ON THE GLOBAL REGULARITY OF AXISYMMETRIC NAVIER-STOKES-BOUSSINESQ SYSTEM

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Abstract. In this paper we prove a global well-posedness result for tridimensional Navier-Stokes-Boussinesq system with axisymmetric initial data. This system couples Navier-Stokes equations with a transport equation governing the density.

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1. Introduction

The purpose of this paper is to study the global well-posedness for three-dimensional Boussinesq system in the whole space with axisymmetric initial data. This system is described as follows,

$$
\begin{align*}
  \partial_t v + v \cdot \nabla v - \Delta v + \nabla p &= \rho e_z, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
  \partial_t \rho + v \cdot \nabla \rho &= 0, \\
  \text{div } v &= 0, \\
  v|_{t=0} &= v_0, \quad \rho|_{t=0} = \rho_0.
\end{align*}
$$

(1)

Here, the velocity $v = (v^1, v^2, v^3)$ is a three-component vector field with zero divergence. The scalar function $\rho$ denotes the density which is transported by the flow and acting for the first equation of (1) only in the vertical direction given by $e_z$. The pressure $p$ is a scalar function related to the unknowns $v$ and $\rho$ through an elliptic equation. Remark that when the initial density $\rho_0$ is identically zero then the above
system is reduced to the classical Navier-Stokes system which is widely studied:

\[
\begin{aligned}
    \partial_t v + v \cdot \nabla v - \Delta v + \nabla p &= 0 \\
    \text{div } v &= 0 \\
    v|_{t=0} &= v_0.
\end{aligned}
\]

(2)

Recall that the existence of global weak solutions in the energy space for (2) goes back to J. Leray [23] in the last century. However the uniqueness of these solutions is only known in space dimension two. It is also well-known that smooth solutions are global in dimension two and for higher dimensions when the data are small in some critical spaces; see for instance [21] for more detailed discussions. Although the breakdown of smooth solutions with large initial data is still now an open problem some partial results are known outside the context of small data. We refer for instance to recent papers of J.-Y. Chemin and I. Gallagher [10, 11] where global existence in dimension three is established for special structure of initial data which are not small in any critical space.

There is an interesting case of global existence for (2) corresponding to large initial data but with special geometry, called axisymmetric without swirl. Before going further in the details let us give some general statements about Navier-Stokes system in space dimension three. First, we start with introducing the vorticity which is a physical quantity that plays a significant role in the theory of global existence; for a given vector field \( v \) the vorticity \( \omega \) is the vector defined by \( \omega = \text{curl } v \). Thus we get from (2) the vorticity equation

\[
\partial_t \omega + v \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla v.
\]

According to Beale-Kato-Majda criterion the formation of singularities in finite time is due to the accumulation of the vorticity. In other words, to have global existence it suffices to bound for every time the quantity \( \| \omega(t) \|_{L^\infty} \). However the main difficulty arising in dimension three is the lack of information about the manner that the vortex-stretching term \( \omega \cdot \nabla v \) affects the dynamic of the fluid.

For the geometry of axisymmetric flows without swirl we have a cancellation in the stretching term giving rise to new conservation laws. We say that a vector field \( v \) is axisymmetric if it has the form:

\[ v(t, x) = v^r(t, r, z)e_r + v^z(t, r, z)e_z, \quad x = (x_1, x_2, z), \quad r = (x_1^2 + x_2^2)^{1/2}, \]

where \((e_r, e_\theta, e_z)\) is the cylindrical basis of \( \mathbb{R}^3 \) and the components \( v^r \) and \( v^z \) do not depend on the angular variable. The main feature of axisymmetric flows arises in the vorticity which takes the form,

\[ \omega = (\partial_z v^r - \partial_r v^z)e_\theta := \omega_\theta e_\theta \]

and satisfies

\[ \partial_t \omega + v \cdot \nabla \omega - \Delta \omega = \frac{v^r}{r} \omega. \]
Since the Laplacian operator has the form \( \Delta = \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{rr} + \partial_{zz} \) in the cylindrical coordinates then the component \( \omega_\theta \) of the vorticity will satisfy
\[
(3) \quad \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \Delta \omega_\theta + \frac{\omega_\theta}{r^2} = \frac{v^r}{r} \omega_\theta.
\]
Consequently, the quantity \( \Gamma := \frac{\omega_\theta}{r} \) obeys to the equation
\[
\partial_t \Gamma + v \cdot \nabla \Gamma - \Delta \Gamma - \frac{2}{r} \partial_r \Gamma = 0.
\]
Obviously, we have for \( p \geq 1, \int \frac{2}{r} \partial_r \Gamma |\Gamma|^{p-1} (\text{sign } \Gamma) dx \leq 0 \) and then we deduce that for all \( p \in [1, \infty] \)
\[
\|\Gamma(t)\|_{L^p} \leq \|\Gamma_0\|_{L^p}.
\]
It was shown by M. Ukhoviskii and V. Yudovich [25] and independently by O. A. Ladyzhenskaya [20] that these conservation laws are strong enough to prevent the formation of singularities in finite time for axisymmetric flows. More precisely, the system (2) has a unique global solution for \( v_0 \in H^1 \) such that \( \omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty \).

Remark that in term of Sobolev regularities these assumptions are satisfied when \( v_0 \in H^s \) with \( s > \frac{7}{2} \). Few decades afterwards, S. Leonardi, J. Málek, J. Necás and M. Pokorný [22] weakened the initial regularity for \( v_0 \in H^2 \). This result was recently improved by H. Abidi [1] for \( v_0 \in H^{\frac{5}{2}} \).

Let us now come back to our first problematic which is the study of global well-posedness for the Boussinesq system (1). In space dimension two many papers are recently devoted to this problem and the study seems to be in a satisfactory state. More precisely we have global existence in different function spaces and for different viscosities, we refer for example to [2, 7, 8, 13, 14, 15, 16, 17, 18, 19].

In the case of space dimension three few results are known about global existence. We recall the result of R. Danchin and M. Paicu [14]. They proved a global well-posedness result for small initial data belonging to some critical Lorentz spaces.

Our goal here is to study the global existence for the system (1) with axisymmetric initial data, which means that the velocity \( v_0 \) is assumed to be an axisymmetric vector field without swirl and the density \( \rho_0 \) depends only on \((r, z)\). It should be mentioned that this structure is preserved for strong solutions in their lifespan.

Before stating our main result we denote by \( \Pi_z \) the orthogonal projector over the axis \((Oz)\). Our result reads as follows.

**Theorem 1.1.** Let \( v_0 \in H^1 \) be an axisymmetric vector field with zero divergence and such that \( \frac{\omega_0}{r} \in L^2 \). Let \( \rho_0 \in L^2 \cap L^\infty \) depending only on \((r, z)\) and such that \( \text{supp } \rho_0 \) does not intersect the axis \((Oz)\) and \( \Pi_z(\text{supp } \rho_0) \) is a compact set. Then the system (1) has a unique global solution \((v, \rho)\) such that
\[
\begin{align*}
v &\in C(\mathbb{R}^+; H^1) \cap L^1_{\text{loc}}(\mathbb{R}^+; W^{1,\infty}), \\
\frac{\omega}{r} &\in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2), \\
\rho &\in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2 \cap L^\infty).
\end{align*}
\]
The assumption that the density $\rho$ is zero in some region of the space $\mathbb{R}^3$ is not very meaningful from a physical point of view. However we can relax this hypothesis and extend our result to more general case: the initial density is assumed to be constant near the axis $(Oz)$ and for large value of $z$. More precisely, we have the following result.

**Corollary 1.2.** Let $v_0 \in H^1$ be an axisymmetric vector field with zero divergence and such that $\frac{\omega_0}{r} \in L^2$. Let $\rho_0 \in L^\infty$ depending only on $(r, z)$ such that $\rho_0 \equiv c_0$, for some constant $c_0$ in a region of type $\{x; r \leq r_0 \text{ or } |z| \geq |z_0|\}$, with $r_0 > 0$ and $\rho_0 - c_0 \in L^2$. Then the system (1) has a unique global solution $(v, \rho)$ such that

$$v \in C(\mathbb{R}_+; H^1) \cap L^1_{loc}(\mathbb{R}_+; W^{1,\infty}),$$

$$\frac{\omega}{r} \in L^\infty_{loc}(\mathbb{R}_+; L^2), \quad \rho - c_0 \in L^\infty_{loc}(\mathbb{R}_+; L^2 \cap L^\infty).$$

The proof of this corollary is an immediate consequence of Theorem 1.1. Indeed, we set $\bar{\rho}(t, x) = \rho(t, x) - c_0$, then the system (1) is reduced to

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p - c_0 e_z = \bar{\rho} e_z \\
\partial_t \rho + v \cdot \nabla \rho = 0 \\
\text{div} v = 0 \\
v|_{t=0} = v_0, \quad \bar{\rho}|_{t=0} = \rho_0 - c_0.
\end{cases}$$

Now by changing the pressure $p$ to $\bar{p} = p - c_0 z$ we get the same system (1) and therefore we can apply the results of Theorem 1.1.

Next we shall briefly discuss the new difficulties that one should deal with compared to the system (2). First we start with writing the analogous equation to (3) for the vorticity. An easy computation gives

$$\text{curl}(\rho e_z) = \begin{pmatrix} \partial_2 \rho \\ \partial_1 \rho \\ 0 \end{pmatrix} = -(\partial_r \rho) e_\theta.$$

This yields to

$$\partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \Delta \omega_\theta + \frac{\omega_\theta}{r^2} = \frac{v^r}{r} \omega_\theta - \partial_r \rho.$$ 

(4)

It follows that the evolution of the quantity $\Gamma := \frac{\omega_\theta}{r}$ is governed by the equation

$$\partial_t \Gamma + v \cdot \nabla \Gamma - \Delta \Gamma - \frac{2}{r} \partial_r \Gamma = -\frac{\partial_r \rho}{r}.$$ 

(5)

Since the density $\rho$ satisfies a transport equation then the only conserved quantities that one should use are $\|\rho(t)\|_{L^p}$ for every $p \in [1, \infty]$. Loosely speaking, the source term $\frac{\partial_r \rho}{r}$ in the equation (5) that one need to estimate has the scale of $\Delta \rho$ and then to estimate $\Gamma$ in some Lebesgue spaces one can try for example the maximal regularity of the heat semigroup. However it is not at all clear whether we can prove a suitable maximal regularity because the involved elliptic operator $-\Delta - \frac{2}{r} \partial_r$ has...
singular coefficients. Now, by taking the $L^2$-inner product of (5) with $\Gamma$ we are led to estimate the quantity $\|\rho/r\|_{L^2}$ which has the scaling of $\|\nabla\rho\|_{L^2}$. In other words we have

$$\|\Gamma(t)\|_{L^2}^2 + \int_0^t \|\Gamma(\tau)\|_{H^1}^2 d\tau \leq \|\Gamma_0\|_{L^2}^2 + \int_0^t \|((\rho/r)(\tau))\|_{L^2}^2 d\tau.$$ 

Remark that the quantity $\|\rho/r\|_{L^2}$ is not well-defined if we don’t assume at least that $\rho(t, 0, z) = 0$. As we have seen previously in the statement of Theorem 1.1 we need more than this latter condition: the initial density $\rho_0$ has to be supported far from the axis ($Oz$). To follow the evolution of the quantity $\|\rho(t)/r\|_{L^2}$ the approach that consists in writing the equation of $\rho(t)$ gives an exponential growth,

$$\|\rho(t)/r\|_{L^2} \leq \|\rho_0/r\|_{L^2} e^\|v/r\|_{L^1 L^\infty}.$$ 

Our idea to perform this growth relies in studying some dynamic aspects of the support of $\rho(t)$ which is nothing but the transported of the initial support by the flow. We need particularly to get a proper low bound about the distance from the axis ($Oz$) of the support of $\rho(t)$, see Proposition 3.2. To reach this target and once again the axisymmetry property of the flow will play a crucial role since the trajectories of the particles are contained in the meridional plane. This approach allows us to improve the exponential growth for $\|\rho(t)/r\|_{L^2}$ to a quadratic estimate. Roughly speaking we obtain the estimate

$$\|\rho(t)/r\|_{L^2} \leq C_0 \|v/r\|_{L^1 L^\infty} (1 + \|v\|_{L^1 L^\infty}).$$ 

The rest of the paper is organized as follows. In section 2 we recall some basic ingredients of Littlewood-Paley theory. In section 3 we study some qualitative and analytic properties of the flow associated to an axisymmetric vector field. In section 4 we give some global a priori estimates. The proof of Theorem 1.1 is done in Section 5.

2. Preliminaries

Throughout this paper, $C$ stands for some real positive constant which may be different in each occurrence and $C_0$ for a positive constant depending on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$.

Let us start with a classical dyadic decomposition of the whole space (see [9]): there exist two radial functions $\chi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \setminus \{0\})$ such that

1. $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \ \forall \xi \in \mathbb{R}^3$,
2. $\text{supp } \varphi(2^{-p}\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \emptyset$, if $|p - q| \geq 2$,
3. $q \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-q}) = \emptyset$. 


For every \( u \in \mathcal{S}'(\mathbb{R}^3) \) we define the nonhomogeneous Littlewood-Paley operators by,
\[
\Delta_{-1} u = \chi(D)u; \quad \forall q \in \mathbb{N}, \quad \Delta_q u = \varphi(2^{-q}D)u \quad \text{and} \quad S_q u = \sum_{-1 \leq j \leq q-1} \Delta_j u.
\]

One can easily prove that for every tempered distribution \( u \),
\[
(6) \quad u = \sum_{q \geq -1} \Delta_q u.
\]

In the sequel we will make an extensive use of Bernstein inequalities (see for example [9]).

**Lemma 2.1.** There exists a constant \( C \) such that for \( k \in \mathbb{N} \), \( 1 \leq a \leq b \) and \( u \in L^a \), we have
\[
\sup_{|\alpha|=k} \| \partial^\alpha S_q u \|_{L^b} \leq C^k 2^{q(k+3(\frac{1}{a} - \frac{1}{b})}) \| S_q u \|_{L^a},
\]
and for \( q \in \mathbb{N} \)
\[
C^{-k} 2^{qk} \| \Delta_q u \|_{L^a} \leq \sup_{|\alpha|=k} \| \partial^\alpha \Delta_q u \|_{L^a} \leq C^k 2^{qk} \| \Delta_q u \|_{L^a}.
\]

Let us now introduce the basic tool of the paradifferential calculus which is Bony’s decomposition [6]. It distinguishes in a product \( uv \) three parts as follows:
\[
uv = T_u v + T_v u + \mathcal{R}(u, v),
\]
where
\[
T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \text{and} \quad \mathcal{R}(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v,
\]
with \( \tilde{\Delta}_q = \sum_{i=-1}^1 \Delta_{q+i} \).

\( T_u v \) is called paraproduct of \( v \) by \( u \) and \( \mathcal{R}(u, v) \) the remainder term.

Let \( (p, r) \in [1, +\infty]^2 \) and \( s \in \mathbb{R} \), then the nonhomogeneous Besov space \( B^s_{p,r} \) is the set of tempered distributions \( u \) such that
\[
\| u \|_{B^s_{p,r}} := \left( 2^{qs} \| \Delta_q u \|_{L^p} \right)_{\ell^r} < +\infty.
\]

We remark that the usual Sobolev space \( H^s \) agrees with Besov space \( B^s_{2,2} \). Also, by using the Bernstein inequalities we get easily
\[
B^s_{p_1, r_1} \hookrightarrow B^s_{p_2, r_2} := B^s_{\frac{p_2 r_2}{r_2 - \frac{1}{r_1}}, p_1 \leq p_2 \quad \text{and} \quad r_1 \leq r_2}.
\]
3. Study of the flow map

The main goal of this section is to study some geometric and analytic properties of the generalized flow map associated to an axisymmetric vector field.

\[ \psi(t, s, x) = x + \int_s^t v(\tau, \psi(\tau, s, x))d\tau. \]

This part will be the cornerstone of the proof of Theorem 1.1. We need first to recall some basic results about the generalized flow. If the vector field \( v \) belongs to \( L^1_{\text{loc}}(\mathbb{R}, C^1_b) \) then the generalized flow is uniquely determined and exists globally in time. Here \( C^1_b \) denotes the space of functions with continuous bounded gradient. In addition, for every \( t, s \in \mathbb{R} \), \( \psi(t, s) \) is a diffeomorphism that preserves Lebesgue measure when \( \text{div} \ v = 0 \) and

\[ \psi^{-1}(t, s, x) = \psi(s, t, x). \]

Now we define the distance from a given point \( x \) to a subset \( A \subset \mathbb{R}^3 \) by

\[ d(x, A) := \inf_{y \in A} \| x - y \|, \]

where \( \| \cdot \| \) is the usual Euclidian norm. The distance between two subsets \( A \) and \( B \) of \( \mathbb{R}^3 \) is defined by

\[ d(A, B) := \inf_{x \in A, y \in B} \| x - y \|. \]

The diameter of a bounded subset \( A \subset \mathbb{R}^3 \) is defined by

\[ \text{diam} \ A = \sup_{x, y \in A} \| x - y \|. \]

Our first result is the following.

**Proposition 3.1.** Let \( v \) be a smooth axisymmetric vector field and \( \psi(t, s) \) its flow. Let \( x \notin (Oz) \) and \( r(x) := d(x, (Oz)) \), then

1. For every \( s \in \mathbb{R} \), the trajectory \( \Gamma_{x,s} := \{ \psi(t, s, x), t \in \mathbb{R} \} \) is a smooth curve contained in the meridional plan.
2. For every \( s \in \mathbb{R} \), the trajectory \( \Gamma_{x,s} \) does not intersect the axis \( (Oz) \), that is \( \Gamma_{x,s} \cap (Oz) = \emptyset \). More precisely,

\[ r(x)e^{-\int_s^t \| \nabla \psi(\tau) \|_{L^\infty}d\tau} \leq d(\psi(t, s, x), (Oz)) \leq r(x)e^{|\int_s^t \| \nabla \psi(\tau) \|_{L^\infty}d\tau|.} \]

**Proof.** (1) We start with the decomposition of the vector \( x \) in the cylindrical basis:

\[ x = r(x) \begin{pmatrix} \cos \theta_x \\ \sin \theta_x \\ 0 \end{pmatrix} + z_x e_z, \quad \text{with} \quad r_x > 0. \]
We decompose also the generalized flow $\psi(t,s,x)$ in the cylindrical coordinates

$$\psi(t,s,x) = r(t,s,x) \begin{pmatrix} \cos(\theta(t,s,x)) \\ \sin(\theta(t,s,x)) \\ 0 \end{pmatrix} + z(t,s,x)e_z.$$ 

Since $v$ is smooth in space-time variables then the flow map $\psi(t,s,x)$ is also smooth. Now we intend to prove that $r(t,s,x)$ remains strictly positive for all $t,s \in \mathbb{R}$. Assume that there exist $t_1, s_1 \in \mathbb{R}$ such that $\psi(t_1,s_1,x)$ belongs to the axis $(Oz)$. As the restriction of the vector field $v$ on the axis $(Oz)$ satisfies $v(t,0,z) = v^z(t,0,z)e_z$ then the trajectory for every $x_0 \in (Oz)$ lies in this same axis

$$\psi(t,s,x_0) = x_0 + \left(0,0, \int_s^t v^z(\tau,s,x_0)d\tau \right).$$

Clearly one can choose $x_0$ such that $\psi(t_1,s_1,x_0) = \psi(t_1,s_1,x)$ and this contradicts the fact that the flow map is an homeomorphism. Consequently and from the smoothness of the generalized flow map one can prove easily that the functions $r(t,s,x)$, $\theta(t,s,x)$ and $z(t,s,x)$ are also smooth in each variable. This allows us to justify the following computation

$$\partial_t \psi(t,s,x) = \partial_t r(t,s,x) \begin{pmatrix} \cos(\theta(t,s,x)) \\ \sin(\theta(t,s,x)) \\ 0 \end{pmatrix} + r(t,s,x) \partial_t \theta(t,s,x) \begin{pmatrix} -\sin(\theta(t,s,x)) \\ \cos(\theta(t,s,x)) \\ 0 \end{pmatrix} + \partial_t z(t,s,x)e_z.$$ 

Since the vector field $v$ is axisymmetric without swirl then

$$v(t,\psi(t,s,x)) = v^r(t,r(t,s,x),z(t,s,x)) \begin{pmatrix} \cos(\theta(t,s)) \\ \sin(\theta(t,s)) \\ 0 \end{pmatrix} + v^z(t,r(t,s,x),z(t,s,x))e_z.$$ 

Thus we get by identification

$$\partial_t r(t,s,x) = v^r(t,r(t,s,x),z(t,s,x)),
\partial_t \theta(t,s) = 0,
\partial_t z(t,s,x) = v^z(t,r(t,s,x),z(t,s,x)).$$

From the above discussion we have $r(t,s,x) > 0$ for all $t,s \in \mathbb{R}$ and therefore we get

$$\theta(t,s,x) = \theta(s,s,x) = \theta_x, \forall t,s \in \mathbb{R}.$$ 

It follows that for every $s \in \mathbb{R}$ the trajectory $\Gamma_{x,s} := \{\psi(t,s,x), t \in \mathbb{R}\}$ lies in the meridional plan.
(2) From the first equation of (7) we get

\[ r(t, s, x) = r(x) + \int_s^t r(\tau, s, x)^{-1} v^r(\tau, r(\tau, s), z(\tau, s)) r(\tau, s, x) d\tau. \]

Using Gronwall lemma we get

\[ r(t, s, x) \leq r(x)e^{\int_s^t \|\psi^r(\tau)\|_{L^\infty} d\tau}. \]

This gives the second inequality of the r.h.s since \(d(\psi(t, s, x), (Oz)) = r(t, s, x)\).

Now we apply the above inequality by taking \(\psi(s, t, x)\) instead of \(x\). Since \(\psi(t, s, \psi(s, t, x)) = x\) then

\[ r(x) \leq r(s, t, x)e^{\int_s^t \|\psi^r(\tau)\|_{L^\infty} d\tau}. \]

Interchanging \((s, t)\) and \((t, s)\) gives

\[ r(x)e^{-\int_s^t \|\psi^r(\tau)\|_{L^\infty} d\tau} \leq r(t, s, x). \]

This achieves the proof of Proposition 3.1.

Proposition 3.1 will be of much use in deriving some a priori estimates about solutions of transport equations. The first application is given below,

**Proposition 3.2.** Let \(v\) be a smooth axisymmetric vector field and \(\rho\) a solution of the transport equation

\[
\begin{cases}
\partial_t \rho + v \cdot \nabla \rho = 0 \\
\rho|_{t=0} = \rho_0.
\end{cases}
\]

(1) Assume that \(d(\text{supp} \, \rho_0, (Oz)) = r_0 > 0\). Then we have for every \(t \geq 0\)

\[ d(\text{supp} \, \rho(t), (Oz)) \geq r_0 e^{-\int_0^t \|\psi^r(\tau)\|_{L^\infty} d\tau}. \]

(2) Denote by \(\Pi_z\) the orthogonal projector over the axis \((Oz)\). We assume that \(\Pi_z(\text{supp} \, \rho_0)\) is compact set with diameter \(d_0\). Then for every \(t \geq 0\), \(\Pi_z(\text{supp} \, \rho(t))\) is compact set with diameter \(d(t)\) such that

\[ d(t) \leq d_0 + 2\int_0^t \|v(\tau)\|_{L^\infty} d\tau. \]

**Proof.** (1) The solution \(\rho\) of the transport equation is completely described through the usual flow \(\psi\), that is, \(\rho(t, x) = \rho_0(\psi^{-1}(t, x))\). Here \(\psi(t, x) := \psi(t, 0, x)\) where \(\psi(t, s, x)\) is the generalized flow introduced in the beginning of this section. Thus it follows that \(\text{supp} \, \rho(t) = \psi(t, \text{supp} \, \rho_0)\). Let now \(y \in \text{supp} \, \rho(t)\) then by definition \(y = \psi(t, x)\) with \(x \in \text{supp} \, \rho_0\). Therefore \(d(y, (Oz)) = r(t, x)\) with \(r(t, x) := r(t, 0, x)\) and so from Proposition 3.1 we get

\[ d(y, (Oz)) \geq d(x, (Oz)) e^{-\int_0^t \|\psi^r(\tau)\|_{L^\infty} d\tau} \]
\[ \geq d(\text{supp} \, \rho_0, (Oz)) e^{-\int_0^t \|\psi^r(\tau)\|_{L^\infty} d\tau} \]
\[ \geq r_0 e^{-\int_0^t \|\psi^r(\tau)\|_{L^\infty} d\tau}. \]
This concludes the first part.

(2) Let \( x, \tilde{x} \in \text{supp} \, \rho_0 \) and denote by \( y(t) = \psi(t, x) \) and \( \tilde{y}(t) = \psi(t, \tilde{x}) \). We set successively \( z(t) \) and \( \tilde{z}(t) \) the last component of \( y(t) \) and \( \tilde{y}(t) \). Using the equation (7) with \( s = 0 \) we get

\[
\dot{z}(t) = v^z(t, r(t, x), z(t)).
\]

Integrating this differential equation we get

\[
z(t) = z(0) + \int_0^t v^z(\tau, r(\tau, x), z(\tau)) d\tau.
\]

It follows that

\[
|z(t) - \tilde{z}(t)| \leq |z(0) - \tilde{z}(0)| + 2 \int_0^t \|v(\tau)\|_{L^\infty} d\tau.
\]

This yields

\[
\text{diam}(\Pi_z(\text{supp} \, \rho(t))) \leq \text{diam}(\Pi_z(\text{supp} \, \rho_0)) + 2 \int_0^t \|v(\tau)\|_{L^\infty} d\tau.
\]

The proof is now completed. \( \square \)

Now we are in a position to give an estimate of the quantity \( \|\rho(t)/r\|_{L^2} \) which is very crucial in the proof of Theorem 1.1. The growth that we will establish is quadratic and this improves the exponential growth that one can easily obtain by writing the equation of \( \rho/r \). More precisely we have

**Corollary 3.3.** Let \( v \) be a smooth axisymmetric vector field with zero divergence, \( \rho_0 \in L^2 \cap L^\infty \) and \( \rho \) be a solution of the transport equation

\[
\begin{align*}
\partial_t \rho + v \cdot \nabla \rho &= 0 \\
\rho|_{t=0} &= \rho_0.
\end{align*}
\]

Assume in addition that

\[
d(\text{supp} \, \rho_0, (Oz)) := r_0 > 0 \quad \text{and} \quad \text{diam}(\Pi_z(\text{supp} \, \rho_0)) := d_0 < \infty.
\]

Then we have

\[
\int_{\mathbb{R}^3} \frac{\rho^2(t, x)}{r^2} dx \leq \frac{1}{r_0^2} \|\rho^0\|^2_{L^2} + 2\pi \|\rho_0\|^2_{L^\infty} \int_0^t \|(v^r/r)(\tau)\|_{L^\infty} d\tau \left( d_0 + 2 \int_0^t \|v(\tau)\|_{L^\infty} d\tau \right),
\]

with \( r = (x_1^2 + x_2^2)^{\frac{1}{2}} \).
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Proof. We have from the definition
\[ \|(\rho/r(t))\|_{L^2}^2 = \int_{r \geq r_0} \frac{\rho^2(t, x)}{r^2} dx + \int_{r \leq r_0} \frac{\rho^2(t, x)}{r^2} dx \]
\[ \leq \frac{1}{r_0^2} \|\rho(t)\|_{L^2}^2 + \|\rho(t)\|_{L^\infty}^2 \int_{\{r \leq r_0\} \cap \supp \rho(t)} \frac{1}{r^2} dx \]
\[ \leq \frac{1}{r_0} \|\rho_0\|_{L^2}^2 + \|\rho_0\|_{L^\infty}^2 \int_{\{r \leq r_0\} \cap \supp \rho(t)} \frac{1}{r^2} dx. \]

We have used the conservation of the \(L^\infty\)-norm of \(\rho\). Now using Proposition 3.2 we get
\[ \int_{\{r \leq r_0\} \cap \supp \rho(t)} \frac{1}{r^2} dx \leq 2\pi \left( \int_{r_0 e^{-\frac{1}{2} \|v\|_{L^\infty} d\tau \leq r_0}} \frac{1}{r} dr \right) \left( \int_{\Pi_z(\supp \rho(t))} dz \right) \]
\[ \leq 2\pi \int_0^t \|v\|_{L^\infty} d\tau \left( d_0 + 2 \int_0^t \|v\|_{L^\infty} d\tau \right). \]

This concludes the proof. \(\square\)

4. A PRIORI ESTIMATES

This section is devoted to the a priori estimates needed for the proof of Theorem 1.1. We distinguish especially two kinds: the first one deals with some easy estimates that one can obtain by energy estimates. However the second one is concerned with some strong estimates which are the heart of the proof of our main result.

4.1. Weak a priori estimates. We will prove the following energy estimates.

**Proposition 4.1.** Let \(v_0 \in L^2\) be a vector field with zero divergence and \(\rho_0 \in L^2 \cap L^\infty\). Then every smooth solution of (1) satisfies
\[ \|\rho(t)\|_{L^2 \cap L^\infty} \leq \|\rho_0\|_{L^2 \cap L^\infty}. \]
\[ \|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(\tau)\|^2_{L^2} d\tau \leq C_0 (1 + t^2). \]

**Proof.** The first estimate is obvious since the flow preserves Lebesgue measure. For the second one we take the \(L^2\)-inner product of the velocity equation with \(v\). Then we get after some integration by parts
\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{L^2} + \|
abla v(t)\|^2_{L^2} \leq \|v(t)\|_{L^2} \|\rho(t)\|_{L^2}. \]

After simplification that one can rigorously justify, the last inequality leads to
\[ \frac{d}{dt} \|v(t)\|_{L^2} \leq \|\rho(t)\|_{L^2}. \]
Integrating in time this inequality yields
\[ \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^t \|\rho(\tau)\|_{L^2} d\tau. \]

Since \( \|\rho(t)\|_{L^2} = \|\rho_0\|_{L^2} \), then
\[ \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + t\|\rho_0\|_{L^2}. \]

Putting this estimate into (8) gives
\[ \frac{1}{2} \|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{2} \|v_0\|_{L^2}^2 + (\|v_0\|_{L^2} + t\|\rho_0\|_{L^2}) \|\rho_0\|_{L^2} t. \]

This gives the desired estimate and the demonstration of the proposition is now accomplished. \(\square\)

The next proposition describes some estimates linking the velocity to the vorticity by the use of the so-called Bio-Savart law.

**Proposition 4.2.** Let \( v \) be a smooth axisymmetric vector field with zero divergence and denote \( \omega = \omega_\theta e_\theta \) its curl. Then

1. \[ \|v\|_{L^\infty} \leq C \|\omega_\theta\|_{L^2}^{\frac{1}{2}} \|\omega_\theta\|_{H^\frac{1}{2}}. \]
2. \[ \|v/|r|\|_{L^\infty} \leq C \|\omega_\theta/|r|\|_{L^2}^{\frac{1}{2}} \|\omega_\theta/|r|\|_{H^\frac{1}{2}}. \]

**Proof.** To start with, recall the classical Biot-Savart law
\[ v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(y - x) \wedge \omega(y)}{|y - x|^3} dy. \]

It follows that
\[ |v(x)| \leq \frac{1}{4\pi} \left( \frac{1}{|\cdot|^2} * |\omega| \right)(x) \]
\[ \lesssim \frac{1}{|\cdot|^2} * |\omega_\theta| := J(x). \]

Let \( \lambda > 0 \) be a real number that will be fixed later. Decompose the convolution integral into two parts as follows
\[ J(x) = \int_{|x - y| \leq \lambda} \frac{|\omega_\theta(y)|}{|x - y|^2} dy + \int_{|x - y| \geq \lambda} \frac{|\omega_\theta(y)|}{|x - y|^2} dy \]
\[ := J_1(x) + J_2(x). \]
Now using Hölder inequalities we get
\[ \| J_1 \|_{L^\infty} \leq \| \omega_\theta \|_{L^6} \left( \int_{|x| \leq \lambda} \frac{1}{|x|^{\frac{12}{5}}} dx \right)^{\frac{5}{2}} \]
\[ \lesssim \| \omega_\theta \|_{L^6} \lambda^{\frac{1}{2}}. \]

For the second integral we use again Hölder inequalities
\[ \| J_2 \|_{L^\infty} \leq \| \omega_\theta \|_{L^2} \left( \int_{|x| \geq \lambda} \frac{1}{|x|^4} dx \right)^{\frac{1}{2}} \]
\[ \lesssim \| \omega_\theta \|_{L^2} \lambda^{-\frac{1}{2}}. \]

This yields
\[ \| v \|_{L^\infty} \lesssim \| \omega_\theta \|_{L^6} \lambda^{\frac{1}{2}} + \| \omega_\theta \|_{L^2} \lambda^{-\frac{1}{2}}. \]

By choosing \( \lambda = \frac{\| \omega_\theta \|_{L^6}^2}{\| \omega_\theta \|_{L^6}} \) we get
\[ \| v \|_{L^\infty} \lesssim \| \omega_\theta \|_{L^2}^\frac{1}{2} \| \omega_\theta \|_{L^6}^\frac{1}{2}. \]

It suffices now to use Sobolev embedding \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6 \) to get the desired result.

For the second estimate we recall the following inequality, see for instance [3, 24],
\[ |v^r/r| (x) \lesssim \left( \frac{1}{|x|^2} \ast |\omega_\theta/r| \right) (x). \]

Now we conclude similarly to the first estimate. \( \square \)

4.2. **Strong a priori estimates.** The task now is to find some global estimates about more strong regularities of the solutions of (1). The estimates developed below will be the basic ingredient of the proof of Theorem 1.1.

**Proposition 4.3.** Let \( v_0 \in H^1 \) be an axisymmetric vector field with zero divergence such that \( \frac{\text{curl} v_0}{r} \in L^2 \). Let \( \rho_0 \in L^2 \cap L^\infty \) depending only on \( r, z \) such that \( \text{supp} \rho_0 \) does not intersect the axis \( (Oz) \) and \( \Pi_z(\text{supp} \rho_0) \) is a compact set. Then every smooth solution \( (v, \rho) \) of the system (1) satisfies for every \( t \geq 0 \),
\[ \| v(t) \|_{H^1}^2 + \int_0^t \| v(\tau) \|_{H^2}^2 d\tau \leq C_0 e^{\exp C_0 t^9}, \]
\[ \| \frac{\omega}{r}(t) \|_{L^2}^2 + \| \frac{\omega}{r} \|_{L^2}^2 \leq C_0 e^{\exp C_0 t^9} \]
Moreover, for every \( p \in ]3, \infty[ \) we have
\[ \| v \|_{L^p_t B^{\frac{3}{p}+1}_{p,1}} + \| \nabla v \|_{L^1_t L^\infty} \leq C_0 e^{\exp C_0 t^9}. \]

The constant \( C_0 \) depends on the initial data.
Proof. Taking the $L^2$-inner product of the equation (4) with $\omega_\theta$ we get

$$\frac{1}{2} \frac{d}{dt} \|\omega_\theta\|^2_{L^2} + \|\nabla \omega_\theta\|^2_{L^2} + \left\| \frac{\omega_\theta}{r} \right\|^2_{L^2} = \int v^r \frac{\omega_\theta}{r} \omega_\theta dx - \int \partial_r \rho \omega_\theta dx.$$ 

For the first integral term of the r.h.s we use Hölder inequalities

$$\int v^r \frac{\omega_\theta}{r} \omega_\theta dx \leq \|v\|_{L^6} \left\| \frac{\omega_\theta}{r} \right\|_{L^2} \|\omega_\theta\|_{L^3}.$$ 

For the second term we integrate by parts taking into account that the density $\rho$ vanishes on the axis ($Oz$) accordingly to Corollary (3.3)

$$-\int \partial_r \rho \omega_\theta dx = -2\pi \int \partial_r \rho r \omega_\theta dz$$

$$= 2\pi \int \rho \partial_r \omega_\theta r dz + 2\pi \int \rho \frac{\omega_\theta}{r} r dz$$

$$= \int \rho (\partial_r \omega_\theta + \frac{\omega_\theta}{r}) dx.$$ 

It follows from Hölder inequalities

$$-\int \partial_r \rho \omega_\theta dx \leq \|\rho\|_{L^2} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^2} + \|\partial_r \omega_\theta\|_{L^2} \right).$$

Putting together these estimates yields

$$\frac{1}{2} \frac{d}{dt} \|\omega_\theta\|^2_{L^2} + \|\nabla \omega_\theta\|^2_{L^2} + \left\| \frac{\omega_\theta}{r} \right\|^2_{L^2} \leq \|v\|_{L^6} \left\| \frac{\omega_\theta}{r} \right\|_{L^2} \|\omega_\theta\|_{L^3}$$

$$+ \|\rho\|_{L^2} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^2} + \|\partial_r \omega_\theta\|_{L^2} \right).$$

Using the inequality $|ab| \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$ for the last term we obtain

$$\|\rho\|_{L^2} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^2} + \|\partial_r \omega_\theta\|_{L^2} \right) \leq \|\rho\|_{L^2}^2 + \frac{1}{2} \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 + \frac{1}{2} \|\partial_r \omega_\theta\|_{L^2}^2$$

$$\leq \|\rho\|_{L^2}^2 + \frac{1}{2} \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 + \frac{1}{2} \|\nabla \omega_\theta\|_{L^2}^2.$$ 

We have used in the last line the fact that in the cylindrical coordinates

$$\|\nabla \omega_\theta\|_{L^2}^2 = \|\partial_r \omega_\theta\|_{L^2}^2 + \|\partial_z \omega_\theta\|_{L^2}^2.$$ 

Therefore we get

$$\frac{d}{dt} \|\omega_\theta\|^2_{L^2} + \|\nabla \omega_\theta\|_{L^2}^2 + \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 \leq 2 \|v\|_{L^6} \left\| \frac{\omega_\theta}{r} \right\|_{L^2} \|\omega_\theta\|_{L^3} + 2 \|\rho\|_{L^2}^2.$$ 

(10)
Combining the interpolation estimate \( \| \omega \|_{L^4} \lesssim \| \omega \|_{L^2}^{\frac{1}{2}} \| \nabla \omega \|_{L^2}^{\frac{1}{2}} \) with Young inequality \(|ab| \leq C_0 a^{\frac{1}{2}} + b^\frac{1}{2}, \forall \eta \in [0, 1]| \) leads to

\[
\| v \|_{L^p} \| \nabla \omega \|_{L^2} \| \omega \|_{L^2} \leq C \| v \|_{L^p} \| \nabla \omega \|_{L^2} \| \omega \|_{L^2} + \frac{1}{4} \| \nabla \omega \|_{L^2}^2
\]

(11)

Inserting (11) into (10) and using the estimate \( \| \rho(t) \|_{L^2} \leq \| \rho_0 \|_{L^2} \)

\[
\frac{d}{dt} \| \omega \|_{L^2}^2 + \frac{1}{2} \| \nabla \omega \|_{L^2}^2 + \| \frac{\omega \theta}{r} \|_{L^2}^2 \leq C \| \omega \|_{L^2}^2 + C \| v \|_{L^2} \| \frac{\omega \theta}{r} \|_{L^2}^2 + 2 \| \rho_0 \|_{L^2}^2.
\]

Using Gronwall inequality and Proposition 4.1 we get

\[
\| \omega(t) \|_{L^2}^2 + \| \nabla \omega \|_{L^2}^2 + \| \frac{\omega \theta}{r} \|_{L^2}^2 \leq C e^{Ct} \left( \| \rho_0 \|_{L^2}^2 + \int_0^t \| \nabla \omega \|_{L^2}^2 + \| \frac{\omega \theta}{r} \|_{L^2}^2 \right) + C \left( \| \rho_0 \|_{L^2}^2 + \int_0^t \| v \|_{L^2} \right)^2 + C e^{Ct} \left( 1 + \| \frac{\omega \theta}{r} \|_{L^2}^2 \right).
\]

(12)

Let us now move to the equation (5) and try to estimate in a proper way the quantity \( \Gamma := \frac{\omega}{r} \). Then taking the \( L^2 \)-inner product of (5) with \( \Gamma \) and integrating by parts using the incompressibility of the flow and \( \rho(t, 0, z) = 0 \) gives

\[
\frac{1}{2} \frac{d}{dt} \| \Gamma \|_{L^2}^2 + \| \partial_r \Gamma \|_{L^2}^2 + \| \partial_r \Gamma \|_{L^2}^2 - 4\pi \int \partial_r (\Gamma) \Gamma r dz = -2\pi \int \partial_r \rho \Gamma r dz
\]

\[
= 2\pi \int \frac{\rho}{r} \partial_r \Gamma r dr dz
\]

\[
\leq \| \frac{\rho}{r} \|_{L^2} \| \partial_r \Gamma \|_{L^2}.
\]

Since

\[
4\pi \int \partial_r (\Gamma) \Gamma r dz = 2\pi \int_\mathbb{R} \int_0^{+\infty} \partial_r (\Gamma)^2 r dz \leq 0,
\]

then and once again by Young inequality yields

\[
\frac{d}{dt} \| \frac{\omega}{r} \|_{L^2}^2 + \| \nabla \left( \frac{\omega}{r} \right) \|_{L^2}^2 \leq \| \frac{\rho}{r} \|_{L^2}^2.
\]

Integrating in time this differential inequality we get

\[
\| \frac{\omega}{r} (t) \|_{L^2}^2 + \int_0^t \| \nabla \left( \frac{\omega}{r} \right) (\tau) \|_{L^2}^2 d\tau \leq \| \frac{\omega}{r} (0) \|_{L^2}^2 + \int_0^t \| \frac{\rho}{r} (\tau) \|_{L^2}^2 d\tau.
\]

(13)

To estimate the last term of the above inequality we will use Corollary 3.3

\[
\| \frac{\rho}{r} (t) \|_{L^2}^2 \leq C_0 + C_0 \int_0^t \| v_r (\tau) \|_{L^\infty} d\tau \left( 1 + \int_0^t \| v (\tau) \|_{L^\infty} d\tau \right).
\]
Thus
\[
\left\| \frac{\omega_{\theta}}{r}(t) \right\|^2_{L^2} + \left\| \nabla \left( \frac{\omega_{\theta}}{r} \right) \right\|^2_{L^2_t L^2} \leq C_0 (1 + t) \\
+ C_0 \int_0^t \left\{ \int_0^{t'} \left\| \frac{v^r}{r}(\tau) \right\|_{L^\infty} \, d\tau \left( 1 + \int_0^{t'} \left\| v(\tau) \right\|_{L^\infty} \, d\tau \right) \right\} \, dt' \\
\leq C_0 (1 + t) + C_0 t \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|_{L^\infty} \, d\tau \\
+ C_0 t \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|_{L^\infty} \, d\tau \int_0^t \left\| v(\tau) \right\|_{L^\infty} \, d\tau.
\]
From Young and Hölder inequalities we get successively
\[
C_0 t \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|_{L^\infty} \, d\tau \leq \int_0^t \left( C_0 t^2 + \left\| \frac{v^r}{r}(\tau) \right\|^2_{L^\infty} \right) \, d\tau \\
\leq C_0 t^3 + \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|^2_{L^\infty} \, d\tau
\]
and
\[
C_0 t \left\| \frac{v^r}{r} \right\|_{L_t^1 L^\infty} \left\| v \right\|_{L_t^1 L^\infty} \leq C_0 t^2 \left\| \frac{v^r}{r} \right\|_{L_t^2 L^\infty} \left\| v \right\|_{L_t^2 L^\infty} \\
\leq C_0 t^4 \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|^2_{L^\infty} \, d\tau + \int_0^t \left\| v(\tau) \right\|^2_{L^\infty} \, d\tau.
\]
Putting together these estimates yields
\[
\left\| \frac{\omega_{\theta}}{r}(t) \right\|^2_{L^2} + \left\| \nabla \left( \frac{\omega_{\theta}}{r} \right) \right\|^2_{L^2_t L^2} \leq C_0 (1 + t^4) \left( 1 + \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|^2_{L^\infty} \, d\tau \right) \\
+ \int_0^t \left\| v(\tau) \right\|^2_{L^\infty} \, d\tau.
\] (14)
Recall from Proposition 4.2 that
\[
\left\| \frac{v^r}{r} \right\|_{L^\infty} \lesssim \left\| \frac{\omega_{\theta}}{r} \right\|_{L^2} \left\| \nabla \left( \frac{\omega_{\theta}}{r} \right) \right\|_{L^2}.
\]
Accordingly, we get by Young inequalities
\[
C_0 (1 + t^4) \int_0^t \left\| \frac{v^r}{r}(\tau) \right\|^2_{L^\infty} \, d\tau \leq C_0 (1 + t^8) \int_0^t \left\| \frac{\omega_{\theta}}{r}(\tau) \right\|^2_{L^2} \, d\tau + \frac{1}{2} \int_0^t \left\| \nabla \left( \frac{\omega_{\theta}}{r} \right)(\tau) \right\|^2_{L^2} \, d\tau.
\]
Inserting this estimate into (14)
\[
\left\| \frac{\omega_{\theta}}{r}(t) \right\|^2_{L^2} + \left\| \nabla \left( \frac{\omega_{\theta}}{r} \right) \right\|^2_{L^2_t L^2} \leq C_0 (1 + t^8) \int_0^t \left\| \frac{\omega_{\theta}}{r}(\tau) \right\|^2_{L^2} \, d\tau \\
+ \int_0^t \left\| v(\tau) \right\|^2_{L^\infty} \, d\tau.
\] (15)
Now using Gronwall inequality we find

\[ \| \frac{\omega}{r}(t) \|_{L^2}^2 + \| \nabla \left( \frac{\omega}{r} \right) \|_{L^2}^2 \leq C_0 e^{C_0 t^9} \left( 1 + \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau \right). \]

Putting (16) into (12) yields

\[ \| \omega(t) \|_{L^2}^2 + \| \nabla \omega(t) \|_{L^2}^2 + \| \frac{\omega}{r}(t) \|_{L^2}^2 \leq C_0 e^{C_0 t^9} \left( 1 + \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau \right). \]

To estimate the last term in the r.h.s of (17) we use Proposition 4.2

\[ \| v(\tau) \|_{L^\infty}^2 \lesssim \| \omega(t) \|_{L^2}^2 \| \nabla \omega(t) \|_{L^2}^2. \]

It follows

\[ \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau \leq \int_0^t \| \omega(t) \|_{L^2}^2 \| \nabla \omega(t) \|_{L^2}^2 d\tau. \]

Therefore we get from (17) and Young inequality

\[ \| \omega(t) \|_{L^2}^2 + \| \nabla \omega(t) \|_{L^2}^2 + \| \frac{\omega}{r}(t) \|_{L^2}^2 \leq C_0 e^{C_0 t^9} \left( 1 + \int_0^t \| \omega(\tau) \|_{L^2}^2 \| \nabla \omega(\tau) \|_{L^2}^2 d\tau \right) \]

\[ \leq C_0 e^{C_0 t^9} \left( 1 + \int_0^t \| \omega(\tau) \|_{L^2}^2 d\tau \right) + \frac{1}{2} \| \nabla \omega(t) \|_{L^2}^2. \]

It suffices now to use Gronwall inequality

\[ \| \omega(t) \|_{L^2}^2 + \| \nabla \omega(t) \|_{L^2}^2 + \| \frac{\omega}{r}(t) \|_{L^2}^2 \leq C_0 e^{\exp C_0 t^9}. \]

Since in cylindrical coordinates we have \( \| \nabla \omega \|_{L^2}^2 = \| \nabla \omega(t) \|_{L^2}^2 + \| \frac{\omega}{r(t)} \|_{L^2}^2 \) then

\[ \| \omega(t) \|_{L^2}^2 + \| \nabla \omega \|_{L^2}^2 \leq C_0 e^{\exp C_0 t^9}. \]

Inserting this estimate into (18) gives

\[ \| v \|_{L^2 L^\infty} \leq C_0 e^{\exp C_0 t^9}. \]

Combining this estimate with (16) and (19) yields

\[ \| \frac{\omega}{r}(t) \|_{L^2}^2 + \| \nabla \left( \frac{\omega}{r} \right)(t) \|_{L^2}^2 \leq C_0 e^{\exp C_0 t^9}. \]

This concludes the first part of Proposition 4.3. Let us now show how to prove the estimate (9). Let \( q \in \mathbb{N} \) and set \( v_q := \Delta_q v \). Then localizing in frequency the first equation of (1) and using Duhamel formula we get

\[ v_q(t) = e^{\Delta t} v_q(0) + \int_0^t e^{(t-\tau)\Delta} \Delta_q (\mathcal{P}(v \cdot \nabla)v)(\tau) d\tau + \int_0^t e^{(t-\tau)\Delta} \Delta_q (\mathcal{P}(\rho e_z))(\tau) d\tau, \]
where $\mathcal{P}$ denotes Leray’s projector over solenoidal vector fields. Now we will use a local version of the smoothing effects of the heat semigroup, for the proof see for example [9],

$$
\|e^{t\Delta_q} \mathcal{P} f\|_{L^p} \leq C e^{-ct^{2q}} \|\Delta_q f\|_{L^p}, \quad \forall p \in [1, \infty].
$$

Combining this estimate with Bernstein inequality gives

$$
\|v_q(t)\|_{L^p} \lesssim e^{-ct^{2q}} \|v_q(0)\|_{L^p} + 2^q \int_0^t e^{-c(t-\tau)^{2q}} \|\Delta_q(v \otimes v)(\tau)\|_{L^p} d\tau
$$

$$
+ \int_0^t e^{-c(t-\tau)^{2q}} \|\Delta_q \rho(\tau)\|_{L^p} d\tau.
$$

Integrating in time and using convolution inequalities lead to

$$
\|v_q\|_{L^p_t L^p} \lesssim 2^{-2q} \|v_q(0)\|_{L^p} + 2^{-q} \int_0^t \|\Delta_q(v \otimes v)(\tau)\|_{L^p} d\tau + 2^{-2q} \int_0^t \|\Delta_q \rho(\tau)\|_{L^p} d\tau.
$$

It follows that

$$
\|v\|_{L^1_t B^{\frac{3}{p}+1}_{p,1}} \leq \|\Delta_{-1}v\|_{L^1_t L^p} + \|v_0\|_{B^{\frac{3}{p}-1}_{p,1}} + \int_0^t \|(v \otimes v)(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau
$$

$$
+ \int_0^t \|\rho(\tau)\|_{B^{\frac{3}{p}-1}_{p,1}} d\tau.
$$

For the first term of the r.h.s we combine Bernstein inequality with Proposition 4.1 ($p \geq 2$)

$$
\|\Delta_{-1}v\|_{L^1_t L^p} \lesssim t \|v\|_{L^\infty_t L^2}
$$

$$
\leq C_0(1 + t^2).
$$

For the last term of the r.h.s we use the embedding $L^p \hookrightarrow B^{\frac{3}{p}-1}_{p,1}$, for $p > 3$, combined with Proposition 4.1

$$
\int_0^t \|\rho(\tau)\|_{B^{\frac{3}{p}-1}_{p,1}} d\tau \lesssim t \|\rho\|_{L^\infty_t L^p}
$$

$$
\leq C_0 t.
$$

On the other hand we have from Besov embedding

$$
\|v_0\|_{B^{\frac{1}{p}-1}_{p,1}} \lesssim \|v_0\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|v_0\|_{H^1}.
$$

Putting together these inequalities we find

$$
\|v\|_{L^1_t B^{\frac{3}{p}+1}_{p,1}} \leq C_0(1 + t^2) + \int_0^t \|(v \otimes v)(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau.
$$
Now we use Besov embeddings, law products and interpolation results
\[ \| v \otimes v \|_{B^{\frac{3}{2}}_{p,1}} \lesssim \| v \otimes v \|_{B^{\frac{3}{2}}_{2,1}} \]
\[ \lesssim \| v \|_{L^\infty} \| v \|_{B^{\frac{3}{2}}_{2,1}} \]
\[ \lesssim \| v \|_{L^\infty} \| v \|_{L^2} + \| v \|_{L^\infty} \| \omega \|_{B^{\frac{1}{2}}_{2,1}} \]
\[ \lesssim \| v \|_{L^\infty} \| v \|_{L^2} + \| v \|_{L^\infty} \| \omega \|_{L^2} \| \omega \|_{H^1}. \]

This yields according to Hölder inequalities
\[ \| v \otimes v \|_{L^1_t B^{\frac{3}{2}}_{p,1}} \lesssim t^{\frac{1}{2}} \| v \|_{L^2_t L^\infty} \| v \|_{L^\infty_t L^2} + t^{\frac{1}{4}} \| v \|_{L^2_t L^\infty} \| \omega \|_{L^2_t L^2} \| \omega \|_{L^2_t H^1}. \]
\[ \lesssim t^{\frac{1}{2}} \| v \|_{L^2_t L^\infty} \| v \|_{L^\infty_t L^2} + t^{\frac{1}{4}} \| v \|_{L^2_t L^\infty} \| \omega \|_{L^\infty_t L^2} \| \omega \|_{L^2_t H^1}. \]

It suffices now to use (19) and (20)
\[ \| v \otimes v \|_{L^1_t B^{\frac{3}{2}}_{p,1}} \leq C_0 e^{\exp C_0 t^q}. \]

Hence we obtain
\[ \| v \|_{L^1_t B^{\frac{3}{2}+1}_{p,1}} \leq C_0 e^{\exp C_0 t^q}. \]

The last estimate of Proposition (9) is a direct consequence of the embedding \( B^{\frac{3}{2}+1}_{p,1} \hookrightarrow W^{1,\infty} \) and the proof is then accomplished. \( \square \)

5. Proof of the main result

The existence part can be done in a classical way for example by smoothing out the initial data. However we have to use an appropriate approximation that does not alter never the initial geometric structure nor the uniform estimates in the space of initial data. We will work with the approximation of the identity and show that it has the requested properties: let \( \phi \) be a smooth positive radial function with support contained in \( B(0,1) \) and such that \( \phi \equiv 1 \) in a neighborhood of zero. We assume that \( \int_{\mathbb{R}^3} \phi(x)dx = 1 \). For every \( n \in \mathbb{N}^* \) we set \( \phi_n(x) = n^3 \phi(nx) \) and we define the family
\[ v_{0,n} = \phi_n * v_0 \quad \text{and} \quad \rho_{0,n} = \phi_n * \rho_0. \]

We will start with the following stability results.

Lemma 5.1. (1) Let \( v_0 \in H^1 \) be an axisymmetric vector field with zero divergence and such that \( (\text{curl } v_0)/r \in L^2 \). Then for every \( n \in \mathbb{N}^* \) the vector field \( v_{0,n} \) is axisymmetric with zero divergence. Moreover, there exists a constant \( C \) depending on \( \phi \) such that
\[ \| v_{0,n} \|_{H^1} \leq \| v_0 \|_{H^1}, \quad \| (\text{curl } v_{0,n})/r \|_{L^2} \leq C \| (\text{curl } v_0)/r \|_{L^2}. \]
(2) Let $v$ be a smooth axisymmetric vector field with zero divergence and $\rho$ be a solution of the transport equation
\[
\begin{cases}
\partial_t \rho^n + v \cdot \nabla \rho^n = 0 \\
\rho^n |_{t=0} = \rho_{0,n}.
\end{cases}
\]
Assume that $d(\text{supp } \rho_0, (Oz)) := r_0 > 0$ and $\text{diam } (\Pi_z(\text{supp } \rho_0)) := d_0 < \infty$.
Then there exists constants $n_0$ and $C$ depending only on $r_0, d_0$ and $\phi$ such that
\[
\int_{\mathbb{R}^3} \left( \frac{\rho^n}{r} \right)^2(t, x) dx \leq \frac{1}{r_0^2} \rho_0^2(t) + 2\|\rho_0\|_{L^\infty} \left( \ln 2 + \|v^r/r\|_{L^1_t L^\infty} \right) \left( d_0 + \|v\|_{L^1_t L^\infty} \right).
\]

Remark 5.2. The estimate of the part (2) of the above lemma is little bit different from part (2) of Corollary 3.3 because we have an additional linear term $\|v\|_{L^1_t L^\infty}$.
Nevertheless, the presence of this term does not deeply affect the calculus seen before in the a priori estimates; we obtain at the end the same estimates of Proposition 4.3.

Proof. (1) The fact that the vector field $v_{0,n}$ is axisymmetric is due to the radial property of the functions $\phi_n$, for more details see [3]. The estimate of $v_{0,n}$ in $H^1$ is easy to obtain by using the classical properties of the convolution operation combined with $\|\phi_n\|_{L^1} = 1$. Concerning the demonstration of the second estimate, it is more subtle and we refer to [5] where it is proven for more general framework Lebesgue space, that is $L^p$, for all $p \in [1, \infty]$.
(2) In order to establish the desired estimate we need to check that the estimates of Proposition 3.2 are stable with respect to $n$. More precisely, we will prove that for sufficiently large $n \geq n_0$
\[
d(\text{supp } \rho^n(t), (Oz)) \geq \frac{r_0}{2} e^{-\int_0^t \|v^r/r\|_{L^\infty} d\tau}
\]
and
\[
d_n(t) \leq 2d_0 + 2 \int_0^t \|v(t)\|_{L^\infty} d\tau \quad \text{with} \quad d_n(t) := \text{diam } (\Pi_z(\text{supp } \rho^n(t))).
\]
Assume that we have the inequalities (21) and (22) then reproducing the proof of Corollary 3.3 we get
\[
\|\rho_n/r(t)\|^2_{L^2} \leq \frac{1}{r_0^2} \|\rho_{0,n}\|^2_{L^2} + \|\rho_{0,n}\|^2_{L^\infty} \int_{\{r \leq r_0\} \cap \text{supp } \rho_n(t)} \frac{1}{r^2} dx
\]
\[
\leq \frac{1}{r_0^2} \|\rho_0\|^2_{L^2} + \|\rho_0\|^2_{L^\infty} \int_{\{r \leq r_0\} \cap \text{supp } \rho_n(t)} \frac{1}{r^2} dx
\]
\[
\leq \frac{1}{r_0^2} \|\rho_0\|^2_{L^2} + \|\rho_0\|^2_{L^\infty} \left( \ln(2) + \|v^r/r\|_{L^1_t L^\infty} \right) \left( 2d_0 + 2 \int_0^t \|v(t)\|_{L^\infty} d\tau \right).
\]
This is what we want. Now let us come back to the proof of the inequality (21). Following the same proof of Proposition 3.2 it appears that one needs only to establish the stability conditions of the support, that is, for large $n \geq n_0$,

$$d(\text{supp } \rho^n_0, (Oz)) \geq \frac{r_0}{2}. \quad (23)$$

From the definition we have

$$\text{supp } \rho^n_0 \subset \{ x; \| x - y \| \leq \frac{1}{n} \text{ for some } , \| y' \| \geq r_0 \}, \quad \text{with } y = (y', y_3), y' \in \mathbb{R}^2.$$ 

It is obvious that for $x \in \text{supp } \rho^n_0$ we have $\| x' - y' \| \leq \frac{1}{n}$ and then by the triangular inequality

$$\| x' \| \geq r_0 - \frac{1}{n}.$$ 

Thus there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\| x' \| \geq \frac{r_0}{2}$. This proves that $\text{supp } \rho^n_0 \subset \{ x, \| x' \| \geq \frac{r_0}{2} \}$ and then we deduce (23).

Let us now show how to get the estimate (22). According to part (2) of Proposition 3.2 we have

$$d_n(t) \leq d_{0,n} + 2 \int_0^t \| v(\tau) \|_{L^\infty} d\tau,$$

with $d_{0,n} := \text{diam } (\Pi_z (\text{supp } \rho_{0,n}))$. Now it is easy to see from the support properties of the convolution that

$$d_{0,n} \leq d_0 + \frac{2}{n}.$$ 

Taking $n$ large enough, $n \geq n_1$ for some $n_1$, yields $d_{0,n} \leq 2d_0$. Now to be sure that both inequalities (21) and (22) are simultaneously satisfied we take $n \geq \max \{ n_0, n_1 \}$. \[\square\]

Let us now come back to the proof of the existence part of Theorem 1.1. We have just seen in Lemma 5.1 that the initial structure of axisymmetry is preserved for every $n$ and the involved norms are uniformly controlled with respect to this parameter $n$. Thus we can construct locally in time a unique solution $(v_n, \theta_n)$ that does not blow up in finite time since the Lipschitz norm of the velocity is well controlled as it was stated in Proposition 4.3. By standard arguments we can show that this family converges to $(v, \theta)$ which satisfies in turn our IVP. We omit here the details which are very classical and we will next focus on the uniqueness part. Set

$$\mathcal{X}_T := \left( L^\infty_T L^2 \cap L^2_T H^1 \right) \times L^\infty_T H^{-1}.$$
Let \((v^i, \rho^i) \in \mathcal{X}_T, 1 \leq i \leq 2\) be two solutions of the system (1) with the same initial data \((v_0, \theta_0)\) and denote \(\delta v = v^2 - v^1, \delta \theta = \theta^2 - \theta^1\). Then

\[
\begin{align*}
&\partial_t \delta v + v^2 \cdot \nabla \delta v - \Delta \delta v + \nabla \delta \rho = -\delta v \cdot \nabla v^1 + \delta \rho e_z \\
&\partial_t \delta \rho + v^2 \cdot \nabla \delta \rho = -\delta v \cdot \nabla \rho^1
\end{align*}
\]  

Taking the \(L^2\)-inner product of the first equation with \(\delta v\) and integrating by parts

\[
\frac{1}{2} \frac{d}{dt} \|\delta v\|_{L^2}^2 + \|\nabla \delta v\|_{L^2}^2 = -\int \delta v \cdot \nabla v^1 \delta v \, dx + \int \delta \rho e_z \delta v \, dx
\]

\[
\leq \|\delta v\|_{L^2} \|v^1\|_{L^\infty} \|\nabla \delta v\|_{L^2} + \|\delta \rho\|_{H^{-1}} \|\delta v\|_{H^1}
\]

\[
\lesssim \left( \|\delta v\|_{L^2} \|v^1\|_{L^\infty} + \|\delta \rho\|_{H^{-1}} \right) \|\nabla \delta v\|_{L^2} + \|\delta \rho\|_{H^{-1}} \|\delta v\|_{L^2}.
\]

Applying Young inequality leads to

\[
\frac{d}{dt} \|\delta v\|_{L^2}^2 + \|\nabla \delta v\|_{L^2}^2 \lesssim \|\delta v\|_{L^2}^2 \left( \|v^1\|_{L^\infty}^2 + 1 \right) + \|\delta \rho\|_{H^{-1}}^2.
\]

To estimate \(\|\delta \rho\|_{H^{-1}}\), we will use Proposition 3.1 of [4]: for every \(p \in [2, \infty[\)

\[
\|\delta \rho(t)\|_{H^{-1}} \leq C \|\delta v \cdot \nabla \rho^1\|_{L^1_t B^{\frac{3}{p}, 1}_p} \exp \left( C \|\nabla v^2\|_{L^1_t B^{\frac{3}{p}, 1}_p} \right).
\]

As \(\text{div} \, \delta v = 0\), then

\[
\|\delta v \cdot \nabla \rho^1\|_{L^1_t H^{-1}} \leq \|\delta v \rho^1\|_{L^1_t L^2} \leq \|\rho_0\|_{L^\infty} \|\delta v\|_{L^1_t L^2}.
\]

Inserting this estimate into (25), we obtain

\[
\frac{d}{dt} \|\delta v(t)\|_{L^2}^2 + \|\nabla \delta v(t)\|_{L^2}^2 \leq C \|\delta v(t)\|_{L^2}^2 \left( \|v^1(t)\|_{L^\infty}^2 + 1 \right)
\]

\[
+ C \exp \left( C \|\nabla v^2\|_{L^1_t B^{\frac{3}{p}, 1}_p} \right) \|\rho_0\|_{L^\infty}^2 \|\delta v\|_{L^1_t L^2}^2.
\]

Integrating this differential inequality we get

\[
\|\delta v\|_{L^\infty_t L^2}^2 \leq C \int_0^t \|\delta v(\tau)\|_{L^2}^2 \left( \|v^1(\tau)\|_{L^\infty}^2 + 1 \right) d\tau
\]

\[
+ C \exp \left( C \|\nabla v^2\|_{L^1_t B^{\frac{3}{p}, 1}_p} \right) \|\rho_0\|_{L^\infty}^2 \int_0^t \|\delta v\|_{L^1_t L^2}^2 d\tau
\]

\[
\leq C \int_0^t \|\delta v\|_{L^\infty_t L^2}^2 \left( \|v^1(\tau)\|_{L^\infty}^2 + 1 \right) d\tau
\]

\[
+ C \exp \left( C \|\nabla v^2\|_{L^1_t B^{\frac{3}{p}, 1}_p} \right) \|\rho_0\|_{L^\infty}^2 \int_0^t \|\delta v\|_{L^\infty_t L^2}^2 d\tau.
\]

It suffices now to use Gronwall inequality.
ON THE AXISYMMETRIC BOUSSINESQ SYSTEM

REFERENCES


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