ON THE GLOBAL WELL-POSEDNESS OF THE EULER-BOUSSINESQ SYSTEM WITH FRACTIONAL DISSIPATION

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Abstract. We study the global well-posedness of the Euler-Boussinesq system with term dissipation $|\mathrm{D}|^\alpha$ on the temperature equation. We prove that for $\alpha > 1$ the coupled system has global unique solution for initial data with critical regularities.

1. Introduction

In this paper we deal with the two-dimensional *Euler-Boussinesq* system given by

$$(\mathrm{B}_\alpha)\qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla\pi = \theta e_2 \\ \partial_t\theta + v\cdot\nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = 0 \\ \mathrm{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

Here, the unknowns are the velocity $v = (v^1, v^2) \in \mathbb{R}^2$, the pressure π and the temperature θ. The vector e_2 is given by $(0,1)$, α is a real number in $]0,2]$ and $\kappa \geq 0$ is called the molecular diffusivity. The fractional Laplacian $|\mathrm{D}|^\alpha$ is defined as follows

$$|\mathrm{D}|^\alpha f(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}|\xi|^\alpha \widehat{f}(\xi)d\xi.$$

The fractional Laplacian serves to model many physical phenomena such as overdriven detonations in gases [9] or anomalous diffusion in semiconductor growth [26]. It is also used in some mathematical models in hydrodynamics, molecular biology and finance mathematics, see [14, 20, 22].
In space dimension two the vorticity is defined by the scalar $\omega = \partial_1 v^2 - \partial_2 v^1$. Thus the system (B_α) can be written under the vorticity-temperature formulation as follows:

$$(1.1)\qquad \begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta \\ \partial_t\theta + v\cdot\nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = 0 \\ v = \nabla^\perp\Delta^{-1}\omega. \end{cases}$$

In the case of zero diffusivity this system can be seen as an hyperbolic quasi-linear system and thus it is locally well-posed in Sobolev spaces H^s with $s > 2$. Nevertheless the question of whether smooth solutions develop singularities in finite time or not remains till now an outstanding open problem. For $\kappa > 0$ and $\alpha = 2$ the question of global existence is solved

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recently in a serie of papers [4, 12, 13, 16]. In [4], Chae proved the global existence and uniqueness for initial data $(v^0,\theta^0)\in H^s\times H^s$, with $s>2$. This result has been recently improved in [16] by Hmidi and Keraani for initial data $v^0\in B_{p,1}^{\frac{2}{p}+1}$ and $\theta^0\in B_{p,1}^{\frac{2}{p}-1}\cap L^r$, with $r\in]2,\infty[$. It seems that the only smoothing effects due to the transport-diffusion equation governing the temperature is sufficient to counterbalance the amplification of the vorticity. More recently, the study of global existence of Yudovich solutions for this system has been done in [12]. We mention that in [13] Danchin and Paicu proved that if we have only an horizontal viscosity, that is $\partial_{11}^2\theta$ instead of $\Delta\theta$, then Euler-Boussinesq system admits global unique solution.

In this paper, we aim at solving the question of global existence for less dissipative term $|\mathrm{D}|^\alpha\theta$. As we shall see the difficulty depends on the parameter α and it appears that the system shares some properties with the 2d quasi-geostrophic equation (QG) described by

$$\partial_t\theta+v\cdot\nabla\theta+\kappa|\mathrm{D}|^\alpha\theta=0,\quad v=(-\partial_2|\mathrm{D}|^{-1}\theta,\partial_1|\mathrm{D}|^{-1}\theta).$$

Indeed, the velocity in the second equation of (B_α) has basically the same regularity as the temperature but it is given through a complex dynamical system. In a similar way to the quasi-geostrophic equation we shall call critical the value $\alpha=1$. It corresponds to the fact that the likely amplification of the vorticity due to the term $\partial_1\theta$ and the dissipation have the same rate. Thus we expect for the sub-critical case $\alpha>1$ to have global existence since the dissipation is much stronger than the amplification. This will be the main goal of this paper. We emphasize that our method does not give any answer to the global existence for the critical case. On the other hand the approach developed by Kiselev, Nazarov and Volberg [21] to settle global existence for the critical (QG) equation does not work here because the relation between the velocity and the temperature is not local in time. Likewise there is no hope with the method used by Caffarelli and Vasseur [3] since we have not sufficient estimates on the velocity like $v\in L^\infty([0,T];\mathrm{BMO})$. We recall that the BMO space is the set of functions of bounded mean oscillation introduced by John and Nirenberg.

Now, we state the main result of this paper.

Theorem 1.1. *Let $(\alpha,p)\in]1,2]\times]1,\infty[$, $v^0\in B_{p,1}^{1+\frac{2}{p}}$ be a divergence free vector-field of $\mathbb{R}^2$ and $\theta^0\in B_{p,1}^{-\alpha+1+\frac{2}{p}}\cap L^r$, with $\frac{2}{\alpha-1}<r<\infty$. Then there exists a unique global solution (v,θ) for the system (B_α) such that*

$$v\in\mathscr{C}\big(\mathbb{R}_+;B_{p,1}^{1+\frac{2}{p}}\big)\quad\textit{and}\quad\theta\in\mathscr{C}\big(\mathbb{R}^+;B_{p,1}^{-\alpha+1+\frac{2}{p}}\cap L^r\big)\cap L^1_{loc}\big(\mathbb{R}_+;\mathrm{Lip}\big)$$

Remark 1.2. Notice that the regularity assumption on the velocity is in some sense critical. Indeed, Euler equations are often seen as an hyperbolic system and then for establishing strong solutions we need the velocity to be Lipschitzian. The spaces used here for the velocity are $B_{p,1}^{\frac{2}{p}+1}$ and they are considered as the best Besov spaces embedded in Lipschitz class. It is proved in [25] that the incompressible Euler system has global unique solution for initial data lying in theses spaces. It is then legitimate to try to obtain a similar result for our system, which is the subject of this paper.

Remark 1.3. The Besov regularity $B_{p,1}^{-\alpha+1+\frac{2}{p}}$ of the temperature is also optimal with respect to the regularity of the velocity. Indeed, to estimate the quantity $\|v(t)\|_{B_{p,1}^{\frac{2}{p}+1}}$ we need to control $\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}}$. Now from the maximal smoothing effect of the fractional heat equation the best space of the initial temperature should be $B_{p,1}^{-\alpha+1+\frac{2}{p}}$. It is obvious that when p is sufficiently large then the temperature will not be necessary in any Lebesgue space and thus there is no plain conservation laws. For this reason we need to put the initial temperature in some regular Lebesgue space L^r, with $r > \frac{2}{\alpha-1}$, but we do not know whether we can improve or remove this technical condition.

The proof of our main theorem relies heavily on some smoothing effects of the transport-diffusion equation governing the evolution of the temperature, see Propositions 3.2 and 3.4. This is the crucial ingredient of the proof and it allows us to control the growth of the vorticity by the quantity $\partial_1\theta$.
Our paper is organized as follows. The second section deals with some basic notions of Littlewood-Paley theory and we recall some useful lemmas. In the third one we are interested in studying a transport-diffusion equation. We prove basically two kinds of estimates: some smoothing effects and a commutator estimate type. The proof of our main result is given in the fourth section.

2. Preliminaries

Throughout this paper, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. We denote by C a harmless constant whose value may vary from line to line.
We will gather in this section some definitions and tools frequently used along this paper. We start with the so-called Littlewood-Paley operators which allow us to define the Besov spaces. For the following assertion we can for example see [6].

Proposition 2.1. *There exists two radial positive functions $\chi \in \mathscr{D}(\mathbb{R}^2)$ and $\varphi \in \mathscr{D}(\mathbb{R}^2\backslash\{0\})$ such that*

1. $\chi(\xi) + \sum_{q\ge 0}\varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$
2. $\sum_{q\in\mathbb{Z}}\varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2\backslash\{0\},$
3. $|j-q| \ge 2 \Rightarrow \textit{supp}\ \varphi(2^{-j}\cdot) \cap \textit{supp}\ \varphi(2^{-q}\cdot) = \varnothing,$
4. $q \ge 1 \Rightarrow \textit{supp}\ \chi \cap \textit{supp}\ \varphi(2^{-q}\cdot) = \varnothing.$

Now we define the the Littlewood-Paley operators as follows: for every tempered distribution $v \in \mathscr{S}'$, we set

$$\Delta_{-1}v = \chi(\mathrm{D})v; \quad \forall q \in \mathbb{N}, \quad \Delta_q = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad S_q v = \sum_{-1\le j\le q-1} \Delta_j v.$$

It is easy to see that the operators Δ_q and S_q map continuously L^p into itself uniformly with respect to q and p. We can also define the homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u \quad \text{and} \quad \dot{S}_q = \sum_{j \leq q-1} \dot{\Delta}_j v$$

According to [2] we can split the product uv into three parts:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p} \Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(2^{qs} \|\dot{\Delta}_q u\|_{L^p} \Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

In the case $(s, p, r) \in]0, 1[\times [1, \infty]^2$ we have an other characterization of the Besov space, (for the proof see [24]),

$$C^{-1} \|v\|_{\dot{B}^s_{p,r}} \leq \Big(\int_{\mathbb{R}^d} \frac{\|v(\cdot - x) - v(\cdot)\|^r_{L^p}}{|x|^{sr}} \frac{dx}{|x|^2} \Big)^{\frac{1}{r}} \leq C \|v\|_{\dot{B}^s_{p,r}}. \tag{2.1}$$

Let $T > 0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^p} \Big)_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^\rho_T L^p} \Big)_{\ell^r} < +\infty.$$

By a direct application of the Minkowski inequality, we have the following links between these spaces. Let $\varepsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \text{ if } \quad r \geq \rho,$$

$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \text{ if } \quad \rho \geq r.$$

The Bernstein inequalities are detailed below.

Lemma 2.2. *There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$ and for every function v and every $q \in \mathbb{Z}$, we have*

$$\begin{aligned}\sup_{|\alpha|=k}\|\partial^\alpha S_q v\|_{L^b} &\leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\|S_q v\|_{L^a},\\ C^{-k}2^{qk}\|\dot\Delta_q v\|_{L^a} &\leq \sup_{|\alpha|=k}\|\partial^\alpha\dot\Delta_q v\|_{L^a}\leq C^k 2^{qk}\|\dot\Delta_q v\|_{L^a}.\end{aligned}$$

Notice that Bernstein inequalities remain true if we change the derivative ∂^α by the fractional derivative $|\mathrm{D}|^\alpha$. The next proposition deals with some commutator estimates.

Proposition 2.3. *Let u be a smooth function and v be a smooth vector-field of $\mathbb{R}^2$ with zero divergence. Then for every $q\geq -1$, we have*

$$\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^p}\lesssim\|u\|_{L^p}\Big(\big\|\nabla\Delta_{-1}v\big\|_{L^\infty}+(q+2)\|\mathrm{curl}\, v\|_{L^\infty}\Big).$$

Besides we have for every $s\geq -1$

$$\sum_{q\geq -1}2^{qs}\|[\Delta_q, v\cdot\nabla]u\|_{L^p}\leq\|\nabla v\|_{L^\infty}\|u\|_{B^s_{p,1}}+\|\nabla u\|_{L^\infty}\|v\|_{B^s_{p,1}}1_{[1,\infty[}(s).$$

The first estimate is proved in [18]. However the second one is classical and its proof can be found for example in [6].
Next we recall a logarithmic estimate proven first by Vishik in [25] for the particular case of Besov space $B^0_{\infty,1}$. The proof for more general case can be found in [19].

Proposition 2.4. *Let $(p,r)\in[1,\infty]^2$, v be a divergence free vector-field belonging to the space $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip}(\mathbb{R}^2))$ and let a be a smooth solution of the following transport equation,*

$$\begin{cases}\partial_t a+v\cdot\nabla a=f\\ a_{|t=0}=a^0.\end{cases}$$

If the initial data $a^0\in B^0_{p,r}$, then we have for all $t\in\mathbb{R}_+$

$$\|a\|_{\widetilde L^\infty_t B^0_{p,r}}\lesssim\big(\|a^0\|_{B^0_{p,r}}+\|f\|_{\widetilde L^1_t B^0_{p,r}}\big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big).$$

Let us now end this section with a classical result about incompressible Euler equation, see for instance [5].

Proposition 2.5. *Let v be a solution of the incompressible Euler system,*

$$\partial_t v+v\cdot\nabla v+\nabla\pi=f,\quad v_{|t=0}=v^0,\quad \mathrm{div} v=0.$$

Then for $s>-1,(p,r)\in]1,\infty[\times[1,\infty]$ we have

$$\|v\|_{\widetilde L^\infty_t B^s_{p,r}}\leq Ce^{CV(t)}\Big(\|v^0\|_{B^s_{p,r}}+\int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with $V(t)=\|\nabla v\|_{L^1_t L^\infty}$.

3. Study of a transport-diffusion equation

This section is devoted to some estimates for the following transport-diffusion model

$$(\mathrm{TD}_\alpha)\qquad \begin{cases} \partial_t\theta + v\cdot\nabla\theta + |\mathrm{D}|^\alpha\theta = f\\ \theta_{|t=0}=\theta^0.\end{cases}$$

The first estimate deals with the L^p estimates, see [11].

Lemma 3.1. *Let $\alpha\in[0,2]$, v be a smooth divergence free vector-field. We assume that θ is a smooth solution of the equation* (TD_α)*. Then for $p\in[1,\infty]$*

$$\|\theta(t)\|_{L^p}\le\|\theta^0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}d\tau.$$

Now we intend to discuss some important smoothing effects which are the cornerstone of the proof of Theorem 1.1.

3.1. Smoothing effects.

We will discuss here two kinds of smoothing effects. The first one is described by the following proposition.

Proposition 3.2. *Let $\alpha\in[0,2]$, $p\in[2,\infty[, \rho\in[1,\infty]$ and v be a smooth divergence free vector-filed of $\mathbb{R}^2$. Let θ be a smooth solution of* (TD_α) *with a zero force f. Then we have for every $t\ge0, q\in\mathbb{N}\cup\{-1\}$*

$$2^{q\frac{\alpha}{\rho}}\|\Delta_q\theta\|_{L^\rho_tL^p}\lesssim\|\theta^0\|_{L^p}\Big(1+t+(q+2)\|\mathrm{curl}\,v\|_{L^1_tL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big).$$

Remark 3.3. If the velocity belongs to $L^1_t\mathrm{Lip}$ then the previous estimate becomes

$$2^{q\frac{\alpha}{\rho}}\|\Delta_q\theta\|_{L^\rho_tL^p}\lesssim\|\theta^0\|_{L^p}\big(1+t+\|\nabla v\|_{L^1_tL^\infty}\big).$$

Although we have not a frequency-logarithmic loss in this case, the estimate seems to be not very convenient for our context due to the term $\|\nabla v\|_{L^\infty}$. As we shall see, it is much harder to estimate this quantity rather than the vorticity.

Proof. First, let $q\in\mathbb{N}^*$ and define $\theta_q\stackrel{def}{=}\Delta_q\theta$. Then applying the Littlewood-Paley operator Δ_q to the equation, we get

$$\partial_t\theta_q+v\cdot\nabla\theta_q+|\mathrm{D}|^\alpha\theta_q=-\big[\Delta_q,v\cdot\nabla\big]\theta.\tag{3.1}$$

Multiplying the above equation by $|\theta_q|^{p-2}\theta_q$ and using Hölder inequalities we get

$$\frac1p\frac{d}{dt}\|\theta_q\|^p_{L^p}+\int_{\mathbb{R}^2}(|\mathrm{D}|^\alpha\theta_q)|\theta_q|^{p-2}\theta_q dx\le\|\theta_q\|^{p-1}_{L^p}\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}.$$

Now recall from [8] the following generalized Bernstein inequality

$$c2^{q\alpha}\|\theta_q\|^p_{L^p}\le\int_{\mathbb{R}^2}(|\mathrm{D}|^\alpha\theta_q)|\theta_q|^{p-2}\theta_q dx,$$

where the constant c depends on p. Inserting this estimate in the previous one yields

$$\frac1p\frac{d}{dt}\|\theta_q(t)\|^p_{L^p}+c2^{q\alpha}\|\theta_q(t)\|^p_{L^p}\lesssim\|\theta_q(t)\|^{p-1}_{L^p}\|[\Delta_q,v\cdot\nabla]\theta(t)\|_{L^p}.$$

Thus we find
$$\frac{d}{dt}\|\theta_q(t)\|_{L^p}+c2^{q\alpha}\|\theta_q(t)\|_{L^p}\lesssim\|[\Delta_q,v\cdot\nabla]\theta(t)\|_{L^p}.$$
Hence
$$\frac{d}{dt}\Big(e^{ct2^{q\alpha}}\|\theta_q(t)\|_{L^p}\Big)\lesssim e^{ct2^{q\alpha}}\|[\Delta_q,v\cdot\nabla]\theta(t)\|_{L^p}.$$
Integrating in time this differential inequality leads to
$$\|\theta_q(t)\|_{L^p}\leq\|\theta_q^0\|_{L^p}e^{-ct2^{q\alpha}}+\int_0^t e^{-c(t-\tau)2^{q\alpha}}\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.$$
Recall from Proposition 2.3 that
$$\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}\leq\|\theta\|_{L^p}\big((q+2)\|\omega\|_{L^\infty}+\|\nabla\Delta_{-1}v\|_{L^\infty}\big).$$
Integrating once again in time and using the convolution inequalities we find the desired result. □

The second main goal of this section is to establish another kind of smoothing effects for any solution of (TD_α).

Proposition 3.4. *Let $\rho\in[1,\infty],\alpha\in[0,2],\ p\in[1,\infty],s>-1$ and v be a smooth divergence free vector-field of $\mathbb{R}^2$. Let θ be a smooth solution of (TD_α) with a zero force f. Then for $t\geq0$*
$$\|\theta\|_{\widetilde L^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}}\leq Ce^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}}(1+t^{\frac{1}{\rho}})+\int_0^t\Gamma_s(\tau)d\tau\Big),$$
with
$$V(t)\overset{def}{=}\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau,\quad \Gamma_s(t)\overset{def}{=}\|\nabla\theta(t)\|_{L^\infty}\|v(t)\|_{B^s_{p,1}}1_{[1,\infty[}(s).$$

Remark 3.5. Using the method described in the proof of Proposition 3.2 one can establish the above proposition for $p\in[2,\infty[$ but the estimates include some constants which blow up when p goes to infinity. On the other hand the method of [8] does not work for $p\in[1,2[$ due to some composition laws which are not valid in this case although we expect the final result to be true. In order to cover all the value $p\in[1,\infty]$ we will use a different approach based on the Lagrangian coordinates.

Proof. The proof will be done in the spirit of [1, 15]. Roughly speaking, it consists first in localizing in frequency the evolution equation and second in rewriting the equation in Lagrangian coordinates. This will lead to some technical difficulties, especially, when we have to treat a commutator term coming from the commutation between the fractional Laplacian and the regularized flows.

Let $q\in\mathbb{N}$ and define $\theta_q\overset{def}{=}\Delta_q\theta$. Then localizing in frequency the equation we get
$$\partial_t\theta_q+S_qv\cdot\nabla\theta_q+|\mathrm{D}|^\alpha\theta_q=(S_qv-v)\cdot\nabla\theta_q-\big[\Delta_q,v\cdot\nabla\big]\theta:=\mathscr{R}_q. \tag{3.2}$$
Applying Lemma 3.1 yields
$$\|\theta(t)\|_{L^p}\leq\|\theta_q^0\|_{L^p}+\int_0^t\|\mathscr{R}_q(\tau)\|_{L^p}d\tau.$$

It follows that

$$\|\theta(t)\|_{B^s_{p,1}}\le\|\theta^0\|_{B^s_{p,1}}+\int_0^t\sum_q 2^{qs}\|\mathscr{R}_q(\tau)\|_{L^p}d\tau.$$

Let us recall from Proposition 2.3 the following estimate

$$\sum_q 2^{qs}\|\mathscr{R}_q(\tau)\|_{L^p}\le\|\nabla v\|_{L^\infty}\|\theta\|_{B^s_{p,1}}+\|\nabla\theta\|_{L^\infty}\|v\|_{B^s_{p,1}}1_{[1,\infty[}(s).\tag{3.3}$$

Combining together these estimates and using Gronwall's inequality we thus find

$$\|\theta\|_{\widetilde{L}^\infty_t B^s_{p,1}}\le Ce^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}}+\int_0^t e^{-CV(\tau)}\Gamma_s(\tau)d\tau\Big).\tag{3.4}$$

This achieves the proof for the particular case $\rho=+\infty$. Let us now move to the smoothing effect. We define by ψ_q the flow of the regularized velocity S_qv, given by the integral equation,

$$\psi_q(t,x)=x+\int_0^t S_qv\big(\tau,\psi_q(\tau,x)\big)d\tau.$$

Set

$$\bar{\theta}_q(t,x)=\theta_q(t,\psi_q(t,x))\quad\text{and}\quad\bar{\mathscr{R}}_q(t,x)=\mathscr{R}_q(t,\psi_q(t,x)).$$

It is easily seen that

$$\partial_t\bar{\theta}_q+|\mathrm{D}|^\alpha\bar{\theta}_q=\bar{\mathscr{R}}_q+|\mathrm{D}|^\alpha\big(\theta_q\circ\psi_q\big)-\big(|\mathrm{D}|^\alpha\theta_q\big)\circ\psi_q:=\mathscr{R}^1_q.\tag{3.5}$$

We will use the following estimate

$$\||\mathrm{D}|^\alpha\big(\theta_q\circ\psi_q\big)-\big(|\mathrm{D}|^\alpha\theta_q\big)\circ\psi_q\|_{L^p}\le Ce^{CV(t)}V(t)2^{\alpha q}\|\theta_q\|_{L^p}.\tag{3.6}$$

The proof of this estimate is postponed at the end of this section. Now, since the flow ψ_q preserves Lebesgue measure then we get by (3.6)

$$\big\|\mathscr{R}^1_q(t)\big\|_{L^p}\ \lesssim\ e^{CV(t)}V(t)2^{\alpha q}\|\theta_q\|_{L^p}+\|\mathscr{R}_q(t)\|_{L^p}.\tag{3.7}$$

At this stage of the proof one can remark that the function $\bar{\theta}_q$ is not necessarily localized in frequency. Thus in order to quantify the smoothing effects we need once again to localize the equation (3.5). Now, let $j\in\mathbb{N}$ then applying the operator Δ_j to the equation (3.5) yields

$$\partial_t\Delta_j\bar{\theta}_q+|\mathrm{D}|^\alpha\Delta_j\bar{\theta}_q=\Delta_j\mathscr{R}^1_q.$$

The frequency description of the smoothing effect of the fractional heat semigroup can be summarized in the following estimate

$$\|e^{-t|\mathrm{D}|^\alpha}\Delta_jf\|_{L^p}\lesssim e^{-ct2^{j\alpha}}\|\Delta_jf\|_{L^p}.$$

The proof of this inequality can be found for example in [7] for the case $\alpha = 2$ and [17] for $\alpha \in]0,2[$. Combining this estimate with Duhamel formula and (3.7) one obtains

$$
\begin{aligned}
\|\Delta_j\bar{\theta}_q(t)\|_{L^p} &\lesssim e^{-ct2^{j\alpha}}\|\Delta_j\theta_q^0\|_{L^p}\\
&+ 2^{q\alpha}e^{CV(t)}V(t)\int_0^t e^{-c(t-\tau)2^{j\alpha}}\|\theta_q(\tau)\|_{L^p}d\tau\\
&+ \int_0^t e^{-c(t-\tau)2^{j\alpha}}\big\|\mathscr{R}_q(\tau)\big\|_{L^p}d\tau. \qquad (3.8)
\end{aligned}
$$

By integrating in time and using convolution inequalities, we get for $j \in \mathbb{N}$

$$
\begin{aligned}
\|\Delta_j\bar{\theta}_q\|_{L^\rho_tL^p} &\lesssim 2^{-j\frac{\alpha}{\rho}}\|\Delta_j\theta_q^0\|_{L^p} + 2^{\alpha(q-j)}e^{CV(t)}V(t)\|\theta_q\|_{L^\rho_tL^p}\\
&+ 2^{-j\frac{\alpha}{\rho}}\big\|\mathscr{R}_q\big\|_{L^1_tL^p}. \qquad (3.9)
\end{aligned}
$$

Now, let $N \in \mathbb{N}$ be a fixed number that will be chosen later. Since ψ_q preserves Lebesgue measure then we get

$$
\begin{aligned}
&2^{q(s+\frac{\alpha}{\rho})}\big\|\theta_q(t)\|_{L^\rho_tL^p} = 2^{q(s+\frac{\alpha}{\rho})}\big\|\bar{\theta}_q(t)\|_{L^\rho_tL^p}\\
&\le 2^{q(s+\frac{\alpha}{\rho})}\Big(\sum_{|j-q|\le N}\big\|\Delta_j\bar{\theta}_q\big\|_{L^\rho_tL^p} + \sum_{|j-q|>N}\big\|\Delta_j\bar{\theta}_q\big\|_{L^\rho_tL^p}\Big)\\
&:= \mathrm{I}_q + \mathrm{II}_q.
\end{aligned}
$$

If $q \ge N$, then it follows from (3.9),

$$
\mathrm{I}_q \lesssim 2^{qs}\|\theta_q^0\|_{L^p} + e^{CV(t)}V(t)2^{\alpha N}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_tL^p} + 2^{N\frac{\alpha}{\rho}}2^{qs}\|\mathscr{R}_q\|_{L^1_tL^p}. \qquad (3.10)
$$

To estimate the second term II_q we use the following result due to Vishik [25]

$$
\big\|\Delta_j\bar{\theta}_q\big\|_{L^p} \lesssim 2^{-|q-j|}e^{CV(t)}\|\theta_q\|_{L^p}.
$$

Hence

$$
\mathrm{II}_q \lesssim 2^{-N}e^{CV(t)}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_tL^p}. \qquad (3.11)
$$

For low frequencies, $q \le N$, we have from Hölder's inequality

$$
\sum_{q\le N}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_tL^p} \lesssim 2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}\|\theta\|_{\widetilde{L}^\infty_tB^s_{p,1}}.
$$

It suffices now to use (3.4), leading to

$$
\sum_{q\le N}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_tL^p} \lesssim 2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}e^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}} + \int_0^t e^{-CV(\tau)}\Gamma_s(\tau)d\tau\Big). \qquad (3.12)
$$

Putting together (3.10), (3.11), (3.3) and (3.12), we get

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}} &\leq C\|\theta^0\|_{B^s_{p,1}}(1+2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}e^{CV(t)})\\ &+ Ce^{CV(t)}\Big(V(t)\,2^{N\alpha}+2^{-N}\Big)\|\theta\|_{\widetilde{L}^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}}\\ &+ C2^{N\frac{\alpha}{\rho}}(1+t^{\frac{1}{\rho}})e^{CV(t)}\int_0^t \Gamma_s(\tau)d\tau.\end{aligned}$$

It is easy to check that there exists two absolute constants $N\in\mathbb{N}$ and $C_1>0$ such that

$$V(t)\leq C_1 \Rightarrow Ce^{CV(t)}\Big(V(t)2^{\alpha N}+2^{-N}\Big)\leq \frac{1}{2}.$$

Indeed, we start with taking t such that $V(t)\leq 1$, which is possible since $\lim_{t\to 0}V(t)=0$. Next, we choose N in order to have $Ce^C 2^{-N}\leq \frac{1}{4}$. Now, we take $V(t)$ sufficiently small such that $Ce^{CV(t)}V(t)2^{\alpha N}\leq \frac{1}{4}$. This proves the above assertion. Under this assumption $V(t)\leq C_1$, we get

$$\|\theta\|_{\widetilde{L}^\rho_t B^{s+\alpha}_{p,1}}\lesssim (1+t^{\frac{1}{\rho}})\Big(\|\theta^0\|_{B^s_{p,1}}+\int_0^t\Gamma_s(\tau)d\tau\Big).$$

This gives the desired result for small time. In order to get the estimate for arbitrary time $t>0$, we consider a partition $(t_i)_{1\leq i\leq K}$ of $[0,t]$ such that

$$\int_{t_i}^{t_{i+1}}\big\|\nabla v(\tau)\big\|_{L^\infty}d\tau\approx C_1.$$

Then reproducing the same calculation we get

$$\|\theta\|_{\widetilde{L}^\rho([t_i,t_{i+1}]B^{s+\alpha}_{p,1})}\lesssim (1+(t_{i+1}-t_i)^{\frac{1}{\rho}})\Big(\|\theta(t_i)\|_{B^s_{p,1}}+\int_{t_i}^{t_{i+1}}\Gamma_s(\tau)d\tau\Big).$$

Since

$$\|\theta\|_{\widetilde{L}^\rho([0,t],B^{s+\alpha}_{p,1})}\leq\sum_{i=0}^{K-1}\|\theta\|_{\widetilde{L}^\rho([t_i,t_{i+1}]B^{s+\alpha}_{p,1})}$$

then

$$\|\theta\|_{\widetilde{L}^\rho([0,t],B^{s+\alpha}_{p,1})}\leq(1+t^{\frac{1}{\rho}})\Big(\sum_{i=0}^{K-1}\|\theta(t_i)\|_{B^s_{p,1}}+\int_0^t\Gamma_s(\tau)d\tau\Big).$$

It suffices now to combine this estimate with (3.4). □

3.2. **Proof of** (3.6)**.**

Proof. Note that the case $\alpha\in]0,1]$ was treated in [1, 16]. We will use here the method developed in these papers to extend the estimate for $\alpha\in[1,2]$. The case $\alpha=2$ can be done explicitly by Leibniz formula and some estimates of the flow. It is plain that

$$\begin{aligned}|\mathrm{D}|^{\alpha}(f_q\circ\psi_q)-(|\mathrm{D}|^{\alpha}f_q)\circ\psi_q &= |\mathrm{D}|^{\frac{\alpha}{2}}\{(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q\}-\{|\mathrm{D}|^{\frac{\alpha}{2}}(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\}\circ\psi_q\\ &\quad+|\mathrm{D}|^{\frac{\alpha}{2}}\{|\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q\}\\ &= \mathrm{I}_q+\mathrm{II}_q.\end{aligned}$$

For the term I_q, it suffices to apply Proposition 3.1 of [17], with $\frac{\alpha}{2}$ and $f=|\mathrm{D}|^{\frac{\alpha}{2}}f_q$. Thus we get,

$$\|\mathrm{I}_q\|_{L^p}\lesssim \max\Big(\big|1-\|\nabla\psi_q^{-1}\|_{L^\infty}^{2+\frac{\alpha}{2}}\big|;\big|1-\|\nabla\psi_q\|_{L^\infty}^{-2-\frac{\alpha}{2}}\big|\Big)\|\nabla\psi_q\|_{L^\infty}^{\frac{\alpha}{2}}\|F_q\|_{\dot B^{\frac{\alpha}{2}}_{p,1}}.$$

The flows $\psi_q^1\overset{def}{=}\psi_q$ and ψ_q^{-1} satisfy the classical estimates

$$e^{-CV(t)}\le\|\nabla\psi_q^{\pm1}\|_{L^\infty}\le e^{CV(t)}. \tag{3.13}$$

It follows from Bernstein inequality

$$\|\mathrm{I}_q\|_{L^p}\lesssim e^{CV(t)}(e^{CV(t)}-1)2^{\alpha q}\|f_q\|_{L^p}. \tag{3.14}$$

For the second term we use the following representation of the fractional Laplacian

$$|\mathrm{D}|^{\frac{\alpha}{2}}f(x)=C\int_{\mathbb{R}^2}\frac{f(x)-f(y)}{|x-y|^{2+\frac{\alpha}{2}}}dy.$$

Since the flow ψ_q preserves Lebesgue measure then we get easily

$$\begin{aligned}|\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)(x)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q(x) &= C\int_{\mathbb{R}^2}\frac{f_q(\psi_q(x))-f_q(\psi_q(y))}{|x-y|^{2+\frac{\alpha}{2}}}\\ &\qquad\times\Big(1-\frac{|x-y|^{2+\frac{\alpha}{2}}}{|\psi_q(x)-\psi_q(y)|^{2+\frac{\alpha}{2}}}\Big)dy.\end{aligned}$$

Set $g_q(x)=f_q(\psi_q(x))$ and use the change of variables $y\mapsto h=x-y$,

$$|\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)(x)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q(x) = C\int_{\mathbb{R}^2}\frac{g_q(x)-g_q(x-h)}{|h|^{2+\frac{\alpha}{2}}}\bar\psi_q(x,h)dh,$$

with

$$\bar\psi_q(x,h)=1-\frac{|h|^{2+\frac{\alpha}{2}}}{|\psi_q(x)-\psi_q(x-h)|^{2+\frac{\alpha}{2}}}.$$

It is not hard to see from Bony's decomposition that for $s>0$ we have the following law product:

$$\|fg\|_{\dot B^s_{p,1}}\lesssim\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,1}}+\|f\|_{\dot B^s_{\infty,1}}\|g\|_{L^p}.$$

Combining this estimate with the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we find

$$\begin{aligned}
\|\mathrm{II}_q\|_{L^p} &\leq \big\||\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q\big\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}\\
&\leq C\|\bar{\psi}_q\|_{L^\infty(\mathbb{R}^4)}\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot)-g_q(\cdot-h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh\\
&+ C\sup_{h\in\mathbb{R}^2}\|\bar{\psi}_q(\cdot,h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}}\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot)-g_q(\cdot-h)\|_{L^p}dh\\
&= J^1_q+J^2_q.
\end{aligned}$$

To estimate J^1_q we use the mean value Theorem,

$$\frac{1}{\|\nabla\psi\|^{2+\frac{\alpha}{2}}_{L^\infty}}\leq\frac{|h|^{2+\frac{\alpha}{2}}}{|\psi(x)-\psi(x-h)|^{2+\frac{\alpha}{2}}}\leq\|\nabla\psi^{-1}\|^{2+\frac{\alpha}{2}}_{L^\infty}.$$

Therefore we get by the definition of $\bar{\psi}_q$ and the above estimate,

$$\|\bar{\psi}_q\|_{L^\infty}\leq\max\Big(|1-\|\nabla\psi_q^{-1}\|^{2+\frac{\alpha}{2}}_{L^\infty}|;|1-\|\nabla\psi_q\|^{-2-\frac{\alpha}{2}}_{L^\infty}|\Big).$$

It follows from (3.13) that

$$\|\bar{\psi}_q\|_{L^\infty}\leq Ce^{CV(t)}(e^{CV(t)}-1). \tag{3.15}$$

Using the definition of Besov spaces and the commutation of Δ_j with translation operators one finds

$$\begin{aligned}
&\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot)-g_q(\cdot-h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh\\
&\qquad\leq\sum_j 2^{\frac{\alpha}{2}j}\int_{\mathbb{R}^2}|h|^{-\frac{\alpha}{2}}\|\Delta_j g_q(\cdot)-(\Delta_j g_q)(\cdot-h)\|_{L^p}\frac{dh}{|h|^2}.
\end{aligned}$$

The characterization of Besov spaces (2.1) yields

$$\begin{aligned}
\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot)-g_q(\cdot-h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh &\leq C\sum_j 2^{\frac{\alpha}{2}j}\|\Delta_j g_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}\\
&\leq C\sum_{|j-k|\leq1}2^{j\frac{\alpha}{2}}2^{\frac{\alpha}{2}k}\|\Delta_j\Delta_k g_q\|_{L^p}\\
&\leq C\|g_q\|_{\dot{B}^{\alpha}_{p,1}}.
\end{aligned}$$

Now we use the following interpolation result,

$$\|g_q\|_{\dot{B}^{\alpha}_{p,1}}\lesssim\|g_q\|^{1-\frac{\alpha}{2}}_{L^p}\|\Delta g_q\|^{\frac{\alpha}{2}}_{L^p}.$$

It is easy to check from the chain rule that

$$\Delta g_q=\Delta(f_q\circ\psi_q)=\sum_{i=1}^d\langle(\nabla^2 f_q)\circ\psi_q\cdot\partial_i\psi_q,\partial_i\psi_q\rangle+(\nabla f_q)\circ\psi_q\cdot\Delta\psi_q.$$

Applying Bernstein inequality and (3.13) we get

$$\|\Delta g_q\|_{L^p} \lesssim e^{CV(t)} 2^{2q} \|f_q\|_{L^p} + 2^q \|f_q\|_{L^p} \|\Delta\psi_q\|_{L^\infty}.$$

The derivative of the flow equation with respect to x and the use of Gronwall and Bernstein inequalities give

$$\begin{aligned} \|\nabla^2\psi_q(t)\|_{L^\infty} &\lesssim e^{CV(t)} \int_0^t \|\nabla^2 S_q v(\tau)\|_{L^\infty} d\tau \\ &\lesssim e^{CV(t)} V(t) 2^q. \end{aligned} \tag{3.16}$$

Combining both last estimates we obtain

$$\|\Delta g_q\|_{L^p} \lesssim e^{CV(t)} 2^{2q} \|f_q\|_{L^p}. \tag{3.17}$$

Putting together (3.15) and (3.17)

$$\|J_q^1(t)\|_{L^p} \lesssim e^{CV(t)}(e^{CV(t)} - 1) 2^{q\alpha} \|f_q\|_{L^p}.$$

Let us now turn to the second term J_q^2. The integral term can be estimated from (2.1) as follows

$$\int_{\mathbb{R}^2} |h|^{-2-\frac{\alpha}{2}} \|g_q(\cdot) - g_q(\cdot - h)\|_{L^p} dh \lesssim \|g_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}.$$

Using the composition result proven in [23]

$$\begin{aligned} \|g_q(t)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} &\lesssim \|\nabla\psi_q\|_{L^\infty}^{\frac{\alpha}{2}} \|f_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} \\ &\lesssim e^{CV(t)} 2^{q\frac{\alpha}{2}} \|f_q\|_{L^p}. \end{aligned} \tag{3.18}$$

In order to estimate $\bar{\psi}_q$ we use the interpolation inequality

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \lesssim \|\bar{\psi}_q(\cdot, h)\|_{L^\infty}^{1-\frac{\alpha}{2}} \|\nabla_x \bar{\psi}_q(\cdot, h)\|_{L^\infty}^{\frac{\alpha}{2}}.$$

This leads in view of (3.15) to

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \leq C e^{CV(t)} \{V(t)\}^{1-\frac{\alpha}{2}} \|\nabla_x \bar{\psi}_q(\cdot, h)\|_{L^\infty}^{\frac{\alpha}{2}}. \tag{3.19}$$

The derivative of $\bar{\psi}_q$ with respect to x yields

$$\begin{aligned} |\nabla_x \bar{\psi}_q(x, h)| &\lesssim \frac{|h|^{3+\frac{\alpha}{2}}}{|\psi_q(x) - \psi_q(x-h)|^{3+\frac{\alpha}{2}}} \frac{|\nabla_x\psi_q(x) - \nabla_x\psi_q(x-h)|}{|h|} \\ &\lesssim \|\nabla\psi_q^{-1}\|_{L^\infty}^{3+\frac{\alpha}{2}} \|\nabla^2\psi_q\|_{L^\infty}. \end{aligned}$$

Combining (3.13) and (3.16), we obtain

$$\|\nabla_x \bar{\psi}_q(t)\|_{L^\infty(\mathbb{R}^4)} \lesssim e^{CV(t)} V(t)\, 2^q. \tag{3.20}$$

Plugging (3.20) into (3.19) we find

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \lesssim e^{CV(t)} V(t)\, 2^{q\frac{\alpha}{2}}. \tag{3.21}$$

Thus we deduce from (3.21) and (3.18) that

$$\|J_q^2(t)\|_{L^p} \le Ce^{CV(t)}V(t)\,2^{q\alpha}\|f_q(t)\|_{L^p}.$$

This achieves the proof of the desired estimate.

□

4. Proof of Theorem 1.1

The aim of this section is to prove our main theorem. It will be done in several steps. In the first step we establish some significant *a priori* estimates. In the second one we prove the uniqueness part and the construction of the solution is described in the third step. the last step is devoted to the continuity-in-time of the solution.

4.1. **a Priori Estimates.**

The *a priori* estimates will be described in several propositions. We start with the following one,

Proposition 4.1. *Let* $\alpha \in]1,2]$, $(p,r) \in [1,\infty[\times]\frac{2}{\alpha-1},\infty[$ *and define* $\bar{p} = \max\{p,r\}$. *If* $\omega^0 \in L^\infty \cap L^p$ *and* $\theta^0 \in L^r$ *then any smooth solution of the Boussinesq system* (B_α) *satisfies*

(1)
$$\|\theta(t)\|_{L^r} \le \|\theta^0\|_{L^r}.$$

(2)
$$\|\omega(t)\|_{L^\infty\cap L^{\bar{p}}} + \|\nabla\theta\|_{L^1_tL^\infty} \le C_0e^{C_0t}.$$

Proof. To prove the inequality (1) it is enough to apply Lemma 3.1. Notice that we do not have any restriction on the value of r for this estimate. To establish the second estimate (2), we start with the vorticity equation

$$\partial_t\omega + v\cdot\nabla\omega = \partial_1\theta.$$

It is clear that

$$\|\omega(t)\|_{L^\infty} \le \|\omega^0\|_{L^\infty} + \|\nabla\theta\|_{L^1_tL^\infty}.$$

Using the classical embedding $B^{1+\frac{2}{r}}_{r,1} \hookrightarrow \mathrm{Lip}(\mathbb{R}^2)$, one can easily obtain

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_tB^{1+\frac{2}{r}}_{r,1}}. \tag{4.1}$$

Combining now Proposition 3.4 with Bernstein inequalities, we get for $\epsilon > 0$

$$\begin{aligned}\|\theta\|_{\widetilde{L}^1_tB^{\alpha-\epsilon}_{r,\infty}} &\lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty}+\|\Delta_{-1}\nabla v\|_{L^1_tL^\infty}\Big)\\ &\lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty}+\|\nabla v\|_{L^1_tL^{\bar{p}}}\Big),\end{aligned}$$

with $\bar{p} \overset{def}{=} \max\{p,r\}$. Take ϵ such that $1+\frac{2}{r} < \alpha-\epsilon$, then we have $\widetilde{L}^1_tB^{\alpha-\epsilon}_{r,\infty} \hookrightarrow L^1_tB^{1+\frac{2}{r}}_{r,1}$. Thus we find,

$$\|\theta\|_{L^1_tB^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty}+\|\nabla v\|_{L^1_tL^{\bar{p}}}\Big).$$

On the other hand, the classical Calderón-Zygmund estimate $\|\nabla v\|_{L^{\bar p}} \approx \|\omega\|_{L^{\bar p}}$ yields

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_t L^\infty}+\|\omega\|_{L^1_t L^{\bar p}}\Big). \tag{4.2}$$

The estimate of the $L^{\bar p}$ norm of the vorticity can be done similarly to the L^∞ estimate

$$\begin{aligned}\|\omega(t)\|_{L^{\bar p}} &\leq \|\omega^0\|_{L^{\bar p}} + \|\nabla\theta\|_{L^1_t L^{\bar p}}\\ &\lesssim \|\omega^0\|_{L^{\bar p}} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}.\end{aligned} \tag{4.3}$$

Set $f(t) \stackrel{def}{=} \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}$, then combining (4.1), (4.2) and (4.3) leads to

$$f(t) \lesssim \|\theta^0\|_{L^r}(1+t+t\|\omega^0\|_{L^\infty\cap L^{\bar p}}) + \|\theta^0\|_{L^r}\int_0^t f(\tau)d\tau.$$

It follows from Gronwall's inequality that,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leq C_0 e^{C_0 t}, \tag{4.4}$$

where C_0 is a constant depending on the initial data. This gives in view of Besov embeddings

$$\|\nabla\theta\|_{L^1_t L^\infty} \leq C_0 e^{C_0 t}.$$

From (4.1) and (4.2), we deduce

$$\|\omega(t)\|_{L^\infty\cap L^{\bar p}} \leq C_0 e^{C_0 t}.$$

This completes the proof of the proposition. □

Next, we will establish the following proposition.

Proposition 4.2. *Under the same assumptions of Proposition 4.1 and if in addition* $\omega^0 \in B^0_{\infty,1}$ *then we have for every* $t \in \mathbb{R}_+$

$$\|\omega(t)\|_{B^0_{\infty,1}} + \|\nabla v(t)\|_{L^\infty} \leq C_0 e^{\exp C_0 t}.$$

Proof. Applying Proposition 2.4 to the vorticity equation and using Besov embeddings,

$$\begin{aligned}\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} &\lesssim \big(\|\omega^0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t B^1_{\infty,1}}\big)\big(1+\|\nabla v\|_{L^1_t L^\infty}\big)\\ &\lesssim \big(\|\omega^0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}\big)\big(1+\|\nabla v\|_{L^1_t L^\infty}\big).\end{aligned} \tag{4.5}$$

On the other hand we have

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v(t)\|_{L^\infty} + \sum_{q\in\mathbb{N}}\|\Delta_q\nabla v(t)\|_{L^\infty}\\ &\lesssim \|\nabla\Delta_{-1}v(t)\|_{L^{\bar p}} + \|\omega(t)\|_{B^0_{\infty,1}}\\ &\lesssim \|\omega(t)\|_{L^{\bar p}} + \|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}}.\end{aligned} \tag{4.6}$$

Putting together (4.4), (4.5) and (4.6) and using Gronwall's inequality we deduce

$$\|\nabla v(t)\|_{L^\infty} + \|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \leq C_0 e^{\exp C_0 t}. \tag{4.7}$$

□

Let us now see how to propagate the initial regularities. The Lipschitz estimate on the velocity will be very crucial.

Proposition 4.3. *Let $p \in]1, \infty[, v^0 \in B_{p,1}^{1+\frac{2}{p}}$ be a divergence free vector-field of $\mathbb{R}^2$ and $\theta^0 \in B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r$, with $1 < 1 + \frac{2}{r} < \alpha$. Then for every $\rho \geq 1$ such that $1 + \frac{2}{r} < \frac{\alpha}{\rho}$ and for $t \in \mathbb{R}_+$,*

$$\|\nabla\theta\|_{L_t^\rho L^\infty} + \|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}-\alpha}} + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} + \|v\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}}.$$

Proof. From Remark 3.3 and (4.7) we have

$$\|\theta\|_{\widetilde{L}_t^\rho B_{r,\infty}^{\frac{\alpha}{\rho}}} \leq C_0 e^{\exp C_0 t}.$$

Now we use Besov embedding $B_{r,\infty}^{\frac{\alpha}{\rho}} \hookrightarrow B_{\infty,\infty}^{\frac{\alpha}{\rho}-\frac{2}{r}}$. Since $1+\frac{2}{r} < \frac{\alpha}{\rho}$ then we have the embeddings

$$\widetilde{L}_t^\rho B_{r,\infty}^{\frac{\alpha}{\rho}} \hookrightarrow L_t^\rho B_{\infty,1}^1 \hookrightarrow L_t^\rho \mathrm{Lip}.$$

Hence, it follows that

$$\|\nabla\theta\|_{L_t^\rho L^\infty} \leq C_0 e^{\exp C_0 t}.$$

To establish the second estimate of the proposition we distinguish two cases: the first one is $-\alpha + 1 + \frac{2}{p} < 1$ and the second one is $-\alpha + 1 + \frac{2}{p} \geq 1$.

• Case $-\alpha + 1 + \frac{2}{p} < 1$. We apply Proposition 3.4 to the temperature equation,

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}-\alpha}} + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} (1+t) e^{CV(t)}.$$

It suffices now to combine this estimate with the Lipschitz bound of the velocity (4.7).

• Case $-\alpha + 1 + \frac{2}{p} \geq 1$. Applying once again Proposition 3.4 we get

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}-\alpha}} + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} \Big(1 + t + \|\nabla\theta\|_{L_t^1 L^\infty} \|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}}\Big) e^{CV(t)}.$$

Hence we obtain from Proposition 4.1 and (4.7)

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \big(1 + \|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big).$$

Applying Proposition 2.5 we get

$$\begin{aligned} \|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}} &\lesssim e^{CV(t)} \big(\|v^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big) \\ &\leq C_0 e^{e^{\exp C_0 t}} \big(1 + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big). \end{aligned}$$

Thus

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \big(1 + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big).$$

Iterating this procedure we get for $n \in \mathbb{N}$

$$\|\theta\|_{L^1_t B^{1+\frac{2}{p}}_{p,1}} \leq C_0 e^{e^{\exp C_0 t}}\big(1+\|\theta\|_{L^1_t B^{-n\alpha+1+\frac{2}{p}}_{p,1}}\big).$$

To conclude it is enough to choose n such that $-(n+1)\alpha+1+\frac{2}{p}<1$ and then we can apply the first case. Finally we get

$$\|\theta\|_{L^1_t B^{1+\frac{2}{p}}_{p,1}} \leq C_0 e^{e^{\exp C_0 t}}.$$

We get also

$$\|\theta\|_{\widetilde{L}^\infty_t B^{1+\frac{2}{p}-\alpha}_{p,1}} \leq C_0 e^{e^{\exp C_0 t}}.$$

Applying again Proposition 2.5 we get

$$\begin{aligned}\|v\|_{\widetilde{L}^\infty_t B^{1+\frac{2}{p}}_{p,1}} &\lesssim e^{CV(t)}\big(\|v^0\|_{B^{1+\frac{2}{p}}_{p,1}}+\|\theta\|_{L^1_t B^{1+\frac{2}{p}}_{p,1}}\big)\\ &\leq C_0 e^{e^{\exp C_0 t}}.\end{aligned}$$

□

4.2. **Uniqueness.** We will prove a uniqueness result in the following space

$$\mathcal{X}_T=(L^\infty_T L^p\cap L^1_T\mathrm{Lip})\times(L^\infty_T L^r\cap L^1_T\mathrm{Lip}),\quad r>2.$$

Without loss of generality we can suppose that $p\in[r,\infty[$. Indeed, let $\bar p=\max\{p,r\}$ then from Besov embedding we have $B^{1+\frac{2}{p}}_{p,1}\hookrightarrow B^{1+\frac{2}{\bar p}}_{\bar p,1}\hookrightarrow L^{\bar p}$. Let (v^1,π^1,θ^1) and (v^2,π^2,θ^2) be two solutions of (B_α) belonging to the space $\mathcal{X}_T$ and denote

$$v=v^2-v^1,\quad \theta=\theta^2-\theta^1\quad\text{and}\quad \pi=\pi^2-\pi^1.$$

Then we have the equations

$$\begin{cases}\partial_t v+v^2\cdot\nabla v=-\nabla\pi-v\cdot\nabla v^1+\theta e_2\\ \partial_t\theta+v^2\cdot\nabla\theta+|\mathrm{D}|^\alpha\theta=-v\cdot\nabla\theta^1\\ v_{|t=0}=v^0,\quad \theta_{|t=0}=\theta^0.\end{cases}$$

The L^p estimate of the velocity is given by

$$\|v(t)\|_{L^p}\leq\|v^0\|_{L^p}+\int_0^t\|v(\tau)\|_{L^p}\|\nabla v^1(\tau)\|_{L^\infty}+\|\nabla\pi(\tau)\|_{L^p}d\tau+\|\theta\|_{L^1_tL^p}.$$

From the incompressibility condition we get

$$\nabla\pi\;=\;\nabla\Delta^{-1}\mathrm{div}\big(-v\cdot\nabla v^1+\theta e_2\big)-\nabla\Delta^{-1}\mathrm{div}(v^2\cdot\nabla v).$$

Now due to the identity $\mathrm{div}(v^2\cdot\nabla v)=\mathrm{div}(v\cdot\nabla v^2)$, one obtains

$$\nabla\pi=\nabla\Delta^{-1}\mathrm{div}\big(-v\cdot\nabla(v^1+v^2)+\theta e_2\big).$$

Using the continuity of Riesz transform on L^p with $p\in]1,\infty[$ we get

$$\|\nabla\pi\|_{L^p}\lesssim\|v\|_{L^p}\big(\|\nabla v^1\|_{L^\infty}+\|\nabla v^2\|_{L^\infty}\big)+\|\theta\|_{L^p}.$$

Combining this estimate with the L^p estimate of the velocity we get

$$\|v(t)\|_{L^p}\lesssim\|v^0\|_{L^p}+\int_0^t\|v(\tau)\|_{L^p}\big(\|\nabla v^1(\tau)\|_{L^\infty}+\|\nabla v^2(\tau)\|_{L^\infty}\big)d\tau+\|\theta\|_{L^1_tL^p}.$$

Now, we apply Proposition 3.4 with $s=-1+\epsilon$ and $\epsilon\in]0,1[$ we get

$$\begin{aligned}\|\theta(t)\|_{L^1_tL^p} &\lesssim \|\theta\|_{L^1_tB^{-1+\epsilon+\alpha}_{p,1}}\\ &\lesssim e^{C\|\nabla v^2\|_{L^1_tL^\infty}}\Big(\|\theta^0\|_{B^{-1+\epsilon}_{p,1}}+\int_0^t\|v\cdot\nabla\theta^1(\tau)\|_{B^{-1+\epsilon}_{p,1}}d\tau\Big)\\ &\lesssim e^{C\|\nabla v^2\|_{L^1_tL^\infty}}\Big(\|\theta^0\|_{B^{-1+\epsilon}_{p,1}}+\int_0^t\|v\cdot\nabla\theta^1(\tau)\|_{L^p}d\tau\Big)\\ &\lesssim e^{C\|\nabla v^2\|_{L^1_tL^\infty}}\Big(\|\theta^0\|_{L^r}+\int_0^t\|v(\tau)\|_{L^p}\|\nabla\theta^1(\tau)\|_{L^\infty}d\tau\Big).\end{aligned}$$

We have used in the last inequality the embeddings $L^r\hookrightarrow B^{-1+\epsilon+\frac{2}{r}}_{r,1}\hookrightarrow B^{-1+\epsilon}_{p,1}$ valid for ϵ satisfying $-1+\epsilon+\frac{2}{r}<0$. This is possible since $r>2$. Finally we get

$$\begin{aligned}\|v(t)\|_{L^p} &\lesssim e^{C\|\nabla v^2\|_{L^1_tL^\infty}}\Big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\\ &+\int_0^t\|v(\tau)\|_{L^p}\big(\|\nabla v^1(\tau)\|_{L^\infty}+\|\nabla v^2(\tau)\|_{L^\infty}+\|\nabla\theta^1\|_{L^\infty}\big)d\tau\Big).\end{aligned}$$

Using Gronwall's inequality we find

$$\|v(t)\|_{L^p}\lesssim e^{C\|(v^1,v^2,\theta^2)\|_{L^1_t\mathrm{Lip}}}\big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\big). \tag{4.8}$$

This gives in turn

$$\|\theta\|_{L^1_tL^p}\lesssim e^{C\|(v^1,v^2,\theta^2)\|_{L^1_t\mathrm{Lip}}}\big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\big)(1+t). \tag{4.9}$$

This concludes the proof of the uniqueness part.

4.3. **Existence.** We consider the following system

$$(\mathrm{B}_n)\qquad \begin{cases}\partial_t v_n+v_n\cdot\nabla v_n+\nabla\pi_n=\theta_n e_2\\ \partial_t\theta_n+v_n\cdot\nabla\theta_n+|\mathrm{D}|^\alpha\theta_n=0\\ \mathrm{div}v_n=0\\ v_{n|t=0}=S_nv^0,\quad \theta_{n|t=0}=S_n\theta^0.\end{cases}$$

By using the method of [18] we can prove that this system has a unique local smooth solution (v_n,θ_n). The global existence of this solution is governed by the quantity $\|\nabla v_n\|_{L^1_TL^\infty}$. Now from the *a priori* estimates the Lipschitz norm can not blow up in finite time and then the solution (v_n,θ_n) is globally defined. Once again from the *a priori* estimates we have

$$\|v_n\|_{\widetilde L^\infty_TB^{1+\frac{2}{p}}_{p,1}}+\|\theta_n\|_{\widetilde L^\infty_TB^{-\alpha+1+\frac{2}{p}}_{p,1}}+\|\theta_n\|_{L^\infty_TL^r}+\|\nabla\theta_n\|_{L^\rho_TL^\infty}\le\Phi_3(T).$$

Consequently, up to an extraction the sequence (v_n, θ_n) converges weakly to (v, θ) belonging to $\widetilde{L}^\infty_T B^{1+\frac{2}{p}}_{p,1} \times \Big(\widetilde{L}^\infty_T B^{-\alpha+1+\frac{2}{p}}_{p,1} \cap L^\infty_T L^r \cap L^\rho_T \text{Lip}\Big)$.
For $(n, m) \in \mathbb{N}^2$ we set $v_{n,m} = v_n - v_m$ and $\theta_{n,m} = \theta_n - \theta_m$ then according to the estimates (4.8) and (4.9) we get

$$\|v_{n,m}\|_{L^\infty_T L^{\bar p}} + \|\theta_{n,m}\|_{L^1_T L^{\bar p}} \le C_0 e^{e^{\exp C_0 t}} \big(\|S_n v^0 - S_m v^0\|_{L^p} + \|S_n\theta^0 - S_m\theta^0\|_{L^r}\big),$$

with $\bar p = \max\{p, r\}$. This proves that the sequence (v_n, θ_n) converges strongly to (v, θ) in $L^\infty_T L^{\bar p} \times L^1_T L^{\bar p}$. This allows us to pass to the limit in the system (B_n) and then we get that (v, θ) is a solution of the Boussinesq system (B_α).

4.4. **Continuity-in-time.** Let us first sketch the proof of the continuity in time of the velocity. Let $\epsilon > 0, N \in \mathbb{N}^*$ and $T > 0$, then for every $0 \le \tau \le t \le T$,

$$\begin{aligned}
\|v(t) - v(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} &\le \sum_{q\le N} 2^{q(1+\frac{2}{p})}\|\Delta_q v(t) - \Delta_q v(\tau)\|_{L^p} + 2\sum_{q>N} 2^{q(1+\frac{2}{p})}\|\Delta_q v\|_{L^\infty_T L^p}\\
&\lesssim 2^{N(1+\frac{2}{p})}\|v(t) - v(\tau)\|_{L^p} + \sum_{q>N} 2^{q(1+\frac{2}{p})}\|\Delta_q v\|_{L^\infty_T L^p}.
\end{aligned}$$

Since $v \in \widetilde{L}^\infty_T B^{1+\frac{2}{p}}_{p,1}$, then there exists N sufficiently large such that

$$\sum_{q>N} 2^{q(1+\frac{2}{p})}\|\Delta_q v\|_{L^\infty_T L^p} \le \epsilon.$$

Thus we get

$$\|v(t) - v(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} \lesssim 2^{N(1+\frac{2}{p})}\|v(t) - v(\tau)\|_{L^p} + \epsilon.$$

On the other hand from the equation of the velocity we get

$$v(t, x) - v(\tau, x) = \int_\tau^t \mathcal{P}(v\cdot\nabla v)(t', x)dt' + \int_\tau^t \mathcal{P}(\theta e_2)(t', x)dt',$$

where $\mathcal{P}$ denotes Leray's projector. Since $\mathcal{P}$ acts continuously on L^p for $p \in]1, \infty[$ then

$$\begin{aligned}
\|v(t) - v(\tau)\|_{L^p} &\lesssim \int_\tau^t \|\nabla v(t')\|_{L^\infty}\|v(t')\|_{L^p}dt' + \int_\tau^t \|\theta(t')\|_{L^p}dt'\\
&\lesssim |t-\tau|\|\nabla v\|_{L^\infty_T L^\infty}\|v\|_{L^\infty_T L^p} + \int_\tau^t \|\theta(t')\|_{L^p}dt'.
\end{aligned}$$

Let $\rho > \alpha$, then using Hölder inequality, Besov embeddings and Proposition 3.4 we get

$$\begin{aligned}
\int_\tau^t \|\theta(t')\|_{L^p}dt' &\le (t-\tau)^{1-\frac{1}{\rho}}\|\theta\|_{\widetilde{L}^\rho_t B^0_{p,1}}\\
&\lesssim e^{CV(t)}(t-\tau)^{1-\frac{1}{\rho}}(1 + t^{\frac{1}{\rho}})\|\theta^0\|_{B^{-\frac{\alpha}{\rho}}_{p,1}}.
\end{aligned}$$

Choose ρ such that $-\alpha+1+\frac{2}{p}\geq -\frac{\alpha}{\rho}$, which is possible for ρ close to α. Then

$$\int_\tau^t \|\theta(t')\|_{L^p}dt' \lesssim e^{CV(t)}(t-\tau)^{1-\frac{1}{\rho}}(1+t^{\frac{1}{\rho}})\|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}}.$$

Finally we obtain

$$\|v(t)-v(\tau)\|_{L^p} \lesssim |t-\tau|\|\nabla v\|_{L^\infty_T L^\infty}\|v\|_{L^\infty_T L^p}+e^{CV(t)}|t-\tau|^{1-\frac{1}{\rho}}(1+t^{\frac{1}{\rho}})\|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}}.$$

This ensures the continuity in time of the velocity.
Let us now move to the proof of the continuity-in-time of the temperature. We will first prove that $\theta\in\mathscr{C}([0,T],B_{p,1}^{-\alpha+1+\frac{2}{p}})$. Similarly to the velocity we write since $\theta\in\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}$ that for large N

$$\begin{aligned}\|\theta(t)-\theta(\tau)\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} &\leq \sum_{q\leq N}2^{q(-\alpha+1+\frac{2}{p})}\|\Delta_q\theta(t)-\Delta_q\theta(\tau)\|_{L^p}\\ &+ 2\sum_{q>N}2^{q(-\alpha+1+\frac{2}{p})}\|\Delta_q\theta\|_{L^\infty_T L^p}\\ &\leq \sum_{q\leq N}2^{q(-\alpha+1+\frac{2}{p})}\|\Delta_q\theta(t)-\Delta_q\theta(\tau)\|_{L^p}+\epsilon.\end{aligned}$$

From the equation of θ and Bernstein inequality we find for $0\leq\tau\leq t$

$$\begin{aligned}\|\Delta_q\theta(t)-\Delta_q\theta(\tau)\|_{L^p} &\leq \int_\tau^t\|\Delta_q(v\cdot\nabla\theta)(t')\|_{L^p}dt'+2^{q\alpha}(t-\tau)\|\Delta_q\theta\|_{L^p}\\ &\lesssim \|v\|_{L^\infty_T L^p}\int_\tau^t\|\nabla\theta(t')\|_{L^\infty}dt'+2^{q(2\alpha-1-\frac{2}{p})}(t-\tau)\|\theta\|_{\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\theta(t)-\theta(\tau)\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} &\leq 2^{N(1+\frac{2}{p})}\|v\|_{L^\infty_T L^p}\int_\tau^t\|\nabla\theta(t')\|_{L^\infty}dt'\\ &+ 2^{N\alpha}(t-\tau)\|\theta\|_{\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}}+\epsilon.\end{aligned}$$

This proves that $\theta\in\mathscr{C}([0,T],B_{p,1}^{-\alpha+1+\frac{2}{p}})$.
Let us now prove that $\theta\in\mathscr{C}([0,T],L^r)$. Denote by $S(t)=e^{-t|\mathrm{D}|^\alpha}$ and $f=-v\cdot\nabla\theta$. Then from Duhamel formulae we get

$$\begin{aligned}\theta(t,x)-\theta(\tau,x) &= [S(t)-S(\tau)]\theta^0(x)+S(t)\Big(\int_\tau^t S(-t')f(t')dt'\Big)\\ &+ [S(t)-S(\tau)]\Big(\int_0^\tau S(-t')f(t')dt'\Big).\end{aligned}$$

To conclude we use the fact that $(S(t))_{t\geq 0}$ is a C_0-semigroup of contractions with positive kernels combined with the estimates

$$\int_\tau^t \|f(t')\|_{L^r} dt' \leq \|v\|_{L^\infty_T L^\infty} \int_\tau^t \|\nabla\theta(t')\|_{L^r} dt'$$

and

$$\Big|[S(t)-S(\tau)]\Big(\int_0^\tau S(-t')f(t',x)dt'\Big)\Big| \leq [S(t)-S(\tau)]\Big(\int_0^T |S(-t')f(t',x)|dt'\Big).$$

This integral $\int_0^t \|f(t')\|_{L^p} dt'$ is bounded according to (4.4) and Proposition 4.3.

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