GLOBAL WELL-POSEDNESS FOR A BOUSSINESQ-NAVIER-STOKES SYSTEM WITH CRITICAL DISSIPATION

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Abstract. In this paper we study a fractional diffusion Boussinesq model which couples a Navier-Stokes type equation with fractional diffusion for the velocity and a transport equation for the temperature. We establish global well-posedness results with rough initial data.

Contents

1. Introduction

The aim of this paper is to study the global well-posedness for the Boussinesq system with partial fractional viscosity

$$\begin{cases} \partial_t v + v\cdot\nabla v + |\mathrm{D}|v + \nabla p = \theta e_2 \\ \partial_t\theta + v\cdot\nabla\theta = 0 \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases} \tag{1.1}$$

Here, we focus on the two-dimensional case, the space variable $x = (x_1, x_2)$ is in $\mathbb{R}^2$, the velocity field v is given by $v = (v^1, v^2)$ and the pressure p and the temperature θ are scalar functions. The factor θe_2 in the velocity equation, the vector e_2 being given by $(0, 1)$, models for example the effect of gravity on the fluid motion. The operator $|\mathrm{D}|$ stands for the multiplication by $|\xi| := \sqrt{\xi_1^2 + \xi_2^2}$ in the Fourier space.

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If we take $\theta^0 = 0$ then the system (1.1) is reduced to the generalized Navier-Stokes system which was studied in a series of papers [24, 25, 26] for all space dimension $d \geq 2$. In particular, for the generalized Navier-Stokes system in dimension two, from the Beale-Kato-Majda criterion [3] and the maximum principle for the vorticity [10], smooth solutions are global in time.

The system (1.1) can be seen as part of the class of generalized Boussinesq systems. These systems under the form

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \theta e_2 + \mathcal{D}_v v \\ \partial_t \theta + v\cdot\nabla\theta = \mathcal{D}_\theta \theta \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0 \end{cases} \tag{1.2}$$

are simple models widely used in the modelling of oceanic and atmospheric motions. They also appear in many physical problems, we refer for example to [6] for more details about the modelling issues.

The operators $\mathcal{D}_v$ and $\mathcal{D}_\theta$ whose form may vary are used to take into account the possible effects of diffusion and dissipation in the fluid motion.

Mathematically, the simplest model to study is the fully viscous model when $\mathcal{D}_v = \Delta$, $\mathcal{D}_\theta = \Delta$. The properties of the system are very similar to the one of the two-dimensional Navier-Stokes equation and similar global well-posedness results can be obtained.

The most difficult model for the mathematical study is the inviscid one, i.e. when $\mathcal{D}_v = \mathcal{D}_\theta = 0$. A local existence result of smooth solution can be proven as for symmetric hyperbolic quasilinear systems, nevertheless, it is not known if smooth solutions can develop singularities in finite time. Indeed, the temperature θ is the solution of a transport equation and the vorticity $\omega = \operatorname{curl} v = \partial_1 v^2 - \partial_2 v^1$ solves the equation

$$\partial_t\omega + v\cdot\nabla\omega = \partial_1\theta. \tag{1.3}$$

The main difficulty is that to get an L^∞ estimate on ω which is crucial to prove global existence of smooth solutions for Euler type equation, one needs to estimate $\int_0^T ||\partial_1\theta||_{L^\infty}$ and, unfortunately, no *a priori* estimate on $\partial_1\theta$ is known.

In order to understand the coupling between the two equations in Boussinesq type systems, there have been many recent works studying Boussinesq systems with partial viscosity i.e. with a viscous term acting only in one equation. We refer for example to [2, 7, 12, 13, 14, 15, 16].

In this paper, we shall focus on system (1.1) which corresponds to the case where the heat conductivity is neglected and $\mathcal{D}_v = -|\mathrm{D}|$.

When considering the usual Navier-Stokes equation for the velocity, i.e. for $\mathcal{D}_v = \Delta$, global well-posedness results were recently established in various functional spaces. In [7], Chae proved the global well-posedness for large initial data $v^0, \theta^0 \in H^s$ with $s > 2$. This result was improved by the first two authors [15] for less regular initial data, that is, $v^0, \theta^0 \in H^s$, with $s > 0$. The uniqueness in the energy space L^2 was recently proved in [13]. According to a recent work of Danchin and Paicu [14] one can construct global unique solution when the dissipation acts only in the horizontal direction: $\mathcal{D}_v = \partial_{11}$.

To explain the new difficulties that appear when a weaker diffusion $\mathcal{D}_v = -|\mathrm{D}|^\alpha$, $\alpha < 2$ is considered, let us write the system under the vorticity formulation. By using the vorticity defined as the scalar $\omega = \partial_1 v^2 - \partial_2 v^1$, we have to study the system

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega + |\mathrm{D}|^\alpha\omega = \partial_1\theta, \\ \partial_t\theta + v\cdot\nabla\theta = 0, \\ \omega_{|t=0} = \mathrm{curl}\ v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

The standard L^2 energy estimate for this system gives

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \frac{1}{2}\|\omega\|_{\dot H^{\frac{\alpha}{2}}}^2 \le \|\theta\|_{\dot H^{1-\frac{\alpha}{2}}}, \quad \|\theta(t)\|_{L^2} = \|\theta_0\|_{L^2}.$$

When $\alpha = 2$, the combination of these two estimates provides the useful information that $\omega \in L^\infty_{loc}L^2 \cap L^2_{loc}\dot H^1$. When $\alpha < 2$ since no *a priori* estimate on $\|\theta\|_{\dot H^{1-\frac{\alpha}{2}}}$ is known some additional work is needed in order to estimate ω. As in [15], the idea would be to use maximal regularity estimates for the semi-group $e^{-t|D|^\alpha}$ in order to compensate the loss of one derivative for θ. Nevertheless, some restriction will appear in order to control the nonlinear term. For the sake of clarity, we shall focus on the most difficult case $\alpha = 1$ where this approach would fail: we consider (1.1) for which the vorticity form of the system is

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega + |\mathrm{D}|\omega = \partial_1\theta, \\ \partial_t\theta + v\cdot\nabla\theta = 0, \\ \omega_{|t=0} = \mathrm{curl}\ v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

This is the critical case in the sense that the gain of one derivative by the diffusion term roughly compensates exactly the loss of one derivative in θ in the vorticity equation and has the same order as the convection term.
The main result of this paper is a global well-posedness result for the system (1.1) (see section 2 for the definitions and the basic properties of Besov spaces).

Theorem 1.1. *Let $\theta^0 \in L^2 \cap B^0_{\infty,1}$ and v^0 be a divergence-free vector field belonging to $H^1 \cap \dot W^{1,p}$ with $p \in]2, +\infty[$. Then the system* (1.1) *has a unique global solution (v, θ) such that*

$$v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1 \cap \dot W^{1,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+; B^1_{\infty,1}) \quad \textit{and} \quad \theta \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2 \cap B^0_{\infty,1}).$$

A few remarks are in order.

Remark 1.2. From the *a priori* estimates that we shall get in the proof of Theorem 1.1, it is possible to obtain the existence of various types of global weak solutions, this is discussed in section 5.
At first, it is possible to get the existence without uniqueness of Leray type weak solutions of (1.1) in the energy espace

$$(v, \theta) \in L^\infty_{loc}(\mathbb{R}_+, L^2) \cap L^2_{loc}(\mathbb{R}_+ \dot H^{\frac{1}{2}}) \times L^\infty_{loc}(\mathbb{R}_+, L^2)$$

by using the energy estimate for the system (1.1). We refer to Proposition 5.1.
We also point out that this system has the following scaling: if (v, θ) is a solution of (1.1) and $\lambda > 0$ then $(v_\lambda, \theta_\lambda)$ is also a solution, where $v_\lambda(t, x) := v(\lambda t, \lambda x)$ and $\theta_\lambda(t, x) := \lambda\theta(\lambda t, \lambda x)$. As a consequence, the space of initial data $\dot H^1 \times L^2$ which is invariant under this transformation is critical.
From the *a priori* estimates that we shall establish (see Proposition 5.3), we can also get the existence of global weak solutions (but stronger than the previous ones) by assuming only that $v^0 \in H^1$ and $\theta^0 \in L^2 \cap L^r$ with $r > 4$ for example i.e. we can get global weak solutions which almost have a critical regularity. Nevertheless

the uniqueness for such solutions which do not satisfy the additional regularity assumptions stated in Theorem 1.1 remains unsolved.

Remark 1.3. Our proof gives more time integrability on the velocity v. More precisely , we have $v \in \widetilde{L}^{\rho}_{\text{loc}}(\mathbb{R}_+; B^1_{\infty,1})$ for all $\rho \in [1, \frac{p}{2}[$.

Let us say a few words about the main difficulties encountered in the proof of Theorem 1.1. Even if we neglect the nonlinear terms in the Boussinesq system, which reduces the system to the following one

$$\partial_t \omega + |\mathrm{D}|\omega = \partial_1 \theta, \quad \partial_t \theta = 0,$$

it is not clear how to perform the standard energy estimate on ω without using neither the control of higher order derivatives of θ nor the maximal regularities of the heat kernel $e^{-t|\mathrm{D}|}$ which would not be compatible with the nonlinear problem. For example for the L^2 energy estimate, one gets

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2} + \|\omega\|^2_{\dot{H}^{\frac12}} = \int_{\mathbb{R}^2} \partial_1 \theta \, \omega \, dx, \quad \|\theta(t)\|_{L^2} = \|\theta_0\|_{L^2},$$

from which no conclusion can be made.
The main idea in the proof of Theorem 1.1, that was also successfully used in the study of the Euler-Boussinesq system [19] is to really use the structural properties of the system solved by (ω, θ). We note that the symbol of the system which is given by

$$\mathcal{A}(\xi) = \begin{pmatrix} -|\xi| & -i\xi_1 \\ 0 & 0 \end{pmatrix}$$

is diagonalizable for $\xi \neq 0$ with eigenvalues 0 and $-|\xi|$ which are real and simple. By using the Riesz transform $\mathcal{R} = \frac{\partial_1}{|\mathrm{D}|}$, one gets that the diagonal form of the system is given by

$$\partial_t\big(\omega - \mathcal{R}\theta\big) + |\mathrm{D}|\big(\omega - \mathcal{R}\theta\big) = 0, \quad \partial_t \theta = 0.$$

This last form of the system is much more convenient in order to perform *a priori* estimates. For example one gets immediately from the continuity on L^2 of $\mathcal{R}$ that

$$\|\omega(t)\|_{L^2} + \|\theta(t)\|_{L^2} \leq C\big(\|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}\big).$$

To prove Theorem 1.1, we shall use the same idea, we shall diagonalize the linear part of the system and then get *a priori* estimates from the study of the new system. The main technical difficulty in this program when one takes the nonlinear terms into account is to evaluate in a sufficiently sharp way the commutator $[\mathcal{R}, v\cdot\nabla]$ between the Riesz transform and the convection operator. Such commutator estimates are stated and proven in section 3 of the paper.
The remaining of the paper is organized as follows.
Section 2 is devoted to the definition of the needed functional spaces and the statement of some of their useful properties. Some technical lemmas are also given. Section 3 is devoted to the study of some commutator estimates involving the Riesz transform. Section 4 is dedicated to the study of linear transport-(fractional) diffusion equation. Basically two kind of estimates are given: smoothing effects and logrithmic estimate. In section 5 we discuss a first set of a priori estimates and the issue of weak solutions. Section 6 is dedicated to the proof of Theorem 1.1. The last section is devoted to the proof of some technical lemmas.

2. Notations and preliminaries

2.1. **Notations.** Throughout this work we will use the following notations.
• For any positive A and B the notation $A \lesssim B$ means that there exist a positive harmless constant C such that $A \leq CB$.
• For any tempered distribution u both $\widehat{u}$ and $\mathcal{F}u$ denote the Fourier transform of u.
• Pour every $p \in [1, \infty]$, $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p.
• The norm in the mixed space time Lebesgue space $L^p([0,T], L^r(\mathbb{R}^d)$ is denoted by $\|\cdot\|_{L^p_T L^r}$ (with the obvious generalization to $\|\cdot\|_{L^p_T \mathcal{X}}$ for any normed space $\mathcal{X}$).
• For any pair of operators P and Q on some Banach space $\mathcal{X}$, the commutator $[P, Q]$ is given by $PQ - QP$.
• For $p \in [1, \infty]$, we denote by $\dot{W}^{1,p}$ the space of distributions u such that $\nabla u \in L^p$.

2.2. **Functional spaces.** Let us introduce the so-called Littlewood-Paley decomposition and the corresponding cut-off operators. There exists two radial positive functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall\, q \geq 1, \text{supp}\, \chi \cap \text{supp}\, \varphi(2^{-q}) = \varnothing$

ii) $\text{supp}\, \varphi(2^{-j}\cdot) \cap \text{supp}\, \varphi(2^{-k}\cdot) = \varnothing$, if $|j - k| \geq 2$.

For every $v \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1} v = \chi(\mathrm{D}) v \;;\; \forall q \in \mathbb{N},\; \Delta_q v = \varphi(2^{-q}\mathrm{D}) v \quad \text{and} \quad S_q = \sum_{-1 \leq p \leq q-1} \Delta_p.$$

The homogeneous operators are defined by

$$\dot{\Delta}_q v = \varphi(2^{-q}\mathrm{D}) v, \quad \dot{S}_q v = \sum_{j \leq q-1} \dot{\Delta}_j v, \quad \forall q \in \mathbb{Z}.$$

From [5] we split the product uv into three parts:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

Let us now define inhomogeneous Besov spaces. For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(2^{qs} \|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

Notice that the usual Sobolev spaces H^s coincide with $B^s_{2,2}$ for every $s \in \mathbb{R}$ and that the homogeneous spaces $\dot{H}^s$ coincide with $\dot{B}^s_{2,2}$.
We shall also use need some mixed space-time spaces. Let $T > 0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^\rho_T L^p}\Big)_{\ell^r} < +\infty.$$

By a direct application of the Minkowski inequality, we have the following links between these spaces.
Let $\varepsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \text{ if } \quad r \geq \rho,$$
$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \text{ if } \quad \rho \geq r.$$

We will make continuous use of Bernstein inequalities (see [8] for instance).

Lemma 2.1. *There exists a constant C such that for $q, k \in \mathbb{N}$, $1 \leq a \leq b$ and for $f \in L^a(\mathbb{R}^d)$,*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} \leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q f\|_{L^a},$$
$$C^{-k}2^{qk}\|\Delta_q f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{qk}\|\Delta_q f\|_{L^a}.$$

The following result generalizes the classical Gronwall inequality, see Lemma 5.2.1 [8] for the proof. It will be very useful in the proof of the uniqueness part of Theorem 1.1.

Lemma 2.2 (Osgood lemma)**.** *Let $\gamma \in L^1_{\text{loc}}(\mathbb{R}_+;\mathbb{R}_+)$, μ a continuous non decreasing function, $a \in \mathbb{R}_+$ and α a measurable function satisfying*

$$0 \leq \alpha(t) \leq a + \int_0^t \gamma(\tau)\mu(\alpha(\tau))d\tau, \qquad \forall t \in \mathbb{R}_+.$$

If we assume that $a > 0$ then

$$-\mathcal{M}(\alpha(t)) + \mathcal{M}(a) \leq \int_0^t \gamma(\tau)d\tau \quad \text{with} \quad \mathcal{M}(x) := \int_x^1 \frac{dr}{\mu(r)}.$$

If we assume $a = 0$ and $\lim_{x\to 0^+} \mathcal{M}(x) = +\infty$, then $\alpha(t) = 0, \forall t \in \mathbb{R}_+$.

Remark 2.3. In the particular case $\mu(r) = r(1 - \log r)$ one can show the following estimate, see Theorem 5.2.1 [8]: for every $t \in \mathbb{R}_+$

$$a \leq e^{1-\exp\int_0^t \gamma(\tau)d\tau} \Longrightarrow \alpha(t) \leq a^{\exp -\int_0^t \gamma(\tau)d\tau} e^{1-\exp\,(-\int_0^t \gamma(\tau)d\tau)}.$$

3. Riesz transform and commutators

A crucial step in the implementation of the strategy exposed in the introduction for the proof of Theorem 1.1, is the study of commutators between the Riesz transform $\mathcal{R} = \partial_1/|D|$ and the convection operator $v \cdot \nabla$. The results of this section hold for all space dimension $d \geq 2$.
Let us first recall some well-known properties of the Riez operator.

Proposition 3.1. *Let $\mathcal{R}$ be the Riez operator $\mathcal{R} = \partial_1/|D|$. Then the following hold true.*
(1) *For every $p \in]1, +\infty[$,*

$$\|\mathcal{R}\|_{\mathcal{L}(L^p)} \lesssim 1.$$

(2) Let $\mathcal{C}$ be a fixed ring. Then, there exists $\psi \in \mathcal{S}$ whose spectum does not meet the origin such that

$$\mathcal{R}f = 2^{dq}\psi(2^q\cdot)\star f$$

for every f with Fourier transform supported in $2^q\mathcal{C}$. In particular, $\mathcal{R}\Delta_q$ is uniformly bounded (with respct to $q \in \mathbb{N}$) in L^p for every $p \in [1,+\infty]$.

The property (1) is a classical Calderón-Zygmund theorem (see [21] for instance) and (2) is obvious.
The proof of the next lemma is easy and can be found in [19].

Lemma 3.2. *Let $p \in [1,\infty]$, f, g and h be three functions such that $\nabla f \in L^p, g \in L^\infty$ and $xh \in L^1$. Then,*

$$\|h\star(fg) - f(h\star g)\|_{L^p} \le \|xh\|_{L^1}\|\nabla f\|_{L^p}\|g\|_{L^\infty}.$$

Here is the main result of this section.

Theorem 3.3. *Let v be is a smooth divergence-free vector field. Then the following hold true.*
(1) For every $s \in]0,1[$

$$\|[\mathcal{R},v]\theta\|_{H^s} \lesssim_s \|\nabla v\|_{L^2}\|\theta\|_{B^{s-1}_{\infty,2}} + \|v\|_{L^2}\|\theta\|_{L^2},$$

for every smooth scalar function θ.
(2) For every $p \in [2,\infty]$

$$\|[\mathcal{R}, v\cdot\nabla]\theta\|_{B^0_{p,\infty}} \lesssim_p \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}} + \|v\|_{L^2}\|\theta\|_{L^2},$$

for every smooth scalar function θ.

Proof of Theorem 3.3. (1) We split the commutator into three parts, according to Bony's decomposition

$$\begin{aligned}[\mathcal{R},v]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R},S_{q-1}v]\Delta_q\theta + \sum_{q\in\mathbb{N}}[\mathcal{R},\Delta_q v]S_{q-1}\theta \\ &+ \sum_{q\ge -1}[\mathcal{R},\Delta_q v]\widetilde{\Delta}_q\theta \\ &= \sum_{q\in\mathbb{N}}\mathrm{I}_q + \sum_{q\in\mathbb{N}}\mathrm{II}_q + \sum_{q\ge -1}\mathrm{III}_q \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

• *Estimation of I.* According to the point (2) of Proposition 3.1 there exists $h \in \mathcal{S}$ whose Fourier transform does not contain the origin such that

$$\mathrm{I}_q(x) = h_q\star(S_{q-1}v\Delta_q\theta) - S_{q-1}v(h_q\star\Delta_q\theta),$$

where $h_q(x) = 2^{dq}h(2^qx)$. Applying Lemma 3.2 with $p = 2$ we infer

$$\begin{aligned}\|\mathrm{I}_q\|_{L^2} &\lesssim \|xh_q\|_{L^1}\|\nabla S_{q-1}v\|_{L^2}\|\Delta_q\theta\|_{L^\infty} \\ &\lesssim 2^{-q}\|\nabla v\|_{L^2}\|\Delta_q\theta\|_{L^\infty}. \qquad (3.1)\end{aligned}$$

In the last line we have used the fact that $\|xh_q\|_{L^1} = 2^{-q}\|xh\|_{L^1}$. Since for every $q \in \mathbb{N}$ the Fourier transform of I_q is supported in a ring of size 2^q then

$$\|\mathrm{I}\|^2_{H^s} \simeq \sum_q 2^{2qs}\|\mathrm{I}_q\|^2_{L^2}.$$

Combined with (3.1) this yields

$$\|\mathrm{I}\|_{H^s}^2 \lesssim \|\nabla v\|_{L^2}^2 \|\theta\|^2_{B^{s-1}_{\infty,2}}.$$

• *Estimation of II.* As before we can write

$$\mathrm{II}_q(x) = h_q \star (S_{q-1}\theta \Delta_q v) - S_{q-1}\theta (h_q \star \Delta_q v),$$

and again by Lemma 3.2 with $p = 2$ we obtain

$$\begin{aligned} \|\mathrm{II}_q\|_{L^2} &\lesssim \|\nabla v\|_{L^2} 2^{-q} \|S_{q-1}\theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^2} 2^{-q} \sum_{j \le q-2} \|\Delta_j \theta\|_{L^\infty}. \end{aligned}$$

Thus,

$$2^{qs} \|\mathrm{II}_q\|_{L^2} \lesssim \|\nabla v\|_{L^2} \big(2^{(1-s)\cdot} \star 2^{(1-s)\cdot} \|\Delta_\cdot \theta\|_{L^\infty}\big)(q).$$

where $\star$ is the discrete convolution in $\mathbb{N} \cup \{-1\}$.
Here again the Fourier transform of II_q is supported in a ring of size 2^q and by consequences $\|\mathrm{II}\|_{H^s} \simeq \big\|2^{s\cdot} \|\mathrm{I}_\cdot\|_{L^2}\big\|_{\ell^2}$. Combined with (**??**) and discrete Young inequalities (remember $s < 1$) this yields

$$\begin{aligned} \|\mathrm{II}\|_{H^s} &\lesssim_s \|\nabla v\|_{L^2} \big\|2^{(s-1)\cdot} \|\Delta_\cdot \theta\|_{L^\infty}\big\|_{\ell^2} \\ &\simeq \|\nabla v\|_{L^2} \|\theta\|_{B^{s-1}_{\infty,2}}. \end{aligned}$$

• *Estimation of III.* We distinguish two parts

$$\mathrm{III} = \sum_{q \ge 1} [\mathcal{R}, \Delta_q v] \widetilde{\Delta}_q \theta + \sum_{q \le 0} [\mathcal{R}, \Delta_q v] \widetilde{\Delta}_q \theta := J_1 + J_2.$$

J_1 contains only terms whose Fourier transform are localized away from zero. For them, we can use Proposition 3.1 and Lemma 3.2 as before. This gives

$$\|[\mathcal{R}, \Delta_q v] \widetilde{\Delta}_q \theta\|_{L^2} \lesssim 2^{-q} \|\nabla v\|_{L^2} \|\widetilde{\Delta}_q \theta\|_{L^\infty}.$$

Note that we have used the fact that for $q \ge 1$, we have $\|\mathcal{R}\widetilde{\Delta}_q \theta\|_{L^\infty} \lesssim \|\widetilde{\Delta}_q \theta\|_{L^\infty}$. Now we have

$$2^{js} \|\Delta_j J_1\|_{L^2} \lesssim \|\nabla v\|_{L^2} \sum_{q \ge j-4} 2^{(j-q)s} 2^{q(s-1)} \|\widetilde{\Delta}_q \theta\|_{L^\infty}$$

Since $s > 0$ then the convolution inequality leads to

$$\|J_1\|_{H^s} \lesssim \|\nabla v\|_{L^2} \|\theta\|_{B^{s-1}_{\infty,2}}.$$

J_2 contains a finite number of terms with low frequencies and it can be handled without using the commutator structure. Indeed, from Bernstein inequalities and the L^2-continuity of the Riesz transform we obtain for $q \le 0$

$$\begin{aligned} \|[\mathcal{R}, \Delta_q v] \Delta_q \theta\|_{L^2} &\lesssim \|\Delta_q v\|_{L^2} \big(\|\Delta_q \theta\|_{L^\infty} + \|\mathcal{R}\Delta_q \theta\|_{L^\infty}\big) \\ &\lesssim \|v\|_{L^2} \big(\|\theta\|_{L^2} + \|\mathcal{R}\theta\|_{L^2}\big) \\ &\lesssim \|v\|_{L^2} \|\theta\|_{L^2}. \end{aligned}$$

Thus we get for every $s \in \mathbb{R}$

$$\|J_2\|_{H^s} \lesssim \|v\|_{L^2} \|\theta\|_{L^2}.$$

This ends the proof of (1).

(2) We use again Bony's decomposition to write

$$\begin{aligned}[\mathcal{R}, v\cdot\nabla]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R}, S_{q-1}v\cdot\nabla]\Delta_q\theta+\sum_{q\in\mathbb{N}}[\mathcal{R}, \Delta_q v\cdot\nabla]S_{q-1}\theta\\ &+ \sum_{q\geq -1}[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Similarly to (3.1) we have

$$\begin{aligned}\big\|[\mathcal{R}, S_{q-1}v\cdot\nabla]\Delta_q\theta\big\|_{L^p} &\lesssim 2^{-q}\|\nabla v\|_{L^p}\|\Delta_q\nabla\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\Delta_q\theta\|_{L^\infty}.\end{aligned}$$

Thus we get

$$\|\mathrm{I}\|_{B^0_{p,\infty}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.$$

For the second term we have

$$\begin{aligned}\big\|[\mathcal{R}, \Delta_q v\cdot\nabla]S_{q-1}\theta\big\|_{L^p} &\lesssim 2^{-q}\|\nabla\Delta_q v\|_{L^p}\|S_{q-1}\nabla\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{j\leq q-2}2^{j-q}\|\Delta_q\theta\|_{L^\infty}.\end{aligned}$$

It follows that

$$\|\mathrm{II}\|_{B^0_{p,\infty}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.$$

To estimate the remainder term we use the embedding $L^p\hookrightarrow B^0_{p,\infty}$,

$$\|\mathrm{III}\|_{B^0_{p,\infty}}\lesssim\sum_{q\leq 1}\|[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta\|_{L^p}+\big\|\sum_{q\geq 2}\mathrm{div}[\mathcal{R}, \Delta_q v]\widetilde{\Delta}_q\theta\big\|_{B^0_{p,\infty}}.$$

For the first term of the RHS we use Bernstein inequalities ($p\geq 2$) and the L^2 continuity of $\mathcal{R}$ to get

$$\begin{aligned}\sum_{q\leq 1}\|[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta\|_{L^p} &\lesssim \|v\|_{L^2}\sum_{q\leq 1}(\|\nabla\widetilde{\Delta}_q\theta\|_{L^\infty}+\|\mathcal{R}\nabla\widetilde{\Delta}_q\theta\|_{L^\infty})\\ &\lesssim \|v\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

The second term is estimated as follows

$$\begin{aligned}\|\sum_{q\geq 2}[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta\|_{B^0_{p,\infty}} &\lesssim \|\nabla v\|_{L^p}\sup_j\sum_{q\geq j-4}2^{j-q}\|\widetilde{\Delta}_q\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

This ends the proof of Theorem 3.3. □

4. Transport-Diffusion models

This section contains some estimates needed in the proof of Theorem 1.1. We start with the following Besov space estimate for the transport equation, for the proof see for example [2].

Proposition 4.1. *Let v be a smooth divergence-free vector field. Then, every scalar solution ψ of the equation*

$$\partial_t\psi+v\cdot\nabla\psi=f,\quad \psi_{|t=0}=\psi^0,$$

satifies, for every $p\in[1,+\infty]$,

$$\|\psi(t)\|_{B^{-1}_{p,\infty}}\leq C\exp\big(C\int_0^t\|v(\tau)\|_{B^1_{\infty,1}}d\tau\big)\Big(\|\psi^0\|_{B^{-1}_{p,\infty}}+\int_0^t\|f(\tau)\|_{B^{-1}_{p,\infty}}d\tau\Big).$$

The second proposition is dedicated to logarithmic and L^p estimates.
The second proposition is dedicated to logarithmic estimates and L^p estimates.

Proposition 4.2. *Let v be a smooth divergence-free vector field, $\kappa\in\mathbb{R}_+$ and $(p,r)\in[1,\infty]^2$. Then there exists $C>0$, such that every scalar solution of*

$$\partial_t\psi+v\cdot\nabla\psi+\kappa|\mathrm{D}|\psi=f,\quad \psi_{|t=0}=\psi^0, \tag{4.1}$$

satisfies

$$\|\psi\|_{\widetilde{L}^\infty_tB^0_{p,r}}\leq C\Big(\|\psi^0\|_{B^0_{p,r}}+\|f\|_{\widetilde{L}^1_tB^0_{p,r}}\Big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big),$$

and

$$\|\psi(t)\|_{L^p}\leq\|\psi^0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}d\tau.$$

The first result was first proved by Vishik in [23] for the case $\kappa=0$ by using the special structure of the transport equation. In [17] the first two authors have generalized Vishik result for a transport-diffusion equation where the dissipation term has the form $-\kappa\Delta a$. The method described in [17] can be easily adapted to the model (4.1), for more details the complete proof can be found in [19]. The L^p estimates are proved in [10].
In the proof of the uniqueness part of the main theorem we shall also need some estimates for the linearized velocity equation.

Proposition 4.3. *Let v a smooth divergence free vector field, $s\in(-1,1)$ and $\rho\in[1,\infty]$. Let u be a smooth solution of the system*

$$\partial_tu+v\cdot\nabla u+|\mathrm{D}|u+\nabla p=f,\qquad \mathrm{div}u=0. \tag{LB}$$

Then, we have for every $t\in\mathbb{R}_+$

$$\|u\|_{L^\infty_tB^s_{2,\infty}}\leq Ce^{CV(t)}\Big(\|u^0\|_{B^s_{2,\infty}}+\|f\|_{\widetilde{L}^\rho_tB^{s-1+\frac{1}{\rho}}_{2,\infty}}(1+t^{1-\frac{1}{\rho}})\Big),$$

where $V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau$.

Proof of Proposition 4.3. For $q\in\mathbb{N}$ we apply the operator Δ_q to (LB)

$$\partial_tu_q+v\cdot\nabla u_q+|\mathrm{D}|u_q+\nabla p_q=-[\Delta_q,v\cdot\nabla]u+f_q.$$

Taking the L^2 inner product of this equation with u_q we get due to the incompressibility of v and u_q

$$\frac{1}{2}\frac{d}{dt}\|u_q\|^2_{L^2}+\int_{\mathbb{R}^2}(|\mathrm{D}|u_q)u_qdx\leq\|u_q\|_{L^2}\Big(\|[\Delta_q,v\cdot\nabla]u\|_{L^2}+\|f_q\|_{L^2}\Big).$$

From Parseval identity we get

$$C2^q\|u_q\|^2_{L^2}\leq\int_{\mathbb{R}^2}(|\mathrm{D}|u_q)u_qdx.$$

Thus we get

$$\frac{d}{dt}\|u_q(t)\|_{L^2}+c2^q\|u_q(t)\|_{L^2}\leq\|[\Delta_q,v\cdot\nabla]u(t)\|_{L^2}+\|f_q(t)\|_{L^2}.$$

Integrating in time this differential inequality we obtain

$$\|u_q(t)\|_{L^2} \lesssim e^{-ct2^q}\|u_q^0\|_{L^2} + \int_0^t e^{-c2^q(t-\tau)}\Big(\|[\Delta_q, v\cdot\nabla]u(\tau)\|_{L^2} + \|f_q(\tau)\|_{L^2}\Big)d\tau$$

Using Hölder inequalities we get

$$\|u_q\|_{L^\infty_t L^2} \lesssim \|u_q^0\|_{L^2} + \int_0^t \|[\Delta_q, v\cdot\nabla]u(\tau)\|_{L^2}d\tau + 2^{q(-1+\frac{1}{\rho})}\|f_q(\tau)\|_{L^\rho_t L^2}$$

Multiplying by 2^{qs} and taking the supremum over $q\in\mathbb{N}$ we find

$$\sup_{q\in\mathbb{N}} 2^{qs}\|u_q\|_{L^\infty_t L^2} \lesssim \|u^0\|_{B^s_{2,\infty}} + \int_0^t \sup_{q\in\mathbb{N}} 2^{qs}\|[\Delta_q, v\cdot\nabla]u(\tau)\|_{L^2}d\tau + \|f\|_{\widetilde{L}^\rho_t B^{s-1+\frac{1}{\rho}}_{2,\infty}}. \tag{4.2}$$

Let us recall the following classical commutator estimate (see [8] for instance)

$$\sup_{q\geq -1} 2^{qs}\|[\Delta_q, v\cdot\nabla]u(\tau)\|_{L^2} \lesssim_s \|\nabla v\|_{L^\infty}\|u\|_{B^s_{2,\infty}}, \qquad \forall s\in(-1,1).$$

Combined with (4.2) this yields

$$\sup_{q\in\mathbb{N}} 2^{qs}\|u_q\|_{L^\infty_t L^2} \lesssim \|u^0\|_{B^s_{2,\infty}} + \int_0^t \|\nabla v(\tau)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,\infty}}d\tau + \|f\|_{\widetilde{L}^\rho_t B^{s-1+\frac{1}{\rho}}_{2,\infty}}.$$

For the low frequency block, we have from the energy estimate of the localized equation

$$\begin{aligned}\frac{d}{dt}\|\Delta_{-1}u(t)\|_{L^2} &\leq \|[\Delta_{-1}, v\cdot\nabla]u(t)\|_{L^2} + \|\Delta_{-1}f(t)\|_{L^2}\\ &\lesssim \|\nabla v(t)\|_{L^\infty}\|u(t)\|_{B^s_{2,\infty}} + \|\Delta_{-1}f(t)\|_{L^2}\end{aligned}$$

It follows

$$\begin{aligned}\|\Delta_{-1}u(t)\|_{L^2} &\leq \|\Delta_{-1}u^0\|_{L^2} + \int_0^t \|\nabla v(\tau)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,\infty}}d\tau + \|\Delta_{-1}f\|_{L^1_t L^2}\\ &\lesssim \|u^0\|_{B^s_{2,\infty}} + \int_0^t \|\nabla v(t)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,\infty}}d\tau + t^{1-\frac{1}{\rho}}\|f\|_{\widetilde{L}^\rho_t B^{s-1+\frac{1}{\rho}}_{2,\infty}}.\end{aligned}$$

The outcome is

$$\|u(t)\|_{B^s_{2,\infty}} \lesssim \|u^0\|_{B^s_{2,\infty}} + \int_0^t \|\nabla v(t)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,\infty}}d\tau + (1+t^{1-\frac{1}{\rho}})\|f\|_{\widetilde{L}^\rho_t B^{s-1+\frac{1}{\rho}}_{2,\infty}}.$$

A Gronwall inequality gives the claimed result. □

The proof of the next proposition can be done in a similar way as Theorem 1.2 in [1].

Proposition 4.4. *Let v be a smooth divergence-free vector field and ψ be a smooth solution of the equation*

$$\partial_t\psi + v\cdot\nabla\psi + |\mathrm{D}|\psi = f, \qquad \psi_{|t=0} = \psi^0.$$

Then, for every $s\in]-1;1[$ and $(\rho,p,r)\in[1,\infty]^3$ there exists $C>0$ such that

$$\|\psi\|_{\widetilde{L}^\infty_t B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|\psi^0\|_{B^s_{p,r}} + (1+t^{1-\frac{1}{\rho}})\|f\|_{\widetilde{L}^\rho_t B^{s-1+\frac{1}{\rho}}_{p,r}}\Big), \qquad \forall t\in\mathbb{R}_+,$$

where $V(t) = \|\nabla v\|_{L^1_t L^\infty}$.

5. Weak solutions

Throughout the coming sections we use the notation Φ_k to denote any function of the form

$$\Phi_k(t) = C_0 \underbrace{\exp(... \exp}_{k\, times}(C_0 t)...),$$

where C_0 depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentionning it) of the following trivial facts

$$\int_0^t \Phi_k(\tau) d\tau \leq \Phi_k(t) \qquad \text{and} \qquad \exp(\int_0^t \Phi_k(\tau) d\tau) \leq \Phi_{k+1}(t).$$

In this section we shall establish a first set of *a priori* estimates and discuss some results about weak solutions which are easy consequences. These *a priori* estimates are also needed in order to construct the global strong solutions as stated in Theorem 1.1.

The first result is concerned with weak solutions in the energy space.

Proposition 5.1. *Let $(v^0, \theta^0) \in L^2 \times L^2$ then there exists a global weak solution of (1.1) in the space $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+; \dot{H}^{\frac12}) \times L^\infty(\mathbb{R}_+; L^2)$ such that*

$$\begin{aligned} \|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{\dot{H}^{\frac12}}^2 d\tau &\leq C_0(1+t^2) \\ \|\theta(t)\|_{L^2} &\leq \|\theta^0\|_{L^2}. \end{aligned}$$

If in addition, $\theta^0 \in L^p$ for some $p \in [1, \infty]$, then there is a weak solution which satisfies also

$$\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p}.$$

Proof of Proposition 5.1. The estimate of θ in L^p is a consequence of the incompressibility of the flow. The L^2 energy estimate for the velocity can be obtained by taking the L^2-inner product of the velocity equation in the Boussinesq system with v,

$$\begin{aligned} \frac12 \frac{d}{dt}\|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{\dot{H}^{\frac12}}^2 d\tau &\leq \|v(t)\|_{L^2}\|\theta(t)\|_{L^2} \\ &\leq \|v(t)\|_{L^2}\|\theta^0\|_{L^2}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \|v(t)\|_{L^2} &\leq \|v^0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2} d\tau \\ &\leq \|v^0\|_{L^2} + \|\theta^0\|_{L^2} t. \end{aligned}$$

Inserting this inequality into the previous one leads to the desired estimate. Now to construct global solution we can proceed in a classical way using the Friedrichs method. We omit here the details and for complete description of this method we refer for example to [12]. □

Remark 5.2. Due to the weak regularity of the velocity, the uniqueness problem for these weak solutions seems an interesting widely open problem.

Now we aim at constructing global weak solutions for more regular initial data near the scaling space $\dot{H}^1 \times L^2$ described in the introduction of this paper.

Proposition 5.3. *Let* $(v^0, \theta^0) \in H^1 \times L^2 \cap L^r$, *with* $r \in]4, \infty]$. *Then, there exists a global weak solution* (v, θ) *for the system* (1.1) *such that*

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|(\omega - \mathcal{R}\theta)(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leq \Phi_1(t),$$

where $\omega = curl\ v$.

Remark 5.4. From the estimate of Proposition 5.3 we see that we have an additional smoothing effect for the quantity $\omega - \mathcal{R}\theta$ since θ belongs only to the space $L_t^\infty(L^2 \cap L^r)$. This phenomenon illustrates the strong coupling between the velocity and the temperature.

Proof of Proposition 5.3. We shall, here again, restrict ourselves to the proof of the *a priori* estimates. The construction of global solutions can be done by following [12]. As explained in the introduction we do not have any available obvious L^p estimate for the vorticity. Thus in order to get some L^p estimates we write the Boussinesq system under its diagonal form. For this purpose we set $\Gamma = \omega - \mathcal{R}\theta$. Then we get from (1.1)

$$(\partial_t + v \cdot \nabla + |\mathrm{D}|)\Gamma = [\mathcal{R}, v \cdot \nabla]\theta. \tag{5.1}$$

By using the identity $[\mathcal{R}, v \cdot \nabla]\theta = \mathrm{div}\big([\mathcal{R}, v]\theta\big)$ and a standard L^2 energy estimate, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 &= \int_{\mathbb{R}^2} \mathrm{div}\big([\mathcal{R}, v]\theta\big)(t,x)\Gamma(t,x)dx \\ &\leq \big\|[\mathcal{R}, v]\theta(t)\big\|_{\dot{H}^{\frac{1}{2}}} \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}.\end{aligned}$$

From Theorem 3.3 and Proposition 5.1 we have

$$\begin{aligned}\big\|[\mathcal{R}, v]\theta(t)\big\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \|\nabla v(t)\|_{L^2}\|\theta(t)\|_{B_{\infty,2}^{-\frac{1}{2}}} + \|v(t)\|_{L^2}\|\theta(t)\|_{L^2} \\ &\lesssim \|\omega(t)\|_{L^2}\|\theta(t)\|_{L^r} + (1+t). \\ &\lesssim \|\omega(t)\|_{L^2}\|\theta^0\|_{L^r} + (1+t).\end{aligned}$$

Here, we have used that $\|\nabla v\|_{L^2} \approx \|\omega\|_{L^2}$ and the continuous embedding $L^r \hookrightarrow B_{\infty,2}^{-\frac{1}{2}}$, for every $r > 4$. Thus we get

$$\big\|[\mathcal{R}, v]\theta(t)\big\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\omega(t)\|_{L^2} + (1+t).$$

However, the L^2-continuity of $\mathcal{R}$ and the conservation of the L^2 norm of θ yield together

$$\begin{aligned}\|\omega(t)\|_{L^2} &\leq \|\Gamma(t)\|_{L^2} + \|\mathcal{R}\theta(t)\|_{L^2} \\ &\lesssim \|\Gamma(t)\|_{L^2} + \|\theta^0\|_{L^2}.\end{aligned}$$

Collecting the previous estimates, we find

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \big(\|\Gamma(t)\|_{L^2} + 1 + t\big)\|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}.$$

It follows from the Young inequality that

$$\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C_0\|\Gamma(t)\|_{L^2}^2 + C_0(1+t^2).$$

After an Integration in time and the use of the Gronwall inequality we obtain

$$\begin{aligned}\|\Gamma(t)\|_{L^2}^2+\int_0^t\|\Gamma(\tau)\|_{\dot H^{\frac12}}^2d\tau &\leq C_0\big(1+t^2\big)e^{C_0t}\\ &\leq \Phi_1(t).\end{aligned}$$

This ends the proof of the claimes *a priori* estimate in Proposition 5.3.

□

6. Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in three steps. First we prove other *a priori* estimates for the equations (1.1). Second, we prove the uniqueness part. Finally, we discuss the construction of the solutions.

6.1. **A priori estimates.** We have already proven some *a priori* estimates in Proposition 5.1 and Proposition 5.3. In the sequel we shall prove some other estimates. To estimate the L^p norm of ω we have to distinguish two cases. If $p\in]2,4[$ then we can prove this estimate in one step. If not, we establish first an estimate of the L^r norms of ω for all $r\in]2,4[$ and then[1] use it to get a smoothing effect on the quantity $(\omega-\mathcal{R}\theta)$. This will in turn yield the crucial estimate on the lipschitz norm of the velocity. Finally, by using the estimate on the lipschitz norm of the velocity, one can easily propagate the L^p norm of the vorticity and conclude the subsection about the *a priori* estimates.

Proposition 6.1. *Let* (v,θ) *be a solution of the Boussinesq system* (1.1) *such that* $v^0\in H^1\cap\dot W^{1,p}$ *and* $\theta^0\in L^2\cap L^\infty$ *with* $p\in]2,+\infty[$. *Then,*

$$\|\omega(t)\|_{L^r}^r+\int_0^t\|(\omega-\mathcal{R}\theta)(\tau)\|_{L^{2r}}^r d\tau\leq\Phi_1(t),$$

for every $r\in[2,4[\cap[2,p]$.

Proof of Proposition 6.1. Multiplying (5.1) by $|\Gamma|^{r-2}\Gamma$ and integrating in space variable we get for every $s\in]0,1[$ (s will be carefully chosen later)

$$\begin{aligned}\frac1r\frac{d}{dt}\|\Gamma(t)\|_{L^r}^r+\int_{\mathbb{R}^2}(|\mathrm{D}|\Gamma)\,|\Gamma|^{r-2}\Gamma dx &= \int_{\mathbb{R}^2}\mathrm{div}\big([\mathcal{R},v]\theta\big)|\Gamma|^{r-2}\Gamma dx\\ &\leq \big\|[\mathcal{R},v]\theta(t)\big\|_{\dot H^{1-s}}\||\Gamma|^{r-2}\Gamma(t)\|_{\dot H^s}.\end{aligned}\tag{6.1}$$

According to Lemma 3.3 in [18] one has

$$\frac2r\||\Gamma|^{\frac r2}\|_{\dot H^{\frac12}}^2\leq\int_{\mathbb{R}^2}(|\mathrm{D}|\Gamma)\,|\Gamma|^{r-2}\Gamma dx.$$

Combining this estimate with the Sobolev embedding $\dot H^{\frac12}\hookrightarrow L^4$, we find

$$\|\Gamma\|_{L^{2r}}^r\lesssim\int_{\mathbb{R}^2}(|\mathrm{D}|\Gamma)\,|\Gamma|^{r-2}\Gamma dx.\tag{6.2}$$

To estimate the RHS of (6.1), we use the following lemma (see the appendix for the proof).

Lemma 6.2.

[1] Notice that, concerning the velocity, the assumptions of Theorem 1.1 are equivalent to $v^0\in L^2$ and $\omega^0\in L^r$ for all $r\in[2,p]$, for some $p\in]2,+\infty[$.

Let $\beta\in[2,+\infty[$ *and* $s\in]0,1[$. *Then, we have*

$$\||u|^{\beta-2}u\|_{\dot H^s}\lesssim\|u\|_{L^{2\beta}}^{\beta-2}\|u\|_{\dot H^{s+1-\frac{2}{\beta}}},$$

for every smooth function u.

Combined with (6.1) and (6.2) this lemma yields

$$\frac{d}{dt}\|\Gamma(t)\|_{L^r}^r+c\|\Gamma(t)\|_{L^{2r}}^r\lesssim\big\|[\mathcal{R},v]\theta(t)\big\|_{\dot H^{1-s}}\|\Gamma\|_{L^{2r}}^{r-2}\|\Gamma\|_{\dot H^{s+1-\frac{2}{r}}}.$$

We choose $s\in]0,1[$ such that $s+1-\frac{2}{r}=\frac12$ which means that $s=\frac{2}{r}-\frac12$, this is possible if $r\in[2,4[$. Thus we get

$$\frac{d}{dt}\|\Gamma(t)\|_{L^r}^r+c\|\Gamma(t)\|_{L^{2r}}^r\lesssim\big\|[\mathcal{R},v]\theta(t)\big\|_{\dot H^{1-s}}\|\Gamma\|_{L^{2r}}^{r-2}\|\Gamma\|_{\dot H^{\frac12}}.$$

Theorem 3.3 and Proposition 5.3 yield

$$\begin{aligned}\big\|[\mathcal{R},v]\theta(t)\big\|_{\dot H^{1-s}}&\le\big\|[\mathcal{R},v]\theta(t)\big\|_{H^{1-s}}\\&\lesssim\|\nabla v\|_{L^2}\|\theta\|_{B^{-s}_{\infty,2}}+\|v(t)\|_{L^2}\|\theta(t)\|_{L^2}\\&\lesssim\|\omega(t)\|_{L^2}\|\theta(t)\|_{B^{-s}_{\infty,2}}+C_0(1+t^2)\\&\le\Phi_1(t)\|\theta(t)\|_{B^{-s}_{\infty,2}}+C_0(1+t^2).\end{aligned}$$

It suffices now to use the embedding $L^\infty\hookrightarrow B^{-s}_{\infty,2}$ for $s>0$. and Proposition 5.1 to get that

$$\big\|[\mathcal{R},v]\theta(t)\big\|_{\dot H^{1-s}}\le\Phi_1(t).$$

Therefore

$$\frac{d}{dt}\|\Gamma(t)\|_{L^r}^r+c\|\Gamma(t)\|_{L^{2r}}^r\le\Phi_1(t)\|\Gamma\|_{L^{2r}}^{r-2}\|\Gamma\|_{\dot H^{\frac12}}.$$

From the Young inequality

$$|ab|\le C|a|^{\frac{r}{2}}+\frac{c}{2}|b|^{\frac{r}{r-2}}$$

we get

$$\frac{d}{dt}\|\Gamma(t)\|_{L^r}^r+\|\Gamma(t)\|_{L^{2r}}^r\le\Phi_1(t)\|\Gamma(t)\|_{\dot H^{\frac12}}^{\frac{r}{2}}.$$

After integration in time and the application of the Hölder inequality (remember that $r<4$), we find

$$\begin{aligned}\|\Gamma(t)\|_{L^r}^r+\int_0^t\|\Gamma(\tau)\|_{L^{2r}}^r d\tau&\le\Phi_1(t)\Big(\int_0^t\|\Gamma(\tau)\|_{\dot H^{\frac12}}^2d\tau\Big)^{r/4}+\|\Gamma^0\|_{L^r}^r\\&\le\Phi_1(t).\end{aligned}\tag{6.3}$$

In the last line we have used Proposition 5.3. □

Next we give a smoothing effect for the quantity Γ which will be the keystone of a Lipschitz control of the velocity.

Proposition 6.3. *Under the assumptions of Proposition 6.1 we have*

$$\|\omega-\mathcal{R}\theta\|_{\widetilde L^\rho_t B^{\frac{2}{r}}_{r,1}}\le\Phi_1(t),$$

for every $r\in[2,4[\cap[2,p]$ *and* $\rho\in[1,\frac{r}{2}[$.

Proof of Proposition 6.3. For $q \in \mathbb{N}$ we set $\Gamma_q = \Delta_q \Gamma$. Then, we localize in frequencies the equation (5.1) for Γ to get

$$\begin{aligned}\partial_t \Gamma_q + v\cdot\nabla\Gamma_q + |\mathrm{D}|\Gamma_q &= -[\Delta_q, v\cdot\nabla]\Gamma + \Delta_q([\mathcal{R}, v\cdot\nabla]\theta)\\ &:= f_q\end{aligned}$$

Since Γ_q is a real-valued function then multiplying the above equation by $|\Gamma_q|^{r-2}\Gamma_q$ and integrating in the space variable we find

$$\frac{1}{r}\frac{d}{dt}\|\Gamma_q(t)\|_{L^r}^r + \int_{\mathbb{R}^2}(|\mathrm{D}|\Gamma_q)\,|\Gamma_q|^{r-2}\Gamma_q dx \le \|\Gamma_q(t)\|_{L^r}^{r-1}\|f_q(t)\|_{L^r}.$$

From [9], we have the following generalized Bernstein inequality

$$\int_{\mathbb{R}^d}(|\mathrm{D}|\Gamma_q)|\Gamma_q|^{r-2}\Gamma_q\,dx \ge c2^q\|\Gamma_q\|_{L^r}^r,$$

for some $c>0$ independent of q and hence we find

$$\frac{1}{r}\frac{d}{dt}\|\Gamma_q(t)\|_{L^r}^r + c2^q\|\Gamma(t)\|_{L^r}^r \le \|\Gamma_q(t)\|_{L^r}^{r-1}\|f_q(t)\|_{L^r}.$$

This yields

$$\|\Gamma_q(t)\|_{L^r} \le e^{-ct2^q}\|\Gamma_q^0\|_{L^r} + \int_0^t e^{-c(t-\tau)2^q}\|f_q(\tau)\|_{L^r}d\tau.$$

By taking the $L^\rho[0,t]$ norm and by using convolution inequalities, we find

$$\begin{aligned}2^{q\frac{2}{r}}\|\Gamma_q\|_{L^\rho_t L^r} \lesssim 2^{q(\frac{2}{r}-\frac{1}{\rho})}\|\Gamma_q^0\|_{L^r} &+ 2^{q(\frac{2}{r}-\frac{1}{\rho})}\int_0^t \big\|[\Delta_q, v\cdot\nabla]\Gamma(\tau)\big\|_{L^r}d\tau\\ &+ 2^{q(\frac{2}{r}-\frac{1}{\rho})}\int_0^t \big\|\Delta_q([\mathcal{R}, v\cdot\nabla]\theta)(\tau)\big\|_{L^r}d\tau.\end{aligned} \tag{6.4}$$

To estimate the second integral of the RHS we use the part (2) of Theorem 3.3, Proposition 5.1 and Proposition 6.1 to get, for every $q\in\mathbb{N}$

$$\begin{aligned}\big\|\Delta_q([\mathcal{R}, v\cdot\nabla]\theta)\big\|_{L^r} &\lesssim \|\nabla v\|_{L^r}\|\theta^0\|_{L^\infty} + \|v\|_{L^2}\|\theta\|_{L^2}\\ &\lesssim \|\omega\|_{L^r}\|\theta^0\|_{L^\infty} + \|v\|_{L^2}\|\theta\|_{L^2}\\ &\le \Phi_1(t).\end{aligned} \tag{6.5}$$

To estimate the first integral of the RHS we use the following lemma (see the appendix for the proof).

Lemma 6.4. *Let v be a smooth divergence-free vector field and f be a smooth scalar function. Then, for all $\alpha \in [1,\infty]$ and $q \ge -1$,*

$$\|[\Delta_q, v\cdot\nabla]f\|_{L^\alpha} \lesssim \|\nabla v\|_{L^r}\|f\|_{B^{\frac{2}{\alpha}}_{\alpha,1}}.$$

Using this lemma and Proposition 6.1 we infer

$$\begin{aligned}\big\|[\Delta_q, v\cdot\nabla]\Gamma\big\|_{L^r} &\lesssim \|\nabla v\|_{L^r}\|\Gamma\|_{B^{\frac{2}{r}}_{r,1}}\\ &\lesssim \|\omega\|_{L^r}\|\Gamma\|_{B^{\frac{2}{r}}_{r,1}}\\ &\le \Phi_1(t)\|\Gamma\|_{B^{\frac{2}{r}}_{r,1}}\end{aligned}$$

Let $N \in \mathbb{N}$ to be chosen later. We shall split the sum that we have to estimate into two parts $q < N$ and $q \geq N$. Since $4 > r > 2\rho$ we get

$$\sum_{q\geq N} 2^{q(\frac{2}{r}-\frac{1}{\rho})}\big\|[\Delta_q, v\cdot\nabla]\Gamma\big\|_{L^r} \leq 2^{-N(\frac{1}{\rho}-\frac{2}{r})}\Phi_1(t)\|\Gamma\|_{B^{\frac{2}{r}}_{r,1}}. \tag{6.6}$$

To estimate the low frequencies, we first use the crude estimate

$$\begin{aligned}\sum_{q<N} 2^{q\frac{2}{r}}\|\Gamma_q\|_{L^\rho_t L^r} &\lesssim 2^{\frac{2}{r}N}\|\Gamma\|_{L^\rho_t L^r}\\ &\leq 2^{\frac{2}{r}N}\Phi_1(t).\end{aligned} \tag{6.7}$$

In the last line we have used (6.3). Gathering (6.4), (6.5), (6.6) and (6.7) together yield

$$\begin{aligned}\|\Gamma\|_{\widetilde{L}^\rho_t B^{\frac{2}{r}}_{r,1}} &= \sum_{q<N} 2^{q\frac{2}{r}}\|\Delta_q\Gamma\|_{L^\rho_t L^r} + \sum_{q\geq N} 2^{q\frac{2}{r}}\|\Delta_q\Gamma\|_{L^\rho_t L^r}\\ &\leq 2^{\frac{2}{r}N}\Phi_1(t) + 2^{-N(\frac{1}{\rho}-\frac{2}{r})}\Phi_1(t)\|\Gamma\|_{L^1_t B^{\frac{2}{r}}_{r,1}}\\ &\leq 2^{\frac{2}{r}N}\Phi_1(t) + 2^{-N(\frac{1}{\rho}-\frac{2}{r})}\Phi_1(t)t^{1-\frac{1}{\rho}}\|\Gamma\|_{L^\rho_t B^{\frac{2}{r}}_{r,1}}\\ &\leq 2^{\frac{2}{r}N}\Phi_1(t) + 2^{-N(\frac{1}{\rho}-\frac{2}{r})}\Phi_1(t)\|\Gamma\|_{\widetilde{L}^\rho_t B^{\frac{2}{r}}_{r,1}}\end{aligned}$$

We choose N such that

$$2^{-N(\frac{1}{\rho}-\frac{2}{r})}\Phi_1(t) \approx \frac{1}{2}$$

and we finally obtain

$$\|\Gamma\|_{\widetilde{L}^\rho_t B^{\frac{2}{r}}_{r,1}} \leq \Phi_1(t).$$

This ends the proof of the desired result. □

Remark 6.5. From the embedding $B^{\frac{2}{r}}_{r,1} \hookrightarrow B^0_{\infty,1}$ we immediately get from the above estimate that for $t \in \mathbb{R}_+$

$$\|\Gamma\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} \leq \Phi_1(t). \tag{6.8}$$

The previous propositions allow us to prove a crucial *a priori* estimate on the gradient of v and to propagate the Besov norm of θ. This will be particularly important for the uniqueness part of the proof of the main theorem.

Proposition 6.6. *Let (v,θ) be a smooth solution of the system* (1.1). *Let $v^0 \in H^1 \cap \dot{W}^{1,p}$ with $p \in]2, +\infty[$ and $\theta^0 \in L^2 \cap B^0_{\infty,1}$. Then, we have*

$$\|v\|_{\widetilde{L}^\rho_t B^1_{\infty,1}} + \|\theta(t)\|_{B^0_{\infty,1}} \leq \Phi_1(t),$$

for every $\rho \in [1,2[\cap[1,\frac{p}{2}[$.

Proof of Proposition 6.6. Using the definition of Γ and (6.8) for $\rho = 1$ we get

$$\begin{aligned}\|\omega\|_{L^1_t B^0_{\infty,1}} &\leq \|\Gamma\|_{L^1_t B^0_{\infty,1}} + \|\mathcal{R}\theta\|_{L^1_t B^0_{\infty,1}}\\ &\leq \Phi_1(t) + \|\mathcal{R}\theta\|_{L^1_t B^0_{\infty,1}}.\end{aligned} \tag{6.9}$$

Now from Bernstein inequality, Proposition 3.1 and Proposition 5.1, we find

$$\begin{aligned} \|\mathcal{R}\theta(t)\|_{B^0_{\infty,1}} &\le \|\Delta_{-1}\mathcal{R}\theta(t)\|_{L^\infty} + \|\theta(t)\|_{B^0_{\infty,1}} \\ &\lesssim \|\theta^0\|_{L^2} + \|\theta(t)\|_{B^0_{\infty,1}}. \end{aligned} \tag{6.10}$$

By applying Proposition 4.2 to the second equation of (1.1) we get

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \le \|\theta^0\|_{B^0_{\infty,1}}\Big(1 + \|\nabla v\|_{L^1_t L^\infty}\Big). \tag{6.11}$$

However, Bernstein inequality and the estimate $2^q\|\Delta_q v\|_{L^\infty} \approx \|\Delta_q\omega\|_{L^\infty}$ for very $q \in \mathbb{N}$ yield together

$$\begin{aligned} \|v\|_{L^1_t B^1_{\infty,1}} &\lesssim \|\Delta_{-1} v\|_{L^1_t L^\infty} + \|\omega\|_{L^1_t B^0_{\infty,1}} \\ &\lesssim \|v\|_{L^1_t L^2} + \|\omega\|_{L^1_t B^0_{\infty,1}} \\ &\le C_0(1+t^2) + C_0\int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}} d\tau. \end{aligned} \tag{6.12}$$

In the last line we have used Proposition 5.1.

We set $X(t) = \displaystyle\int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}} d\tau$. Combining (6.9), (6.10), (6.11) and (6.12) we get

$$X(t) \le \Phi_1(t) + C_0\int_0^t X(\tau)d\tau.$$

and the Gronwall inequality yields

$$\int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}} d\tau \le \Phi_1(t).$$

Form the above estimate and (6.12) we also get

$$\begin{aligned} \|v\|_{L^1_t B^1_{\infty,1}} &\lesssim C_0(1+t^2) + C_0 X(t) \\ &\le \Phi_1(t). \end{aligned}$$

By combining this estimate with (6.10), we find

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \le \Phi_1(t).$$

This leads to

$$\|\mathcal{R}\theta\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \le \Phi_1(t). \tag{6.13}$$

Finally (6.8) yields for every $\rho \in [1, \frac{p}{2}[$

$$\begin{aligned} \|\omega\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} &\le \|\Gamma\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} + \|\mathcal{R}\theta\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} \\ &\le \Phi_1(t) + \|\mathcal{R}\theta\|_{\widetilde{L}^\rho_t B^0_{\infty,1}}. \end{aligned}$$

By using Hölder inequality and (6.13) we find

$$\begin{aligned} \|\mathcal{R}\theta\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} &\lesssim t^{\frac{1}{\rho}}\|\mathcal{R}\theta\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \\ &\le \Phi_1(t). \end{aligned}$$

It follows that

$$\|\omega\|_{\widetilde{L}^\rho_t B^0_{\infty,1}} \le \Phi_1(t),$$

and then

$$\|v\|_{\widetilde{L}^\rho_t B^1_{\infty,1}} \leq \Phi_1(t).$$

□

It remains finally to propagate the L^p norm of the vorticity (when $p \geq 4$).

Proposition 6.7. *Under the hypotheses of Proposition 6.6 we have*

$$\|\omega(t)\|_{L^p} \leq \Phi_2(t),$$

for every $t \in \mathbb{R}_+$.

Proof of Proposition 6.7. Recall that the quantity $\Gamma = \omega - \mathcal{R}\theta$ satisfies

$$\partial_t \Gamma + v \cdot \nabla \Gamma + |\mathrm{D}|\Gamma = [\mathcal{R}, v \cdot \nabla]\theta.$$

Using Proposition 4.2 we find

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \big\|[\mathcal{R}, v \cdot \nabla]\theta(\tau)\big\|_{L^p} d\tau.$$

Recall now the following commutator result proven in [19]: for $p \in [2, \infty[$ we have

$$\big\|[\mathcal{R}, v \cdot \nabla]\theta\big\|_{B^0_{p,1}} \lesssim \|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,1}} + \|\theta\|_{L^p}\big)$$

It follows from the Calderòn-Zygmund Theorem and Proposition 6.6 that

$$\big\|[\mathcal{R}, v \cdot \nabla]\theta(t)\big\|_{B^0_{p,1}} \lesssim \Phi_1(t)\|\omega(t)\|_{L^p}.$$

On the other hand we have

$$\begin{aligned} \|\omega(t)\|_{L^p} &\leq \|\Gamma(t)\|_{L^p} + \|\mathcal{R}\theta(t)\|_{L^p} \\ &\lesssim \|\Gamma(t)\|_{L^p} + \|\theta^0\|_{L^p}. \end{aligned}$$

By putting together these estimates, we find

$$\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \Phi_1(\tau)\|\omega(\tau)\|_{L^p} d\tau.$$

It suffices to use the Gronwall inequality to end the proof. □

Remark 6.8. Taking this estimate into account, we can also trivially extend the results of Proposition 6.3 and Proposition 6.6 to every $\rho \in [1, \frac{p}{2}[$.

6.2. **Uniqueness.** We will now prove a uniqueness result for the system (1.1) in the following space

$$\mathcal{X}_T := L^\infty_T H^1 \cap L^1_T B^1_{\infty,1} \times L^\infty_T\big(L^2 \cap B^0_{\infty,1}\big).$$

Let (v^i, θ^i) two solutions of the system (1.1) with initial data (v^0_i, θ^0_i) and lying in the space $\mathcal{X}_T$. We set $v = v^1 - v^2$, $\theta = \theta^1 - \theta^2$. Then

$$\begin{aligned} \partial_t v + v^2 \cdot \nabla v + |\mathrm{D}|v + \nabla p &= -v \cdot \nabla v^1 + \theta e_2 \\ \partial_t \theta + v^2 \cdot \nabla \theta &= -v \cdot \nabla \theta^1. \end{aligned}$$

To estimate v, we shall use Proposition 4.3. By considering the equation for v as a linear equation with a right hand-side which is made of the sum of two terms, we can write $v = V_1 + V_2$ where V_i solves

$$\partial_t V_i + v^2 \cdot \nabla V_i + |\mathrm{D}|V_i + \nabla p_i = F_i$$

with $F_1 = -v\cdot\nabla v^1$, and $F_2 = \theta e_2$. To estimate V_1, we use Proposition 4.3 for $\rho = 1$ and $s = 0$ while to estimate V_2, we use Proposition 4.3 for $\rho = +\infty$, $s = 0$. This yields for every $t \in [0, T]$

$$\|v(t)\|_{B^0_{2,\infty}} \lesssim e^{CV_2(t)}\Big(\|v^0\|_{B^0_{2,\infty}} + \|v\cdot\nabla v^1\|_{L^1_t B^0_{2,\infty}} + \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}}(1+t)\Big). \tag{6.14}$$

From Lemma 6.10 in the appendix we get

$$\|v\cdot\nabla v^1\|_{B^0_{2,\infty}} \lesssim \|v^1\|_{B^1_{\infty,1}}\|v\|_{L^2}.$$

Now, by using the logarithmic interpolation inequality of Lemma 6.11 combined with easy computations, we find

$$\begin{aligned}\|v\|_{L^2} &\lesssim \|v\|_{B^0_{2,\infty}}\log\Big(e + \frac{\|v\|_{H^1}}{\|v\|_{B^0_{2,\infty}}}\Big)\\ &\lesssim \|v\|_{B^0_{2,\infty}}\log\Big(e + \frac{1}{\|v\|_{B^0_{2,\infty}}}\Big)\log\big(e + \|v\|_{H^1}\big).\end{aligned}$$

Thus we get

$$\|v\cdot\nabla v^1\|_{B^0_{2,\infty}} \lesssim \|v^1\|_{B^1_{\infty,1}}\log\big(e + \|v\|_{H^1}\big)\mu(\|v\|_{B^0_{2,\infty}}). \tag{6.15}$$

where $\mu(x) = x\log(e+1/x)$. On the other hand, applying Proposition 4.1 with $p = 2$ to θ yields

$$\|\theta\|_{L^\infty_t B^{-1}_{2,\infty}} \lesssim e^{C\|v^2\|_{L^1_t B^1_{\infty,1}}}\Big(\|\theta^0\|_{B^{-1}_{2,\infty}} + \int_0^t \|v\cdot\nabla\theta^1(\tau)\|_{B^{-1}_{2,\infty}}d\tau\Big). \tag{6.16}$$

To estimate the right hand-side, we use the following product estimate (see Lemma 6.10 in the appendix)

$$\|v\cdot\nabla\theta^1\|_{B^{-1}_{2,\infty}} \lesssim \|v\|_{L^2}\|\theta^1\|_{B^0_{\infty,1}}.$$

The combination of this estimate with Lemma 6.11 yield

$$\|v\cdot\nabla\theta^1\|_{B^{-1}_{2,\infty}} \lesssim \|\theta^1\|_{B^0_{\infty,1}}\log\big(e + \|v\|_{H^1}\big)\mu(\|v\|_{B^0_{2,\infty}}). \tag{6.17}$$

We set $X(t) = \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}} + \|v\|_{L^\infty_t B^0_{2,\infty}}$. Putting together (6.14), (6.15), (6.16) and (6.17) gives

$$X(t) \le f(t)\Big(X(0) + \int_0^t \|v^1(\tau)\|_{B^1_{\infty,1}}\mu(X(\tau))d\tau\Big),$$

with f a known function depending continuously and increasingly on the quantities $\|(v^i, \theta^i)\|_{\mathcal{X}_t}$ and on the variable time. Now from Lemma 2.2 we get the uniqueness. Finally, let us now give some quantified estimates that will be used later for the construction of the solutions. Applying Remark 2.3 we get

$$X(0) \le \alpha(T) \Longrightarrow X(t) \le \beta(T)\big(X(0)\big)^{\gamma(T)}, \tag{6.18}$$

where α, β, γ are explicit functions depending continuously on $\|(v^i, \theta^i)\|_{\mathcal{X}_T}$ and T.

6.3. **Existence.** We consider the following system

$$(\mathrm{B}_n)\qquad \begin{cases} \partial_t v_n + v_n\cdot\nabla v_n + |\mathrm{D}|v_n + \nabla p_n = \theta_n e_2\\ \partial_t\theta_n + v_n\cdot\nabla\theta_n = 0\\ \mathrm{div} v_n = 0\\ v_{n|t=0} = S_n v^0, \quad \theta_{n|t=0} = S_n\theta^0 \end{cases}$$

First remark that $S_n v^0, S_n\theta^0 \in H^s, \forall s\in\mathbb{R}$ since $v^0,\theta^0\in L^2$. As in the classical theory of quasi-linear hyperbolic systems, we can prove the local well-posedness of the system (B_n). The global well-posedness is related to the following criterion: the solution can be continued beyond the time T if the quantity $\|\nabla v_n\|_{L^1_T L^\infty}$ is finite. Now from the *a priori* estimates, in particular Proposition 6.6, the Lipschitz norm of the velocity can not blow up in finite time and hence the solution (v_n,θ_n) is globally defined. Once again from the *a priori* estimates we have for $1\le\rho<p/2$

$$\|v_n\|_{L^\infty_T(H^1\cap\dot W^{1,p})} + \|v_n\|_{L^\rho_T B^1_{\infty,1}} \le \Phi_2(T),$$

and

$$\|\theta_n\|_{L^\infty_T(L^2\cap B^0_{\infty,1})}\le\Phi_2(T).$$

It follows that up to the extraction of a subsequence (v_n,θ_n) is weakly convergent to (v,θ) satisfying the same estimate as above. Now using (6.18) we get the following: if we have

$$a_{n,m} = \|(S_n-S_m)v^0\|_{B^0_{2,\infty}} + \|(S_n-S_m)\theta^0\|_{B^{-1}_{2,\infty}} \le \alpha(T)$$

then

$$\|v_n-v_m\|_{L^\infty_T B^0_{2,\infty}} + \|\theta_n-\theta_m\|_{L^\infty_T B^{-1}_{2,\infty}} \le \beta(T)\big(a_{n,m}\big)^{\gamma(T)}.$$

This proves that (v_n) is a Cauchy sequence and hence that it converges strongly to v in the space $L^\infty_T B^0_{2,\infty}$. By interpolation we can easily get the strong convergence of v_n to v in $L^2([0,T]\times\mathbb{R}^2)$. This implies that $v_n\otimes v_n$ converges in $L^1([0,T]\times\mathbb{R}^2)$. But since θ_n converges to θ weakly in $L^2([0,T]\times\mathbb{R}^2)$ then, by weak strong convergence, we have also that $v_n\,\theta_n$ converges weakly to $v\,\theta$.
This allows us to pass to the limit in the system (B_n) and to get that (v,θ) is a solution of our initial problem.

Appendix: some technical lemmas

Here we restate and prove Lemma 6.2.

Lemma 6.9.
Let $\beta\in[2,+\infty[$, $s\in]0,1[$ and $u\in L^{2\beta}\cap\dot H^{s+1-\frac{2}{\beta}}$. Then we have

$$\||u|^{\beta-2}u\|_{\dot H^s}\lesssim\|u\|^{\beta-2}_{L^{2\beta}}\|u\|_{\dot H^{s+1-\frac{2}{\beta}}}.$$

Proof. We shall actually establish the more accurate estimate:

$$\||u|^{\beta-2}u\|_{\dot H^s}\le C\|u\|^{\beta-2}_{L^{2\beta}}\|u\|_{\dot B^s_{\beta,2}}.$$

Once this estimate is established, the result follows from the embedding $\dot H^{s+1-\frac{2}{\beta}}\hookrightarrow\dot B^s_{\beta,2}$, for $\beta\ge2$ which is an easy consequence of Bernstein inequalities. For $0<s<1$, we can use the characterization of the homogeneous Sobolev space $\dot{H}^s$,

$$\big\||u|^{\beta-2}u\big\|_{\dot{H}^s}^2 \approx \int_{\mathbb{R}^2} \frac{\||u|^{\beta-2}u(x-\cdot)-|u|^{\beta-2}u(\cdot)\|_{L^2}^2}{|x|^{2+2s}}dx. \tag{6.19}$$

On the other hand there exists C depending on β such that for every $a, b\in\mathbb{R}$

$$\big||a|^{\beta-2}a-|b|^{\beta-2}b\big| \leq C|a-b|\big(|a|^{\beta-2}+|b|^{\beta-2}\big).$$

Thus using this inequality and integrating in y we get by Cauchy-Schwarz inequality

$$\||u|^{\beta-2}u(x-\cdot)-|u|^{\beta-2}u(\cdot)\|_{L^2} \leq C\|u(x-\cdot)-u(\cdot)\|_{L^\beta}\|u\|_{L^{2\beta}}^{\beta-2}.$$

Inserting this estimate into (6.19) and using the characterization of Besov space leads to

$$\begin{aligned}\||u|^{\beta-2}u\|_{\dot{H}^s}^2 &\lesssim \|u\|_{L^{2\beta}}^{2\beta-4}\int_{\mathbb{R}^2}\frac{\|u(x-\cdot)-u(\cdot)\|_{L^\beta}^2}{|x|^{2+2s}}dx\\ &\lesssim \|u\|_{L^{2\beta}}^{2\beta-4}\|u\|_{\dot{B}^s_{\beta,2}}^2.\end{aligned}$$

This concludes the proof. □

Lemma 6.10 (Commutators estimates)**.** *Let v be a smooth divergence-free vector field and f be a smooth function then*

(1) *for every $q\geq -1$*

$$\|[\Delta_q, v\cdot\nabla]f\|_{L^p} \lesssim \|\nabla v\|_{L^p}\|f\|_{B^0_{\infty,\infty}}.$$

(2) *For every $s\in[-1,0]$*

$$\|v\cdot\nabla f\|_{B^s_{2,\infty}} \lesssim \|v\|_{L^2}\|f\|_{B^{1+s}_{\infty,1}}.$$

Proof. (1) We shall actually prove the refined estimate

$$\|[\Delta_q, v\cdot\nabla]\theta\|_{L^p} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.$$

The desired estimate will follow from the embedding $B^{\frac{2}{p}}_{p,1}\hookrightarrow B^0_{\infty,\infty}$. We have from Bony's decomposition

$$\begin{aligned}[\Delta_q, v\cdot\nabla]\theta &= \sum_{|j-q|\leq 4}[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j\theta + \sum_{|j-q|\leq 4}[\Delta_q,\Delta_j v\cdot\nabla]S_{j-1}\theta\\ &+ \sum_{j\geq q-4}[\Delta_q,\Delta_j v\cdot\nabla]\widetilde{\Delta}_j\theta\\ &:= \mathrm{I}_q+\mathrm{II}_q+\mathrm{III}_q.\end{aligned}$$

Observe first that

$$\mathrm{I}_q = \sum_{|j-q|\leq 4} h_q\star(S_{j-1}v\cdot\nabla\Delta_j\theta) - S_{j-1}v\cdot(h_q\star\nabla\Delta_j\theta)$$

where $\hat{h}_q(\xi)=\varphi(2^{-q}\xi)$. Thus, Lemma 3.2 and Bernstein inequalities yield

$$\begin{aligned}\|\mathrm{I}_q\|_{L^p} &\lesssim \sum_{|j-q|\leq 4}\|xh_q\|_{L^1}\|\nabla S_{j-1}v\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|h_0\|_{L^1}\sum_{|j-q|\leq 4}2^{j-q}\|\Delta_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

To estimate the second term we use once again Lemma 3.2

$$\begin{aligned}\|\mathrm{II}_q\|_{L^p} &\lesssim \sum_{|j-q|\leq 4} 2^{-q}\|\Delta_j \nabla v\|_{L^p}\|\nabla S_{j-1}\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{\substack{|j-q|\leq 4\\ k\leq j-2}} 2^{k-q}\|\Delta_k\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

Let us now move to the remainder term. We separate it into two terms: high frequencies and low frequencies.

$$\begin{aligned}\mathrm{III}_q &= \sum_{\substack{j\geq q-4\\ j\in\mathbb{N}}} [\Delta_q\partial_i, \Delta_j v^i]\widetilde{\Delta}_j\theta + [\Delta_q, \Delta_{-1}v\cdot\nabla]\widetilde{\Delta}_{-1}\theta\\ &:= \mathrm{III}_q^1 + \mathrm{III}_q^2.\end{aligned}$$

For the first term we don't need to use the structure of the commutator. We estimate separately each term of the commutator by using Bernstein inequalities.

$$\begin{aligned}\|\mathrm{III}_q^1\|_{L^p} &\lesssim \sum_{\substack{j\geq q-4\\ j\in\mathbb{N}}} 2^q\|\Delta_j v\|_{L^p}\|\widetilde{\Delta}_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\sum_{j\geq q-4} 2^{q-j}\|\widetilde{\Delta}_j\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

For the second term we use Lemma 3.2 combined with Bernstein inequalities.

$$\begin{aligned}\|\mathrm{III}_q^2\|_{L^p} &\lesssim \|\nabla\Delta_{-1}v\|_{L^p}\|\nabla\widetilde{\Delta}_{-1}\theta\|_{L^\infty}\\ &\lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.\end{aligned}$$

(2) According to by Bony's decompostion and div$v=0$

$$v\cdot\nabla u = T_{v^i}\partial_i u + T_{\partial_i u}v^i + \partial_i\mathcal{R}(v^i,u).$$

To estimate the first term we write by definition

$$2^{qs}\|\Delta_q(T_{v^i}\partial_i u)\|_{L^2} \lesssim \sum_{|j-q|\leq 4} 2^{j+qs}\|S_{j-1}v\|_{L^2}\|\Delta_j u\|_{L^\infty}.$$

Now we use the inequality

$$\|S_{j-1}v\|_{L^2}\leq\|v\|_{L^2}.$$

Thus we get

$$\sup_q 2^{qs}\|\Delta_q(T_{v^i}\partial_i u)\|_{L^2}\lesssim\|v\|_{L^2}\|u\|_{B^{1+s}_{\infty,\infty}}.$$

Straightforward calculus gives since $1+s\leq 0$,

$$\begin{aligned}2^{qs}\|\Delta_q(T_{\partial_i u}v^i)\|_{L^2} &\lesssim \sum_{\substack{|j-q|\leq 4\\ k\leq j-1}} 2^{j(1+s)}\|\Delta_k u\|_{L^\infty}\|\Delta_j v\|_{L^2}\\ &\lesssim \|v\|_{L^2}\|u\|_{B^{1+s}_{\infty,1}}.\end{aligned}$$

and

$$\|\partial_i\mathcal{R}(v^i,u)\|_{B^s_{2,\infty}}\lesssim\|v\|_{L^2}\|u\|_{B^{1+s}_{\infty,1}}.$$

□

Lemma 6.11. *Let $v \in H^1$ then we have*

$$\|v\|_{L^2} \lesssim \|v\|_{B^0_{2,\infty}} \log\Big(e + \frac{\|v\|_{H^1}}{\|v\|_{B^0_{2,\infty}}}\Big).$$

Proof. Let $N \in \mathbb{N}^*$ be a fixed number then using the dyadic decomposition we get

$$\begin{aligned}\|v\|_{L^2} &\leq \sum_{q\leq N-1} \|\Delta_q v\|_{L^2} + \sum_{q\geq N} \|\Delta_q v\|_{L^2}\\ &\lesssim N\|v\|_{B^0_{2,\infty}} + 2^{-N}\|v\|_{H^1}.\end{aligned}$$

Choosing

$$N = \Big[\log_2\Big(e + \|v\|_{H^1}/\|v\|_{B^0_{2,\infty}}\Big)\Big]$$

gives the desired result. □

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