Holomorphic families of complex projective structures on compact Riemann surfaces

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In this expository text we give necessary definitions and the proof to the following result:

Theorem 0.1 Let $\pi : \mathcal{X} \to C$ be a holomorphic family of compact Riemann surfaces parametrized by a compact Riemann surface $C$. Suppose that there is a holomorphically family of complex projective structures on the fibers of $\pi$. Then the fibers are all isomorphic Riemann surfaces and the complex projective structures are the same.

This theorem follows immediately from the computation of a cocycle. Based on some basic theory of theta functions on Riemann surfaces in [Fay73], this computation is easy and appeared already in [Tuy78] and [BKN17]. The key to Theorem 0.1 is the construction of the Bergman projective connection from theta functions which can be traced back to [HS66] or even to Klein [Kle90].
1 Projective structures

Definition 1.1 A complex projective structure, or a \((\text{PGL}_2(\mathbb{C}), \mathbb{P}^1(\mathbb{C}))\)-structure on a orientable real surface \(\Sigma\) is a maximal atlas \(\{(U_i, \varphi_i)\}\) of charts such that

- \(\varphi_i : U_i \to \mathbb{P}^1\) is a homeomorphism onto its image;
- the transition functions \(\psi_{ij} = \varphi_i \circ \varphi_j^{-1}\) are restrictions of elements of \(\text{PGL}_2(\mathbb{C})\).

A complex affine structure, or a \((\text{Aff}_1(\mathbb{C}), \mathbb{C})\)-structure on \(\Sigma\) is a maximal atlas \(\{(U_i, \varphi_i)\}\) of charts such that

- \(\varphi_i : U_i \to \mathbb{C}\) is a homeomorphism onto its image;
- the transition functions \(\psi_{ij} = \varphi_i \circ \varphi_j^{-1}\) are restrictions of elements of \(\text{Aff}_1(\mathbb{C})\).

A complex affine structure induces a complex projective structure. A complex projective structure induces a complex structure on \(\Sigma\), we denote by \(X\) the corresponding Riemann surface. The sphere has a unique complex structure \(\mathbb{P}^1(\mathbb{C})\), and \(\mathbb{P}^1(\mathbb{C})\) has an obvious complex projective structure (we will soon see that it is unique). An elliptic curve is the quotient of \(\mathbb{C}\) by some lattice \(\mathbb{Z}1 + \mathbb{Z}\tau\), thus has naturally a complex affine structure. The following examples show that every compact Riemann surface is the underlying Riemann surface of some complex projective structure.

Example 1.2 (classical Kleinian groups) Let \(\Omega\) be a finitely generated discrete subgroup of \(\text{PGL}_2(\mathbb{C})\) without torsion, acting freely and properly discontinuously on some non-empty open subset of \(\mathbb{P}^1(\mathbb{C})\). There is a unique maximal such open subset \(U\), called the discontinuity set of \(\Omega\): it is a union of finitely many connected open subsets, each called a component. The complementary subset \(L = \mathbb{P}^1(\mathbb{C}) \setminus U\) is called the limit set of \(\Omega\). Let \(U^0\) be a component of the discontinuity set, then \(X = U^0/\Omega\) is a Riemann surface with a complex projective structure induced by the covering map \(U^0 \to X\). If the limit set \(L\) is a round circle, then \(\Omega\) is called a Fuchsian group. Poincaré-Koebe’s uniformization theorem says that every hyperbolic Riemann surface is the quotient by some Fuchsian group, thus has a complex projective structure; this unique determined complex projective structure will be called Fuchsian. If \(L\) is a Jordan curve, then it is called a quasi-circle and \(\Omega\) is called a quasi-Fuchsian group. Quasi-Fuchsian groups are obtained as deformations of Fuchsian groups. In general \(\Omega\) is called a Kleinian group and \(L\) can be very complicated.

Definition/Proposition 1.3 Let \(X\) be a Riemann surface with a complex projective structure. Denote by \(\tilde{X}\) the universal cover of \(X\) and \(\pi\) the quotient map. There exist a homomorphism \(\text{Hol} : \pi_1(X) \to \text{PGL}_2(\mathbb{C})\) and a \(\pi_1(X)\)-equivariant holomorphic map \(\text{Dev} : \tilde{V} \to \mathbb{P}^1(\mathbb{C})\) such that

\[
\forall \gamma \in \pi_1(X), \text{Dev} \circ \gamma = \text{Hol}(\gamma) \circ \text{Dev}.
\]

If \((\text{Hol}', \text{Dev}')\) is another such pair, then there exists \(\sigma \in \text{PGL}_2(\mathbb{C})\) such that \(\text{Hol}' = \sigma \text{Hol} \sigma^{-1}\) and \(\text{Dev}' = \sigma \circ \text{Dev}\). The morphism \(\text{Hol}\) is called holonomy.
representation and Dev is called developping map. A complex projective structure is determined by its holonomy representation and developping map (up to composition).

A complex projective structure on \( \mathbb{P}^1(\mathbb{C}) \) gives rise to a developping map \( \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \) which is locally biholomorphic thus must be an automorphism. This proves that the obvious complex projective structure on \( \mathbb{P}^1(\mathbb{C}) \) is the unique one. With a little bit more work, we can prove that a complex projective structure on an elliptic curve is always reduced to a complex affine structure, and that a complex projective structure on a compact hyperbolic Riemann surface is never reduced to an affine one. Furthermore the monodromy representation associated with a complex projective structure on a compact hyperbolic Riemann surface always lifts to a representation into \( \text{SL}_2(\mathbb{C}) \) (see [Gun67] for these assertions).

2 Schwarzian derivative

Let \( f(z) \) be a holomorphic function with nowhere vanishing derivative defined in a domain \( D \subset \mathbb{C} \). The Schwarzian derivative of \( f \) is the holomorphic function

\[
S(f; z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

The following properties are easy to prove and can be found in any reference on Schwarzian derivative.

Proposition 2.1

- Under a change of variables \( z = g(t) \), we have \( S(f(g); t)(dt)^2 = S(f; z)(dz)^2 + S(g; t)(dt)^2 \).
- \( S(z; f)(df)^2 = -S(f; z)(dz)^2 \).
- \( S \left(f, \frac{az+b}{cz+d} \right) \frac{(ad-bc)^2}{(cz+d)^2} = S(f; z) \).
- Let \( q(z) \) be a holomorphic function of \( z \). Then any solution \( f \) of the Schwarzian differential equation \( S(f; z) = q(z) \) equals to \( g_1/g_2 \) where \( (g_1, g_2) \) is a pair of independent solutions of the linear differential equation \( g'' + \frac{2}{z}g = 0 \). Conversely if \( (g_1, g_2) \) is a pair of independent solutions of \( g'' + \frac{2}{z}g = 0 \), then \( g_1/g_2 \) is a solution of the Schwarzian equation.
- \( S(f; z) = 0 \) if and only if \( f(z) = \frac{az+b}{cz+d} \).

The third property says that the Schwarzian derivative transforms as a quadratic differential under fractional-linear transformation of the domain. The fourth property implies in particular that a Schwarzian equation always has a solution and that the solution is unique up to post-composition by a fractional-linear transformation. The fifth property is a consequence of the fourth and it means
that in some sense the Schwarzian derivative measures to which extent a function is different from being fractional-linear.

For later use we now perform some seemingly contoured computations that lead to the mysterious expression of Schwarzian derivative. We expand \( \frac{f(x) - f(y)}{x - y} \) as a series in \( x_1 = x - z, y_1 = y - z \):

\[
\frac{f(x) - f(y)}{x - y} = f'(z) + \frac{1}{2}(x_1 + y_1) + \frac{1}{6}f'''(z)(x_1^2 + x_1y_1 + y_1^2) + \ldots.
\]

We expand also \( \log \left( \frac{f(x) - f(y)}{x - y} \right) \):

\[
\log \left( \frac{f(x) - f(y)}{x - y} \right) = \log \left( f'(z) \right) + \frac{f'''(z)}{2f'(z)}(x_1 + y_1) +
\]

\[
+ \left( \frac{f'''(z)}{6f'(z)} \right) \left( \frac{1}{8} \left( \frac{f''(z)}{f'(z)} \right)^2 \right) (x_1^2 + y_1^2) - \left( \frac{f'''(z)}{6f'(z)} \right) - \frac{1}{4} \left( \frac{f''(z)}{f'(z)} \right)^2 \right) x_1y_1 + \ldots \tag{1}
\]

We apply \( \frac{\partial^2}{\partial x \partial y} \) to both sides of Equation (1):

\[
\frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2} = \frac{f'''(z)}{6f'(z)} - \frac{1}{4} \left( \frac{f''(z)}{f'(z)} \right)^2 + R(x_1, y_1), \tag{2}
\]

where \( R(x_1, y_1) \) is a sum of terms in \( x_1, y_1 \) of degree \( > 0 \). We denote by \( p = f(x), q = f(y) \) the target variables and by \( S(p, q; x, y) \) the symmetric expression

\[
S(p, q; x, y) = \frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2}. \tag{3}
\]

If \( x = g(v), y = g(w) \) are themselves the target variables of a holomorphic function \( g \), then we have the additional formula:

\[
S(p, q; v, w)dvdw = S(p, q; x, y)dxdy + S(x, y; v, w)dvdw. \tag{4}
\]

Let \( x = y = z \) in Equation (3), we get the Schwarzian derivative:

\[
S(p, p; x, x) = \frac{1}{6} \left( \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right) = \frac{1}{6} \left( \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right) = \frac{1}{6}S(f; z). \tag{5}
\]

3 Projective connections

Definition 3.1 Let \( X \) be a Riemann surface and let \( U_i = \{U_i, z_i\}_{i \in I} \) be an atlas of holomorphic coordinates. A projective connection on \( X \) with respect to \( U_i \) is a collection of holomorphic functions \( \{h_i\} \) such that on each \( U_i \cap U_j \) we have

\[
S(z_i; z_j) = h_i \left( \frac{dz_i}{dz_j} \right)^2 - h_j. \tag{6}
\]
For \( \{ h_i \} \) and \( \{ h'_i \} \) two projective connections with respect to \((U)_j\), their difference is a quadratic differential on \( X \) because \((h_i - h'_i)(dz_i)^2 = (h_j - h'_j)(dz_j)^2\) on \( U_i \cap U_j \).

The set of projective connections on a Riemann surface \( X \)

\[ 2.1 \]

shows that projective connections on \( X \) have a special form. The Schwarzian derivatives \( S(f; z_i) = -h_i \) are nowhere vanishing derivative so that we can assume it is injective up to shrinking \( U_i \).

Then the new coordinates \( \{ U_i, g_i \circ z_i \} \) define the same complex structure on \( X \).

The Schwarzian derivatives \( S(g_i \circ z_i; g_j \circ z_j) \) are easily seen to be zero by Proposition 2.1. This implies, again by Proposition 2.1, that \( \{ U_i, g_i \circ z_i \} \) is an atlas of complex projective structure. A different choice of atlas or a different collection of solutions \( g_i' \) would define the same complex projective structure.

Conversely if \( \{ V_j, f_j \} \) is an atlas of complex projective structure, then for any atlas \( \{ U_i \} \) of holomorphic coordinates, \( \{ V_j \cap U_i, S(f_j; z_i) \} \) defines a projective connection.

Because the difference of two projective connections is a quadratic differential, i.e., a holomorphic section of the line bundle \( K_X^2 \), the square of the canonical bundle (bundle of differential forms) of \( X \), and because we know the existence of at least one complex projective structure on each Riemann surface by uniformization, we have the following:

Proposition 3.4 The set of projective connections on a Riemann surface \( X \), or the set of complex projective structures on \( X \), has a structure of principal homogeneous space over \( \text{H}^0(X, K_X^2) \).

Let \( \{ U_i, z_i \} \) be an atlas of holomorphic coordinates. The collection \( \{ S(z_i; z_j) \} \) defines a cocycle with values in \( K_X^2 \) because \( S(z_i; z_k) = S(z_i; z_j) \left( \frac{dz_j}{dz_i} \right)^2 + S(z_j; z_k) \) by Proposition 2.1. If \( X \) is a compact hyperbolic Riemann surface, then \( \text{H}^1(X, K_X^2) = \{ 0 \} \) so that the cocyle \( \{ S(z_i; z_j) \} \) is necessarily exact, that is, there is a projective connection \( \{ h_i \} \) such that \( S(z_i; z_j) = h_i \left( \frac{dz_j}{dz_i} \right)^2 - h_j \). This gives a non-constructive proof that every compact hyperbolic Riemann surface admits a complex projective structure. Proposition 3.4 shows that projective connections on \( X \) form an affine space for the vector space \( \text{H}^0(X, K_X^2) \); the complex dimension is \( 3g - 3 \) provided with \( g > 1 \) by Riemann-Roch’s Theorem.

Definition 3.5 Let \( X \) be a compact Riemann surface. A meromorphic 2-form \( \omega(x, y) \) on \( X \times X \) is called a bidifferential of the second kind if \( \omega(x_0, y) \) for all \( x_0 \in X \).
gives a 1-form on $X$ with a single pole of order two at $x_0$, and if the same things holds if we fix the second variable. It is called symmetric if $\omega(x, y) = \omega(y, x)$.

We denote by $S$ the product surface $X \times X$, and by $\Delta$ the diagonal divisor in $S$. The canonical bundle of $S$ is $K_S = p_1^*K_X \otimes p_2^*K_X$ where $p_1, p_2$ are projections onto the two factors. A bidifferential of the second kind $\omega$ is a section of the line bundle $K_S(2\Delta)$. In local coordinates around the diagonal $\omega$ has the form

$$\frac{\alpha dx dy}{(x - y)^2} + H(x, y) dx dy,$$

where $\alpha \in \mathbb{C}^*$ and $H(x, y)$ is holomorphic. The complex number $\alpha$ does not depend on the coordinate and is called the biresidue of $\omega$. We will see in Section 3.7 that there always exists a symmetric bidifferential of second kind with biresidue 1.

Proposition 3.6 Let $\omega$ be a bidifferential of the second kind with biresidue 1 and let $\{H_i(x, y)\}$ be the collection of its regular part in local coordinates as in Section 3.7. The functions $h_i^\alpha(z) = -6H_i(z, z)$ form a projective connection on $X$.

Proof Let $x = g(v), y = g(w)$ be a change of coordinates. Then using Formula (6) we can write

$$H_j(v, w) dv dw = H_i(x, y) dx dy + S(x, y; v, w) dv dw.$$ 

Putting $v = w = z_j$ and $x = y = z_i$, by Equation (6) we get $h_i^\alpha(z_j) dz_j^2 = h_i^\alpha(z_i) dz_i^2 + S(z_i; z_j) dz_j^2$, which is the definition of a projective connection. \(\square\)

Corollary 3.7 Two bidifferentials of the second kind with biresidue 1 define the same projective connections if and only if their difference vanishes on the diagonal $\Delta$.

Let us remark that, in order to get a projective connection, we only need the bidifferential $\omega \in H^0(S, K_S(2\Delta))$ to be defined locally in a neighborhood of $\Delta$. In fact by Corollary 3.7 we only need $\omega$ to be some section of the line bundle $K_S(2\Delta)$ on the scheme-theoretic infinitesimal neighborhood $3\Delta$. Such a nowhere non-vanishing section defines a trivialisation of $K_S(2\Delta)$ on $3\Delta$ while the condition that the bidifferential is of second kind with biresidue 1 means that when restricted to $2\Delta$ the trivialisation is a fixed one given by the term $1/(x - y)^2$ in (7). From the exact sequence

$$0 \to K_X^2 \to K_S(2\Delta) \to K_S(2\Delta)|_{3\Delta} \to K_S(2\Delta)|_{2\Delta} \to 0$$

we see that the set of trivialisations of $K_S(2\Delta)|_{3\Delta}$ which induce a fixed trivialisation of $K_S(2\Delta)|_{2\Delta}$ is an affine space for $H^0(X, K_X^2)$. The above discussion thus shows

Proposition 3.8 (Biswas-Raina [BR96]) The affine space of all trivialisations of $K_S(2\Delta)|_{3\Delta}$ which on restriction to $2\Delta$ give the trivialisation corresponding to $dx dy/(x - y)^2$ is canonically isomorphic to the affine space of complex projective structures on $X$.  

6
Remark 3.9 A crucial point in the above discussion is the existence of a canonical bidifferential of the second kind with biresidue 1, which gives the canonical trivialisation \( dx dy / (x - y)^2 \) over \( 2\Delta \). This is non-trivial and will be shown in Section 5.

The difference of two bidifferentials of the second kind with biresidue 1 is a section of \( K_S \) and the space of bidifferentials of the second kind with biresidue 1 on \( S \) is an affine space for \( H^0(S, K_S) \). The trivialisations of \( K_S(2\Delta)|_{2\Delta} \) coming from global bidifferentials of the second kind with biresidue 1 form an affine space for the image vector space of the restriction map

\[
H^0(X, K_X)^2 = H^0(S, K_S) \to H^0(\Delta, K_S|_{2\Delta}) = H^0(X, K_2^2).
\]

This restriction map can be identified with the product map from \( H^0(X, K_X)^2 \) to \( H^0(X, K_2^2) \), which is surjective if \( X \) is non-hyperelliptic by Noether’s Theorem (see [GH78]). Thus we obtain

Proposition 3.10 (Tyurin [Tyu78]) If \( X \) is a non-hyperelliptic compact Riemann surface, then every complex projective structure on \( X \) is induced by a bidifferential of the second kind with biresidue 1.

4 Theta functions

Let \( A = C^g / (Id_m Z^g + \tau Z^g) \) be a principally polarized abelian variety, where \( \tau \) is a symmetric \( g \times g \)-matrix with positive definite imaginary part, that is, a point in the Siegel half space \( \mathcal{H}_g \). Riemann’s theta function is the following holomorphic function on \( C^g \times \mathcal{H}_g ^{2} \):

\[
\vartheta(z, \tau) = \sum_{m \in \mathbb{Z}^g} \exp(i\pi m^\top \tau m + 2i\pi m^\top z).
\]

For \( a, b \in \mathbb{R}^g \), the theta function with characteristics \( \begin{bmatrix} a \\ b \end{bmatrix} \) is

\[
\vartheta \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) (z, \tau) = \sum_{m \in \mathbb{Z}^g} \exp[i\pi (m + a)^\top \tau (m + a) + 2i\pi (m + a)^\top (z + b)].
\]

Then \( \vartheta(z, \tau) = \vartheta \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) (z, \tau) \). For \( u, v \in \mathbb{Z}^g \), we have

\[
\vartheta \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) (z + v + \tau u, \tau) = \exp[2i\pi (a^\top v - u^\top (z + b))] \vartheta \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) (z + b + \tau a, \tau)
\]

\[
\vartheta \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) (z + v + \tau u, \tau) = \exp[2i\pi (a^\top v - u^\top (z + b) - i\pi u^\top \tau u)] \vartheta \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) (z, \tau)
\]

For fixed \( \tau \), the theta function \( \vartheta \) is a multi-valued holomorphic function on the abelian variety \( A \) which can be viewed as a section of some line bundle \( L_\vartheta \) on
A. The zero divisor $\Theta$ of $\vartheta$ is well defined on $A$ and is called the theta divisor; we have $L_\Theta = \mathcal{O}_A(\Theta)$. The multi-valued function $\vartheta \begin{pmatrix} a \\ b \end{pmatrix} (z, \tau)$ is a section of $L_\Theta$ translated by $\tau a + b$. We refer to [Mum83] for the following transformation formula for theta functions when we transform $\tau$ into $\tau' = (A\tau + B)(C\tau + D)^{-1}$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathbb{Z})$:

$$\vartheta \begin{pmatrix} a' \\ b' \end{pmatrix} (Mz, \tau') = \kappa_4 (\text{det} M)^{1/2} \exp \left( \frac{1}{2} \sum_{j,k} \frac{z_j z_k}{z} \frac{\partial \log \text{det} M}{\partial \tau_{jk}} \right) \vartheta \begin{pmatrix} a \\ b \end{pmatrix} (z, \tau)$$  \hspace{1cm} (8)

with $\kappa_4 \in \mathbb{C}^*$, $M = C\tau + D$ and

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \text{diagonal}(CD^T) \\ \text{diagonal}(AB^T) \end{pmatrix}$$  \hspace{1cm} (9)

where diagonal takes the diagonal of a square matrix as a column vector.

We are particularly interested in those characteristics $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \frac{1}{2} \mathbb{Z}^g$. They correspond to 2-torsion points on $A$ and are called half-period characteristics. We will denote the $4^g$ half-period characteristics by $\delta_1, \ldots, \delta_{4g}$, and the corresponding theta functions by $\vartheta[\delta]$. A theta function with half-period characteristics is either even or odd; a half-period is called even or odd if the corresponding function is so. There are $2^g(2^g + 1)$ even half-periods and $2^g(2^g - 1)$ odd ones. We will need the following two embedding theorems concerning the theta functions with half-period characteristics, that we state without giving proofs. Basically the two theorems say that either for fixed $z$ or for fixed $\tau$, the theta functions with half-period characteristics form a very ample linear system.

Theorem 4.1 (Lefschetz embedding theorem [Mum83]) The map

$$A \to \mathbb{P}^{4g-1}(\mathbb{C}), z \mapsto [\vartheta[\delta_1](4z, 4\tau); \cdots; \vartheta[\delta_{4g}](4z, 4\tau)]$$

is an embedding.

Theorem 4.2 (Jun-Ichi Igusa [Igu72]) The map

$$\mathcal{H}_g \to \mathbb{P}^{4g-1}(\mathbb{C}), \tau \mapsto [\vartheta[\delta_1](0, \tau); \cdots; \vartheta[\delta_{4g}](0, \tau)]$$

induces an embedding from $\mathcal{H}_g / \Gamma$ into $\mathbb{P}^{4g-1}(\mathbb{C})$. Here $\Gamma$ is a non-principal congruence subgroup of $\text{Sp}_2(\mathbb{Z})$ so that $\mathcal{H}_g / \Gamma$ is a finite cover of the moduli space of principally polarized abelian varieties.

Now let $X$ be a compact Riemann surface of genus $g > 0$. Fix a Torelli marking on $X$, that is, a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $H_1(X, \mathbb{Z})$ such that the intersection matrix has the form $\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ in this base. Let $v_1, \ldots, v_g$ be a basis of $H^0(X, K_X)$ such that the period matrix with respect to $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$
is \((Id_{g}, \tau)\) with \(\tau \in \mathcal{H}_g\). The Jacobian variety \(J(X)\) is the principally polarized abelian variety \(C^g/(Id_{g}Z^g + \tau Z^g)\) which is identified with the group of divisors on \(X\) of degree 0 modulo principal divisors, and also with the group of line bundles on \(X\) of degree 0. For \(d \in \mathbb{Z}\), denote by \(J_d(X)\) the set of line bundles on \(X\) of degree \(d\): it is a principal homogeneous space for \(J(X)\). For a fixed \(x \in X\) the Abel-Jacobi map from \(X\) to \(J(X)\) is \(y \mapsto y - x\). We can pull back theta functions on \(J(X)\) to get multi-valued functions on \(X\). We refer to [GH78] for the proof of the following important theorem:

**Theorem 4.3** (Riemann) There is a divisor class \(\Xi \in J_{g-1}(X)\) such that \(2\Xi = K_X \in J_{2g-2}(X)\), and we have for any \(x \in X\), \(e \in J(X)\):

1. If \(\vartheta(e, \tau) \neq 0\), then the zero divisor of \(\vartheta([y-x]-e, \tau)\), as a function in \(y\), is an effective divisor \(C\) of degree \(g\) on \(X\) such that \(\dim H^0(X, \mathcal{O}(C)) = 1\) and \(e = [C-x] - \Xi\).

2. \(\vartheta(e, \tau) = 0\) if and only if there exists an effective divisor \(D\) of degree \(g-1\) such that \(e = [D] - \Xi\).

Furthermore \(e\) is a smooth point of the theta divisor \(\Theta\) if and only if \(\dim H^0(X, \mathcal{O}(D)) = 1\). If \(\dim H^0(X, \mathcal{O}(D)) = 1\) and \(\dim H^0(X, \mathcal{O}(D+x)) = 1\), then the divisor of \(\vartheta([y-x]-e, \tau)\) is \(x + D\); otherwise \(\vartheta([y-x]-e, \tau)\) vanishes identically on \(X\).

**Corollary 4.4** Let \(\delta \in J(X)\) be an odd half-period characteristic. There exists a divisor \(D_{\delta}\) such that \(\delta = [D_{\delta}] - \Xi\) and \([2D_{\delta}] = K_X\).

**Proof** The odd characteristic \(\delta\) is on the theta divisor because \(\vartheta(\delta, \tau) = \vartheta(\delta)(0, \tau) = 0\).

By Theorem 4.1 there exists at least one odd half-period characteristic \(\delta\) such that \(d\vartheta(\delta)_{|z=0} \neq 0\). This means that \(d\vartheta|_{z=0} \neq 0\), that is, \(\delta\) is a non-singular point of the theta divisor.

**Proposition 4.5** Let \(\delta \in J(X)\) be a non-singular odd half-period characteristic and \(D_{\delta}\) the corresponding divisor. Then \(2D_{\delta}\) is the divisor of the holomorphic differential

\[
\omega_{\delta} = \sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}(\delta, \tau) v_j.
\]

**Proof** By Theorem 4.3 \(\vartheta([D] - \Xi)\) vanishes for all effective divisor \(D = x_1 + \cdots + x_{g-1}\). For all \(k\) differentiating with respect to \(x_k\) we get

\[
\sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}([D] - \Xi, \tau) v_j(x_k) = 0.
\]
Putting \([D_\delta] - \Xi = \delta\) in the equality we deduce that \(\omega_\delta\) vanishes on \(D_\delta\). By Theorem 4.3 and the fact that \(\delta\) is non-singular, we have \(\dim H^0(X, \mathcal{O}(D_\delta)) = 1\). By Riemann-Roch’s Theorem, we have \(\dim H^0(X, \mathcal{O}(K_X - D_\delta)) = 1\). This means that the divisor \(D_\delta\) does not move in a linear system and that up to multiplication by a constant \(\omega_\delta\) is the only holomorphic differential vanishing on \(D_\delta\). The conclusion follows.

\[\vartheta[\delta)((y-x), \tau)\] 

Definition 4.6 (John Fay) Let \(\delta\) be a non-singular odd half-period. Let \(r_\delta\) be the section of \(\mathcal{O}(D_\delta)\) such that \(r_\delta^2 = \omega_\delta\). The prime form is the following multi-valued function on \(X \times X\):

\[E(x,y) = \frac{\vartheta[\delta][\left[(y-x), \tau\right]}{r_\delta(x)r_\delta(y)}\]

which is a section of the line bundle \(p_1^*\mathcal{O}(D_\delta)^{-1} \otimes p_2^*\mathcal{O}(D_\delta)^{-1} \otimes \xi^*(L_\delta)\) where \(\xi\) is the map from \(X \times X\) to \(J(X)\) sending \((x,y)\) to \([y-x]\).

Proposition 4.7

1. \(E(x,y) = -E(y,x)\).
2. The divisor of \(E\) is the diagonal \(\Delta\).
3. The multi-valued function \(E\) is invariant under cycles \(L_1, \cdots, L_g\); along \(\beta_k\) it transforms as

\[E(\beta_k(x), y) = \exp(-i\pi \tau_{2k} - 2i\pi \int_x^y \nu_k)E(x,y).\]

4. It does not depend on the non-singular odd characteristic \(\delta\).

5. For \(x_1, \cdots, x_n, y_1, \cdots, y_n \in X\), the divisor of the meromorphic function \(\prod_{j=1}^n \frac{E(x,y)}{E(x_j, y)}\) is the divisor \(\sum_{j=1}^n y_j - \sum_{j=1}^n x_j\).

Proof When \(x\) tends to \(y\), \(\vartheta[\delta][\left[(y-x), \tau\right]\) is equivalent to \(\vartheta(\delta - [y-x])\). By Theorem 4.3 the divisor of the latter are the diagonal \(\Delta\) and also the \(\{x_j\} \times X, X \times \{x_j\}\) where \(\sum_{j=1}^{n-1} x_j = D_\delta\). However \(r_\delta(x)\) vanishes exactly on \(D_\delta\) so that for \(E\) the only remaining zero is \(\Delta\). This proves the second assertion.

The first assertion holds because \(\vartheta[\delta]\) is an odd function. The third assertion follows from the transformation formulas for theta functions. The fourth assertion follows from the third one. The fifth follows from the second one.

5 Bergman and Wirtinger projective connections

Though the prime form \(E(x,y)\) is only a multi-valued function on \(X \times X\), the partial derivative

\[\omega_f(x,y) = \frac{\partial^2}{\partial x \partial y} \log E(x,y)dxdy\]
is a well-defined meromorphic differential on $X \times X$, by the third formula in Proposition 4.7. It is also equal to for any non-singular point $e \in \Theta$ (cf. [Fay73]):

$$\frac{\partial^2}{\partial x \partial y} \log \theta([y-x] - e, \tau) dx dy.$$

We call $\omega_f$ the fundamental bidifferential; such bidifferentials appeared already in [Kle90]. Moreover the Proposition 4.7 implies:

Proposition 5.1 $\omega_f(x,y)$ is a symmetric bidifferential of the second kind with biresidue 1. For fixed $x \in X$ and all $j$

$$\int_{a_j} \omega_f(x,y) = 0 \quad \text{and} \quad \int_{\beta_j} \omega_f(x,y) = v_j(x).$$

Remark that $\omega_f$ depends on the period matrix $\tau$, that is, it depends on the Torelli marking fixed on $X$. If we change the Torelli marking by a matrix $\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z})$, then $\omega_f$ becomes

$$\omega'_f(x,y) = \omega_f(x,y) - \frac{1}{2} \sum_{j \leq k} \frac{\partial}{\partial \tau_{jk}} \log \det(C \tau + D)[v_j(x)v_k(y) + v_k(x)v_j(y)], \quad (10)$$

which follows from the corresponding transformation formulas for theta functions (see Equation (8)).

By Proposition 3.6, the fundamental bidifferential gives a projective connection $h_B$ on $X$; the projective connection $h_B$ is called the Bergman projective connection. We refer to [Fay73] for the following explicit expression of $h_B$ that we do not need in the sequel:

$$h_B(z)dz^2 = S \left( \int_{\delta_0} T_1^\tau(z) dz^2 + \frac{3}{2} \left( \frac{T_2^\tau}{T_1^\tau} \right)^2 (z) - 2 \frac{T_3^\tau}{T_1^\tau} (z) \right)$$

where $e$ is an arbitrary non-singular point of the theta divisor and

$$T_1^\tau(z) = \sum_{j=1}^g \frac{\partial}{\partial z_j}(e)v_j(z)$$

$$T_2^\tau(z) = \sum_{j,k=1}^g \frac{\partial^2}{\partial z_j \partial z_k}(e)v_j(z)v_k(z)$$

$$T_3^\tau(z) = \sum_{j,k,l=1}^g \frac{\partial^3}{\partial z_j \partial z_k \partial z_l}(e)v_j(z)v_k(z)v_l(z).$$

The Bergman projective connection $h_B$ depends on the Torelli marking too; the corresponding transformation formula follows from Equation (10):

$$h'_B(z) = h_B(z) - \frac{1}{2} \sum_{j,k=1}^g v_j(z)v_k(z) \frac{\partial}{\partial \tau_{jk}} \log \det(C \tau + D) \quad (11)$$
Formula (9) gives an action of $\text{Sp}_{2g}(\mathbb{Z})$ on the set of $4^g$ half-period characteristics. For $\delta$ a half-period characteristic, we denote by $\Gamma_\delta$ the subgroup of $\text{Sp}_{2g}(\mathbb{Z})$ fixing $\delta$. It follows from Equations (8) and (11) that

$$h_\delta = h_B + \sum_{j,k=1}^g \left( \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \vartheta[\delta](0, \tau) \right) v_j v_k$$

is a projective connection invariant under changes of marking by $\Gamma_\delta$. We call $h_\delta$ a partial Wirtinger projective connection; it is only defined for even $\delta$ such that $\vartheta[\delta](0, \tau) \neq 0$. Since the partial derivatives of an even function vanish at $z = 0$, we obtain by Equation (8) the following expression of the difference between two partial Wirtinger connections:

$$h_\delta - h_{\delta'} = \sum_{j,k=1}^g v_j v_k \left( \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \vartheta[\delta](0, \tau) - \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \vartheta[\delta'](0, \tau) \right)$$

(12)

The Wirtinger projective connection (firstly appeared in [Wir44]) is the following connection invariant under $\text{Sp}_{2g}(\mathbb{Z})$:

$$h_W = h_B + \frac{2}{4^g + 2^g} \sum_{\delta \text{ even}} \left( \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \left( \prod_{\text{even } \delta} \vartheta[\delta](0, \tau) \right) \right) v_j v_k$$

$$= \frac{2}{4^g + 2^g} \sum_{\delta \text{ even}} h_\delta;$$

it is only defined on those Riemann surfaces $X$ for which $\vartheta[\delta](0,\tau) \neq 0$ for all even $\delta$.

Example: elliptic curves. Let us consider the case where $z, \tau \in \mathbb{C}$ and $\Re \tau > 0$. Let $E = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ be the corresponding elliptic curve. There are four theta functions with half-period characteristics:

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) = \vartheta(z, \tau) = \sum_{j \in \mathbb{Z}} \exp(i \pi j^2 \tau + 2i \pi j z)$$

and $\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (z, \tau), \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z, \tau), \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau)$. Each theta function has exactly one zero on $E$. The zeros of $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ are respectively $\frac{1}{2} + \frac{\tau}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ and 0. The only odd half-period is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. We have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau) = \varphi(z) + \epsilon$$
where $\wp$ is the Weierstrass function and $\varepsilon \in \mathbb{C}$ is a constant. The fundamental bidifferential is

$$\frac{\partial^2}{\partial x \partial y} \log \vartheta([y-x], \tau) dxdy = (\wp(y-x) + \varepsilon) dxdy.$$ 

A computation using formulas of theta functions shows that the bidifferential which gives the invariant Wirtinger connection is

$$\wp(x-y) dxdy.$$ 

Since $\wp(z) = \frac{1}{z^2} + O(z^2)$, we see by Corollary 3.7 that the invariant Wirtinger connection is trivial with respect to the original coordinate on $E$, that is, the complex projective structure determined by the invariant Wirtinger connection is just the natural complex affine structure on $E$.

6 Moduli space of projective connections

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a fibration between complex manifolds whose set-theoretic fibers are compact Riemann surfaces. For $b \in \mathcal{B}$ we denote by $X_b$ the fiber. We want to define how complex projective structures vary in a holomorphic way when the underlying complex structures vary.

Definition 6.1

- A relative complex projective structure on $\mathcal{X}$ is a maximal relative atlas $\{U_i, w_i\}$ where the $U_i$ form an open cover of $\mathcal{X}$ and the $w_i : U_i \to \mathbb{P}^1(\mathbb{C})$ are holomorphic maps which are biholomorphism onto their images when restricted to fibers of $\pi$, such that for each $b \in \mathcal{B}$, the fiber restriction $\{U_i|_b, w_i|_b\}$ is an atlas of complex projective structure on $X_b$.

- A relative projective connection on $\mathcal{X}$ is given by a collection of local holomorphic functions $H_i$ so that when restricted to $X_b$ for all $b \in \mathcal{X}$ they form a projective connection on $X_b$.

Definition/Proposition 6.2 (Hubbard [Hub81]) Let $\mathcal{P}_\mathcal{B}(\mathcal{X})$ bet the set of pairs $(b, h)$ where $b \in \mathcal{B}$ and $h$ is a projective connection on $X_b$. Then there is a unique structure of complex manifold on $\mathcal{P}_\mathcal{B}(\mathcal{X})$ such that

- the projection $\rho : \mathcal{P}_\mathcal{B}(\mathcal{X}) \to \mathcal{B}, (b, h) \mapsto b$ is holomorphic;

- relative projective connections on $\mathcal{X}$ correspond to holomorphic sections of $\rho$;

- the action of the bundle of quadratic differentials $\mathcal{Q}_B(\mathcal{X})$ on $\mathcal{P}_\mathcal{B}(\mathcal{X})$ given by $((b, q), (b, h)) \mapsto (b, h+q)$ makes $\mathcal{P}_\mathcal{B}(\mathcal{X})$ into a holomorphic affine bundle for the vector bundle $\mathcal{Q}_B(\mathcal{X})$.
The Bergman projective connections on the fibers vary holomorphically in coordinates and give a relative projective connection, at least locally over some open sets of $B$. Then the action of $\mathcal{D}_{B}(X)$ on $\mathcal{P}_{B}(X)$ can be used to transport locally the complex structure on $\mathcal{D}_{B}(X)$ to $\mathcal{P}_{B}(X)$. The proofs of other statements are left to the reader.

An affine bundle is determined by the corresponding vector bundle and a class in the first cohomology group of the vector bundle; we denote by $\zeta_{X} \in H^{1}(\mathcal{B}, \mathcal{D}_{B}(X))$ the cohomology class determining $\mathcal{P}_{B}(X)$. A representative cocycle can be constructed as follows: let $\{V_{i}\}$ be an open covering of $B$ such that for all $i$ there is a holomorphic section $s_{i}$ of $\mathcal{P}_{B}(X)$ over $U_{i}$, then $\{s_{i} - s_{j}\}$ is the desired cocycle. Thus we have

Proposition 6.3 $X$ has a relative complex projective structure if and only if $\zeta_{X} = 0$.

Given a relative projective structure on $X$ and a section of $\pi$, there are a holomorphic map $D$ from the universal covering of $X$ to $\mathbb{P}^{1}(\mathbb{C})$ and a holomorphic family of representations $r_{b}$ from the fundamental group of a fiber into $\text{PGL}_{2}(\mathbb{C})$ parametrized by the universal covering of $B$, such that $D_{b}, r_{b}$ are the developing map and holonomy representation of the complex projective structure on $X_{b}$.

There is a tautological family over $\mathcal{P}_{B}(X)$ with a tautological relative projective connection where the fiber over $(b, h)$ is $X_{b}$ equipped with the projective connection $h$. This tautological family satisfies a universal property: the map which associates to any holomorphic mapping $f : T \rightarrow \mathcal{P}_{B}(X)$ the pulled-back family of projective connections on the family of Riemann surfaces pulled back by $\pi \circ f$ is a bijection from the set of holomorphic maps from $T$ to $\mathcal{P}_{B}(X)$ onto the set of relative projective connection on $(\pi \circ f)^{*}X$.

We will be interested in the case where $B = \mathcal{M}_{g}$ is the moduli space of compact Riemann surfaces of genus $g$ and $X = \mathcal{X}_{g}$ is the universal curve. We will denote $\mathcal{P}_{\mathcal{M}_{g}}(\mathcal{X}_{g})$ simply by $\mathcal{P}_{g}$; it is an affine bundle for the bundle of quadratic differentials $\mathcal{D}_{g}$ which is identified with the cotangent bundle $\mathcal{T}_{g}$ of $\mathcal{M}_{g}$ via Kodaira-Spencer theory. Actually for $\mathcal{P}_{g}$ we need an orbifold (or stack) version of the above discussion. Though unfortunately the author is unable to find a reference for or work out himself a rigorous presentation in this more general setting, we will deal with $\mathcal{P}_{g}$ as in many other papers, for example $[\text{BKN17}]$. We denote by $\zeta_{g} \in H^{1}(\mathcal{M}_{g}, \mathcal{D}_{g})$ the class determining the affine bundle structure of $\mathcal{P}_{g}$.

7 Determine the cocycle

We prove Theorem 0.1 in this section. Let $\pi : \mathcal{X} \rightarrow C$ be a non-isotrivial family of compact Riemann surfaces of genus $g$ over a compact Riemann surface $C$. The family is induced by a non-constant morphism $f : C \rightarrow \mathcal{M}_{g}$. To prove Theorem 0.1, it suffices to prove $\zeta_{\mathcal{X}} \neq 0$ by Proposition 6.3. The class $\zeta_{\mathcal{X}} \in$
$H^1(C, \mathcal{D}_C) = H^1(C, f^* \mathcal{T}_g^*)$ is obtained by pulling back $\zeta_g \in H^1(\mathcal{M}_g, \mathcal{D}) = H^1(\mathcal{M}_g, \mathcal{T}_g^*)$ by $f$.

If a line bundle $L$ on a variety $Y$ is given by a cocycle $\{\alpha_{ij}\} \in H^1(Y, \mathcal{O}_Y^*)$, then its Chern class $c(L)$ is the element of $H^1(Y, \mathcal{T}_g^*)$ represented by the cocycle $\{\alpha_{ij} \cdot d\alpha_{ij}\}$. Now if $L$ is a line bundle on $\mathcal{M}_g$, we have $f^* c(L) = c(f^* L)$.

Lemma 7.1 If $\zeta_g = dc(L)$ for some non-zero number $d$ and for an ample line bundle $L$ on $\mathcal{M}_g$, then $\zeta_X = 0$.

Proof We have a morphism $\nu: H^1(C, f^* \mathcal{T}_g^*) \to H^1(C, \mathcal{T}_C^*)$ induced by pulling back differentials on $\mathcal{M}_g$ to $C$. We have $\nu(\zeta_X) = dc(f^* L)$. Since $f: C \to \mathcal{M}_g$ is non-constant, $f^* L$ is ample on $C$. As $C$ is compact, the Chern class of an ample line bundle in $H^1(C, \mathcal{T}_C^*) = H^1, 1(C, C)$ is non-zero. Therefore Theorem 0.1 is a consequence of the following proposition

Proposition 7.2 $\zeta_g$ is proportional to $c(L)$ for some ample line bundle $L$ on $\mathcal{M}_g$.

We give two computations which lead to Proposition 7.2. The two are basically the same: the first one is straightforward and the second one is more explicit. We need

Proposition 7.3 (Ahlfors-Rauch Formula \[\text{Ahlf60}\] \[\text{Rau59}\]) Consider the entry $\tau_{jk}$ of the period matrix as a local function on $\mathcal{M}_g$. Under the identification of the cotangent bundle of $\mathcal{M}_g$ with the bundle of quadratic differentials, the differential $d\tau_{jk}$ is the family of quadratic differentials $v_j v_k$.

Using Bergman connections. By Equation 11 the class $\zeta_g$ can be represented by the Cech cocycle $\{\lambda_{\tau\tau'}\}$ where

$$\lambda_{\tau\tau'} = \frac{1}{2} \sum_{j,k=1}^{g} v_j(z)v_k(z) \frac{\partial}{\partial \tau_{jk}} \log \det(C\tau + D).$$

Using Ahlfors-Rauch Formula, we get

$$\lambda_{\tau\tau'} = \frac{1}{2} \frac{d \det(C\tau + D)}{\det(C\tau + D)}$$

and $\zeta_g = \frac{1}{2} c(t^* L)$ where $t: \mathcal{M}_g \to \mathcal{A}_g$ is the Torelli map from $\mathcal{M}_g$ into the moduli space of principally polarized abelian varieties, and $L$ is the line bundle on $\mathcal{A}_g$ represented by the cocycle $\{\det(C\tau + D)\}$ whose sections are Siegel modular forms with weight one half.

Using partial Wirtinger connections. This computation is made by Tyurin in \[\text{Tyu78}\]. Consider the Torelli map from $t: \mathcal{M}_g \to \mathcal{A}_g$. Theorem 4.2 gives an embedding of $\mathcal{A}_g'$, a finite cover of $\mathcal{A}_g$, into some $\mathbb{P}^d$ by using theta constants. Composing this embedding with the Torelli map, we get an injective
morphism $F : \mathcal{M}_g' \to \mathbb{P}^n$ from a finite cover of $\mathcal{M}_g$ into $\mathbb{P}^n$. We will deal with $\zeta'_g \in H^1(\mathcal{M}_g',\mathcal{O}^*\mathcal{M}_g')$, the class pulled back from $\zeta_g$.

Let us consider the line bundle $L = F^*\mathcal{O}(1)$. It is determined by the Cech cocyle $\{\alpha_{\delta\delta'}\}$ where $\delta, \delta'$ are even half-period characteristics and

$$\alpha_{\delta\delta'} = \frac{\vartheta[\delta](0,\tau)}{\vartheta[\delta']}(0,\tau).$$

The Chern class $c(L)$ is represented by the Cech cocyle $\{\alpha_{\delta\delta'}^{-1}d\alpha_{\delta\delta'}\}$ where

$$\alpha_{\delta\delta'}^{-1}d\alpha_{\delta\delta'} = \sum_{j,k=1}^g \frac{\partial}{\partial \tau_{jk}} \left( \log \frac{\vartheta[\delta](0,\tau)}{\vartheta[\delta']}(0,\tau) \right) d\tau_{jk}.$$

The theta functions satisfy the heat equation (see [Mum83]):

$$\frac{\partial}{\partial \tau_{jk}} \vartheta[\delta](0,\tau) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \vartheta[\delta](0,\tau).$$

Therefore

$$\alpha_{\delta\delta'}^{-1}d\alpha_{\delta\delta'} = \sum_{j,k=1}^g \left( \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \vartheta[\delta](0,\tau) - \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \vartheta[\delta'](0,\tau) \right) d\tau_{jk}.$$

Using Ahlfors-Rauch Formula and Equation (12), we see that this is just the cocyle determined by the differences of partial Wirtinger connections. Therefore we have $c(L) = \zeta'_g$.

8 References


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