty \subset \mathbb{R}^d$, such that, up to extracting a subsequence, we have

$$v_n(\cdot + x_n^1) \rightharpoonup V^1 \qquad \text{in} \quad H^1.$$

We set

$$v_n^1 = v_n - V^1(\cdot - x_n^1).$$

Since $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ weakly in H^1, we get

$$\begin{aligned} \|v_n\|_{L^2}^2 &= \|V^1\|_{L^2}^2 + \|v_n^1\|_{L^2}^2 + o(1), \\ \|\nabla v_n\|_{L^2}^2 &= \|\nabla V^1\|_{L^2}^2 + \|\nabla v_n^1\|_{L^2}^2 + o(1), \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Now, we replace $\mathbf{v}$ by $\mathbf{v}^1$ and repeat the same process. If $\eta(\mathbf{v}^1) > 0$ we get V^2, $\{x_n^2\}_{n=1}^\infty$ and $\mathbf{v}^2$. Moreover, we have

$$|x_n^1 - x_n^2| \longrightarrow \infty, \qquad \text{as} \quad n \to \infty$$

Otherwise, up to extracting of subsequence, we get

$$x_n^1 - x_n^2 \longrightarrow x_0$$

for some $x_0 \in \mathbb{R}^d$. Since

$$v_n^1(\cdot + x_n^2) = v_n^1(\cdot + (x_n^2 - x_n^1) + x_n^1)$$

and $v_n^1(\cdot + x_n^1)$ converge weakly to 0, then $V^2 = 0$. Thus $\eta(\mathbf{v}^1) = 0$, a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family $\{\mathbf{x}^j\}_{j=1}^\infty$ and $\{V^j\}_{j=1}^\infty$ satisfying the claims above. Furthermore, the convergence of the series $\sum_{j=1}^\infty \|V^j\|_{H^1}^2$ implies that

$$\|V^j\|_{H^1} \underset{j\to\infty}{\longrightarrow} 0.$$

However, by construction, we have

$$\eta(\mathbf{v}^j) \leq \|V^{j-1}\|_{H^1},$$

which proves that $\eta(\mathbf{v}^j) \to 0$ as claimed. To complete the proof of Proposition 2.3, (2.4) remains to be proved. For that purpose let us introduce $\chi_R \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \leq \hat{\chi}_R \leq 1$ and

$$\hat{\chi}_R(\xi) = 1 \quad \text{if} \quad |\xi| \leq R, \quad \hat{\chi}_R(\xi) = 0 \quad \text{if} \quad |\xi| \geq 2R.$$

Here $\hat{}$ denotes the Fourier transform. One has

$$v_n^\ell = \chi_R * v_n^\ell + (\delta - \chi_R) * v_n^\ell,$$

where $*$ stands for the convolution. Let $p \in]2, 2^*[$ to be fixed. On the one hand, in view of Sobolev embedding, we get

$$\|(\delta - \chi_R) * v_n^\ell\|_{L^p} \lesssim \|(\delta - \chi_R) * v_n^\ell\|_{\dot{H}^\beta} \lesssim R^{\beta-1}\|v_n^\ell\|_{H^1},$$

for $\beta = d(\frac{1}{2} - \frac{1}{p}) < 1$. On the other hand, one can estimate

$$\begin{aligned} \|\chi_R * v_n^\ell\|_{L^p} &\lesssim \|\chi_R * v_n^\ell\|_{L^2}^{2/p}\|\chi_R * v_n^\ell\|_{L^\infty}^{1-2/p} \\ &\lesssim \|v_n^\ell\|_{L^2}^{2/p}\|\chi_R * v_n^\ell\|_{L^\infty}^{1-2/p}. \end{aligned}$$

Now, observe that

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} = \sup_{\{x_n\}} \limsup_{n\to+\infty} |\chi_R * v_n^\ell(x_n)|.$$

Thus, in view of the definition of $\mathcal{V}(\mathbf{v}^\ell)$, we infer

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \leq \sup\{|\int_{\mathbb{R}^d} \chi_R(-x)V(x)dx|, \quad V \in \mathcal{V}(\mathbf{v}^\ell)\}.$$

Therefore, by Hölder's inequality, it follows that

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \leq C_2 \sup\{\|V\|_{L^2(\mathbb{R}^d)}, \quad V \in \mathcal{V}(\mathbf{v}^\ell)\}.$$

Here, $C_2(R) = \|\chi_R\|_{L^2(\mathbb{R}^d)}$. Thus, we obtains

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \leq C_2\eta(\mathbf{v}^\ell)$$

for every $\ell \geq 1$. Finally, we get

$$\|v_n^\ell\|_{L^p(\mathbb{R}^d)} \lesssim R^{\beta-1}\|v_n^\ell\|_{H^1} + C(R)\|v_n^\ell\|_{L^2}^{2/p}\eta(\mathbf{v}^\ell)^{1-2/p}.$$

Now, we let ℓ go to infinity , then R go to infinity and since $\eta(\mathbf{v}^\ell) \underset{\ell\to\infty}{\longrightarrow} 0$ and the family of sequences $\{v_n^\ell\}$ are uniformly bounded in $H^1(\mathbb{R}^d)$, we obtain

$$\limsup_{n\to\infty} \|v_n^\ell\|_{L^p} \underset{\ell\to\infty}{\longrightarrow} 0$$

as claimed. This closes the proof of the proposition 2.3. □

Let us return to the proof of Theorem 1.1. According to Proposition 2.3, the sequence $\{v_n\}_{n=1}^\infty$ can be written, up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x)$$

such that (2.4) and (2.5) hold. This implies, in particular,

$$m^{\frac{4}{d}+2} \leq \limsup_{n\to\infty} \|\sum_{j=1}^{\infty} V^j(\cdot - x_n^j)\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

The elementary inequality

$$||\sum_{j=1}^{l} a_j|^{4/d+2} - \sum_{j=1}^{l} |a_j|^{4/d+2}| \leq C \sum_{j\neq k} |a_j||a_k|^{4/d+1}.$$

and the pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ leads the mixed terms in the sum above to vanish and we get

$$m^{\frac{4}{d}+2} \leq \sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

On the one hand, in view of Galiardo-Nirenberg inequality (1.3), we have

$$\sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d \sup\{\|V^j\|_{L^2}^{4/d}, j \geq 1\} \sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2.$$

On the other hand, from (2.4), we get

$$\sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2 \leq \limsup_{n\to\infty} \|\nabla v_n\|_{L^2}^2 \leq M^2.$$

Therefore,

$$\sup_{j\geq 1} \|V^j\|_{L^2}^{4/d} \geq \frac{m^{\frac{4}{d}+2}}{(M^2 C_d)^{d/4}}.$$

Since the series $\sum \|V^j\|_{L^2}^2$ converges then the supremum above is attained. In particular, there exists j_0, such that

$$\|V^{j_0}\|_{L^2} \geq \frac{m^{\frac{2}{d}+1}}{(C_d M)^{d/4}} = (\frac{d}{d+2})^{d/4} \frac{m^{\frac{2}{d}+1}}{M^{d/2}} \|Q\|_{L^2}.$$

On the other hand, a change of variables gives

$$v_n(x + x_n^{j_0}) = V^{j_0}(x) + \sum_{\substack{1 \leq j \leq \ell \\ j \neq j_0}} V^j(x + x_n^{j_0} - x_n^j) + \tilde{v}_n^{\ell}(x),$$

where $v_n^{\ell}(x) = \tilde{v}_n^{\ell}(x + x_n^{j_0})$. The pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ implies

$$V^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0 \quad \text{weakly}$$

for every $j \neq j_0$. Hence, we get

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{v}^{\ell},$$

where $\tilde{v}^{\ell}$ denote the weak limit of $\{\tilde{v}_n^{\ell}\}_{n=1}^{\infty}$. However, we have

$$\|\tilde{v}^{\ell}\|_{L^{\frac{4}{d}+2}} \leq \limsup_{n\to\infty} \|\tilde{v}_n^{\ell}\|_{L^{\frac{4}{d}+2}} = \limsup_{n\to\infty} \|v_n^{\ell}\|_{L^{\frac{4}{d}+2}} \underset{l\to\infty}{\longrightarrow} 0.$$

Thereby, by uniqueness of weak limit, we get

$$\tilde{v}^{\ell} = 0$$

for every $\ell \geq j_0$. This yields

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0}.$$

The sequence $\{x_n^{j_0}\}_{n=1}^{\infty}$ and the function V^{j_0} fulfill the conditions of Theorem 1.1. ∎

References

[1] H. Bahouri and P. Gérard: *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), no. 1, 131–175.

[2] V. Banica: *Remarks on the blow-up for the Schrdinger equation with critical mass on a plane domain*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) *3* (2004), no. 1, 139–170.

[3] T. Cazenave: *Semilinear Schrdinger equations*, Courant Lecture Notes in Mathematics, **10**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[4] P. Gérard: *Description du défaut de compacité de l'injection de Sobolev*, ESAIM.COCV, Vol **3**, (1998) 213-233.

[5] J. Ginibre, G. Velo: *On a class of nonlinear Schrdinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. **32** (1979), no. 1, 1–32.

[6] S. Keraani: *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*, J. Differential Equations **175** (2001), no. 2, 353–392.

[7] M.K. Kwong: *uniqueness of positive solutions to* $\Delta u - u + u^p = 0$*, in* $\mathbb{R}^n$, Arch. Rat. Mech. Anal **105** (1989), n. 3, 243-266.

[8] P-L. LIONS: *The concentration-compactness principle in the calculus of variations. The compact case. Part 1*, Ann. Inst. Henri Poincaré, Analyse non linéaire **1** (1984), 109-145.

[9] F. Merle: *Determination of blowup solutions with minimal mass for nonlinear Schrödinger equations with critical power*, Duke Math. J. **69**, (1993),no. 2, 203-254.

[10] ———: *Construction of solutions with exactly k blowup points for nonlinear Schrödinger equations with critical nonlinearity*, Comm.Math. Phys. **129** (1990), no.2, 223-240.

[11] ———: *Blow-up phenomena for critical nonlinear Schrödinger and Zakharov equations.* Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998). Doc. Math. 1998, Extra Vol. III, 57–66

[12] F. Merle, Y. Tsutsumi: L^2 *concentration of blowup solutions for the nonlinear Schrdinger equation with critical power nonlinearity*, J. Differential Equations **84** (1990), no. 2, 205–214.

[13] C. Sulem, P-L. Sulem: *The nonlinear Schrdinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.

[14] Y. Tsutsumi: *Rate of* L^2 *concentration of blowup solutions for the nonlinear Schrdinger equation with critical power*, Nonlinear Anal. **15** (1990), no. 8, 719–724.

[15] M. I. Weinstein: *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Physi. **87** (1983), 567-567.

[16] ———: *On the structure of singularities in solutions to the nonlinear dispersive evolution equations* , Comm. Partial Differential Equations, **11** (1984), 545-565.

[17] ———: *The nonlinear Schrdinger equation—singularity formation, stability and dispersion. The connection between infinite-dimensional and finite-dimensional dynamical systems*, Contemp. Math., 99, Amer. Math. Soc., Providence, RI, 1989, 213–232.

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France

E-mail address: `thmidi@univ-renne1.fr`

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France

E-mail address: `sahbi.keraani@univ-rennes1.fr`