ON THE GLOBAL SOLUTIONS OF THE SUPER-CRITICAL 2D QUASI-GEOSTROPHIC EQUATION IN BESOV SPACES

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ABSTRACT. In this paper we study the super-critical 2D dissipative quasi-geostrophic equation. We obtain some regularization effects allowing us to prove global well-posedness result for small initial data lying in critical Besov spaces constructed over Lebesgue spaces L^p, with $p \in [1, \infty]$. Local results for arbitrary initial data are also given.

1. INTRODUCTION

This paper deals with the Cauchy problem for the two-dimensional dissipative quasi-geostrophic equation

$$(\mathrm{QG}_\alpha) \begin{cases} \partial_t \theta + v \cdot \nabla \theta + |\mathrm{D}|^\alpha \theta = 0 \\ \theta_{|t=0} = \theta^0, \end{cases}$$

where the scalar function θ represents the potential temperature and $\alpha \in [0, 2]$. The velocity $v = (v^1, v^2)$ is determined by θ through a stream function ψ, namely

$$v = (-\partial_2 \psi, \partial_1 \psi), \quad \text{with} \quad |\mathrm{D}|\psi = \theta.$$

Here, the differential operator $|\mathrm{D}| = \sqrt{-\Delta}$ is defined in a standard fashion through its Fourier transform: $\mathcal{F}(|D|u) = |\xi|\mathcal{F}u$. The above relations can be rewritten as

$$v = (-\partial_2 |\mathrm{D}|^{-1}\theta, \partial_1 |\mathrm{D}|^{-1}\theta) = (-R_2\theta, R_1\theta),$$

where $R_i (i = 1, 2)$ are Riesz transforms.
First we notice that solutions for (QG_α) equation are scaling invariant in the following sense: if θ is a solution and $\lambda > 0$ then $\theta_\lambda(t, x) = \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x)$ is also a solution of (QG_α) equation. From the definition of the homogeneous Besov spaces, described in next section, one can show that the norm of θ_λ in the space $\dot{B}_{p,r}^{1+\frac{2}{p}-\alpha}$, with p, $r \in [1, \infty]$, is quasi-invariant. That is, there exists a pure constant $C > 0$ such that for every $\lambda, t > 0$

$$C^{-1}\|\theta_\lambda(t)\|_{\dot{B}_{p,r}^{1+\frac{2}{p}-\alpha}} \le \|\theta(\lambda^\alpha t)\|_{\dot{B}_{p,r}^{1+\frac{2}{p}-\alpha}} \le C\|\theta_\lambda(t)\|_{\dot{B}_{p,r}^{1+\frac{2}{p}-\alpha}}.$$

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Besides its intrinsic mathematical importance the (QG_α) equation serves as a 2D models arising in geophysical fluid dynamics, for more details about the subject see [6, 15] and the references therein. Recently the (QG_α) equation has been intensively investigated and much attention is carried to the problem of global existence. For the sub-critical case $(\alpha > 1)$ the theory seems to be in a satisfactory state. Indeed, global existence and uniqueness for arbitrary initial data are established in various function spaces (see for example [8, 16]). However in the critical case, that is $\alpha = 1$, Constantin et *al.* [7] showed the global existence in Sobolev space H^1 under smallness assumption of the L^∞-norm of the initial temperature θ^0 but the uniqueness is proved for initial data in H^2. Many other relevant results can be found in [9, 13, 14]. The super-critical case $\alpha < 1$ seems harder to deal with and work on this subject has just started to appear. In [2] the global existence and uniqueness are established for data in critical Besov space $B^{2-\alpha}_{2,1}$ with a small $\dot{B}^{2-\alpha}_{2,1}$ norm. This result was improved by N. Ju [12] for small initial data in H^s with $s \geq 2-\alpha$. We would like to point out that all these spaces are constructed over Lebesgue space L^2 and the same problem for general Besov space $B^s_{p,r}$ is not yet well explored and few results are obtained in this subject. In [20], Wu proved the global existence and uniqueness for small initial data in $C^r \cap L^q$ with $r > 1$ and $q \in]1, \infty[$, which is not a scaling space. We can also mention the paper [21] in which global well-posedness is established for small initial data in $B^s_{2,\infty} \cap B^s_{p,\infty}$, with $s > 2-\alpha$ and $p = 2^N$.
The main goal of the present paper is to study existence and uniqueness problems in the super-critical case when initial data belong to inhomogeneous critical Besov spaces $B^{1+\frac{2}{p}-\alpha}_{p,1}$, with $p \in [1, \infty]$.
Our first main result reads as follows.

Theorem 1.1. *Let $\alpha \in [0, 1[$, $p \in [1, \infty]$ and $s \geq s^p_c$, with $s^p_c = 1 + \frac{2}{p} - \alpha$ and define*

$$\mathcal{X}^s_p = \begin{cases} B^s_{p,1}, & \text{if} \quad p < \infty, \\ B^s_{\infty,1} \cap \dot{B}^0_{\infty,1}, & \text{otherwise.} \end{cases}$$

Then for $\theta^0 \in \mathcal{X}^s_p$ there exists $T > 0$ such that the (QG_α) equation has a unique solution θ belonging to $C([0,T]; \mathcal{X}^s_p) \cap L^1([0,T]; \dot{B}^{s+\alpha}_{p,1})$.
In addition, there exists an absolute constant $\eta > 0$ such that if

$$\|\theta^0\|_{\dot{B}^{1-\alpha}_{\infty,1}} \leq \eta,$$

then one can take $T = +\infty$.

Remark 1. *We observe that in our global existence result we make only a smallness assumption of the data in Besov space $\dot{B}^{1-\alpha}_{\infty,1}$ which contains the increasing Besov chain spaces $\{\dot{B}^{s^p_c}_{p,1}\}_{p\in[1,\infty]}$.*

Remark 2. *In the case of $s > s_c^p$ we have the following lower bound for the local time existence. There exists a nonnegative constant C such that*

$$T \geq C\|\theta^0\|_{\dot{B}^{s-\frac{2}{p}}_{\infty,1}}^{\frac{-\alpha}{s-s_c^p}}.$$

However in the critical case $s = s_c^p$ the local time existence is bounded from below by

$$\sup\Big\{t \geq 0, \sum_{q\in\mathbb{Z}} (1 - e^{-ct2^{q\alpha}})^{\frac{1}{2}} 2^{q(1-\alpha)} \|\Delta_q \theta^0\|_{L^\infty} \leq \eta\Big\},$$

where η is an absolute nonnegative constant.

The proof relies essentially on some new estimates for transport-diffusion equation

$$(\mathrm{TD}_\alpha)\begin{cases} \partial_t\theta + v\cdot\nabla\theta + |\mathrm{D}|^\alpha\theta = f \\ \theta_{|t=0} = \theta^0, \end{cases}$$

where the unknown is the scalar function θ. Our second main result reads as follows

Theorem 1.2. *Let $s \in]-1, 1[$, $\alpha \in [0, 1[$, $(p, r) \in [1, +\infty]^2$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}^s_{p,1})$ and v be a divergence free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^d))$. We consider a smooth solution θ of the transport-diffusion equation (TD_α) , then there exists a constant C depending only on s and α such that*

$$\|\theta\|_{\widetilde{L}^r_t \dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \leq Ce^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}\Big(\|\theta^0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1_t \dot{B}^s_{p,1}}\Big).$$

Besides if $v = \nabla^\perp|\mathrm{D}|^{-1}\theta$ then the above estimate is valid for all $s > -1$.

We use for the proof a new approach based on Lagrangian coordinates combined with paradifferential calculus and a new commutator estimate. This idea has been recently used by the first author to treat the two-dimenional Navier-Stokes vortex patches [11].

Remark 3. *The estimates of Theorem 1.2 hold true for Besov spaces $\dot{B}^s_{p,m}$, with $m \in [1, \infty]$. The proof can be done strictly in the same line as the case $m = 1$. It should be also mentioned that we can derive similar results for inhomogeneous Besov spaces.*

Notation: Throughout the paper, C stands for a constant which may be different in each occurrence. We shall sometimes use the notation $A \lesssim B$ instead of $A \leq CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.
The rest of this paper is structured as follows. In next section we recall some basic results on Littlewood-Paley theory and we give some useful lemmas. Section 3 is devoted to the proof of a new commutator estimate while sections 4 and 5 are dealing successively with the proofs of Theorem 1.2 and 1.1. We give in the end of this paper an appendix.

2. Preliminaries

In this preparatory section, we provide the definition of some function spaces based on the so-called Littlewood-Paley decomposition and we review some important lemmas that will be used constantly in the following pages.
We start with the dyadic decomposition. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be supported in the ring $\mathcal{C} := \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

We define also the function $\chi(\xi) = 1 - \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi)$. Now for $u \in \mathcal{S}'$ we set

$$\Delta_{-1}u = \chi(\mathrm{D})u;\, \forall q \in \mathbb{N},\ \Delta_q u = \varphi(2^{-q}\mathrm{D})u \quad \text{and}\ \forall\, q \in \mathbb{Z},\ \dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})u.$$

The following low-frequency cut-off will be also used:

$$S_q u = \sum_{-1\leq j\leq q-1} \Delta_j u \quad \text{and} \quad \dot{S}_q u = \sum_{j\leq q-1} \dot{\Delta}_j u.$$

We caution that we shall sometimes use the notation Δ_q instead of $\dot{\Delta}_q$ and this will be tacitly understood from the context.
Let us now recall the definition of Besov spaces through dyadic decomposition.
Let $(p, m) \in [1, +\infty]^2$ and $s \in \mathbb{R}$, then the inhomogeneous space $B^s_{p,m}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p,m}} := \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^m} < \infty.$$

To define the homogeneous Besov spaces we first denote by $\mathcal{S}'/\mathcal{P}$ the space of tempered distributions modulo polynomials. Thus we define the space $\dot{B}^s_{p,r}$ as the set of distribution $u \in \mathcal{S}'/\mathcal{P}$ such that

$$\|u\|_{\dot{B}^s_{p,m}} := \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^m} < \infty.$$

We point out that if $s > 0$ then we have $B^s_{p,m} = \dot{B}^s_{p,m} \cap L^p$ and

$$\|u\|_{B^s_{p,m}} \approx \|u\|_{\dot{B}^s_{p,m}} + \|u\|_{L^p}.$$

Another characterization of homogeneous Besov spaces that will be needed later is given as follows (see [18]). For $s \in]0, 1[, p, m \in [1, \infty]$

$$C^{-1}\|u\|_{\dot{B}^s_{p,m}} \leq \Big(\int_{\mathbb{R}^d} \frac{\|u(\cdot - x) - u(\cdot)\|^m_{L^p}}{|x|^{sm}} \frac{dx}{|x|^d}\Big)^{\frac{1}{m}} \leq C\|u\|_{\dot{B}^s_{p,m}}, \tag{1}$$

with the usual modification if $m = \infty$.
In our next study we require two kinds of coupled space-time Besov spaces. The

first one is defined in the following manner: for $T>0$ and $m\geq 1$, we denote by $L^r_T\dot{B}^s_{p,m}$ the set of all tempered distribution u satisfying

$$\|u\|_{L^r_T\dot{B}^s_{p,r}} := \Big\|\Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^m}\Big\|_{L^r_T} < \infty.$$

The second mixed space is $\widetilde{L}^r_T\dot{B}^s_{p,m}$ which is the set of tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^r_T\dot{B}^s_{p,m}} := \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^r_T L^p}\Big)_{\ell^m} < \infty.$$

We can define by the same way the spaces $L^r_T B^s_{p,m}$ and $\widetilde{L}^r_T B^s_{p,m}$.
The following embeddings are a direct consequence of Minkowski's inequality.
Let $s\in\mathbb{R}$, $r\geq 1$ and $\big(p,m\big)\in[1,\infty]^2$, then we have

$$\begin{aligned} L^r_T\dot{B}^s_{p,m} &\hookrightarrow \widetilde{L}^r_T\dot{B}^s_{p,m}, \text{ if } \quad m\geq r \quad \text{and}\\ \widetilde{L}^r_T\dot{B}^s_{p,m} &\hookrightarrow L^r_T\dot{B}^s_{p,m}, \text{ if } \quad r\geq m. \end{aligned} \tag{2}$$

Another classical result that will be frequently used here is the so-called Bernstein inequalities (see [3] and the references therein): there exists C such that for every function u and for every $q\in\mathbb{Z}$, we have

$$\begin{aligned} \sup_{|\alpha|=k}\|\partial^\alpha S_q u\|_{L^b} &\leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q u\|_{L^a}, \quad \text{for} \quad b\geq a,\\ C^{-k}2^{qk}\|\dot{\Delta}_q u\|_{L^a} &\leq \sup_{|\alpha|=k}\|\partial^\alpha \dot{\Delta}_q u\|_{L^a}\leq C^k 2^{qk}\|\dot{\Delta}_q u\|_{L^a}. \end{aligned}$$

It is worth pointing out that the above inequalities hold true if we replace the derivative ∂^α by fractional derivative $|\mathrm{D}|^\alpha$. According to Bernstein inequalities one can show the following embeddings

$$\dot{B}^s_{p,m}\hookrightarrow \dot{B}^{s-d(\frac{1}{p}-\frac{1}{p_1})}_{p_1,m_1}, \quad \text{for} \quad p\leq p_1 \quad \text{and} \quad m\leq m_1.$$

Now let us we recall the following commutator lemma (see [3, 10] and the references therein).

Lemma 2.1. *Let $p,r\in[1,\infty], 1=\frac{1}{r}+\frac{1}{\bar{r}}, \rho_1<1, \rho_2<1$ and v be a divergence free vector field of $\mathbb{R}^d$. Assume in addition that*

$$\rho_1+\rho_2+d\min\{1,2/p\}>0 \quad \textit{and} \quad \rho_1+d/p>0.$$

Then we have

$$\sum_{q\in\mathbb{Z}} 2^{q(\frac{d}{p}+\rho_1+\rho_2-1)}\big\|[\dot{\Delta}_q, v\cdot\nabla]u\big\|_{L^1_tL^p}\lesssim \|v\|_{\widetilde{L}^r_t\dot{B}^{\frac{d}{p}+\rho_1}_{p,1}}\|u\|_{\widetilde{L}^{\bar{r}}_t\dot{B}^{\frac{d}{p}+\rho_2}_{p,1}}.$$

Moreover we have for $s\in]-1,1[$

$$\sum_{q\in\mathbb{Z}} 2^{qs}\big\|[\dot{\Delta}_q, v\cdot\nabla]u\big\|_{L^p}\lesssim \|\nabla v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,1}}.$$

In addition this estimate holds true for all $s > -1$ if $v = \nabla^\perp |\mathrm{D}|^{-1} u$.

The following result describes the action of the semi-group operator $e^{t|\mathrm{D}|^\alpha}$ on distributions whose Fourier transform is supported in a ring.

Proposition 2.2. *Let $\mathcal{C}$ be a ring and $\alpha \in \mathbb{R}_+$. There exists a positive constant C such that for any $p \in [1; +\infty]$, for any couple (t, λ) of positive real numbers, we have*

$$\mathrm{supp}\mathcal{F}u \subset \lambda\mathcal{C} \Rightarrow \|e^{t|\mathrm{D}|^\alpha} u\|_{L^p} \leq Ce^{-C^{-1}t\lambda^\alpha} \|u\|_{L^p}.$$

Proof. We will imitate the same idea of [4]. Let $\phi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$, radially and whose value is identically 1 near the ring $\mathcal{C}$. Then we have

$$e^{t|\mathrm{D}|^\alpha} u = \phi(\lambda^{-1}|\mathrm{D}|)u = h_\lambda * u,$$

where

$$h_\lambda(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\lambda^{-1}\xi) e^{-t|\xi|^\alpha} e^{i<x,\xi>} d\xi.$$

We set

$$\bar{h}_\lambda(t, x) := \lambda^{-d} h(t, \lambda^{-1}x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\xi) e^{-t\lambda^\alpha|\xi|^\alpha} e^{i<x,\xi>} d\xi.$$

Now to prove the proposition it suffices to show that $\|\bar{h}_\lambda(t)\|_{L^1} \leq Ce^{-C^{-1}t\lambda^\alpha}$. For this purpose we write with the aid of an integration by parts

$$(1 + |x|^2)^d \bar{h}_\lambda(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\mathrm{Id} - \Delta_\xi)^d \big(\phi(\xi) e^{-t\lambda^\alpha|\xi|^\alpha}\big) e^{i<x,\xi>} d\xi.$$

From Leibnitz's formula, we have

$$(\mathrm{Id} - \Delta_\xi)^d \big(\phi(\xi) e^{-t\lambda^\alpha \xi^\alpha}\big) = \sum_{\substack{|\gamma| \leq 2d \\ \beta \leq \gamma}} C_{\gamma,\beta} \partial^{\gamma-\beta} \phi(\xi) \partial^\beta e^{-t\lambda^\alpha \xi^\alpha}.$$

As ϕ is supported in a ring that does not contain some neighbourhood of zero then we get for $\xi \in \mathrm{supp}\ \phi$

$$\begin{aligned} |\partial^\beta e^{-t\lambda^\alpha|\xi|^\alpha}| &\leq C_\beta (1 + t\lambda^\alpha)^{|\beta|} e^{-t\lambda^\alpha|\xi|^\alpha}, \ \forall \xi \in \ \mathrm{supp}\ \phi \\ &\leq C_\beta e^{-C^{-1}t\lambda^\alpha}. \end{aligned}$$

Thus we find that

$$\big|(\mathrm{Id} - \Delta_\xi)^d \big(\phi(\xi) e^{-t\lambda^\alpha \xi^\alpha}\big)\big| \leq Ce^{-C^{-1}t\lambda^\alpha} \sum_{\substack{|\gamma| \leq 2d \\ \beta \leq \gamma}} C_{\gamma,\beta} |\partial^{\gamma-\beta} \phi(\xi)|.$$

Since the term of the right-hand side belongs to $L^1(\mathbb{R}^d)$, then we deduce that

$$(1 + |x|^2)^d |\bar{h}_\lambda(x)| \leq Ce^{-C^{-1}t\lambda^\alpha}.$$

This completes the proof of the proposition. □

3. Commutator estimate

The main result of this section is the following estimate that will play a crucial role for the proof of Theorem 1.2.

Proposition 3.1. *Let $f \in \dot{B}^{\alpha}_{p,1}$ with $\alpha \in [0,1[$ and $p \in [1,+\infty]$, and let ψ be a Lipshitz measure-preserving homeomorphism on $\mathbb{R}^d$. Then there exists $C := c(\alpha)$ such that*

$$\big\||\mathrm{D}|^{\alpha}(f\circ\psi)-(|\mathrm{D}|^{\alpha}f)\circ\psi\big\|_{L^p} \leq C\max\Big(|1-\|\nabla\psi^{-1}\|^{d+\alpha}_{L^\infty}|;|1-\|\nabla\psi\|^{-d-\alpha}_{L^\infty}|\Big) \|\nabla\psi\|^{\alpha}_{L^\infty}\|f\|_{\dot{B}^{\alpha}_{p,1}}.$$

Proof. First we rule out the obvious case $\alpha=0$ and let us recall the following formula detailed in [9] which tells us that for all $\alpha \in]0,2[$

$$|\mathrm{D}|^{\alpha}f(x)=C_\alpha\,\mathrm{P.\,V.}\int\frac{f(x)-f(y)}{|x-y|^{d+\alpha}}dy.$$

Now we claim from (1) that if $g\in\dot{B}^{\alpha}_{p,1}$, with $\alpha\in]0,1[$, then the above identity holds as an L^p equality

$$|\mathrm{D}|^{\alpha}f(x)=C_\alpha\int_{\mathbb{R}^d}\frac{f(x)-f(y)}{|x-y|^{d+\alpha}}dy,\quad a.e.w. \tag{3}$$

and moreover,

$$\||\mathrm{D}|^{\alpha}f\|_{L^p}\lesssim\|f\|_{\dot{B}^{\alpha}_{p,1}}. \tag{4}$$

Indeed, the L^p norm of the integral function satisfies in view of Minkowski inequalities

$$\Big\|\int_{\mathbb{R}^d}\frac{f(\cdot)-f(y)}{|\cdot-y|^{d+\alpha}}dy\Big\|_{L^p}\leq\int_{\mathbb{R}^d}\frac{\|f(\cdot)-f(\cdot-y)\|_{L^p}}{|y|^{d+\alpha}}dy\approx\|f\|_{\dot{B}^{\alpha}_{p,1}}.$$

Thus we find that the left integral term is finite almost every where.
Inasmuch as the flow preserves Lebesgue measure then the formula (3) yields

$$(|\mathrm{D}|^{\alpha}f)\circ\psi(x)=C_\alpha\int_{\mathbb{R}^d}\frac{f(\psi(x))-f(y)}{|\psi(x)-y|^{d+\alpha}}dy=C_\alpha\int_{\mathbb{R}^d}\frac{f(\psi(x))-f(\psi(y))}{|\psi(x)-\psi(y)|^{d+\alpha}}dy.$$

Applying again (3) with $f\circ\psi$, we obtain

$$|\mathrm{D}|^{\alpha}(f\circ\psi)(x)=C_\alpha\int_{\mathbb{R}^d}\frac{f(\psi(x))-f(\psi(y))}{|x-y|^{d+\alpha}}dy.$$

Thus we get

$$|\mathrm{D}|^{\alpha}(f\circ\psi)(x)-(|\mathrm{D}|^{\alpha}f)\circ\psi(x) = C_\alpha\int_{\mathbb{R}^d}\frac{f(\psi(x))-f(\psi(y))}{|x-y|^{d+\alpha}}\times\Big(1-\frac{|x-y|^{d+\alpha}}{|\psi(x)-\psi(y)|^{d+\alpha}}\Big)dy.$$

Taking the L^p norm and using (1) we obtain

$$\big\||\mathrm{D}|^\alpha(f\circ\psi)-(|\mathrm{D}|^\alpha f)\circ\psi\big\|_{L^p}\lesssim\|f\circ\psi\|_{\dot B^\alpha_{p,1}}\sup_{x,y}\big|1-\frac{|x-y|^{d+\alpha}}{|\psi(x)-\psi(y)|^{d+\alpha}}\big|. \tag{5}$$

According to [17] one has the following composition result

$$\|f\circ\psi\|_{\dot B^\alpha_{p,1}}\le c_\alpha\|\nabla\psi\|^\alpha_{L^\infty}\|f\|_{\dot B^\alpha_{p,1}},\quad\text{for}\quad\alpha\in]0,1[.$$

Therefore (5) becomes

$$\begin{aligned}\big\||\mathrm{D}|^\alpha(f\circ\psi)-(|\mathrm{D}|^\alpha f)\circ\psi\big\|_{L^p}\ &\le\ C\|\nabla\psi\|^\alpha_{L^\infty}\|f\|_{\dot B^\alpha_{p,1}}\times\\ &\qquad\sup_{x,y}\big|1-\frac{|x-y|^{d+\alpha}}{|\psi(x)-\psi(y)|^{d+\alpha}}\big|.\end{aligned}$$

It is plain from mean value Theorem that

$$\frac{1}{\|\nabla\psi\|^{d+\alpha}_{L^\infty}}\le\frac{|x-y|^{d+\alpha}}{|\psi(x)-\psi(y)|^{d+\alpha}}\le\|\nabla\psi^{-1}\|^{d+\alpha}_{L^\infty},$$

which gives easily the inequality

$$\sup_{x,y}\big|1-\frac{|x-y|^{d+\alpha}}{|\psi(x)-\psi(y)|^{d+\alpha}}\big|\le\max\Big(|1-\|\nabla\psi^{-1}\|^{d+\alpha}_{L^\infty}|;|1-\|\nabla\psi\|^{-d-\alpha}_{L^\infty}|\Big).$$

This concludes the proof. □

4. Proof of Theorem 1.2

We shall divide our analysis into two cases: $r=+\infty$ and r is finite. The first case is more easy and simply based upon a maximum principle and a commutator estimate. Before we move on let us mention that in what follows we will work with the homogeneous Littlewood-Paley operators but we take the same notation of the inhomogeneous operators.

Set $\theta_q:=\Delta_q\theta$, then localizing the (QG_α) equation through the operator Δ_q gives

$$\partial_t\theta_q+v\cdot\nabla\theta_q+|\mathrm{D}|^\alpha\theta_q=-[\Delta_q,v\cdot\nabla]\theta+f_q:=\mathcal{R}_q. \tag{6}$$

According to Proposition 6.2 we have

$$\|\theta_q(t)\|_{L^p}\le\|\theta^0_q\|_{L^p}+\int_0^t\|\mathcal{R}_q(\tau)\|_{L^p}d\tau. \tag{7}$$

Multiplying both sides by 2^{qs} and summing over q

$$\|\theta\|_{\widetilde L^\infty_t\dot B^s_{p,1}}\le\|\theta^0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t\dot B^s_{p,1}}+\int_0^t\sum_q 2^{qs}\|\mathcal{R}_q(\tau)\|_{L^p}d\tau.$$

This yields in view of Lemma 2.1

$$\|\theta\|_{\widetilde L^\infty_t\dot B^s_{p,1}}\le\|\theta^0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t\dot B^s_{p,1}}+C\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|\theta\|_{\widetilde L^\infty_\tau\dot B^s_{p,1}}d\tau.$$

To achieve the proof in the case of $r=\infty$, it suffices to use Gronwall's inequality.

We shall now turn to the proof of the finite case $r<\infty$ which is more technical. Let ψ denote the flow of the velocity v and set

$$\bar\theta_q(t,x)=\theta_q(t,\psi(t,x))\quad\text{and}\quad \bar{\mathcal{R}}_q(t,x)=\mathcal{R}_q(t,\psi(t,x)).$$

Since the flow preserves Lebesgue measure then we obtain

$$\|\bar{\mathcal{R}}_q\|_{L^p}\le\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}+\|f_q\|_{L^p}. \tag{8}$$

It is not hard to check that the function $\bar\theta_q$ satisfies

$$\partial_t\bar\theta_q+|\mathrm{D}|^\alpha\bar\theta_q=|\mathrm{D}|^\alpha(\theta_q\circ\psi)-(|\mathrm{D}|^\alpha\theta_q)\circ\psi+\bar{\mathcal{R}}_q:=\bar{\mathcal{R}}_q^1. \tag{9}$$

From Proposition 3.1 we find that for $q\in\mathbb{Z}$

$$\begin{aligned}\||\mathrm{D}|^\alpha(\theta_q\circ\psi)-(|\mathrm{D}|^\alpha\theta_q)\circ\psi\|_{L^p}&\le Ce^{CV(t)}\Big(e^{CV(t)}-1\Big)\|\theta_q(t)\|_{\dot B^\alpha_{p,1}}\\&\le Ce^{CV(t)}\Big(e^{CV(t)}-1\Big)2^{q\alpha}\|\theta_q\|_{L^p},\end{aligned}\tag{10}$$

where $V(t):=\|\nabla v\|_{L^1_tL^\infty}$. Notice that we have used here the classical estimates

$$e^{-CV(t)}\le\|\nabla\psi^{\mp1}(t)\|_{L^\infty}\le e^{CV(t)}.$$

Putting together (8) and (10) yield

$$\|\bar{\mathcal{R}}^1_q(t)\|_{L^p}\le\|f_q(t)\|_{L^p}+\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}+Ce^{CV(t)}(e^{CV(t)}-1)2^{q\alpha}\|\theta_q(t)\|_{L^p}.$$

Applying the operator Δ_j, for $j\in\mathbb{Z}$, to the equation (9) and using Proposition 2.2

$$\begin{aligned}\|\Delta_j\bar\theta_q(t)\|_{L^p}&\le Ce^{-ct2^{j\alpha}}\|\Delta_j\theta^0_q\|_{L^p}+C\int_0^te^{-c(t-\tau)2^{j\alpha}}\|f_q(\tau)\|_{L^p}d\tau\\&+Ce^{CV(t)}(e^{CV(t)}-1)2^{q\alpha}\int_0^te^{-c(t-\tau)2^{j\alpha}}\|\theta_q(\tau)\|_{L^p}d\tau\\&+C\int_0^te^{-c(t-\tau)2^{j\alpha}}\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.\end{aligned}\tag{11}$$

Integrating this estimate with respect to the time and using Young's inequality

$$\begin{aligned}\|\Delta_j\bar\theta_q\|_{L^r_tL^p}&\le C2^{-j\alpha/r}\big((1-e^{-crt2^{j\alpha}})^{\frac1r}\|\Delta_j\theta^0_q\|_{L^p}+\|f_q\|_{L^1_tL^p}\big)\\&+Ce^{CV(t)}(e^{CV(t)}-1)2^{(q-j)\alpha}\|\theta_q\|_{L^r_tL^p}\\&+C2^{-j\alpha/r}\int_0^t\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.\end{aligned}\tag{12}$$

Since the flow ψ preserves Lebesgue measure then one writes

$$\begin{aligned}2^{q(s+\alpha/r)}\|\theta_q\|_{L^r_tL^p} &= 2^{q(s+\alpha/r)}\|\bar\theta_q\|_{L^r_tL^p}\\ &\leq 2^{q(s+\alpha/r)}\Big(\sum_{|j-q|>N}\|\Delta_j\bar\theta_q\|_{L^r_tL^p}+\sum_{|j-q|\leq N}\|\Delta_j\bar\theta_q\|_{L^r_tL^p}\Big)\\ &:= \mathrm{I}+\mathrm{II}.\end{aligned}$$

To estimate the term I we make appeal to Lemma 6.1

$$\begin{aligned}\|\Delta_j\bar\theta_q\|_{L^r_tL^p} &\leq C2^{-|q-j|}e^{\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}\|\theta_q\|_{L^r_tL^p}\\ &\leq C2^{-|q-j|}e^{V(t)}\|\theta_q\|_{L^r_tL^p}.\end{aligned}$$

Therefore we get

$$\mathrm{I}\leq C2^{-N}e^{V(t)}2^{q(s+\alpha/r)}\|\theta_q\|_{L^r_tL^p}. \tag{13}$$

In order to bound the second term II we use (12)

$$\begin{aligned}\mathrm{II} &\leq C(1-e^{-crt2^{q\alpha}})^{\frac1r}2^{qs}\|\theta^0_q\|_{L^p}+C2^{N\frac{\alpha}{r}}2^{qs}\|f_q\|_{L^1_tL^p}\\ &+ C2^{N\alpha}e^{CV(t)}(e^{CV(t)}-1)2^{q(s+\alpha/r)}\|\theta_q\|_{L^r_tL^p}\\ &+ C2^{N\alpha/r}2^{qs}\int_0^t\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.\end{aligned} \tag{14}$$

Denote $Z^r_q(t):=2^{q(s+\alpha/r)}\|\theta_q\|_{L^r_tL^p}$, then we obtain in view of (13) and (14)

$$\begin{aligned}Z^r_q(t) &\leq C(1-e^{-crt2^{q\alpha}})^{\frac1r}2^{qs}\|\theta^0_q\|_{L^p}+C2^{N\frac{\alpha}{r}}2^{qs}\|f_q\|_{L^1_tL^p}\\ &+ C\big(2^{N\alpha}e^{CV(t)}(e^{CV(t)}-1)+2^{-N}e^{CV(t)}\big)Z^r_q(t)\\ &+ C2^{N\alpha/r}2^{qs}\int_0^t\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.\end{aligned}$$

We can easily show that there exists two pure constants N and C_0 such that

$$V(t)\leq C_0\Rightarrow C2^{-N}e^{CV(t)}+C2^{N\alpha}e^{CV(t)}(e^{CV(t)}-1)\leq\frac12\cdot$$

Thus we obtain under this condition

$$\begin{aligned}Z^r_q(t) &\leq C(1-e^{-crt2^{q\alpha}})^{\frac1r}2^{qs}\|\theta^0_q\|_{L^p}+C2^{qs}\|f_q\|_{L^1_tL^p}\\ &+ C2^{qs}\int_0^t\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.\end{aligned} \tag{15}$$

Summing over q and using Lemma 2.1 lead for $V(t)\leq C_0$,

$$\begin{aligned}\|\theta\|_{\widetilde L^r_t\dot B^{s+\frac{\alpha}{r}}_{p,1}} &\leq C\|\theta^0\|_{\dot B^s_{p,1}}+C\|f\|_{L^1_t\dot B^s_{p,1}}+C\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|\theta(\tau)\|_{\dot B^s_{p,1}}d\tau\\ &\leq C\|\theta^0\|_{\dot B^s_{p,1}}+C\|f\|_{L^1_t\dot B^s_{p,1}}+CV(t)\|\theta\|_{L^\infty_t\dot B^s_{p,1}}.\end{aligned}$$

Thus we get in view of the estimate of the case $r=\infty$

$$\|\theta\|_{\widetilde{L}^r_t\dot{B}^{s+\frac{\alpha}{r}}_{p,1}}\leq C\|\theta^0\|_{\dot{B}^s_{p,1}}+C\|f\|_{L^1_t\dot{B}^s_{p,1}}. \tag{16}$$

This gives the result for a short time.
For an arbitrary positive time T we make a partition $(T_i)_{i=0}^M$ of the interval $[0,T]$, such that $\int_{T_i}^{T_{i+1}}\|\nabla v(\tau)\|_{L^\infty}d\tau\approx C_0$. Then proceeding for (16), we obtain

$$\|\theta\|_{\widetilde{L}^r_{[T_i,T_{i+1}]}\dot{B}^{s+\frac{\alpha}{r}}_{p,1}}\leq C\|\theta(T_i)\|_{\dot{B}^s_{p,1}}+C\int_{T_i}^{T_{i+1}}\|f(\tau)\|_{\dot{B}^s_{p,1}}d\tau.$$

Applying the triangle inequality gives

$$\|\theta\|_{\widetilde{L}^r_T\dot{B}^{s+\frac{\alpha}{r}}_{p,1}}\leq C\sum_{i=0}^{M-1}\|\theta(T_i)\|_{\dot{B}^s_{p,1}}+C\int_0^T\|f(\tau)\|_{\dot{B}^s_{p,1}}d\tau.$$

On the other hand the estimate proven in the case $r=\infty$ allows us to write

$$\|\theta\|_{\widetilde{L}^r_T\dot{B}^{s+\frac{\alpha}{r}}_{p,1}}\leq CM\big(\|\theta^0\|_{\dot{B}^s_{p,1}}+\|f\|_{L^1_T\dot{B}^s_{p,1}}\big)e^{CV(T)}+\|f\|_{L^1_T\dot{B}^s_{p,1}}.$$

Thus the following observation $C_0M\approx 1+V(t)$ completes the proof of the theorem.

5. Proof of Theorem 1.1

For the sake of a concise presentation, we shall just provide the *a priori* estimates supporting the claims of the theorem. To achieve the proof one must combine in a standard way these estimates with a standard approximation procedure such as the following iterative scheme

$$\begin{cases}\partial_t\theta_{n+1}+v^n\cdot\nabla\theta_{n+1}+|\mathrm{D}|^\alpha\theta_{n+1}=0,\\ v_n=(-R_2\theta_n,R_1\theta_n),\\ \theta_{n+1}(0,x)=S_n\theta^0(x),\\ (\theta_0,v_0)=(0,0).\end{cases}$$

5.1. **Global existence.** It is plain from Theorem 1.2 that to derive global *a priori* estimates it is sufficient to bound globally in time the quantity $V(t):=\|\nabla v\|_{L^1_tL^\infty}$. First, the embedding $\dot{B}^0_{\infty,1}\hookrightarrow L^\infty$ combined with the fact that Riesz transform maps continuously homogeneous Besov space into itself

$$\|\nabla v\|_{L^1_tL^\infty}\leq\|\nabla v\|_{L^1_t\dot{B}^0_{\infty,1}}\leq C\|\theta\|_{L^1_t\dot{B}^1_{\infty,1}}. \tag{17}$$

Combined with Theorem 1.2 this yields

$$V(t)\leq C\|\theta^0\|_{\dot{B}^{1-\alpha}_{\infty,1}}e^{CV(t)}.$$

Since the function V depends continuously in time and $V(0)=0$ then we can deduce that for small initial data V does not blow up, and there exists $C_1,\eta>0$ such that

$$\|\theta^0\|_{\dot B^{1-\alpha}_{\infty,1}}<\eta\Rightarrow\|\nabla v\|_{L^1(\mathbb{R}_+;L^\infty)}\leq C_1\|\theta^0\|_{\dot B^{1-\alpha}_{\infty,1}},\forall t\in\mathbb{R}_+. \tag{18}$$

Let us now show how to derive the *a priori* estimates. Take $s\geq s_c^p:=1+\frac{2}{p}-\alpha$. Then combining Theorem 1.2 with (18) we get

$$\begin{aligned}\|\theta\|_{\widetilde L^\infty_{\mathbb{R}_+}\dot B^s_{p,1}}+\|\theta\|_{L^1_{\mathbb{R}_+}\dot B^{s+\alpha}_{p,1}} &\leq C\|\theta^0\|_{\dot B^s_{p,1}}e^{C\|\theta^0\|_{\dot B^{1-\alpha}_{\infty,1}}}\\ &\leq C\|\theta^0\|_{\dot B^s_{p,1}}.\end{aligned}$$

On the other hand we have from Proposition 6.2

$$\forall t\in\mathbb{R}_+,\ \|\theta(t)\|_{L^p}\leq\|\theta^0\|_{L^p}.$$

Therefore we get an estimate of θ in the inhomogeneous Besov space as follows

$$\|\theta\|_{\widetilde L^\infty_{\mathbb{R}_+}B^s_{p,1}}\leq C\|\theta^0\|_{B^s_{p,1}}.$$

Using again Theorem 1.2 yields

$$\|\theta\|_{\widetilde L^\infty_t\dot B^0_{\infty,1}}\leq C\|\theta^0\|_{\dot B^0_{\infty,1}}e^{CV(t)}\leq C\|\theta^0\|_{\dot B^0_{\infty,1}},$$

Thus we obtain for $p\in[1,\infty]$

$$\|\theta\|_{\widetilde L^\infty_{\mathbb{R}_+}\mathcal{X}^s_p}\leq C\|\theta^0\|_{\mathcal{X}^s_p}. \tag{19}$$

For the velocity we have the following result.

Lemma 5.1. *For $p\in]1,\infty]$ there exists C_p such that*

$$\|v\|_{\widetilde L^\infty_{\mathbb{R}_+}B^s_{p,1}}\leq C_p\|\theta^0\|_{\mathcal{X}^s_p}.$$

However, for $p=1$ we have

$$\|v\|_{\widetilde L^\infty_{\mathbb{R}_+}\dot B^s_{1,1}}+\|v\|_{L^\infty_{\mathbb{R}_+}L^{p_1}}\leq C_{p_1}\|\theta^0\|_{B^s_{1,1}},\ \forall p_1>1.$$

Proof. Let $p\in]1,\infty[$. Then we can write in view of (19)

$$\begin{aligned}\|v\|_{\widetilde L^\infty_{\mathbb{R}_+}B^s_{p,1}} &\leq \|v\|_{\widetilde L^\infty_{\mathbb{R}_+}\dot B^s_{p,1}}+\|\Delta_{-1}v\|_{L^\infty_{\mathbb{R}_+}L^p}\\ &\leq C\|\theta\|_{\widetilde L^\infty_{\mathbb{R}_+}\dot B^s_{p,1}}+C\|v\|_{L^\infty_{\mathbb{R}_+}L^p}\\ &\leq C\|\theta^0\|_{B^s_{p,1}}+C\|v\|_{L^\infty_{\mathbb{R}_+}L^p}.\end{aligned}$$

Combining the boundedness of Riesz transform with the maximum principle

$$\|v\|_{L^\infty_{\mathbb{R}_+}L^p}\leq C_p\|\theta^0\|_{L^p}.$$

Thus we obtain

$$\|v\|_{\widetilde L^\infty_{\mathbb{R}_+}B^s_{p,1}}\leq C_p\|\theta^0\|_{B^s_{p,1}}.$$

To treat the case $p=\infty$ we write according to the embedding $\dot{B}^0_{\infty,1}\hookrightarrow L^\infty$ and the continuity of Riesz transform

$$\|\Delta_{-1}v(t)\|_{L^\infty}\leq C\|\theta(t)\|_{\dot{B}^0_{\infty,1}}.$$

Combining this estimate with (19) yields

$$\|v\|_{\widetilde{L}^\infty_{\mathbb{R}_+}B^s_{\infty,1}}\leq C\|\theta^0\|_{B^s_{\infty,1}\cap\dot{B}^0_{\infty,1}}.$$

Hence we get for all $p\in]1,\infty]$

$$\|v\|_{\widetilde{L}^\infty_{\mathbb{R}_+}B^s_{p,1}}\leq C_p\|\theta^0\|_{\mathcal{X}^s_p}. \tag{20}$$

Let us now move to the case $p=1$. Since $B^s_{1,1}\hookrightarrow L^{p_1}$ for all $p_1\geq 1$ then we get in view of Bernstein's inequality and the maximum principle

$$\|\Delta_{-1}v\|_{L^{p_1}}\leq C_{p_1}\|\theta^0\|_{L^{p_1}}\leq C_{p_1}\|\theta^0\|_{B^s_{1,1}}.$$

We eventually find that $v\in\widetilde{L}^\infty_{\mathbb{R}_+}\dot{B}^s_{1,1}\cap L^\infty_{\mathbb{R}_+}L^{p_1}$.

□

Let us now briefly sketch the proof of the continuity in time, that is $\theta\in C(\mathbb{R}_+;\mathcal{X}^s_p)$. We should only treat the finite case of p and similarly one can show the case $p=\infty$. From the definition of Besov spaces we have

$$\|\theta(t)-\theta(t')\|_{B^s_{p,1}}\leq\sum_{q<N}2^{qs}\|\theta_q(t)-\theta_q(t')\|_{L^p}+2\sum_{q\geq N}2^{qs}\|\theta_q\|_{L^\infty_{\mathbb{R}_+}L^p}$$

Let $\epsilon>0$ then we get from (19) the existence of a number N such that

$$\sum_{q\geq N}2^{qs}\|\theta_q\|_{L^\infty_{\mathbb{R}_+}L^p}\leq\frac{\epsilon}{4}.$$

Thanks to Taylor's formula

$$\begin{aligned}\sum_{q<N}2^{qs}\|\theta_q(t)-\theta_q(t')\|_{L^p} &\leq |t-t'|\sum_{q<N}2^{qs}\|\partial_t\theta_q\|_{L^\infty_{\mathbb{R}_+}L^p}\\ &\leq C|t-t'|2^N\|\partial_t\theta\|_{L^\infty_{\mathbb{R}_+}B^{s-1}_{p,1}}.\end{aligned}$$

To estimate the last term we write

$$\partial_t\theta=-|\mathrm{D}|^\alpha\theta-v\cdot\nabla\theta.$$

In one hand we have $|\mathrm{D}|^\alpha\theta\in B^{s-\alpha}_{p,1}\hookrightarrow B^{s-1}_{p,1}$. On the other hand since the space $B^s_{p,1}$ is an algebra ($s>\frac{2}{p}$) and v is zero divergence then

$$\|v\cdot\nabla\theta\|_{B^{s-1}_{p,1}}\leq C\|v\,\theta\|_{B^s_{p,1}}\leq C\|v\|_{B^s_{p,1}}\|\theta\|_{B^s_{p,1}}.$$

Thus we get $\partial_t\theta\in L^\infty_{\mathbb{R}_+}B^{s-1}_{p,1}$ and this allows us to finish the proof of the continuity.

5.2. **Local existence.** The local time existence depends on the control of the quantity $V(t) := \|\nabla v\|_{L^1_t L^\infty}$. In our analysis we distinguish two cases:

• *First case:* $s > s_c^p = 1 + \frac{2}{p} - \alpha$.

We observe first that there exists $r > 1$ such that $1 + \frac{2}{p} - \frac{\alpha}{r} \leq s$. From (17) and according to the said Hölder's inequality we have

$$\begin{aligned} V(t) &\leq C\|\theta\|_{L^1_t \dot{B}^1_{\infty,1}} \\ &\leq Ct^{\frac{1}{\bar{r}}}\|\theta\|_{L^r_t \dot{B}^1_{\infty,1}}. \end{aligned}$$

Using Theorem (1.2) we obtain

$$V(t) \leq Ct^{\frac{1}{\bar{r}}}\|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} e^{CV(t)}.$$

Thus we conclude that there exists C_0, $\eta > 0$ such that

$$t^{\frac{1}{\bar{r}}}\|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \leq \eta \Rightarrow V(t) \leq C_0, \tag{21}$$

and this gives from Theorem 1.2

$$\|\theta\|_{L^\infty_t B^s_{p,1}} + \|\theta\|_{L^1_t \dot{B}^{1+\frac{2}{p}}_{p,1}} \leq C\|\theta^0\|_{B^s_{p,1}}. \tag{22}$$

We point out that one can deduce from (21) that the time existence is bounded below

$$T \gtrsim \|\theta^0\|^{-\bar{r}}_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}}.$$

• *Second case:* $s = s_c^p = 1 + \frac{2}{p} - \alpha$.

By applying (15) to the (QG_α) equation with $r = 1$, $p = \infty$ and $s = 1 - \alpha$ we have under the condition $V(t) \leq C_0$

$$\|\theta\|_{L^1_t \dot{B}^1_{\infty,1}} \leq C\sum_{q\in\mathbb{Z}}(1 - e^{-ct2^{q\alpha}})2^{q(1-\alpha)}\|\theta_q^0\|_{L^\infty} + C\sum_{q\in\mathbb{Z}} 2^{q(1-\alpha)}\|[\Delta_q, v\cdot\nabla]\theta\|_{L^1_t L^\infty}.$$

The second term of the right-hand side can be estimated from Lemma 2.1 as follows

$$\begin{aligned} \sum_{q\in\mathbb{Z}} 2^{q(1-\alpha)}\|[\Delta_q, v\cdot\nabla]\theta\|_{L^1_t L^\infty} &\leq C\|v\|_{\widetilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}}\|\theta\|_{\widetilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \\ &\leq C\|\theta\|^2_{\widetilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}}. \end{aligned} \tag{23}$$

Notice that we have used in the above inequality the fact that Riesz transform maps continuously homogeneous Besov space into itself. Hence we get

$$\|\theta\|_{L^1_t \dot{B}^1_{\infty,1}} \leq C\sum_{q\in\mathbb{Z}}(1 - e^{-ct2^{q\alpha}})2^{q(1-\alpha)}\|\theta_q^0\|_{L^\infty} + C\|\theta\|^2_{\widetilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}}. \tag{24}$$

Using again (15) with $r=2$, $p=\infty$ and $s=1-\alpha$, we obtain

$$\|\theta\|_{\widetilde{L}^2_t\dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq C\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}+C\sum_{q\in\mathbb{Z}}2^{q(1-\alpha)}\|[\Delta_q, v\cdot\nabla]\theta\|_{L^1_tL^\infty}.$$

Thus (23) yields

$$\|\theta\|_{\widetilde{L}^2_t\dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq C\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}+C\|\theta\|^2_{\widetilde{L}^2_t\dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}}.$$

By Lebesgue theorem we have

$$\lim_{t\to 0^+}\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}=0.$$

Let η be a sufficiently small constant and define

$$T_0:=\sup\Big\{t>0, \sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}\leq\eta\Big\}.$$

Then we have under the assumptions $t\leq T_0$ and $V(t)\leq C_0$

$$\|\theta\|_{\widetilde{L}^2_t\dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq 2C\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}.$$

Inserting this estimate into (24) gives

$$\begin{aligned} V(t)\leq C\|\theta\|_{L^1_t\dot{B}^1_{\infty,1}} &\leq C\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty} \\ &+ C\Big(\sum_{q\in\mathbb{Z}}(1-e^{-ct2^{q\alpha}})^{\frac{1}{2}}2^{q(1-\alpha)}\|\theta^0_q\|_{L^\infty}\Big)^2. \end{aligned}$$

For sufficiently small η we obtain $V(t)<C_0$ and this allows us to prove that the time T_0 is actually a local time existence. Thus we obtain from Theorem 1.2

$$\|\theta\|_{\widetilde{L}^\infty_T B^{s^p_c}_{p,1}}+\|\theta\|_{L^1_T\dot{B}^{1+\frac{2}{p}}_{p,1}}\leq C\|\theta^0\|_{B^{s^p_c}_{p,1}}.$$

5.3. **Uniqueness.** We shall give the proof of the uniqueness result which can be formulated as follows. There exists at most one solution for the system (QG$_\alpha$) in the functions space $X_T:=L^\infty_T\dot{B}^0_{\infty,1}\cap L^1_T\dot{B}^1_{\infty,1}$. We stress out that the space $L^\infty_TX^s_p\cap L^1_T\dot{B}^{s+\alpha}_{p,1}$, with $p\in[1,\infty]$, is continuously embedded in X_T.
Let $\theta^i, i=1,2$ (and v^i the corresponding velocity) be two solutions of the (QG$_\alpha$) equation with the same initial data and belonging to the space X_T. We set $\theta=\theta^1-\theta^2$ and $v=v^1-v^2$, then it is plain that

$$\partial_t\theta+v^1\cdot\nabla\theta+|\mathrm{D}|^\alpha\theta=-v\cdot\nabla\theta^2,\ \theta_{|t=0}=0.$$

Applying Theorem 1.2 to this equation gives

$$\|\theta(t)\|_{\dot B^0_{\infty,1}} \le Ce^{C\|\nabla v^1\|_{L^1_tL^\infty}}\int_0^t \|v\cdot\nabla\theta^2(\tau)\|_{\dot B^0_{\infty,1}}d\tau. \tag{25}$$

We will now make use of the following law product and its proof will be given later.

$$\|v\cdot\nabla\theta^2\|_{\dot B^0_{\infty,1}} \le C\|v\|_{\dot B^0_{\infty,1}}\|\theta^2\|_{\dot B^1_{\infty,1}}. \tag{26}$$

Since Riesz transform maps continuously $\dot B^0_{\infty,1}$ into itself, then we get

$$\|v\cdot\nabla\theta^2\|_{\dot B^0_{\infty,1}} \le C\|\theta\|_{\dot B^0_{\infty,1}}\|\theta^2\|_{\dot B^1_{\infty,1}}.$$

Inserting this estimate into (25) and using Gronwall's inequality give the wanted result.

Let us now turn to the proof of (26) which is based on Bony's decomposition

$$v\cdot\nabla\theta^2 = T_v\nabla\theta^2 + T_{\nabla\theta^2}v + R(v,\nabla\theta^2), \quad \text{with}$$

$$T_v\nabla\theta^2 = \sum_{q\in\mathbb Z}\dot S_{q-1}v\cdot\nabla\dot\Delta_q\theta^2 \quad \text{and} \quad R(v,\nabla\theta^2) = \sum_{\substack{q\in\mathbb Z\\ i\in\{\mp1,0\}}}\dot\Delta_q v\cdot\dot\Delta_{q+i}\nabla\theta^2.$$

Using the quasi-orthogonality of the paraproduct terms one obtains

$$\begin{aligned}\|T_v\nabla\theta^2\|_{\dot B^0_{\infty,1}} &\le C\sum_{q\in\mathbb Z}\|\dot S_{q-1}v\|_{L^\infty}\|\dot\Delta_q\nabla\theta^2\|_{L^\infty}\\ &\le C\|v\|_{\dot B^0_{\infty,1}}\|\theta^2\|_{\dot B^1_{\infty,1}}.\end{aligned}$$

By the same way we get

$$\begin{aligned}\|T_{\nabla\theta^2}v\|_{\dot B^0_{\infty,1}} &\le C\sum_{q\in\mathbb Z}\|\dot S_{q-1}\nabla\theta^2\|_{L^\infty}\|\dot\Delta_q v\|_{L^\infty}\\ &\le C\|\nabla\theta^2\|_{L^\infty}\|v\|_{\dot B^0_{\infty,1}}\\ &\le C\|\theta^2\|_{\dot B^1_{\infty,1}}\|v\|_{\dot B^0_{\infty,1}}.\end{aligned}$$

For the remainder term we write in view of the incompressibility of the velocity and the convolution inequality

$$\begin{aligned}\|R(v,\nabla\theta^2)\|_{\dot B^0_{\infty,1}} &= \sum_{j\in\mathbb Z}\|\dot\Delta_j R(v,\nabla\theta^2)\|_{L^\infty} \le C\sum_{\substack{q\ge j-3\\ i\in\{\mp1,0\}}}2^j\|\dot\Delta_q v\|_{L^\infty}\|\dot\Delta_{q+i}\theta^2\|_{L^\infty}\\ &\le C\sum_{\substack{q\ge j-3\\ i\in\{\mp1,0\}}}2^{j-q}\|\dot\Delta_q v\|_{L^\infty}2^q\|\dot\Delta_{q+i}\theta^2\|_{L^\infty}\\ &\le C\|\nabla\theta^2\|_{L^\infty}\|v\|_{\dot B^0_{\infty,1}}\\ &\le C\|\theta^2\|_{\dot B^1_{\infty,1}}\|v\|_{\dot B^0_{\infty,1}}.\end{aligned}$$

This completes the proof of (26).

6. Appendix

The following result is due to Vishik [19] and was used in a crucial way for the proof of Theorem 1.2. For the convenience of the reader we will give a short proof based on the duality method.

Lemma 6.1. *Let f be a function in Schwartz class and ψ a diffeomorphism preserving Lebesgue measure, then we have for all $p \in [1, +\infty]$ and for all $j, q \in \mathbb{Z}$,*

$$\|\dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} \le C 2^{-|j-q|} \|\nabla \psi^{\epsilon(j,q)}\|_{L^\infty} \|\dot{\Delta}_q f\|_{L^p},$$

with

$$\epsilon(j, q) = sign(j - q).$$

We shall begin with the proof of Lemma 6.1

Proof. We distinguish two cases: $j \ge q$ and $j < q$. For the first one we simply use Bernstein's inequality

$$\|\dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} \lesssim 2^{-j} \|\nabla \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p}.$$

It suffices now to combine Leibnitz formula again with Bernstein's inequality

$$\begin{aligned}\|\nabla \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} &\lesssim \|\nabla \dot{\Delta}_q f\|_{L^p} \|\nabla \psi\|_{L^\infty} \\ &\lesssim 2^q \|\dot{\Delta}_q f\|_{L^p} \|\nabla \psi\|_{L^\infty}.\end{aligned}$$

This yields to the desired inequality. Let us now move to the second case and use the following duality result

$$\|\dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} = \sup_{\|g\|_{L^{\bar{p}}} \le 1} \big|\langle \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi), g\rangle\big|, \text{ with } \quad \frac{1}{p} + \frac{1}{\bar{p}} = 1. \tag{27}$$

Let $\bar{\varphi} \in C_0^\infty(\mathbb{R}^d)$ be supported in a ring and taking value 1 on the ring $\mathcal{C}$ (see the definition of the dyadic decomposition). We set $\bar{\dot{\Delta}}_q f := \bar{\varphi}(2^{-q}\mathrm{D}) f$. Then we can see easily that $\dot{\Delta}_q f = \bar{\dot{\Delta}}_q \dot{\Delta}_q f$. Combining this fact with Parseval's identity and the preserving measure by the flow

$$\big|\langle \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi), g\rangle\big| = \big|\langle \dot{\Delta}_q f, \bar{\dot{\Delta}}_q\big((\dot{\Delta}_j g) \circ \psi^{-1}\big)\rangle\big|.$$

Therefore we obtain

$$\big|\langle \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi), g\rangle\big| \le \|\dot{\Delta}_q f\|_{L^p} \|\bar{\dot{\Delta}}_q\big((\dot{\Delta}_j g) \circ \psi^{-1}\big)\|_{L^{\bar{p}}}.$$

This implies in view of the first case

$$\begin{aligned}\big|\langle \dot{\Delta}_j(\dot{\Delta}_q f \circ \psi), g\rangle\big| &\lesssim \|\dot{\Delta}_q f\|_{L^p} 2^{j-q} \|\nabla \psi^{-1}\|_{L^\infty} \|\dot{\Delta}_j g\|_{L^{\bar{p}}} \\ &\lesssim \|\dot{\Delta}_q f\|_{L^p} 2^{j-q} \|\nabla \psi^{-1}\|_{L^\infty} \|g\|_{L^{\bar{p}}}.\end{aligned}$$

Thus we get in view of (27) the wanted result. □

Next we give a maximum principle estimate for the equation (TD_α) extending a recent result due to [9] for the partial case $f = 0$. The proof uses the same idea and will be briefly described.

Proposition 6.2. *Let v be a smooth divergence free vector field and f be a smooth function. We assume that θ is a smooth solution of the equation*

$$\partial_t\theta + v\cdot\nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = f, \quad \text{with} \quad \kappa \geq 0 \quad \text{and} \quad \alpha \in [0,2].$$

Then for $p \in [1,+\infty]$ we have

$$\|\theta(t)\|_{L^p} \leq \|\theta(0)\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau.$$

Proof. Let $p \geq 2$, then multiplying the equation by $|\theta|^{p-2}\theta$ and integrating by parts lead to

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \kappa\int |\theta|^{p-2}\theta\,|\mathrm{D}|^\alpha\theta dx = \int f|\theta|^{p-2}\theta dx.$$

On the other hand it is shown in [9] that

$$\int |\theta|^{p-2}\theta\,|\mathrm{D}|^\alpha\theta dx \geq 0.$$

Now using Hölder's inequality for the right-hand side

$$\int f|\theta|^{p-2}\theta dx \leq \|f\|_{L^p}\|\theta\|_{L^p}^{p-1}.$$

Thus we obtain

$$\frac{d}{dt}\|\theta(t)\|_{L^p} \leq \|f(t)\|_{L^p}.$$

We can deduce the result by integrating in time. The case $p \in [1,2[$ can be obtained through the duality method. □

Remark 4. *When this paper was finished we had been informed that similar results were obtained by Chen et al* [5]. *In fact they obtained global well-posedness result for small initial data in $\dot{B}^{s_c^p}_{p,q}$, with $p \in [2,\infty[$ and $q \in [1,\infty[$. For the particular case $q = \infty$ our result is more precise. Indeed, first, we can extend their result to $p \in [1,\infty]$ and second our smallness condition is given in the space $\dot{B}^{1-\alpha}_{\infty,1}$ which contains Besov spaces $\{\dot{B}^{s_c^p}_{p,1}\}_{p\in[1,\infty]}$.*

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