INVISCID LIMIT FOR AXISYMMETRIC NAVIER-STOKES SYSTEM

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Abstract. We are interested in the global well posedness of the axisymmetric Navier-Stokes system with initial data belonging to the critical Besov spaces $B^{1+\frac{3}{p}}_{p,1}$. We obtain uniform estimates of the viscous solutions (v_ν) with respect to the viscosity in the spirit of the work [2] concerning the axisymmetric Euler equations. We provide also a strong convergence result in the L^p norm of the viscous solutions (v_ν) to the Eulerian one v.

1. Introduction

In this paper we deal with the incompressible Navier-Stokes system described by

$$(\mathrm{NS}_\nu)\qquad \begin{cases} \partial_t v_\nu + v_\nu\cdot\nabla v_\nu - \nu\Delta v_\nu = -\nabla p_\nu\\ \mathrm{div}\ v_\nu = 0\\ v_{\nu|t=0} = v_0. \end{cases}$$

This system models the flow of a homogeneous incompressible viscous fluid of viscosity $\nu > 0$. The velocity v_ν is a three-dimensional vector field, the pressure $p_\nu = p_\nu(t,x)$ is a real scalar. The condition div $v_\nu = 0$ means that the fluid is incompressible.

The mathematical theory of (NS_ν) was started by J. Leray in his pioneering work [13]. He proved the global existence of weak solutions in energy space by using a compactness method. Nevertheless, the uniqueness of weak solutions is only known in space dimension two. According to the work of H. Fujita and T. Kato [6], we can prove local well posedness for initial data lying in the critical Sobolev space $\dot H^{\frac12}$. Similar results are established in various functional spaces like L^3, $\dot B^{-1+\frac{3}{p}}_{p,\infty}$ and BMO^{-1}. We refer to [11] for

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more details about the subject. The global existence of these solutions for arbitrary initial data is an outstanding open problem in PDEs.

When the viscosity $\nu = 0$, the Navier-Stokes system is reduced to the Euler system (E) which has a local theory in a satisfactory state. We will restrict ourselves to some significant results: in [9] Kato proved local well posedness for initial data in H^s, with $s > \frac{5}{2}$. We can even obtain the same result in critical Besov spaces $B_{p,1}^{1+\frac{3}{p}}$, see for example D. Chae [4].

There is an interesting case of special initial data leading to global existence for both Navier-Stokes and Euler systems. These data are called axisymmetric, which means that they have, in cylindrical coordinates $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$, the following structure: $v(t,x) = v_r(t,r,z)\vec{e}_r + v_z(t,r,z)\vec{e}_z$. For these flows the vorticity ω takes the form $\omega = (\partial_z v^r - \partial_r v^z)\vec{e}_\theta$ and satisfies the equation

$$\partial_t\omega - \nu\Delta\omega + (v\cdot\nabla)\omega = \frac{v_r}{r}\omega.$$

For the axisymmetric Navier-Stokes system, M. Ukhoviskii and V. Yudovich [19] proved global well posedness for initial data $v^0 \in H^1$ such that $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$, with uniform bounds on the viscosity. In [12], S. Leonardi, J. Màlek, J. Necǎs and M. Pokorný proved global well posedness for initial data $v^0 \in H^2$. This result was recently improved by H. Abidi [1] for $v^0 \in H^{\frac{1}{2}}$ and external axisymmetric forces $f \in L^2_{loc}\big(\mathbb{R}_+; H^\beta\big)$, with $\beta > \frac{1}{4}$.

In the case of axisymmetric Euler system, M. Ukhoviskii and V. Yudovich [19] proved global well posedness for initial data $v^0 \in H^s$, with $s > \frac{7}{2}$. This result was relaxed by Yanagisawa [17] for Kato's solutions, $v^0 \in H^s, s > \frac{5}{2}$. We point out that their proofs are based on B-K-M criterion. More recently, H. Abidi, T. Hmidi and S. Keraani [2] proved a similar result for critical Besov spaces $B_{p,1}^{1+\frac{3}{p}}$, with $1 \le p \le \infty$. To overcome the non-validity of B-K-M criterion they use the special geometric structure of the vorticity leading to a new decomposition of the vorticity. This allows them to bound the Lipschitz norm of the velocity.

In this paper we study the persistence of the Besov regularity $B_{p,1}^{1+\frac{3}{p}}$ for Navier-Stokes solutions uniformly with respect to the viscosity. The inviscid limit problem is also treated. We notice that this problem was studied by Majda for smooth initial data in all dimensions, see [14]. In space dimension two we refer to the papers of T. Hmidi and S. Keraani [7, 8] where they proved the uniform persistence in critical Besov spaces $B_{p,1}^{1+\frac{2}{p}}, p \in [1,\infty]$.

Here are the main results of this paper.

Theorem 1.1 (Uniform boundedness of the velocity)**.** *Let $p\in[1,+\infty]$ and v_0 be an axisymmetric divergence free vector-field. Assume that*

(A1) $v_0\in B_{p,1}^{1+\frac{3}{p}}$,
(A2) $\frac{\omega_0}{r}\in L^{3,1}$.

Then there exists a unique global solution $v_\nu\in\mathcal{C}\big(\mathbb{R}_+;B_{p,1}^{1+\frac{3}{p}}\big)$ to the Navier-Stokes system, such that

$$\|v_\nu(t)\|_{B_{p,1}^{1+\frac{3}{p}}}\le C_0e^{e^{\exp C_0t}},$$

where C_0 depends only on the initial data and not on the viscosity.

The proof relies on the uniform estimate of the Lipschitz norm of the velocity. For this purpose we use the method developed in [2] for the inviscid case. However, the situation in the viscous case is more complicated because of the dissipative term. We especially have to check that it doesn't undermine some geometric properties of the vorticity.

Remark 1.2. For $p\in[1,3[$ the second condition (A2) is a consequence of the first one (A1). More precisely, we have

$$\Big\|\frac{\omega}{r}\Big\|_{L^{3,1}}\le C\|v\|_{B_{p,1}^{1+\frac{3}{p}}}.$$

Our second main result deals with the inviscid limit.

Theorem 1.3 (Rate convergence)**.** *Let v_ν and v be respectively the solution of the Navier-Stokes and Euler systems with the same initial data $v^0\in B_{p,1}^{1+\frac{3}{p}}$. Then we have the rate of convergence*

$$\|v_\nu-v\|_{B^0_{\max(p,3),1}}\le C_0e^{\exp C_0t}(\nu t)^{\frac12+\frac{3}{2\max(p,3)}},\quad p\in[1,\infty].$$

We use for the proof the uniform bounds in Besov spaces combined with some smoothing effects on the viscous vorticity.

The paper is organized as follows: Section 2 is devoted to some basic tools: we introduce the functional framework needed for the proofs and we recall some useful lemmas. We discuss in Section 3 the persistence of some important geometric properties for a vorticity like equation. This part is essential for the proof of the main results. We give in Section 4 some a priori estimates and we prove a new decomposition of vorticity which allows us to prove the result of Theorem 1.1. The proof of the inviscid limit is given in Section 5. We end this paper with an appendix where we give the proof of a technical lemma.

2. Preliminaries

We recall in this section some functional spaces and tools frequently used in this paper. We begin with the usual Lebesgue space L^p defined as the set of $p-$ integrable functions, endowed with the following norm

$$\|v\|_{L^p} = \Big(\int_{\mathbb{R}^3} |v(x)|^p dx\Big)^{\frac{1}{p}}.$$

We recall now Lorentz spaces.

Definition 2.1. *Let $1 < p < \infty$ and $q \in [1, \infty]$. The Lorentz space $L^{p,q}$ can be defined by the real interpolation theory,*

$$L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p},q}.$$

The spaces $L^{p,q}$ have the following properties:

1) $L^{p,p} = L^p$,
2) $L^{p,q_0} \hookrightarrow L^{p,q_1}$ for all $1 \le q_0 \le q_1 \le \infty$,
3) $\|uv\|_{p,q} \le \|u\|_\infty \|v\|_{p,q}$.

Now, we give the Littlewood-Paley operators based on a dyadic partition of unity; for more details we refer the reader to J.-Y. Chemin [5].

Proposition 2.2. *There exist two radial functions $\chi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \backslash \{0\})$ such that*

(i) $\chi(\xi) + \sum_{q \ge 0} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3$,
(ii) $\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3 \backslash \{0\}$,
(iii) $|j - q| \ge 2 \Rightarrow supp\ \varphi(2^{-j}\cdot) \cap supp\ \varphi(2^{-q}\cdot) = \varnothing$*, and*
(iv) $q \ge 1 \Rightarrow supp\ \chi \cap supp\ \varphi(2^{-q}\cdot) = \varnothing$.

Definition 2.3. *For every v in $\mathcal{S}'$, we define the Littlewood-Paley operators by*

$$\Delta_{-1} v = \chi(D) v; \quad \forall q \in \mathbb{N} \quad \Delta_q v = \varphi(2^{-q} D) v, \quad S_q v = \sum_{-1 \le j \le q-1} \Delta_j v.$$

Consequently

$$\begin{aligned} \Delta_{-1} v(x) = g \star v(x) &= \int_{\mathbb{R}^3} g(y) v(x-y) dy \\ \forall q \in \mathbb{N}, \quad \Delta_q v(x) = 2^{3q} h(2^q .) \star v(x) &= 2^{3q} \int_{\mathbb{R}^3} h(2^q y) v(x-y) dy, \end{aligned}$$

where $\hat{g} \equiv \chi$ and $\hat{h} \equiv \varphi$. Let us notice that the operators Δ_q and S_q map continuously L^p into itself uniformly on q and p. The homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j \leq q-1} \dot{\Delta}_j v.$$

Now, we will recall the definition of the Besov spaces.

Definition 2.4. *Let $s \in \mathbb{R}, p, r \in [1, \infty]$. The inhomogeneous Besov space $B^s_{p,r}$ (respectively the homogeneous Besov space $\dot{B}^s_{p,r}$) is the set of all tempered distributions $v \in \mathcal{S}'$ (respectively $v \in \mathcal{S}'_{|P}$) such that*

$$\|v\|_{B^s_{p,r}} \stackrel{def}{=} \big(2^{qs}\|\Delta_q v\|_{L^p}\big)_{\ell^r} < \infty.$$
$$\big(resp.\ \|v\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \big(2^{qs}\|\dot{\Delta}_q v\|_{L^p}\big)_{\ell^r(\mathbb{Z})} < \infty\big).$$

We have denoted by P the set of polynomials.

Let us recall the Bony decomposition [3]. For $u, v \in \mathcal{S}'$. The product of uv is formally defined by

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v \stackrel{def}{=} \sum_q S_{q-1}\Delta_q v, \quad R(u, v) \stackrel{def}{=} \sum_q \Delta_q u \widetilde{\Delta}_q v,$$

with $\widetilde{\Delta}_q = \sum_{i=1}^{-1} \Delta_{q+i}$. The terms $T_u v$ and $T_v u$ are called paraproducts and the third term $R(u, v)$ is the remainder. We need the following mixed spaces. Fix $T > 0$ and $a \geq 1$. We define the spaces $L^a_T B^s_{p,r}$ and $\widetilde{L^a_T} B^s_{p,r}$ as

$$L^a_T B^s_{p,r} = \Big\{v : [0, T] \to \mathcal{S}' : \|v\|_{L^a_T B^s_{p,r}} \stackrel{def}{=} \big\|\big(2^{qs}\|\Delta_q v\|_{L^p}\big)_{\ell^r}\big\|_{L^a_T} < \infty\Big\},$$
$$\widetilde{L^a_T} B^s_{p,r} = \Big\{v : [0, T] \to \mathcal{S}' : \|v\|_{\widetilde{L^a_T} B^s_{p,r}} \stackrel{def}{=} \big(2^{qs}\|\Delta_q v\|_{L^a_T L^p}\big)_{\ell^r} < \infty\Big\}.$$

We have the following embeddings:

$$\begin{cases} L^a_T B^s_{p,r} \hookrightarrow \widetilde{L^a_T} B^s_{p,r} & \text{if } a \leq r, \\ \widetilde{L^a_T} B^s_{p,r} \hookrightarrow L^a_T B^s_{p,r} & \text{if } a \geq r. \end{cases}$$

In addition, we have the interpolation result: let $T > 0$, $s_1 < s < s_2$ and $\kappa \in (0, 1)$ such that $s = \kappa s_1 + (1 - \kappa)s_2$. Then we have

$$\|v\|_{\widetilde{L^a_T} B^s_{p,r}} \leq C\|v\|^{\kappa}_{\widetilde{L^a_T} B^{s_1}_{p,\infty}} \|v\|^{1-\kappa}_{\widetilde{L^a_T} B^{s_2}_{p,\infty}}. \tag{2.1}$$

Next, we state the following proposition which deals with the persistence of Besov regularities in a transport-diffusion equation. For the proof, we refer for example to [2].

Proposition 2.5. *Let v be a smooth divergence free vector-field and f be a smooth solution of the transport-diffusion equation*

$$\begin{cases} \partial_t f - \nu\Delta f + v\cdot\nabla f = g \\ f_{|t=0} = f_0, \end{cases} \tag{TD$_\nu$}$$

where $f_0 \in B^s_{p,r}, g \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$ and $(s,r,p) \in (-1,1)\times[1,\infty]^2$. Then we have, for $t \geq 0$,

$$\|f(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau,$$

and C a constant which depends only on s and not on the viscosity. For the limit case

$$s=-1, r=\infty \text{ and } p\in[1,\infty] \quad \text{or} \quad s=1, r=1 \text{ and } p\in[1,\infty],$$

the above estimate remains true even if we change $V(t)$ by

$$Z(t) \overset{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}.$$

In addition, if $f = \operatorname{curl} v$, then the above estimate holds true for all $s \in [1,+\infty)$.

In the sequel, we denote by C a harmless constant whose value may vary from line to line. The notation $X \lesssim Y$ means that $X \leq CY$ for some constant C.

3. Study of a vorticity like equation

In this section, we study some geometrical properties of any solution satisfying a vorticity like equation, given by

$$\partial_t\Gamma - \nu\Delta\Gamma + (v\cdot\nabla)\Gamma = (\Gamma\cdot\nabla)v, \qquad \Gamma_{|t=0} = \Gamma^0, \tag{3.1}$$

where $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ is an unknown vector-valued function. Our main result in this section reads as follows.

Proposition 3.1. *Let v be an axisymmetric smooth vector field with zero divergence and Γ be the solution of* (3.1) *with smooth initial data Γ^0. Then we have the following properties:*

(i) *If* $\text{div}\,\Gamma_0 = 0$, *then* $\text{div}\,\Gamma(t) = 0$, *for all* $t \in \mathbb{R}_+$.
(ii) *If* $\Gamma^0 = \Gamma^0_\theta(r,z)\,\vec{e}_\theta$, *then* $\Gamma(t,x) = \Gamma_\theta(t,r,z)\,\vec{e}_\theta$, *for all* $t \in \mathbb{R}_+$.
(iii) *Under assumption (ii) we have* $\Gamma_1(t,x_1,0,z) = \Gamma_2(t,0,x_2,z)$ *and*

$$\partial_t\Gamma_\theta + (v\cdot\nabla)\Gamma_\theta - \nu\Big[\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big]\Gamma_\theta = \frac{v^r}{r}\Gamma_\theta.$$

Proof. (i) We apply the divergence operator to the equation (3.1) and, by easy computations, using the incompressibility of v, we get

$$\partial_t \text{div}\,\Gamma - \nu\Delta\text{div}\,\Gamma + (v\cdot\nabla)\text{div}\,\Gamma = 0, \qquad \text{div}\,\Gamma_{|t=0} = 0. \tag{3.2}$$

From the maximum principle, we obtain

$$\|\text{div}\Gamma\|_{L^\infty} \leq \|\text{div}\Gamma^0\|_{L^\infty}.$$

This gives the desired result.

(ii) Let $(\Gamma_r, \Gamma_\theta, \Gamma_z)$ denote the coordinates of Γ in a cylindrical basis. The result will be done in two steps: we show first that the cylindrical components of Γ do not depend on the angular parameter θ. We prove in the second step that the components Γ_r and Γ_z are zero.

To establish the first point it suffices to prove that (3.1) is stable under rotation transforms. For this purpose we will check that, for every $\alpha \in \mathbb{R}$, the quantity $\Gamma_\alpha(t,x) = \mathcal{R}_\alpha^{-1}\Gamma(t,\mathcal{R}_\alpha x)$ satisfies also (3.2). Here, $\mathcal{R}_\alpha$ is a rotation with angle α and axis (oz); i.e.,

$$\mathcal{R}_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is well known that the operator Δ commutes with rotations; i.e.,

$$\Delta(\Gamma(t,\mathcal{R}_\alpha x)) = (\Delta\Gamma)(t,\mathcal{R}_\alpha x).$$

This gives

$$\Delta\Gamma_\alpha(t,x) = \mathcal{R}_\alpha^{-1}(\Delta\Gamma)(t,\mathcal{R}_\alpha x). \tag{3.3}$$

For the advection term we write

$$(\mathcal{R}_\alpha^{-1}(v\cdot\nabla\Gamma))(t,\mathcal{R}_\alpha x) = (v\cdot\nabla\,\mathcal{R}_\alpha^{-1}\Gamma)(t,\mathcal{R}_\alpha x).$$

On the other hand we have

$$v(t,x)\cdot\nabla\,(\mathcal{R}_\alpha^{-1}\Gamma(t,\mathcal{R}_\alpha x)) = \Big(\mathcal{R}_\alpha v(t,\mathcal{R}_\alpha^{-1}x)\cdot\nabla\mathcal{R}_\alpha^{-1}\Gamma\Big)(t,\mathcal{R}_\alpha(x)).$$

As the velocity is axisymmetric it follows that

$$v(t,x)\cdot\nabla\,\Gamma_\alpha(t,x)=\big(v\cdot\nabla\mathcal{R}_\alpha^{-1}\Gamma\big)(t,\mathcal{R}_\alpha(x)).$$

Combining these estimates we find

$$(\mathcal{R}_\alpha^{-1}(v\cdot\nabla\Gamma))(t,\mathcal{R}_\alpha x)=v(t,x)\cdot\nabla\,\Gamma_\alpha(t,x). \tag{3.4}$$

For the stretching term we write by the same way as before

$$\begin{aligned}(\mathcal{R}_\alpha^{-1}(\Gamma\cdot\nabla v))(t,\mathcal{R}_\alpha x)&=(\Gamma\cdot\nabla\,\mathcal{R}_\alpha^{-1}v)(t,\mathcal{R}_\alpha x) \\ &=\big(\Gamma\cdot\nabla\,v(t,\mathcal{R}_\alpha^{-1}x\big)(t,\mathcal{R}_\alpha x)=\Gamma_\alpha\cdot\nabla\,v(t,x).\end{aligned} \tag{3.5}$$

Plugging together the equation (3.1) with the identities (3.3), (3.4) and (3.5) yields

$$\partial_t\Gamma_\alpha+v\cdot\nabla\,\Gamma_\alpha-\nu\Delta\Gamma_\alpha=\Gamma_\alpha\cdot\nabla v.$$

Since $\Gamma^0_\alpha(x)=\Gamma^0(x)$, by uniqueness we get $\Gamma_\alpha(t,x)=\Gamma(t,x)$. This shows that the components of Γ do not depend on the angle θ and, thus, the first step is achieved.

It remains now to prove that the components Γ_r and Γ_z are zero. We will begin by writing their equations. We take the $\mathbb{R}^3$-inner product of the equation of Γ with $\vec{e}_r$.

Since v is axisymmetric, we get by straightforward computations

$$\begin{aligned}(v\cdot\nabla\Gamma)\cdot\vec{e}_r&=v_r\partial_r\Gamma_r+v_z\partial_z\Gamma_r=v\cdot\nabla\Gamma_r,\\ (\Gamma\cdot\nabla v)\cdot\vec{e}_r&=\Gamma_r\partial_r v_r+\frac{1}{r}\Gamma_\theta\partial_\theta v_r+\Gamma_z\partial_z v_r=\Gamma_r\partial_r v_r+\Gamma_z\partial_z v_r.\end{aligned}$$

For the dissipative term we have by definition and by the first step,

$$\begin{aligned}\nu\Delta\Gamma\cdot\vec{e}_r&=\nu\Big[\partial_r^2\Gamma\cdot\vec{e}_r+\frac{1}{r}\partial_r\Gamma_r\cdot\vec{e}_r+\frac{1}{r^2}\partial_\theta^2\Gamma\cdot\vec{e}_r+\partial_z^2\Gamma\cdot\vec{e}_r\Big]\\ &=\nu\Big[\partial_r^2(\Gamma\cdot\vec{e}_r)+\frac{1}{r}\partial_r(\Gamma\cdot\vec{e}_r)\\ &\qquad+\frac{1}{r^2}\Big(\partial_\theta^2(\Gamma\cdot\vec{e}_r)+(\Gamma\cdot\vec{e}_r)-2\partial_\theta\Gamma\cdot\vec{e}_\theta\Big)+\partial_z^2(\Gamma\cdot\vec{e}_r)\Big]\\ &=\nu\Big[\partial_r^2\Gamma_r+\frac{1}{r}\partial_r\Gamma_r+\frac{1}{r^2}\partial_\theta^2\Gamma_r-\frac{1}{r^2}\Gamma_r-\frac{2}{r^2}\partial_\theta\Gamma_\theta+\partial_z^2\Gamma_r\Big]\\ &=\nu\big[\Delta\Gamma_r-\frac{1}{r^2}\Gamma_r\big].\end{aligned}$$

It follows that

$$\partial_t\Gamma_r+v\cdot\nabla\Gamma_r-\nu\big[\Delta\Gamma_r-\frac{1}{r^2}\Gamma_r\big]=\Gamma_r\partial_r v_r+\Gamma_z\partial_z v_r,\quad \Gamma_{r|t=0}=\Gamma^0_r. \tag{3.6}$$

By the same method, we can find that the component Γ_z satisfies the following equation:

$$\partial_t\Gamma_z + v\cdot\nabla\Gamma_z - \nu\Delta\Gamma_z = \Gamma_r\partial_r v_z + \Gamma_z\partial_z v_z, \quad \Gamma_{z|t=0} = 0. \tag{3.7}$$

We multiply (3.6) by $|\Gamma_r|^{p-2}\Gamma_r$; integrating by parts and using the fact that $\operatorname{div} v = 0$,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\Gamma_r(t)\|_{L^p}^p + \nu(p-1)\int_{\mathbb{R}^3}|\nabla\Gamma_r|^2|\Gamma_r|^{p-2}dx + \nu\int_{\mathbb{R}^3}\frac{|\Gamma_r|^p}{r^2}dx\\ \le \int_{\mathbb{R}^3}|\Gamma_r|^p\partial_r v_r dx + \int_{\mathbb{R}^3}\Gamma_z|\Gamma_r|^{p-2}\Gamma_r\partial_z v_r dx\\ \le \Big(\|\Gamma_r\|_{L^p}^p + \|\Gamma_z\|_{L^p}\|\Gamma_r\|_{L^p}^{p-1}\Big)\|\nabla v\|_{L^\infty},\end{aligned}$$

where we have used the Hölder inequality. This gives

$$\|\Gamma_r(t)\|_{L^p} \le \|\Gamma_r^0\|_{L^p} + \int_0^t\big(\|\Gamma_r(\tau)\|_{L^p} + \|\Gamma_z(\tau)\|_{L^p}\big)\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

Applying the same argument to (3.7), we get

$$\|\Gamma_z(t)\|_{L^p} \le \|\Gamma_z^0\|_{L^p} + \int_0^t\big(\|\Gamma_r(\tau)\|_{L^p} + \|\Gamma_z(\tau)\|_{L^p}\big)\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

It suffices now to use Gronwall's lemma, implying that, for every $p\in[2,\infty]$,

$$\|\Gamma_r(t)\|_{L^p} + \|\Gamma_z(t)\|_{L^p} \le \Big(\|\Gamma_r^0\|_{L^p} + \|\Gamma_z^0\|_{L^p}\Big)e^{2\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}.$$

As $\Gamma_r^0 = \Gamma_z^0 = 0$ we then deduce that $\Gamma_r(t) = \Gamma_z(t) = 0$ for all $t\in\mathbb{R}_+$. The proof of part (ii) is now completed.

(iii) The first statement is a direct consequence of $\Gamma\wedge\vec{e}_\theta = \vec{0}$. Let's give the equation governing the angular component Γ_θ. By easy computations one has

$$\Delta\Gamma\cdot\vec{e}_\theta = \Delta(\Gamma_\theta\vec{e}_\theta)\cdot\vec{e}_\theta = \Delta\Gamma_\theta - \frac{\Gamma_\theta}{r^2},$$

and

$$(v\cdot\nabla\Gamma)\cdot\vec{e}_\theta = v\cdot\nabla\Gamma_\theta, \quad (\Gamma\cdot\nabla v)\cdot\vec{e}_\theta = \frac{v_r}{r}\Gamma_\theta.$$

Then taking the angular component in the system (3.1) we get

$$\partial_t\Gamma_\theta + v\cdot\nabla\Gamma_\theta - \nu\big[\Delta\Gamma_\theta - \frac{\Gamma_\theta}{r^2}\big] = \frac{v_r}{r}\Gamma_\theta., \quad \Gamma_{\theta|t=0} = \Gamma_\theta^0. \tag{3.8}$$

This achieves the proof. □

We need the following properties of the vorticity, see for example [2].

Proposition 3.2. *Assume that v is an axisymmetric vector field with zero divergence and $\omega = \nabla \wedge v$ its vorticity. Then the following properties hold true:*

$(i)'$ *The vector ω satisfies*

$$\vec{\omega} \wedge \vec{e}_\theta = \vec{0}.$$

In particular, for every (x_1, x_2, z) in $\mathbb{R}^3$ we have

$$\omega_3 = 0, \quad \omega_1(x_1, 0, z) = \omega_2(0, x_2, z) = 0.$$

$(ii)'$ *For every $q \geq -1$, $\Delta_q v$ is axisymmetric and*

$$\Delta_q \omega \wedge \vec{e}_\theta = \vec{0}.$$

4. Proof of Theorem 1.1

4.1. **Some a priori estimates.** In this section we give some elementary estimates.

Proposition 4.1. *Let v be an axisymmetric solution of the Navier-Stokes system. Then we have for all $t \in \mathbb{R}_+$*

(i) $\|r^{-1}\omega(t)\|_{L^{3,1}} \leq \|r^{-1}\omega_0\|_{L^{3,1}}$;
(ii) $\|r^{-1}v_r(t)\|_{L^\infty} \leq C\|r^{-1}\omega_0\|_{L^{3,1}}$;
(iii) $\|\omega(t)\|_{L^\infty} \leq C\|\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}$;
(iv) $\|v(t)\|_{L^\infty} \leq C\big(\|v_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big) e^{\exp(Ct\|\frac{\omega_0}{r}\|_{L^{3,1}})}$.

The constant C does not depend on the viscosity.

proof. (i) We set $\eta = \frac{\omega_\theta}{r}$, then we have

$$\partial_t \eta + v \cdot \nabla \eta - \nu\big[\Delta \eta + \frac{3}{r}\partial_r \eta\big] = 0.$$

The dissipative term has a good sign and thus we have for all $p \in [1, \infty]$ (see also [19]) $\|\eta(t)\|_{L^p} \leq \|\eta_0\|_{L^p}$. By interpolation we get, for $1 < p < \infty$ and $q \in [1, \infty]$, $\|\eta(t)\|_{L^{p,q}} \leq \|\eta_0\|_{L^{p,q}}$.

(ii) We use the following inequality due to T. Shirota and T. Yanagisawa [17]:

$$\frac{|v^r|}{r} \lesssim \frac{1}{|.|^2} \star \Big|\frac{\omega_\theta}{r}\Big|,$$

As $\frac{1}{|.|^2} \in L^{\frac{3}{2},\infty}$, then from the convolution laws $L^{p,q} \star L^{p',q'} \to L^\infty$ we have

$$\Big\|\frac{v_r(t)}{r}\Big\|_{L^\infty} \lesssim \Big\|\frac{\omega(t)}{r}\Big\|_{L^{3,1}}.$$

It suffices now to combine this estimate with (i).

(iii) Since ω satisfies

$$\partial_t\omega + v\cdot\nabla\omega - \nu\Delta\omega = \frac{v_r}{r}\omega, \quad \omega_{|t=0} = \omega^0,$$

the maximum principle and the estimate of (ii) yield

$$\begin{aligned}\|\omega(t)\|_{L^\infty} &\leq \|\omega^0\|_{L^\infty} + \int_0^t \Big\|\frac{v_r(\tau)}{r}\omega(\tau)\Big\|_{L^\infty} d\tau\\ &\leq \|\omega_0\|_{L^\infty} + \Big\|\frac{\omega_0}{r}\Big\|_{L^{3,1}}\int_0^t \|\omega(\tau)\|_{L^\infty} d\tau.\end{aligned}$$

This gives in view of Granwall's inequality

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

The desired estimate is then proved.

(iv) We will use an argument due to P. Serfati [16] and applied for the Euler case. From the homogeneous Littlewood-Paley decomposition,

$$\|v(t)\|_{L^\infty} \leq \|\dot S_{-N}v\|_{L^\infty} + \sum_{q\geq -N}\|\dot\Delta v\|_{L^\infty},$$

where N is a parameter that will be judiciously chosen later. Using Bernstein's inequality, we get

$$\sum_{q\geq -N}\|\dot\Delta v\|_{L^\infty} \lesssim 2^N\|\omega\|_{L^\infty}. \tag{4.1}$$

Since $\dot S_{-N}v$ satisfies the equation

$$(\partial_t - \nu\Delta)\dot S_{-N}v = -\mathbb{P}\big(v\cdot\nabla v\big),$$

we get easily

$$\|\dot S_{-N}v\|_{L^\infty} \leq \|\dot S_{-N}v_0\|_{L^\infty} + \int_0^t \|\dot S_{-N}\mathbb{P}(v\cdot\nabla v)(\tau)d\tau\|_{L^\infty},$$

where $\mathbb{P}$ is Leray's projector of degree zero and $\dot\Delta_q\mathbb{P}$ maps continuously L^∞ into itself. Thus, we deduce that

$$\|\dot S_{-N}v\|_{L^\infty} \leq \|\dot S_{-N}v_0\|_{L^\infty} + 2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau. \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\|v(t)\|_{L^\infty} \lesssim \|v_0\|_{L^\infty} + 2^N\|\omega\|_{L^\infty} + 2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau. \tag{4.3}$$

If we choose N such that

$$2^{2N} \approx 1 + \|\omega\|_{L^\infty}^{-1} \int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau,$$

then the estimate (4.3) becomes

$$\|v(t)\|_{L^\infty}^2 \lesssim \|v_0\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty} \int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau.$$

We apply again the Gronwall inequality

$$\|v(t)\|_{L^\infty} \lesssim \big(\|v_0\|_{L^\infty} + \|\omega(t)\|_{L_t^\infty L^\infty}\big) e^{Ct\|\omega\|_{L^\infty_{t,x}}}. \tag{4.4}$$

Inserting the estimate (iii) of Proposition 4.1 into (4.4),

$$\|v(t)\|_{L^\infty} \lesssim \big(\|v_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big) e^{\exp Ct \|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

The proof is now completed. □

4.2. **Vorticity decomposition and Lipschitz bound.** The following result is the main step to bounding the Lipschitz norm of the velocity. We will establish a new decomposition of the vorticity based on the special structure of axisymmetric flows. We mention that this result was first proved for the Euler case [2] and we generalize it here for the viscous case uniformly with respect to the viscosity.

Proposition 4.2. *Let ω be the vorticity of the viscous axisymmetric solution. Then there exists a decomposition $\{\tilde\omega_q\}_{q\geq -1}$ of the vorticity ω such that, for every $t \in \mathbb{R}_+$,*

- (B1) $\omega(t,x) = \sum_{q\geq -1} \tilde\omega_q(t,x)$;
- (B2) $\operatorname{div} \tilde\omega_q(t,x) = 0$;
- (B3) *for all* $q \geq -1$, $\|\tilde\omega_q(t)\|_{L^\infty} \leq \|\Delta_q \omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}$;
- (B4) *there exists a constant $C > 0$ independent of the viscosity such that for every $k, q \geq -1$*

$$\|\Delta_k \tilde\omega_q(t)\|_{L^\infty} \leq C 2^{-|k-q|} e^{CZ(t)} \|\Delta_q \omega_0\|_{L^\infty},$$

with $Z(t) \stackrel{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$.

Proof. For $q \geq -1$ we define $\tilde\omega_q$ as the solution of the following linear Cauchy problem:

$$\partial_t \tilde\omega_q - \nu\Delta\tilde\omega_q + (v\cdot\nabla)\tilde\omega_q = \tilde\omega_q \cdot \nabla v, \quad \tilde\omega_q|_{t=0} = \Delta_q \omega^0. \tag{4.5}$$

By linearity and uniqueness, we see that

$$\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x).$$

Since div $\Delta_q\omega^0 = 0$, Proposition 3.1 gives div $\tilde{\omega}_q(t) = 0$. From Proposition 3.2, we have $\Delta_q\omega^0 \wedge \vec{e}_\theta = \vec{0}$. It follows from Proposition 3.1 that this property is preserved in time and

$$\partial_t\tilde{\omega}_q - \nu\Delta\tilde{\omega}_q + (v\cdot\nabla)\tilde{\omega}_q = \frac{v^r}{r}\tilde{\omega}_q, \quad \tilde{\omega}_q|_{t=0} = \Delta_q\omega_0. \tag{4.6}$$

Applying the maximum principle, we obtain

$$\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q\omega_0\|_{L^\infty} + \int_0^t \Big\|\frac{v^r}{r}\Big\|_{L^\infty}\|\tilde{\omega}_q\|_{L^\infty}d\tau.$$

Therefore we get, from Gronwall's lemma and (ii) of Proposition 4.1,

$$\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega^0}{r}\|_{L^{3,1}}}.$$

The proof of (B4) is equivalent to

$$\|\Delta_k\tilde{\omega}_q(t)\|_{L^\infty} \leq C2^{k-q}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty}, \tag{4.7}$$

and

$$\|\Delta_k\tilde{\omega}_q(t)\|_{L^\infty} \leq C2^{q-k}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty}. \tag{4.8}$$

To prove (4.7) we apply Proposition 2.5:

$$e^{-CZ(t)}\|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} \leq C\Big(\|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}} + \int_0^t e^{-CZ(\tau)}\|\tilde{\omega}_q\cdot\nabla v(\tau)\|_{B^{-1}_{\infty,\infty}}d\tau\Big).$$

According to Bony's decomposition,

$$\tilde{\omega}_q\cdot\nabla v = T_{\tilde{\omega}_q}\cdot\nabla v + T_{\nabla v}\cdot\tilde{\omega}_q + R(\tilde{\omega}_q^i, \partial_i v).$$

Thus we have

$$\begin{aligned}\|\tilde{\omega}_q\cdot\nabla v\|_{B^{-1}_{\infty,\infty}} &\leq \|T_{\tilde{\omega}_q}\cdot\nabla v\|_{B^{-1}_{\infty,\infty}} + \|T_{\nabla v}\cdot\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} + \|R(\tilde{\omega}_q\cdot, \nabla v)\|_{B^{-1}_{\infty,\infty}}\\ &\lesssim \|\nabla v\|_{L^\infty}\|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} + \|R(\tilde{\omega}_q^i, \partial_i v)\|_{B^{-1}_{\infty,\infty}}.\end{aligned}$$

Using (B2) we obtain

$$\begin{aligned}\|R(\tilde{\omega}_q^i, \partial_i v)\|_{B^{-1}_{\infty,\infty}} = \|\partial_i R(\tilde{\omega}_q^i, v)\|_{B^{-1}_{\infty,\infty}} &\lesssim \sup_k \sum_{j\geq k-3}\|\Delta_j\tilde{\omega}_q\|_{L^\infty}\|\tilde{\Delta}_j v\|_{L^\infty}\\ &\lesssim \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}}\|v\|_{B^1_{\infty,1}}.\end{aligned}$$

Consequently,

$$\|\tilde{\omega}_q\cdot\nabla v\|_{B^{-1}_{\infty,\infty}}\lesssim\|v\|_{B^1_{\infty,1}}\|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}}.$$

We finally obtain

$$e^{-CZ(t)}\|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}}\lesssim\|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}}+\int_0^t\|v(\tau)\|_{B^1_{\infty,1}}e^{-CZ(\tau)}\|\tilde{\omega}_q(\tau)\|_{B^{-1}_{\infty,\infty}}d\tau.$$

Then Gronwall's inequality yields

$$\|\tilde{\omega}(t)\|_{B^{-1}_{\infty,\infty}}\lesssim\|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}}e^{CZ(t)}\lesssim 2^{-q}\|\Delta_q\omega_0\|_{L^\infty}e^{CZ(t)}.$$

It follows that

$$\|\Delta_k\tilde{\omega}_q(t)\|_{L^\infty}\le C2^{k-q}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty}. \tag{4.9}$$

Let us now move to the estimate (4.8). As $v^\theta=0$, it follows that

$$\frac{v^r}{r}=\frac{v^1}{x_1}=\frac{v^2}{x_2},$$

where $(v^1,v^2,0)$ are the components of v in a Cartesian basis. According to Proposition 3.1 the vector-valued solution $\tilde{\omega}_q$ has two components in the Cartesian basis $\tilde{\omega}_q^1$ and $\tilde{\omega}_q^2$. We restrict ourselves to the proof of the estimate of the first component. The second one is done the same way. We have

$$\partial_t\tilde{\omega}_q^1-\nu\Delta\tilde{\omega}_q^1+(v\cdot\nabla)\tilde{\omega}_q^1=\frac{v^2}{x_2}\tilde{\omega}_q^1,\quad \tilde{\omega}_q^1|_{|t=0}=\Delta_q\omega_0^1. \tag{4.10}$$

By Proposition 3.2, we have

$$e^{-CZ(t)}\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}\lesssim\|\Delta_q\omega_0^1\|_{B^1_{\infty,1}}+\int_0^t e^{-CZ(\tau)}\Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}(\tau)\Big\|_{B^1_{\infty,1}}d\tau. \tag{4.11}$$

For the last term of the right-hand side we have

$$\begin{aligned}\Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}}&\le\Big\|T_{\frac{\tilde{\omega}_q^1}{x_2}}v^2\Big\|_{B^1_{\infty,1}}+\Big\|T_{v^2}\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}}+\Big\|R\Big(v^2,\frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{B^1_{\infty,1}}\\&\overset{def}{=}F_1+F_2+F_3.\end{aligned} \tag{4.12}$$

To estimate F_1 we use the definitions of paraproducts and Besov spaces:

$$F_1\lesssim\sum_{k\ge-1}2^k\Big\|S_{k-1}\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{L^\infty}\|\Delta_k v^2\|_{L^\infty}\lesssim\|v\|_{B^1_{\infty,1}}\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty}. \tag{4.13}$$

Similarly, we have, for (F_3),

$$F_3 \lesssim \sum_{l\geq k-3} 2^k \|\Delta_l v^2\|_{L^\infty} \Big\|\tilde{\Delta}_l \frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty} \lesssim \|v\|_{B^1_{\infty,1}} \Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty}. \tag{4.14}$$

The estimate of F_2 is more subtle:

$$F_2 \lesssim \sum_{l\in\mathbb{N}} 2^l \Big\|S_{l-1}v^2(x)\Delta_l\Big(\frac{\tilde{\omega}_q^1(x)}{x_2}\Big)\Big\|_{L^\infty}. \tag{4.15}$$

It is easily seen that

$$\begin{aligned}&\Big\|S_{l-1}v^2(x)\Delta_l\Big(\frac{\tilde{\omega}_q^1(x)}{x_2}\Big)\Big\|_{L^\infty}\\ &\leq \Big\|S_{l-1}v^2(x)\frac{\Delta_l\tilde{\omega}_q^1(x)}{x_2}\Big\|_{L^\infty} + \Big\|S_{l-1}v^2(x)\Big[\Delta_l, \frac{1}{x_2}\Big]\tilde{\omega}_q^1\Big\|_{L^\infty}.\end{aligned}$$

By Proposition 3.2, $S_{l-1}v$ is axisymmetric and $S_{l-1}v^2(0,x_2,z) = 0$. Thus, we obtain, by Taylor's formula,

$$\Big\|S_{l-1}v^2(x)\frac{\Delta_l\tilde{\omega}_q^1(x)}{x_2}\Big\|_{L^\infty} \leq \|\nabla v\|_{L^\infty}\|\Delta_l\tilde{\omega}_q^1\|_{L^\infty}.$$

Therefore, we get

$$\sum_{l\in\mathbb{N}} 2^l \Big\|S_{l-1}v^2(x)\frac{\Delta_l\tilde{\omega}_q^1(x)}{x_2}\Big\|_{L^\infty} \leq \|\nabla v\|_{L^\infty}\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}. \tag{4.16}$$

To treat the commutator term we write, by definition,

$$\begin{aligned}S_{l-1}v^2(x)\Big[\Delta_l, \frac{1}{x_2}\Big]\tilde{\omega}_q^1 &= \frac{S_{l-1}v^2}{x_2} 2^{3l}\int_{\mathbb{R}^3} h(2^l(x-y))(x_2-y_2)\frac{\tilde{\omega}_q^1(y)}{y_2}dy\\ &= 2^{-l}\Big(\frac{S_{l-1}v^2}{x_2}\Big)2^{3l}\tilde{h}(2^l.)\star\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)(x),\end{aligned}$$

where $\tilde{h}(x) = x_2 h(x)$. The following identity holds true for every $f \in S'(\mathbb{R}^3)$:

$$2^{3l}\tilde{h}(2^l.)\star f = \sum_{|l-k|\leq 1} 2^{3l}\tilde{h}(2^l.)\star\Delta_k f.$$

Indeed, we have $\widehat{\tilde{h}} = i\partial_{\xi_2}\hat{h} = i\partial_{\xi_2}\varphi(\xi)$. This implies that $\mathrm{supp}\widehat{\tilde{h}} \subset \mathrm{supp}\varphi$, and so

$$2^{3l}\tilde{h}(2^l.)\star\Delta_l f \equiv 0, \ \text{ for } |l-k|\geq 2.$$

Consequently,

$$\begin{aligned}&\sum_{l\in\mathbb{N}}2^l\Big\|S_{l-1}v^2(x)\Big[\Delta_l,\frac{1}{x_2}\Big]\tilde{\omega}_q^1\Big\|_{L^\infty}\\&\lesssim\sum_{|l-k|\leq1}\Big\|\frac{S_{l-1}v^2}{x_2}\Big\|_{L^\infty}\Big\|\Delta_k\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{L^\infty}\lesssim\|\nabla v\|_{L^\infty}\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}.\end{aligned}\tag{4.17}$$

Thus, it follows from (4.16) and (4.17) that

$$F_2\lesssim\|\nabla v\|_{L^\infty}\Big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}+\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}\Big).\tag{4.18}$$

Putting together (4.13), (4.14) and (4.18), we get

$$\Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\lesssim\|\nabla v\|_{B^1_{\infty,1}}\Big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}+\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}\Big).$$

Thanks to (4.11) and the above estimate one has

$$\begin{aligned}&e^{-CZ(t)}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}\\&\lesssim\|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}}+\int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}d\tau\\&+\int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\Big\|\frac{\tilde{\omega}_q^1(\tau)}{x_2}\Big\|_{B^0_{\infty,1}}d\tau.\end{aligned}\tag{4.19}$$

In order to estimate the quantity $\big\|\frac{\tilde{\omega}_q^1}{x_2}\big\|_{B^0_{\infty,1}}$ we will make use of Lemma 6.1 (see appendix): first of all, we have, due to Proposition 3.2, $\tilde{\omega}_q^1(x_1,0,z)=0$. Hence we get, by Taylor's formula and Lemma 6.1,

$$\begin{aligned}\frac{\tilde{\omega}_q^1}{x_2}&=\int_0^1\partial_y\tilde{\omega}_q^1(x_1,\mu x_2,x_3)d\mu\\\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}&\lesssim\int_0^1\|\partial_y\tilde{\omega}_q^1\|_{B^0_{\infty,1}}(1-\log\mu)d\mu\\&\lesssim\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}\int_0^1(1-\log\mu)d\mu\lesssim\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}.\end{aligned}$$

Then the estimate (4.19) becomes

$$e^{-CZ(t)}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}\lesssim\|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}}+\int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}d\tau.$$

It follows from Gronwall's lemma that

$$\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}} \leq C\|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}} e^{CZ(t)}.$$

This gives in particular the estimate (4.8):

$$\|\Delta_k \tilde{\omega}_q(t)\|_{L^\infty} \leq C 2^{q-k} e^{CZ(t)} \|\Delta_q \omega_0\|_{L^\infty}.$$

The proof of (B4) is now completed. □

Now, we achieve the proof of Theorem 1.1 by giving the persistence of the initial regularity uniformly on the viscosity.

Proposition 4.3. *Let $p \in [1,\infty]$ and v be the solution of (NS_ν) with initial data $v_0 \in B^{1+\frac{3}{p}}_{p,1}$ such that $\frac{\omega_0}{r} \in L^{3,1}$. Then we have the following.*

(E1) *Case $p = +\infty$.*

$$\forall t \geq 0, \quad \|\omega(t)\|_{B^0_{\infty,1}} + \|v(t)\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t}.$$

(E2) *Case $1 \leq p < +\infty$.*

$$\forall t \geq 0, \quad \|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} + \|v(t)\|_{B^{1+\frac{3}{p}}_{p,1}} \leq C_0 e^{e^{\exp C_0 t}},$$

where the constant C_0 depends on the norm of v_0 but not on the viscosity.

Proof. To prove (E1), we fix an integer N which will be judiciously chosen later. By virtue of (B1) of Proposition 4.2, we have

$$\begin{aligned} \|\omega(t)\|_{B^0_{\infty,1}} &\leq \sum_j \Big\|\Delta_j \sum_q \tilde{\omega}_q(t)\Big\|_{L^\infty} \\ &\leq \sum_{|j-q|\geq N} \|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} + \sum_{|j-q|<N} \|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} \stackrel{def}{=} H_1 + H_2. \end{aligned} \tag{4.20}$$

To estimate the first member H_1 we apply the last part (B4) of Proposition 4.2:

$$H_1 \leq 2^{-N} \|\omega_0\|_{B^0_{\infty,1}} e^{CZ(t)}. \tag{4.21}$$

For H_2, we use (B3) of Proposition 4.2:

$$H_2 \lesssim e^{C_0 t} \sum_{|j-q|\leq N} \|\Delta_q \omega_0(t)\|_{L^\infty} \lesssim e^{C_0 t} N \|\omega_0\|_{B^0_{\infty,1}}.$$

Combining this estimate with (4.21) we obtain

$$H_1 + H_2 \lesssim 2^{-N} e^{CZ(t)} + N e^{C_0 t}.$$

It suffices now to take N equal to $\big[CZ(t)\big]+1$,

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \big(Z(t)+1\big)e^{C_0 t}.$$

On the other hand, we have

$$\|v\|_{B^1_{\infty,1}} \lesssim \|v\|_{L^\infty} + \|\omega\|_{B^0_{\infty,1}}.$$

Thus, it follows from (iii) and (iv) of Proposition 4.1 that

$$\|v\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t} + C_0 e^{C_0 t}\int_0^t \|v(\tau)\|_{B^1_{\infty,1}} d\tau.$$

Hence, we get by Gronwall's lemma

$$\|v\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t},$$

and consequently we have

$$\|\omega\|_{B^0_{\infty,1}} \lesssim C_0 e^{\exp C_0 t}.$$

This gives (E1).
For (E2), we apply Proposition 2.5 to the vorticity equation

$$e^{-CV(\tau)}\|\omega\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CV(\tau)}\|\omega\cdot\nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau. \tag{4.22}$$

We want to prove that

$$\|\omega\cdot\nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega\|_{B^{\frac{3}{p}}_{p,1}}\|\nabla v\|_{L^\infty}. \tag{4.23}$$

According to Bony's decomposition we have

$$\|\omega\cdot\nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \leq \|T_{\nabla v}\cdot\omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega\cdot\nabla v\|_{B^{\frac{3}{p}}_{p,1}} + \|R\big(\omega^i,\partial_i v\big)\|_{B^{\frac{3}{p}}_{p,1}}. \tag{4.24}$$

By definition of $\|R\big(\omega^i,\partial_i v\big)\|_{B^{\frac{3}{p}}_{p,1}}$, we have

$$\begin{aligned}\|R\big(\omega,\nabla v\big)\|_{B^{\frac{3}{p}}_{p,1}} &\leq \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\sum_{j\geq q-3}\|\Delta_j\omega\|_{L^p}\|\Delta_j\nabla v\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^\infty}\sum_{j\geq q-3} 2^{(q-j)\frac{3}{p}}2^{j\frac{3}{p}}\|\Delta_j\omega\|_{L^p} \lesssim \|\nabla v\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

We can get easily the following:

$$\|T_{\nabla v}\cdot\omega\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\nabla v\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.$$

For the third term, we write

$$\|T_\omega\cdot\nabla v\|_{B^{\frac{3}{p}}_{p,1}}\lesssim\sum_{q\in\mathbb{N}}2^{q\frac{3}{p}}\|S_{q-1}\omega\|_{L^\infty}\|\nabla\Delta_q v\|_{L^p}$$
$$\lesssim\|\omega\|_{L^\infty}\sum_{q\in\mathbb{N}}2^{q\frac{3}{p}}\|\Delta_q\omega\|_{L^p}\lesssim\|\nabla v\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.$$

Thus, we get

$$e^{-CV(t)}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}\lesssim\|\omega_0\|_{B^{\frac{3}{p}}_{p,1}}+\int_0^t e^{-CV(t)}\|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{B^{\frac{3}{p}}_{p,1}}d\tau.$$

Using Gronwall's lemma, we get

$$\|\omega\|_{B^{\frac{3}{p}}_{p,1}}\leq\|\omega_0\|_{B^{\frac{3}{p}}_{p,1}}e^{CV(t)}\leq C_0e^{e^{\exp C_0t}}.$$

By definition,

$$\|v(t)\|_{B^{1+\frac{3}{p}}_{p,1}}\lesssim\|\Delta_{-1}v\|_{B^{\frac{3}{p}}_{p,1}}+\sum_{q\in\mathbb{N}}2^{q\frac{3}{p}}2^q\|\Delta_q v\|_{L^p}\ \lesssim\|v(t)\|_{L^p}+\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}}.$$

It remains to estimate $\|v(t)\|_{L^p}$. Since, for $p\in(1,\infty)$, the Riesz transform maps continuously L^p into itself, it follows that

$$\|v(t)\|_{L^p}\leq\|v^0\|_{L^p}+\int_0^t\|v(\tau)\cdot\nabla v(\tau)\|_{L^p}d\tau$$
$$\lesssim\|v^0\|_{L^p}+\int_0^t\|v(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

Applying again Gronwall's lemma,

$$\|v(t)\|_{L^p}\lesssim\|v^0\|_{L^p}e^{CV(t)}\leq C_0e^{e^{\exp C_0t}}.$$

For the case $p=1$ we write

$$\|v(t)\|_{L^1}\leq\|\dot S_0v(t)\|_{L^1}+\sum_{q\geq0}\|\dot\Delta_q v(t)\|_{L^1}$$
$$\lesssim\|\dot S_0v(t)\|_{L^1}+\sum_{q\geq0}2^{-q}\|\dot\Delta_q\nabla v(t)\|_{L^1}$$
$$\lesssim\|\dot S_0v(t)\|_{L^1}+\|\omega(t)\|_{L^1}\stackrel{def}{=}Y_1+Y_2.$$

To estimate Y_1 we use the Navier-Stokes equations,

$$Y_1\leq\|\dot S_0v^0\|_{L^1}+\sum_{q\leq-1}\|\dot\Delta_q\mathbb{P}\big((v\cdot\nabla)v(t)\big)\|_{L^1}$$

$$\begin{aligned}&\lesssim \|v^0\|_{L^1} + \sum_{q\leq -1} 2^q \|\dot{\Delta}_q(v\otimes v(t))\|_{L^1}\\ &\lesssim \|v^0\|_{L^1} + \|v(t)\|_{L^2}^2 \lesssim \|v^0\|_{L^1} + \|v^0\|_{L^2}^2.\end{aligned}$$

For Y_2, we have

$$Y_2 \leq \|\omega\|_{B^0_{1,1}} \leq \|\omega\|_{B^3_{1,1}}.$$

Combining Y_1 and Y_2, we get

$$\|v(t)\|_{L^1} \leq C_0 e^{e^{\exp C_0 t}}.$$

This finishes the proof. □

5. The Rate Convergence

Before stating the proof of Theorem 1.3, we need the following lemmata; for more details we refer to T. Hmidi and S. Keraani [8].

Lemma 5.1. *Let $s \in (-1,1)$, $(p_1,p_2,a)\in[1,\infty]^3$ and $v \in L^1\big(\mathbb{R}_+; Lip(\mathbb{R}^3)\big)$ be a divergence free vector field. Then, there exists a constant C which depends on s and such that the following holds true: let f be a smooth solution of (TD_ν), then for all $t\in\mathbb{R}_+$*

$$\nu^{\frac1a}\|f\|_{\widetilde{L^a_t}B^{s+\frac2a}_{p_1,p_2}} \leq Ce^{CV(t)}(1+\nu t)^{\frac1a}\big(\|f_0\|_{B^s_{p_1,p_2}} + \int_0^t \|g(\tau)\|_{B^s_{p_1,p_2}}d\tau\big),$$

where $V(t) = \|\nabla v\|_{L^1_tL^\infty}$ and C is a constant that does not depend on the viscosity.

Lemma 5.2. *Let $v\in B^0_{p,1}$ be a divergence free vector field and $w\in B^1_{\infty,1}$. Then we have the following.*

(1) $\|v\cdot\nabla w\|_{\dot{B}^0_{p,1}} \lesssim \|v\|_{B^0_{p,1}}\|w\|_{B^1_{\infty,1}}$.

If, in addition, $v\equiv w$, then we have

(2) $\|v\cdot\nabla v\|_{\dot{B}^0_{p,1}} \lesssim \|v\|_{L^p}\|v\|_{B^1_{\infty,1}}$.

Proof of Theorem 1.3. We distinguish in the proof three cases: $p<3$, $p=3$ and $3<p$.

Case $p\in(3,\infty]$**.** We set $z_\nu = v_\nu - v$ and $P_\nu = p_\nu - p$. We can easily check that z_ν satisfies the system

$$\partial_t z_\nu + (v_\nu\cdot\nabla)z_\nu = \nu\Delta v_\nu - (z_\nu\cdot\nabla)v + \nabla P_\nu, \qquad z_{|t=0}=0. \qquad (\widetilde{\text{NS}}_\nu)$$

By virtue of Proposition 2.5 and Proposition 4.3, we have

$$\|z_\nu\|_{B^0_{p,1}} \leq C_0 e^{e^{\exp C_0 t}} \Big(\int_0^t \Big(\nu \|\Delta v_\nu(\tau)\|_{B^0_{p,1}} + \|z_\nu \cdot \nabla v(\tau)\|_{B^0_{p,1}} + \|\nabla P_\nu(\tau)\|_{B^0_{p,1}} \Big) \Big) d\tau \stackrel{def}{=} C_0 e^{e^{\exp C_0 t}} (X_1 + X_2 + X_3). \tag{5.1}$$

To estimate X_1, we use the relation

$$\Delta v_\nu = \nabla \wedge \omega_\nu.$$

Therefore, we get from the interpolation inequality (2.1),

$$X_1 \leq C \|\nu\omega_\nu\|_{\widetilde{L}^1_t B^1_{p,1}} \leq C \|\nu\omega_\nu\|^{\frac{1}{2}+\frac{3}{2p}}_{\widetilde{L}^1_t B^{\frac{3}{p}}_{p,\infty}} \|\nu\omega_\nu\|^{\frac{1}{2}-\frac{3}{2p}}_{\widetilde{L}^1_t B^{2+\frac{3}{p}}_{p,\infty}}.$$

For the first term of the right-hand side, we apply Hölder's inequality and Proposition 4.3:

$$\|\nu\omega_\nu\|^{\frac{1}{2}+\frac{3}{2p}}_{\widetilde{L}^1_t B^{\frac{3}{p}}_{p,\infty}} \leq (\nu t)^{\frac{1}{2}+\frac{3}{2p}} \|\omega_\nu\|^{\frac{1}{2}+\frac{3}{2p}}_{\widetilde{L}^\infty_t B^{\frac{3}{p}}_{p,\infty}} \leq C_0 (\nu t)^{\frac{1}{2}+\frac{3}{2p}} e^{e^{\exp C_0 t}}. \tag{5.2}$$

For the second term we use Lemma 5.1:

$$\|\nu\omega_\nu\|_{\widetilde{L}^1_t B^{2+\frac{3}{p}}_{p,\infty}} \leq C(1+\nu t) e^{CV_\nu(t)} \Big(\|\omega^0_\nu\|_{B^{\frac{3}{p}}_{p,\infty}} + \|\omega_\nu \cdot \nabla v_\nu\|_{L^1_t B^{\frac{3}{p}}_{p,1}} \Big). \tag{5.3}$$

We have the law product

$$\|\omega_\nu \cdot \nabla v_\nu\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_\nu\|_{B^{\frac{3}{p}}_{p,1}} \|v_\nu\|_{B^{1+\frac{3}{p}}_{p,1}}.$$

Indeed, for $p < \infty$ this is a direct consequence of the algebra structure of the space $B^{\frac{3}{p}}_{p,1}$. But for $p = \infty$, we use the incompressibility of the vorticity ($\text{div}\, \omega_\nu = 0$). Thanks to (E2) of Proposition 4.3 and (5.2),

$$\|\nu\omega_\nu\|_{\widetilde{L}^1_t B^{2+\frac{3}{p}}_{p,\infty}} \leq C_0 e^{e^{\exp C_0 t}} (1+\nu t). \tag{5.4}$$

Thus, we get

$$X_1 \leq C_0 e^{e^{\exp C_0 t}} (\nu t)^{\frac{1}{2}+\frac{3}{2p}} (1+\nu t)^{\frac{1}{2}-\frac{3}{2p}}. \tag{5.5}$$

Concerning X_2, we use (1) of Lemma 5.2, from which it follows that

$$X_2 \lesssim \|z_\nu\|_{B^0_{p,1}} \|v\|_{B^1_{\infty,1}}. \tag{5.6}$$

Applying the differential operator div to $(\widetilde{\text{NS}}_\nu)$, we deduce

$$-\Delta P_\nu = \text{div}(z_\nu \cdot \nabla(v_\nu + v)).$$

Since $\dot{B}^0_{p,1} \hookrightarrow B^0_{p,1}$ and Riesz transforms act continuously on homogeneous Besov spaces,

$$\|\nabla P_\nu\|_{B^0_{p,1}} \leq C\|z_\nu \cdot \nabla(v_\nu + v)\|_{\dot{B}^0_{p,1}}.$$

Combining this estimate with Lemma 5.2 and Proposition 4.3, we find

$$\|\nabla P_\nu\|_{B^0_{p,1}} \leq C_0 e^{\exp C_0 t}\|z_\nu\|_{B^0_{p,1}}.$$

This leads to

$$X_3 \leq C_0 e^{\exp C_0 t}\int_0^t \|z_\nu(\tau)\|_{B^0_{p,1}} d\tau. \tag{5.7}$$

Inserting (5.5), (5.6) and (5.7) in (5.1), we get

$$\|z_\nu\|_{B^0_{p,1}} \lesssim C_0 e^{e^{\exp C_0 t}}(\nu t)^{\frac{1}{2}+\frac{3}{2p}}(1+\nu t)^{\frac{1}{2}-\frac{3}{2p}} + C_0 e^{\exp C_0 t}\int_0^t \|z_\nu(\tau)\|_{B^0_{p,1}} d\tau.$$

Gronwall's lemma yields

$$\|z_\nu\|_{B^0_{p,1}} \leq C_0 e^{e^{\exp C_0 t}}(\nu t)^{\frac{1}{2}+\frac{3}{2p}}.$$

Case $p = 3$. We reproduce the same calculation by changing only the interpolation inequality before (5.2) by

$$\|\nu\omega\|_{\widetilde{L^1}B^1_{3,1}} \leq (\nu t)\|\omega_\nu\|_{\widetilde{L^\infty}B^1_{3,1}}.$$

We get finally

$$\|z_\nu\|_{\widetilde{L^1}B^1_{3,1}} \leq C_0 e^{e^{\exp C_0 t}}(\nu t).$$

For $1 \leq p < 3$**,** we use the Besov embedding $B^{1+\frac{3}{p}}_{p,1} \hookrightarrow B^1_{3,1}$. It follows that

$$\|z_\nu\|_{B^0_{3,1}} \leq C_0 e^{e^{\exp C_0 t}}(\nu t).$$

This achieves the proof of the theorem. □

6. Appendix

We have the following result which was proved in [2]; for the convenience of the reader we will give here the proof.

Lemma 6.1. *Let* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *be a function in* $B^0_{\infty,1}$ *and take* $h_\mu(x_1, x_2, x_3) = h(\mu x_1, x_2, x_3)$ *with* $\mu \in (0, 1)$*. Then, there exists an absolute constant* C *such that the following inequality holds:*

$$\|h_\mu\|_{B^0_{\infty,1}} \leq C(1 - \log\mu)\|h\|_{B^0_{\infty,1}}.$$

Proof. Let μ in $(0,1)$ and take $h_{q,\mu} = (\Delta_q h)_\mu$ for $q \geq -1$. It is obvious that $h_\mu = \sum_{q\geq -1} h_{q,\mu}$. By definition we have

$$\begin{aligned}\|h_\mu\|_{B^0_{\infty,1}} &= \|\Delta_{-1}h_\mu\|_{L^\infty} + \sum_{j\in\mathbb{N}} \|\Delta_j h_\mu\|_{L^\infty}\\ &\leq C\|h\|_{L^\infty} + \sum_{j\in\mathbb{N},\ q\geq -1} \|\Delta_j h_{q,\mu}\|_{L^\infty}.\end{aligned}$$

For $j, q \in \mathbb{N}$, the Fourier transform of $\Delta_j h_{q,\lambda}$ is supported in the set

$$\Big\{|\xi_1| + |\xi'| \approx 2^j \quad \text{and} \quad \mu^{-1}|\xi_1| + |\xi'| \approx 2^q\Big\},$$

where $\xi' = (\xi_2, \xi_3)$. A direct consideration shows that this set is empty if $2^q \lesssim 2^j$ or $2^{j-q} \lesssim \mu$. For $q = -1$ the set is empty if $j \geq n_0$; this last number is absolute. Thus, we get for an integer n_1

$$\begin{aligned}\|h_\mu\|_{B^0_{\infty,1}} &\lesssim \|h\|_{L^\infty} + \sum_{\substack{q-n_1+\log\mu\leq j\\ j\leq q+n_1}} \|\Delta_j h_{q,\mu}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1 - \log\mu)\sum_q \|h_{q,\mu}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1 - \log\mu)\sum_q \|h_q\|_{L^\infty} \lesssim (1-\log\mu)\|h\|_{B^0_{\infty,1}}.\end{aligned}$$

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