GLOBAL WELL-POSEDNESS FOR THE NAVIER-STOKES-BOUSSINESQ SYSTEM WITH AXISYMMETRIC DATA

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Abstract. In this paper we prove the global well-posedness for a three-dimensional Boussinesq system with axisymmetric initial data. This system couples the Navier-Stokes equation with a transport-diffusion equation governing the temperature. Our result holds uniformly with respect to the heat conductivity coefficient $\kappa \geq 0$ which may vanish.

1. Introduction

The Boussinesq system is widely used to model the dynamics of the ocean or the atmosphere. It arises from the density dependent incompressible Navier-Stokes equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This system writes

$$
(1) \qquad \begin{cases} \partial_t v + v\cdot\nabla v - \Delta v + \nabla p = \rho e_z, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \partial_t\rho + v\cdot\nabla\rho = \kappa\Delta\rho,\\ \operatorname{div} v = 0,\\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases}
$$

Here, the velocity $v = (v^1, v^2, v^3)$ is a three-component vector field with zero divergence, the scalar function ρ denotes the density or the temperature and p the pressure of the fluid. The coefficient $\kappa \geq 0$ is a Reynolds number which takes into account the strength of heat conductivity. Note that we have assumed that the viscosity coefficient is one, one can always reduce the problem to this situation by a change of scale (as soon as the fluid is assumed to be viscous) which is not important for global well-posedness issues with data of arbitrary size that we shall consider. The term ρe_z where $e_z = (0, 0, 1)^t$ takes into account the influence of the gravity and the stratification on the motion of the fluid. Note that when the initial density ρ_0 is identically zero (or constant) then the above system reduces to the classical incompressible Navier-Stokes equation:

$$
(2) \qquad \begin{cases} \partial_t v + v\cdot\nabla v - \Delta v + \nabla p = 0\\ \operatorname{div} v = 0\\ v_{|t=0} = v_0. \end{cases}
$$

From this observation, one cannot expect to have a better theory for the Boussinesq system than for the Navier-Stokes equations. The existence of global weak solutions in the energy space for (2) goes back to J. Leray [28]. However the uniqueness of these solutions is only known in space dimension two. It is also well-known that smooth solutions are global in dimension two and for higher dimensions when the data are small in some critical spaces; see for instance [26] for more detailed discussions. In a similar way, the global well-posedness for two-dimensional Boussinesq systems which has recently drawn a lot of attention seems to be in a satisfactory state. More precisely global well-posedness has been shown in various function spaces and for different viscosities, we refer for example to [2, 9, 11, 17, 18, 19, 20, 22, 23, 24]. For three-dimensional systems few results are known about global existence. We can quote the result of R. Danchin and M. Paicu [18] who proved a global well-posedness result for small initial data belonging to some critical Lorentz spaces.

Let us recall that it is still not known if smooth solutions with large initial data for the Navier-Stokes equations can blow-up in finite time in dimension 3. Only some partial results are known. For example, in a recent series of papers [13, 14, 15] global existence in dimension three is established for initial data which are not small in any critical space but which have some special structure (oscillations or slow variations in one direction). Another interesting case of global existence for (2) corresponding to large initial data but with special structure is the more classical case of axisymmetric solutions without swirl. Our aim in this paper is to establish the corresponding global well-posedness result for the three-dimensional Boussinesq system.

Before stating our main result, let us describe the classical result for the Navier-Stokes equation. It is well-known that the control of the vorticity ω which is the vector defined by $\omega = \operatorname{curl} v$ and solving the vorticity equation

$$\partial_t \omega + v \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla v$$

is crucial in order to get global well-posedness results. According to the classical Beale-Kato-Majda criterion [6] the control of the vorticity in $L^1_{loc}(\mathbb{R}_+, L^\infty)$ is sufficient to get the global existence of smooth solutions. The main difficulty arising in dimension three is the lack of information about the influence of the vortex-stretching term $\omega \cdot \nabla v$ on the motion of the fluid. Let us now consider a vector field v which is axisymmetric without swirl, this means that it has the form:

$$v(t,x) = v^r(t,r,z)e_r + v^z(t,r,z)e_z, \quad x = (x_1, x_2, z), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}}, \tag{3}$$

where $\big(e_r, e_\theta, e_z\big)$ is the local orthonormal basis of $\mathbb{R}^3$ corresponding to cylindrical coordinates. Note that we assume that the velocity is invariant by rotation around the vertical axis (axisymmetric flow) and that the component v^θ of v about e_θ identically vanishes (without swirl). For these flows, we have:

$$\omega = (\partial_z v^r - \partial_r v^z) e_\theta := \omega_\theta e_\theta, \quad \omega \cdot \nabla v = \frac{v^r}{r}\omega.$$

In particular ω_θ satisfies the equation

$$\partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \Delta \omega_\theta + \frac{\omega_\theta}{r^2} = \frac{v^r}{r}\omega_\theta. \tag{4}$$

The crucial fact is then that the quantity $\zeta := \frac{\omega_\theta}{r}$ evolves according to the equation

$$\partial_t \zeta + v \cdot \nabla \zeta - \big(\Delta + \frac{2}{r}\partial_r\big)\zeta = 0$$

from which we get that for all $p \in [1, \infty]$

$$\|\zeta(t)\|_{L^p} \leq \|\zeta_0\|_{L^p}.$$

It was shown by M. Ukhoviskii and V. Yudovich [30] and independently by O. A. Ladyzhenskaya [25] that these new a priori estimates are strong enough to prevent the formation of singularities in finite time for axisymmetric flows without swirl: the system (2) has a unique global solution for $v_0 \in H^1$ such that $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$. We point out that the result is

uniform with respect to vanishing viscosity and thus there is no blowup even for the Euler equation. Note that in term of Sobolev regularities these assumptions are satisfied when $v_0 \in H^s$ with $s > \frac{7}{2}$. This regularity assumption is not optimal and has been weakened in [27] for $v_0 \in H^2$ and more recently in [1] for $v_0 \in H^{\frac{1}{2}}$.
Our aim here is to extend these classical results for the Navier-Stokes equation to Boussinesq systems. The equation for $\zeta = \omega_\theta / r$ becomes

$$\partial_t \zeta + v \cdot \nabla \zeta - \big(\Delta\Gamma + \frac{2}{r}\partial_r\big)\zeta = -\frac{\partial_r \rho}{r} \tag{5}$$

and thus the difficulty is to use some a priori estimates on ρ to control the term in the right-hand side of (5). The rough idea is that on the axis $r = 0$ the singularity $\frac{1}{r}$ scales as a derivative and hence that the forcing term $\partial_r \rho / r$ can be thought as a Laplacian of ρ and thus one may try to use smoothing effects to control it. Nevetheless, when $\kappa = 0$, since there is no smoothing effects on ρ, one can hope to compensate the loss of derivatives in the right hand-side of (5) only by using the full smoothing effects of the heat type equation in the left hand side. Note that this kind of estimates does not follow from energy estimates and hence the problem that one has to face is that the convection term that has to be handled in the process is not negligible: this approach naturally leads to some restriction on the size of the data.
In [3], a global existence result for the system (1), with $\kappa = 0$, was established but under some restrictive conditions on the support of the initial density namely that it does not intersect the axis $r = 0$. More precisely,

Theorem 1.1. *Let $v_0 \in H^1$ be an axisymmetric divergence free vector field without swirl and such that $\frac{\omega_0}{r} \in L^2$. Let $\rho_0 \in L^2 \cap L^\infty$ axisymmetric and such that supp ρ_0 does not intersect the axis (Oz) and $\Pi_z(\text{supp } \rho_0)$ is a compact set. Then the system (1), with $\kappa = 0$, has a unique global solution (v, ρ) such that*

$$v \in C(\mathbb{R}_+; H^1) \cap L^1_{\text{loc}}(\mathbb{R}_+; W^{1,\infty}),$$
$$\frac{\omega}{r} \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2), \quad \rho \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2 \cap L^\infty).$$

Here Π_z denotes the orthogonal projector over (Oz).

Since to bound the quantity $\|\omega/r\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}$ one needs to estimate $\|\rho/r(t)\|_{L^2}$. The idea of the proof was to get an estimate from below for the distance of the support of ρ to the vertical axis (note that this distance remains positive as long as the solution remains smooth because of the assumption on the initial density). Note that for this approach, it is crucial to have a transport equation for the density, it fails for $\kappa > 0$ because ρ cannot be supported away from the axis even if the initial data is.
In this paper, by using a different approach, which uses more deeply the structure of the coupling between the two equations of (1), we remove the assumption on the support of the density and we give a global well-posedness result with uniform bounds with respect to the heat conductivity κ. Our main result reads as follows.

Theorem 1.2. *Consider the Boussinesq system (1) for $\kappa \geq 0$. Let $v_0 \in H^1$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and let $\rho_0 \in L^2 \cap B^0_{3,1}$ an axisymmetric function. Then there is a unique global solution (v, ρ) such that*

$$v \in \mathcal{C}(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^2) \cap L^1_{\text{loc}}(\mathbb{R}_+; B^1_{\infty,1}),$$

$$\frac{\omega}{r} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2), \quad \rho \in \mathcal{C}(\mathbb{R}_+; L^2 \cap B^0_{3,1}).$$

Moreover the estimates in the above spaces are uniform for $\kappa \geq 0$ in any bounded set of $\mathbb{R}_+$.

The definition of the Besov spaces $B^s_{p,q}$ is recalled below.
Let us give a few comments about our result.

Remark 1.3. We find at most an exponential growth for the velocity: there exists $C_0 > 0$ depending only on the data such that for every $\kappa \in [0,1]$, we have

$$\|v\|_{L^\infty_t H^1 \cap L^2_t H^2} \leq C_0 e^{C_0 t}.$$

Remark 1.4. The uniformity of the norms with respect to the conductivity κ can be established for every $\kappa \in [0, A]$ for every $A > 0$. Nevertheless, as we shall see below, we need to use a different transformation of the equation when κ is close to one.

Remark 1.5. The result of the theorem remains true if we change the Besov space $B^0_{3,1}$ for the Lebesgue space L^m for $m > 3$. We have not tried to get the best result in terms of the regularity of the velocity. At the price of more technicality, it is probably possible as in [1] to get the same result by assuming only that $v_0 \in H^{\frac{1}{2}}$.

Remark 1.6. By using the control of ∇v in $B^0_{\infty,1}$ which is given by Theorem 1.2, on can easily propagate by classical arguments higher order regularity for example higher H^s Sobolev regularity.

Let us explain our strategy for the proof. The crucial part in the proof consists now in finding suitable a priori estimates for (ζ, ρ). Let us describe the idea in the case that $\kappa = 0$ (this is the one that one needs to understand in priority because of the lack of smoothing effect). As we have already noticed the coupling between the two equations does not make the original Boussinesq system (1) well-suited for a priori estimates. Since the right hand side of (5) behaves roughly as a Laplacian, we need to fully use the left hand side to control it. The main idea is to use an approach related to the one used for the study of two-dimensional systems with a critical dissipation, see [23, 24]. It consists in diagonalizing the linear part of the system satisfied by ζ and ρ. We introduce a new unknown Γ which formally reads

$$\Gamma = \zeta - \big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\frac{\partial_r}{r}\rho := \zeta - \mathcal{L}\rho$$

and we study the system satisfied by (Γ, ρ) which is given by:

$$\partial_t \Gamma + v \cdot \nabla \Gamma - \big(\Delta + \frac{2}{r}\partial_r\big)\Gamma = -\big[\mathcal{L}, v \cdot \nabla \rho\big], \quad \partial_t \rho + v \cdot \nabla \rho = 0$$

where $\big[\mathcal{L}, v \cdot \nabla\big]$ is the commutator defined by

$$[\mathcal{L}, v \cdot \nabla] f = \mathcal{L}(v \cdot \nabla f) - v \cdot \nabla \mathcal{L} f.$$

We can thus get a priori estimates for Γ and ρ (note that they are obvious if we negect the commutator) and then use them to deduce estimates for ζ. The main difficulties that one has to deal with are twofold. The first one is the study of the operator $\mathcal{L}$, to make this argument rigorous, we need to prove that $\mathcal{L}$ is well-defined and is a bounded operator on L^p. The second one is the study of the commutator. Once estimates on ζ are obtained, the situation gets close to the standard axisymmetric Navier-Stokes equations. When κ is non zero, it turns out that the same kind of transformation can be used and that it depends

smoothly on κ. The new unknown $\Gamma_\kappa = (1-\kappa)\zeta - \mathcal{L}\rho$ solves the same convection diffusion equation as Γ. This is due to a surprising commutation property between Δ and $\mathcal{L}$ which is stated in Lemma 3.4. Consequently, we are again able to obtain and estimate for ζ from an estimate for Γ_κ as soon as $\kappa \neq 1$. The case that κ is close to one (which is easier since there is a non vanishing smoothing effect in the density equation) can be handled by using a different transformation.
The paper is organized as follows. In section 2 we fix the notations, give the definitions of the functional spaces that we shall use and state some useful inequalities. Next, in section 3, we study the operator $\mathcal{L}$, this amounts to study an elliptic equation with singular coefficients. In section 4, we obtain a priori estimates for sufficiently smooth solutions of (1), they are obtained by using the procedure that we have just described. Finally, in section 5, we give the proof of Theorem 1.2: we obtain the existence part by using the a apriori estimates and an approximation argument and then we prove the uniqueness part.

2. Preliminaries

Throughout this paper, C stands for some real positive constant which may be different in each occurrence and C_0 for a positive constant depending on the initial data. Moreover, both are assumed to be independent of κ for $\kappa \geq 0$ in a bounded set. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of the type $X \leq CY$.
For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^3)$ the standard Sobolev spaces: u belongs to H^s if u is a tempered distribution and

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < \infty.$$

We shall also use the homogeneous version $\dot{H}^s(\mathbb{R}^3)$: for $s < 3/2$, $u \in \dot{H}^s$ if $u \in L^1_{loc}$ and

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

Now to introduce Besov spaces which are a generalization of Sobolev spaces we need to recall the dyadic decomposition of the whole space (see [12]).

Proposition 2.1. *There exist two positive radial functions $\chi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \backslash \{0\})$ such that*

(1) $\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \frac{1}{3} \leq \chi^2(\xi) + \sum_{q \in \mathbb{N}} \varphi^2(2^{-q}\xi) \leq 1 \ \ \forall \xi \in \mathbb{R}^3,$

(2) $\operatorname{supp} \varphi(2^{-p}\cdot) \cap \operatorname{supp} \varphi(2^{-q}\cdot) = \varnothing$, *if* $|p-q| \geq 2$,

(3) $q \geq 1 \Rightarrow \operatorname{supp}\chi \cap \operatorname{supp} \varphi(2^{-q}) = \varnothing$.

For every $u \in \mathcal{S}'(\mathbb{R}^3)$ we define the nonhomogeneous Littlewood-Paley operators by,

$$\Delta_{-1} u = \chi(\mathrm{D})u;\ \forall q \in \mathbb{N},\ \Delta_q u = \varphi(2^{-q}\mathrm{D})u \quad \text{and} \quad S_q u = \sum_{-1 \leq j \leq q-1} \Delta_j u.$$

One can easily prove that for every tempered distribution u, we have

$$u = \sum_{q \geq -1} \Delta_q u. \tag{6}$$

In the sequel we will frequently use Bernstein inequalities (see for example [12]).

Lemma 2.2. *There exists a constant C such that for $k \in \mathbb{N}$, $1 \le a \le b$ and $u \in L^a$, we have*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \;\; \le \;\; C^k\, 2^{q(k+3(\frac{1}{a}-\frac{1}{b}))} \|S_q u\|_{L^a},$$

and for $q \in \mathbb{N}$

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \;\; \le \;\; \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \le C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

Let $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, then the Besov space $B^s_{p,r}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p} \Big)_{\ell^r} < +\infty.$$

We remark that the Sobolev space H^s agrees with the Besov space $B^s_{2,2}$. Also, a straightforward consequence of the Bernstein inequalities is the following continous embedding:

$$B^s_{p_1,r_1} \hookrightarrow B^{s+3(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}, \qquad p_1 \le p_2 \quad \text{and} \quad r_1 \le r_2. \tag{7}$$

For any Banach space X with norm $\|\cdot\|_X$ and fonctions $f(t,x)$ such that for every t, $f(t,\cdot) \in X$, we shall use the notation $\|f\|_{L^p_t X} = \| \|f\|_X \|_{L^p([0,T])}$.
A useful application of Besov spaces is the following logarithmic estimate for convection diffusion equations.

Proposition 2.3. *There exists $C > 0$ such that for every $\kappa \ge 0$, $p \in [1,\infty]$ and for every ρ solution of*

$$(\partial_t + v \cdot \nabla - \kappa \Delta)\rho = f, \quad \rho(0,x) = \rho_0(x)$$

with v a divergence free vector field, the following estimate holds true

$$\|\rho(t)\|_{B^0_{p,1}} \le C\Big(\|\rho_0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}} \Big)\Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \Big), \quad \forall t \ge 0.$$

We refer to [21] for the proof. Note that the amplification factor is only linear in $\|\nabla v\|_{L^\infty}$.

3. About an elliptic problem

The aim of this section is the study of the operator $\mathcal{L} = (\Delta + \frac{2}{r})^{-1} \frac{\partial_r}{r}$. This is the heart of the paper since this is crucial to make rigorous the argument sketched in the introduction. This amounts to study the regularity of the solution of an elliptic equation with singular coefficients. This is the goal of the following proposition.

Proposition 3.1. *Let $\rho \in H^2(\mathbb{R}^3)$ axisymmetric, then there exists a unique axisymmetric solution $f \in H^2$ of the elliptic problem*

$$\Big(\Delta + \frac{2}{r} \partial_r \Big) f = \frac{\partial_r \rho}{r}. \tag{8}$$

Moreover, for every $p \in [2,+\infty)$, there exists an absolute constant $C_p > 0$ such that:

$$\|f\|_{L^p} \le C_p \|\rho\|_{L^p}. \tag{9}$$

The important fact in this proposition is the L^p estimate (9) which only involves the L^p norm of ρ. An immediate consequence of Proposition 3.1 is that $\mathcal{L}$ defines a bounded operator on L^p for every $p \in [2, +\infty)$. The additional H^2 regularity is used to give a meaning to the equation. Because of the $1/r$ singularity on the axis, we cannot give a meaning to the term $\partial_r f/r$ as a distribution when f is merely L^p. When f is in H^2, there is no problem we have that $\partial_r f/r \in L^1_{loc}$ since for every compact set $K \subset \mathbb{R}^3$

$$\|\frac{\partial_r f}{r}\|_{L^1(K)} \leq \|\nabla f\|_{L^6} \, \|\frac{1}{r}\|_{L^{\frac{6}{5}}(K)} < +\infty$$

thanks to the Sobolev embedding $H^1 \subset L^6$ in dimension 3.

Proof. Let us first prove the existence of a solution satisfying the required properties. We can first assume that $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^3)$ and then conclude by density. Since the elliptic operator has some singular coefficients, we shall use an approximation argument. Since we have by definition that $r\partial_r = x_h \cdot \nabla$ with the notation $x_h = (x_1, x_2, 0)$, we shall consider for $\varepsilon > 0$ the elliptic problem

$$\big(\Delta + \frac{2}{r^2+\varepsilon} x_h \cdot \nabla\big) f = \frac{1}{r^2+\varepsilon} x_h \cdot \nabla \rho. \tag{10}$$

Since the coefficients are not singular any more, there is a unique solution f^ε for this problem given by the classical methods. By standard regularity arguments, this solution is in the Schwartz class and hence the following a priori estimates are justified. Moreover, since ρ is axisymmetric, f^ε is also axisymmetric.
We shall first prove that the solution f^ε of (10) satisfies the estimate (9) with a constant independent of ε. In the proof of the a priori estimate, we shall denote f^ε by f for notational convenience. By taking the $L^2(\mathbb{R}^3)$ scalar product of (10) with $(r^2+\varepsilon)|f|^{p-1}\mathrm{sign}(f)$, we find

$$\begin{aligned} &\int_{\mathbb{R}^3} \Delta f \, |f|^{p-1}\mathrm{sign}(f)(r^2+\varepsilon)\, dx + 2\int_{\mathbb{R}^3} (x_h \cdot \nabla f)\, |f|^{p-1}\mathrm{sign}(f)\, dx \\ &= \int_{\mathbb{R}^3} x_h \cdot \nabla \rho \, |f|^{p-1}\mathrm{sign}(f)\, dx. \end{aligned} \tag{11}$$

For the first term in the left-hand side, an integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^3} \Delta f \, |f|^{p-1}\mathrm{sign}(f)\, (r^2+\varepsilon) dx &= -(p-1)\int_{\mathbb{R}^3} |f|^{p-2}\, |\nabla f|^2\, (r^2+\varepsilon) dx \\ &\quad -2\int_{\mathbb{R}^3} |f|^{p-1}\mathrm{sign}(f)\, x_h \cdot \nabla f \, dx. \end{aligned}$$

Consequently, we get that

$$\begin{aligned} &\int_{\mathbb{R}^3} \Delta f \, |f|^{p-1}\mathrm{sign}(f)\, dx + 2\int_{\mathbb{R}^3} (x_h \cdot \nabla f)\, |f|^{p-1}\mathrm{sign}(f)\, dx \\ &= -(p-1)\int_{\mathbb{R}^3} |f|^{p-2}\, |\nabla f|^2\, (r^2+\varepsilon) dx. \end{aligned} \tag{12}$$

For the right-hand side of (11), we also obtain from an integration by parts that

$$\begin{aligned}\int_{\mathbb{R}^3} x_h\cdot\nabla\rho\,|f|^{p-1}\mathrm{sign}(f)\,dx &= -2\int_{\mathbb{R}^3}\rho|f|^{p-1}\mathrm{sign}(f)\,dx\\ &\quad -(p-1)\int_{\mathbb{R}^3}\rho\,x_h\cdot\nabla f\ |f|^{p-2}\,dx.\end{aligned}$$

By using the Hölder inequality, this yields

$$\Big|\int_{\mathbb{R}^3} x_h\cdot\nabla\rho\,|f|^{p-1}\mathrm{sign}(f)\,dx\Big| \leq 2\,\|\rho\|_{L^p}\,\|f\|_{L^p}^{p-1}+(p-1)\|f\|_{L^p}^{\frac{p-2}{2}}\,\|\rho\|_{L^p}\Big|\int_{\mathbb{R}^3}|\nabla_h f|^2\,|f|^{p-2}\,r^2\,dx\Big|^{\frac{1}{2}}$$

where $\nabla_h=(\partial_1,\partial_2)^t$. By using this last estimate, (11) and (12), we obtain that for every $\varepsilon>0$

$$\begin{aligned}(p-1)\int_{\mathbb{R}^3}|f|^{p-2}\,|\nabla f|^2\,r^2\,dx &\leq 2\,\|\rho\|_{L^p}\,\|f\|_{L^p}^{p-1}\\ &\quad +(p-1)\|f\|_{L^p}^{\frac{p-2}{2}}\,\|\rho\|_{L^p}\Big|\int_{\mathbb{R}^3}|\nabla_h f|^2\,|f|^{p-2}\,r^2\,dx\Big|^{\frac{1}{2}}.\end{aligned}$$

Consequently, by using the Young inequality, we get that

$$\int_{\mathbb{R}^3}|f|^{p-2}\,|\nabla f|^2\,r^2\,dx\leq C\Big(\|\rho\|_{L^p}\,\|f\|_{L^p}^{p-1}+\|\rho\|_{L^p}^2\,\|f\|_{L^p}^{p-2}\Big) \tag{13}$$

for some $C>0$ independent of ε. To conclude, we can use the following inequality: for $p\in[2,\infty[$, we have

$$\|f\|_{L^p}^p\leq\frac{p^2}{4}\int|\nabla_h f|^2|f|^{p-2}\,r^2dx. \tag{14}$$

This is a special case of Caffarelli-Kohn-Nirenberg inequality [10] (we shall recall the proof will be given in the end of the proof of the proposition). From (13) and (14), we obain that

$$\|f\|_{L^p}^2\leq C\Big(\|\rho\|_{L^p}\,\|f\|_{L^p}+\|\rho\|_{L^p}^2\Big)$$

where C is independent of ε and and thus we obtain the estimate (9) for the solution of (10) by using the Young inequality.

When ε goes to zero, we get the existence of $f\in L^p$ and a subsequence ε_n such that the solution f^{ε_n} of (10) converges weakly to f which satisfies the desired estimate

$$\|f\|_{L^p}\leq C_p\|\rho\|_{L^p}.$$

In order to get that f solves the equation (8), we need more information on f^ε. Indeed, the difficulty is to give a meaning to the term $\partial_r f/r$ which is not well-defined as a distribution when f is merely L^p. We shall use a uniform H^2 estimate for the solution of (10) but which also involves the H^2 norm of ρ. This is why we have required more regularity on ρ. At first, let us notice that since we assume that ρ is in L^2, we have that f^ε is uniformly bounded in L^2. Next, we multiply (10) by Δf, we get that

$$\int_{\mathbb{R}^3}|\Delta f|^2\,dx+2\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla f}{r^2+\varepsilon}\Delta f\,dx=\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla\rho}{r^2+\varepsilon}\,\Delta f\,dx. \tag{15}$$

The right-hand side can be estimated by

$$\Big|\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla\rho}{r^2+\varepsilon}\,\Delta f\,dx\Big| \;\;\le\;\; \|\frac{\partial_r\rho}{r}\|_{L^2}\|\Delta f\|_{L^2}.$$

Since we assume that ρ is axisymmetric, we can use that

$$\frac{\partial_r\rho}{r}=\Delta\rho-\partial_{zz}^2-\partial_r^2\rho=\frac{x_2^2}{r^2}\partial_1^2\rho+\frac{x_1^2}{r^2}\partial_2^2\rho-2\frac{x_1x_2}{r^2}\partial_{12}^2\rho$$

and hence that we have the estimate

$$\|\frac{\partial_r\rho}{r}\|_{L^2}\le 4\|\rho\|_{H^2}.$$

This yields

$$\Big|\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla\rho}{r^2+\varepsilon}\,\Delta f\,dx\Big|\le 4\|\rho\|_{H^2}\|\Delta f\|_{L^2}. \tag{16}$$

Next, we can study the second term in the left-hand side of (15). We have by integration by parts that

$$\begin{aligned}2\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla f}{r^2+\varepsilon}\Delta f\,dx &= -\int_{\mathbb{R}^3}\frac{x_h}{r^2+\varepsilon}\cdot\nabla\big(|\nabla f|^2\big)-2\int_{\mathbb{R}^3}\Big(\nabla\big(\frac{x_h}{r^2+\varepsilon}\big)\cdot\nabla\Big)f\cdot\nabla f\\ &= \int_{\mathbb{R}^3}\nabla\cdot\big(\frac{x_h}{r^2+\varepsilon}\big)\,|\nabla f|^2-2\int_{\mathbb{R}^3}\Big(\nabla\big(\frac{x_h}{r^2+\varepsilon}\big)\cdot\nabla\Big)f\cdot\nabla f dx.\end{aligned}$$

Next, we infer

$$\nabla\cdot\big(\frac{x_h}{r^2+\varepsilon}\big)=\frac{2\varepsilon}{(r^2+\varepsilon)^2}$$

and since f is axisymmetric

$$\Big(\nabla\big(\frac{x_h}{r^2+\varepsilon}\big)\cdot\nabla\Big)f\cdot\nabla f=\Big(\frac{1}{r^2+\varepsilon}-\frac{2r^2}{(r^2+\varepsilon)^2}\Big)|\partial_r f|^2.$$

This yields

$$2\int_{\mathbb{R}^3}\frac{x_h\cdot\nabla f}{r^2+\varepsilon}\Delta f\,dx=2\int_{\mathbb{R}^3}\frac{r^2}{(r^2+\varepsilon)^2}|\partial_r f|^2\,dx\ge 0. \tag{17}$$

Consequently, we get from (15), (16) and (17) that

$$\|\Delta f\|_{L^2}\le 4\|\rho\|_{H^2}$$

and hence that

$$\|f\|_{H^2}\le C\|\rho\|_{H^2} \tag{18}$$

with C independent of ε. From this uniform H^2 estimate for f^ε, we get that f^{ε_n} (up to a subsequence not relabelled) converges weakly in H^2 to some $f\in H^2$ and then that f is a weak solution of (8). This ends the proof of the existence of a solution. The uniqueness is a consequence of the standard energy inequality.

Let us now come back to the proof of (14). Since $\mathrm{div}(x_h) = 2$, by integrating by parts, we obtain

$$\begin{aligned} 2\|f\|_{L^p}^p &= -\int_{\mathbb{R}^3} x_h \cdot \nabla_h(|f|^p)dx \\ &= -p\int_{\mathbb{R}^3} (r\partial_r f)|f|^{p-1}\mathrm{sign}(f)dx \\ &\leq p\Big(\int_{\mathbb{R}^3} |r\partial_r f|^2|f|^{p-2}dx\Big)^{\frac{1}{2}}\|f\|_{L^p}^{\frac{p}{2}}. \end{aligned}$$

Therefore we find the desired estimate,

$$\|f\|_{L^p}^p \leq \frac{p^2}{4}\int_{\mathbb{R}^3} |r\partial_r f|^2|f|^{p-2}dx.$$

This ends the proof of Proposition 3.1.

□

In the proof of the main result, we shall also need to use the operator $\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\partial_z/r$. The aim of the following proposition is to define rigorously this operator.

Proposition 3.2. *Let $\rho \in L^2(\mathbb{R}^3)$ be axisymmetric such that $\partial_z\rho/r \in L^2(\mathbb{R}^3)$, then there exists a unique axisymmetric solution $f \in H^2$ of the elliptic problem*

$$\Big(\Delta + \frac{2}{r}\partial_r\Big)f = \frac{\partial_z\rho}{r}. \tag{19}$$

Moreover, for every $p \in [2, +\infty)$, there exists an absolute constant $C_p > 0$ such that:

$$\|f\|_{L^p} \leq C_p\|\rho\|_{L^p}. \tag{20}$$

Again, the important fact is the estimate (20) which only involves the L^p norm of ρ. From this estimate, we get that the operator $\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\partial_z/r$ is a bounded operator on L^p. The additional regularity $\rho \in L^2$, $\partial_z\rho/r \in L^2$ is again only used to get the H^2 regularity on the solution which allows to give a meaning to the equation.
Note that the assumptions on ρ here are different from the one of Proposition 3.1. This comes from the fact that we shall need to use $\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\frac{\partial_r}{r}\rho$ when ρ is a smooth solution of (1) whereas, we shall only need to use $\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\frac{\partial_z}{r}\big(v^r\rho\big)$ where (v, ρ) is a smooth solution of (1). In the first case, as we have seen in the proof, the regularity on ρ ensures that $\partial_r\rho/r$ is in L^2. In a similar way, in the second case for a smooth solution of (1) such that $\omega/r \in L^2$, we indeed have that $\frac{\partial_z}{r}(v^r\rho) \in L^2$.

Proof. The proof follows the same lines as the proof of Proposition 3.1. Consequently, we shall just indicate the main difference. We now consider the regularized problem

$$(\Delta + \frac{2}{r^2+\varepsilon}x_h \cdot \nabla_h)f = \frac{\partial_z\rho}{\sqrt{r^2+\varepsilon}}. \tag{21}$$

By multiplying the equation with $(r^2+\varepsilon)|f|^{p-1}\mathrm{sign}(f)$, we get by using again (12) and an integration by parts for the right-hand side that

$$\int_{\mathbb{R}^3} |f|^{p-2}\,|\nabla f|^2(r^2+\varepsilon)\,dx \leq \int_{\mathbb{R}^3} \sqrt{r^2+\varepsilon}\,|\rho|\,|f|^{p-2}\,|\partial_z f|\,dx.$$

From the Hölder inequality, we obtain that

$$\Big(\int_{\mathbb{R}^3} |f|^{p-2}\,|\nabla f|^2 r^2\,dx\Big)^{\frac{1}{2}} \leq \|\rho\|_{L^p}\,\|f\|_{L^p}^{\frac{p-2}{2}}$$

and hence by using again (14), we finally get that

$$\|f\|_{L^p} \leq C_p\|\rho\|_{L^p}.$$

This will give the estimate (20) by passing to the limit.
To prove an H^2 estimate on f, we again multiply (21) by Δf, since f is axisymmetric, we get from (17) that

$$\|\Delta f\|_{L^2} \leq \|\frac{\partial_z\rho}{r}\|_{L^2}$$

and this provides the H^2 estimate for f. We can then pass to the limit to get a solution of (19) with the claimed properties. The uniqueness follows from the classical energy estimate. This ends the proof of Proposition 3.2.

□

The aim of the next two lemmas is to prove some identities involving the operator $\mathcal{L} = \big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\frac{\partial_r}{r}$ which will be useful to get the equation satisfied by $\mathcal{L}\rho$ in order to diagonalize the system.
At first, we have

Lemma 3.3. *For any smooth axisymmetric function f we have the identity :*

$$\mathcal{L}\,\partial_r f = \frac{f}{r} - \mathcal{L}(\frac{f}{r}) - \partial_z\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}\frac{\partial_z f}{r}.$$

Proof. We first obtain that

$$\begin{aligned}
\frac{1}{r}\partial_{rr}f &= \partial_r(\frac{1}{r}\partial_r f) + \frac{1}{r^2}\partial_r f \\
&= \partial_r\big(\partial_r(\frac{f}{r}) + \frac{f}{r^2}\big) + \frac{1}{r^2}\partial_r f \\
&= \partial_{rr}(\frac{f}{r}) + \frac{2}{r^2}\partial_r f - \frac{2}{r^3}f \\
&= \big(\partial_{rr} + \frac{2}{r}\partial_r\big)(\frac{f}{r}) \\
&= \big(\Delta + \frac{2}{r}\partial_r\big)(\frac{f}{r}) - \frac{1}{r}\partial_r(\frac{f}{r}) - \partial_{zz}(\frac{f}{r}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\frac{1}{r}\partial_{rr}\,f) &= \frac{f}{r} - \big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\frac{1}{r}\partial_r(\frac{f}{r})) - \big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\partial_{zz}(\frac{f}{r})) \\
&= \frac{f}{r} - \mathcal{L}(\frac{f}{r}) - \partial_z\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\frac{\partial_z f}{r}).
\end{aligned}$$

Note that we have used the fact that the operators $\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}$ and ∂_z commute since the coefficients of the operator $\Delta + \frac{2}{r}\partial_r$ do not depend on the variable z. This is the desired identity and therefore, this ends the proof of Lemma 3.3. □

We shall also use the following:

Lemma 3.4. *For every smooth axisymmetric function ρ, we have the identity:*

$$\mathcal{L}\,\Delta\rho = \big(\Delta + \frac{2}{r}\partial_r\big)\,\mathcal{L}\rho.$$

Proof. At first, by direct computations, we find that

$$\big[\Delta, \frac{1}{r}\partial_r\big] = -2\big(\frac{1}{r^2}\partial_r^2 - \frac{1}{r^3}\partial_r\big) = -\frac{2}{r}\partial_r\big(\frac{1}{r}\partial_r\,\cdot\,\big). \tag{22}$$

Now, let us consider $f = \mathcal{L}\,\rho$. By definition, f solves the elliptic equation

$$\big(\Delta + \frac{2}{r}\partial_r\big)f = \frac{1}{r}\partial_r\rho.$$

Consequently, we get that $u = \big(\Delta + \frac{2}{r}\partial_r\big)f$ solves the elliptic equation:

$$\big(\Delta + \frac{2}{r}\partial_r\big)u = \big(\Delta + \frac{2}{r}\partial_r\big)\big(\frac{1}{r}\partial_r\rho\big).$$

By using the formula (22), we obtain for the right-hand side

$$\big(\Delta + \frac{2}{r}\partial_r\big)\big(\frac{1}{r}\partial_r\rho\big) = \frac{1}{r}\partial_r\big(\Delta\rho + \frac{2}{r}\partial_r\rho\big) + \big[\Delta, \frac{1}{r}\partial_r\big]\rho = \frac{1}{r}\partial_r\big(\Delta\rho + \frac{2}{r}\partial_r\rho - \frac{2}{r}\partial_r\rho\big) = \frac{1}{r}\partial_r\Delta\rho.$$

This proves that u solves the equation

$$\big(\Delta + \frac{2}{r}\partial_r\big)u = \frac{1}{r}\partial_r\Delta\rho$$

and hence that $u = \mathcal{L}\Delta\rho$. Since $u = \big(\Delta + \frac{2}{r}\partial_r\big)f = \big(\Delta + \frac{2}{r}\partial_r\big)\mathcal{L}\,\rho$, this ends the proof of Lemma 3.4.

□

4. A priori estimates

This section is devoted to the a priori estimates needed for the proof of Theorem 1.2. We shall prove two results: the first one deals with some basic energy estimates. The second one which is more difficult deals with the control of some stronger norms.

Proposition 4.1. *Let (v, ρ) be a smooth solution of (1) then*

(1) *for $p \in [1, \infty]$ and $t \in \mathbb{R}_+$, we have*

$$\|\rho(t)\|_{L^p} \le \|\rho_0\|_{L^p},$$

(2) *for $v_0 \in L^2, \rho_0 \in L^2$ and $t \in \mathbb{R}_+$ we have*

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \le C_0(1 + t^2),$$

where C_0 depends only on $\|v_0\|_{L^2}$ and $\|\rho_0\|_{L^2}$.

Note that the axisymmetric assumption is not needed in this proposition

Proof. The first estimate is classical for convection diffusion equations with a divergence free vector field. For $p = \infty$ it is just the maximum principle, while for finite p, it is a consequence of the fact that

$$\big(\partial_t + v\cdot\nabla - \kappa\Delta\big)|\rho|^p \le 0.$$

For the second one we take the $L^2(\mathbb{R}^3)$ scalar product of the velocity equation with v. From integration by parts and the fact that v is divergence free, we get

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \le \|v(t)\|_{L^2}\|\rho(t)\|_{L^2}. \tag{23}$$

This yields

$$\frac{d}{dt}\|v(t)\|_{L^2} \le \|\rho(t)\|_{L^2}.$$

By integration in time, we find that

$$\|v(t)\|_{L^2} \le \|v_0\|_{L^2} + \int_0^t \|\rho(\tau)\|_{L^2} d\tau.$$

Since $\|\rho(t)\|_{L^2} \le \|\rho_0\|_{L^2}$, we infer

$$\|v(t)\|_{L^2} \le \|v_0\|_{L^2} + t\|\rho_0\|_{L^2}.$$

Plugging this estimate into (23) gives

$$\frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|v_0\|_{L^2}^2 + \big(\|v_0\|_{L^2} + t\|\rho_0\|_{L^2}\big)\|\rho_0\|_{L^2} t.$$

This gives the second claimed estimate and ends the proof of the proposition. □

We shall next prove the following result.

Proposition 4.2. *Let $v_0 \in H^1$, with $\omega_0/r \in L^2$ and $\rho_0 \in L^2 \cap L^3$. Then any smooth solution (v,ρ) of (1) with ρ axisymmetric and v axisymmetric without swirl satisfies:*

(1) *for every $t \in \mathbb{R}_+$*

$$\Big\|\frac{\omega}{r}(t)\Big\|_{L^2} \le C_0 e^{C_0 t},$$

(2) *for every $t \in \mathbb{R}_+$*

$$\|v(t)\|_{H^1}^2 + \int_0^t \|v(\tau)\|_{H^2}^2 d\tau \le C_0 e^{C_0 t},$$

where C_0 depends only on the norms of the initial data.

Note that the axi-symmetry is crucial in this proposition. The estimates are uniform for $\kappa \ge 0$ in a bounded set.

Proof. We shall use the notation $\zeta = \omega_\theta/r$ where the vorticity ω is given by $\omega = \omega_\theta e_\theta$ since the flow is axisymmetric. The equation for ζ reads

$$\big(\partial_t + v\cdot\nabla\big)\zeta - \big(\Delta + \frac{2}{r}\partial_r\big)\zeta = -\frac{\partial_r \rho}{r}\cdot \tag{24}$$

By using the operator $\mathcal{L}$, this equation can be written as

$$\big(\partial_t + v\cdot\nabla\big)\zeta - \big(\Delta + \frac{2}{r}\partial_r\big)\big(\zeta - \mathcal{L}\rho\big) = 0. \tag{25}$$

Applying $\mathcal{L}$ to the equation for ρ we first get

$$\partial_t \mathcal{L}\rho + v\cdot\nabla\mathcal{L}\rho - \kappa\mathcal{L}\Delta\rho = -[\mathcal{L}, v\cdot\nabla]\rho$$

and hence by using Lemma 3.4, we find

$$\partial_t \mathcal{L}\rho + v\cdot\nabla\mathcal{L}\rho - \kappa\big(\Delta + \frac{2}{r}\partial_r\big)\mathcal{L}\rho = -[\mathcal{L}, v\cdot\nabla]\rho. \tag{26}$$

In view of (25) and (26), we can set

$$\Gamma := (1-\kappa)\zeta - \mathcal{L}\rho.$$

We find that Γ solves the equation

$$\big(\partial_t + v\cdot\nabla\big)\Gamma - \big(\Delta + \frac{2}{r}\partial_r\big)\Gamma = \mathrm{div}\,(v\,\mathcal{L}\rho) - \mathcal{L}(v\cdot\nabla\rho).$$

Taking the $L^2(\mathbb{R}^3)$ inner product with Γ and integrating by parts in the usual way we get since v is divergence free that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2} + \|\nabla\Gamma(t)\|_{L^2}^2 &\le \|\nabla\Gamma(t)\|_{L^2}\|v\mathcal{L}\rho\|_{L^2}\\ &- \int_{\mathbb{R}^3}\mathcal{L}(v\cdot\nabla\rho)\Gamma dx\\ &= \mathrm{I}+\mathrm{II}.\end{aligned} \tag{27}$$

To estimate the first term we use successively the Hölder inequality, Proposition 3.1 and the first estimate of Proposition 4.1 to get

$$\|v\,\mathcal{L}\rho\|_{L^2} \le \|v\|_{L^6}\|\mathcal{L}\rho\|_{L^3} \lesssim \|v\|_{L^6}\|\rho\|_{L^3} \lesssim \|v\|_{L^6}\|\rho_0\|_{L^3}$$

Now from the Young inequality, we get

$$\mathrm{I} \le \frac{1}{4}\|\nabla\Gamma\|_{L^2}^2 + C\|v\|_{L^6}^2\|\rho_0\|_{L^3}^2. \tag{28}$$

Towards the estimate of the second term in the right-hand side of (27), we can use since v is divergence free that

$$v\cdot\nabla\rho = \partial_r(v^r\rho) + \frac{v^r}{r}\rho + \partial_z(v^z\rho).$$

Thus we obtain

$$\begin{aligned}\mathcal{L}(v\cdot\nabla\rho) &= \mathcal{L}\partial_r(v^r\rho) + \mathcal{L}(\frac{v^r}{r}\rho)\\ &+ \partial_z\mathcal{L}(v^z\rho).\end{aligned}$$

Next, we can use Lemma 3.3 to get

$$\begin{aligned}\mathcal{L}(v\cdot\nabla\rho) &= \frac{v^r}{r}\rho - \partial_z\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\frac{1}{r}\partial_z(v^r\rho))\\ &+ \partial_z\mathcal{L}(v^z\rho).\end{aligned}$$

Thus we find that

$$\begin{aligned}\mathrm{II} &= -\int\frac{v^r}{r}\rho\,\Gamma dx + \int\big(\Delta + \frac{2}{r}\partial_r\big)^{-1}(\frac{1}{r}\partial_z(v^r\rho))\partial_z\Gamma dx\\ &- \int\mathcal{L}(v^z\rho)\partial_z\Gamma dx\\ &= J_1 + J_2 + J_3.\end{aligned}$$

From Cauchy-Schwarz inequality, we first obtain that

$$|J_2+J_3| \leq \|\partial_z\Gamma\|_{L^2}\Big(\Big\|(\Delta+\frac{2}{r}\partial_r)^{-1}(\frac{\partial_z}{r})(v^r\rho)\Big\|_{L^2}+\|\mathcal{L}(v^z\rho)\|_{L^2}\Big) \leq \|\nabla\Gamma\|_{L^2}\big(\|v^r\rho\|_{L^2}+\|v^z\rho\|_{L^2}\big)$$

where the last estimate comes from Proposition 3.2 and Proposition 3.1.
Next, by using successively the Hölder inequality, the Sobolev inequality

$$\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2} \tag{29}$$

in dimension 3, and the first estimate of Proposition 4.1, we infer

$$\begin{aligned} |J_2+J_3| &\leq \|\nabla\Gamma\|_{L^2}\|v\|_{L^6}\|\rho\|_{L^3} \\ &\leq \|\nabla\Gamma\|_{L^2}\|\nabla v\|_{L^2}\|\rho_0\|_{L^3}. \end{aligned}$$

In a similar way, we estimate J_1:

$$\begin{aligned} |J_1| &\leq \|\Gamma\rho\|_{L^{\frac{6}{5}}}\|v^r/r\|_{L^6} \\ &\leq \|\Gamma\|_{L^2}\|\rho\|_{L^3}\|v^r/r\|_{L^6} \\ &\leq \|\Gamma\|_{L^2}\|\rho_0\|_{L^3}\|v^r/r\|_{L^6}. \end{aligned}$$

To estimate $\|v^r/r\|_{L^6}$, we can use the inequality

$$|v^r/r| \lesssim \frac{1}{|\cdot|^2}\star(|\omega_\theta/r|)$$

which is very useful in the field of axisymmetric solution of incompressible fluid mechanics equations, we refer to [29, 16] for the proof. Next, since $\frac{1}{|x|^2}\in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, we get from the classical Hardy-Littlewood-Sobolev inequality that

$$\|v^r/r\|_{L^6} \lesssim \|\omega_\theta/r\|_{L^2} = \|\zeta\|_{L^2}. \tag{30}$$

From the definition of Γ we have that

$$\|\zeta\|_{L^2} \leq |\kappa-1|^{-1}\Big(\|\Gamma\|_{L^2}+\|\mathcal{L}\rho\|_{L^2}\Big). \tag{31}$$

Note that this estimate is uniform for $\kappa\geq 0$ and far from $\kappa=1$. We will see later how to obtain uniform estimates around the value $\kappa=1$. Therefore, by using again Proposition 3.1 and the first estimate of Proposition 4.1, we obtain

$$\|v^r/r\|_{L^6} \lesssim \|\Gamma\|_{L^2}+\|\mathcal{L}\rho\|_{L^2} \lesssim \|\Gamma\|_{L^2}+\|\rho\|_{L^2} \lesssim \|\Gamma\|_{L^2}+\|\rho_0\|_{L^2}.$$

Consequently

$$|J_1| \lesssim \|\Gamma\|_{L^2}\|\rho_0\|_{L^3}\big(\|\Gamma\|_{L^2}+\|\rho_0\|_{L^2}\big).$$

Combining theses estimates with the Young inequality we get

$$\begin{aligned} |\mathrm{II}| &\lesssim \|\nabla\Gamma\|_{L^2}\|\nabla v\|_{L^2}\|\rho_0\|_{L^3}+\|\Gamma\|_{L^2}\|\rho_0\|_{L^3}\big(\|\Gamma\|_{L^2}+\|\rho_0\|_{L^2}\big) \\ &\leq \frac{1}{4}\|\nabla\Gamma\|_{L^2}^2+C\|\nabla v\|_{L^2}^2\|\rho_0\|_{L^3}^2+C\|\rho_0\|_{L^3}\|\Gamma\|_{L^2}^2+C\|\Gamma\|_{L^2}\|\rho_0\|_{L^3}\|\rho_0\|_{L^2}. \end{aligned} \tag{32}$$

It follows from (27), (28) and (32) that

$$\begin{aligned} \frac{d}{dt}\|\Gamma(t)\|_{L^2}^2+\|\nabla\Gamma(t)\|_{L^2}^2 &\lesssim \|\nabla v\|_{L^2}^2\|\rho_0\|_{L^3}^2+\|\rho_0\|_{L^3}\|\Gamma\|_{L^2}^2+\|\Gamma\|_{L^2}\|\rho_0\|_{L^3}\|\rho_0\|_{L^2} \\ &\lesssim \|\rho_0\|_{L^3}\|\Gamma\|_{L^2}^2+\|\nabla v\|_{L^2}^2\|\rho_0\|_{L^3}^2+\|\rho_0\|_{L^3}\|\rho_0\|_{L^2}^2. \end{aligned}$$

We can then integrate in time and use the energy inequality of Proposition 4.1-(2) to get

$$\begin{aligned}\|\Gamma(t)\|_{L^2}^2+\int_0^t\|\nabla\Gamma(\tau)\|_{L^2}^2d\tau &\lesssim \|\rho_0\|_{L^3}^2\|\nabla v\|_{L^2_tL^2}^2+\|\rho_0\|_{L^3}\|\rho_0\|_{L^2}^2t\\ &+ \|\rho_0\|_{L^3}\int_0^t\|\Gamma(\tau)\|_{L^2}^2d\tau\\ &\le C_0(1+t^2)+C_0\int_0^t\|\Gamma(\tau)\|_{L^2}^2d\tau.\end{aligned}$$

By using the Gronwall inequality we find

$$\|\Gamma(t)\|_{L^2}^2+\int_0^t\|\nabla\Gamma(\tau)\|_{L^2}^2d\tau\le C_0e^{C_0t}.$$

It follows from a new use of (31) that

$$\|\zeta(t)\|_{L^2}\le C_0e^{C_0t}. \tag{33}$$

This proves (1) in Proposition 4.2.
To prove (2), we can now perform an energy estimate on the equation (4) satisfied by ω_θ. By taking the $L^2(\mathbb{R}^3)$ scalar product of (4) with ω_θ we get

$$\frac{1}{2}\frac{d}{dt}\|\omega_\theta(t)\|_{L^2}^2+\|\nabla\omega_\theta\|_{L^2}^2+\|\omega_\theta/r\|_{L^2}^2 \le \int_{\mathbb{R}^3}v^r(\omega_\theta/r)\omega_\theta dx-\int_{\mathbb{R}^3}\partial_r\rho\omega_\theta dx.$$

Thanks to an integration by parts, we have that

$$\Big|\int_{\mathbb{R}^3}\partial_r\rho\omega_\theta dx\Big|\le\|\rho\|_{L^2}\big(\|\nabla\omega_\theta\|_{L^2}+\|\omega_\theta/r\|_{L^2}\big)$$

and hence, by using the Holder inequality and the Sobolev inequality (29), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\omega_\theta(t)\|_{L^2}^2+\|\nabla\omega_\theta\|_{L^2}^2+\|\omega_\theta/r\|_{L^2}^2\\ \le\ & \|v\|_{L^3}\|\omega_\theta/r\|_{L^2}\|\omega_\theta\|_{L^6}+\|\rho\|_{L^2}\big(\|\nabla\omega_\theta\|_{L^2}+\|\omega_\theta/r\|_{L^2}\big)\\ \lesssim\ & \|v\|_{L^3}\|\omega_\theta/r\|_{L^2}\|\nabla\omega_\theta\|_{L^2}+\|\rho\|_{L^2}\big(\|\nabla\omega_\theta\|_{L^2}+\|\omega_\theta/r\|_{L^2}\big).\end{aligned}$$

Thus from (33), the first estimate of Proposition 4.1 and the Young inequality, we obtain

$$\frac{d}{dt}\|\omega_\theta(t)\|_{L^2}^2+\frac{1}{2}\Big(\|\nabla\omega_\theta\|_{L^2}^2+\|\omega_\theta/r\|_{L^2}^2\Big)\lesssim\|\rho_0\|_{L^2}^2+C_0e^{C_0t}\|v(t)\|_{L^3}^2. \tag{34}$$

By interpolation, the Sobolev embedding (29) and the second estimate of Proposition 4.1 we have

$$\int_0^t\|v\|_{L^3}^2 \le \int_0^t\|v\|_{L^2}\|v\|_{L^6}\lesssim\int_0^t\|v\|_{L^2}\|\nabla v\|_{L^2}\le C_0(1+t^2).$$

Therefore we get by integrating (34) in time that

$$\|\omega_\theta(t)\|_{L^2}^2+\int_0^t\big(\|\nabla\omega_\theta(\tau)\|_{L^2}^2+\|\frac{\omega_\theta}{r}(\tau)\|_{L^2}^2\big)d\tau\le e^{C_0t}.$$

Since we have $\|\omega\|_{L^2}=\|\omega_\theta\|_{L^2}$ and

$$\|\nabla\omega\|_{L^2}^2=\|\nabla\omega_\theta\|_{L^2}^2+\|\omega_\theta/r\|_{L^2}^2,$$

we finally obtain that

$$\|\omega(t)\|_{L^2}^2+\int_0^t\|\nabla\omega(\tau)\|_{L^2}^2d\tau\leq e^{C_0t}.$$

This ends the proof of Proposition 4.2 where κ belongs to a compact set that does not contain 1. Let us now see how to get uniform bounds around $\kappa=1$ which is more easy and does not require the use of the operator $\mathcal{L}$. We write the equation of the density under the form

$$\partial_t\rho+v\cdot\nabla\rho-(\Delta+\frac{2}{r}\partial_r)\rho=(1-\kappa)\Delta\rho-\frac{2}{r}\partial_r\rho.$$

We set $\Gamma_1=\zeta-\frac{\rho}{2}$. Then combining this equation with (24) we find

$$\partial_t\Gamma_1+v\cdot\nabla\Gamma_1-(\Delta+\frac{2}{r}\partial_r)\Gamma_1=\frac{\kappa-1}{2}\Delta\rho.$$

Taking the L^2 scalar product of this equation with Γ_1 and integrating by parts we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Gamma_1(t)\|_{L^2}^2+\|\nabla\Gamma_1(t)\|_{L^2}^2 &\leq \frac{k-1}{2}\|\nabla\rho\|_{L^2}\|\nabla\Gamma_1\|_{L^2}\\ &\leq \frac{1}{2}(\frac{\kappa-1}{2})^2\|\nabla\rho\|_{L^2}^2+\frac{1}{2}\|\nabla\Gamma_1\|_{L^2}^2.\end{aligned}$$

Integrating in time yields

$$\|\Gamma_1(t)\|_{L^2}^2+\|\nabla\Gamma_1\|_{L^2_tL^2}^2\leq C(\kappa-1)^2\|\nabla\rho\|_{L^2_tL^2}^2.$$

Combining this estimate with the energy estimate

$$\kappa^{\frac{1}{2}}\|\nabla\rho\|_{L^2_tL^2}\leq\|\rho_0\|_{L^2}$$

gives

$$\|\Gamma_1(t)\|_{L^2}^2+\|\nabla\Gamma_1\|_{L^2_tL^2}^2\leq C\frac{(\kappa-1)^2}{\kappa^{\frac{1}{2}}}\|\rho_0\|_{L^2}^2.$$

This gives the desired result. □

The following proposition gives some more precise information than stated in Theorem 1.2 about the solution. This will be useful to prove the uniqueness result.

Proposition 4.3. *Let $v_0\in H^1$ be a divergence free axisymmetric without swirl vector field such that $\omega_0/r\in L^2$ and $\rho_0\in L^2\cap L^m, m>3$ or $\rho_0\in L^2\cap B^0_{3,1}$ an axisymmetric function. Then any smooth solution (v,ρ) of the system (1) satisfies for every $p\in]3,\infty]$*

$$\|v\|_{L^1_tB^{1+\frac{3}{p}}_{p,1}}+\|\nabla v\|_{L^1_tL^\infty}\leq C_0e^{C_0t}.$$

The estimate is uniform with respect to κ lying in a bounded set.

Proof. We first prove the result in the case of $\rho_0\in L^2\cap B^0_{3,1}$. Let $q\in\mathbb{N}$ and set $v_q:=\Delta_qv$. Then applying the operator Δ_q from the Littlewood-Paley decomposition to the velocity equation and using Duhamel formula we get

$$v_q(t)=e^{t\Delta}v_q(0)+\int_0^te^{(t-\tau)\Delta}\Delta_q\mathcal{P}(v\cdot\nabla v)(\tau,x)d\tau+\int_0^te^{(t-\tau)\Delta}\Delta_q\mathcal{P}(\rho e_z)(\tau,x)d\tau, \tag{35}$$

where $\mathcal{P}$ is the Leray projection on divergence free vector fields. Now we will use two estimates: the first one is proved in [12]

$$\|e^{t\Delta}\Delta_qf\|_{L^p}\leq Ce^{-ct2^{2q}}\|\Delta_qf\|_{L^p}.$$

The second estimate is

$$\|\Delta_q \mathcal{P} f\|_{L^p} \leq C \|\Delta_q f\|_{L^p}.$$

This last estimate is a consequence of the fact that $\Delta_q \mathcal{P} = \psi(2^{-q}\mathrm{D})$ with $\psi \in \mathcal{D}(\mathbb{R}^3)$. Therefore, we get from (35) that

$$\|v_q\|_{L^1_t L^p} \lesssim 2^{-2q}\|v_q(0)\|_{L^p} + 2^{-2q}\int_0^t \|\Delta_q(v\cdot\nabla v)(\tau)\|_{L^p} d\tau + 2^{-2q}\|\Delta_q \rho\|_{L^1_t L^p}.$$

It follows from the above inequality, the Besov embeddings (7) and Proposition 4.1 that

$$\begin{aligned}
\|v\|_{L^1_t B^{1+\frac{3}{p}}_{p,1}} &\leq \|\Delta_{-1} v\|_{L^1_t L^p} + \|v_0\|_{B^{-1+\frac{3}{p}}_{p,1}} + \|v\cdot\nabla v\|_{L^1_t B^{-1+\frac{3}{p}}_{p,1}} + \|\rho\|_{L^1_t B^{-1+\frac{3}{p}}_{p,1}} \\
&\leq Ct\|v\|_{L^\infty_t L^2} + \|v_0\|_{H^1} + \|v\otimes v\|_{L^1_t B^{\frac{3}{2}}_{2,1}} + \|\rho\|_{L^1_t B^0_{3,1}} \\
&\leq C_0(1+t^2) + \|v\otimes v\|_{L^1_t B^{\frac{3}{2}}_{2,1}} + \|\rho\|_{L^1_t B^0_{3,1}}.
\end{aligned}$$

Since $B^{\frac{3}{2}}_{2,1}$ is an algebra, we have

$$\|v\otimes v\|_{L^1_t B^{\frac{3}{2}}_{2,1}} \leq C\|v\|^2_{L^2_t B^{\frac{3}{2}}_{2,1}}.$$

Moreover, the embedding $H^2 \hookrightarrow B^{\frac{3}{2}}_{2,1}$ combined with the second estimate of Proposition 4.2 gives

$$\|v(t)\|_{L^2_t B^{\frac{3}{2}}_{2,1}} \leq C_0 e^{C_0 t}.$$

Consequently, we obtain

$$\|v\|_{L^1_t B^{1+\frac{3}{p}}_{p,1}} \leq C_0 e^{C_0 t} + \|\rho\|_{L^1_t B^0_{3,1}}. \tag{36}$$

It remains to estimate the norm of the density. For this purpose, we first use the logarithmic estimate described in Proposition 2.3 and (7) to get

$$\begin{aligned}
\|\rho(t)\|_{B^0_{3,1}} &\leq C\|\rho_0\|_{B^0_{3,1}}\Big(1+\int_0^t \|v(\tau)\|_{B^1_{\infty,1}} d\tau\Big) \\
&\leq C\|\rho_0\|_{B^0_{3,1}}\Big(1+\int_0^t \|v(\tau)\|_{B^{1+\frac{3}{p}}_{p,1}} d\tau\Big).
\end{aligned}$$

Set $V(t) := \|v\|_{L^1_t B^{\frac{3}{p}}_{p,1}}$, then combining this estimate with (36) yields

$$V(t) \leq C_0 e^{C_0 t} + \|\rho_0\|_{B^0_{3,1}} \int_0^t V(\tau) d\tau.$$

We conclude now by Gronwall lemma.
Let us now come back to the case that $\rho_0 \in L^2 \cap L^m$, with $m > 3$ which is more easy than the previous case. The same proof as above for the velocity yields

$$\|v\|_{L^1_t B^{1+\frac{3}{p}}_{p,1}} \leq C_0 e^{C_0 t} + \|\rho\|_{L^1_t B^{-1+\frac{3}{p}}_{p,1}}$$

and it still remains to estimate the density. Let us set $m_1 := \min(m,p) > 3$ then by Besov embeddings and the first estimate of Proposition 4.1, we get

$$\|\rho\|_{L^1_t B^{-1+\frac{3}{p}}_{p,1}} \lesssim \|\rho\|_{L^1_t B^{-1+\frac{3}{m_1}}_{m_1,1}} \lesssim \|\rho\|_{L^1_t L^{m_1}} \lesssim t\|\rho_0\|_{L^{m_1}} \lesssim t\|\rho_0\|_{L^2\cap L^m}.$$

Note that the last estimate holds by interpolation. This ends the proof. □

5. Proof of the main result

For the existence part of Theorem 1.2 we smooth out the initial data as follows

$$v_{0,n} = S_n v_0, \rho_{0,n} = S_n \rho_0,$$

where S_n is the cut-off in frequency defined in section 2. We start with the following stability results.

Lemma 5.1. *Let v_0 be a free divergence axisymmetric vector-field without swirl and ρ_0 an axisymmetric scalar function. Then*

(1) *for every $n \in \mathbb{N}$, $v_{0,n}$ and $\rho_{0,n}$ are axisymmetric and* $\operatorname{div} v_{0,n} = 0$.
(2) *If $v_0 \in H^1$ is such that $(\operatorname{curl} v_0)/r \in L^2$ and $\rho_0 \in L^2 \cap B^0_{3,1}$. Then there exists a constant C independent of n such that*

$$\|v_{0,n}\|_{H^1} \le \|v_0\|_{H^1}, \quad \big\|(\operatorname{curl} v_{0,n})/r\big\|_{L^2} \le C\big\|(\operatorname{curl} v_0)/r\big\|_{L^2},$$

$$\|\rho_{0,n}\|_{L^2} \le \|\rho_0\|_{L^2}, \quad \|\rho_{0,n}\|_{B^0_{3,1}} \le C\|\rho_0\|_{B^0_{3,1}}.$$

Proof. We have $v_{0,n} = 2^{3n}\chi(2^n\cdot) \star v_0$. The fact that the vector field $v_{0,n}$ is axisymmetric is due to the radial property of the functions χ, for more details see [4]. The estimate of $v_{0,n}$ in H^1 is easy to obtain by using the classical properties of the convolution laws. The proof of the second estimate for the velocity is more subtle and we refer to [7] where it is proven in the general framework of Lebesgue spaces, that is in L^p, for all $p \in [1,\infty]$. The estimates for the density follow by standard convolution inequalities. □

We have just seen in Lemma 5.1 that the initial structure of axisymmetry is preserved for every n and the involved norms are uniformly controlled with respect to this parameter n. Thus we can construct locally in time a unique solution (v_n, ρ_n). This solution is globally defined since the Lipschitz norm of the velocity does not blow up in finite time as it was stated in Proposition 4.3. Note that since $\rho_0 \in B^0_{3,1}$ (or L^m, $m > 3$), the L^3 norm of $\rho_{0,n}$ is also uniformly bounded thanks to the continuous embedding $B^0_{3,1} \subset L^3$. By standard arguments we can show that this family (v_n, ρ_n) converges to (v, ρ) which satisfies in turn our initial problem. We omit here the details and we will next focus on the uniqueness part. Set

$$\mathcal{X}_T := \big(L^\infty_T H^1 \cap L^2_T H^2 \cap L^1_T B^{1+\frac{3}{p}}_{p,1}\big) \times L^\infty_T H^{-1}, \quad \text{for some} \quad p \in [3,\infty[.$$

Let $(v^i, \rho^i) \in \mathcal{X}_T, 1 \le i \le 2$ be two solutions of the system (1) with the same initial data (v_0, θ_0) and denote $\delta v = v^2 - v^1, \delta\theta = \theta^2 - \theta^1$. Then

$$\begin{cases} \partial_t \delta v - \Delta \delta v = -\mathcal{P}(v^2 \cdot \nabla \delta v) - \mathcal{P}(\delta v \cdot \nabla v^1) + \mathcal{P}(\delta\rho\, e_z) \\ \partial_t \delta\rho + v^2 \cdot \nabla \delta\rho - \kappa\Delta\, \delta\rho = -\delta v \cdot \nabla \rho^1 \\ \operatorname{div} v^i = 0. \end{cases} \tag{37}$$

Using the maximal smoothing of the heat operator combined with Hölder inequality we get

$$
\begin{aligned}
\|\delta v\|_{L^\infty_t \dot H^1} &\lesssim \|v^2\cdot\nabla\delta v\|_{L^2_tL^2}+\|\delta v\cdot\nabla v^1\|_{L^2_tL^2}+\|\delta\rho\|_{L^1_t\dot H^{-1}}\\
&\lesssim \Big(\int_0^t\|v^2(\tau)\|^2_{L^\infty}\|\delta v(\tau)\|^2_{\dot H^1}d\tau\Big)^{\frac12}+\Big(\int_0^t\|\delta v(\tau)\|^2_{\dot H^1}\|v^1(\tau)\|^2_{\dot H^{\frac32}}d\tau\Big)^{\frac12}\\
&\qquad +\|\delta\rho\|_{L^1_t\dot H^{-1}}.
\end{aligned}\tag{38}
$$

We have used the classical law product

$$\|fg\|_{L^2}\lesssim\|f\|_{\dot H^1}\|g\|_{\dot H^{\frac12}}.$$

To estimate $\|\delta\rho\|_{H^{-1}}$ we will use Proposition 3.1 of [5] : for every $p\in[2,\infty[$

$$\|\delta\rho(t)\|_{\dot H^{-1}}\;\leq\; C\|\delta v\cdot\nabla\rho^1\|_{L^1_t\dot H^{-1}}\exp\big(C\|\nabla v^2\|_{L^1_tB^{\frac3p}_{p,1}}\big).$$

We remark that the proof of this result was done in the inviscid case but it can be extended to the viscous case with uniform bounds with respect to the parameter κ.
As $\operatorname{div}\delta v=0$, then using Hölder inequality and Sobolev embedding $\dot H^1\hookrightarrow L^6$ yield

$$
\begin{aligned}
\|\delta v\cdot\nabla\rho^1\|_{L^1_t\dot H^{-1}} &\leq \|\delta v\,\rho^1\|_{L^1_tL^2}\\
&\leq \|\delta v\|_{L^1_tL^6}\|\rho^1\|_{L^\infty_tL^3}\\
&\leq \|\rho_0\|_{L^3}\|\delta v\|_{L^1_t\dot H^1}.
\end{aligned}
$$

Thus we get

$$\|\delta\rho(t)\|_{\dot H^{-1}}\;\leq\; C\|\rho_0\|_{L^3}\exp\big(C\|\nabla v^2\|_{L^1_tB^{\frac3p}_{p,1}}\big)\|\delta v\|_{L^1_t\dot H^1}.$$

By plugging this estimate into (38), we finally get

$$
\begin{aligned}
\|\delta v\|^2_{L^\infty_t\dot H^1} &\lesssim \Big(\int_0^t\|v^2(\tau)\|^2_{L^\infty}\|\delta v(\tau)\|^2_{\dot H^1}d\tau+\int_0^t\|\delta v(\tau)\|^2_{\dot H^1}\|v^1(\tau)\|^2_{\dot H^{\frac32}}d\tau\\
&+\;\|\rho_0\|^2_{L^3}\exp\big(C\|\nabla v^2\|_{L^1_tB^{\frac3p}_{p,1}}\big)\,t\int_0^t\|\delta v(\tau)\|^2_{\dot H^1}d\tau.
\end{aligned}
$$

Since $v^2\in L^2_tL^\infty$ and $v^1\in L^2_t\dot H^{\frac32}$ then we get the uniqueness by using Gronwall inequality.

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