Quantitative ergodicity for some switched dynamical systems

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Abstract

We provide quantitative bounds for the long time behavior of a class of Piecewise Deterministic Markov Processes with state space $\mathbb{R}^d \times E$ where E is a finite set. The continuous component evolves according to a smooth vector field that switches at the jump times of the discrete coordinate. The jump rates may depend on the whole position of the process. Under regularity assumptions on the jump rates and stability conditions for the vector fields we provide explicit exponential upper bounds for the convergence to equilibrium in terms of Wasserstein distances.

Keywords. Coupling; Ergodicity; Linear Differential Equations; Piecewise Deterministic Markov Process; Switched dynamical systems; Wasserstein distance.

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1 Introduction and main results

Piecewise deterministic Markov processes (PDMPs in short) are intensively used in many applied areas (molecular biology [31], storage modelling [7], Internet traffic [19, 22, 23], neuronal activity [29, 8], populations growth models [27]...). Roughly speaking, a Markov process is a PDMP if its randomness is only given by the jump mechanism: in particular, it admits no diffusive dynamics. This huge class of processes has been introduced by Davis (see [14, 15]) in a general framework. Several works [11, 18, 12] deal with their long time behavior (existence of an invariant probability measure, Harris recurrence, exponential ergodicity...). In particular, it is shown in [13] that the behavior of a general PDMP can be related to the one of the discrete time Markov chain made of the positions at the jump times of the process and of an additional independent Poisson process. Nevertheless, this general approach does not seem to provide quantitative bounds for the convergence to equilibrium. Recent papers have tried to establish such estimates for some specific PDMPs (see [10, 20, 4]) or continuous time Markov chains (see [9]).

In the present paper, we investigate the long time behavior of an interesting subclass of PDMPs that plays a role in molecular biology (see [31, 8]). We consider a PDMP on $\mathbb{R}^d \times E$ where E is a finite set. The first coordinate moves continuously on $\mathbb{R}^d$ according to a smooth vector field that depends on the second coordinate whereas the second coordinate jumps with a rate that may depend on the first one. This class of Markov processes is reminiscent of the so-called iterated random functions in the discrete time setting (see [17] for a good review of this topic).

Let E be a finite set, $(a(\cdot, i, j))_{i,j\in E^2}$ be n^2 nonnegative continuous functions on $\mathbb{R}^d$, and, for any $i \in E$, $F^i : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a smooth vector field such that the ordinary

differential equation

$$\begin{cases} x_t' = F^i(x_t), \quad t > 0; \\ x_0 = x, \end{cases}$$

has a unique and global solution $t \mapsto \varphi_t^i(x)$ on $[0, \infty)$ for any initial condition $x \in \mathbb{R}^d$. Let us consider the Markov process

$$(Z_t)_{t \geqslant 0} = ((X_t, I_t))_{t \geqslant 0} \text{ on } \mathbb{R}^d \times E$$

defined by its extended generator L as follows:

$$Lf(x,i) = \Big\langle F^i(x), \nabla_x f(x,i) \Big\rangle + \sum_{j \in E} a(x,i,j)(f(x,j) - f(x,i)) \tag{1}$$

for any smooth function $f : \mathbb{R}^d \times E \to \mathbb{R}$ (see [15] for full details on the domain of L). A simple case occurs when $a(x,i,j)$ can be written as $\lambda(x,i)P(i,j)$ for $(\lambda(\cdot,i))_{i \in E}$ a set of nonnegative continuous functions, and P an irreducible stochastic matrix, in which case:

$$Lf(x,i) = \Big\langle F^i(x), \nabla_x f(x,i) \Big\rangle + \lambda(x,i) \sum_{j \in E} P(i,j)(f(x,j) - f(x,i)) \tag{2}$$

Let us describe the dynamics of this process in this simple case, the general case being similar. Assume that $(X_0, I_0) = (x,i) \in \mathbb{R}^d \times E$. Before the first jump time T_1 of I, the first component X is driven by the vector field F^i and then $X_t = \varphi_t^i(x)$. The time T_1 can be defined by:

$$T_1 = \inf \left\{ t > 0 \, : \, \int_0^t \lambda(X_s, i) \, ds \geqslant E_1 \right\},$$

where E_1 is an exponential random variable with parameter 1. Since the paths of X are deterministic between the jump times of I, the randomness of T_1 comes from the one of E_1 and

$$T_1 = \inf \left\{ t > 0 \, : \, \int_0^t \lambda(\varphi_s^i(x), i) \, ds \geqslant E_1 \right\}.$$

Remark 1.1. *Notice that $\mathbb{P}_{(x,i)}(T_1 = +\infty) > 0$ if and only if*

$$\int_0^{+\infty} \lambda(\varphi_s^i(x), i) \, ds < +\infty.$$

If we assume that $\underline{\lambda} := \inf_{(x,i)} \lambda(x,i) > 0$ then the process I jumps infinitely often.

At time T_1, the second coordinate I performs a jump with the law $P(i, \cdot)$ and the vector field that drives the evolution of X is switched...

Remark 1.2. *In general, I is not a Markov process on its own since its jump rates depend on X. In this paper, we will study both the simple — Markov — case and the general case.*

The main goal of the present work is to provide quantitative bounds for the long time behavior of ergodic processes driven by (2) thanks to the construction of explicit couplings. For the first component X of the process, we will provide quantitative bounds in terms of the Wasserstein coupling distance rather than total variation one. Recall that for every $p \geqslant 1$, the Wasserstein distance W_p between two laws μ and $\tilde{\mu}$ on $\mathbb{R}^d$ with finite p^{th} moment is defined by

$$W_p(\mu, \tilde{\mu}) = \left(\inf_{\Pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - \tilde{x}|^p \, \Pi(dx, d\tilde{x}) \right)^{1/p} \tag{3}$$

where the infimum runs over all the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and $\tilde{\mu}$ (such measures are called couplings of μ and $\tilde{\mu}$). It is well known that for any $p \geqslant 1$, the convergence in W_p Wasserstein distance is equivalent to weak convergence together with convergence of all moments up to order p. However, two probability measures can be both very close in the W_p sense and singular. Choose for example $\mu = \delta_0$ and $\tilde{\mu} = \delta_\varepsilon$. In this case,

$$W_p(\mu, \tilde{\mu}) = \varepsilon \quad \text{and} \quad \|\mu - \tilde{\mu}\|_{\mathrm{TV}} = 1.$$

See e.g. [30, 34] for further details and properties for Wasserstein distances.

Estimates for the Wasserstein distances do not require (nor provide) any information about the support of the invariant measure (which is the set of the recurrent points). This set may be difficult to determine and if the initial distribution of X is not supported by this set, the law of X_t and the invariant measure may be singular. To illustrate this fact, one can consider the following trivial example:

$$E = \{0, 1\}, \quad \lambda(x, i) = 1, \quad F^i(x) = -(x - ia) \quad \text{with } a = (1, 0).$$

The process (X, I) is ergodic and the first marginal μ of its invariant measure is supported by the segment $\{\rho a \ ; \ \rho \in [0, 1]\}$ (it is shown in [7] that μ is a Beta distribution). Despite its extreme simplicity, this process does not in general converge in total variation: if $X_0 = (0, 1)$, the law of X_t is singular with the invariant measure for any $t \geqslant 0$, so $\|\mathcal{L}(X_t) - \mu\|_{\mathrm{TV}}$ is always equal to 1. On the contrary, $W_p(\mathcal{L}(X_t), \mu)$ goes to 0 exponentially fast (see below).

In [2], the authors provide Hörmander-like conditions on the vectors fields $(F^i)_{i \in E}$ that ensure, for constant jump rate $(\lambda_i)_{i \in E}$, the uniqueness and the absolute continuity of the invariant measure provided that it exists. The main drawback of this result is that these regularity assumptions have to be checked at a point that can be reached starting from any other point. It can be hard to determine the set of such points and it can be empty. Several examples are studied in [6] that underline this fact. In particular, even if the process evolves in a compact set, the process (X, I) may admit one or several recurrent classes (and invariant measures) depending only on the values of jump rates $(\lambda_i)_{i \in E}$. Moreover, even when convergence results can be obtained (as in [6]), the rates of convergence are given by compactness arguments and are not explicit.

We are able to get explicit rates of convergence in two situations. Firstly, if the jump rates of I does not depend on X, then the vector fields $(F^i)_{i \in E}$ will be assumed to satisfy an averaged exponential stability. Secondly, if the jump rates of I are assumed to be Lipschitz functions of X, then the vector fields $(F^i)_{i \in E}$ will be assumed to satisfy a uniform exponential stability.

In the sequel, μ_t stands for the first marginal law of $Z_t = (X_t, I_t)$.

1.1 Constant jump rates

If the jump rates of I do not depend on X, then $(I_t)_{t \geqslant 0}$ is a Markov process on the finite space E and $(X_s)_{0 \leqslant s \leqslant t}$ is a deterministic function of $(I_s)_{0 \leqslant s \leqslant t}$. Many results are available both in the discrete time setting (see [26, 33, 21, 25, 1]) and in the continuous time setting (see [24, 16, 3]). Morevover [5] provides a simple example of surprising phase transition for a switching of two exponentially stable flows that can be explosive (when the jump rates are sufficiently large).

Assumption 1.3. *Assume that the jump rates $(a(\cdot,i,j))_{i\in E}$ do not depend on x and that I is an irreducible Markov process on E. Let us denote by ν its invariant probability measure.*

Remark 1.4. *If $a(\cdot,i,j)$ does not depend on x, we can always write $a(i,j)=\lambda(i)P(i,j)$ with $P(\cdot,\cdot)$ a Markov transition matrix.*

Assumption 1.5. *Assume that for any $i\in E$, there exists $\alpha(i)\in\mathbb{R}$ such that,*

$$\left\langle x-\tilde{x}, F^i(x)-F^i(\tilde{x})\right\rangle \leqslant -\alpha(i)|x-\tilde{x}|^2, \quad x,\tilde{x}\in\mathbb{R}^d,$$

and that

$$\sum_{i\in E}\alpha(i)\nu(i)>0$$

where ν is defined in Assumption 1.3.

Firstly, one can establish that the process X is bounded in some L^p space.

Lemma 1.6. *Under Assumptions 1.3 and 1.5, there exists $\kappa>0$ such that, for any $q<\kappa$, the function $t\mapsto\mathbb{E}(|X_t|^q)$ is bounded as soon as $\mathbb{E}(|X_0|^q)$ is finite. More precisely, there exists $M(q,m)$ such that*

$$\sup_{t\geqslant 0}\mathbb{E}(|X_t|^q)\leqslant M(q,m),$$

as soon as $\mathbb{E}(|X_0|^q)\leqslant m$.

Let us now turn to the long time behavior estimate.

Theorem 1.7. *Assume that Assumptions 1.3 and 1.5 hold. Let $p<q<\kappa$ and denote by s the conjugate of q: $q^{-1}+s^{-1}=1$. Assume that μ_0 and $\tilde{\mu}_0$ admit a finite q^{th} moment smaller than m. Then,*

$$W_p(\mu_t,\tilde{\mu}_t)\leqslant 2^{p+1}M(q,m)^{p/q}C_2(p)\exp\left(-\frac{\eta_p}{1+s\eta_p/\rho}t\right),$$

where ρ and η_p are positive constants depending only on the Markov chain I.

The constants ρ, η_p and $C_2(p)$ are given below in equations (6), (7) and (8).

Corollary 1.8. *Under Assumptions 1.3 and 1.5, the process Z admits a unique invariant measure μ and*

$$W_p(\mu_t,\mu)\leqslant 2^{p+1}M(q,m)^{p/q}C_2(p)\exp\left(-\frac{\eta_p}{1+s\eta_p/\rho}t\right).$$

1.2 Non constant jump rates

Let us now turn to the case when the jump rates of I depend on X. We will assume that the $a(x,i,j)$ are smooth in the x variable and that each vector field F^i has a unique stable point.

Assumption 1.9. *There exist $\underline{a}>0$ and $\kappa>0$ such that, for any $x,\tilde{x}\in\mathbb{R}^d$ and $i,j\in E$,*

$$a(x,i,j)\geqslant\underline{a}\quad\text{and}\quad\sum_{j\in E}|a(x,i,j)-a(\tilde{x},i,j)|\leqslant\kappa|x-\tilde{x}|,$$

The lower bound condition insures that the second — discrete — coordinate of Z changes often enough (so that the second coordinates of two independent copies of Z coincide sufficiently often). This is a rather strong condition that can be relaxed in certain situations. If for example if $a(x,i,j) = \lambda(x,i)P_{ij}$ as in (2), Lipschitz continuous λ bounded from below and an aperiodic P are sufficient to get a similar rate of convergence.

Assumption 1.10. *Assume that there exists $\alpha > 0$ such that,*

$$\left\langle x - \tilde{x}, F^i(x) - F^i(\tilde{x}) \right\rangle \leqslant -\alpha |x - \tilde{x}|^2, \quad x, \tilde{x} \in \mathbb{R}^d, \ i \in E. \tag{4}$$

Assumption 1.10 ensures that, for any $i \in E$,

$$\left| \varphi_t^i(x) - \varphi_t^i(\tilde{x}) \right| \leqslant e^{-\alpha t} |x - \tilde{x}|, \quad x, \tilde{x} \in \mathbb{R}^d.$$

As a consequence, the vector fields F^i has exactly one critical point $\sigma(i) \in \mathbb{R}^d$. Moreover it is exponentially stable since, for any $x \in \mathbb{R}^d$,

$$\left| \varphi_t^i(x) - \sigma(i) \right| \leqslant e^{-\alpha t} |x - \sigma(i)|.$$

In particular, X cannot escape from a sufficiently large ball (this implies that the (continuous) functions $a(\cdot, i, j)$ are also bounded from above along the trajectories of X). More precisely, the following estimate holds.

Lemma 1.11. *Under Assumptions 1.9 and 1.10, the process Z cannot escape from the compact set $\bar{B}(0,r) \times E$ where $\bar{B}(0,r)$ is the (closed) ball centered in $0 \in \mathbb{R}^d$ with radius r given by*

$$r = \frac{\max_{i \in E} |F^i(0)|}{\alpha}. \tag{5}$$

Moreover, if $|X_0| > r$ then

$$\left(|X_t|^2 - r^2 \right)^+ \leqslant e^{-\alpha t} \left(|X_0|^2 - r^2 \right)^+.$$

In particular the support of any invariant measure is included in $\bar{B}(0,r)$.

Let us now state our main result which establishes the quantitative exponential ergodicity of the process Z under Assumptions 1.9 and 1.10.

Theorem 1.12. *Assume that Assumptions 1.9 and 1.10 hold and that the supports of μ_0 and $\tilde{\mu}_0$ are included in the ball $\bar{B}(0,r)$ where r is given by (5). Then there exists positive constants c and γ such that*

$$W_1(\mu_t, \tilde{\mu}_t) \leqslant 2r(1 + ct) \exp\left(-\frac{\alpha}{1 + \alpha/\gamma} t \right)$$

where α is given by (4).

The constants obtained in the proof are the following

$$\gamma = \frac{(\alpha + b) - \sqrt{(\alpha + b)^2 - 4bp\alpha}}{2} \quad \text{and} \quad c = \frac{\alpha}{\alpha + \gamma} \frac{ep\alpha b}{\sqrt{(\alpha + b)^2 - 4bp\alpha}},$$

with $p = e^{-2r\kappa/\alpha}$ and $e = \exp(1)$ and where b depends on the coalescence time of two independent processes defined on E as the second coordinates of independent copies of Z.

To obtain this result we couple two copies of our process and compare their distance to a real-valued process that can pass instantly (less and less often) from small to large values ($2r$). This may seem rough, but in the general case, nothing much better can be done : if one of the flow is very strongly attractive, two trajectories may indeed very rapidly diverge. For particular examples, or under additional assumptions on the flows, it must be possible to get better rates.

Corollary 1.13. *Under Assumptions 1.9 and 1.10, the process Z admits a unique invariant measure μ and*

$$W_1(\mu_t, \mu) \leqslant 2r(1 + ct) \exp\left(-\frac{\alpha}{1 + \alpha/\gamma} t\right).$$

Section 2 is dedicated to the proof of Theorem 1.7. Theorem 1.12 is established in Section 3.

2 Constant jump rates

The aim of this section is to prove Theorem 1.7. Assumption 1.3 ensures that $(I_t)_{t\geqslant 0}$ is an irreducible Markov process on the finite space E. Its generator A is the matrix defined by $A(i,i) = -\lambda(i)$ and $A(i,j) = \lambda(i)P(i,j)$ for $i \neq j$. Let us denote by ν its unique invariant probability measure. The study of the long time behavior of I is classical: since I takes its values in a finite set, it is quite simple to construct a coalescent coupling of two processes starting from different points.

Lemma 2.1 ([32])**.** *If Assumption 1.3 holds then there exists $\rho > 0$ such that for any $i, j \in E$,*

$$\mathbb{P}(T > t | I_0 = i,\ \tilde{I}_0 = j) \leqslant e^{-\rho t} \tag{6}$$

where $(I_t)_{t\geqslant 0}$ and $(\tilde{I}_t)_{t\geqslant 0}$ are two independent Markov processes with infinitesimal generator A starting respectively at i and j and $T = \inf\left\{t \geqslant 0 \,:\, I_t = \tilde{I}_t\right\}$ is the first intersection time.

Remark 2.2. *If $E = \{1, 2\}$, then the first intersection time is distributed as an exponential random variable with parameter $\lambda(1) + \lambda(2)$ and Equation (6) holds with $\rho = \lambda(1) + \lambda(2)$.*

The proof of Theorem 1.7 is made of two steps. Firstly we couple two processes starting respectively from (x, i) and $(\tilde{x}, i)$ to get a simple estimate as time goes to infinity. Then we use this estimate and Lemma 2.1 to manage the general case.

2.1 Moments estimates

In this section we prove Lemma 1.6 and get an L^p estimate for $|X_t|$. For any $p \geqslant 2$ and $\varepsilon > 0$,

$$\begin{aligned}
\frac{d}{dt}|X_t|^p &= p|X_t|^{p-2}\Big\langle X_t, F^{I_t}(X_t)\Big\rangle \\
&= p|X_t|^{p-2}\Big\langle X_t, F^{I_t}(X_t) - F^{I_t}(0\Big\rangle + p|X_t|^{p-2}\Big\langle X_t, F^{I_t}(0)\Big\rangle \\
&\leqslant -(p\alpha(I_t) - \varepsilon)|X_t|^p + C(p, \varepsilon).
\end{aligned}$$

Thanks to Gronwall's Lemma, we get that

$$\mathbb{E}(|X_t|^p) \leqslant C(p,\varepsilon) \int_0^t \mathbb{E}\left(e^{-\int_s^t (p\alpha(I_u) - \varepsilon)\, du}\right) ds + \mathbb{E}(|X_0|^p)\mathbb{E}\left(e^{-\int_0^t (p\alpha(I_u) - \varepsilon)\, du}\right).$$

Remark 2.3. *A similar estimate can be obtained for $p \geqslant 1$ using a regularization of the application $x \mapsto |x|^p$.*

The right hand side depends only on $\mathbb{E}(|X_0|^p)$ and $(I_s)_{0\leqslant s\leqslant t}$. Thus, it is sufficient to investigate the behavior of $e(p,t)$ defined for any $t \geqslant 0$ by

$$e(p,t) = \max_{i\in E} \mathbb{E}_i\Big(\exp\Big(-\int_0^t p\alpha(I_u)\,du\Big)\Big).$$

This study has been already performed in [3]. Let us state the precise result. We denote by A_p the matrix $A - pB$ where A is the infinitesimal generator of I and B is the diagonal matrix with diagonal $(\alpha(1), \ldots, \alpha(n))$ and associate to A_p the quantity

$$\eta_p := -\max_{\gamma\in \mathrm{Spec}(A_p)} \mathrm{Re}\ \gamma. \tag{7}$$

The long time behavior of $e(p,t)$ is characterised by η_p as follows. For any $p > 0$, there exist $0 < C_1(p) < 1 < C_2(p) < +\infty$ such that, for any any $t > 0$,

$$C_1(p)e^{-\eta_p t} \leqslant e(p,t) \leqslant C_2(p)e^{-\eta_p t}. \tag{8}$$

Moreover the following dichotomy holds:

- if $\underline{\alpha} \geqslant 0$, then $\eta_p > 0$ for all $p > 0$,
- if $\underline{\alpha} < 0$, there is $\kappa \in (0, \min\{A_{ii}/\alpha(i) \,:\, \alpha(i) < 0\})$ such that $\eta_p > 0$ for $p < \kappa$ and $\eta_p < 0$ for $p > \kappa$.

See [3] for further details.

Corollary 2.4. *If $p < \kappa$ then $t \mapsto \mathbb{E}(|X_t|^p)$ is bounded as soon as $\mathbb{E}(|X_0|^p)$ is finite.*

2.2 Convergence rate

Let us now get the upper bound for the Wasserstein distance W_p for some $p < \kappa$. Assume firstly that the initial law are two Dirac masses at (x,i) and $(\tilde{x},i)$. It is easy to construct a good coupling of the two processes (X,I) and $(\tilde{X},\tilde{I})$: since the jump rates of I do not depend on X, one can choose I and $\tilde{I}$ equal! As a consequence, for any $p \geqslant 2$,

$$\begin{aligned}\frac{d}{dt}|X_t - \tilde{X}_t|^p &= p|X_t - \tilde{X}_t|^{p-2}\Big\langle X_t - \tilde{X}_t, F^{I_t}(X_t) - F^{I_t}(\tilde{X}_t)\Big\rangle \\ &\leqslant -p\alpha(I_t)|X_t - \tilde{X}_t|^p.\end{aligned}$$

As a consequence,

$$\begin{aligned}\mathbb{E}\Big(|X_t - \tilde{X}_t|^p\Big) &\leqslant \mathbb{E}_i\Big(\exp\Big(-p\int_0^t \alpha(I_s)\,ds\Big)\Big)|x - \tilde{x}|^p \\ &\leqslant e^{-\eta_p t}|x - \tilde{x}|^p.\end{aligned}$$

Let us now turn to a general initial condition. Choose (x,i) and $(\tilde{x},j)$ in $\mathbb{R}^d \times E$ and consider the following coupling: the two processes evolve independently until the intersection time T of the second coordinates. Then, I and $\tilde{I}$ are chosen to be equal for ever. Now fix $t > 0$ and $\beta \in (0,1)$ and decompose:

$$\mathbb{E}\Big(|X_t - \tilde{X}_t|^p\Big) = \mathbb{E}\Big(|X_t - \tilde{X}_t|^p \mathbb{1}_{\{T>\beta t\}}\Big) + \mathbb{E}\Big(|X_t - \tilde{X}_t|^p \mathbb{1}_{\{T\leqslant\beta t\}}\Big)$$

Choose $q \in (p, \kappa)$ and define $r = q/p$ and s as the conjugate of r. The Hölder inequality ensures that

$$\begin{aligned}\mathbb{E}\Big(|X_t - \tilde{X}_t|^p \mathbb{1}_{\{T > \beta t\}}\Big) &\leqslant \mathbb{E}\Big(|X_t - \tilde{X}_t|^q\Big)^{p/q} \mathbb{P}(T \geqslant \beta t)^{1/s} \\ &\leqslant 2^p M(q,m)^{p/q} e^{-(\beta\rho/s)t}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\Big(|X_t - \tilde{X}_t|^p \mathbb{1}_{\{T \leqslant \beta t\}}\Big) &= \mathbb{E}\Big(|X_T - \tilde{X}_T|^p \mathbb{E}_{I_T}\Big(\exp\Big(-p\int_T^t \alpha(I_s)\,ds\Big)\Big)\mathbb{1}_{\{T \leqslant \beta t\}}\Big) \\ &\leqslant 2^p M(p,m) C_2(p) e^{-\eta_p(1-\beta)t}.\end{aligned}$$

At last, one has to optimize over $\beta \in (0,1)$. With

$$\beta = \frac{\eta_p}{\eta_p + \rho/s},$$

one has

$$\mathbb{E}\Big(|X_t - \tilde{X}_t|^p\Big) \leqslant 2^{p+1} M(q,m)^{p/q} C_2(p) \exp\left(-\frac{\rho/s}{\eta_p + \rho/s}\eta_p t\right).$$

This concludes the proof of Theorem 1.7.

3 Non constant jump rates

Let us now turn to the proof of Theorem 1.12. In this section we do not assume that the jump rates depend only on the discrete component. Thus, the coupling is more subtle since once I and $\tilde{I}$ are equal, they can go apart with positive probability. The main idea is the following. If I and $\tilde{I}$ are equal, the distance between X and $\tilde{X}$ decreases exponentially fast and then it should be more and more easier to make the processes I and $\tilde{I}$ jump simultaneously (since the jump rates are Lipschitz functions of X). This idea has been used in a different framework in [10, 4].

This section is organized as follows. Firstly we prove Lemma 1.11 that ensures that the process X cannot escape from a sufficiently large ball. In particular, the support of the invariant law of X is included in this ball. Then we construct the coupling of two processes (X, I) and $(\tilde{X}, \tilde{I})$ driven by the same infinitesimal generator (2) with different initial condition. At last we compare the distance between X and $\tilde{X}$ to an companion process that goes to 0 exponentially fast.

Proof of Lemma 1.11. Setting $\tilde{x} = 0$ in (4) ensures that, for $\varepsilon \in (0, \alpha)$,

$$\Big\langle F^i(x), x\Big\rangle \leqslant -\alpha|x|^2 + \Big\langle F^i(0), x\Big\rangle \leqslant -(\alpha - \varepsilon)|x|^2 + M/(4\varepsilon),$$

if $M = \max_{i \in E} \big|F^i(0)\big|^2$. In other words,

$$|X_t|^2 - |X_s|^2 = \int_s^t 2\Big\langle F^{I_u}(X_u), X_u\Big\rangle\, du \leqslant -2(\alpha - \varepsilon)\int_s^t |X_u|^2\, du + \frac{M}{2\varepsilon}(t - s).$$

As a consequence,

$$|X_t|^2 \leqslant \frac{M}{4\varepsilon(\alpha - \varepsilon)}(1 - e^{-2(\alpha-\varepsilon)t}) + |X_0|^2 e^{-2(\alpha-\varepsilon)t}.$$

Choosing $\varepsilon = \alpha/2$ ensures that

$$\left(|X_t|^2 - \frac{M}{\alpha^2}\right)^+ \leqslant e^{-\alpha t}\left(|X_0|^2 - \frac{M}{\alpha^2}\right)^+.$$

In particular, X cannot escape from the centered closed ball with radius $r = \sqrt{M}/\alpha$. $\square$

3.1 The coupling

Let us construct a Markov process on $(\mathbb{R}^d \times E)^2$ with marginals driven by (2) starting respectively from (x, i) and $(\tilde{x}, j)$. This is done *via* its infinitesimal generator $\mathfrak{L}$ which is defined as follows:

- if $i \neq j$

$$\begin{aligned}\mathfrak{L}f(x,i,\tilde{x},j) =& \Big\langle F^i(x), \nabla_x f(x,i,\tilde{x},j)\Big\rangle + \Big\langle F^j(\tilde{x}), \nabla_{\tilde{x}} f(x,i,\tilde{x},j)\Big\rangle \\ &+ \sum_{i' \in E} a(x,i,i')(f(x,i',\tilde{x},j) - f(x,i,\tilde{x},j)) \\ &+ \sum_{j' \in E} a(\tilde{x},j,j')(f(x,y,\tilde{x},j') - f(x,y,\tilde{x},j)).\end{aligned}$$

- if $i = j$:

$$\begin{aligned}\mathfrak{L}f(x,i,\tilde{x},j) =& \Big\langle F^i(x), \nabla_x f(x,i,\tilde{x},i)\Big\rangle + \Big\langle F^i(\tilde{x}), \nabla_{\tilde{x}} f(x,i,\tilde{x},i)\Big\rangle \\ &+ \sum_{i' \in E} (a(x,i,i') \wedge a(\tilde{x},i,i'))(f(x,i',\tilde{x},i') - f(x,i,\tilde{x},i)) \\ &+ \sum_{i' \in E} (a(x,i,i') - a(\tilde{x},i,i'))^+(f(x,i',\tilde{x},i) - f(x,i,\tilde{x},i)) \\ &+ \sum_{i' \in E} (a(x,i,i') - a(\tilde{x},i,i'))^-(f(x,i,\tilde{x},i') - f(x,i,\tilde{x},i)).\end{aligned}$$

Notice that if f depends only on (x, i) or on $(\tilde{x}, j)$, then $\mathfrak{L}f = Af$. Let us explain how this coupling works. When I and $\tilde{I}$ are different, the two processes (X, I) and $(\tilde{X}, \tilde{I})$ evolve independently. If $I = \tilde{I}$ then two jump processes are in competition: a single jump *vs* two simultaneous jumps. The rate of arrival of a single jump is given by $\sum_{i' \in E} |a(x,i,i') - a(\tilde{x},i,i')|$. It is bounded above by $\kappa|x - \tilde{x}|$. The rate of arrival of a simultaneous jump is given by $\sum_{i' \in E}(a(x,i,i') \wedge a(\tilde{x},i,i'))$.

Assume firstly that X_0 and $\tilde{X}_0$ belong to the ball $\bar{B}(0, r)$ where r is given by (5). Let us define D_t as the distance between X_t and $\tilde{X}_t$ for any $t \geqslant 0$. The process $(D_t)_{t \geqslant 0}$ is not Markovian. Nevertheless, as long as $I = \tilde{I}$, D_t decreases with a rate which is greater than α. If it is no longer the case, then D_t can increase. Nevertheless it is still smaller than $d = 2r$. After the coalescent time T_c of two independent independent copies of I, D decreases once again. There exists $b > 0$ such that T_c is (stochastically) smaller than $\mathcal{E}(b)$ (for example, if $E = \{0, 1\}$, then T_c is equal to the minimum of the jump times of the two independent processes which are both stochastically smaller than a random variable of law $\mathcal{E}(\underline{a})$ and T_c is stochastically smaller than $\mathcal{E}(2\underline{a})$). Then $\mathbb{E}(D_t) \leqslant \mathbb{E}(U_t)$ where the Markov process $(U_t)_{t \geqslant 0}$ on $[0, d] \cup \{d + \varepsilon\}$ is driven by the infinitesimal generator

$$Gf(x) = \begin{cases} -\alpha x f'(x) + \kappa x(f(d+\varepsilon) - f(x)) & \text{if } x \in [0, d], \\ b(f(d) - f(d+\varepsilon)) & \text{if } x = d + \varepsilon. \end{cases}$$

3.2 The companion process

Theorem 3.1. *For any $t \geqslant 0$,*

$$\mathbb{E}(U_t|U_0 = d) \leqslant \left(d + (d+\varepsilon)\left(\frac{p\alpha be}{\sqrt{(\alpha+b)^2 - 4p\alpha b}}\right)\frac{\alpha t}{\alpha+\gamma}\right)\exp\left(-\frac{1}{1+\alpha/\gamma}\alpha t\right) \tag{9}$$

where

$$p = e^{-d\kappa/\alpha} \quad \textit{and} \quad \gamma = \frac{(\alpha+b) - \sqrt{(\alpha+b)^2 - 4p\alpha b}}{2} = \frac{(\alpha+b) - \sqrt{(\alpha-b)^2 + 4(1-p)\alpha b}}{2}.$$

Remark 3.2. *If α goes to ∞, then γ goes to d whereas $\gamma \sim p\alpha/b$ if b goes to ∞.*

Proof. Starting from $d+\varepsilon$, the process U jumps after a random time with law $\mathcal{E}(b)$ to d and then goes to zero exponentially fast until it (possibly) goes back to $d+\varepsilon$. The first jump time T starting from d can be constructed as follows: let E be a random variable with law $\mathcal{E}(1)$. Then

$$T \overset{\mathcal{L}}{=} \begin{cases} -\frac{1}{\alpha}\log\left(1 - \frac{\alpha E}{d\kappa}\right) & \text{if } E < \frac{d\kappa}{\alpha}, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, conditionally on $\{U_0 = d\}$,

$$\int_0^t \lambda(V_s)ds = \int_0^t d\kappa e^{-\alpha s}\, ds = \frac{d\kappa}{\alpha}(1 - e^{-\alpha t}).$$

In other words, the cumulative distribution function F_T of T is such that, for any $t \geqslant 0$,

$$1 - F_T(t) = \mathbb{P}(T > t) = \exp\left(-\frac{d\kappa}{\alpha}(1 - e^{-\alpha t})\right).$$

Let us define $p = e^{-d\kappa/\alpha}$. The law of T is the mixture with respective weights p and $1-p$ of a Dirac mass at $+\infty$ and a probability measure on $\mathbb{R}$ with density

$$f \,:\, t \mapsto f(t) = \frac{d\kappa}{1-p}e^{-\alpha t}e^{-\frac{d\kappa}{\alpha}(1-e^{-\alpha t})}\mathbb{1}_{(0,+\infty)}(t) \tag{10}$$

and cumulative distribution function

$$F \,:\, t \mapsto F(t) = \left(\frac{1 - e^{-\frac{d\kappa}{\alpha}(1-e^{-\alpha t})}}{1 - e^{-\frac{d\kappa}{\alpha}}}\right)\mathbb{1}_{(0,+\infty)}(t).$$

Starting at d, U will return to d with probability $1-p$. The Markov property ensures that the number N of returns of U to 1 is a random variable with geometric law with parameter p. The length of a finite loop from d to d can be written as the sum $S+E$ where the law of S has the density function f given in (10), the law of E is the exponential measure with parameter b and S and E are independent.

Lemma 3.3. *The variable S is stochastically smaller than an exponential random variable with parameter α* i.e. *for any $t \geqslant 0$, $F(t) \geqslant F_\alpha(t)$ where $F_\alpha(t) = (1 - e^{-\alpha t})\mathbb{1}_{\{t>0\}}$.*

Proof. For any $t \geqslant 0$,

$$1 - F(t) = \frac{e^{\frac{d\kappa}{\alpha}e^{-\alpha t}} - 1}{e^{\frac{d\kappa}{\alpha}} - 1} \leqslant e^{-\alpha t} = 1 - F_\alpha(t).$$

This ensures the stochastic bound. □

As a consequence, the Laplace transform L_S of S with density f is smaller than the one of an exponential variable with parameter α: for any $s < \alpha$,

$$L_S(s) \leqslant \frac{\alpha}{\alpha - s}.$$

If L_e is the Laplace transform of $S + E$, then, for any $s < \alpha \wedge b$, we have

$$L_e(s) \leqslant \frac{\alpha}{\alpha - s}\frac{b}{b - s}.$$

Let us denote by H the last hitting time of 1 *i.e.* the last jump time of X and by L its Laplace transform. Let us introduce $N \sim \mathcal{G}(p)$, $(S_i)_{i\geqslant 1}$ with density f and $(E_i)_{i\geqslant 1}$ with law $\mathcal{E}(b)$. All the random variables are assumed to be independent. Then

$$H \overset{\mathcal{L}}{=} \sum_{i=1}^{N}(S_i + E_i).$$

Classically, for any $s \in \mathbb{R}$ such that $(1-p)L_e(s) < 1$, one has

$$L(s) = \mathbb{E}\Big(e^{sH}\Big) = \frac{pL_e(s)}{1-(1-p)L_e(s)} = \frac{p}{1-p}\bigg(\frac{1}{1-(1-p)L_e(s)} - 1\bigg).$$

Let us denote by

$$\gamma = \frac{(\alpha+b) - \sqrt{(\alpha+b)^2 - 4p\alpha b}}{2} \quad \text{and} \quad \tilde{\gamma} = \frac{(\alpha+b) + \sqrt{(\alpha+b)^2 - 4p\alpha b}}{2}$$

the two roots of $\xi^2 - (\alpha+b)\xi + p\alpha b = 0$. Notice that $\gamma < \alpha \wedge b < \tilde{\gamma}$. For any $s < \gamma$, one has $(1-p)L_e(s) < 1$ and

$$L(s) \leqslant \frac{p\alpha b}{(\gamma - s)(\tilde{\gamma} - s)} \leqslant \frac{p\alpha b}{\tilde{\gamma} - s}\frac{1}{\gamma - s}. \tag{11}$$

Let us now turn to the control of $\mathbb{E}(U_t|U_0 = d)$. The idea is to discuss wether $H > \beta t$ or not for some $\beta \in (0,1)$ (and then to choose β as good as possible):

- if $H < \beta t$, then $U_t \leqslant e^{-(1-\beta)\alpha t}$,
- the event $\{H \geqslant \beta t\}$ has a small probability for large t since H has a finite Laplace transform on a neighbourhood of the origin.

For any $\beta \in (0,1)$ and $s > 0$,

$$\begin{aligned}\mathbb{E}(U_t|U_0 = d) &= \mathbb{E}\Big(U_t \mathbb{1}_{\{T\leqslant \beta t\}}\Big) + \mathbb{E}\Big(U_t \mathbb{1}_{\{T > \beta t\}}\Big) \\ &\leqslant de^{-(1-\beta)\alpha t} + (d+\varepsilon)L(s)e^{-s\beta t}. \end{aligned} \tag{12}$$

From Equation (11), we get that, for any $s < \gamma$, $\log L(s) - \beta t s \leqslant h(s)$ where

$$h(s) = \log\left(\frac{p\alpha b}{\tilde{\gamma} - \gamma}\right) - \log(\gamma - s) - \beta t s.$$

The function h reaches its minimun at $s(t) = \gamma - (\beta t)^{-1}$ and

$$h(s(t)) = \log\left(\frac{p\alpha b}{\tilde{\gamma} - \gamma}\right) + \log(\beta t) + 1 - \gamma\beta t.$$

For $t > 0$ and $\beta \in (0,1)$, choose $s(t) = \gamma - (\beta t)^{-1}$ in (12) to get

$$\begin{aligned}\mathbb{E}(U_t) &\leqslant de^{-(1-\beta)\alpha t} + (d+\varepsilon)e^{h(\gamma(t))} \\ &\leqslant de^{-(1-\beta)\alpha t} + (d+\varepsilon)\left(\frac{p\alpha be}{\tilde{\gamma}-\gamma}\right)\beta t e^{-\gamma\beta t}.\end{aligned}$$

At last, one can choose $\beta = \alpha(\alpha+\gamma)^{-1}$ in order to have $(1-\beta)\alpha = \gamma\beta$. This ensures that

$$\mathbb{E}(U_t) \leqslant \left(d + (d+\varepsilon)\left(\frac{p\alpha be}{\tilde{\gamma}-\gamma}\right)\frac{\alpha t}{\alpha+\gamma}\right)\exp\left(-\frac{\alpha\gamma}{\alpha+\gamma}t\right).$$

Replacing $\tilde{\gamma} - \gamma$ by its expression as a function of α, b and p provides (9). □

4 Example

The Morris–Lecar model introduced in [28] studies the evolution in time of the electric potential $V(t)$ in a neuron. The neuron exchanges different ions with its environment via ion channels which may be open or closed. In the original — deterministic — model, the proportion of open channels of different types are coded by two functions $m(t)$ and $n(t)$, and the three quantities m, n and V evolve through the flow of an ordinary differential equation.

Various stochastic versions of this model exist. Here we focus on a model described in [35], to which we refer for additional information. This model is motivated by the fact that m and n, being proportions of open channels, are better coded as discrete variables. More precisely, we fix a large integer K (the total number of channels) and define a PDMP (V, u_1, u_2) with values in $\mathbb{R} \times \{0, 1/K, 2/K \dots 1\}^2$ as follows.

Firstly, the potential V evolves according to

$$\frac{dV(t)}{dt} = \frac{1}{C}\left(I - \sum_{i=1}^{3} g_i u_i(t)(V - V_i)\right) \tag{13}$$

where C and I are positive constants (the capacitance and input current), the g_i and V_i are positive constants (representing conductances and equilibrium potentials for different types of ions), $u_3(t)$ is equal to 1 and $u_1(t)$, $u_2(t)$ are the (discrete) proportions of open channels for two types of ions.

These two discrete variables follow birth-death processes on $\{0, 1/K, \dots 1\}$ with birth rates α_1, α_2 and death rates β_1, β_2 that depend on the potential V :

$$\begin{aligned}\alpha_i(V) &= c_i \cosh\left(\frac{V - V_i'}{2V_i''}\right)\left(1 + \tanh\left(\frac{V - V_i'}{V_i''}\right)\right) \\ \beta_i(V) &= c_i \cosh\left(\frac{V - V_i'}{2V_i''}\right)\left(1 - \tanh\left(\frac{V - V_i'}{V_i''}\right)\right)\end{aligned} \tag{14}$$

where the c_i and V_i', V_i'' are constants.

Let us check that our main result can be applied in this example. Formally the process is a PDMP with $d = 1$ and the finite set $E = \{0, 1/K, \dots 1\}^2$. The discrete process (u_1, u_2) plays the role of the index $i \in E$, and the fields $F^{(u_1,u_2)}$ are defined (on $\mathbb{R}$) by (13) by setting $u_1(t) = u_1$, $u_2(t) = u_2$.

The constant term $u_3 g_3$ in (13) ensures that the uniform dissipation property (4) is satisfied: for all (u_1, u_2),

$$\left\langle V - \tilde{V}, F^{(u_1,u_2)}(V) - F^{(u_1,u_2)}(\tilde{V}) \right\rangle = -\frac{1}{C} \sum_{i=1}^{3} u_i g_i (V - \tilde{V})^2$$
$$\leqslant -\frac{1}{C} u_3 g_3 (V - \tilde{V})^2.$$

The Lipschitz character and the bound from below on the rates are not immediate. Indeed the jump rates (14) are not bounded from below if V is allowed to take values in $\mathbb{R}$.

However, a direct analysis of (13) shows that V is essentially bounded : all the fields $F^{(u_1,u_2)}$ point inward at the boundary of the (fixed) line segment $\mathcal{S} = [0, \max(V_1, V_2, V_3 + (I+1)/g_3 u_3)]$, so if $V(t)$ starts in this region it cannot get out. The necessary bounds all follow by compactness, since $\alpha_i(V)$ and $\beta_i(V)$ are $\mathcal{C}^1$ in $\mathcal{S}$ and strictly positive.

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