REMARKS ON THE BLOWUP FOR THE L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

TAOUFIK HMIDI AND SAHBI KERAANI

Abstract. This note is dedicated to the formation of singularities of the solutions of L^2-critical nonlinear Schrödinger equation. We prove a refined version of compactness lemma adapted to the blowup analysis and we use it to improve the recent result by Colliander, Raynor, Sulem and Wright [5] on the concentration in the L^2-critical nonlinear Schrödinger equation below H^1.

1. Introduction

We consider fdfdf the L^2-critical nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplace operator on $\mathbb{R}^d$, $u : \mathbb{R}_t \times \mathbb{R}_x^d \longrightarrow \mathbb{C}$. It is well-known (see [4] for instance) that Cauchy problem (1.1) is locally well-posed in H^s for every $s \geq 0$. The unique solution has the following conservation law

$$\int_{\mathbb{R}^d} |u(t,x)|^2 dx = \int_{\mathbb{R}^d} |u_0(x)|^2 dx. \tag{1.2}$$

Also, if $s \geq 1$, the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \frac{d}{4+2d} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{4}{d}+2} dx$$

is conserved as t varies. For $s > 0$ the equation (1.1) is subcritical: the lifespan of the solution depends only on the H^s norm of the data. This yields the following blowup alternative: either $T^* = \infty$ or $T^* < +\infty$ and

$$\lim_{t \uparrow T^*} \|u(t,\cdot)\|_{H^s} = +\infty.$$

The space L^2 and the equation have the same scaling. More precisely, if u solves (1.1), then for every $\lambda > 0$, so does $u_\lambda(x,t) = \lambda^{d/2} u(\lambda^2 t, \lambda x)$ with data $u_\lambda(0,x) = \lambda^{d/2} u_0(\lambda x)$. But $\|u_\lambda(0,\cdot)\|_{L^2} = \|u_0\|_{L^2}$ and from this point of view (1.1) is L^2-critical. In this case the situation is more subtle and the time of existence depends on shape of of the data.

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The local theory relies heavily on some integrability properties of the solution of the associated linear Schrödinger equation

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u = 0, \\ u(0,x) = u_0, \end{array} \right. \tag{1.3}$$

called Strichartz estimates. In fact, by using Fourier analysis, in connections with the work by Tomas [19], as in [18] or an abstract operators theory as in [8], it was proved that $e^{it\Delta}u_0$, solution of (1.3), satisfies

$$\|e^{it\Delta}u_0\|_{L^{\frac{4}{d}+2}(\mathbb{R}^{d+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

The local solution follows from solving the equivalent integral equation

$$u(t,x) = e^{it\Delta}u_0(x) + i\int_0^t e^{i(t-s)\Delta}|u|^{\frac{4}{d}}u(s,x)ds,$$

by a standard Picard iteration method. The small data theory asserts that there exists a $\delta > 0$ (related to the constant C in (1.4)) such that if

$$\|u_0\|_{L^2(\mathbb{R}^d)} < \delta,$$

the initial values problem (1.1) has unique global solution. This follows by solving the Cauchy problem (1.1) directly in the whole space (the first step of the iteration method suffices to reach $T^* = \infty$). However, for a large data *blowup* may occur. The blowup or "wave collapse" corresponds to self-trapping of beams in laser propagation. A lot of theoretical and numerical works are dedicated to this subject when the initial data belongs to H^1. In fact, in this space energy arguments apply and a blowup theory has been developed in the two last decays (see [4], [17], [14] and the references therein). This theory is mainly connected to the notion of ground state: the unique positive radial solution of the elliptic problem

$$\Delta Q - Q + |Q|^{\frac{4}{d}}Q = 0.$$

In [21], M. I. Weinstein exhibited the following refined Galiardo-Nirenberg inequality

$$\|\psi\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d\|\psi\|_{L^2}^2\|\nabla\psi\|_{L^2}^2, \qquad \forall\psi \in H^1, \tag{1.5}$$

with $C_d = \frac{d+2}{d}\frac{1}{\|Q\|_{L^2}^2}$. Combined with the conservation of energy, this implies that $\|Q\|_{L^2}$ is the critical mass for the formation of singularities: for every $u_0 \in H^1$ such that

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

the solution of (1.1) with initial data u_0 is global. Also, this bound is optimal. In fact, by using the conformal invariance, one constructs

$$u(t,x) = |T-t|^{-1}e^{[(-i/(T-t))+(i|x|^2/T-t)]}Q(\frac{x}{T-t})$$

a blowing up solution of (1.1) with $\|u\|_{L^2} = \|Q\|_{L^2}$. F. Merle [12] has proved that, up the invariants of (1.1), this is the only blowing up solution with minimal mass. It is also proved (see [15] and [20]) that at the blowup there is a concentration phenomenon in L^2 norm: there exits a continuous functions $x(t)$ such that

$$\forall R > 0, \quad \liminf_{t \to T^*} \int_{|x - x(t)| \le R} |u(t,x)|^2 dx \ge \int Q^2. \tag{1.6}$$

For the case $u_0 \in H^s$, with $0 \le s < 1$, the classical energy arguments don't work. However, the general consensus is that for every $0 \le s < 1$ there is a concentration phenomenon in L^2 which occurs at the blowup time, at least greater than the minimal mass $\|Q\|_{L^2}$. Recently, Colliander, Raynor, Sulem and Wright [5] have confirmed this conjecture in the radial case for $s > s_Q$[1]. Their proof is based on the so called I-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao [6].
In this note we prove a compactness lemma adapted to the analysis of the blowup phenomenon of the nonlinear Schroödinger equation. The main tools of the proof of this result is an argument of profile decomposition by P.Gérad [7] and the sharp Galiardo-Nirenberg inequalities (1.5). As an application we improve the results of [5]: we remove the assumption of radial symmetry of the initial data and we prove that Q is a profile for the singular solutions with minimal mass.
We prove the following result.

Theorem 1.1. *Let $\{v_n\}_{n=1}^{\infty}$ be a bounded family of $H^1(\mathbb{R}^d)$, such that*

$$\liminf_{n \to \infty} \|\nabla v_n\|_{L^2} \le M$$

and

$$\limsup_{n \to \infty} \|v_n\|_{L^{\frac{4}{d}+2}} \ge m.$$

Then, there exits $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ such that, up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V$$

with $\|V\|_{L^2} \ge \sqrt{\frac{d}{d+2}} \frac{m^{\frac{2}{d}+1}}{M} \|Q\|_{L^2}$.

Remark 1.2. *The lower-bound on the L^2 norm of V is optimal. In fact, if we take $v_n = Q$ then we get equality.*

Remark 1.3. *In this theorem we can interchange the roles of $\|\nabla v_n\|_{L^2}$ and $\|v_n\|_{L^2}$. More precisely, if we assume $\liminf \|v_n\|_{L^2} \le M$, we get $\|\nabla V\|_{L^2} \ge \sqrt{\frac{d}{d+2}} \frac{m^{\frac{2}{d}+1}}{M} \|Q\|_{L^2}$.*

Remark 1.4. *In the H^1 context this theorem allows us to obtain easily the results on the concentration and uniqueness of the profile of concentration yet proved by M.I. Weinstein [22] using concentration-compactness lemma by P-L. Lions [11]. To see this take $u_0 \in H^1$ such that the corresponding*

[1]Some non optimal index $\le \frac{1+\sqrt{11}}{5}$.

solution u of (1.1) *blows up in finite time $T^* > 0$ and $t_n \uparrow T^*$ as $n \to \infty$. We set*

$$v_n(x) = \lambda_n^{d/2} u(t_n, \lambda_n x), \qquad \lambda_n = 1/\|\nabla u(t_n, .)\|_{L^2}.$$

Using conservation of the energy, we get trivially that $\{v^n\}_{n=1}^\infty$ satisfies the assumptions of Theorem 1.1 with $M = 1$ and $m = (\frac{d+2}{d})^{\frac{d}{2d+4}}$, which implies that

$$\lambda_n^{d/2} u(t_n, \lambda_n(\cdot - x_n)) \rightharpoonup V,$$

with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. This yields, in particular, the concentration estimate (1.6)*. If we assume, in addition, that $\|u_0\|_{L^2} = \|Q\|_{L^2}$ then the limit above becomes strong in H^1 and the variational characterization of the ground state Q implies the universality of the profile of the blowup solutions with minimal mass. It is worthy to note that these arguments are indeed standard and the novelty is just the use of Theorem 1.1 to ovoid to the discussion of concentration, vanishing and dichotomy cases of the concentration-compactness Lemma by Lions and then simplify the proof.*

As an application of Theorem 1.1 and the results of [5] we obtain

Theorem 1.5. *Assume $d = 2$ and $s > s_Q$. Let $u_0 \in H^s(\mathbb{R}^2)$ such that the corresponding solution u of* (1.1) *blows up in finite time $T^* > 0$. Then there exists a sequence $t_n \longrightarrow T^*$ such that the following holds true: there exists a function $V \in H^1$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$, and a sequence $\{\rho_n, x_n\}_{n=1}^\infty \subset \mathbb{R}_+^* \times \mathbb{R}^2$ satisfying*

$$\rho_n \leq A(T^* - t_n)^{s/2}$$

for some $A > 0$, such that

$$\rho_n u(t_n, \rho_n x + x_n) \rightharpoonup V \quad \text{weakly.}$$

Remark 1.6. *Note that despite the fact that $\{u(t_n, \cdot)\}_{n=1}^\infty$ belongs to H^s the blowup profile V is in H^1. This fact corroborates the expectation that $V = Q$.*

Theorem 1.5 and the variational characterization of the ground state allow to prove the following theorem:

Theorem 1.7. *Assume $d = 2$ and $s > s_Q$. Let $u_0 \in H^s(\mathbb{R}^2)$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ such that the solution u of* (1.1) *blows up in finite time $T^* > 0$. Then there exists a sequence $t_n \longrightarrow T^*$ satisfying*

$$\rho_n \leq A(T^* - t_n)^{s/2}$$

for some $A > 0$, a sequence $\{\theta_n, \rho_n, x_n\}_{n=1}^\infty \subset \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}^2$ such that

$$\rho_n e^{i\theta_n} u(t_n, \rho_n x + x_n) \longrightarrow Q$$

strongly in $H^{\tilde{s}-}$ where $\tilde{s} = \frac{s+1}{4-2s}$.

As an application one obtain

Corollary 1.8. *Under the assumptions of Theorem 1.5. Let $\lambda(t) > 0$, such that $\frac{(T-t)^{s/2}}{\lambda(t)} \longrightarrow 0$ as $t \to T^*$. There exits $x(t) \in \mathbb{R}^2$, such that*

$$\limsup_{t\to T^*} \int_{|x-x(t)|\leq\lambda(t)} |u(t,x)|^2 dx \geq \int Q^2.$$

Remark 1.9. *As already remarked in* [5]*, the fact that Theorem 1.5 and Theorem 1.7 hold for only a time sequence $\{t_n\}$ and the* lim sup *(instead of* lim inf*) in Corollary 1.8 are due to the lack of informations on the monotonicity of the H^s-norm of the blowing up solutions near the collapse time.*

The rest of this note is structured as follows. In section 2 we prove Theorem 1.1. Section 3 is devoted to the proofs of blowup results.

2. Proof of Theorem 1.1

In the sequel we put $2^* = \infty$ if $d = 1, 2$, and $2^* = \frac{2d}{d-2}$ if $d \geq 3$. Theorem 1.1 is a consequence of a profile decomposition of the bounded sequences in H^1 following the work by P. Gérard [7] (see also [1],[9]). In [7] it has been proved that the defect of compactness of the Sobolv embedding $\dot{\mathbf{H}}^s \hookrightarrow L^{\dot{p}}$, with $\dot{p} = \frac{2d}{d-2s}$, is generated by invariance by translation and scaling. More precisely, every bounded sequence of $\dot{\mathbf{H}}^s(\mathbb{R}^d)$ can be written, up to a subsequence, as an almost orthogonal sum of sequences of the type $h_n^{-d/\dot{p}} V(\frac{x-x_n}{h_n})$ with a small remainder term in $L^{\dot{p}}$ norm. In our case we deal with the inhomogeneous embedding $H^1 \hookrightarrow L^p$ ($p < 2^*$) for which the defect of compactness is only due to the invariance by translation. Thus, we have the following

Proposition 2.1. *Let $\{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then there exist a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted $\{v_n\}_{n=1}^\infty$), a family $\{\mathbf{x}^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^d$ and a sequence $\{V^j\}_{j=1}^\infty$ of H^1 functions, such that*

i) *for every $k \neq j$, $|x_n^k - x_n^j| \underset{n\to\infty}{\longrightarrow} +\infty$;*

ii) *for every $\ell \geq 1$ and every $x \in \mathbb{R}^d$, we have*

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x),$$

with

$$\limsup_{n\to\infty} \|v_n^\ell\|_{L^p(\mathbb{R}^d)} \underset{\ell\to\infty}{\longrightarrow} 0 \tag{2.1}$$

for every $2 < p < 2^*$.

Moreover, we have

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o(1), \tag{2.2}$$

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^\ell\|_{L^2}^2 + o(1), \tag{2.3}$$

as $n \to \infty$.

Proof. Let $\mathbf{v} = \{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Let $\mathcal{V}(\mathbf{v})$ be the set of functions obtained as weak limits in H^1 of subsequences of the translated $v_n(.+x_n)$ with $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$. We denote

$$\eta(\mathbf{v}) = \sup\{\|V\|_{H^1}, \quad V \in \mathcal{V}(\mathbf{v})\}.$$

Clearly

$$\eta(\mathbf{v}) \le \limsup_{n\to\infty} \|v_n\|_{H^1}.$$

We shall prove the existence of a sequence $\{V^j\}_{j=1}^\infty$ of $\mathcal{V}(\mathbf{v})$ and a family $\{\mathbf{x}^j\}_{j=1}^\infty$ of sequences of $\mathbb{R}^d$, such that

$$k \neq j \Longrightarrow |x_n^k - x_n^j| \underset{n\to\infty}{\longrightarrow} \infty,$$

and, up to extracting a subsequence, the sequence $\{v_n\}_{n=1}^\infty$ can be written as

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x), \quad \eta(\mathbf{v}^\ell) \underset{\ell\to\infty}{\longrightarrow} 0,$$

such that the identities (2.2)-(2.3) hold. Indeed, if $\eta(\mathbf{v}) = 0$, we can take $V^j \equiv 0$ for all j, otherwise we choose $V^1 \in \mathcal{V}(\mathbf{v})$, such that

$$\|V^1\|_{H^1} \ge \frac{1}{2}\eta(\mathbf{v}) > 0.$$

By definition, there exists some sequence $\{x_n^1\}_{n=1}^\infty$ of $\mathbb{R}^d$, such that, up to extracting a subsequence, we have

$$v_n(\cdot + x_n^1) \rightharpoonup V^1 \qquad \text{in} \quad H^1.$$

We set

$$v_n^1 = v_n - V^1(\cdot - x_n^1).$$

Since $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ weekly in H^1, we get

$$\begin{aligned} \|v_n\|_{L^2}^2 &= \|V^1\|_{L^2}^2 + \|v_n^1\|_{L^2}^2 + o(1), \\ \|\nabla v_n\|_{L^2}^2 &= \|\nabla V^1\|_{L^2}^2 + \|\nabla v_n^1\|_{L^2}^2 + o(1), \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Now, we replace $\mathbf{v}$ by $\mathbf{v}^1$ and repeat the same process. If $\eta(\mathbf{v}^1) > 0$ we get V^2, $\{x_n^2\}_{n=1}^\infty$ and $\mathbf{v}^2$. Moreover, we have

$$|x_n^1 - x_n^2| \longrightarrow \infty, \qquad \text{as} \quad n \to \infty$$

Otherwise, up to extracting of subsequence, we get

$$x_n^1 - x_n^2 \longrightarrow x_0$$

for some $x_0 \in \mathbb{R}^d$. Since

$$v_n^1(\cdot + x_n^2) = v_n^1(\cdot + (x_n^2 - x_n^1) + x_n^1)$$

and $v_n^1(\cdot + x_n^1)$ converge weakly to 0, then $V^2 = 0$. Thus $\eta(\mathbf{v}^1) = 0$, a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family $\{\mathbf{x}^j\}_{j=1}^{\infty}$ and $\{V^j\}_{j=1}^{\infty}$ satisfying the claims above. Furthermore, the convergence of the series $\sum_{j=1}^{\infty} \|V^j\|_{H^1}^2$ implies that

$$\|V^j\|_{H^1} \underset{j\to\infty}{\longrightarrow} 0.$$

However, by construction, we have

$$\eta(\mathbf{v}^j) \le \|V^{j-1}\|_{H^1},$$

which proves that $\eta(\mathbf{v}^j) \to 0$ as claimed. To complete the proof of Proposition 2.1, (2.1) remains to be proved. For that purpose let us introduce $\chi_R \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \le \hat{\chi}_R \le 1$ and

$$\hat{\chi}_R(\xi) = 1 \quad \text{if} \quad |\xi| \le R, \quad \hat{\chi}_R(\xi) = 0 \quad \text{if} \quad |\xi| \ge 2R.$$

Here $\hat{}$ denotes the Fourier transform. One has

$$v_n^\ell = \chi_R * v_n^\ell + (\delta - \chi_R) * v_n^\ell,$$

where $*$ stands for the convolution. Let $p \in]2, 2^*[$ to be fixed. On the one hand, in view of Sobolev embedding, we get

$$\|(\delta - \chi_R) * v_n^\ell\|_{L^p} \lesssim \|(\delta - \chi_R) * v_n^\ell\|_{\dot{H}^\beta} \lesssim R^{\beta-1}\|v_n^\ell\|_{H^1},$$

for $\beta = d(\frac{1}{2} - \frac{1}{p}) < 1$. On the other hand, one can estimate

$$\begin{aligned} \|\chi_R * v_n^\ell\|_{L^p} &\lesssim \|\chi_R * v_n^\ell\|_{L^2}^{2/p} \|\chi_R * v_n^\ell\|_{L^\infty}^{1-2/p} \\ &\lesssim \|v_n^\ell\|_{L^2}^{2/p} \|\chi_R * v_n^\ell\|_{L^\infty}^{1-2/p}. \end{aligned}$$

Now, observe that

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} = \sup_{\{x_n\}} \limsup_{n\to+\infty} |\chi_R * v_n^\ell(x_n)|.$$

Thus, in view of the definition of $\mathcal{V}(\mathbf{v}^\ell)$, we infer

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \le \sup\{|\int_{\mathbb{R}^d} \chi_R(-x)V(x)dx|, \quad V \in \mathcal{V}(\mathbf{v}^\ell)\}.$$

Therefore, by Hölder's inequality, it follows that

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \le C_2 \sup\{\|V\|_{L^2(\mathbb{R}^d)}, \quad V \in \mathcal{V}(\mathbf{v}^\ell)\}.$$

Here, $C_2(R) = \|\chi_R\|_{L^2(\mathbb{R}^d)}$. Thus, we obtains

$$\limsup_{n\to\infty} \|\chi_R * v_n^\ell\|_{L^\infty(\mathbb{R}^d)} \le C_2 \eta(\mathbf{v}^\ell)$$

for every $\ell \geq 1$. Finally, we get

$$\|v_n^\ell\|_{L^p(\mathbb{R}^d)} \precsim R^{\beta-1}\|v_n^\ell\|_{H^1} + C(R)\|v_n^\ell\|_{L^2}^{2/p}\eta(\mathbf{v}^\ell)^{1-2/p}.$$

Now, we let ℓ go to infinity , then R go to infinity and since $\eta(\mathbf{v}^\ell) \underset{\ell\to\infty}{\longrightarrow} 0$ and the family of sequences $\{v_n^\ell\}$ are uniformly bounded in $H^1(\mathbb{R}^d)$, we obtain

$$\limsup_{n\to\infty} \|v_n^\ell\|_{L^p} \underset{\ell\to\infty}{\longrightarrow} 0$$

as claimed. This closes the proof of the proposition 2.1. □

Let us return to the proof of Theorem 1.1. According to Proposition 2.1, the sequence $\{v_n\}_{n=1}^\infty$ can be written, up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x)$$

such that (2.1) and (2.2) hold. This implies, in particular,

$$m^{\frac{4}{d}+2} \leq \limsup \|\sum_{j=1}^{\infty} V^j(\cdot - x_n^j)\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

The elementary inequality

$$||\sum_{j=1}^{l} a_j|^{4/d+2} - \sum_{j=1}^{l} |a_j|^{4/d+2}| \leq C\sum_{j\neq k} |a_j||a_k|^{4/d+1}.$$

and the pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^\infty$ leads the mixed terms in the sum above to vanish and we get

$$m^{\frac{4}{d}+2} \leq \sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

On the one hand, in view of Galiardo-Nirenberg inequality (1.5), we have

$$\sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d \sup\{\|V^j\|_{L^2}^2, j \geq 1\} \sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2.$$

On the other hand, from (2.1), we get

$$\sum_{j=1}^{\infty} \|\nabla V^j\|_{L^2}^2 \leq \limsup_{n\to\infty} \|\nabla v_n\|_{L^2}^2 \leq M^2.$$

Therefore,

$$\sup_{j\geq 1} \|V^j\|_{L^2}^2 \geq \frac{m^{\frac{4}{d}+2}}{M^2 C_d}.$$

Since the series $\sum \|V^j\|_{L^2}^2$ converges then the supremum above is attained. In particular, there exists j_0, such that

$$\|V^{j_0}\|_{L^2} \geq \frac{m^{\frac{2}{d}+1}}{\sqrt{C_d}M}.$$

On the other hand, a change of variables gives

$$v_n(x+x_n^{j_0}) = V^{j_0}(x) + \sum_{\substack{1 \leq j \leq \ell \\ j \neq j_0}} V^j(x+x_n^{j_0}-x_n^j) + \tilde{v}_n^\ell(x),$$

where $v_n^\ell(x) = \tilde{v}_n^\ell(x+x_n^{j_0})$. The pairwise orthogonality of the family $\{\mathbf{x}^j\}_{j=1}^\infty$ implies

$$V^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0 \quad \text{weakly}$$

for every $j \neq j_0$. Hence, we get

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{v}^\ell,$$

where $\tilde{v}^\ell$ denote the weak limit of $\{\tilde{v}_n^\ell\}_{n=1}^\infty$. However, we have

$$\|\tilde{v}^\ell\|_{L^{\frac{4}{d}+2}} \leq \limsup_{n\to\infty} \|\tilde{v}_n^\ell\|_{L^{\frac{4}{d}+2}} = \limsup_{n\to\infty} \|v_n^\ell\|_{L^{\frac{4}{d}+2}} \underset{l\to\infty}{\longrightarrow} 0.$$

Thereby, by uniqueness of weak limit, we get

$$\tilde{v}^\ell = 0$$

for every $\ell \geq j_0$. So that

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0}.$$

The sequence $\{x_n^{j_0}\}_{n=1}^\infty$ and the function V^{j_0} fulfill the conditions of Theorem 1.1. □

3. Proof of the main results

3.1. **The modified energy.** Here we recall the result of almost conservation of the modified energy proved in [5].
I_N stands for the smoothing operators[2] $I_N : H^s \to H^1$:

$$\widehat{I_N u}(\xi) = m(\xi)\hat{u}(\xi)$$

where

$$m(\xi) = \begin{cases} 1, & |\xi| \leq N \\ (\frac{|\xi|}{N})^{s-1}, & |\xi| > 3N \end{cases}$$

with $m(\xi)$ smooth, radial and monotone in $|\xi|$. The following properties of I_N are easily verified

$$\begin{aligned} \|I_N u\|_{L^2} &\leq \|u\|_{L^2} \\ \|u\|_{H^s} \leq &\|I_N u\|_{H^1} \leq N^{1-s}\|u\|_{H^s}. \end{aligned} \tag{3.1}$$

[2]See [6] for more properties of this operator.

The *blowup parameter* associated to the H^s-norm of the solution is

$$\Lambda(t) = \sup_{0\le\tau\le t} \|u(t)\|_{H^s}.$$

The following proposition is a restatement of the part of [5] which is relevant for us.

Proposition 3.1 ([5])**.** *There exists $s_Q \le \frac{1}{5}+\frac{1}{5}\sqrt{11}$ such that for all $s > s_Q$ there exists $p(s) < 2$ with the following holding true: If $H^s \ni u_0 \longmapsto u(t)$ solves (1.1) on a maximal (forward) finite existence interval $[0, T^*)$ then for all $T < T^*$ there exists $N = N(T)$ such that*

$$|E[I_{N(T)}u(T)]| \le C_0(\Lambda(T))^{p(s)} \tag{3.2}$$

with $C_0 = C_0(s, T^*, \|u_0\|_{H^s})$. Moreover, $N(T) = C(\Lambda(T))^{\frac{p(s)}{2(1-s)}}$.

In [5] $p(s)$ is explicitly given by

$$p(s) = \frac{6^+}{2^- - 4^+(1-s)}2(1-s), \tag{3.3}$$

where $\alpha^{\pm} = \alpha \pm 0$.

3.2. **Proof of Theorem 1.5.** As in [5], we choose $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $t_n \uparrow T^*$ and for each t_n

$$\|u(t_n)\|_{H^s} = \Lambda(t_n).$$

We set

$$\psi_n = \rho_n I_N u(t_n, \rho_n x),$$

where

$$\rho_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla I_N u(t_n,\cdot)\|_{L^2}}.$$

The estimate (3.1) yields

$$\rho_n \le \frac{1}{\|u(t_n,\cdot)\|_{H^s}} = \frac{1}{\Lambda(t_n)}.$$

Also, from Corollary 3.6 in [5], it holds that

$$\rho_n \le A(T^* - t_n)^{s/2}$$

for some constant $A > 0$. The sequence $\{\psi_n\}_{n=1}^{\infty}$ satisfies:

$$\|\psi_n\|_{L^2} \le \|u_0\|_{L^2}, \quad \|\nabla\psi_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

Furthermore, in view of Proposition 3.1,

$$E(\psi_n) = \rho_n^2 E[I_{N(t_n)}u(t_n)] \le \rho_n^2(\Lambda(t_n))^{p(s)} \le (\Lambda(t_n))^{p(s)-2}.$$

Since $\|u(t_n)\|_{H^s} \longrightarrow +\infty$ and $p(s) < 2$, it holds that

$$E(\psi_n) \longrightarrow 0, \qquad \text{as} \quad n \to \infty$$

which yields, in particular,

$$\|\psi_n\|_{\mathrm{L}^4}^4 \longrightarrow 2\|\nabla Q\|_{L^2}^2, \qquad \text{as} \quad n \to \infty.$$

The family $\{\psi_n\}_{n=1}^{\infty}$ satisfies the conditions of the lemma above with

$$m = (2\|\nabla Q\|_{L^2}^2)^{1/4} \qquad \text{and} \qquad M = \|\nabla Q\|_{L^2}.$$

Thus, there exists exits $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that, up to a subsequence,

$$\psi_n(\cdot + x_n) \rightharpoonup V$$

with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. Coming back to $\{\psi_n\}_{n=1}^{\infty}$, one obtains

$$\rho_n I_N u(t_n, \rho_n x + x_n) = V + \varepsilon_n,$$

where $\varepsilon_n \rightharpoonup 0$ in H^1.
However, for every $\bar{s} < s$, one has

$$\begin{aligned} \|\rho_n(I_N u(t_n) - u(t_n))(\rho_n \cdot + x_n)\|_{\dot{H}^{\bar{s}}(\mathbb{R}^2)} &\leq \rho_n^{\bar{s}} N^{\bar{s}-s} \|u(t_n, \cdot)\|_{H^s(\mathbb{R}^2)} \\ &\leq (\Lambda(t_n))^{\frac{p(s)(\bar{s}-s)}{2(1-s)} + 1 - \bar{s}}. \end{aligned}$$

Using the explicit formula of $p(s)$ an easy calculus yields that

$$\frac{p(s)(\bar{s}-s)}{2(1-s)} + 1 - \bar{s} < 0 \Longleftrightarrow \bar{s} < \tilde{s} := \frac{s+1}{4-2s}.$$

Under this choice, we get

$$\|\rho_n(I_N u(t_n) - u(t_n))(\rho_n \cdot + x_n)\|_{H^{\tilde{s}-}(\mathbb{R}^2)} \longrightarrow 0, \qquad \text{as} \quad n \to \infty.$$

Thus

$$\rho_n u(t_n, \rho_n x + x_n) = V + h_n \tag{3.4}$$

where $h_n \rightharpoonup 0$ in $H^{\tilde{s}-}$. This concludes the proof of Theorem 1.5.

3.3. **Proof of Theorem 1.7.** In the context of the proof of Theorem 1.5. Since $\|\psi_n\|_{L^2} \leq \|u_0\|_{L^2} = \|Q\|_{L^2}$, then

$$\|V\|_{L^2} = \|Q\|_{L^2}.$$

Thus, $\psi_n(\cdot - x_n) \to V$ in L^2. Also, $\psi_n(\cdot - x_n) \to V$ in L^4. This yields, in view, of Galiardo-Nirenberg inequality

$$\|\nabla V\|_{L^2} \geq \|\nabla Q\|_{L^2}.$$

Since $\|\nabla V\|_{L^2} \leq \limsup \|\nabla \psi_n\|_{L^2} = \|\nabla Q\|_{L^2}$, then

$$\|\nabla \psi_n\|_{L^2} \longrightarrow \|\nabla V\|_{L^2}.$$

This means that the strong convergence holds in H^1. This fact implies, in particular,

$$E(V) = 0.$$

Let us summarize the properties of the limit V:

$$V \in H^1, \quad \|V\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla V\|_{L^2} = \|\nabla Q\|_{L^2} \quad \text{and} \quad E(V) = 0.$$

The variational characterization of the ground state implies that

$$V(x) = e^{i\theta} Q(x + x_0),$$

for some $\theta \in [0, 2\pi[$ and $x_0 \in \mathbb{R}^2$. Coming back to (3.4), one obtains

$$\rho_n u(t_n, \rho_n x + x_n) = \lambda e^{i\theta} Q(\lambda x + x_0) + \varepsilon_n,$$

where $\varepsilon_n \to 0$ in $H^{\tilde{s}-}$. This ends the proof of Theorem 1.7.

3.4. **Proof of Corollary 1.8.** Let u to be a blowup solution of (1.1) at finite time $T^* > 0$. According to Theorem 1.5, there exist a time sequence such that $t_n \longrightarrow T^*$, a profile $V \in L^2(\mathbb{R}^2)$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$ and a sequence $\{\rho_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \mathbb{R}^2$ such that,

$$\rho_n u(t_n, \rho_n x + x_n) \rightharpoonup V \tag{3.5}$$

and

$$\lim_{n\to\infty} \frac{\rho_n}{(T^* - t_n)^{s/2}} \leq A \tag{3.6}$$

for some $A \geq 0$. From (3.5), it follows that

$$\lim_{n\to\infty} \frac{1}{(\rho_n)^2} \int_{|x|\leq R} |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x|\leq R} |V|^2 dx,$$

for every $R > 0$. Thus,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{|x-y|\leq R\rho_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |V|^2 dx. \tag{3.7}$$

Since $\frac{(T^*-t)^{s/2}}{\lambda(t)} \to 0$ as $t \to T^*$, it follows from (3.6) that $\frac{\rho_n}{\lambda(t_n)} \to 0$ and then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{|x-y|\leq\lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |V|^2 dx$$

for every R. We let R goes to infinity to obtain

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{\{|x-y|\leq\lambda(t_n)\}} |u(t_n, x)|^2 dx \geq \int_{\mathbb{R}^2} |V|^2 dx \geq \|Q\|_{L^2}^2.$$

This yields finally

$$\limsup_{t\uparrow T^*} \sup_{y\in\mathbb{R}^2} \int_{\{|x-y|\leq\lambda(t)\}} |u(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$

Since, for every t, the function $y \longmapsto \int_{\{|x-y|\leq\lambda(t)\}} |u(t,x)|^2 dx$ is continuous and goes to 0 at infinity, then there exits a family $x(t)$ such that

$$\sup_{y\in\mathbb{R}^2} \int_{\{|x-y|\leq\lambda(t)\}} |u(t, x)|^2 dx = \int_{\{|x-x(t)|\leq\lambda(t)\}} |u(t, x)|^2 dx,$$

which concludes the proof of Corollary 1.8. ■

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IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France
E-mail address: thmidi@univ-renne1.fr

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France
E-mail address: sahbi.keraani@univ-rennes1.fr