EXAMPLES OF QUASI-HYPERBOLIC DYNAMICAL SYSTEMS WITH SLOW DECAY OF CORRELATIONS

by Stéphane Le Borgne

Abstract. — We give examples of quasi-hyperbolic dynamical systems with the following properties : polynomial decay of correlations, convergence in law toward a non gaussian law of the ergodic sums (divided by $n^{3/4}$) associated to non degenerated regular functions.

Introduction

In the litterature dealing with stochastic properties of quasi-hyperbolic transformations of manifolds one establish most often exponential decay of correlations and central limit theorem. For a few years several examples of dynamical systems were found for which the correlations and the behaviour in law are different ([4], [10], [11], [13], [15]). Here we present a new family of examples (of systems that can be invertible) constructed in a very simple way as skew products.

Let (X, A, μ) be an Anosov system or an Axiom-A system and (Y, ϕ_t, ν) an Anosov flow both with exponential decay of correlations. Let us fix a regular function f from X to $\mathbb{R}$. On the product $X \times Y$ défine the transformation

$$T : X \times Y \longrightarrow X \times Y$$
$$(x, y) \longmapsto (Ax, \phi_{f(x)}y).$$

The hyperbolic transformation A acts on the first coordinate ; the flow acts on the second one during a time which depends on the first coordinate via f. The product measure $\mu \otimes \nu$ is T-invariant : $(X \times Y, T, \mu \otimes \nu)$ is a dynamical system.

Let $S_n f(x)$ denote $\sum_{k=0}^{n-1} f(A^k x)$ $(S_0 = 0)$ the ergodic sums associated to the action of A on the function f. A simple computation shows that, for all $n \geq 0$, the expression of T^n

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is:

$$T^n(x, y) = (A^n x, g_{S_n f(x)} y)$$

As f is centered the ergodic $S_n f$ sums grows almost surely less rapidly than $n^{1/2+\varepsilon}$ (for all $\varepsilon > 0$). So transformation T has a null Lyapounoff exponent in the tangent spaces of the submanifold $\{x\} \times Y$.

0.1. Decay of correlation. — Let $\sigma(f)$ denote the non-negative quantity defined by

$$\sigma^2(f) = \int_X f^2 d\lambda + 2 \sum_{k \geq 1} < f, A^k f > .$$

THEOREM 0.1. — *If the function f is aperiodic and if $\varphi : X \times Y \to \mathbb{R}$ is a centered hölder-continuous function such that the function $y \longmapsto \int_X \varphi(a, y) d\mu(a)$ is not a coboundary in the sense of the flow the the number*

$$\Sigma^2(\varphi) = \int_{\mathbb{R}} \int_Y \left(\int_X \varphi(a, g_b y) d\mu(a) \right) \left(\int_X \varphi(c, y) d\mu(c) \right) d\nu(y) \ db$$

is positive and the $< \varphi \circ T^k, \varphi >$ is equivalent to

$$\frac{1}{\sqrt{2\pi k}\sigma(f)} \overset{2}{\Sigma}(\varphi).$$

Proof : We're interested in

$$< \varphi \circ T^k, \varphi > = \mathbb{E}(\varphi(T^k., g_{S_k f(.)}.)\varphi(.,.))$$
$$= \int_{X \times Y} \varphi(T^k x, g_{S_{k-1} f(x)} y)\varphi(x, y) d\mu(x) d\nu(y).$$

Let us consider the function F :

$$X \times \mathbb{R} \times X \longrightarrow \mathbb{R}$$
$$(a, b, c) \longmapsto \int_Y \varphi(a, g_b y)\varphi(c, y) d\nu(y) - (\int_Y \varphi(a, y) d\nu(y))(\int_Y \varphi(c, y) d\nu(y))$$

The correlation $< \varphi \circ T^k, \varphi >$ is

$$\int_X F(T^k x, S_{k-1} f(x), x) d\mu(x) + \int_X (\int_Y \varphi(T^k x, y) d\nu(y))(\int_Y \varphi(x, y) d\nu(y)) d\mu(x).$$

The action of A is exponentially mixing thus the second element of this sum tends exponentially fast toward 0.

The action of the flow is exponentially mixing. The function F decrease exponentially fast when b grows. The local limit theorem of Guivarc'h and Hardy assure that the integral

$$\int_X F(T^k x, S_{k-1} f(x), x) d\mu(x)$$

is equivalent to,

$$\frac{1}{\sqrt{2k\pi}\sigma(f)} \int_{X \times \mathbb{R} \times X} F(a, b, c) d\mu(a) \, db \, d\mu(c)$$

$$= \frac{1}{\sqrt{2\pi k}\sigma(f)} \int_{\mathbb{R}} \left(\int_X \int_X \int_Y \varphi(a, g_b y)\varphi(c, y)d\nu(y) \, d\mu(a) \, d\mu(c) \right.$$

$$\left. - \underbrace{\int_X \int_X \int_Y \varphi(a, y)d\nu(y) \int_Y \varphi(c, y)d\nu(y) \, d\mu(a) \, d\mu(c)}_{0} \right) db$$

$$= \frac{1}{\sqrt{2\pi k}\sigma(f)} \underbrace{\int_{\mathbb{R}} \int_Y \left(\int_X \varphi(a, g_b y)d\mu(a) \right) \left(\int_X \varphi(c, y)d\mu(c) \right) d\nu(y) \, db}_{\sum^2(\varphi).}$$

The quantity $\sum^2(\varphi)$ is zero if and only if the function $y \longmapsto \int_X \varphi(a, y)d\mu(a)$ is a coboundary in the sense of the flow (see [3] for more details) that is if and only if there exists a function $\varphi : Y \to \mathbb{R}$ such that for almost every $y \in Y$ one has

$$\int_X \varphi(a, y)d\mu(a) = \lim_{t \to 0} \frac{\psi(g + y) - \varphi(y)}{t}.$$

0.2. Variances of the ergodic sums. —

COROLLARY 0.1. — *1 If* $\sum^2(\varphi)$ *is not zero the variance of* $\sum_{k=0}^{n-1} \varphi \circ T^k)$ *is equivalent to*

$$\frac{8}{3} \frac{\sum^2(\varphi)}{\sqrt{2\pi}\sigma(f)} n^{3/2}.$$

Proof : This is a direct computation using the theorem.

Remark : An analogous construction is possible in continuous time. Let (Z, ψ_t, ρ) be another Anosov flow. Define

$$\chi_t : Z \times Y \longrightarrow Z \times Y$$
$$(z, y) \longmapsto (\chi_t z, \phi_{\int_0^t f(\psi_s z)ds} y).$$

One obtain a quasi-hyperbolic flow with decay of correlations in $t^{-1/2}$. To show this one has to use the results of Waddington [14].

0.3. Convergence in law. — Once we know the behaviour of the variances, we can ask the question of the convergence in law. We treat this problem on an example. Many technical difficulties arise when we try to generalise our result.

Our example is the following one. Let us consider

- the group $G = PSL(2, \mathbb{R})$,

- Γ a cocompact lattice in G,

- the one-parameter group $g_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$,

- the two dimensional torus $2, \mathbb{T}^2$,

- a centered function f from $\mathbb{T}^2$ to $\mathbb{R}$,

- the matrix $A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

In 1988 Rudolph [12] has introduced the application

$$T : \mathbb{T}^2 \times G/\Gamma \longrightarrow \mathbb{T}^2 \times G/\Gamma$$
$$(x, y) \longmapsto (Ax, g_{f(x)}y).$$

An example of concrete function for which the system has good properties is the function :

$$f : \quad \mathbb{T}^2 \longrightarrow \mathbb{R}$$
$$(x_1, x_2) \longmapsto sin(2\pi x_1).$$

The transformation T has two non zero Lyapounoff exponents, $\lambda = \frac{3+\sqrt{5}}{2}$ and $\frac{1}{\lambda}$, associated to two directions one expanded the other contracted uniformly by T and a zero exponent in the tangent spaces to the submanifolds $\{x\} \times G/\Gamma$. Let μ and ν be the Haar measure and the measure coming from the Haar measure respectively.

The system $(\mathbb{T}^2 \times G/\Gamma, T, \mu \otimes \nu)$ is a regular version of the T, T^{-1}-transformation. Both are examples of K-systems that do not have the Benoulli property ([7], [12]) .

In $\mathbb{R}^2$ let $\tilde{x}_0$ (resp.$\tilde{x}_{-1}$ denote the intersection point of the contracted right line passing through the point $(1, 0)$ (resp. $(1, -1)$) and the expanded right line passing through the point $(1, 1)$. These are a homoclinic points : in the torus $T^k\tilde{x}_0$ tend and $T^k\tilde{x}_{-1}$ to zero when k tends toward $+\infty$ and toward $-\infty$.

THEOREM 0.2. — *Let f be a hölder continuous function such that*

$$2\sum_{-\infty}^{\infty}(f(T^k\tilde{x}_0) - f(0)) \neq \sum_{-\infty}^{\infty}(f(T^k\tilde{x}_{-1}) - f(0)).$$

Let $\varphi : \mathbb{T}^2 \times G/\Gamma \longrightarrow \mathbb{R}$ be a hölder continuous function such that $\sum^2(\varphi)$ is positive. There exist three brownian motion (non necessarily reduced) W, W_+, W_- such that, if $L_t(x)$ denotes the local time of W in x, one has :

$$\frac{1}{n^{3/4}}\sum_{k=0}^{n-1}\varphi \circ T^k \overset{\mathcal{L}}{\longrightarrow} \int_0^{+\infty}L_1(x)dW_+(x) + \int_0^{+\infty}L_1(-x)dW_-(x)$$

Remark : One easily verify that if f is a coboundary the condition on f isn't satisfied. For regular function and for the automorphism A, to be aperiodic simply means not to be a coboundary. The condition of the theorem is *a priori* stronger than aperiodicity. Is

it really stronger ? Another question : for which systems and which functions the notions of periodicity and coboundaricity do coincide ?

One always has $\Sigma^2(\varphi - \int_{\mathbb{T}^2} \varphi(a,.)d\mu(a)) = 0$. The study of the convergence in law of $\sum \varphi \circ T^k$ is thus reduced to the one of $\sum(\int_{\mathbb{T}^2} \varphi(a,.)d\mu(a)) \circ T^k$. So one can restrict ourselves to the case where φ depends uniquely on the second coordinate. That what we're doing from now on.

We'll use the following result of Kesten and Spitzer.

THEOREM 0.3. — *(Kesten, Spitzer) Let X_i an i.i.d sequence of centered random variables with values in $\mathbb{Z}$. Let (ξ_i) an i.i.d sequence of centered random variables independent of the X_i variables. There exist three brownian motion (non necessarily reduced) W, W_+, W_- such that, if $L_t(x)$ denotes the local time of W in x, one has :*

$$\frac{1}{n^{3/4}} \sum_{k=0}^{n-1} \xi_{X_i+\ldots+X_k} \xrightarrow{\mathcal{L}} \int_0^{+\infty} L_1(x)dW_+(x) + \int_0^{+\infty} L_1(-x)dW_-(x)$$

The theorem is a direct consequence of the two following propositions and of the result of Kesten and Spitzer.

Define

$$N(n,p) = \sharp\{k \in \{0,\ldots,n-1\}|/S_k f \in [p,p+1[\}.$$

PROPOSITION 0.3.1. —

$$\frac{1}{n^{3/4}} \sum_{p \in \mathbb{Z}} [N(n,p) \int_p^{p+1} \varphi(g_t \cdot)dt - \sum_{k:S_k f \in [p,p+1[} \varphi(g_{S_k f} \cdot)] \xrightarrow{\mathbb{P}} 0$$

Let Φ denote the function $\Phi(y) = \int_0^1 \varphi(g_t y)dt$. One has $\int_p^{p+1} \varphi(g_t y)dt = \Phi \circ g_p(y)$.

PROPOSITION 0.3.2. — *There exists two independent sequences of i.i.d centered random variables with values in $\mathbb{Z}$, (X_i), (ξ_i) such that :*

$$\mathbb{E}(exp(it\frac{1}{n^{3/4}} \sum_{p \in \mathbb{Z}} N(n,p)\Phi \circ g_p)) - \mathbb{E}(exp(it\frac{1}{n^{3/4}} \sum_{k=0}^{n-1} \xi_{X_1+\ldots+X_2})) \xrightarrow[n \to +\infty]{} 0.$$

To prove these two propositions we use the three following lemmas.

LEMMA 0.3.3. — *(Moderated deviation) There exist $C > 0, c > 0$ such that, for very n, every $\beta \in]0, \frac{1}{2}[$, one has*

$$\mathbb{P}(S_n f > n^{1-\beta}) \leq Ce^{-cn^{1-2\beta}}$$

(Here the probability $\mathbb{P}$ is the Lebesgue measure on $\mathbb{T}^2$).

LEMMA 0.3.4. — *(Multiple mixing for the geodesic flow)*

Let m and m' be two integers, $(\Phi_i)_{i=i}^{m+m'}$ hölder-continuous functions defined on $G/\Gamma, t_1, \leq \ldots \leq t_m \leq 0 \leq s_1 \ldots \leq s_{m'}, T > 0$ real numbers. There exists $C > 0$, and $\delta \in]0, 1[$, such that

$$Cov\left(\prod_{i=1}^{m}\Phi_i\circ g_{t_i},\prod_{j=1}^{m'}\Phi_j\circ g_{s_i+T}\right)\leq C\left(\prod_{i=1}^{m+m'}\|\Phi_i\|_\infty+\sum_j[\Phi_j]\sum_{i\neq j}\|\phi_j\|_\infty\right)\delta^T.$$

LEMMA 0.3.5. — *Let*

- *I be an interval of length 1,*

- *$\varepsilon > 0$ a real number,*

- *J, K be two subintervals of I of length $\frac{1}{[n^\varepsilon]}$.*

Let $N(n, I)$ denote the quantity :

$$N(n,I)=\sharp\{k\in\{0,-n-1\}/S_kf\in I\}.$$

We define in the same way $N(n, K)$ and $N(n, J)$.

Under the hypotheses of the theorem there exist $\xi > 0, C > 0$ such that :

$$\mathbb{E}(N(n,I)^2)-[n^\varepsilon]\mathbb{E}(N(n,I)N(n,J))|\leq Cn^{1-\xi}$$

$$\mathbb{E}(N(n,J)N(n,K))[n^\varepsilon]^2-\mathbb{E}(N(n,I))|\leq Cn^{1-\xi}$$

$$\mathbb{E}(N(n,I)^3)|\leq Cn^{3/2}$$

For the first and second lemmas see [2] et [9]. The third one is deduced from a result of speed of convergence in the local limit theorem for the sums S_nf.

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